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Moore matrices and Ulrich bundles on an elliptic curve

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Abstract. We give normal forms of determinantal representations of a smooth projective plane cubic in terms of *Moore matrices*. Building on this, we exhibit matrix factorizations for all indecomposable vector bundles of rank 2 and degree 0 without nonzero sections, also called *Ulrich bundles*, on such curves.

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To Frank-Olaf Schreyer on the occasion of his sixtieth birthday

1. Linear determinantal representations of smooth Hesse cubics and statement of the results

1.1. Let K be a field and $f \in S = K[x_0, \dots, x_n]$ a nonzero homogeneous polynomial of degree $d \geqslant 1$. A matrix $M = (m_{ij})_{i,j=0,\dots,d-1} \in \mathsf{Mat}_{d \times d}(S)$ is *linear* if its entries are linear homogeneous polynomials and it provides a *linear determinantal representation* of f if $\det M \doteq f$, where $\doteq$ means that the two quantities are equal up to a nonzero scalar from K. More generally, if M is a linear matrix for which there exists a matrix M' with $M \cdot M' = f \,\mathrm{id}_d = M' \cdot M$, then M is an *Ulrich matrix* and $\mathbf{F} = \operatorname{Coker} M$, necessarily annihilated by f, is an *Ulrich module* over $R = S/(f)$.

If $V(f) = \{f = 0\} \subset \mathbb{P}^n$, the projective hypersurface defined by f, is smooth, then the sheafification of $\mathbf{F}$ is a vector bundle, called an *Ulrich bundle*.

Our aim here is to give normal forms for all Ulrich bundles of rank 1 or 2 over a plane elliptic curve in Hesse form.

Linear determinantal representations of hypersurfaces have been studied since, at least, the middle of the 19th century and for a very recent comprehensive treatment for curves and surfaces see Dolgachev's monograph [5]. This reference contains as well a detailed study of the geometry of smooth cubic curves, especially of those in Hesse form and we refer to it for background material.

For smooth plane projective curves, the state-of-the-art result is due to Beauville.

1.2 Theorem ([4, Proposition 3.1]). *Let $C = V(f)$ be a smooth plane projective curve of degree d defined by an equation $f = 0$ in $\mathbb{P}^2$ over K. With $g = \frac{1}{2}(d-1)(d-2)$ the genus of C, one has the following:*

(a) *Let L be a line bundle of degree $g-1$ on C with $H^0(X, L) = 0$. Then there exists a $d \times d$ linear matrix M such that $f = \det M$ and an exact sequence of $\mathcal{O}_{\mathbb{P}^2}$-modules*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^d \longrightarrow L \longrightarrow 0.$$

(b) *Conversely, let M be a $d \times d$ linear matrix such that $f \doteq \det M$. Then the cokernel of $M\colon \mathcal{O}_{\mathbb{P}^2}(-2)^d \to \mathcal{O}_{\mathbb{P}^2}(-1)^d$ is a line bundle L on C of degree $g-1$ with $H^0(X, L) = 0$.* ■

In this result, the matrix M clearly determines L uniquely up to isomorphism, but L determines M only up to equivalence of matrices in that the cokernel of every matrix PMQ^{-1} for $P, Q \in \mathrm{GL}(d, K)$ yields a line bundle isomorphic to L. The so obtained action on these matrices of the group $\mathsf{G}_d = \mathrm{GL}(d, K) \times \mathrm{GL}(d, K)/\mathbb{G}_m(K)$, with $\mathbb{G}_m(K)$ the diagonally embedded subgroup of nonzero multiples of the identity matrix, is free and proper; see [4, Proposition 3.3]. The geometric quotient by this group action identifies with the affine variety $\mathrm{Jac}^{g-1}(C) \setminus \Theta$, the Jacobian of C of line bundles of degree $g-1$ minus the theta divisor Θ of those line bundles of that degree that have a nonzero section.

There is therefore the issue of finding useful representatives, or normal forms, for such linear representations in a given orbit. Realizing hyperelliptic curves as double covers of $\mathbb{P}^1$, Mumford [14, IIIa, Section 2] exhibited canonical presentations for such line bundles, which motivated Beauville's work [3], and Laza–Pfister–Popescu in [11] found such representative matrices for the Fermat cubic, while for general elliptic curves in Weierstraß form Galinat [9] determines normal forms of those linear representations.

Our first result yields the following normal forms for plane elliptic curves in Hesse form. Here, and in the sequel, we denote $[a_0 : \cdots : a_n] \in \mathbb{P}^n(K)$ the K-rational point with homogeneous coordinates $a_i \in K$, not all zero.

The name Moore matrix in the context of elliptic curves was introduced by K. Ranestad [16] analogously to Moore matrix for abelian varieties.

In the paper we assume that K is an algebraically closed field.[1]

1.3 Theorem A. *If* char $K \neq 2, 3$*, each linear determinantal representation of the smooth plane projective curve* E *with equation*

$$x_0^3 + x_1^3 + x_2^3 + \lambda x_0x_1x_2 = 0, \quad \lambda \in K, \; \lambda^3 + 27 \neq 0,$$

is equivalent to a Moore matrix

$$M_{(a_0,a_1,a_2),\mathbf{x}} = (a_{i+j}x_{i-j})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = \begin{pmatrix} a_0x_0 & a_1x_2 & a_2x_1 \\ a_1x_1 & a_2x_0 & a_0x_2 \\ a_2x_2 & a_0x_1 & a_1x_0 \end{pmatrix}$$

with $\mathbf{a} = [a_0 : a_1 : a_2] \in E$ *and* $a_0a_1a_2 \neq 0$.

Two such Moore matrices $M_{(a_0,a_1,a_2),\mathbf{x}}$ *and* $M_{(a_0',a_1',a_2'),\mathbf{x}}$ *yield equivalent linear determinantal representations of* E *if, and only if,*

$$3 \cdot_E \mathbf{a} = 3 \cdot_E \mathbf{a}',$$

where $3 \cdot_E \mathbf{a} = \mathbf{a} +_E \mathbf{a} +_E \mathbf{a}$ *is calculated with respect to the group law* $+_E$ *on* E *whose identity element is an inflection point of* E.

1.4 Remarks. We make the following two remarks.

(a) Tripling a point $\mathbf{a}$ in the group law on E results in the same point, no matter which inflection point is chosen as origin.

Furthermore, choosing an inflection point as origin for the group law, the exceptional points $\mathbf{a} \in E$ with $a_0a_1a_2 = 0$ are precisely the 3-torsion points, equivalently, the inflection points of E. They form the subgroup

$$E[3] = \{[1 : -\omega : 0], [0 : 1 : -\omega], [-\omega : 0 : 1] \mid \omega \in K, \omega^3 = 1\} \subseteq E,$$

isomorphic to the elementary abelian 3-group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ of rank 2.

(b) In geometric terms, the preceding result states that the map $\mathbf{a} \mapsto \operatorname{coker} M_{a,x}$ from $E \setminus E[3]$ to its punctured Jacobian $\operatorname{Jac}^0(E) \setminus \{[\mathcal{O}_E]\} \cong E \setminus E[1]$ of line bundles of degree 0 without nonzero sections is well defined and is (isomorphic to the restriction of) the isogeny[2] of degree 9 that is given by multiplication by 3 on E.

Building on the previous result, our second contribution is as follows.

[1]Likely, it suffices that K contains six distinct sixth roots of unity, which forces char $K \neq 2, 3$. However, one key ingredient in the proof (see (2.19) below), is stated in the literature only over algebraically closed fields, thus, we are compelled to make that assumption too.

[2]We thank Steve Kudla for suggesting this interpretation.

1.5 Theorem B. *Let E be the smooth plane cubic curve from above.*

(a) *Let F be an indecomposable vector bundle of rank 2 and degree 0 on E. If $H^0(E,F)=0$, then there exists $a=(a_0,a_1,a_2)\in K^3$ representing a point $\mathbf{a}\in E$ with $a_0a_1a_2\neq 0$ such that the sequence of $\mathcal{O}_{\mathbb{P}^2}$-modules*

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^6 \xrightarrow{\begin{pmatrix} M_{a,x} & M_{b,x}\\ 0 & M_{a,x}\end{pmatrix}} \mathcal{O}_{\mathbb{P}^2}(-1)^6\longrightarrow F\longrightarrow 0,$$

with $b=(b_0,b_1,b_2)\in K^3$ representing $2\cdot_E \mathbf{a}$, is exact.

(b) *Conversely, if a represents $\mathbf{a}\in E$ with $a_0a_1a_2\neq 0$ and b represents $2\cdot_E\mathbf{a}$, then the cokernel of the block matrix*

$$\begin{pmatrix} M_{a,x} & M_{b,x}\\ 0 & M_{a,x}\end{pmatrix}:\mathcal{O}_{\mathbb{P}^2}(-2)^6\rightarrow\mathcal{O}_{\mathbb{P}^2}(-1)^6$$

is an indecomposable vector bundle F of rank 2 and degree 0 on E that has no nonzero sections, $H^0(E,F)=0$.

