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Periodicity in the cumulative hierarchy

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Abstract. We investigate the structure of rank-to-rank elementary embeddings at successor rank, working in ZF set theory without the Axiom of Choice. Recall that the set-theoretic universe is naturally stratified by the cumulative hierarchy, whose levels V_α are defined via iterated application of the power set operation, starting from $V_0 = \emptyset$, setting $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and taking unions at limit stages. Assuming that

$$j : V_{\alpha+1} \to V_{\alpha+1}$$

is a (non-trivial) elementary embedding, we show that V_α is fundamentally different from $V_{\alpha+1}$: we show that j is definable from parameters over $V_{\alpha+1}$ iff $\alpha + 1$ is an odd ordinal. The definability is uniform in odd $\alpha + 1$ and j. We also give a characterization of elementary $j : V_{\alpha+2} \to V_{\alpha+2}$ in terms of ultrapower maps via certain ultrafilters.

For limit ordinals λ, we prove that if $j : V_\lambda \to V_\lambda$ is Σ_1-elementary, then j is not definable over V_λ from parameters, and if $\beta < \lambda$ and $j : V_\beta \to V_\lambda$ is fully elementary and $\in$-cofinal, then j is likewise not definable.

If there is a Reinhardt cardinal, then for all sufficiently large ordinals α, there is indeed an elementary $j : V_\alpha \to V_\alpha$, and therefore the cumulative hierarchy is eventually *periodic* (with period 2).

Keywords. Large cardinal, Reinhardt cardinal, rank-to-rank, elementary embedding, definability, periodicity, cumulative hierarchy, Axiom of Choice

1. Introduction

The universe V of all sets is the union of the *cumulative hierarchy* $\langle V_\alpha \rangle_{\alpha \in \mathrm{OR}}$. Here OR denotes the class of all ordinals, and the sets V_α are obtained by iterating the power set operation $X \mapsto \mathcal{P}(X)$ transfinitely, starting with $V_0 = \emptyset$, setting $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and $V_\eta = \bigcup_{\alpha<\eta} V_\alpha$ for limit ordinals η.

Before Cantor's discovery of the transfinite ordinals, mathematicians typically considered only sets within the first few infinite levels of the cumulative hierarchy (below $V_{\omega+5}$

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say). Since then our understanding much higher in the hierarchy has deepened extensively. In a sense to be clarified, however, most research has been focused below a certain threshold, due to its interaction with the Axiom of Choice. This paper investigates certain features of the hierarchy which first appear just beyond this threshold.

After some distance, finite intervals in the cumulative hierarchy have the appearance of uniformity: for large infinite limit ordinals γ and large natural numbers n and m, one might expect not to find natural set-theoretic properties which differentiate between $V_{\gamma+n}$ and $V_{\gamma+m}$: one might expect $V_{\gamma+813}$, for example, to be essentially structurally indistinguishable from $V_{\gamma+814}$. But the key result of this paper shows that assuming γ is *very large* – so large, in fact, that the Axiom of Choice must be violated – $V_{\gamma+813}$ and $V_{\gamma+814}$ display fundamental structural differences. More generally, the properties of $V_{\gamma+n}$ depend on the parity of n.

Exactly how large must γ be for these differences to arise? To answer this question requires introducing some basic concepts from the theory of *large cardinals*, one of the main areas of research in modern set theory. The simplest example of a large cardinal[1] is an *inaccessible cardinal*. An uncountable ordinal κ is inaccessible if every function from V_α to κ where $\alpha < \kappa$ is bounded strictly below κ.[2,3] So inaccessible cardinals are "unreachable from below", and form a natural kind of closure point of the set-theoretic universe. If κ is inaccessible then V_κ models all of the ZF axioms, as does V_α for unboundedly many ordinals $\alpha < \kappa$. So by Gödel's Incompleteness Theorem, inaccessible cardinals cannot be proven to exist in ZF, and inaccessibility somehow "transcends" ZF. (The *Zermelo–Fraenkel* axioms, denoted ZF, are the usual axioms of set theory, without the Axiom of Choice AC. And ZFC denotes ZF augmented with AC.)

Inaccessibles are just the beginning. Further up in the hierarchy, large cardinals are typically exhibited by some form of non-identity *elementary embedding*

$$j : V \to M$$

from the universe V of all sets to some transitive[4] class $M \subseteq V$. *Elementarity* demands that j preserve the truth of all first-order statements in parameters between V and M (see §1.1 for details). One can show that there is an ordinal κ such that $j(\kappa) > \kappa$, and the least such ordinal is called the *critical point* $\mathrm{crit}(j)$ of j; if ZFC[5] holds then such a critical point is known as a *measurable cardinal*. The critical point of an elementary embedding is inaccessible, and in fact there are unboundedly many inaccessible cardinals $\eta < \kappa$. So

[1]There is no general formal definition of "large cardinal".

[2]An ordinal α is formally equal to the set of ordinals $\beta < \alpha$, so if $\pi : X \to \kappa$, then π is bounded strictly below κ iff there is $\alpha < \kappa$ such that $\pi(\beta) < \alpha$ for all $\beta \in X$.

[3]Assuming the Axiom of Choice AC, inaccessibility is usually defined slightly differently, but under AC, the definitions are equivalent. The definition we give here is the appropriate one when one does not assume AC.

[4]That is, for all $x \in M$, we have $x \subseteq M$.

[5]Under ZFC, this notion is equivalent to measurability, but the notions are not equivalent in general under ZF alone.

such critical points transcend inaccessible cardinals. Critical points are transcended by still larger large cardinals.

Large cardinal axioms are some of the most widely accepted and well-studied principles extending the standard axioms of set theory.[6] One of the main reasons for this is the empirical fact that large cardinal axioms are arranged in an essentially linear hierarchy of strength, with each large cardinal notion typically transcending all the preceding ones.[7] There is no known example of a pair of incompatible large cardinal axioms, and the linearity phenomenon suggests that none will ever arise.

The strength of a large cardinal notion $j : V \to M$ depends in large part on the extent to which M resembles V and contains fragments of j. So taking the notion to its logical extreme, William Reinhardt suggested in his doctoral dissertation taking $M = V$; that is, a (non-identity) elementary embedding

$$j : V \to V.$$

The critical point of such an embedding became known as a *Reinhardt cardinal*. But Kunen proved in [12] (see also [6] and [9]) that, under ZFC, such embeddings do not exist. In fact, suppose $j : V \to M$ is elementary where $M \subseteq V$ is a transitive class and j is not the identity. Let $\kappa_0 = \text{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$; then because j is order-preserving on ordinals (an easy consequence of elementarity),

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots .$$

Let their supremum be $\lambda = \sup_{n<\omega} \kappa_n$. Write $\kappa_n(j) = \kappa_n$ and $\kappa_\omega(j) = \lambda$. Kunen proved in [12] (from ZFC) that $V_{\lambda+1} \not\subseteq M$, and in fact that $j``\lambda \notin M$. He also proved that there is no ordinal λ and elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Following Kunen's discovery (and in the primary AC context), set-theorists turned their focus to embeddings just below this level (and continued investigating others further below), with the upper echelons including $j : V_{\lambda+i} \to V_{\lambda+i}$ with $i = 0$ or $i = 1$ (these axioms are known as I_3 and I_1 respectively). Much detailed structure in the hierarchy of large cardinals is now understood, and continuing to be revealed, but because of Kunen's result, AC enforces a rather abrupt upper limit.

Now it has remained a mystery whether AC is actually needed to prove there can be no elementary $j : V \to V$. Suzuki [20] showed in ZF alone that such a j *cannot* be definable from parameters over V. Recall that a *class* is some collection $C \subseteq V$, and j is such. But what exactly is permitted as a class? In the most restrictive formulation, classes are all definable from parameters, so in this setting, Suzuki's result rules out an elementary

[6] An example of a large cardinal axiom is the assertion that there is an inaccessible cardinal or the assertion that there is a critical point cardinal. While there is no formal definition of the term "large cardinal axiom", there is little controversy over which principles qualify as large cardinal axioms.

[7] This is a bit of an oversimplification.

$j : V \to V$ from ZF alone, and the matter is settled (not the $k : V_{\lambda+2} \to V_{\lambda+2}$ matter, although a variant of Suzuki's argument *will* give key information about such k).[8] But one can also formulate classes more generally, and appropriately formulated, there is no known way to disprove the existence of $j : V \to V$ without AC. For the most part in this paper, we focus anyway on embeddings of set size, so the precise definition of classes is not so important for us here.[9]

One can state Kunen's result from a different angle: if $j : V \to V$ is elementary and $\lambda = \kappa_\omega(j)$, then there is a failure of (the axiom of) choice within $V_{\lambda+2}$. In this sense, very strong elementary embeddings limit the extent of validity of choice, and set theory under choice can be seen as focusing on sets inside V_λ, below the threshold where choice breaks down. There is, however, another natural mathematical interpretation which seems reasonable: choice holds (all throughout V), but there are *inner models* $M \subseteq V$ such that M models ZF + "There is an elementary $j : V_{\lambda+2} \to V_{\lambda+2}$", or stronger. The latter is indeed analogous to a common view of the relationship between the Axiom of Determinacy and choice.

In the last few years, there has been growing interest in investigating large cardinals without choice, particularly with notions like $j : V \to V$ and beyond (often augmented with fragments of AC).[10] This paper sits within that line of investigation, just beyond the level which violates choice, focusing on elementary, or at least Σ_1-elementary,[11] embeddings of the form

$$j : V_\alpha \to V_\alpha$$

with α an ordinal. Generalizing some standard terminology, we call these *rank-to-rank* embeddings,[12] because V_α is a *rank initial segment* of V.

We primarily consider the following question, with ZF as background theory. Let α be an ordinal and $j : V_\alpha \to V_\alpha$ be elementary. Is j definable from parameters over V_α? That is, we investigate whether there is $p \in V_\alpha$ and some formula φ in the language of set theory (with binary predicate symbol $\in$ for membership) such that for all $x, y \in V_\alpha$, we have

$$j(x) = y \iff V_\alpha \models \varphi(p, x, y),$$

where $\models$ is the usual model-theoretic truth satisfaction relation.

It turns out that there is a very simple answer to this question, generalizing Suzuki's theorem, but with a twist. We say that an ordinal α is *even* iff $\alpha = \eta + 2n$ for some $n < \omega$, with $\eta = 0$ or η a limit ordinal. Naturally, *odd* means not even.

[8] It will be used to show that k cannot be definable from parameters over $V_{\lambda+2}$ for even λ.

[9] In §1.1 we discuss the theory ZF(A); one can formulate $j : V \to V$ formally in ZF(j).

[10] See for example [1–5, 8, 10, 16, 18–21].

[11] That is, $V_\alpha \models \varphi(\vec{x})$ iff $V_\alpha \models \varphi(j(\vec{x}))$ for all Σ_1 formulas φ and $\vec{x} \in V_\alpha^{<\omega}$.

[12] In the ZFC context, by Kunen's Theorem, the only rank-to-rank embeddings in this strict sense are $k : V_\lambda \to V_\lambda$ or $k : V_{\lambda+1} \to V_{\lambda+1}$ where $\lambda = \lambda(k)$ (his proof does rule out a Σ_1-elementary $k : V_{\lambda+2} \to V_{\lambda+2}$). The I_0 embeddings $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ are also traditionally known as *rank-to-rank* embeddings, even if the terminology does not seem to quite match reality in that case. We adopt the same *rank-to-rank* terminology for Σ_1-elementary $j : V_\alpha \to V_\alpha$ in general because it is very natural.

Theorem 1.1.[13] *Let* $j : V_\alpha \to V_\alpha$ *be fully elementary, with* $j \neq \mathrm{id}$. *Then* j *is definable from parameters over* V_α *iff* α *is odd.*

The proof appears at the end of §3, and then a second, slightly different proof is sketched in Remark 4.8.

So if there is an elementary $j : V_{\eta+184} \to V_{\eta+184}$ (and hence an elementary embedding from $V_{\eta+183}$ to $V_{\eta+183}$, namely $j \upharpoonright V_{\eta+183}$), then $V_{\eta+183}$ and $V_{\eta+184}$ are indeed different (but $V_{\eta+182}$ is analogous to $V_{\eta+184}$, etc.). The proof will also yield much information about such embeddings, and demonstrate strong structural differences between those at even and odd levels. A consequence of Theorem 6.1 will be that if there is a Reinhardt cardinal, and $j : V \to V$ is an elementary embedding witnessing this, then for *all* ordinals $\eta \geq \kappa_\omega(j)$ there is an elementary $j : V_\eta \to V_\eta$, and so this *periodicity* phenomenon holds from λ onward.

One further point should be noted. Periodicity phenomena (with period 2) are of course a familiar feature of logical quantifiers: $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots$ They are pervasive in descriptive set theory (in particular in the Periodicity Theorems, see [15]). But in such cases, which arise in the analysis of complexity classes and so forth arising from quantifier alternation, the periodicity is built into the definitions in the first place. This particular instance of periodicity shows up more subtly in inner model theory, in particular regarding the canonical inner model M_n with n *Woodin cardinals*, where n is finite;[14] Woodin cardinals are beyond measurables, but well below those we consider in this paper. It turned out that n Woodin cardinals corresponds tightly to n alternations of quantifiers over real numbers, and this has the consequence that many important features of M_n depend on the parity of n. However, the basic definition of M_n (and similarly for n measurable cardinals, etc.) does not have any obvious dependence on parity built into it. The periodicity present in Theorem 1.1 is in this sense analogous to the case of M_n. The periodicity in the V_α's also seems to manifest certain "$\forall/\exists$" features, although the full nature of this is probably as of yet not understood.

In §4 we present a different perspective on elementary $j : V_{\alpha+2} \to V_{\alpha+2}$, relating such elementary embeddings to ultrapower embeddings via associated ultrafilters, and sketch the proof of Theorem 1.1 for successor ordinals again, from this new perspective. We also establish a characterization of such j in terms of ultrapower embeddings.[15] The results here also demonstrate that, although $j : V_{\alpha+2} \to V_{\alpha+2}$ is incompatible with AC,

[13] This theorem is also proved in [8], where the theorem is then applied in generalizing Woodin's I_0 theory. In the present paper, we focus on Theorem 1.1 and closely related results, some of which are lemmas toward its proof, and some of which extend it. There is more discussion of those at the end of this introduction.

[14] M_0 is just Gödel's constructible universe L.

[15] There is an important subtlety here. We will identify a certain ultrafilter U and form the ultrapower $U = \mathrm{Ult}(V_{\alpha+2}, U)$, and define $i : V_{\alpha+2} \to U$ to be the ultrapower map. We will show that $i = j$, i.e., these maps have the same graph. If $\alpha + 2$ is even, we will also show $U = V_{\alpha+2}$. But if $\alpha + 2$ is odd, then $U \subsetneq V_{\alpha+2}$.

the existence of such embeddings does actually *imply* certain weaker choice principles (see Remark 4.11).[16]

In §5 we prove some more general results in the limit case; in particular:

Theorem (5.7, 5.9). *Let $\beta \leq \delta$ be limit ordinals and $j : V_\beta \to V_\delta$ be Σ_1-elementary and $\in$-cofinal, and suppose that either $\beta = \delta$, or j is fully elementary. Then j is not definable over V_δ from parameters.*

Note that the $\beta < \delta$ case of this theorem applies to embeddings which are compatible with choice, in fact just around the level of extendible cardinals.

Finally, in §6, we discuss a folklore result: if there is a Reinhardt cardinal, then there is an ordinal λ such that for *every* $\alpha \geq \lambda$, there is an elementary $j : V_\alpha \to V_\alpha$. So above λ, Theorem 1.1 applies, showing that the cumulative hierarchy (and correspondingly, the power set operation) is eventually periodic in nature.

§1.1 and §2 cover background material.

We note some history on the development of the work. The results on the limit case in §3.1 and §5 are due to the second author, and most of that material appeared in the informal notes [17] (Theorems 5.7 (2), and 5.9 came later). The analysis of embeddings $j : V_{\lambda+n} \to V_{\lambda+n}$ for limit λ and $n = 2$ in terms of Reinhardt ultrafilters, in §4, was discovered in some form by the first author in 2017, and he communicated this to the second author shortly after the release of [17]. The first author then discovered Theorem 1.1, and used this to generalize Woodin's I_0-theory to higher levels (see [8]). A few months later, also attempting to generalize the first author's analysis of embeddings to $n > 2$, the second author rediscovered Theorem 1.1. Our two proofs of non-definability in the even successor case (Theorem 3.12) were different; the one we give here is that due to the second author. The original one, due to the first author, can be seen in [8].

