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The Karoubi–Weibel complexity for groups

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Abstract. Let G be a finitely presented group. A new complexity called *Karoubi–Weibel complexity* or *covering type*, is defined for G. The construction is inspired by recent work of Karoubi and Weibel (2016), initially applied to topological spaces. We introduce a similar notion in combinatorial form in order to apply it to finitely presentable groups. Some properties of this complexity as well as a few examples of calculation/estimation for certain classes of finitely presentable groups are considered. Finally, we give a few applications of complexity to some geometric problems, namely to the systolic area and the volume entropy of groups.

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1. Introduction

Topological complexity has been a hot subject in pure and applied topology over the last few years, although it does not yet admit a universal definition. A crude measure of the complexity of a space X is the size of a finite open covering of X by simple subsets, i.e., contractible subspaces. This can be reformulated as follows: what is the minimum number of simple subsets, satisfying some natural conditions on their intersections, into which X decomposes?

For the case of triangulated spaces we can consider all triangulations on X. In this case, it is possible to measure the complexity of X, by calculating, for example, the minimal number of highest-dimensional simplices ($\dim(X)$-simplices) needed to triangulate X. This complexity measures the combinatorial volume of X, compare with [3]. Dual complexity concerns the smallest number of vertices necessary to triangulate this space.

The category of Lusternik–Schnirelmann, known for almost a hundred years (see [22]), can be used to measure the complexity of the topological space X. This complexity is a very important tool in variational calculus and it is equal to the minimal number of open subsets, contractible in X that form a cover of the space.

The complexity recently introduced by Karoubi and Weibel [17], called covering type and denoted by "ct", formally resembles to that of Lusternik–Schnirelmann, but it takes into account mutual intersections of open subsets of a covering. The requirement that intersections be contractible gives a better account of the topology of the space in question.

All topological spaces below are supposed to be path-connected and locally contractible. For a topological space X we consider a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of X. We say that $\mathcal{U}$ is a *good cover* of X if, for all $i \in \{1, \dots, k\}$, the open subspaces U_i are contractible and each of their non-empty intersections is also contractible. The size of $\mathcal{U}$ denoted $|\mathcal{U}|$ is the number of open subspaces of $\mathcal{U}$. We denote by $\operatorname{ss}(X)$ the smallest size of a good cover of X. The ct-complexity (covering type) of X, denoted by $\operatorname{ct}(X)$, is the minimum of the strict covering types of spaces Y homotopy equivalent to X, such that

$$\operatorname{ct}(X) = \min_{Y \sim X} \operatorname{ss}(Y).$$

If there are no finite coverings with the prescribed properties the ct-complexity is equal to infinity by definition.

Let $\mathcal{N}(\mathcal{U})$ be the nerve of the above finite cover $\{U_i\}_{i=1}^k$. The finite simplicial polyhedron $\mathcal{N}(\mathcal{U})$ has k vertices. In the case when $\mathcal{U}$ is a good cover of X, $\mathcal{N}$ and X are homotopy equivalent. Note that $\mathcal{N}$ has a good cover formed by open stars of the vertices. So in the previous definition, we can minimize the number of vertices of a finite simplicial complex Y where Y runs over the set of complexes homotopy equivalent to X.

For an m-dimensional simplicial complex, finding a lower and an upper bounds for the minimal number of m-simplices that make up this complex and for its covering type are quite different. Recently, Adiprasito, Avvakumov and Karasev in [1], gave the following subexponential upper bound for the covering type of the real projective spaces $\mathbb{R}P^m$:

$$\operatorname{ct}(\mathbb{R}P^m) \le \exp\Big\{\Big(\frac{1}{2} + o(1)\Big)\sqrt{m+1}\log(m+1)\Big\}.$$

The polynomial lower bound $\operatorname{ct}(\mathbb{R}P^m) \geq \frac{1}{2}(m+1)(m+2)$ was given by Arnoux and Marin [2, §21]; see also the recent paper of Govc, Marzantowicz and Pavešič [13]. At the same time, remark that the theorem of Bárány and Lovász [5] implies that the number of m-simplices needed to triangulate $\mathbb{R}P^m$ is at least 2^m.