(c) *Replacing a by a' results in a vector bundle isomorphic to F if, and only if, $3\cdot_E\mathbf{a}=3\cdot_E\mathbf{a}'$ on E.*

In the next section we will review some known facts about elliptic curves in Hesse form and will prove Theorem A. In Section 3 we will establish Theorem B.

2. Proof of Theorem A

To lead up to the proof of Theorem A, we first review some ingredients. To begin with, we review how Moore matrices encode conveniently the group law on a smooth Hesse cubic E. We set $S=K[x_0,x_1,x_2]$.

Moore matrices and their rank.

2.1 Definition. With $a=(a_0,a_1,a_2)\in K^3$, and $x=(x_0,x_1,x_2)\in S^3$ the vector of coordinate linear forms, the *Moore matrix* defined by a is

$$M_{a,x}=(a_{i+j}x_{i-j})_{i,j\in\mathbb{Z}/3\mathbb{Z}}=\begin{pmatrix} a_0x_0 & a_1x_2 & a_2x_1\\ a_1x_1 & a_2x_0 & a_0x_2\\ a_2x_2 & a_0x_1 & a_1x_0\end{pmatrix}$$

with *adjugate*, or *signed cofactor matrix*

$$\begin{aligned} M_{a,x}^{\mathrm{adj}} &= (a_{i+j-1}a_{i+j+1}x_{j-i}^2 - a_{i+j}^2 x_{j-i-1}x_{j-i+1})_{i,j\in\mathbb{Z}/3\mathbb{Z}} \\ &= \begin{pmatrix} a_1a_2x_0^2 - a_0^2x_1x_2 & a_0a_2x_1^2 - a_1^2x_0x_2 & a_0a_1x_2^2 - a_2^2x_0x_1 \\ a_0a_2x_2^2 - a_1^2x_0x_1 & a_0a_1x_0^2 - a_2^2x_1x_2 & a_1a_2x_1^2 - a_0^2x_0x_2 \\ a_0a_1x_1^2 - a_2^2x_0x_2 & a_1a_2x_2^2 - a_0^2x_0x_1 & a_0a_2x_0^2 - a_1^2x_1x_2 \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \det M_{a,x} &= M_{a,x}M_{a,x}^{\mathrm{adj}} = M_{a,x}^{\mathrm{adj}}M_{a,x} \\ &= a_0a_1a_2(x_0^3+x_1^3+x_2^3) - (a_0^3+a_1^3+a_2^3)x_0x_1x_2. \end{aligned}$$

If now $a_0a_1a_2 \neq 0$, then set $\lambda = (a_0^3+a_1^3+a_2^3)/a_0a_1a_2 \in K$ to obtain

$$\det M_{a,x} \doteq f := x_0^3 + x_1^3 + x_2^3 - \lambda x_0x_1x_2,$$

whence $M_{a,x}$ indeed yields a determinantal presentation of the cubic curve $C = V(f)$ and the point $\mathbf{a} = [a_0 : a_1 : a_2]$ in $\mathbb{P}^2(K)$ underlying a lies on C.

Note that C will be smooth if, and only if, $\lambda^3 \neq 27$ in K. In the smooth case we write $E = V(f)$ to remind the reader that this curve is *elliptic* over K in that it is smooth of genus 1 and contains at least one point, for example $[0 : -1 : 1]$, defined over K.

The next result is well known and easily established through, say, explicit calculation as in [8, Lemma 3].

2.2 Lemma. *For $E = V(f)$ a smooth cubic in Hesse form and every pair a, b with $\mathbf{a}, \mathbf{b} \in E$, the (specialized) Moore matrix $M_{a,b}$ is of rank 2.* ∎

In the situation of the preceding lemma, basic linear algebra tells us that the one-dimensional null space

$$M_{a,b}^{\perp} = \{(c_0, c_1, c_2) \in K^3 \mid M_{a,b} \cdot (c_0, c_1, c_2)^T = 0\}$$

is spanned by the column vectors of $M_{a,b}^{\mathrm{adj}}$. Following Dolgachev, we denote

$$\mathfrak{l}(M_{a,b}) = \mathbf{c} = [c_0 : c_1 : c_2] \in \mathbb{P}^2(K)$$

the point underlying the one-dimensional space $M_{a,b}^{\perp}$ in $\mathbb{P}^2$.

A less obvious result, part of [8, Theorem 4], is that $\mathbf{c}$ will again be a point of E along with $\mathbf{a}, \mathbf{b}$.

Before we turn to the group structure, let us note that the transpose of a Moore matrix is again a Moore matrix,

$$M_{a,b}^T = (a_{j+i}b_{j-i})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = (a_{i+j}b_{-(i-j)})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = M_{a,\iota(b)},$$

where ι is the involution $\iota(x_0, x_1, x_2) = (x_0, x_2, x_1)$, or, counting indices modulo 3, $\iota(x_j) = x_{-j}$. For use below, we follow again Dolgachev and set

$$\mathfrak{r}(M_{a,b}) = \mathfrak{l}(M_{a,b}^T) = \mathfrak{l}(M_{a,\iota(b)}) \in \mathbb{P}^2(K).$$

The point $\mathfrak{r}(M_{a,b}) = \mathbf{d} = [d_0 : d_1 : d_2]$ thus represents the (row) space ${}^{\perp}M_{a,x}$ of solutions to the linear system of equations $d \cdot M_{a,b} = 0$.

2.3 Theorem (see [8, Theorem 4]). *The assignment* $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{c} = \mathfrak{l}(M_{a,b})$ *for* $\mathbf{a}, \mathbf{b} \in E$ *defines the group law on* E *by setting* $\mathbf{b} -_E \mathbf{a} = \mathbf{c}$. *The identity element is given by* $\mathfrak{o} = [0 : -1 : 1]$ *and the inverse of* $\mathbf{a}$ *is* $-_E \mathbf{a} = \iota(\mathbf{a}) = [a_0 : a_2 : a_1]$. ■

Before continuing towards the proof of Theorem A, we take the opportunity to interpret Moore matrices geometrically in two ways, following Artin–Tate–van den Bergh [1] in the first and Dolgachev [5] in the second.

Geometric interpretation à la Artin–Tate–Van den Bergh.

2.4. In the introduction to [1] the authors consider[3] the trilinear forms

$$f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_0x_0y_0 + a_1x_2y_1 + a_2x_1y_2 \\ a_1x_1y_0 + a_2x_0y_1 + a_0x_2y_2 \\ a_2x_2y_0 + a_0x_1y_1 + a_1x_0y_2 \end{pmatrix}$$

that can as well be interpreted as a system of three linear equations in, at least, two ways:

$$\begin{pmatrix} a_0x_0 & a_1x_2 & a_2x_1 \\ a_1x_1 & a_2x_0 & a_0x_2 \\ a_2x_2 & a_0x_1 & a_1x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$$
$$= \left[\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_0y_0 & a_2y_1 & a_1y_2 \\ a_2y_2 & a_1y_0 & a_0y_1 \\ a_1y_1 & a_0y_2 & a_2y_0 \end{pmatrix} \right]^T,$$

or, shorter, in terms of Moore matrices,

$$M_{a,x} \cdot y^T = f = \left(x \cdot M_{\iota(a),\iota(y)}\right)^T = M_{\iota(a),y} \cdot x^T.$$

[3] Their indexing of the ingredients is different, but that is just due to the fact that the authors of [1] chose the point at infinity $[1 : -1 : 0]$ as the origin for the group law on E.

2.5. Viewing, for a fixed $\mathbf{a} \in E$, the f_i as sections of $\mathcal{O}(1,1)$ on $\mathbb{P}^2_x \times \mathbb{P}^2_{\mathbf{y}}$, these equations imply, by Lemma 2.2, that the subscheme

$$X = V(f_0, f_1, f_2) \subseteq \mathbb{P}^2_{\mathbf{x}} \times \mathbb{P}^2_{\mathbf{y}}$$

is mapped isomorphically by each of the projections $p_{\mathbf{x}}, p_{\mathbf{y}}\colon \mathbb{P}^2_x \times \mathbb{P}^2_{\mathbf{y}} \to \mathbb{P}^2$ onto $E \subseteq \mathbb{P}^2$, and, in light of the preceding theorem, the subscheme X constitutes the graph of the translation by $-\mathbf{a}$ on the elliptic curve in that

$$t_{-\mathbf{a}} = p_{\mathbf{y}}(p_{\mathbf{x}}|_X)^{-1}\colon E \xrightarrow{\cong} E, \quad t_{-\mathbf{a}}(\mathbf{x}) = \mathfrak{l}(M_{a,x}) = \mathbf{x} -_E \mathbf{a} = \mathbf{y}$$

when going from $\mathbb{P}^2_x$ to $\mathbb{P}^2_{\mathbf{y}}$, while it represents the graph of the translation by $\mathbf{a}$,

$$t_{\mathbf{a}} = p_{\mathbf{x}}(p_{\mathbf{y}}|_X)^{-1}\colon E \xrightarrow{\cong} E, \quad t_{\mathbf{a}}(\mathbf{x}) = \mathfrak{l}(M_{\iota(a),x}) = \mathbf{x} +_E \mathbf{a} = \mathbf{y}$$

when going in the opposite direction. In other words,

$$X = \left\{(\mathbf{x}, \mathbf{x} -_E \mathbf{a}) \in \mathbb{P}^2_{\mathbf{x}} \times \mathbb{P}^2_{\mathbf{y}} \mid \mathbf{x} \in E\right\} = \left\{(\mathbf{y} +_E \mathbf{a}, \mathbf{y}) \in \mathbb{P}^2_{\mathbf{x}} \times \mathbb{P}^2_{\mathbf{y}} \mid \mathbf{y} \in E\right\}.$$

Geometric interpretation à la Dolgachev.