1.1. Terminology, notation, basic facts

We will assume the reader is familiar with basic first-order logic and set theory. But much of the material, particularly in the earlier parts of the paper, does not require extensive background in set theory, so we aim to make at least those parts fairly broadly accessible. Therefore we do explain some points in the paper which are standard, and summarize in this section some basic facts for convenience; the reader should refer to texts like [13] for more details.

The language of set theory is the first-order language with the binary relation symbol $\in$.[17] The Zermelo–Fraenkel axioms are denoted by ZF, and ZFC denotes ZF + AC, where AC is the Axiom of Choice. We sometimes discuss ZF($\dot{A}$), where $\dot{A}$ is an extra predicate symbol; this is just like ZF, but in the expanded language with both $\in$ and $\dot{A}$,

[16]This is analogous to the fact that the Axiom of Determinacy, while inconsistent with AC, also implies certain weak choice principles.

[17]This is informal. Formally, the language has a binary relation symbol $\dot{\in}$, and a model of the language is of the form $\mathcal{M} = (M, \in)$, where $\in \subseteq M \times M$ is the interpretation of $\dot{\in}$. Write $\in = \dot{\in}^{\mathcal{M}}$.

and incorporates the Collection and Separation schemata for all formulas in the expanded language. A model of ZF($\dot{A}$) has the form $(V, \in, A)$, abbreviated (V, A), where V is the universe of sets and $A \subseteq V$. In the interesting case, A is not already definable from parameters over V.

We write $\Sigma_0 = \Pi_0 = \Delta_0$ for the class of formulas (in the language of set theory) in which all quantifiers are *bounded*, meaning of the form "$\forall x \in y$" or "$\exists x \in y$". Then Σ_{n+1} formulas are those of the form "$\exists x_1, \ldots, x_n \psi(x_1, \ldots, x_n, \vec{y})$" where ψ is Π_n, and Π_{n+1} formulas are negations of Σ_{n+1}. A relation is Δ_{n+1} if expressed by both Σ_{n+1} and Π_{n+1} formulas.

Given structures $\mathcal{M} = (M, R_1, R_2, \ldots, R_n)$ and $\mathcal{N} = (N, S_1, S_2, \ldots, S_n)$ for the same first-order language $\mathscr{L}$, with universe M and N respectively, a map $\pi : \mathcal{M} \to \mathcal{N}$ (as a function, $\pi : M \to N$) is *elementary*, just in case

$$\mathcal{M} \models \varphi(\vec{x}) \iff \mathcal{N} \models \varphi(\pi(\vec{x})) \tag{1.1}$$

for all first-order formulas φ of $\mathscr{L}$ and all finite tuples $\vec{x} \in M^{<\omega}$. Here $M^{<\omega}$ denotes the set of finite sequences of elements of M.

We can refine this notion by considering formulas of only a certain complexity: We say π is Σ_n*-elementary* iff line (1.1) holds for all $\vec{x} \in M^{<\omega}$ and Σ_n formulas φ. (From now on we may blur the distinction between a structure $\mathcal{M}$ and its universe M.)

An elementary substructure is of course the special case of this in which π is just the inclusion map. We write $M \preccurlyeq N$ for a fully elementary substructure, and $M \preccurlyeq_n N$ for Σ_n-elementary.

Given $X \subseteq M$ and $p \in M$, X is *definable over M from the parameter p* iff there is a formula $\varphi \in \mathscr{L}$ such that for all $x \in M$, we have

$$x \in X \iff M \models \varphi(x, p).$$

This can also be refined to Σ_n*-definable from p*, if we demand φ be a Σ_n formula, and likewise for Π_n. We say that X is *definable over M without parameters* if we can take $p = \emptyset$. We say X is *definable over M from parameters* if X is definable over M from some $p \in M$.

Recall that a set M is

- *transitive* iff $\forall x \in M\ \forall y \in x\ [y \in M]$,
- *extensional* iff $\forall x, y \in M\ [x \neq y \Rightarrow \exists z \in M\ [z \in x \Leftrightarrow z \notin y]]$;

note these notions are Δ_0. The *Mostowski collapsing theorem* asserts that if M is a set and E a binary relation on M which is wellfounded and (M, E) satisfies E*-extensionality*

But most of the time we will blur the distinction between the symbol and the relation, and just write $\in$.

We follow the convention that first order languages all contain the equality symbol $\doteq$, which is automatically interpreted by the true equality relation over the universe of a structure. Some authors do not follow this convention.

(that is, $\forall x, y \in M\ [x \neq y \Rightarrow \exists z \in M\ [zEx \Leftrightarrow \neg zEy]]$), then there is a unique transitive set $\bar{M}$, and a unique map $\pi : \bar{M} \to M$, such that π is an isomorphism

$$\pi : (\bar{M}, \in) \to (M, E);$$

here $\bar{M}$ is called the *Mostowski* or *transitive collapse* of (M, E), and π the *Mostowski uncollapse map*. The most important example of transitive sets in this paper are the segments V_α of the cumulative hierarchy.

A key fact for transitive sets is that of *absoluteness* with respect to Δ_0 truth: Let M be transitive. Then Δ_0 formulas are *absolute* to M, meaning that if ψ is Δ_0 and $\vec{x} \in M^{<\omega}$, then[18]

$$\psi(\vec{x}) \iff [M \models \psi(\vec{x})].$$

Here the blanket assertion "$\psi(\vec{x})$" on the left implicitly means "$V \models \psi(\vec{x})$" where V is the ambient universe in which we are working. This equivalence is proven by induction on the formula length. It follows that if ψ is Δ_0 then

$$[M \models \exists y\ \psi(y, \vec{x})] \implies [\exists y\ \psi(y, \vec{x})]$$

(in fact any witness $y \in M$ also works in V), so conversely,

$$[\forall y\ \psi(y, \vec{x})] \implies [M \models \forall y\ \psi(y, \vec{x})].$$

We write OR for the class of all ordinals. Ordinals α, β are represented as sets in the standard form: $0 = \emptyset$, $\alpha + 1 = \alpha \cup \{\alpha\}$, and we take unions at limit ordinals η. The standard ordering on the ordinals is then $\alpha < \beta \Leftrightarrow \alpha \in \beta$, and this ordering is wellfounded. Being an ordinal is a Δ_0-definable property, because x is an ordinal iff x is transitive and (the elements of) x are linearly ordered by $\in$. Therefore being an ordinal is absolute for transitive sets, and preserved by Σ_0-elementary embeddings between transitive sets. That is, if M, N are transitive and $x \in M$ then

$$x \text{ is an ordinal} \iff M \models x \text{ is an ordinal},$$

and if $j : M \to N$ is also Σ_0-elementary then

$$M \models x \text{ is an ordinal} \iff N \models j(x) \text{ is an ordinal},$$

[18] Actually we are ignoring a technical point here. There are formally two versions of first order language to be considered. One is the usual one, which occurs in the meta-theory, involving formulas one can write down on paper, etc. (meaning that their length is a standard integer). The second is a formalized version of language(s) which appear inside the universe of ZF under consideration. The meta-theory formulas all have formal representations inside the model, but the converse need not be true (and is not precisely if the model contains non-standard integers). When we write, for example, $M \models \varphi$, this might be referring to either one of these notions, and we have not made explicit which. We leave it to the reader to determine which is the relevant notion where. But if the reader is not already familiar with the distinction, then they will not lose much by identifying the two notions.

and therefore,

$$x \text{ is an ordinal} \iff j(x) \text{ is an ordinal.}$$

So this will hold in particular for the elementary embeddings $j : V_\alpha \to V_\beta$ that we consider. Note that transitivity of sets is also a Δ_0-definable property, so absolute. Note that if M is a transitive set then $\mathrm{OR} \cap M$ is also an ordinal, in fact the least ordinal not in M.

If N is a model of ZF (possibly non-transitive), we write

$$\mathrm{OR}^N = \{\alpha \in N : N \models \text{“}\alpha \in \mathrm{OR}\text{”}\}.$$

Similarly, if $\alpha \in \mathrm{OR}^N$ we write

$$V_\alpha^N = \text{ the unique } v \in N \text{ such that } N \models \text{“}v = V_\alpha\text{”}.$$

We use analogous superscript-N notation whenever we have a notion defined using some theory T and $N \models T$. So superscript-N means “as computed/defined in/over N”.

Given a set x, the *rank* of x, denoted $\mathrm{rank}(x)$, denotes the least ordinal α such that $x \subseteq V_\alpha$. (The Axiom of Foundation ensures that rank is well-defined.)

Given a function $f : X \to Y$, $\mathrm{dom}(f)$ denotes the domain of f, $\mathrm{rg}(f)$ the range, and given $A \subseteq X$, $f[A]$ or $f``A$ denotes the pointwise image of A.

Let $j : V \to M$ be elementary, where $M \subseteq V$ and j is non-identity. (When we say this, we mean implicitly that j, M are classes of V; this can be taken to mean that $(V, (M, j))$ models $\mathrm{ZF}(M, j)$.) An argument by contradiction can be used to show that there is an ordinal κ such that $j(\kappa) > \kappa$, and the least such is called the *critical point* of j, denoted $\mathrm{crit}(j)$. The same holds much more generally, for example if $j : V_\alpha \to V_\alpha$ is Σ_1-elementary. Similarly, if M is a transitive set or class and $j : M \to M$ is Σ_1-elementary and j is surjective, then $j = \mathrm{id}$.

If $j : M \to N$ is Σ_1-elementary between transitive sets M, N, then $M \cong \mathrm{rg}(j) \preccurlyeq_1 N$, and $\mathrm{rg}(j)$ is a wellfounded extensional set, and therefore the Mostowski collapsing theorem applies to it. The transitive collapse is just M, and j is the uncollapse map. So from j we can compute $\mathrm{rg}(j)$ (and $M = \mathrm{dom}(j)$), and from $\mathrm{rg}(j)$ we can recover M, j.

Given $j : M \to N$ where M, N are structures for a language including the binary relation symbol E, we say that j is E*-cofinal* iff for every $y \in N$, there is $x \in M$ with $(y, j(x)) \in E^N$.

2. Non-definability of $j : V \to V$

Suzuki proved the following theorem. We will use variants of its proof later, and the proof is short, so for expository purposes we include it as a warm-up. Everything in this section is well known.

Theorem 2.1 (Suzuki, [20]). *Assume* ZF.[19] *Then no class k which is definable from parameters is a non-trivial elementary embedding $k : V \to V$.*

[19] That is, we are assuming that the universe $V \models \mathrm{ZF}$. We often use this language and then make statements which are to be interpreted in/over V.

Here when we say simply "definable from parameters", we mean over V. Of course, the theorem is really a theorem scheme, giving one statement for each possible formula φ being used to define k (from a parameter). In order to give the proof, we need a couple of lemmas. The first is a little easier to consider in the case that α in the proof is a limit ordinal, but the proof goes through in general.

Lemma 2.2. *Let $j : V_\delta \to V_\lambda$ be Σ_1-elementary. Then $j(V_\alpha) = V_{j(\alpha)}$ for all $\alpha < \delta$.*

Proof. Fix $\alpha < \delta$. Note that V_δ satisfies the following statements about the parameters α and V_α:[20]

- "V_α is transitive."
- "For every $X \in V_\alpha$ and every $Y \subseteq X$, we have $Y \in V_\alpha$."
- "V_α satisfies 'For every ordinal β, V_β exists'."[21]

The first statement here is Σ_0 (in parameter V_α), the second is Π_1, and the third Δ_1, so V_λ satisfies the same assertions of the parameter $j(V_\alpha)$. It follows that $j(V_\alpha) = V_\beta$ for some $\beta < \lambda$. But also $\alpha = V_\alpha \cap \mathrm{OR}$, another fact preserved by j (again by Σ_1-elementarity), so $j(\alpha) = j(V_\alpha) \cap \mathrm{OR}$, so $\beta = j(\alpha)$. ■

The following is [11, Proposition 5.1] (though it is stated under the assumption that M, N are transitive proper class inner models there). We will also use a generalization of this result later, due to Gaifman (and note he mentions in [7, Remark 2, p. 55] that Lemma 2.3 was already known, but is not attributed).

Lemma 2.3. *Let $(M, \in^M), (N, \in^N)$ be models of* ZF. *Let $j : M \to N$ be Σ_1-elementary and $\in$-cofinal. Then j is fully elementary.*

Note that we do not need to assume that M and/or N are wellfounded.

Proof of Lemma 2.3. We just write "$\in$", instead of "$\in^M$" and "$\in^N$". We prove by induction on $n < \omega$ that j is Σ_n-elementary.

Because j is Σ_1-elementary, we have $j(V_\alpha^M) = V_{j(\alpha)}^N$ for each α; the proof is essentially the same as that for the previous lemma.

Suppose j is Σ_n-elementary where $n \geq 1$. Let $C_n \subseteq \mathrm{OR}^M$ be the M-class of all α such that $V_\alpha^M \preccurlyeq_n M$. (Note that C_n is as defined over M, without parameters.) ZF proves (via standard model-theoretic methods) that C_n is unbounded in OR.

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[20] When we write "V_α" in the 3 statements, we refer to the object $x = V_\alpha$ as a parameter, as opposed to the object defined as the αth stage of the cumulative hierarchy. But note that the "β" and "V_β" are quantified variables, and here V_β *does* refer to the βth stage of the cumulative hierarchy.

[21] The reader might notice that this needs to be formulated appropriately, because if $\alpha = \beta + 1$, then the standard definition of $\langle V_\gamma \rangle_{\gamma \leq \beta}$ is the function $f : \beta + 1 \to V$ where $f(\gamma) = V_\gamma$, and if we are using the usual representation of functions f as the set of ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ such that $f(x) = y$ (which we are until mentioned otherwise), then $f \notin V_\alpha$. But it is straightforward to reformulate things appropriately. For the case in which $j : V_\delta \to V_\delta$ and δ is a limit, one can also get around these things in other ways, since we can just talk about elements of V_δ, instead of literally talking about something that V_α satisfies.

Let $\alpha \in C_n$. We claim that $j(\alpha) \in C_n^N$ (with C_n^N defined analogously over N; see §1.1). Indeed, suppose $N \models \varphi(x)$ where $x \in V_{j(\alpha)}^N$ and φ is Σ_n, but $V_{j(\alpha)}^N \models \neg\varphi(x)$. The existence of such an x is a Σ_n assertion about the parameter $V_{j(\alpha)}^N$, satisfied by N, so M satisfies the same about V_α^M (by Σ_n-elementarity of j). But $\alpha \in C_n$, a contradiction.

Now suppose that $N \models \varphi(j(x))$, where φ is Σ_{n+1}. Then by the $\in$-cofinality of j and the previous remarks, we may pick $\alpha \in C_n$ such that $x \in V_\alpha^M$ and $V_{j(\alpha)}^N \models \varphi(j(x))$. But then $V_\alpha^M \models \varphi(x)$, and since $\alpha \in C_n$, it follows that $M \models \varphi(x)$, as desired. ∎

Proof of Theorem 2.1. Suppose that $k : V \to V$ is elementary and there is a Σ_n formula φ and $p \in V$ such that for all x, y, we have

$$k(x) = y \iff \varphi(p, x, y).$$

Given any parameter q, attempt to define a function j_q by

$$j_q(x) = y \iff \varphi(q, x, y).$$

Say that q is *bad* iff $j_q : V \to V$ is a Σ_1-elementary, non-identity map. Because j_q is defined using the fixed formula φ and we only demand Σ_1-elementarity, *badness* is a definable notion (without parameters). And p above is bad.

By Lemma 2.3 above, if q is bad then j_q is in fact fully elementary.

Now let κ_0 be the least critical point $\mathrm{crit}(j_q)$ among all bad parameters q. Note then that the singleton $\{\kappa_0\}$ is definable over V, from no parameters. (So there is a formula ψ such that $\psi(x) \Leftrightarrow x = \kappa_0$, for all sets x.)