There exist other complexities of different natures, for example the Schwarz genus of a fiber space [28], which is closely related to the Farber complexity [12]. These complexities measure the minimal number of parts into which the base space of a fiber bundle decomposes, with the condition that above each part, there is a section.

These two types of complexities as well as the Lusternik–Schnirelmann category never exceed the dimension of the space plus one.

Another interesting type of complexity is the so-called Matveev complexity. This is a combinatorial invariant quite useful in 3-dimensional topology (see [23]), we do not discuss it here.

Considering a good cover to study the topology of a space is not new. This goes back to Jean Leray's old work [20], and was mentioned in 1952 by André Weil [29].

Throughout this article, we focus on the complexity of Karoubi–Weibel of finitely presented groups.

Definition 1.1. Let G be a finitely presented group. One defines the KW-complexity be the covering type of G denoted by $\mathrm{KW}(G)$ given by

$$\mathrm{KW}(G) = \min_{\pi_1(X)=G} \mathrm{ct}(X).$$

In the formula, we minimize the covering type of spaces with the given fundamental group G.

Each finitely presented group G is a fundamental group of a finite 2-simplicial complex, so its KW-complexity, $\mathrm{KW}(G)$ is always finite. The following result describes the combinatorial nature of this complexity.

Let K be a finite simplicial complex. We denote by $s_n(K)$ the number of n-simplices in K. For a finitely presented group, we set

$$s_0(G) = \min_{\pi_1(K)=G} s_0(K),$$

where K ranges over the set of 2-dimensional simplicial complexes of the fundamental group G.

Theorem 1.2. *Any finitely presented group G satisfies* $\mathrm{KW}(G) = s_0(G)$.

The simplest example is the group with only one element, its KW-complexity is equal to 1 and its optimal complex is composed with a single vertex.

From Theorem 1.2, it is obvious that the number of isomorphism classes of groups of KW-complexity bounded by T is finite. For a more detailed count, let's make some remarks. Kurosh's decomposition Theorem [19] implies that any group G admits a decomposition in the form

$$G = G_1 * \mathbb{F}_n, \tag{1.1}$$

where $\mathbb{F}_n$ is the free group of rank n and G_1 cannot be decomposed with free factors. The rank n of the free group of the equation (1.1) is uniquely defined and called *free index* of G. We will see below that, rather often, the KW-complexity is little sensitive to the free factor of (1.1). This means that the problem of counting pairwise non-isomorphic groups of bounded KW-complexity should be restricted to the class of groups with free index zero. In the following, we consider only the class of groups of free index equal to 0. Denote by $\mathcal{G}_{\mathrm{KW}}(T)$ the set of free index zero, pairwise non-isomorphic groups of KW-complexity bounded by T. Theorem 1.2 implies the following result:

Corollary 1.3. *For any positive real number T, the number of groups in $\mathcal{G}_{\mathrm{KW}}(T)$ satisfies*

$$|\mathcal{G}_{\mathrm{KW}}(T)| \leq 2^{3T^3 \log_2(T)}.$$

Even for simple groups the exact calculation of its covering type turns out to be a rather difficult technical problem. One can show that for non-free groups the smallest value of $\mathrm{KW}(G)$ is equal to 6. It corresponds to the cyclic group $\mathbb{Z}_2$, compare with [17, Proposition 5.2]. Complexities of abelian groups of finite rank as well as Artin and Coxeter groups are considered in Chapter 3. We give some framing estimate of its KW-complexity. Unfortunately the exact values of KW-complexities of these types of groups remain an open problem.

As yet, we only know two classes of groups whose exact values of KW-complexity are known: the free groups $\mathbb{F}_n$ and the surface groups.