2.6. Applying the treatment from [5, Section 4.1.2] to the special case of plane elliptic curves gives a geometric interpretation of the adjugate of a Moore matrix as follows.

Fixing again $\mathbf{a} \in E$, consider the closed embedding

$$(\mathfrak{l}, \mathfrak{r})_{\mathbf{a}}\colon E \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$
$$(\mathfrak{l}, \mathfrak{r})_{\mathbf{a}}(\mathbf{x}) = (\mathfrak{l}(M_{a,x}), \mathfrak{r}(M_{a,x})) = (\mathbf{x} -_E \mathbf{a}, -_E\mathbf{x} -_E \mathbf{a}),$$

and follow it with the Segre embedding $s_2\colon \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ that sends $(\mathbf{x}, \mathbf{y})$ to the class of the 3×3 matrix $[\mathbf{x}^T \cdot \mathbf{y}]$ viewed as a point in $\mathbb{P}^8$. The composition

$$\psi_{\mathbf{a}} = s_2(\mathfrak{l}, \mathfrak{r})_{\mathbf{a}}\colon E \hookrightarrow \mathbb{P}^8$$

then sends

$$\mathbf{x} \mapsto \left[M^{\mathrm{adj}}_{\mathbf{a},\mathbf{x}}\right] = (\mathbf{x} -_E \mathbf{a})^T \cdot (-_E\mathbf{x} -_E \mathbf{a}),$$

thus, E gets embedded into the Segre variety $s_2(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ through the adjugate of the Moore matrix. As to the image of E in $\mathbb{P}^2 \times \mathbb{P}^2$, if we set $\mathbf{y} = \mathbf{x} -_E \mathbf{a}$, then

$$\mathbf{x} = \mathbf{y} +_E \mathbf{a} \quad \text{and} \quad -_E\, \mathbf{x} -_E \mathbf{a} = -_E\mathbf{y} -_E 2 \cdot_E \mathbf{a}$$

so that $(\mathfrak{l}, \mathfrak{r})_{\mathbf{a}}(E)$ is the graph of the involution[4] $\mathbf{y} \mapsto -_E\mathbf{y} -_E 2 \cdot_E \mathbf{a}$ on E that one may view as the "reflection" in $-_E\mathbf{a}$.

[4]That Moore matrices define an involution on E in this way we learned from Kristian Ranestad who kindly shared his notes [16] with us.

Doubling and tripling points on E.

2.7. As an immediate application of Theorem 2.3 one can easily determine[5] $2\cdot_E \mathbf{a}$ and $3\cdot_E \mathbf{a}$ for $\mathbf{a}\in E$ in that $2\cdot_E \mathbf{a} = \mathfrak{l}(M_{\iota(a),a})$ and $3\cdot_E \mathbf{a} = \mathfrak{l}(M_{\iota(a),b})$, where $\mathbf{b} = 2\cdot_E \mathbf{a}$, and explicit coordinates are obtained from the columns of the corresponding adjugate matrices. Now

$$M_{\iota(\mathbf{a}),\mathbf{a}} = (a_{-i-j}a_{i-j})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = \begin{pmatrix} a_0^2 & a_2^2 & a_1^2 \\ a_2a_1 & a_1a_0 & a_0a_2 \\ a_1a_2 & a_0a_1 & a_2a_0 \end{pmatrix},$$

and so

$$M^{\mathrm{adj}}_{\iota(\mathbf{a}),\mathbf{a}} = (a_{-i-j}a_{i-j})^{\mathrm{adj}}_{i,j\in\mathbb{Z}/3\mathbb{Z}} = \begin{pmatrix} a_0(a_2^3-a_1^3) \\ a_2(a_1^3-a_0^3) \\ a_1(a_0^3-a_2^3) \end{pmatrix}\cdot(0,-1,1),$$

whence $2\cdot_E \mathbf{a} = [a_0(a_2^3-a_1^3) : a_2(a_1^3-a_0^3) : a_1(a_0^3-a_2^3)]$.

Next, set

$$(b_0,b_1,b_2) = (a_0(a_2^3-a_1^3), a_2(a_1^3-a_0^3), a_1(a_0^3-a_2^3))$$

and evaluate through a straightforward, though somewhat lengthy, expansion:

$$\begin{aligned} M^{\mathrm{adj}}_{\iota(a),b} &= (a_{-i-j}b_{i-j})^{\mathrm{adj}}_{i,j\in\mathbb{Z}/3\mathbb{Z}} \\ &= \begin{pmatrix} a_1a_2b_0^2 - a_0^2b_1b_2 & a_0a_1b_1^2 - a_2^2b_0b_2 & a_0a_2b_2^2 - a_1^2b_0b_1 \\ a_0a_1b_2^2 - a_2^2b_0b_1 & a_0a_2b_0^2 - a_1^2b_1b_2 & a_1a_2b_1^2 - a_0^2b_0b_2 \\ a_0a_2b_1^2 - a_1^2b_0b_2 & a_1a_2b_2^2 - a_0^2b_0b_1 & a_0a_1b_0^2 - a_2^2b_1b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_0a_1a_2(a_0^6+a_1^6+a_2^6-a_0^3a_1^3-a_1^3a_2^3-a_0^3a_2^3) \\ a_0^6a_1^3+a_1^6a_2^3+a_2^6a_0^3-3(a_0a_1a_2)^3 \\ a_0^6a_2^3+a_1^6a_0^3+a_2^6a_1^3-3(a_0a_1a_2)^3 \end{pmatrix}\cdot(a_0,a_2,a_1). \end{aligned}$$

For an additional check, note that $[a_0 : a_2 : a_1] = -_E\mathbf{a} = -2_E\mathbf{a} +_E \mathbf{a}$, so that indeed

$$\left[M^{\mathrm{adj}}_{\iota(\mathbf{a}),\mathbf{a}}\right] = (2_E\mathbf{a} +_E \mathbf{a})^T(-2_E\mathbf{a} +_E \mathbf{a}) = (3_E\mathbf{a})^T(-_E\mathbf{a}),$$

as it has to be.

When $a_0a_1a_2 \neq 0$, the case we are interested in, these results can be simplified a bit.

[5] The formulas for $2\cdot_E \mathbf{a}$ are already contained in [8]. We recall them here for completeness and later use.

2.8 Corollary. *For $a = (a_0, a_1, a_2)$, as above, representing a point $\mathbf{a} \in E$ with $a_0a_1a_2 \neq 0$, doubling, respectively tripling $\mathbf{a}$ on E results in*

$$2 \cdot_E \mathbf{a} = \left[\frac{a_2^3 - a_1^3}{a_1a_2} : \frac{a_1^3 - a_0^3}{a_0a_1} : \frac{a_0^3 - a_2^3}{a_0a_2}\right],$$

$$3 \cdot_E \mathbf{a} = \left[\frac{a_0^6 + a_1^6 + a_2^6}{(a_0a_1a_2)^2} - \frac{a_0^3a_1^3 + a_1^3a_2^3 + a_0^3a_2^3}{(a_0a_1a_2)^2} : \right.$$
$$\left. \frac{a_0^6a_1^3 + a_1^6a_2^3 + a_2^6a_0^3}{(a_0a_1a_2)^3} - 3 : \frac{a_0^6a_2^3 + a_1^6a_0^3 + a_2^6a_1^3}{(a_0a_1a_2)^3} - 3\right].$$ ■

2.9 Example. As an immediate application, one obtains the set of 6-torsion points on E in that $2 \cdot_E \mathbf{a}$ is a 3-torsion point if, and only if, $\mathbf{a}$ is a 6-torsion point. Now $E[3] = E \cap V(x_0x_1x_2)$ as was noted above. The formulas for doubling a point thus show that

$$E[6] = E \cap V\big(x_0x_1x_2(x_0^3 - x_2^3)(x_1^3 - x_3^3)(x_2^3 - x_1^3)\big)$$

is the intersection of E with the indicated 12 lines. As the four lines $V(x_0x_1x_2(x_2-x_1))$ cut out the 3- and 2-torsion points, the remaining 8 lines cut out the 24 primitive 6-torsion points as stated in [8].

It follows that $E[6] \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and that all 36 points of $E[6]$ are defined over K, as soon as char $K \neq 2, 3$ and K contains three distinct third roots of unity.