Let q_0 witness the choice of κ_0. As mentioned above, j_{q_0} is in fact fully elementary, and we have $\mathrm{crit}(j_{q_0}) = \kappa_0$. So $j_{q_0}(\kappa_0) > \kappa_0$, whereas $j_{q_0}(\alpha) = \alpha$ for all $\alpha < \kappa_0$. Since j_{q_0} is order-preserving, $\kappa_0 \notin \mathrm{rg}(j_{q_0})$. But by the (full) elementarity of $j_{q_0} : V \to V$ and definability of $\{\kappa_0\}$, we must have $j_{q_0}(\kappa_0) = \kappa_0 \in \mathrm{rg}(j_{q_0})$, a contradiction. ∎

We remark that Suzuki [20, Theorem 3.1] is actually more general, considering elementary embeddings of the form $j : M \to V$ where $M \subseteq V$ is transitive and contains all ordinals, and (j, M) is definable from parameters.

3. Definability of rank-to-rank embeddings

3.1. The limit case

Most investigations of rank-to-rank embeddings to date have focused on elementary embeddings $j : V_\alpha \to V_\alpha$ where $\alpha = \kappa_\omega(j)$ or $\alpha = \kappa_\omega(j) + 1$, since assuming Choice, these are the only rank-to-rank embeddings there could possibly be. The following very simple fact turns out to play a central role in these investigations: if λ is a limit ordinal, an elementary embedding from V_λ to V_λ extends in at most one way to an elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$.

Definition 3.1. For a structure M, $\mathscr{E}(M)$ denotes the set of all elementary embeddings $j : M \to M$.

Definition 3.2. Let λ be a limit ordinal and $j \in \mathscr{E}(V_\lambda)$. The *canonical extension of* j is the function $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ defined $j^+(X) = \bigcup_{\alpha<\lambda} j(X \cap V_\alpha)$.

The canonical extension j^+ is a function $V_{\lambda+1} \to V_{\lambda+1}$. However, it is well known that it can fail to be elementary. (For example, let κ be least such that there is an elementary $j : V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$, and show that j^+ is not elementary.) But if j *does* extend to some $i \in \mathscr{E}(V_{\lambda+1})$, or even just to a Σ_1-elementary $i : V_{\lambda+1} \to V_{\lambda+1}$, then clearly $i(V_\lambda) = V_\lambda$ and $i = j^+$.

Let λ be a limit ordinal. It follows that every $j \in \mathscr{E}(V_{\lambda+1})$ is definable over $V_{\lambda+1}$ from parameters, in fact, from its own restriction $j \restriction V_\lambda$. (Since V_λ is closed under ordered pairs, $j \restriction V_\lambda \in V_{\lambda+1}$.) However, j is not definable over $V_{\lambda+1}$ from any element of V_λ, and no $j \in \mathscr{E}(V_\lambda)$ is definable from parameters over V_λ:

Theorem 3.3. *Let δ be an ordinal, $j \in \mathscr{E}(V_\delta)$ and $p \in V_\delta$, with j definable over V_δ from the parameter p. Then $\delta = \beta + 1$ is a successor and $p \notin V_\beta$ (so $\mathrm{rank}(p) = \beta$).*

Remark 3.4. Richard Matthews independently proved a result related to Theorems 3.3 and 3.12 in the context of AC; see [14, Theorem 5.4].[22]

Proof. Suppose not. We adapt the proof of Suzuki's theorem. Fix (k, φ, β) such that $k < \omega$, φ is a Σ_k formula and $\beta < \delta$, and for some $p \in V_\beta$ we have $j_p \in \mathscr{E}(V_\delta)$ where

$$j_p = \{(x, y) \in V_\delta \times V_\delta : V_\delta \models \varphi(p, x, y)\}.$$

Say that $q \in V_\beta$ is *ω-bad* iff $j_q \in \mathscr{E}(V_\delta)$.

Let μ_0 be the least critical point among all such (fully elementary) embeddings j_q (minimizing over all ω-bad parameters q). Let $p_0 \in V_\beta$ witness this, so $j_{p_0} \in \mathscr{E}(V_\delta)$ and $\mathrm{crit}(j_{p_0}) = \mu_0$.

For $n < \omega$, say that $q \in V_\beta$ is *n-bad* iff $j_q : V_\delta \to V_\delta$ and is Σ_n-elementary. Let $A_n = \{q \in V_\beta : q \text{ is } n\text{-bad}\}$. So $A_n \in V_\delta$ and note that A_n is definable over V_δ from the parameter β.

Since $j = j_{p_0}$ is fully elementary, $j(A_n) \cap V_\beta = A_n$ (note $j(\beta) \geq \beta$). Let $A = \bigcap_{n<\omega} A_n$, so $A \in V_\delta$. Note that $j_q \in \mathscr{E}(V_\delta)$ for every $q \in A$.

The sequence $\langle A_n \rangle_{n<\omega}$ can easily be coded by a set in V_δ (with methods as in the next section; if δ is a limit then it is in fact literally in V_δ), and therefore

$$j(A) = \bigcap_{n<\omega} j(A_n),$$

so $p_0 \in j(A)$. Therefore $V_\delta \models$ "$\exists q \in j(A)$ such that $\mathrm{crit}(j_q) < j(\mu_0)$" (as witnessed by p_0). Pulling this back with the elementarity of j yields $V_\delta \models$ "$\exists q \in A$ such that $\mathrm{crit}(j_q) < \mu_0$." This contradicts the minimality of μ_0. ∎

[22] Matthews considers elementary $j : V \to V$ assuming $(V, j) \models \mathrm{ZFC}^-$, hence giving information about embeddings $j : H_{\lambda^+} \to H_{\lambda^+}$ under choice. But in the choiceless context, we cannot assume that Collection holds in the analogue of H_{λ^+} (a natural variant of $V_{\lambda+1}$; see [8]). And we also consider V_δ for limit δ.

3.2. A flat pairing function

If δ is a limit ordinal then V_δ is closed under pairs $\{x, y\}$, and hence, ordered pairs (x, y), represented in the standard fashion as $(x, y) = \{\{x\}, \{x, y\}\}$. But this fails in the successor case, at least when we use this standard representation: For example, $V_\delta \in V_{\delta+1}$ but $\{V_\delta, \emptyset\} \notin V_{\delta+1}$. It is therefore useful to employ a different representation or *coding* of ordered pairs with the property that for every infinite ordinal α, for all $x, y \in V_\alpha$, the code $\lceil x, y \rceil$ for the pair (x, y) is an element of V_α. In this case, the function $(x, y) \mapsto \lceil x, y \rceil$ is called a *flat pairing function*.

There are many different flat pairing functions, and which one we use will not really be relevant in our applications. All we will really require of the pairing function is that it be a Σ_0-definable injection $\Phi : V \times V \to V$ such that $\Phi``(V_\alpha \times V_\alpha) \subseteq V_\alpha$ for all infinite ordinals α.

Nevertheless, let us define the *Quine–Rosser pairing function*, which is officially the pairing function we employ below. The basic idea is to code a pair (x, y) by a labeled disjoint union of x and y. Somewhat more precisely, we will take two disjoint copies V_0 and V_1 of the universe V and bijections $f_0 : V \to V_0$ and $f_1 : V \to V_1$, which are both rank-preserving over all sets of rank $\geq \omega$. The ordered pair (x, y) is then coded by the set $\lceil x, y \rceil = f_0``x \cup f_1``y$.

To implement this idea without leaving V, let V_0 be the class of sets that do not contain the empty set and let V_1 be the class of sets that do. Let $s : V \to V$ be defined by setting $s(n) = n + 1$ for all $n < \omega$ and $s(u) = u$ for all $u \notin \omega$. Then let $f_0 : V \to V_0$ be defined by $f_0(X) = s``X$ and $f_1 : V \to V_1$ be defined by $f_1(u) = (s``X) \cup \{\emptyset\}$.

Definition 3.5. For sets $x, y \in V$, let

$$\lceil x, y \rceil = f_0``x \cup f_1``y,$$

where f_0 and f_1 are as defined above. The set $\lceil x, y \rceil$ is the *Quine–Rosser pair* coding (x, y).

The Quine–Rosser pairing function $(x, y) \mapsto \lceil x, y \rceil$ establishes a bijection from $V \times V$ to V whose inverse is the function $z \mapsto (f_0^{-1}[z], f_1^{-1}[z])$. It is easy to show that for any set u, $\text{rank}(f_0(u))$ and $\text{rank}(f_1(u))$ are bounded by $1 + \text{rank}(u)$, which implies

$$\text{rank}(\lceil x, y \rceil) \leq 1 + \max\{\text{rank}(x), \text{rank}(y)\}.$$

In particular, for any infinite ordinal α, the Quine–Rosser pairing function restricts to a bijection from $V_\alpha \times V_\alpha$ to V_α. Moreover, this function is Σ_0-definable over the structure $(V_\alpha, \in)$.[23]

From now on, we shift notation, and whenever we talk about ordered pairs, we mean Quine–Rosser pairs, and whenever we talk about binary relations R on V_α (where $\alpha \geq \omega$) we will literally mean that R is a set of Quine–Rosser pairs, and similarly for n-ary rela-

[23] To be clear, the definability means there is a Σ_0 formula φ of three variables such that $\lceil x, y \rceil = z \Leftrightarrow \varphi(x, y, z)$. (Note this can be written without presupposing any coding of 3-tuples (x, y, z).)

tions. Therefore $R \in V_{\alpha+1}$. Moreover, note that there is a Σ_0 formula in the language of set theory such that for any such α and binary relation R on V_α and $x, y \in V_\alpha$, we have xRy iff $V_{\alpha+1} \models \varphi(R, x, y)$. This will be used in particular for (partial) functions $f : V_\alpha \to V_\alpha$.

3.3. The successor case

Our observations so far suggest the following natural questions. Let ξ be an ordinal and $j \in \mathscr{E}(V_{\xi+2})$. Then (i) can j be definable over $V_{\xi+2}$ from some parameter (by Theorem 3.3, necessarily of rank $\xi+1$)? And more specifically, (ii) can j be definable over $V_{\xi+2}$ from $j \restriction V_{\xi+1}$? Note here that, because we are using Quine–Rosser pairs, $j \restriction V_{\xi+1} \in V_{\xi+2}$.

Using Theorem 3.3, we can easily answer question (ii) in the case that ξ is a limit ordinal: In this case j is *not* definable from the parameter $j \restriction V_{\xi+1}$ over $V_{\xi+2}$ (thus giving the first evidence of periodicity). For suppose otherwise. Then j is in fact definable from $j \restriction V_\xi$ over $V_{\xi+2}$, since $j \restriction V_{\xi+1} = (j \restriction V_\xi)^+$, and the canonical extension operation is itself definable. But $j \restriction V_\xi$ has rank $\xi < \xi + 1$, contradicting Theorem 3.3.

But the foregoing argument does not seem to answer question (i) when ξ is a limit, nor generalize to higher successor levels at all. In this section, we look further into these questions, and answer them. In the end, most of the results from the limit case do generalize to the case of arbitrary even ordinals.

At first glance, it seems that the definition of the canonical extension operation (Definition 3.2) makes fundamental use of the assumption that λ is a limit ordinal. In particular, this definition exploits the hierarchy $\langle V_\alpha \rangle_{\alpha<\lambda}$ stratifying V_λ; this hierarchy seems to have no analog at the successor even levels. But on further thought, we could have defined $j^+(X)$ for $X \in V_{\lambda+1}$ as follows:

$$j^+(X) = \bigcup \{j(a) : a \in V_\lambda \text{ and } a \subseteq X\}$$

Thus $j^+(X)$ is the union of the image of j on all the subsets of X that belong to V_λ.

At successor ordinals we must generalize this slightly, instead taking the union of the image of j on all the subsets of X that are *coded* in V_λ.

Definition 3.6. Suppose a and b are sets. For any set x, let $(a)_x$ denote the set $\{y : \ulcorner x, y \urcorner \in a\}$, and let $(a \mid b) = \{(a)_x : x \in b\}$.

Thus for $a, b \in V_\lambda$, $(a \mid b)$ is the subset of V_λ whose elements are the sections $(a)_x \in V_\lambda$ of the binary relation coded by a that are indexed by some $x \in b$. Say a set $X \subseteq V_\lambda$ is *coded in* V_λ if $X = (a \mid b)$ for some $a, b \in V_\lambda$. For λ a limit ordinal, every set coded in V_λ belongs to V_λ, but if λ is a successor ordinal, then the sets coded in V_λ are precisely those $X \subseteq V_\lambda$ such that there is a partial surjection from $V_{\lambda-1}$ onto X. (Given $f : V_{\lambda-1} \to V_\lambda$, let $a \subseteq V_\lambda$ be the set of all pairs $\ulcorner x, y \urcorner$ where $x \in V_{\lambda-1}$ and $y \in f(x)$.)

Definition 3.7. Suppose λ is an ordinal. For any function $j : V_\lambda \to V_\lambda$, the *canonical extension of* j is the function $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ defined by

$$j^+(X) = \bigcup \{(j(a) \mid j(b)) : a, b \in V_\lambda \text{ and } (a \mid b) \subseteq X\}.$$

While j^+ is well-defined for any function j, it is not of much interest unless j has the property that $(j(a) \mid j(b)) = (j(a') \mid j(b'))$ whenever $(a \mid b) = (a' \mid b')$.

Suppose $a, b \in V_\lambda$, $X \in V_{\lambda+1}$, and $(a \mid b) \subseteq X$. The fact that $(a \mid b)$ is included in X is a first-order expressible property of a, b, and X in $V_{\lambda+1}$, so for any $k \in \mathscr{E}(V_{\lambda+1})$, $(k(a) \mid k(b)) \subseteq k(X)$. It follows that $(k \restriction V_\lambda)^+(X) \subseteq k(X)$, whether λ is even or odd. The reverse inclusion, however, will be true if and only if λ is even.

Definition 3.8. Suppose λ is an ordinal. An embedding $j : V_\lambda \to V_\lambda$ is *cofinal* if for any set $c \in V_\lambda$, there exist sets $a, b \in V_\lambda$ such that $c \in (j(a) \mid j(b))$.

Equivalently, $j : V_\lambda \to V_\lambda$ is cofinal if $j^+(V_\lambda) = V_\lambda$. It follows immediately that if $k \in \mathscr{E}(V_{\lambda+1})$ and $k = (k \restriction V_\lambda)^+$, then $k \restriction V_\lambda$ must be cofinal. The converse is also true:

Lemma 3.9. *Suppose $k \in \mathscr{E}(V_{\lambda+1})$ and $k \restriction V_\lambda$ is cofinal. Then $k = (k \restriction V_\lambda)^+$.*

Proof. Fix $X \in V_{\lambda+1}$. Our comments above show that $(k \restriction V_\lambda)^+(X) \subseteq k(X)$. For the reverse inclusion, fix $c \in k(X)$. We will show $c \in (k \restriction V_\lambda)^+(X)$.

Since $k \restriction V_\lambda$ is cofinal, there are sets $a, b \in V_\lambda$ such that $c \in (k(a) \mid k(b))$. Let $b' = \{x \in b : (a)_x \in X\}$, so that $(a \mid b') = (a \mid b) \cap X$. Now

$$c \in (k(a) \mid k(b)) \cap k(X) = k((a \mid b) \cap X) = k\left(a \mid b'\right) \subseteq (k \restriction V_\lambda)^+(X).$$

This shows $k(X) \subseteq (k \restriction V_\lambda)^+(X)$, completing the proof. ■

The periodicity phenomenon is driven by the following lemma:

Lemma 3.10. *Suppose $j : V_{\lambda+2} \to V_{\lambda+2}$ is an elementary embedding such that $(j \restriction V_\lambda)^+ = j \restriction V_{\lambda+1}$. Then j is cofinal.*

Proof. Fix $j : V_{\lambda+2} \to V_{\lambda+2}$ and $C \in V_{\lambda+2}$. We must show that there exist $A, B \in V_{\lambda+2}$ such that $C \in (j(A) \mid j(B))$. Let B consist of those sets $x \in V_{\lambda+1}$ such that the binary relation $\{(a, b) : \ulcorner a, b \urcorner \in x\}$ coded by x is the graph of a function $f_x : V_\lambda \to V_\lambda$. By elementarity, $j(B) = B$.