Let $\mathbb{F}_n$ be a free group of rank n and $k = \mathrm{KW}(\mathbb{F}_n)$. We would like to find a relation between the rank of this group and its KW-complexity. Let K_k be the complete graph with k vertices. Fix a vertex x_0 in K_k and contract the star $St\{x_0\}$, we obtain a bouquet of $(k-1)(k-2)/2$ circles and its fundamental group is free of rank $(k-1)(k-2)/2 \geq n$. So k is the least integer satisfying $n \leq (k-1)(k-2)/2$, and we get

$$\mathrm{KW}(\mathbb{F}_n) = \left\lceil \frac{3 + \sqrt{1 + 8n}}{2} \right\rceil.$$

The notation $\lceil x \rceil$ stands for the least integer $\geq x$.

We denote by S_g the orientable surfaces of genus g and let $\pi^+(g) = \pi_1(S_g)$ be its fundamental group. We denote respectively N_q the non-orientable surfaces of genus q and $\pi^-(q) = \pi_1(N_q)$.

Recall the definition of chromatic number of orientable and non-orientable surfaces (see [26]), it is a lower bound for the number of colors that suffice to color any graph embedded in the surface. We denote by $\mathrm{chr}(S_g)$ and $\mathrm{chr}(N_q)$ the chromatic numbers of these surfaces, such that

$$\mathrm{chr}(S_g) = \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil \quad \text{for all } g \neq 2 \text{ and } \mathrm{chr}(S_2) = 10$$

and

$$\mathrm{chr}(N_q) = \left\lceil \frac{7 + \sqrt{1 + 24q}}{2} \right\rceil \quad \text{for all } q \neq 2, 3 \text{ and } \mathrm{chr}(N_2) = 8,\ \mathrm{chr}(N_3) = 9.$$

The recent paper [6] of Borghini and Minian implies that for KW-complexity of surface groups:

$$\mathrm{KW}(\pi^+(g)) = \mathrm{chr}(S_g) \ \ (g \neq 2) \quad \text{and} \quad \mathrm{KW}(\pi^-(q)) = \mathrm{chr}(N_q).$$

They proved also that there is only one exceptional case where KW-complexity is different from the chromatic number, it is the case of $\pi^+(2)$ where $\mathrm{KW}(\pi^+(2)) = 9$. The exceptional surface group $\pi^+(2)$ is particularly interesting because its KW-complexity is realized by certain 2-simplicial complex, which is not homeomorphic to a surface; see [6]. It is not the case for other surface groups.

In order to give some geometric applications of KW-complexity, recall the definition of the systolic area for groups; see [14]. Let G be a finitely presented group and X be a finite 2-simplicial complex such that $\pi_1(X) = G$. Endow X with a piecewise smooth Riemannian metric h. The systole, denoted by $\mathrm{sys}(X, h)$, is the shortest length of a non-contractible closed curve in the Riemannian polyhedron (X, h). Let $\mathrm{vol}(X, h)$ be the sum of all the h-areas of the 2-simplices of X. We call systolic area of the group G the following quantity:

$$\sigma(G) := \inf_{(X,h)} \frac{\mathrm{vol}(X, h)}{\mathrm{sys}(X, h)^2},$$

where (X, h) ranges over the set of all 2-dimensional Riemannian polyhedra (X, h) such that $\pi_1(X) = G$.

During the last fifteen years, the study of this invariant has been deepened; see [3, 18, 27]. It is known that $\sigma(G) = 0$ if and only if G is a free group. Otherwise, a universal lower bound has been given in [27]:

$$\sigma(G) \geq \frac{\pi}{16}.$$

However, several questions remain open, for example: does every non-free finitely presented group G satisfy

(1) $\sigma(G) \geq 2/\pi$, and

(2) $\sigma(G * \mathbb{Z}) = \sigma(G)$?

The constant in (1) corresponds to the systolic area of $\mathbb{R}P^2$; see [25]. So the first question is equivalent to saying that the systolic area of each simplicial complex with non-free fundamental group is at least $\sigma(\mathbb{R}P^2)$.

The equality in (2) may be generalized to the conjecture that the systolic area does not depend on the free factors.