The algebraic Heisenberg group. It is a classical result in the theory of elliptic curves that translation by a 3-torsion point on a smooth cubic is afforded by a projective linear transformation; see [12, Section 5, case (b)]. We first recall the precise result and then show that the action of the relevant algebraic Heisenberg group lifts to a free action on the Moore matrices.

2.10 Definition. Let K be a field that contains three distinct third roots of unity, $\mu_3(K) = \{1, \omega, \omega^2\} \leqslant K^*$ with $\omega^3 = 1$. In terms of the matrices

$$\Sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in \mathrm{SL}(3, K),$$

of order 3, the *algebraic Heisenberg group* is

$$\mathsf{Heis}_3(K) = \{\mu T^i \Sigma^j \mid \mu \in K^*; i, j \in \mathbb{Z}/3\mathbb{Z}\} \leqslant \mathrm{GL}(3, K).$$

That this is indeed a subgroup is due to the equality $\Sigma T = \omega T \Sigma$. This same equality also shows that H_3, the subset of Heis_3, where μ is restricted to powers of ω, is a finite subgroup of $\mathrm{SL}(3, K)$ of order 27.

We begin the proof of Theorem A with the following crucial property of the algebraic Heiseisenberg group.

2.11 Proposition. *Let* $a = (a_0, a_1, a_2)$*, as before, represent a point* $\mathbf{a} \in E$ *with* $a_0a_1a_2 \neq 0$*. For* $a' = (a'_0, a'_1, a'_2) \in K^3$*, the following are equivalent:*

(1) a' *represents a point* $\mathbf{a}' \in E$ *with* $3 \cdot_E \mathbf{a}' = 3 \cdot_E \mathbf{a}$.

(2) a' *is in the* Heis_3*-orbit of* a*, thus,* $a' \in \mathsf{Heis}_3 \cdot a$.

(3) *The Moore matrices* $M_{a,x}$ *and* $M_{a',x}$ *are equivalent.*

Moreover, the action of Heis_3 *on the Moore matrices is free.*

Proof. The equivalence of (1) and (2) is, of course, classical. For (2) $\Longrightarrow$ (3), it suffices to show that the Moore matrices for $a, T(a), \Sigma(a)$ and $\mu a, \mu \in K^*$, are equivalent. This is obvious for μa as $M_{\mu a,x} = (\mu\,\mathrm{id}_3)M_{a,x}$. For $T(a) = (a_0, \omega a_1, \omega^2 a_2)$, the Moore matrix is

$$M_{T(a),x} = (\omega^{i+j} a_{i+j} x_{i-j})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = (\omega^i a_{i+j} x_{i-j} \omega^j)_{i,j\in\mathbb{Z}/3\mathbb{Z}} = T \cdot M_{a,x} \cdot T.$$

Thus, $M_{T(a),x}$ is equivalent to $M_{a,x}$. For $\Sigma(a) = (a_2, a_0, a_1)$, one verifies

$$\begin{aligned} M_{(a_2,a_0,a_1),x} &= (a_{i+j-1} x_{i-j})_{i,j\in\mathbb{Z}/3\mathbb{Z}} \\ &= (a_{(i+1)+(j+1)} x_{(i+1)-(j+1)})_{i,j\in\mathbb{Z}/3\mathbb{Z}} = \Sigma^{-1} \cdot M_{a,x} \cdot \Sigma, \end{aligned}$$

whence $M_{\Sigma(a),x}$ is indeed as well equivalent to $M_{a,x}$.

It remains to prove (3) $\Longrightarrow$ (1). If $M_{a,x}$ is equivalent to $M_{a',x}$ then these two matrices have the same determinant up to a nonzero scalar. This shows that a' represents a point on E along with a and that $a'_0a'_1a'_2 \neq 0$.

Now write

$$M_{a,x} = M_0x_0 + M_1x_1 + M_2x_2,$$

where $M_i = \partial M_{a,x}/\partial x_i \in \mathsf{Mat}_{3\times3}(K)$. The matrix $M_0 = \mathsf{diag}(a_0, a_2, a_1)$ is invertible by assumption, and we set

$$N_i = M_0^{-1} M_i \in \mathsf{Mat}_{3\times3}(K) \quad \text{and} \quad N = \sum_{i=0}^{2} N_i x_i = M_0^{-1} M_{a,x}.$$

For a' with $\mathbf{a}' \in E$ and $a'_0a'_1a'_2 \neq 0$, define N'_i, N' analogously with a' replacing a. Then the matrices $M_{a,x}, M_{a',x}$ are equivalent under the action of G_3 if, and only if N and N' are equivalent under that action. As $N_0 = N'_0 = \mathbf{1}_3$, the identity matrix, the matrices N, N' are equivalent with respect to G_3 if, and only if, N and N' are equivalent under conjugation by a matrix $P \in \mathrm{GL}(3, K)$, that is, $N' = PNP^{-1}$, equivalently,

$$N'_1 = PN_1P^{-1} \quad \text{and} \quad N'_2 = PN_2P^{-1}.$$

In other words, the pairs of 3×3 matrices (N_1, N_2) and (N_1', N_2') are related by simultaneous conjugation. Clearly the trace functions $\mathrm{tr}(A_1 \cdots A_n)$, for any n-tuple $A_i \in \{U, V\}, i = 1, \ldots, n$, are constant on the class of a pair $(U, V) \in \mathsf{Mat}_{3\times 3}(K)^2$ under simultaneous conjugation. Moreover, Teranishi [17] showed that 11 of these traces suffice to generate the ring of invariants. See [7] for a survey of these results, especially the list of the generating traces on the bottom of page 25.

We will not need any details of that invariant theory, but we easily extract from those classical results the traces that are relevant here.

2.12 Lemma. *With notation as just introduced, set further $a_{ij} = a_i/a_j$ for $i, j = 0, 1, 2$.*

(i) *The matrices N_1, N_2 have the form*

$$N_1 = \begin{pmatrix} 0 & 0 & a_{20} \\ a_{12} & 0 & 0 \\ 0 & a_{01} & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & a_{10} & 0 \\ 0 & 0 & a_{02} \\ a_{21} & 0 & 0 \end{pmatrix}.$$

(ii) *Taking traces yields*

$$\mathrm{tr}\big((N_1N_2)^2\big) = \frac{a_0^6 + a_1^6 + a_2^6}{a_0^2a_1^2a_2^2}, \quad \mathrm{tr}(N_1^2N_2^2) = \frac{a_0^3a_1^3 + a_0^3a_2^3 + a_1^3a_2^3}{a_0^2a_1^2a_2^2},$$
$$\mathrm{tr}(N_1N_2N_1^2N_2^2) = \frac{a_0^6a_1^3 + a_1^6a_2^3 + a_2^6a_0^3}{a_0^3a_1^3a_2^3}.$$

Proof. Straightforward verification. ■

Combining item (ii) in this lemma with Corollary 2.8 shows that equivalence of $M_{a,x}$ and $M_{a',x}$ forces $3 \cdot_E \mathbf{a}' = 3 \cdot_E \mathbf{a}$.

As for the final claim, this follows from Beauville's result that the action of G_3 on linear matrices is free. ■

2.13 Corollary. *The subgroup of G_3 that transforms Moore matrices into such is isomorphic to the algebraic Heisenberg group Heis_3.* ■

2.14 Remark. In light of the preceding result, we sometimes write $M_{\mathbf{a},x}$ to denote any representative of the equivalence class of $M_{a,x}$ under the action of Heis_3, with $\mathbf{a} \in E$ as before representing the point underlying $a \in K^3$.

It is indeed the representation theory of the Heisenberg groups that allows us to finish the proof of Theorem A. Instead of working with the algebraic Heisenberg groups, it suffices to restrict to the finite Heisenberg groups and their representations.

The Schrödinger representations of the finite Heisenberg groups.

2.15. To prove the main statement of Theorem A we need some observations on Heisenberg groups. The general Heisenberg group $\mathsf{H}(R)$ over a commutative ring R is usually understood to be the subgroup of unipotent upper triangular 3×3 matrices in $\mathrm{GL}(3, R)$. For $R = \mathbb{Z}/n\mathbb{Z}$, $n \geqslant 1$ an integer, we call these the *finite Heisenberg groups* and abbreviate $\mathsf{H}_n = \mathsf{H}(\mathbb{Z}/n\mathbb{Z})$. The group H_n is of order n^3 and admits the presentation

$$\mathsf{H}_n = \langle \sigma, \tau \mid [\sigma, [\sigma, \tau]] = [\tau, [\sigma, \tau]] = \sigma^n = \tau^n = 1 \rangle.$$

Each element of H_n has a unique representation as $[\sigma, \tau]^r \sigma^s \tau^t$ with $r, s, t \in \mathbb{Z}/n\mathbb{Z}$.

Note that H_3 as defined here is indeed isomorphic to the group H_3 that we exhibited as a subgroup of Heis_3 above.