Now define $A = \{\ulcorner x, y \urcorner \in B \times V_{\lambda+1} : f_x^+(y) \in C\}$. In other words, for each $x \in B$, $(A)_x = (f_x^+)^{-1}[C]$. Now

$$j(A) = \{\ulcorner x, y \urcorner \in B \times V_{\lambda+1} : f_x^+(y) \in j(C)\}.$$

Let $w = \{\ulcorner a, j(a) \urcorner : a \in V_\lambda\}$, so that $f_w = j \restriction V_\lambda$. Then $w \in B$ and

$$(j(A))_w = (f_w^+)^{-1}[j(C)] = ((j \restriction V_\lambda)^+)^{-1}[j(C)] = (j \restriction V_{\lambda+1})^{-1}[j(C)] = C.$$

Therefore $C = (j(A))_w \in (j(A) \mid B) = (j(A) \mid j(B))$, as desired. ■

Theorem 3.11. *Suppose λ is an even ordinal and $j : V_\lambda \to V_\lambda$ is an elementary embedding. Then j is cofinal. Suppose in addition that j extends to an elementary embedding $k : V_{\lambda+1} \to V_{\lambda+1}$. Then $k = j^+$.*

Proof. We have already established the theorem with respect to limit ordinals λ. We now proceed by induction. So let λ be an even successor ordinal and assume that the theorem holds for all ordinals less than λ. Applying this at $\lambda - 2$, we have $j \restriction V_{\lambda-1} = (j \restriction V_{\lambda-2})^+$, so j is cofinal by Lemma 3.10. Since j is cofinal, Lemma 3.9 implies that if j extends to an elementary embedding $k : V_{\lambda+1} \to V_{\lambda+1}$, then $k = j^+$. This completes the proof. ∎

The requirement that λ be even in the previous theorem is unusual, but one can show that the theorem fails whenever λ is odd. The proof given here is an elaboration on that of Theorem 3.3 (and recall Remark 3.4).

Theorem 3.12. *Suppose α is an ordinal, $j \in \mathscr{E}(V_\alpha)$, and $a, b \in V_\alpha$. Then j is not definable over V_α from parameters in $(j(a) \mid j(b))$.*

Proof. Suppose towards a contradiction that the theorem fails. Then there is a formula $\varphi(v_0, v_1, v_2)$ and a parameter $p \in (j(a) \mid j(b))$ such that

$$j(u) = w \iff V_\alpha \models \varphi(u, w, p)$$

for all $u, w \in V_\alpha$. For $q \in V_\alpha$, define a relation

$$j_q = \{(u, w) \in V_\alpha^2 : V_\alpha \models \varphi(u, w, q)\}.$$

For $n \leq \omega$, say $q \in V_\alpha$ is *n-bad* if $j_q : V_\alpha \to V_\alpha$ is a non-trivial Σ_n-elementary embedding and there exist $a', b' \in V_\alpha$ such that $q \in (j_q(a') \mid j_q(b'))$.

So p is ω-bad. Let $\kappa = \min\{\text{crit}(j_q) : q \text{ is } \omega\text{-bad}\}$. Fix an ω-bad parameter r such that $\text{crit}(j_r) = \kappa$.

Fix $c, d \in V_\alpha$ with $r \in (j_r(c) \mid j_r(d))$. For each $n \leq \omega$, let

$$d_n = \{x \in d : (c)_x \text{ is } n\text{-bad}\}.$$

By the elementarity of j_r,

$$j_r(d_n) = \{x \in j_r(d) : (j_r(c))_x \text{ is } n\text{-bad}\}.$$

Let $e = \{\ulcorner n, x \urcorner : x \in d_n\}$, so that $(e)_n = d_n$. Since $d_\omega = \bigcap_{n<\omega} (e)_n$, we have $j_r(d_\omega) = \bigcap_{n<\omega} (j_r(e))_n = \bigcap_{n<\omega} j_r(d_n)$. It follows that

$$j_r(d_\omega) = \{x \in j_r(d) : (j_r(c))_x \text{ is } \omega\text{-bad}\}.$$

In particular, $r \in (j_r(c) \mid j_r(d_\omega))$ and every $q \in (j_r(c) \mid j_r(d_\omega))$ is ω-bad, so

$$\min\{\text{crit}(j_q) : q \in (j_r(c) \mid j_r(d_\omega))\} = \kappa.$$

Therefore letting $\bar{\kappa} = \min\{\text{crit}(j_q) : q \in (c \mid d_\omega)\}$, we have $j_r(\bar{\kappa}) = \kappa$, which contradicts that κ is the critical point of j_r. ∎

Putting everything together, we can now prove Theorem 1.1; that is, if $j \in \mathscr{E}(V_\lambda)$ is non-trivial, then j is definable from parameters over V_λ iff λ is odd:

Proof of Theorem 1.1. Suppose λ is even. Then by Theorem 3.11, j is cofinal, which means that every $p \in V_\lambda$ belongs to $(j(a) \mid j(b))$ for some $a, b \in V_\lambda$. Therefore by Theorem 3.12, j is not definable from any parameter in V_λ.

On the other hand, if λ is odd, then $j = (j \upharpoonright V_{\lambda-1})^+$ by Theorem 3.11, and therefore j is definable over V_λ from $j \upharpoonright V_\lambda$, or more precisely from the set $\{\ulcorner x, j(x) \urcorner : x \in V_{\lambda-1}\}$, which belongs to V_λ. ■

4. Reinhardt ultrafilters

Solovay's discovery of supercompactness in the late 1960s marked the beginning of the modern era of large cardinal theory. In the context of ZFC, supercompactness has both a combinatorial characterization in terms of normal ultrafilters and a "model-theoretic" characterization in terms of elementary embeddings $j : V \to M$ where M is an inner model. In the choiceless context, however, the equivalence between the usual characterizations is no longer provable, and instead supercompactness splinters into a number of inequivalent but interrelated concepts.

The rank-to-rank embeddings $j : V_\delta \to V_\delta$ studied here exhibit features reminiscent of supercompactness. In this section we evidence this via a characterization in terms of normal ultrafilters in the case that $\delta = \alpha + 2$.[24] But since these embeddings force us into the choiceless realm, we must deal with the subtleties this brings. A key issue in this regard is that one needs to be more careful regarding Łoś's theorem for ultrapowers, given the role of choice in its usual proof.

4.1. Ultrapowers and Łoś's theorem

In this section we give a quick review of some standard background. We assume familiarity with (ultra)filters, which can be found in standard texts. If $\mathscr{F}$ is a filter over a set X (so $X \in \mathscr{F}$) and φ is some property, say that $\varphi(x)$ holds for *$\mathscr{F}$-almost all x* (or just *almost all x*) iff $\{x \in X : \varphi(x)\} \in \mathscr{F}$.

We first recall the definition of *ultrapowers* in our context. Let $\gamma, \beta \in \mathrm{OR}$ and let $\mathscr{F}$ be any ultrafilter over V_γ. Let $\mathscr{U}$ denote the set of all functions $f : V_\gamma \to V_\beta$. We define a binary relation over $\mathscr{U}$ by

$$f \approx_{\mathscr{F}} g \iff \{x \in V_\alpha : f(x) = g(x)\} \in \mathscr{F}.$$

Because $\mathscr{F}$ is a filter, it is easy to see that $\approx_{\mathscr{F}}$ is an equivalence relation; let $[f]^{V_\beta}_{\mathscr{F}}$ be the equivalence class of f, where we just write $[f]$ if there is no ambiguity. We also define the relation

$$f \in_{\mathscr{F}} g \iff \{x \in V_\alpha : f(x) \in g(x)\} \in \mathscr{F}.$$

Then $\in_{\mathscr{F}}$ respects $\approx_{\mathscr{F}}$. The *ultrapower* $\mathrm{Ult}(V_\beta, \mathscr{F})$ of V_β by $\mathscr{F}$ is the structure $(U, \in^U)$, where $U = \{[f] : f \in \mathscr{U}\}$, and $\in^U$ is the binary relation on U induced by $\in_{\mathscr{F}}$. The *ultrapower embedding* $i^{V_\beta}_{\mathscr{F}} : V_\beta \to U$ is defined by $i^{V_\beta}_{\mathscr{F}}(x) = [c_x]$ where $c_x \in \mathscr{U}$ is the constant function with constant value x.

[24] In this section we assume familiarity with ultrapowers as used in set theory; the reader familiar with supercompactness measures should be fine.

Now let us say that *Σ_n-Łoś's theorem for U* holds iff for all Σ_n formulas φ (in the language of set theory) and functions $f_1, \ldots, f_n \in \mathscr{U}$, we have

$$U \models \varphi([f_1], \ldots, [f_n]) \iff V_\beta \models \varphi(f_1(x), \ldots, f_n(x)) \text{ for almost all } x \in V_\gamma.$$

We just say *Łoś's theorem holds for U* if Σ_n-Łoś's theorem holds for all $n < \omega$. For atomic formulas $\varphi(u, v)$ ("$u = v$" and "$u \in v$") the stated equivalence holds by definition. Assuming AC it holds for all formulas, as proved by induction on formula complexity. The only step that uses AC is that for quantifiers: suppose for example that

$$\sigma \in \mathscr{F} \text{ and for all } x \in \sigma \text{ we have } V_\beta \models \exists w\ \varphi(w, f(x)).$$

Then we want $U \models \exists w\ \varphi(w, [f])$, which needs some $w \in \mathscr{U}$ with $U \models \varphi([w], [f])$. So we need $w : V_\gamma \to V_\beta$ and by induction, we need some σ' such that

$$\sigma' \in \mathscr{F} \text{ and for all } x \in \sigma', \text{ we have } V_\beta \models \varphi(w(x), f(x)).$$

Using AC, we can in fact take $\sigma' = \sigma$ and w to be an appropriate choice function. But it is important here that we do not actually require $\sigma' = \sigma$; so even if AC fails and there is no choice function with domain σ, there might be one with a smaller domain $\sigma' \in \mathscr{F}$.

If Łoś's theorem holds for U then the ultrapower embedding $i : V_\beta \to U$ is elementary. (However, a key point is that U need not be wellfounded in general: consider for example non-principal ultrafilters over V_ω.) If U is wellfounded and extensional, then by Mostowski's theorem, it is isomorphic to its (transitive) Mostowski collapse, and following the usual convention in this case, we then identify these two. But we will at times need to deal with ultrapowers without knowing that these properties hold.

In this section we are only actually interested in the case that the ordinal β above is a successor, so from now on, we restrict to this case. In order to analyze ultrapowers and the associated embeddings defined as above, we will observe that the coding apparatus from §3.2 allows us to represent functions $f : V_\gamma \to V_\beta$ where $\gamma < \beta$ (such as those forming the ultrapower above), and simple properties thereof, in a simple manner. That is, although maybe $f \notin V_\beta$, we define the *code* of f as

$$\widetilde{f} = \{\ulcorner x, y \urcorner : x \in V_\gamma \text{ and } y \in f(x)\};$$

note $\widetilde{f} \in V_\beta$ (as $\gamma < \beta$ and β is a successor). Unraveling the coding above and the flat pairing function, it is straightforward to write a Σ_0 formula ψ such that for all such β, γ, f we have

$$\forall x \in V_\gamma\ \forall y \in V_{\beta-1}\ [y \in f(x) \Leftrightarrow V_\beta \models \psi(\widetilde{f}, x, y)].$$

More generally:

Lemma 4.1. *There is a recursive function $\varphi \mapsto \psi_\varphi$ such that for each Σ_0 formula φ, ψ_φ is a Σ_0 formula, and for all successor ordinals $\beta > \omega$ and ordinals $\gamma < \beta$ and all finite tuples $\vec{f} = (f_0, \ldots, f_{n-1})$ of functions $f_i : V_\gamma \to V_\beta$, and all $x \in V_\gamma$ and $z \in V_\beta$,*

$$V_\beta \models \varphi(f_0(x), \ldots, f_{n-1}(x), z) \iff V_\beta \models \psi_\varphi(\widetilde{f_0}, \ldots, \widetilde{f_1}, x, z).$$

We leave the straightforward proof to the reader.

Definition 4.2. For a transitive structure M and $k \leq \omega$, $\mathscr{E}_k(M)$ denotes the set of all Σ_k-elementary maps $j : M \to M$. So $\mathscr{E}_\omega(M) = \mathscr{E}(M)$.

Now suppose β is a successor ordinal and $j \in \mathscr{E}_0(V_\beta)$ and $j(V_\alpha) = V_{j(\alpha)}$ for each $\alpha < \beta$. Let $\alpha + 1 < \beta$ and $s \in V_{j(\alpha)+1}$. The ultrafilter $\mathscr{F}$ over $V_{\alpha+1}$ *derived from* j *with seed* s is defined as follows: For $\sigma \subseteq V_{\alpha+1}$, set

$$\sigma \in \mathscr{F} \iff s \in j(\sigma).$$

Note that $\mathscr{F}$ is principal iff $s \in \mathrm{rg}(j)$.

For $f : V_{\alpha+1} \to V_\beta$, we need not have $f \in V_\beta = \mathrm{dom}(j)$, but we define

$$j(f) : V_{j(\alpha)+1} \to V_\beta$$

to be the function g such that $\widetilde{g} = j(\widetilde{f})$.

Let $U = \mathrm{Ult}(V_\beta, \mathscr{F})$. Define the *natural factor map* $\pi : U \to V_\beta$ by

$$\pi([f]) = j(f)(s).$$

Then π is well-defined. For if $[f] = [g]$ then there is $\sigma \in \mathscr{F}$ such that

$$\forall x \in \sigma\ [f(x) = g(x)],$$

so by Lemma 4.1,

$$\forall x \in j(\sigma)\ [j(f)(x) = j(g)(x)],$$

and since $\sigma \in \mathscr{F}$, we have $j(f)(s) = j(g)(s)$. Similarly, $\pi : U \to \mathrm{rg}(\pi)$ is an isomorphism (with respect $\in^U$ and $\in$). In particular, in this case, U is wellfounded. However, without AC, it is not immediate that U is extensional. That is, suppose $[f] \neq [g]$. To witness extensionality for $[f], [g]$, we need some $h : V_{\alpha+1} \to V_\beta$ such that $[h] \in^U [f]$ iff $[h] \notin^U [g]$; that is, we need $\sigma \in \mathscr{F}$ such that $h(x) \in f(x) \triangle g(x)$ for all $x \in \sigma$ (where $\triangle$ denotes symmetric difference). Because $[f] \neq [g]$, there is indeed $\sigma \in \mathscr{F}$ such that $f(x) \triangle g(x) \neq \emptyset$ for all $x \in \sigma$, but it is not clear whether there is a corresponding choice function (even on some smaller $\sigma' \in \mathscr{F}$).

4.2. *Successor rank-to-rank embeddings as ultrapowers*

In this section we sketch an alternative proof of Theorem 1.1, one which is equivalent to that presented already, but superficially different, and maybe more standard for set theory. We will also consider partial elementarity.

Definition 4.3. Let η be even and $j \in \mathscr{E}_0(V_{\eta+2})$ with $j(V_{\eta+1}) = V_{\eta+1}$. Then μ_j denotes the ultrafilter over $V_{\eta+1}$ derived from j with seed $j \upharpoonright V_\eta$.[25] That is,

$$\mu_j = \{\sigma \subseteq V_{\eta+1} : j \upharpoonright V_\eta \in j(\sigma)\}.$$

This filter is analogous to filters considered in the study of I_0.

[25]Note that by our flat pairing convention, $j \upharpoonright V_\eta \in V_{\eta+1}$.

We will again define for all even ordinals δ a *canonical extension* operation $k \mapsto k^+$, with domain $\mathscr{E}_1(V_\delta)$, such that $k^+ : V_{\delta+1} \to V_{\delta+1}$ (but k^+ is not claimed to be elementary in general), and such that k^+ is the unique candidate for a Σ_0-elementary map $\ell : V_{\delta+1} \to V_{\delta+1}$ such that $k \subseteq \ell$ and $\ell(V_\delta) = V_\delta$. The operation $k \mapsto k^+$, with domain $\mathscr{E}_1(V_\delta)$, will be definable over $V_{\delta+1}$ without parameters, uniformly in δ (meaning that there is a formula ψ such that for all even δ and $k \in \mathscr{E}_1(V_\delta)$ and $x, y \in V_{\delta+1}$, we have

$$k^+(x) = y \iff V_{\delta+1} \models \psi(k, x, y),$$

noting $k \in V_{\delta+1}$ by our flat pairing convention). The definition of $k \mapsto k^+$ for $k \in \mathscr{E}_1(V_\delta)$, and proof of its basic properties, is by induction on $n < \omega$, where $\delta = \lambda + n$ for some limit ordinal λ.