2.16. Over a field K that contains a primitive nth root of unity $\zeta \in K^*$, the group H_n carries the K-linear *Schrödinger representations* $\rho_j \colon \mathsf{H}_n \to \mathrm{GL}(n, K)$, parametrized by $j \in \mathbb{Z}/n\mathbb{Z}$, that in a suitable *Schrödinger basis* $v_i, i \in \mathbb{Z}/n\mathbb{Z}$, of a vector space V of dimension n over K are given by

$$\rho_j(\sigma)(v_i) = v_{i-1}, \quad \rho_j(\tau)(v_i) = \zeta^{ij} v_i, \quad i \in \mathbb{Z}/n\mathbb{Z},$$

and thus, for a general element,

$$\rho_j\big([\sigma, \tau]^r \sigma^s \tau^t\big)(v_i) = \zeta^{j(it+r)} v_{i-s}.$$

In particular, the character χ_j of the representation ρ_j satisfies

$$\chi_j\big([\sigma, \tau]^r \sigma^s \tau^t\big) = \begin{cases} 0 & \text{if } s \not\equiv 0 \bmod n \text{ or } jt \not\equiv 0 \bmod n, \\ n\zeta^{jr} & \text{if } s \equiv jt \equiv 0 \bmod n. \end{cases}$$

2.17. If $d \geqslant 2$ is a divisor of n, say $n = dm$, then the subgroup of H_n generated by σ^m, τ^m is a homomorphic image of H_d, in that surely σ^m and τ^m are of order d, and these elements commute with $[\sigma^m, \tau^m] = [\sigma, \tau]^{m^2}$. If we restrict the Schrödinger representation ρ_j of H_n along the resulting homomorphism $\mathsf{H}_d \to \mathsf{H}_n$, then it decomposes in that the actions of σ^m and τ^m, given by

$$\rho_j(\sigma^m)(v_i) = v_{i-m}, \quad \rho_j(\tau^m)(v_i) = (\zeta^m)^{ij} v_i \quad \text{for } i \in \mathbb{Z}/n\mathbb{Z},$$

yield the H_d-subrepresentations

$$W_{jk} = \bigoplus_{i \equiv k \bmod m} K v_i \subseteq V \quad \text{for } k \in \mathbb{Z}/m\mathbb{Z}.$$

For $i = \alpha m + k$, in the basis $w_\alpha = v_{\alpha m+k}$, for $\alpha = 0, \dots, d-1$, of W_{jk} the action is given by

$$\rho_j(\sigma^m)(w_\alpha) = w_{\alpha-1}, \quad \rho_j(\tau^m)(w_\alpha) = (\zeta^m)^{j(\alpha m+k)} w_\alpha,$$

and, for a general element

$$\begin{aligned}\rho_j.\big([\sigma^m, \tau^m]^r (\sigma^m)^s (\tau^m)^t\big)(w_\alpha) &= \rho_j\big([\sigma, \tau]^{m^2 r} \sigma^{ms} \tau^{mt}\big)(w_\alpha)\\ &= (\zeta^m)^{j((\alpha m+k)t+mr)} w_{\alpha-s}.\end{aligned}$$

The corresponding character is thus

$$\rho_j\big([\sigma^m, \tau^m]^r (\sigma^m)^s (\tau^m)^t\big) = \begin{cases} 0 & \text{if } s \not\equiv 0 \bmod d \text{ or } jmt \not\equiv 0 \bmod d,\\ d(\zeta^m)^{j(kt+mr)} & \text{if } s \equiv jmt \equiv 0 \bmod d.\end{cases}$$

If $\gcd(d, m) = 1$, then $jmt \equiv 0 \bmod d$ if, and only if $jt \equiv 0 \bmod d$ and one recognizes the Schrödinger representation ρ_{jm} of H_d. Therefore, we have the following result.

2.18 Lemma. *For d a positive divisor of n with $\gcd(d, n/d) = 1$, under the group homomorphism $\mathsf{H}_d \to \mathsf{H}_n$ described above the Schrödinger representation ρ_j of H_n restricts to the direct sum of n/d copies of the Schrödinger representation $\rho_{(jn/d) \bmod d}$ of H_d.* ∎

2.19. Returning to elliptic curves, let, more generally, L be an ample line bundle on an abelian variety defined over an algebraically closed field K whose characteristic does not divide the degree $n > 0$ of L. It is a deep result from the theory of abelian varieties; see [13, Proposition 3.6] for the general case or [10] for an explicit treatment over the complex numbers; that then the vector space of sections of L comes naturally equipped with the Schrödinger representation ρ_1 of H_n – in fact, this is the restriction of the Schrödinger representation of the larger algebraic Heisenberg group Heis_n that is defined in analogous fashion to Heis_3.

In case L is a line bundle on an elliptic curve E over K this representation lifts the translation by n-torsion points on E, thus, the action of $E[n] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{P}(H^0(E, L))$ to an action by linear automorphisms on $V = H^0(E, L)$. For an elliptic curve E, embedded as a smooth projective plane cubic curve, the special case $L = \mathcal{O}_E(1)$ with $n = \deg L = 3$ was discussed in detail above.

Using the preceding lemma, the following result is an easy consequence of the fundamental fact just recalled.

2.20 Proposition. *Let $\mathcal{L}, \mathcal{L}'$ be locally free sheaves of degree 3 and $\mathcal{L}''$ a locally free sheaf of degree 6 on an elliptic curve E over an algebraically closed field K whose characteristic does not divide 6.*

(a) *Restricting the translations by* 6*-torsion points to the* 3*-torsion points restricts the representation* ρ_1 *of* H_6 *on* $H^0(E,\mathcal{L}'')$ *to the direct sum of two copies of the Schrödinger representation* ρ_2 *of* H_3.

(b) *The tensor product* $H^0(E,\mathcal{L})\otimes_K H^0(E,\mathcal{L}')$ *of the Schrödinger representations* ρ_1 *of* H_3 *on each of the two factors decomposes as the direct sum of three copies of the Schrödinger representation* ρ_2 *of* H_3.

(c) *With* $\mathcal{L}''=\mathcal{L}\otimes_{\mathcal{O}_E}\mathcal{L}'$*, the natural multiplication map on global sections represents a surjective* H_3*-equivariant homomorphism*

$$H^0(E,\mathcal{L})\otimes_K H^0(E,\mathcal{L}')\to \operatorname{res}\downarrow^{\mathsf{H}_6}_{\mathsf{H}_3} H^0(E,\mathcal{L}'').$$

In particular, the kernel of that homomorphism is a Schrödinger representation ρ_2 *of* H_3.

Proof. Part (a) is Lemma 2.18 applied to the case $j=1, n=6, d=3$, thus $n/d=2$.

For part (b), let (x_0,x_1,x_2) be a Schrödinger basis of $V=H^0(E,\mathcal{L})$ and (y_0,y_1,y_2) be a Schrödinger basis of $V'=H^0(E,\mathcal{L}')$. With $x_iy_j=x_i\otimes y_j$ and $a_0,a_1,a_2\in K$, the tensor

$$f_0=a_0x_0y_0+a_1x_2y_1+a_2x_1y_2$$

is a fixed vector for the action of $\tau'=\rho_1(\tau)\otimes\rho_1(\tau)$ on $V\otimes_K V'$. Abbreviating also $\sigma'=\rho_1(\sigma)\otimes\rho_1(\sigma)$, with $f_{-i}=(\sigma')^i(f_0), i\in\mathbb{Z}/3\mathbb{Z}$, one has

$$\begin{aligned} f_0&=a_0x_0y_0+a_1x_2y_1+a_2x_1y_2, & \tau'(f_0)&=f_0,\\ f_1&=a_2x_2y_0+a_0x_1y_1+a_1x_0y_2, & \tau'(f_1)&=\omega^2 f_1,\\ f_2&=a_1x_1y_0+a_2x_0y_1+a_0x_2y_2, & \tau'(f_2)&=\omega f_2. \end{aligned}$$

Therefore, f_0, f_1, f_2 form indeed a Schrödinger basis for a representation of H_3 that is equivalent to ρ_2 as soon as $(a_0,a_1,a_2)\neq(0,0,0)\in K^3$. Choosing in turn $(a_0,a_1,a_2)=e_i$, for $i\in\mathbb{Z}/3\mathbb{Z}$ and $(e_i)_{i=0,1,2}$ the standard basis of K^3, it follows that indeed $\rho_1\otimes_K\rho_1\cong\rho_2^{\oplus 3}$ as H_3-representations – which, in fact, could have been established as well by just looking at the corresponding group characters. The reader will note that viewed as trilinear forms, the f_i are precisely the forms from Section 2.4 above.

In part (c), surjectivity of the multiplication map is well known and the H_3-equivariance follows as translation is compatible with tensor products,

$$t_{\mathbf{a}}(\mathcal{L})\otimes_{\mathcal{O}_E} t_{\mathbf{a}}(\mathcal{L}')\cong t_{\mathbf{a}}(\mathcal{L}\otimes_{\mathcal{O}_E}\mathcal{L}')$$

for any point $\mathbf{a}\in E$. Applied to 3-torsion or 6-torsion points and using that translations by those points manifest themselves through the Schrödinger representation ρ_1 of H_3, respectively H_6, the proof of the proposition is complete. ∎

End of the proof of Theorem A.

2.21. The statement of the theorem about equivalence of Moore matrices is proved in Proposition 2.11. We need to show that a determinantal presentation of a smooth cubic is given by a Moore matrix. We can finish the proof of the theorem by choosing the right basis in the vector spaces of global sections.

Let L be a line bundle of degree 0 on the smooth cubic curve $E \subset \mathbb{P}^2$ with defining equation $f = 0$. According to Beauville's result stated above in Theorem 1.2, if $H^0(E, L) = 0$ then there exists a 3×3 linear matrix M such that $f = \det M$ and an exact sequence of $\mathcal{O}_{\mathbb{P}^2}$-modules

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^3 \longrightarrow L \longrightarrow 0.$$

Twisting this sequence by $\mathcal{O}_E(2)$, and taking global sections, one can identify this exact sequence as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \otimes_K W \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1) \otimes_K H^0(E, L(1)) \longrightarrow L \longrightarrow 0,$$

where W is the kernel of the H_3-equivariant multiplication map

$$H^0(E, \mathcal{O}_E(1)) \otimes_K H^0(E, L(1)) \to \operatorname{res}\downarrow^{\mathsf{H}_6}_{\mathsf{H}_3} H^0(E, L(2))$$

as in Proposition 2.20 (c) above for $\mathcal{L} = \mathcal{O}_E(1), \mathcal{L}' = L(1)$.

Choosing Schrödinger bases f_0, f_1, f_2 for W, x_0, x_1, x_2 for $H^0(E, \mathcal{O}_E(1))$, and y_0, y_1, y_2 for $H^0(E, L(1))$ as in the proof of Proposition 2.20 (b), M becomes identified with a Moore matrix $M_{a,x} = (a_{i+j}x_{i-j})_{i,j \in \mathbb{Z}/3\mathbb{Z}}$ for some $a = (a_0, a_1, a_2) \in K^3$. As $\det M \doteq f$ by Beauville's result, it follows that $\mathbf{a} \in E$ with $a_0 a_1 a_2 \neq 0$. This completes the proof of Theorem A from the introduction. ∎

3. Proof of Theorem B

The starting point is the following result from Atiyah's seminal paper [2].

A result of Atiyah and Ulrich bundles.

3.1 Theorem (cf. Atiyah [2, Theorem 5 (ii)]). *Let F be an indecomposable vector bundle of rank 2 on an elliptic curve E over a field K. If $\deg F = 0$, then there exists a unique line bundle L of degree 0 that fits into an exact sequence of vector bundles*

$$0 \longrightarrow L \longrightarrow F \longrightarrow L \longrightarrow 0$$

Moreover, $H^0(E, F) = 0$ if, and only if, $H^0(E, L) = 0$.

Conversely, if L is a line bundle of degree 0 then there exists an indecomposable vector bundle F, unique up to isomorphism and necessarily of rank 2 and degree 0, that fits into such an exact sequence. ∎

3.2. By our Theorem A we know that a line bundle of degree 0 with no nonzero sections is obtained as $L = \operatorname{coker} M_{a,x}$, where $M_{a,x}$ is a Moore matrix, $a \in K^3$ representing a point $\mathbf{a} \in E$ on the elliptic curve $E \subset \mathbb{P}^2$ with $a_0a_1a_2 \neq 0$. We fix these data in the following.

Let $S = K[x_0, x_1, x_2]$ be the homogeneous coordinate ring of $\mathbb{P}^2(K)$, with its homogeneous components $S_m = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m))$ the vector space over K of homogeneous polynomials of degree $m \in \mathbb{Z}$.

Applying the functor $\Gamma_* = \oplus_{i\in\mathbb{Z}} H^0(\mathbb{P}^2, (\ )(i))$ to the exact sequence of coherent $\mathcal{O}_{\mathbb{P}^2}$-modules

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^3 \xrightarrow{M_{a,x}} \mathcal{O}_{\mathbb{P}^2}(-1)^3 \longrightarrow L \longrightarrow 0,$$

yields a short exact sequence of graded S-modules

$$0 \longrightarrow S(-2)^3 \xrightarrow{M_{a,x}} S(-1)^3 \longrightarrow \Gamma_*(L) \longrightarrow 0.$$

The module $\mathbf{L} = \Gamma_*(L)$, cokernel of the map between graded free S-modules represented by $M_{a,x}$, is an *Ulrich module* of rank one over the homogeneous coordinate ring $R = S/(f)$ of E, and each Ulrich module over R of rank one (and generated in degree 1) can be so realized by Theorem A.

Matrix factorizations and extensions. In view of Atiyah's result cited above, our aim here is to find a similar description for Ulrich modules over R of rank two, namely the one stated in Theorem B. To simplify notation a bit, we fix for now the point a and set $A = M_{a,x}$, $B = M_{a,x}^{\mathrm{adj}}$, viewed as matrices over S. The pair (A, B) represents a matrix factorization of $f = \det A \in S$ and so, by [6], $\mathbf{L}$ admits the graded R-free resolution

$$0 \longleftarrow \mathbf{L} \longleftarrow R(-1)^3 \xleftarrow{A} R(-2)^3 \xleftarrow{B} R(-4)^3 \xleftarrow{A} R(-5)^3 \xleftarrow{B} \cdots$$

that is 2-periodic up to the shift in degrees by $-\deg f = -3$.

3.3. Now consider an element[6] of $\operatorname{Extgr}^1_R(\mathbf{L}, \mathbf{L}(m))$ for some $m \in \mathbb{Z}$. It can be represented by a homotopy class of morphisms between graded free resolutions and, invoking

[6]We write Extgr_R for the extensions in the category of graded R-modules with degree–preserving R-linear maps.

again [6], such morphisms and their homotopies can again be chosen to be 2-periodic so that one has a diagram as follows:

$$
\begin{array}{ccccccccccccc}
\mathbf{L} & & 0 \longleftarrow & R(-1)^3 & \xleftarrow{A} & R(-2)^3 & \xleftarrow{B} & R(-4)^3 & \xleftarrow{A} & R(-5)^3 & \xleftarrow{B} & \cdots \\
\downarrow & & & \downarrow & \overset{U}{\searrow} & \downarrow{\scriptstyle C} & \overset{V}{\searrow} & \downarrow{\scriptstyle D} & \overset{U}{\searrow} & \downarrow{\scriptstyle C} & \overset{V}{\searrow} & \\
\mathbf{L}(m)[1] & & 0 \longleftarrow & 0 & \longleftarrow & R(m-1)^3 & \xleftarrow[-A]{} & R(m-2)^3 & \xleftarrow[-B]{} & R(m-4)^3 & \xleftarrow[-A]{} & \cdots
\end{array}
$$

Here,

- $C \in \mathsf{Mat}_{3\times 3}(S_{m+1})$, $D \in \mathsf{Mat}_{3\times 3}(S_{m+2})$ are 3×3 matrices whose entries are homogeneous polynomials of the indicated degrees;
- the pair of matrices (C, D) satisfies $AD + CB = \mathbf{0} = DA + BC$ over S, with $\mathbf{0}$ the zero matrix, and so defines a morphism of complexes over R;
- $U, V \in \mathsf{Mat}_{3\times 3}(S_m)$ represent the possible homotopies, in that the morphisms of complexes $\mathbf{L} \to \mathbf{L}(m)$ induced by

$$C' = C + UA - AV, \quad D' = D + VB - BV$$

run through the homotopy class of (C, D) for the various choices of U, V.

3.4. Given a pair of matrices (C, D) with $AD + CB = \mathbf{0} = DA + BC$ as above, the block matrices

$$\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} B & D \\ 0 & B \end{pmatrix}$$

constitute a matrix factorization of f and give rise to the commutative diagram of graded S-modules with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(m-2)^3 & \xrightarrow{A} & S(m-1)^3 & \longrightarrow & \mathbf{L}(m) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle in_1} & & \downarrow{\scriptstyle in_1} & & \downarrow & & \\
0 & \longrightarrow & S(m-2)^3 \oplus S(-2)^3 & \xrightarrow{\left(\begin{smallmatrix} A & C \\ 0 & A \end{smallmatrix}\right)} & S(m-1)^3 \oplus S(-1)^3 & \longrightarrow & \mathbf{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} & & \downarrow & & \\
0 & \longrightarrow & S(-2)^3 & \xrightarrow{A} & S(-1)^3 & \longrightarrow & \mathbf{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with the rightmost column representing the extension defined by (C, D) over R.

The following observation cuts down considerably on the work of finding solutions to the equations $AD + CB = \mathbf{0} = DA + BC$, whenever (A, B) is a matrix factorization of a non-zero-divisor f in a commutative ring S.