If δ is a limit, then k^+ is defined as in Definition 3.2.

Suppose now that $\delta = \eta + 2$ where η is even. Let $j \in \mathscr{E}_0(V_{\eta+2})$ with $j(V_{\eta+1}) = V_{\eta+1}$; we want to define j^+ and prove some facts.[26] Let $\mu = \mu_j$. Let

- $U = \mathrm{Ult}(V_{\eta+2}, \mu)$ and $i_\mu : V_{\eta+2} \to U$ be the ultrapower map,
- $\widetilde{U} = \mathrm{Ult}(V_{\eta+3}, \mu)$ and $\widetilde{i}_\mu : V_{\eta+3} \to \widetilde{U}$ be the ultrapower map.

We will eventually show that $i_\mu = j$ and $j \subseteq \widetilde{i}_\mu$, and define $j^+ = \widetilde{i}_\mu$. We do not yet know $U, \widetilde{U}$ are extensional/wellfounded, so these ultrapowers are at the "representation" level (their elements are equivalence classes $[f]_\mu$).

Consider the hull

$$H = \mathrm{Hull}^{V_{\eta+2}}(\mathrm{rg}(j) \cup \{j \restriction V_\eta\}), \tag{4.1}$$

where $\mathrm{Hull}^M(X)$, for $X \subseteq M$, denotes the set of all $x \in M$ such that x is definable over M from parameters in X. The following claim is a typical feature of ultrapowers via a measure derived from an embedding, although part (1) only holds because $j \restriction V_\eta$ encodes enough information, and for this it is crucial that the canonical extension $(j \restriction V_\eta)^+$ is equal to $j \restriction V_{\eta+1}$, and that this operation is definable over $V_{\eta+1}$, a fact we know by induction.

Lemma 4.4. *Recall $U = \mathrm{Ult}(V_{\eta+2}, \mu)$ and H is defined in (4.1). We have:*

(1) *U is extensional and wellfounded; moreover, $U \cong H = V_{\eta+2}$.*

(2) *$i_\mu = j$, after we identify U with its transitive collapse $V_{\eta+2}$.*

(3) *$j : V_{\eta+2} \to V_{\eta+2}$ is Σ_1-elementary.*

Proof. (1) We first show $H = V_{\eta+2}$. As noted above, from the parameter $j \restriction V_\eta$, $V_{\eta+1}$ (and hence $V_{\eta+2}$) can define $k = (j \restriction V_\eta)^+ = j \restriction V_{\eta+1}$. Now let $x \in V_{\eta+2}$. Then $x \subseteq V_{\eta+1}$ and $x = k^{-1}``j(x)$, and since $j(x) \in \mathrm{rg}(j)$, this suffices.[27]

[26] We will end up seeing that it follows that $j \in \mathscr{E}_1(V_{\eta+2})$.

[27] Note that the proof actually shows that $V_{\eta+2} = \mathrm{Hull}^{V_{\eta+2}}_{\Sigma_1}(\mathrm{rg}(j) \cup \{j \restriction V_\eta\})$, where $\mathrm{Hull}^M_{\Sigma_1}(X)$ is defined like $\mathrm{Hull}^M(X)$, except that it only consists of the $y \in M$ such that for some $\vec{x} \in X^{<\omega}$ and Σ_1 formula φ, y is the unique $y' \in M$ such that $M \models \varphi(\vec{x}, y')$.

Let $\pi : U \to V_{\eta+2}$ be the factor map $\pi([f]_\mu^{V_{\eta+2}}) = j(f)(j \restriction V_\eta)$. By §4.1, π is a well-defined $\in$-isomorphism $U \to \mathrm{rg}(\pi)$. But then $\mathrm{rg}(\pi) = V_{\eta+2}$, because given $x \in V_{\eta+2}$, let

$$f_x : \mathscr{E}(V_\eta) \to V_{\eta+2}$$

be such that $f_x(k) = (k^+)^{-1}``x$, and note by the definability of canonical extension over $V_{\eta+1}$ (and that $j \in \mathscr{E}_0(V_{\eta+2})$ and $j(V_{\eta+1}) = V_{\eta+1}$), we have $j(f_x)(j \restriction V_\eta) = x$.

(2) We have $i_\mu(x) = [c_x]_\mu^{V_{\eta+2}}$. But note $\pi \circ i_\mu = j$, because

$$\pi([c_x]_\mu^{V_{\eta+2}}) = j(c_x)(j \restriction V_\eta) = c_{j(x)}(j \restriction V_\eta) = j(x),$$

since j is Σ_0-elementary. But identifying U with $V_{\eta+2}$, we have $\pi = \mathrm{id}$, so $i_\mu = j$.

(3) Let φ be Σ_0 and $x, y \in V_{\eta+2}$ with $V_{\eta+2} \models \varphi(j(x), y)$. We have $y = j(f_y)(j \restriction V_\eta)$ where f_y is as above. So

$$V_{\eta+2} \models \exists k \in V_{\eta+1}\ [\varphi(j(x), j(f_y)(k))].$$

But since j is Σ_0-elementary and $j(V_{\eta+1}) = V_{\eta+1}$, it follows that

$$V_{\eta+2} \models \exists k \in V_{\eta+1}\ [\varphi(x, f_y(k))],$$

hence $V_{\eta+2} \models \exists z\ \varphi(x, z)$, as desired. ∎

Having analyzed j as an ultrapower map, we now consider extending j to $V_{\eta+3}$. Recall $\widetilde{U} = \mathrm{Ult}(V_{\eta+3}, \mu)$ and $\widetilde{i}_\mu = i_\mu^{V_{\eta+3}}$.

Definition 4.5. Let $R \subseteq \mathscr{E}(V_\eta) \times V$ be a relation. A *μ-uniformization of R* is a function $f : \mathscr{E}(V_\eta) \to V$ such that for μ-measure 1 many $k \in \mathscr{E}(V_\eta)$, if there is x such that $(k, x) \in R$ then $(k, f(k)) \in R$.

The existence of μ-uniformizations is a kind of choice principle.

Lemma 4.6. *We have:*

(1) *$\widetilde{U}$ is wellfounded.*

(2) *The following are equivalent:*

  (a) *$\widetilde{U}$ is extensional,*

  (b) *j extends to a Σ_0-elementary $\ell : V_{\eta+3} \to V_{\eta+3}$,*

  (c) *for all $R \subseteq \mathscr{E}(V_\eta) \times V_{\eta+2}$, there is a μ-uniformization of R.*

(3) *If $\ell : V_{\eta+3} \to V_{\eta+3}$ is a Σ_0-elementary extension of j then identifying $\widetilde{U}$ with its transitive collapse, we have $V_{\eta+2} \subsetneq \widetilde{U} \subseteq V_{\eta+3}$ and $\ell = \widetilde{i}_\mu$ and $\ell(V_{\eta+2}) = V_{\eta+2}$.*[28]

[28] The arXiv:v1 draft of this paper over-asserted here "$\widetilde{U} \subsetneq V_{\eta+3}$, and in fact $\mu \notin \widetilde{U}$", but this is not immediately clear. If ℓ is *fully* elementary, it holds, by Theorem 3.12. And the analogous statement holds with $\eta + 2$ replaced by a limit; see Theorem 5.7.

Proof. (1) By Lemma 4.4, the part of the ultrapower formed by functions with codomain $V_{\eta+2}$ is isomorphic to $V_{\eta+2}$. It follows that $\widetilde{U}$ is wellfounded.

(2) Suppose $j \subseteq \ell \in \mathscr{E}_0(V_{\eta+3})$. We show $\ell(V_{\eta+2}) = V_{\eta+2}$. Clearly $\ell(V_{\eta+2}) \subseteq V_{\eta+2}$, so we just need $V_{\eta+2} \subseteq \ell(V_{\eta+2})$. Let $x \in V_{\eta+2}$. Then $x = j(f_x)(j \restriction V_\eta)$. But

$$V_{\eta+3} \models \text{"}f_x(k) \in V_{\eta+2} \text{ for all } k \in \mathscr{E}(V_\eta)\text{"},$$

which is a Σ_0 statement of the parameters $f_x, V_{\eta+2}, \mathscr{E}(V_\eta)$, and therefore

$$V_{\eta+3} \models \text{"}\ell(f_x)(k) \in \ell(V_{\eta+2}) \text{ for all } k \in \ell(\mathscr{E}(V_\eta))\text{"},$$

but $j \subseteq \ell$, and it follows that $x = \ell(f_x)(j \restriction V_\eta) \in \ell(V_{\eta+2})$.

We next show that $\widetilde{i}_\mu = \ell$. We know $i_\mu = j$ already, so consider $X \in V_{\eta+3} \setminus V_{\eta+2}$, so $X \subseteq V_{\eta+2}$. Let $x \in V_{\eta+2}$. Let

$$D = \{k \in \mathscr{E}(V_\eta) : f_x(k) \in X\}.$$

Then $x \in \widetilde{i}_\mu(X)$ iff $D \in \mu$ iff $j \restriction V_\eta \in j(D) = \ell(D)$ iff (by Σ_0-elementarity) $\ell(f_x)(j \restriction V_\eta) \in \ell(X)$ iff $x = j(f_x)(j \restriction V_\eta) \in \ell(X)$.

Now let us deduce that (c) holds. So let $R \subseteq \mathscr{E}(V_\eta) \times V_{\eta+2}$ and let E be the domain of R, that is,

$$E = \{k \in \mathscr{E}(V_\eta) : \exists x \, [(k, x) \in R]\}.$$

We may assume $E \in \mu$, so $j \restriction V_\eta \in j(E)$. Now $R \in V_{\eta+3}$ and

$$V_{\eta+3} \models \forall k \in E \; \exists x \in V_{\eta+2} \, [(k, x) \in R].$$

So by Σ_0-elementarity and since $\ell(V_{\eta+2}) \subseteq V_{\eta+2}$ (in fact we have equality),

$$V_{\eta+3} \models \forall k \in \ell(E) \; \exists x \in V_{\eta+2} \, [(k, x) \in \ell(R)],$$

and since $E \in \mu$, we can therefore fix $x \in V_{\eta+2}$ with $(j \restriction V_\eta, x) \in \ell(R)$. We claim that f_x is a μ-uniformization of R. Indeed, suppose instead that

$$C = \{k \in \mathscr{E}(V_\eta) : (k, f_x(k)) \notin R\} \in \mu.$$

Then $j \restriction V_\eta \in j(C) = \ell(C)$, and by Σ_0-elementarity, $(j \restriction V_\eta, \ell(f_x)(j \restriction V_\eta)) \notin \ell(R)$, so $(j \restriction V_\eta, x) \notin \ell(R)$, a contradiction.

Now assume (c) holds (μ-uniformization); we will show that $\widetilde{U}$ is extensional and Σ_0-Łoś's theorem holds for $\widetilde{U}$, which implies that

$$\widetilde{i}_\mu : V_{\eta+3} \to \widetilde{U} \subseteq V_{\eta+3}$$

is Σ_0-elementary, and therefore in fact $\widetilde{i}_\mu : V_{\eta+3} \to V_{\eta+3}$ is Σ_0-elementary.

For extensionality, let $f, g : \mathscr{E}(V_\eta) \to V_{\eta+3}$ be such that $[f] \neq [g]$, that is,

$$D = \{k \in \mathscr{E}(V_\eta) : f(k) \neq g(k)\} \in \mu.$$

Then define the relation

$$R = \{(k,x) \in \mathscr{E}(V_\eta) \times V_{\eta+2} : x \in f(k) \triangle g(k)\}.$$

Note that for all $k \in D$, there is x with $(k,x) \in R$. So we can μ-uniformize R with some $h : \mathscr{E}(V_\eta) \to V_{\eta+2}$. Since μ is an ultrafilter, either (i) for μ-measure 1 many k, we have $h(k) \in f(k) \setminus g(k)$, or (ii) vice versa. Suppose (i) holds. Then $[h] \in [f]$ and $[h] \notin [g]$, verifying extensionality for $[f], [g]$.

It follows now that $\widetilde{U}$ is isomorphic to some subset of $V_{\eta+3}$ (and we already know $V_{\eta+2} \subseteq \widetilde{U}$). Now observe that the assumed μ-uniformization is enough for the proof of Σ_0-Łoś's theorem. It follows as usual that $\widetilde{i}_\mu$ is Σ_0-elementary as a map $V_{\eta+3} \to \widetilde{U}$, and hence as a map $V_{\eta+3} \to V_{\eta+3}$, as desired.

Finally, suppose that μ-uniformization as in (c) fails; we will show that $\widetilde{U}$ is not extensional. Let $R \subseteq \mathscr{E}(V_\eta) \times V_{\eta+2}$ be a counterexample to μ-uniformization. We have the constant function $c_\emptyset$. Define $f : \mathscr{E}(V_\eta) \to V_{\eta+3}$ by

$$f(k) = \{x : (k,x) \in R\}.$$

Note that $f(k) \neq \emptyset$ for almost all k. So $[f] \neq [c_\emptyset]$. But there is no g such that $[g] \in [f]$, and therefore $\widetilde{U}$ is non-extensional with respect to $[f], [c_\emptyset]$.

(3) We already saw these things in the proof of (2). ■

Definition 4.7 (Canonical extension via ultrapowers). Let η be even.

For $j \in \mathscr{E}_1(V_{\eta+2})$, we define $j^+ : V_{\eta+3} \to V_{\eta+3}$ as $j^+ = \widetilde{i}_{\mu_j}$, as above.

For $x \in V_{\eta+2}$, let $f_x : \mathscr{E}(V_\eta) \to V_{\eta+2}$ be defined by $f_x(k) = (k^+)^{-1}``x$ (really f_x depends on η, but this should be clear in context).

Remark 4.8. We now reprove Theorem 1.1, by induction, using the canonical extension j^+ just defined. The argument is essentially as before, so we just give a sketch. Let λ be a limit and $j : V_{\lambda+2} \to V_{\lambda+2}$ be elementary. Let $\mu = \mu_j$. By Lemma 4.4, $V_{\lambda+2} = \mathrm{Ult}(V_{\lambda+2}, \mu)$ and $j = i_\mu^{V_{\lambda+2}}$ is the ultrapower map.

We claim j is not definable over $V_{\lambda+2}$ from parameters. Indeed, suppose j is definable over $V_{\lambda+2}$ from $p \in V_{\lambda+2}$. Then $p \in \mathrm{rg}(j(f_p))$, since

$$p = [f_p]_\mu^{V_{\lambda+2}} = j(f_p)(j \restriction V_\lambda).$$

One can now argue as in the proof of Theorem 3.12 (much as in Theorem 3.3) to reach a contradiction.

Next, if $\ell : V_{\lambda+3} \to V_{\lambda+3}$ is elementary and $j = \ell \restriction V_{\lambda+2}$, then $\ell = j^+$ by the preceding lemmas. But $\mu \in V_{\lambda+3}$ (μ as above), and it is straightforward to see that the ultrapower map $j^+ = \widetilde{i}_\mu = i_\mu^{V_{\lambda+3}}$ is definable over $V_{\lambda+3}$ from the parameter μ, or equivalently, from j. So ℓ is definable as desired.

Now suppose $j : V_{\lambda+4} \to V_{\lambda+4}$ is elementary. Let $\mu = \mu_j$ (the measure derived from j with seed $j \restriction V_{\lambda+2}$). Then since $j \restriction V_{\lambda+3} = (j \restriction V_{\lambda+2})^+$, the lemmas show that $\mathrm{Ult}(V_{\lambda+4}, \mu) = V_{\lambda+4}$ and j is the ultrapower map, so as before, we deduce that j is not definable from parameters, etc.

4.3. Reinhardt ultrafilters

Let λ be even. One can abstract out a notion of filter which corresponds precisely to elementary embeddings in $\mathscr{E}(V_{\lambda+2})$, and also filters which correspond to embeddings in $\mathscr{E}_{n+1}(V_{\lambda+2})$, for each $n < \omega$. The filters below are over $V_{\lambda+1}$, but one could consider instead filters over $\mathscr{E}(V_\lambda)$, identifying $j \in \mathscr{E}(V_\lambda)$ with $\mathrm{rg}(j)$, and with a small abuse of notation, we treat the two interchangeably.