3.5 Lemma. *Assume $A, B \in \mathsf{Mat}_{n\times n}(S)$ is a matrix factorization of a non-zero-divisor $f \in S$, in that $AB = f\,\mathrm{id}_n = BA$. For a matrix $C \in \mathsf{Mat}_{n\times n}(S)$, the following are equivalent:*

(a) *There exists a matrix $D \in \mathsf{Mat}_{n\times n}(S)$ such that $AD + CB = \mathbf{0}$.*

(b) *There exists a matrix $D' \in \mathsf{Mat}_{n\times n}(S)$ such that $D'A + BC = \mathbf{0}$.*

(c) *There exists a matrix $D'' \in \mathsf{Mat}_{n\times n}(S)$ such that $fD'' + BCB = \mathbf{0}$. Equivalently, each entry of $BCB \in \mathsf{Mat}_{n\times n}(S)$ is divisible by f.*

If either equivalent condition holds, then $D = D' = D''$ and that matrix is the unique one satisfying $fD = -BCB$. Moreover, one can recover C from D in that C is the unique matrix such that $fC = -ADA$.

Proof. If $AD + CB = \mathbf{0}$ then multiplying from the left with B yields

$$0 = BAD + BCB = fD + BCB,$$

whence $BCB \equiv \mathbf{0} \bmod (f)$ and one can take $D'' = D$. Conversely, if that congruence holds then there exists a matrix D'' with $fD'' = -BCB$. Multiplying this equation with A from the left results in

$$AfD'' + ABCB = f(AD'' + CB) = \mathbf{0}.$$

As f is a non-zero-divisor, this implies $AD'' + CB = \mathbf{0}$, whence one can take $D = D''$. Thus, (a) $\iff$ (c). The equivalence (b) $\iff$ (c) is completely analogous. Uniqueness of D follows as $AD_1 + CB = AD_2 + CB$ implies $A(D_1 - D_2) = 0$. Now the linear map represented by A is injective because $f\,\mathrm{id}_n = BA$ and multiplication with f is injective by assumption. Thus, $D_1 = D_2$ as claimed and, in particular, $D = D''$ must hold. Analogously one must have $D' = D''$.

Concerning the final assertion, multiply the equation $fD = -BCB$ on both sides with A to obtain $ADAf = -fCf$, and thus $fC = -ADA$, as f is a non-zero-divisor. Uniqueness of C follows as above. ∎

If A, as in our case of interest, is a determinantal representation of a reduced polynomial, one can reduce the description of extensions further.

3.6 Lemma. *If the determinant $f = \det A \in S$, for a matrix $A \in \mathsf{Mat}_{n\times n}(S)$, is reduced, then with $B = A^{\mathrm{adj}}$ one has for any matrix $C \in \mathsf{Mat}_{n\times n}(S)$ that*

$$BCB \equiv \mathrm{tr}(BC)B \bmod (f),$$

and $BCB \equiv \mathbf{0} \bmod (f)$ if, and only if, $\mathrm{tr}(BC) \equiv 0 \bmod (f)$.

Proof. As f is reduced, it is generically regular. For a regular point $x \in V(f)$ this implies that rank $A(x) = n - 1$, thus, rank $B(x) = 1$, as the cokernel of A is maximal Cohen–Macaulay, hence is locally free of rank 1 at such point. Accordingly there are vectors $u, v \in k(x)^n$ such that $B(x) = u^T \cdot v$. Therefore,

$$B(x)C(x)B(x) = u^T \cdot v \cdot C(x) \cdot u^T \cdot v.$$

Now $v \cdot C(x) \cdot u^T$ is an element of the residue field $k(x)$ at x and so, considering it as a 1×1 matrix,

$$v \cdot C(x) \cdot u^T = \operatorname{tr}\big(v \cdot C(x) \cdot u^T\big) = \operatorname{tr}\big(u^T \cdot v \cdot C(x)\big) = \operatorname{tr}\big(B(x)C(x)\big).$$

Embedding this observation into the right-hand side of the previous equality, it follows that

$$\begin{aligned} B(x)C(x)B(x) &= u^T \cdot v \cdot C(x) \cdot u^T \cdot v \\ &= \operatorname{tr}\big(B(x)C(x)\big)u^T \cdot v = \operatorname{tr}\big(B(x)C(x)\big)B(x). \end{aligned}$$

Therefore, $BCB - \operatorname{tr}(BC)B$ vanishes at each regular point of $\{f = 0\}$, thus, it vanishes everywhere on that hypersurface. Moreover, $B(x) \neq \mathbf{0}$ at each regular point, whence at such points $BCB(x) = \mathbf{0}$ if, and only if, $\operatorname{tr}(B(x)C(x)) = 0$. The claim follows. ∎

3.7. Putting these facts together, we get the following description of the extension groups we are interested in here:

$$(*) \qquad \operatorname{Extgr}^1_R(\mathbf{L}, \mathbf{L}(m)) \cong \frac{\{C \in \mathsf{Mat}_{3\times 3}(S_{m+1}) \mid \operatorname{tr}(M^{\text{adj}}_{a,x} \cdot C) \equiv 0 \bmod f\}}{\{UM_{a,x} - M_{a,x}V \mid U, V \in \mathsf{Mat}_{3\times 3}(S_m)\}}.$$

This description shows immediately that $\operatorname{Extgr}^1_R(\mathbf{L}, \mathbf{L}(m)) = 0$ when $m < -1$, because there is then no nonzero choice for C. It also shows that there are no nonzero homotopies for $m = -1$, a fact we will exploit below. Concerning shifts by $m \geqslant -1$, we determine the size of the extension group directly in terms of possible Yoneda extensions, that is, short exact sequences, as follows. Given a short exact sequence

$$0 \longrightarrow \mathbf{L}(m) \longrightarrow \mathbf{M} \longrightarrow \mathbf{L} \longrightarrow 0$$

of graded R-modules, sheafifying it yields a short exact sequence of $\mathcal{O}_E$-modules,

$$0 \longrightarrow L(m) \longrightarrow M \longrightarrow L \longrightarrow 0.$$

Conversely, applying Γ_* to such a short exact sequence of $\mathcal{O}_E$-modules with $m \geqslant -1$ returns a short exact sequence of graded R-modules as above in that the connecting homomorphism in cohomology $H^0(E, L(i)) \to H^1(E, L(m+i))$ is 0 for each $i \in \mathbb{Z}$. Indeed, for $i \leqslant 0$, one has

$$H^0(E, L(i)) = 0,$$

while for $i > 0$ one has $i + m \geqslant 0$, whence

$$H^1(E, L(m+i)) = 0.$$

It follows that Γ_* and sheafification yield inverse isomorphisms between $\mathrm{Ext}^1_E(L, L(m))$ and $\mathrm{Extgr}^1_R(\mathbf{L}, \mathbf{L}(m))$ for $m \geqslant -1$.

Further, $\mathrm{Ext}^1_E(L, L(m)) \cong H^1(E, \mathcal{O}_E(m))$ vanishes for $m > 0$, while for $m = -1$ that vector space is Serre-dual to

$$H^0(E, \mathcal{O}_E(1)) \cong S_1,$$

and for $m = 0$, of course,

$$H^1(E, \mathcal{O}_E) \cong K.$$

We thus have the following result.

3.8 Lemma. *Let* $\mathbf{L}$ *be the Ulrich module that is the cokernel of the Moore matrix* $M_{a,x}$ *as in Theorem A. The vector spaces of graded self-extensions of* $\mathbf{L}$ *over* R *satisfy*

$$\dim_K \mathrm{Extgr}^1_R(\mathbf{L}, \mathbf{L}(m)) = \begin{cases} 3 & \textit{for } m = -1, \\ 1 & \textit{for } m = 0, \\ 0 & \textit{else}. \end{cases} \qquad \blacksquare$$

With these preparations we now determine $\mathrm{Extgr}^1_R(\mathbf{L}, \mathbf{L}(-1))$.

3.9 Proposition. *Let* $\mathbf{L}$ *be the cokernel of a Moore matrix* $M_{a,x}\colon S(-2)^3 \to S(-1)^3$ *for* $a \in K^3$ *with* $a_0a_1a_2 \neq 0$ *representing a point* $\mathbf{a} \in E$*. The three-dimensional vector space* $\mathrm{Extgr}^1_R(\mathbf{L}, \mathbf{L}(-1))$ *over* K *is isomorphic to the space of specialized Moore matrices*

$$M_{(a_0(a_2^3-a_1^3), a_2(a_1^3-a_0^3), a_1(a_0^3-a_2^3)), (s,t,u)}, \qquad (s,t,u) \in K^3.$$

Note that $(a_0(a_2^3 - a_1^3), a_2(a_1^3 - a_0^3), a_1(a_0^3 - a_2^3))$ *represents the point* $2 \cdot_E \mathbf{a} \in E$*, whence this set of matrices consists of all* K*-rational specializations of* $M_{2\cdot_E \mathbf{a}, x}$.