Definition 4.9 (Reinhardt ultrafilters). Let λ be even and μ be an ultrafilter over $V_{\lambda+1}$. We say that μ is

(1) *rank-Jónsson* iff $\sigma = \{A : A \preccurlyeq V_\lambda$ and A has transitive collapse $V_\lambda\} \in \mu$,

(2) *fine* iff for each $x \in V_\lambda$, we have $\tau_x = \{A : x \in A \subseteq V_\lambda\} \in \mu$,

(3) *normal* iff for each $\langle \sigma_x \rangle_{x \in V_\lambda} \subseteq \mu$, the diagonal intersection

$$\Delta_{x \in V_\lambda} \sigma_x = \{A : A \subseteq V_\lambda \text{ and } A \in \sigma_x \text{ for each } x \in A\} \text{ is in } \mu,$$

(4) *pre-Reinhardt* iff it is rank-Jónsson, fine and normal,

(5) $\Sigma_1^{\lambda+2}$*-Reinhardt* iff it is pre-Reinhardt and every $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ can be μ-uniformized,

(6) $\Sigma_{n+2}^{\lambda+2}$*-Reinhardt* iff it is pre-Reinhardt and every $R \subseteq V_{\lambda+1} \times V_{\lambda+2}$ which is Π_n-definable over $V_{\lambda+2}$ from parameters can be μ-uniformized,

(7) $\Sigma_\omega^{\lambda+2}$*-Reinhardt* iff it is $\Sigma_{n+1}^{\lambda+2}$-Reinhardt for all $n < \omega$.

Note that if $x \in V_{\lambda+i}$, where $i \leq 1$, then $f_x : \mathscr{E}(V_\lambda) \to V_{\lambda+i}$ where f_x is as in Definition 4.7.

Lemma 4.10. *Let μ be a pre-Reinhardt ultrafilter over $V_{\lambda+1}$. Let $U = \mathrm{Ult}(V_{\lambda+1}, \mu)$. Then U is extensional, wellfounded and isomorphic to $V_{\lambda+1}$. Moreover, $[\mathrm{id}]_\mu = i_\mu$“$V_\lambda$ and $[f_x]_\mu = x$ for each $x \in V_{\lambda+1}$.*

Proof. We start by considering V_λ. Write $[f] = [f]_\mu$.

Claim. $\mathrm{Ult}(V_\lambda, \mu) = V_\lambda$ *and* $x = [f_x]$ *for each* $x \in V_\lambda$.

Proof. Given $x, y \in V_\lambda$, we have

$$([f_x] \in^U [f_y] \Leftrightarrow x \in y) \quad \text{and} \quad ([f_x] =^U [f_y] \Leftrightarrow x = y).$$

For by rank-Jónssonness and fineness, for μ-measure 1 many $k \in \mathscr{E}(V_\lambda)$, we have $x, y \in \mathrm{rg}(k)$, and for all such k, note $f_x(k) = k^{-1}(x)$ and $f_y(k) = k^{-1}(y)$. This yields the stated equivalences. Now let $f : \mathscr{E}(V_\lambda) \to V_\lambda$. We claim that there is $x \in V_\lambda$ such that $[f] =^U [f_x]$. Indeed, suppose not; then for each $x \in V_\lambda$, defining

$$\sigma_x = \{k \in \mathscr{E}(V_\lambda) : f(k) \neq f_x(k)\},$$

we get $\sigma_x \in \mu$. So

$$\sigma = \{k \in \mathscr{E}(V_\lambda) : f(k) \neq f_x(k) \text{ for all } x \in k\text{“}V_\lambda\} \in \mu,$$

so $\sigma \neq \emptyset$. Let $k \in \sigma$ and $\bar{x} = f(k) \in V_\lambda$. Let $x = k(\bar{x})$. Then $f_x(k) = k^{-1}(x) = \bar{x} = f(k)$, a contradiction. ■

Now let $x \in V_{\lambda+1} \setminus V_\lambda$ and $y \in V_\lambda$. Then $[f_y] \in^U [f_x]$ iff $y \in x$, because $y \in \mathrm{rg}(k)$ for μ-almost every k. Note also that $[f_x] \notin^U [f_y]$. It also easily follows that if $x' \in V_{\lambda+1} \setminus V_\lambda$ with $x' \neq x$ then $[f_x] \neq [f_{x'}]$ (consider $[f_y]$ for some $y \in x \triangle x'$).

For extensionality, let $f : \mathscr{E}(V_\lambda) \to V_{\lambda+1}$ and $x = \{y \in V_\lambda : [f_y] \in^U [f]\}$. We claim $[f] =^U [f_x]$. To see this, for each $y \in V_\lambda$, let

$$\sigma_y = \{k \in \mathscr{E}(V_\lambda) : f_y(k) \in f_x(k) \Leftrightarrow f_y(k) \in f(k)\}.$$

Note $\sigma_y \in \mu$. Let $\sigma \in \mu$ be the diagonal intersection, and note $f(k) = f_x(k)$ for each $k \in \sigma$. So $[f] =^U [f_x]$, as desired, and extensionality follows easily.

The fact that $[\mathrm{id}] = i_\mu``V_\lambda$ is a straightforward consequence of fineness and normality. The rest of the lemma now follows easily. ■

Remark 4.11. We now characterize the elements of $\mathscr{E}_{n+1}(V_{\lambda+2})$ as the ultrapower maps given by $\Sigma^{\lambda+2}_{n+1}$-Reinhardt ultrafilters, and hence the elements of $\mathscr{E}(V_{\lambda+2})$ as the ultrapower maps via $\Sigma^{\lambda+2}_{\omega}$-Reinhardt ultrafilters. Note that because of the μ-uniformization aspect of Reinhardt ultrafilters, the theorem shows that weak choice principles follow from the existence of appropriate elementary embeddings.

Theorem 4.12. *Let λ be even and $n < \omega$. Then:*

(1) *If $j \in \mathscr{E}_{n+1}(V_{\lambda+2})$ then μ_j is a $\Sigma^{\lambda+2}_{n+1}$-Reinhardt ultrafilter and $j = i^{V_{\lambda+2}}_{\mu_j}$.*

(2) *Let μ be a $\Sigma^{\lambda+2}_{n+1}$-Reinhardt ultrafilter, $U = \mathrm{Ult}(V_{\lambda+2}, \mu)$ and $j : V_{\lambda+2} \to U$ be the ultrapower map $j = i^{V_{\lambda+2}}_{\mu}$. Then:*
   - (a) *U is extensional and wellfounded, $U = V_{\lambda+2}$, $\mu = \mu_j$, $[\mathrm{id}] = j``V_\lambda$ and $x = [f_x] = j(f_x)(j``V_\lambda)$ for each $x \in V_{\lambda+2}$.*
   - (b) *$j \in \mathscr{E}_{n+1}(V_{\lambda+2})$.*

Proof. (1) Let $\mu = \mu_j$. Rank-Jónssonness and fineness are straightforward. Consider normality, and fix $\vec{\sigma} = \langle \sigma_x \rangle_{x \in V_\lambda} \subseteq \mu$, and let $\langle \sigma'_x \rangle_{x \in V_\lambda} = j(\vec{\sigma})$. Let $B = \Delta_{x \in V_\lambda} \sigma_x$. We must see that

$$j``V_\lambda \in j(B) = \Delta_{x \in V_\lambda} \sigma'_x.$$

But if $y \in j``V_\lambda$ then $y = j(x)$ for some $x \in V_\lambda$, and $\sigma_x \in \mu$, so $j``V_\lambda \in j(\sigma_x) = \sigma'_y$, as desired.

Now let $U = \mathrm{Ult}(V_{\lambda+2}, \mu)$. By Lemma 4.4, $U = V_{\lambda+2}$ and $j = i^{V_{\lambda+2}}_{\mu}$. Let us verify that μ is $\Sigma^{\lambda+2}_1$-Reinhardt. Let $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ and $D \in \mu_j$ be such that for all $k \in D$, there is $x \in V_{\lambda+1}$ with $(k, x) \in R$. Then by Σ_1-elementarity and since $j \upharpoonright V_\lambda \in j(D)$, there is $x \in V_{\lambda+1}$ with $(j \upharpoonright V_\lambda, x) \in j(R)$. Fix such an x. We have $x = j(f_x)(j \upharpoonright V_\lambda)$. So letting D' be the set of all $k \in D$ such that $(k, f_x(k)) \in R$, we have $D' \in \mu$, so we are done.

Now suppose $j \in \mathscr{E}_{n+2}(V_{\lambda+2})$, let ψ be a Π_n formula, and let $p \in V_{\lambda+2}$ and $D \in \mu$ be such that for all $k \in D$, there is $x \in V_{\lambda+2}$ with $V_{\lambda+2} \models \psi(p,k,x)$. The assertion "$\forall k \in D\ \exists x\ \psi(p,k,x)$" is Π_{n+2} in parameters D, p. So by Σ_{n+2}-elementarity and since $j \upharpoonright V_\lambda \in j(D)$, we can fix $x \in V_{\lambda+2}$ such that

$$V_{\lambda+2} \models \psi(j(p), j \upharpoonright V_\lambda, x).$$

Let D' be the set of all $k \in D$ with $V_{\lambda+2} \models \psi(p,k,f_x(k))$. We claim $D' \in \mu$, giving the desired μ-uniformization. So suppose otherwise. Then $E = \mathscr{E}(V_\lambda) \setminus D' \in \mu$, and $V_{\lambda+2} \models \forall k \in E\ [\neg\psi(p,k,f_x(k))]$. But then by Σ_{n+1}-elementarity and since $j \upharpoonright V_\lambda \in j(E)$, we get $V_{\lambda+2} \models \neg\psi(j(p), j \upharpoonright V_\lambda, x)$, a contradiction.

(2)(a) By Lemma 4.10, we already know $V_{\lambda+1} = \mathrm{Ult}(V_{\lambda+1}, \mu)$ (including extensionality and wellfoundedness) and $x = [f_x]$ for all $x \in V_{\lambda+1}$. Note that it follows that $U = \mathrm{Ult}(V_{\lambda+2}, \mu)$ is wellfounded (though we have not yet shown extensionality).

Now μ is $\Sigma_1^{\lambda+2}$-Reinhardt. Using this, extensionality is just as in the proof of Lemma 4.6. So we identify U with its Mostowski collapse, so $V_{\lambda+1} \subseteq U \subseteq V_{\lambda+2}$. Similarly to extensionality, Σ_0-Łoś's theorem holds. The Σ_1-elementarity of $j : V_{\lambda+2} \to U$ follows: if $U \models \exists w\ \varphi(j(x), w)$ where φ is Σ_0, then there is f with $U \models \varphi(j(x), [f])$, and by Σ_0-Łoś, $V_{\lambda+2} \models \varphi(x, f(k))$ for μ-measure 1 many k, so $V_{\lambda+2} \models \exists w\ \varphi(x, w)$. And because $[\mathrm{id}] = j``V_\lambda$ by Lemma 4.10, it is easy to see that $\mu = \mu_j$ (although we have not shown that $U = V_{\lambda+2}$, we can still define μ_j as before).

To see $U = V_{\lambda+2}$, it suffices to see that $[f_x] = x$ for each $x \in V_{\lambda+2}$, and for this, given $y \in V_{\lambda+1}$, we must see that $[f_y] \in^U [f_x]$ iff $y \in x$. To see the latter, it suffices to show that $y \in \mathrm{rg}(k^+)$ for μ-measure 1 many k, because for all such k, we have $y \in x$ iff

$$f_k(y) = (k^+)^{-1}(y) \in (k^+)^{-1}``x = f_k(x).$$

Let D be the set of all $k \in \mathscr{E}(V_\lambda)$ such that $y \in \mathrm{rg}(k^+)$. Then since j is Σ_1-elementary and $j(V_{\lambda+1}) = V_{\lambda+1}$, $j(D)$ is the set of all $k \in \mathscr{E}(V_\lambda)$ such that $j(y) \in \mathrm{rg}(k^+)$. But $j \upharpoonright V_{\lambda+1} = (j \upharpoonright V_\lambda)^+$, so $j \upharpoonright V_\lambda \in j(D)$, so D is μ-measure 1, as desired.

Finally, we already have $[\mathrm{id}] = j``V_\lambda$, and $x = [f_x]$ for each $x \in V_{\lambda+2}$. But then as in the proof of Lemma 4.4, the factor map $\pi : U \to V_{\lambda+2}$, defined by $\pi([f_x]) = j(f_x)(j \upharpoonright V_\lambda)$, is surjective and in fact is the identity, so $x = j(f_x)(j \upharpoonright V_\lambda)$.

(2)(b) For $n = 0$, this was verified above. So suppose $m < \omega$ and μ is $\Sigma_{m+2}^{\lambda+2}$-Reinhardt; we show j is Σ_{m+2}-elementary. Let φ be Π_{m+1} and suppose that $V_{\lambda+2} \models \varphi(j(x), y)$. We have $y = [f_y]$. Let D be the set of all $k \in \mathscr{E}(V_\lambda)$ such that $V_{\lambda+2} \models \varphi(x, f_y(k))$. It suffices to see that $D \in \mu$, so suppose $E = \mathscr{E}(V_\lambda) \setminus D \in \mu$. Let ψ be a Π_m formula such that

$$\neg\varphi(u,v) \iff \exists w\ \psi(u,v,w).$$

So $V_{\lambda+2} \models \forall k \in E\ \exists w\ \psi(x, f_y(k), w)$. Since μ is $\Sigma_{m+2}^{\lambda+2}$-Reinhardt, there are $E' \in \mu$ and $g : E' \to V_{\lambda+2}$ such that $V_{\lambda+2} \models \forall k \in E'\ \psi(x, f_y(k), g(k))$. By induction, j is Σ_{m+1}-elementary, and as $y = j(f_y)(j \upharpoonright V_\lambda)$ and $j \upharpoonright V_\lambda \in j(E')$, we get $V_{\lambda+2} \models \psi(j(x), y, j(g)(j \upharpoonright V_\lambda))$, so $V_{\lambda+2} \models \neg\varphi(j(x), y))$, a contradiction. ∎

5. Σ_1-elementarity at limit rank-to-rank

It is natural to ask whether we can prove a version of Theorem 1.1 when we assume less than full elementarity of the maps. Here we focus on the limit case; the successor case is less clear. If we only demand Σ_0-elementarity, then the embedding can easily be definable, even without parameters:

Example 5.1. Assume ZFC, let μ be a normal measure and $j : V \to \mathrm{Ult}(V, \mu)$ be the ultrapower map, and identify $\mathrm{Ult}(V, \mu)$ with transitive $M \subseteq V$. Then note that in fact, $j : V \to V$ is Σ_0-elementary and definable from the parameter μ. There are models of set theory in which the least measurable cardinal carries a unique normal ultrafilter, and in this case there will be a Σ_0-elementary embedding from V to V that is definable without parameters.

We now consider the case that δ is a limit and $j \in \mathscr{E}_1(V_\delta)$. We need some more standard set-theoretic notions, but expressed appropriately for the ZF context.

Definition 5.2. Let $\kappa \in \mathrm{OR}$. We say κ is *inaccessible* iff whenever $\alpha < \kappa$ and $\pi : V_\alpha \to \kappa$, then $\mathrm{rg}(\pi)$ is bounded in κ. The *cofinality* $\mathrm{cof}(\kappa)$ of κ is the least $\eta \in \mathrm{OR}$ such that there is a map $\pi : \eta \to \kappa$ with $\mathrm{rg}(\pi)$ unbounded in κ. We say κ is *regular* iff $\mathrm{cof}(\kappa) = \kappa$.

A *norm* on a set X is a surjective function $\pi : X \to \eta$ for some $\eta \in \mathrm{OR}$. The associated *prewellorder* on X is the relation R on X given by xRy iff $\pi(x) \leq \pi(y)$. One can also axiomatize prewellorders on X as those relations R on X which are linear, total, reflexive, with wellfounded strict part (the *strict part* is the relation $x <_R y$ iff $[xRy$ and $\neg yRx]$). Note that a prewellorder naturally gives rise to a wellorder of the equivalence classes with respect to the equivalence relation $x \equiv_R y$ iff $xRyRx$. The *ordertype* of R is the ordertype of this wellorder.