Proof. Set $b = (a_0(a_2^3 - a_1^3), a_2(a_1^3 - a_0^3), a_1(a_0^3 - a_2^3))$ and note that

$$M_{b,(s,t,u)} = M_{b,(1,0,0)}s + M_{b,(0,1,0)}t + M_{b,(0,0,1)}u.$$

Because $b \neq (0,0,0)$, the matrices $M_0 = M_{b,(1,0,0)}$, $M_1 = M_{b,(0,1,0)}$, $M_2 = M_{b,(0,0,1)}$ are clearly linearly independent in the vector space $\mathsf{Mat}_{3\times 3}(K)$ in that their nonzero entries are located at different positions in these matrices. Further, as $\mathrm{tr}(B\cdot -)$ is an S-linear function on $\mathsf{Mat}_{3\times 3}(S)$, and $\mathrm{Extgr}^1_R(\mathbf{L},\mathbf{L}(-1))$ is known to be of dimension 3 over K, it suffices to show that for each of the three matrices M_i, one has

$$\mathrm{tr}(BM_i) \equiv 0 \bmod (f),$$

where $B = M^{\mathrm{adj}}_{a,x}$. In fact, as the entries of BM_i are quadratic polynomials, but f is of degree 3, the congruence is equivalent to $\mathrm{tr}(BM_i) = 0$.

One now verifies this easily directly for the three matrices in question.

For a more conceptual explanation of the identities $\mathrm{tr}(BM_i) = 0$, note that, say, $M_0 = M_{b,(1,0,0)} = \mathsf{diag}(b)$ is a diagonal matrix with the coordinates b on the diagonal representing $2\cdot_E \mathbf{a}$. On the other hand, the diagonal elements in $B = M^{\mathrm{adj}}_{a,x}$ involve only the quadratic monomials x_0^2 and x_1x_2, and the coefficients of x_0^2 along the diagonal are the entries from the third row, those of x_1x_2 the entries from the first row of $M_{\iota(a),a}$; see Section 2.7. The column vector b^T spans the kernel of that matrix by construction. As the trace $\mathrm{tr}(BM_0)$ is the scalar product of the two diagonals, the vanishing of the trace becomes obvious. The case of the remaining two matrices yields to analogous arguments. ∎

The selfextensions of an Ulrich line bundle. Next we turn to $\mathrm{Extgr}^1_R(\mathbf{L},\mathbf{L})$, the extension group we are really interested in.

3.10 Definition. With notation as in the preceding proof, set

$$M_{b,y} = M_{b,(1,0,0)}y_0 + M_{b,(0,1,0)}y_1 + M_{b,(0,0,1)}y_2,$$

for $y = (y_0, y_1, y_2) \in S_1^3$, a vector of linear forms from S, and define the *divergence* of $M_{b,y}$ to be

$$\mathsf{div}(M_{b,y}) = \frac{\partial y_0}{\partial x_0} + \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} \in K.$$

Note in particular that $\mathsf{div}(M_{b,x}) = 3 \in K$, thus, is not zero when $\mathrm{char}(K) \neq 3$. For a characteristic-free statement, note that $\mathsf{div}(M_{b,\iota(x)}) = 1 \in K$.

3.11 Theorem. *Let* $\mathbf{L}$ *be the cokernel of a Moore matrix* $M_{a,x}\colon S(-2)^3 \to S(-1)^3$ *for* $a \in K^3$ *representing a point* $\mathbf{a} \in E$ *with* $a_0a_1a_2 \neq 0$ *and set*

$$b = \big(a_0(a_2^3 - a_1^3), a_2(a_1^3 - a_0^3), a_1(a_0^3 - a_2^3)\big),$$

as before. The one-dimensional vector space $\mathrm{Extgr}^1_R(\mathbf{L},\mathbf{L})$ *over* K *can be realized as*

$$\mathrm{Extgr}^1_R(\mathbf{L},\mathbf{L}) \cong \frac{\{M_{b,y} \in \mathsf{Mat}_{3\times 3}(S_1)\}}{\{M_{b,y} \in \mathsf{Mat}_{3\times 3}(S_1)\} \cap \{UM_{a,x} - M_{a,x}V \mid U, V \in \mathsf{Mat}_{3\times 3}(K)\}}$$

and the divergence $M_{b,y} \mapsto \mathsf{div}(M_{b,y}) \in K$ *induces an isomorphism*

$$\mathrm{Extgr}^1_R(\mathbf{L},\mathbf{L}) \xrightarrow[\cong]{\mathsf{div}} K.$$

Proof. As mentioned before, with $B = M_{a,x}^{\mathrm{adj}}$, the function $\operatorname{tr}(B \cdot -)\colon \mathsf{Mat}_{3\times 3}(S) \to S$ is S-linear, whence each matrix $M_{b,y}$ satisfies $\operatorname{tr}(B \cdot M_{b,y}) = 0$ as we know this for the matrices $M_0 = M_{b,(1,0,0)}$, $M_1 = M_{b,(0,1,0)}$, $M_2 = M_{b,(0,0,1)}$ from above. Thus, the vector space $\{M_{b,y} \in \mathsf{Mat}_{3\times 3}(S_1)\}$ is contained in the numerator of the description of $\operatorname{Extgr}^1_R(\mathbf{L}, \mathbf{L})$ in Section 3.7 $(*)$. As we know from Lemma 3.8 that this extension group is one-dimensional, and, say, $\mathsf{div} M_{b,\iota(x)} = 1 \in K$ as noted above, it remains only to show that the denominator in the description here lies in the kernel of the divergence.

To this end, assume $M_{b,y} = UM_{a,x} - M_{a,x}V$ for some linear forms y_i and some $U, V \in \mathsf{Mat}_{3\times 3}(K)$. Differentiating both sides with respect to x_0 and comparing the diagonal entries yields the system of equations

$$a_{2i}\left(a_{2i-1}^3 - a_{2i+1}^3\right)\frac{\partial y_0}{\partial x_0} = a_{2i}(u_{ii} - v_{ii}), \quad \text{for } i \in \mathbb{Z}/3\mathbb{Z}.$$

Dividing by a_{2i}, which is not zero by assumption, and then adding up shows that necessarily

$$\sum_{i\in\mathbb{Z}/3\mathbb{Z}} u_{ii} = \sum_{i\in\mathbb{Z}/3\mathbb{Z}} v_{ii}.$$

Differentiating as well with respect to x_1, x_2, comparing entries on both sides of the matrix equation and eliminating common factors of the form a_i leads to the system of equations

$$\begin{aligned}
(a_2^3 - a_1^3)\frac{\partial y_0}{\partial x_0} &= u_{00} - v_{00}, & (a_0^3 - a_2^3)\frac{\partial y_2}{\partial x_2} &= u_{00} - v_{11},\\
(a_1^3 - a_0^3)\frac{\partial y_1}{\partial x_1} &= u_{00} - v_{22}, & (a_0^3 - a_2^3)\frac{\partial y_1}{\partial x_1} &= u_{11} - v_{00},\\
(a_1^3 - a_0^3)\frac{\partial y_0}{\partial x_0} &= u_{11} - v_{11}, & (a_2^3 - a_1^3)\frac{\partial y_2}{\partial x_2} &= u_{11} - v_{22},\\
(a_1^3 - a_0^3)\frac{\partial y_2}{\partial x_2} &= u_{22} - v_{00}, & (a_2^3 - a_1^3)\frac{\partial y_1}{\partial x_1} &= u_{22} - v_{11},\\
(a_0^3 - a_2^3)\frac{\partial y_0}{\partial x_0} &= u_{22} - v_{22}.
\end{aligned}$$

Now at least one of the terms $(a_i^3 - a_{i-1}^3)$, $i \in \mathbb{Z}/3\mathbb{Z}$, that occur as coefficients on the left-hand sides is nonzero, as not all entries of b are zero. Picking one such nonzero term and using it to solve for $\frac{\partial y_i}{\partial x_i}$, $i = 0, 1, 2$, shows immediately that

$$\mathsf{div}(M_{b,y}) = \frac{\partial y_0}{\partial x_0} + \frac{\partial y_1}{\partial x_1} + \frac{\partial y_0}{\partial x_1} = 0$$

is a necessary condition on $M_{b,y}$ to be representable as $UM_{a,x} - M_{a,x}V$. In fact, we also know that this condition is sufficient. ∎

The proof of Theorem B. With F as in Theorem B (a), Atiyah's result Theorem 3.1 shows that F can be obtained as an extension of a line bundle L by itself, with $\deg L = 0$ and $H^0(E, L) = 0$. Applying Γ_* to such extension and using 3.7 results in an extension of $\mathbf{L}$ by itself. Those extensions were classified in 3.11, it yields a presentation of F as claimed.

Part (b) of Theorem B follows as the cokernel F of the triangular block matrix fits into a short exact sequence $0 \to L \to F \to L \to 0$ with $L = \operatorname{Coker} M_{a,x}$, which yields immediately that F is a vector bundle of rank 2 and degree 0 that has no nonzero sections. Moreover, F is indecomposable as the extension is not split, due to $\mathsf{div}(M_{a,x}) = 3 \neq 0$ in K.

Part (c) of Theorem B follows from Atiyah's result and from Theorem A, as the line bundle L in the short sequence above is uniquely determined by F up to isomorphism. ■

The case of Ulrich bundles of higher rank (and its relation to theta functions) is treated over complex numbers in [15], but a purely algebraic description of canonical forms of Ulrich bundles of ranks greater than two on elliptic curves is yet to be proved.

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