If κ is regular but non-inaccessible, and $\alpha \in \mathrm{OR}$ is least such that there is a cofinal map $\pi : V_\alpha \to \kappa$, then the *Scott ordertype* of κ, denoted $\mathrm{scot}(\kappa)$,[29] is the set of all prewellorders of V_α whose ordertype is κ.

Remark 5.3. Suppose κ is regular but not inaccessible, and let α be as above and $\pi : V_\alpha \to \kappa$ be cofinal. Then $\mathrm{rg}(\pi)$ has ordertype κ, as otherwise κ is singular. Moreover, α is a successor ordinal, for otherwise, by the minimality of α, we get a cofinal function $f : \alpha \to \kappa$ by defining $f(\beta) = \sup(\pi``V_\beta)$ for $\beta < \alpha$, again contradicting regularity.

Definition 5.4. Let δ be a limit and $j \in \mathscr{E}_1(V_\delta)$. Define $j_0 = j$ and for $n \geq 0$ define $j_{n+1} = j^+(j_n)$. Say $x \in V_\delta$ is (j, n)-*stable* iff $j_m(x) = x$ for all $m \in [n, \omega)$.

Say that j is *nicely stable* iff either (i) δ is inaccessible, or (ii) δ is singular and $j(\mathrm{cof}(\delta)) = \mathrm{cof}(\delta)$, or (iii) δ is regular non-inaccessible and $j(\mathrm{scot}(\delta)) = \mathrm{scot}(\delta)$.

For $j : V_\delta \to V_\delta$ and $A, B \subseteq V_\delta$, say $j : (V_\delta, A) \to (V_\delta, B)$ is $(\Sigma_n$-$)$*elementary* iff j is $(\Sigma_n$-$)$elementary in the language $\mathscr{L}_{\dot{A}}$, with $\dot{A}$ interpreted by the predicates A, B respectively.

[29]This is an abbreviation of *Sc*ott *o*rder*t*ype. The second author thanks Asaf Karagila for suggesting the terminology *Scott ordertype*.

The following fact is a special case of Gaifman's [7, Theorem II.1, p. 54]:[30]

Theorem 5.5 (Gaifman). *Let $\delta \in \mathrm{Lim}$ and $j \in \mathscr{E}_1(V_\delta)$ be nicely stable. Then $j \in \mathscr{E}(V_\delta)$. In fact, for each $A \subseteq V_\delta$, the map*

$$j_n : (V_\delta, A) \to (V_\delta, j_n^+(A))$$

is fully elementary.

We will include a proof of Gaifman's theorem later for self-containment. But first we indicate how we will use it:

Theorem 5.6. *Let $\delta \in \mathrm{Lim}$ and $j \in \mathscr{E}_1(V_\delta)$.*[31] *Then:*

(1) *$j_n : V_\delta \to V_\delta$ is Σ_1-elementary; in fact, $j_n : (V_\delta, A) \to (V_\delta, j_n^+(A))$ is Σ_1-elementary for every $A \subseteq V_\delta$.*
(2) *$j_{n+1} = j_n^+(j_n)$.*
(3) *If $x \in V_\delta$ and $j_n(x) = x$ then x is (j, n)-stable.*
(4) *For each $\alpha < \delta$ there is $n < \omega$ such that α is (j, n)-stable.*
(5) *For each $\alpha < \delta$ and $\xi \in \mathrm{OR}$, letting P be the set of all prewellorders of V_α of length ξ, there is $n < \omega$ such that P is (j, n)-stable.*
(6) *There is $n < \omega$ such that j_n is nicely stable.*

Proof. For this proof we just write $j(A)$ instead of $j^+(A)$, and $j_n(A)$ instead of $j_n^+(A)$, for $A \subseteq V_\delta$. Note this is unambiguous when $A \in V_\delta$.

(1) Let $\alpha < \delta$ and $\alpha' = j(\alpha)$ and $j' = j \upharpoonright V_\alpha$. So $j' : V_\alpha \to V_{\alpha'}$ is fully elementary. This fact is preserved by j, by Σ_1-elementarity. Clearly also $j(j)$ maps V_δ to V_δ, and is therefore Σ_0-elementary. But $j(j)$ is also $\in$-cofinal, hence Σ_1-elementary (with respect to $\in$).

For the Σ_1-elementarity of $j_n : (V_\delta, A) \to (V_\delta, j_n(A))$, let $x \in V_\delta$ and φ be Σ_0 (in the expanded language), and suppose

$$(V_\delta, j_n(A)) \models \exists y\ \varphi(j_n(x), y).$$

Let $\alpha < \delta$ be sufficiently large that $x \in V_\alpha$ and

$$(V_{j_n(\alpha)}, j_n(A) \cap V_{j_n(\alpha)}) \models \exists y\ \varphi(j_n(x), y).$$

Then by the Σ_1-elementarity of j_n (just in the language with $\in$),

$$(V_\alpha, A \cap V_\alpha) \models \exists y\ \varphi(x, y),$$

so $(V_\delta, A) \models \exists y\ \varphi(x, y)$ as desired.

[30] Gaifman's theorem is more general, and is not specific to rank-to-rank embeddings.
[31] Recall that by Lemma 2.2, $j(V_\alpha) = V_{j(\alpha)}$ for each $\alpha < \delta$.

(2) For $n = 0$ this is just the definition. For $n = 1$ note that

$$j_2 = j(j_1) = j(j(j)) = (j(j))(j(j)) = j_1(j_1).$$

The rest is similar.

(3) If $x = j(x)$ then $j(x) = j(j(x)) = j(j)(j(x)) = j(j)(x)$.

(4) Suppose not and let $\alpha < \delta$ be least otherwise. We use the argument in [18], which is just a slight variant on the standard proof of linear iterability. For $n < \omega$ let $A_n = \{\beta < \alpha : j_n(\beta) = \beta\}$. So $\alpha = \bigcup_{n<\omega} A_n$ and $\langle A_n \rangle_{n<\omega} \in V_\delta$. Note $j(A_n) = \{\beta < j(\alpha) : j_{n+1}(\beta) = \beta\}$ and

$$j(\alpha) = j\Big(\bigcup_{n<\omega} A_n\Big) = \bigcup_{n<\omega} j(A_n).$$

But $\alpha < j(\alpha)$ by choice of α and (3), so $\alpha \in j(A_n)$ for some n, so $j_{n+1}(\alpha) = \alpha$, a contradiction.

(5) By the above, there is n_0 such that α is (j, n_0)-stable. Now argue as in the previous part from n_0 onward, and using the parameter α, define the collection P of prewellorders of V_α of the form $P = P_\xi$ for some ordinal ξ, with ξ least such that for no $n \in [n_0, \omega)$ is $j_n(P) = P$. Here $\xi \geq \delta$ is possible. Note that the notion of *prewellorder* (regarding relations $R \in V_\delta$) is simple enough that it is preserved by our Σ_1-elementary maps. Likewise, the lengths of two prewellorders can be compared in a simple enough fashion, and hence we always have $j_n(P_\xi) = P_{\xi'_n}$ for some ordinal ξ'_n. In fact, $\xi'_n > \xi$, and one can now argue for a contradiction much as before.

(6) By (4) and (5). ∎

We now include a proof of Theorem 5.5 (it is essentially the same as Gaifman's proof):

Proof of Theorem 5.5. If δ is inaccessible then for every $A \subseteq V_\delta$, $(V_\delta, A) \models \mathrm{ZF}(A)$. By Theorem 5.6 (1), $j : (V_\delta, A) \to (V_\delta, j(A))$ is Σ_1-elementary. Therefore a direct relativization of Lemma 2.3 shows that j is fully elementary in the expanded language.

Claim 1. *j is Σ_2-elementary (with respect to A).*

Proof. Consider first the case that δ is singular and let $\gamma = \mathrm{cof}(\delta)$. By replacing j with an iterate of j, we may assume $j(\gamma) = \gamma$. Let $A \subseteq V_\delta$. We know $j : (V_\delta, A) \to (V_\delta, j(A))$ is Σ_1-elementary.

Let $x \in V_\delta$ and φ be Π_1 and suppose that

$$(V_\delta, j(A)) \models \exists y\, \varphi(j(x), y),$$

and let $\beta < \delta$ be such that some $y \in V_{j(\beta)}$ witnesses this.

Let $f : \gamma \to \delta$ be cofinal and increasing. For $\xi < \gamma$ let

$$B_\xi = \{z \in V_\beta : (V_{f(\xi)}, A \cap V_{f(\xi)}) \models \varphi(x, z)\}.$$

Then note that

$$j(B_\xi) = \{z \in V_{j(\beta)} : (V_{j(f(\xi))}, j(A) \cap V_{j(f(\xi))}) \models \varphi(j(x), z)\}.$$

Therefore $y \in j(B_\xi)$, so in fact $y \in (\bigcap_{\xi<\gamma} j(B_\xi)) \neq \emptyset$. As $\gamma < \delta$, we have $\langle B_\xi \rangle_{\xi<\gamma} \in V_\delta$. Also,

$$\xi_0 < \xi_1 \implies B_{\xi_1} \subseteq B_{\xi_0}.$$

So the same holds of $j(\langle B_\xi \rangle_{\xi<\gamma})$, and since $j(\gamma) = \gamma$, we have $j``\gamma$ cofinal in $j(\gamma)$, and so letting $j(\langle B_\xi \rangle_{\xi<\gamma}) = \langle B'_\xi \rangle_{\xi<\gamma}$,

$$j\Big(\bigcap_{\xi<\gamma} B_\xi\Big) = \bigcap_{\xi<\gamma} B'_\xi = \bigcap_{\xi<\gamma} B'_{j(\xi)} = \bigcap_{\xi<\gamma} j(B_\xi) \neq \emptyset.$$

So $\bigcap_{\xi<\gamma} B_\xi \neq \emptyset$. But letting $z \in \bigcap_{\xi<\gamma} B_\xi$, note $(V_\delta, A) \models \varphi(x, z)$, as desired.

Now suppose instead that δ is regular non-inaccessible. Define $\langle B_\xi \rangle_{\xi<\delta}$ as before, except that now $f(\xi) = \xi$ for $\xi < \delta$. Like before, we just need to see that $\bigcap_{\xi<\delta} B_\xi \neq \emptyset$, so assume otherwise. Then there is no $\xi_0 < \delta$ such that $B_\xi = B_{\xi_0}$ for all $\xi \in [\xi_0, \delta)$. Given $z_0, z_1 \in B = \bigcup_{\xi<\delta} B_\xi$, say that $z_0 <^* z_1$ iff there is $\xi < \delta$ such that $z_1 \in B_\xi$ but $z_0 \notin B_\xi$. Then $<^*$ is a prewellorder on B, and $<^*$ is in V_δ, and by regularity, δ is the ordertype of $<^*$. So let $P = \mathrm{scot}(\delta)$. By assumption $j(P) = P$, which easily implies that $j(<^*)$ also has ordertype δ. The function $z \mapsto B_{\mathrm{rank}^*(z)}$, with domain B, and where $\mathrm{rank}^*(z)$ is the $<^*$-rank of z, is also in V_δ. But then we can argue as before to show $\bigcap_{\xi<\delta} B_\xi \neq \emptyset$, a contradiction. ∎

Now suppose we have Σ_k-elementarity where $k \geq 2$. Define the theory

$$T = T^A_{k-1} = \mathrm{Th}^{(V_\delta, A)}_{\Sigma_{k-1}}(V_\delta);$$

this denotes the set of all pairs (φ, x) such that φ is a Σ_{k-1} formula and $(V_\delta, A) \models \varphi(x)$. The Σ_k-elementarity of j gives

Claim 2. $j(T) = \mathrm{Th}^{(V_\delta, j(A))}_{\Sigma_{k-1}}(V_\delta)$.

Proof. Given $\alpha < \delta$, we have

$$(V_\delta, A) \models \forall x \in V_\alpha \left[\forall\ \Sigma_{k-1} \text{ formulas } \varphi \text{ of } \mathcal{L}_{\dot{A}}\ [\varphi(x) \Leftrightarrow (\varphi, x) \in T \cap V_\alpha]\right],$$

which is a Π_k assertion of parameter $(V_\alpha, T \cap V_\alpha)$, which therefore lifts to $(V_\delta, j(A))$ regarding the parameter $(V_{j(\alpha)}, j(T) \cap V_{j(\alpha)})$. ∎

So by what we have proved above, but with (A, T) replacing A, we deduce that j is Σ_2-elementary as a map

$$j : (V_\delta, (A, T)) \to (V_\delta, (j(A), j(T))). \tag{5.1}$$

Now let φ be Σ_{k-1} and suppose that

$$(V_\delta, j(A)) \models \exists y \forall z\ [\varphi(j(x), y, z)],$$

or equivalently

$$(V_\delta, (j(A), j(T))) \models \exists y\ \forall z\ [(\varphi, (j(x), y, z)) \in j(T)].$$

By the Σ_2-elementarity of j with respect to the structures in (5.1) above, we have

$$(V_\delta, (A, T)) \models \exists y\ \forall z\ [(\varphi, (x, y, z)) \in T],$$

or equivalently $(V_\delta, A) \models \exists y \forall z\ [\varphi(x, y, z)]$, as desired. ∎

Using the preceding theorem, we now improve on Theorem 3.3:

Theorem 5.7. *Let $j \in \mathscr{E}_1(V_\delta)$ where $\delta \in \mathrm{Lim}$. Then:*

(1) *j is not definable from parameters over V_δ.*

(2) *There is no (a, f) with $a \in V_\delta$ and $f : V_\delta \to V_{\delta+1}$ and $j^+(f)(a) = j$.*

Remark 5.8. The reader familiar with extenders will note that in the proof of (2), we are considering $\mathrm{Ult}(V_{\delta+1}, E)$ where E is the V_δ-extender derived from j. As before, we can represent functions $f : V_\delta \to V_{\delta+1}$ via relations $\subseteq V_\delta \times V_\delta$, and hence with elements of $V_{\delta+1}$, and when $j \in \mathscr{E}_1(V_\delta)$, one gets $j^+(f) : V_\delta \to V_{\delta+1}$, making sense of the statement of (2) above.

Proof. (1) Suppose otherwise. Then by Theorem 5.6, there is $n < \omega$ such that $j_n : V_\delta \to V_\delta$ is fully elementary, and since j is definable from parameters over V_δ, so is j_n. This contradicts Theorem 3.3.

(2) Suppose otherwise and fix a counterexample (j, a, f). Then for each $n < \omega$, (j_n, a_n, f_n) is also a counterexample, where $(a_0, f_0) = (a, f)$ and $(a_{k+1}, f_{k+1}) = (j_k(a_k), j_k^+(f_k))$. (Indeed, one can apply j to each initial segment of the sets corresponding to the equation $j^+(f)(a) = j$, and their union yields $j_1^+(j^+(f))(j(a)) = j_1$, so $j_1^+(f_1)(a_1) = j_1$, etc.) So by Theorem 5.6, we may assume j is nicely stable, and so by Theorem 5.5, $j^+ : V_{\delta+1} \to V_{\delta+1}$ is Σ_0-elementary. Let $\mathscr{I}$ be the set of functions $g : V_\delta \to V_{\delta+1}$. We have

$$V_{\delta+1} = \{j^+(g)(a) : g \in \mathscr{I}\},$$

because if $y \in V_{\delta+1}$ then $y = j^{-1}``j^+(y)$, so letting $g(u) = f(u)^{-1}``y$ (where $j^+(f)(a) = j$), we get $y = j^+(g)(a)$. It follows that j^+ is Σ_1-elementary: Let φ be Σ_0 and suppose $V_{\delta+1} \models \varphi(j^+(x), y)$. Let $g : V_\delta \to V_{\delta+1}$ be such that $j^+(g)(a) = y$. Then there is $b \in V_\delta$ such that $V_{\delta+1} \models \varphi(j(x), j^+(g)(b))$, and this can be expressed as a Σ_0 statement about $j(x), j^+(g), V_\delta$ (in particular dealing with the codings of functions), and therefore there is $b \in V_\delta$ such that $V_{\delta+1} \models \varphi(x, g(b))$.

Now if δ is singular, let $p = \mathrm{cof}(\delta)$, and if δ is regular non-inaccessible, let $p = \mathrm{scot}(\delta)$, and otherwise let $p = \emptyset$. Let κ_0 be the least critical point of all $k \in \mathscr{E}_1(V_\delta)$ such that $k(p) = p$ and $k = k^+(h)(c)$ for some $c \in V_\delta$ and $h : V_\delta \to V_{\delta+1}$. Fix j_0, h_0, c_0 witnessing the choice of κ_0. By the preceding discussion, $j_0 \in \mathscr{E}(V_\delta)$ and $j_0^+ \in \mathscr{E}_1(V_{\delta+1})$. We have $p \in \mathrm{rg}(j_0^+)$, but $\kappa_0 \notin \mathrm{rg}(j_0^+)$.

Let $\eta = j_0(\kappa_0)$. Then $V_{\delta+1} \models$ "there are k, μ, h, c such that $k \in \mathscr{E}_1(V_\delta)$ and $\mathrm{crit}(k) = \mu < \eta$ and $k(p) = p$ and $h : V_\delta \to V_{\delta+1}$ and $c \in V_\delta$ and $k^+(h)(c) = k$" (as witnessed by j_0, κ_0, h_0, c_0). Since $j_0(p, \kappa_0) = (p, \eta)$ and by the Σ_1-elementarity of j_0^+, we can fix some such $\mu \in \mathrm{rg}(j_0)$. But note $\kappa_0 \leq \mu < \eta$, by the minimality of κ_0, a contradiction. ∎

Many of the arguments applied in this section to rank-to-rank embeddings also apply more generally, and in particular to embeddings consistent with ZFC. The following is also analogous to Suzuki's [20, Theorem 3.1] on $j : M \to V$.

Theorem 5.9. *Let $\eta < \delta$ be limit ordinals and $j : V_\eta \to V_\delta$ be Σ_1-elementary and $\in$-cofinal. Then:*

(1) *If j is fully elementary then j is not definable over V_δ from parameters.*

(2) *If $\mu = \mathrm{cof}(\eta) < \eta$ and $j(\mu) = \mu$ then for every $A \subseteq V_\eta$, defining*

$$j(A) = \bigcup_{\beta<\eta} j(A \cap V_\beta),$$

the map $j : (V_\eta, A) \to j(V_\delta, j(A))$ is fully elementary.

Proof. (2) This is also due to Gaifman [7]; the proof is very similar to that for the singular case of Theorem 5.5.

(1) Suppose not. Then δ is singular, definably from parameters over V_δ, as witnessed by $j \upharpoonright \eta : \eta \to \delta$. Let $\mu = \mathrm{cof}(\delta) = \mathrm{cof}(\eta)$. Using the elementarity of j, it easily follows that there is $n < \omega$ such that both V_η and V_δ satisfy "There is a function $k : \mu \to \mathrm{OR}$ which is Σ_n-definable from parameters, and μ is least such", and $j(\mu) = \mu$. Note that it also follows that μ is definable over V_δ without parameters.

Now fix a formula φ and $p \in V_\delta$ such that $j(x) = y$ iff $V_\delta \models \varphi(p, x, y)$. For $q \in V_\delta$ let $j_q = \{(x, y) : V_\delta \models \varphi(q, x, y)\}$. Say q is *good* iff there is a limit $\eta' < \delta$ such that $j_q : V_{\eta'} \to V_\delta$ is Σ_1-elementary and $\in$-cofinal and $j_q(\mu) = \mu$. By (2), if q is good then j_q is fully elementary. Then the least critical point among all good j_q is definable over V_δ without parameters, which leads to the usual contradiction. ∎

Of course, in the situation above, the iterates j_n of j are not well-defined (at least not in their earlier form), so we have not ruled out the possibility of $j : V_\eta \to V_\delta$ which is Σ_1-elementary and $\in$-cofinal with $j(\mu) > \mu$, which is definable from parameters.

The following theorem, due to Andreas Lietz and the second author, shows that if a Reinhardt cardinal exists then it is at times necessary to pass from j to j_n to secure full elementarity:[32]

Theorem 5.10 (Lietz and Schlutzenberg). *Suppose $j \in \mathscr{E}(V_{\lambda+})$ where $\lambda = \kappa_\omega(j)$. Then for each $n < \omega$ there is a limit $\delta < \lambda^+$ such that $j``\delta \subseteq \delta$ and $k = j \upharpoonright V_\delta \in \mathscr{E}_1(V_\delta)$, but $k = k_0, k_1, \ldots, k_n \notin \mathscr{E}_2(V_\delta)$.*

Proof. First consider $n = 0$. Let $\kappa = \mathrm{crit}(j)$ and $\delta = \lambda + \kappa$ and $k = j \upharpoonright V_\delta$. Since $j(\lambda) = \lambda$ and $j \upharpoonright \kappa = \mathrm{id}$, we have $k : V_\delta \to V_\delta$, and clearly k is $\in$-cofinal and Σ_0-elementary, hence Σ_1-elementary. But consider the Π_2 formula

$$\varphi(\dot\kappa, \dot\lambda) = \text{“}\forall \alpha < \dot\kappa\ \exists \xi \in \mathrm{OR}\ [\xi = \dot\lambda + \alpha]\text{”}.$$

[32]The second author initially noticed the $n = 1$ example, then Lietz generalized this to $n > 1$ via basically the method at the end of the proof, but from a stronger assumption to secure fixed points, and then the second author observed the claim on fixed points, leading to the version here.

Then $V_{\lambda+\kappa} \models \varphi(\kappa, \lambda)$, but $V_{\lambda+\kappa} \models \neg\varphi(j(\kappa), j(\lambda))$; that is, $V_{\lambda+\kappa} \models \neg\varphi(j(\kappa), \lambda)$, since $\alpha = \kappa < j(\kappa)$, but $\lambda + \kappa \notin V_{\lambda+\kappa}$. For this example, $k_1(\kappa) = \kappa = \mathrm{cof}(\lambda + \kappa)$, so k_1 is fully elementary, by Theorem 5.6.

Now let n be arbitrary. Note that if $n > 0$, it does not suffice to replace κ above with $\kappa_n = \mathrm{crit}(j_n)$, since then $j``(\lambda + \kappa_n) \not\subseteq \lambda + \kappa_n$.

Claim. *j has λ^+-many fixed points $< \lambda^+$.*

Proof. Let $F_n = \{\alpha < \lambda^+ : j_n(\alpha) = \alpha\}$. By Theorem 5.6, $\lambda^+ = \bigcup_{n<\omega} F_n$. The ordertypes α_n of the F_n are then either unbounded in λ^+, or some α_n equals λ^+, since otherwise one easily constructs a surjection $\pi : \lambda \to \lambda^+$ (consider the uncollapse maps $\pi_n : \alpha_n \to F_n$). Now F_0 is unbounded in λ^+. Indeed, suppose not, and let $\sup(F_0) < \beta_0 < \lambda^+$. Let $\pi_0 : \lambda \to \beta_0$ be a surjection. Let $\pi_{n+1} = j(\pi_n)$ and $\beta_{n+1} = \mathrm{rg}(\pi_{n+1}) = j(\beta_n)$. From $\langle \pi_n \rangle_{n<\omega}$ we get a surjection $\lambda \to \beta = \sup_{n<\omega} \beta_n$. Therefore $\beta < \lambda^+$, but note $\mathrm{cof}(\beta) = \omega$, so $j(\beta) = \beta$, contradicting the choice of β_0. Now $\alpha_0 = \lambda^+$.[33] Indeed, suppose not. Then note $\alpha_{n+1} = \sup j``\alpha_n = \sup j_n``\alpha_n$ (using the fact that F_n is cofinal in λ^+). Then letting $\alpha_0 < \eta \in F_0$, we have $\alpha_n < \eta$ for all $n < \omega$, a contradiction. ∎

Now let δ be the supremum of the first $\mathrm{crit}(j_n)$ fixed points of j which are $> \lambda$. Then $j``\delta \subseteq \delta$, so $k = j \upharpoonright V_\delta \in \mathscr{E}_1(V_\delta)$. Let W be a wellorder of λ in ordertype δ (note $\lambda < \delta < \lambda^+$, so W exists). Then

$$V_\delta \models \text{“every proper segment of } W \text{ has ordertype some } \alpha \in \mathrm{OR}\text{”}. \tag{5.2}$$

But for $m \leq n$, $k_m(W)$ is a wellorder of $k_m(\lambda) = \lambda$ in ordertype some δ'_m, and $\delta < \delta'_m$, because (i) the ordertype of W is $\leq$ that of $k_m(W)$, and (ii) $\mathrm{cof}(W) = \mathrm{crit}(k_n)$, so $\mathrm{cof}(k_m(W)) = k_m(\mathrm{crit}(k_n)) = \mathrm{crit}(k_{n+1})$. Since $\delta < \delta'_m$, V_δ does not satisfy (5.2) with W replaced by $k_m(W)$, so k_m is not Σ_2-elementary. ∎

Remark 5.11. There is a variant giving $k : V_\delta \to V_\delta$ with $k_0, \ldots, k_n$ being Σ_{m+1}- but not Σ_{m+2}-elementary, and k_{n+1} elementary, where $1 \leq m < \omega$: Suppose $j : V_\xi \to V_\xi$ is elementary where $V_\xi \models \mathrm{ZF}$ and $\lambda = \kappa_\omega(j) < \xi$. Let γ be the supremum of the first $\mathrm{crit}(j_n)$ fixed points of j which are $\geq \lambda$. Let δ be the γth ordinal $\beta < \xi$ with $V_\beta \preccurlyeq_m V_\xi$. Then $k = j \upharpoonright V_\delta : V_\delta \to V_\delta$ works (cf. the proof of Lemma 2.3).

6. Which ordinals are large enough?

We said in the introduction that if an ordinal η is large enough, then V_η and $V_{\eta+1}$ are very different from each other. We have seen that there are such differences assuming there is an elementary $j : V_{\eta+1} \to V_{\eta+1}$. So we could take this as the definition of “large enough”, but then the term is not very natural, because then $\eta + 1$ need not be also “large enough”.

[33] Note it is not obvious that λ^+ is regular. The first author has results regarding this.

To get a good notion of "large enough", we assume that there is a Reinhardt cardinal. Let then $j : V \to V$ be elementary with $\kappa_\omega(j)$ minimal; recall that $\kappa_\omega(j)$ is the supremum of the critical sequence of j (see the Introduction).[34] Say η is "large enough" iff $\eta \geq \kappa_\omega(j)$. Recall ZF($j$) was defined in §1.1. Working in ZF(j), we can assert that "$j : V \to V$ is elementary" with the single formula "$j : V \to V$ is Σ_1-elementary", by Lemma 2.3. The following theorem was mentioned to the first author by Peter Koellner a few years ago, but may be folklore. There are some further related things in [17]:

Theorem 6.1 (Folklore). *Assume* ZF(j) *and* $j : V \to V$ *is elementary non-identity. Let* $\lambda = \kappa_\omega(j)$. *Then for all* $\alpha \geq \lambda$ *and all* $\eta < \lambda$, *there is an elementary* $k : V_\alpha \to V_\alpha$ *such that* $\text{crit}(k) > \eta$ *and* $\kappa_\omega(k) = \lambda$.

Proof. Suppose not and let (η, α) be the lexicographically least counterexample. Then (η, α) is definable from the parameter λ, and hence fixed by j. But then $j(\alpha) = \alpha$, so $j \restriction V_\alpha : V_\alpha \to V_\alpha$, and $j(\eta) = \eta < \lambda$, so $\eta < \text{crit}(j) = \text{crit}(j \restriction V_\alpha)$, so $j \restriction V_\alpha$ contradicts the choice of (η, α). ■

So above $\lambda = \kappa_\omega(j)$, the cumulative hierarchy is periodic the whole way up.

Remark 6.2. For the reader familiar with [3], note that the property stated of $\lambda = \kappa_\omega(j)$ in the theorem above is just that of a Berkeley cardinal (see [3]) *with respect to rank segments of* V (except that we have also stated it for V_λ itself, although $\lambda \notin V_\lambda$). One could call such a λ a *rank-Berkeley cardinal*. Note that unlike Reinhardtness, rank-Berkeleyness is first-order. If there is a Reinhardt, then which is less, the least Reinhardt or the least rank-Berkeley? If $j : V \to V$ and $\lambda = \kappa_\omega(j)$ is the least rank-Berkeley, then note that for every $k : V \to V$ with $\text{crit}(k) < \lambda$, we have $\kappa_\omega(k) = \lambda$. In particular, if κ is super Reinhardt then the least rank-Berkeley is $< \kappa$. We show next that the least rank-Berkeley, being below the least Reinhardt, has consistency strength beyond that of a Reinhardt.

Every rank-Berkeley is HOD-Berkeley. Is the least HOD-Berkeley cardinal less than the least rank-Berkeley?

Theorem 6.3. *Suppose* $(V, j) \models \text{ZF}(j)$ *and* $j : V \to V$, *and let* $\kappa = \text{crit}(j)$ *and* $\lambda = \kappa_\omega(j)$, *and suppose the least rank-Berkeley is* $\delta < \lambda$. *Let* μ_j *be the normal measure over* κ *derived from* j. *Then* $\delta < \kappa$ *and there is* $\kappa' < \delta$ *such that for* μ_j*-measure* 1 *many* $\gamma < \kappa$, $(V_\gamma, V_{\gamma+1}) \models$ *"*κ' *is a Reinhardt cardinal".*

Proof. Suppose $\delta < \lambda$ is rank-Berkeley, so $\delta < \kappa$. Then there is $k : V_\kappa \to V_\kappa$ which is elementary and not the identity. Let $\kappa' = \text{crit}(k)$. Then κ is inaccessible and $(V_\kappa, V_{\kappa+1}) \models \text{ZF}_2$+"$\kappa'$ is Reinhardt, as witnessed by k". Since $\kappa = \text{crit}(j)$, the theorem follows routinely. ■

[34] Note that the minimization of $\kappa_\omega(j)$ need not be valid in general (why can we quantify over such j?). But we will ignore this here; in fact the minimization is only to define "large enough" unambiguously.

Corollary 6.4. *Suppose* ZF(j) + "$j : V \to V$" *is consistent. Then so is*

$$\mathrm{ZF}(j) + \text{“}j : V \to V\text{”} + \text{“}\kappa_\omega(j) \textit{ is the least rank-Berkeley”}.$$

This also shows that $\lambda = \kappa_\omega(j)$ can be definable over V without parameters. But there is anyway another way to see that $j : V \to V$ with λ non-definable is stronger than just $j : V \to V$. Since λ is a limit of inaccessibles, if λ is non-definable, then V has inaccessibles $\delta > \lambda$, and taking the least such, we have $j(\delta) = \delta$, so we get $(V_\delta, V_{\delta+1}) \models \mathrm{ZF}_2+$"There is a Reinhardt" (actually the latter holds for every inaccessible $\delta > \lambda$, since $j_n(\delta) = \delta$ for some n).

7. Questions and related work

In §5 we ruled out the definability of Σ_1-elementary embeddings $j : V_\delta \to V_\delta$ for δ a limit. Note that we also observed that for δ even, Σ_1-elementary maps $j : V_{\delta+1} \to V_{\delta+1}$ are always definable from the parameter $j \upharpoonright V_\delta$. But what about partially elementary maps $V_{\delta+2} \to V_{\delta+2}$? Can they be definable from parameters over $V_{\delta+2}$? If so, what can one say about the complexity of the definition in relation to the degree of elementarity?

One can also generalize the notion of "definable from parameters" to allow higher-order definitions, such as looking in $L(V_\delta)$. If δ is a limit and $L(V_\delta) \models$ "$\mathrm{cof}(\delta) > \omega$" then $L(V_\delta)$ has no elementary $j : V_\delta \to V_\delta$ (see [16]; the case that δ is inaccessible was established earlier by the first author). There is a little on the cofinality ω case in [16], but this case is much more subtle.

The existence of the canonical extension j^+ of an embedding $j : V_\lambda \to V_\lambda$ for limit λ is of fundamental importance to the analysis of I_0; see for example [22]. But we now have the generalization of this to all even λ. It turns out that much of the I_0 theory generalizes in turn, and this is one of the topics of [8]. Various stark structural differences between V_λ and $V_{\lambda+1}$ are revealed there.

Of course a significant question looming over this work is whether embeddings of the form we are considering can even exist. Some recent progress in this regard, establishing the consistency of ZF + $j : V_{\lambda+2} \to V_{\lambda+2}$ relative to ZFC + I_0, is the topic of [19].

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