Theorem 1.4. *Let G be a finitely presented group, then*

$$\sigma(G) \leq \frac{1}{27\pi}(\mathrm{KW}(G))^3.$$

If in addition G has zero free index, then

$$\frac{1}{576}\,\mathrm{KW}(G) \leq \sigma(G).$$

In order to give the definition of the volume entropy of a group, start by recalling the definition of this entropy for finite simplicial complexes; for more details, see [4]. Consider a finite simplicial complex X of dimension m equipped with a piecewise Riemannian metric h. Denote by $\widehat{X}$ the universal cover of X and by $\widehat{h}$ the lift of h. Take $q \in \widehat{X}$ a point and let $\widehat{B}_q(R)$ be the geodesic ball of radius R centered at q in X. Put

$$\mathrm{ent}(X,h) := \lim_{R\to\infty} \frac{\log\big(\mathrm{vol}(\widehat{B}_q(R))\big)}{R},$$

where the volume means the m-dimensional Hausdorff measure corresponding to the lifted metric $\widehat{h}$.

It is well known that this limit exists and does not depend on the chosen point q. It is also known that $\mathrm{ent}(X,h)$ is strictly positive if and only if $\pi_1(X)$ is a group of exponential growth.

The quantity $\mathrm{ent}(X,h)(\mathrm{vol}(X,h))^{1/m}$ remains invariant under homotheties given by $h \leftrightarrow \lambda^2 h$, and we define the minimal volume entropy of X as follows:

$$\omega(X) := \inf_h \mathrm{ent}(X,h)\big(\mathrm{vol}(X,h)\big)^{1/m}. \tag{1.2}$$

For a non-free finitely presented group G, its volume entropy is then

$$\omega(G) := \inf_{\pi_1(X)=G} \omega(X), \tag{1.3}$$

where X ranges over the set of 2-dimensional complexes of fundamental group G.

Note that formally applied to free groups, this definition gives zero when $m = 2$ in (1.2). The free groups are naturally 1-dimensional objects and to include them in the general concept, we have to reduce ourselves in (1.3) to 1-dimensional complexes, in other words, finite metrized (or weighted) graphs. Minimal volume entropy for arbitrary finite graphs is completely studied by Lim [21] and by Kapovich and Nagnibeda [16] for the case of 3-valented graphs; see also later work of McMullen [24] for an alternative proof. The results of these articles imply directly the explicit value for the volume entropy of free groups:

$$\omega(\mathbb{F}_n) = 3(n-1)\log 2.$$

Theorem 1.5. *Each non-free finitely presented group G satisfies*

$$\omega(G) \leq \frac{1}{3}\log(\mathrm{KW}(G))(\mathrm{KW}(G))^{3/2}. \tag{1.4}$$

Remark 1.6. Note that in general, a universal lower bound cannot exist for the inequality (1.4). There are groups of arbitrarily large KW-complexity and of subexponential growth for which the volume entropy is equal to zero. On the other hand, we currently know several classes of groups whose volume entropy is positive, for example the surface groups. Another recently found class of groups (see [8]) consists of right-angled Artin groups such that the corresponding graph is not a forest, but has no cycles of length 3.

Finding a lower bound depending on the KW-complexity for these last classes of groups remains an open question.

This paper is organized as follows, in the next section, we give the proofs of the first two results presented above and a technical lemma which will be used repeatedly throughout this paper. In Section 3, we study some groups: right-angled, large and extra-large Artin/Coxeter groups, cyclic and abelian groups, and estimate the covering type of each of them. In the last section, we prove our two geometric applications, i.e., Theorems 1.4 and 1.5.

2. Karoubi–Weibel complexity

All the topological spaces considered in the future are supposed path-connected and locally contractible.

Proof of Theorem 1.2. Let G be a finitely presented group and X be a topological space such that $\pi_1(X) = G$. Suppose that $\mathrm{ct}(X) = \mathrm{KW}(G)$, i.e., X admits a cover $\mathcal{U}$ with $\mathrm{KW}(G)$ open contractible subsets and each non-empty intersection of these subsets is also contractible. The Nerve Lemma (see [15] and [11]) implies that there is a weak homotopy equivalence between X and its nerve $\mathcal{N}(\mathcal{U})$. Therefore the 2-skeleton of the nerve, $\mathrm{Sk}^2(\mathcal{N}(\mathcal{U}))$, satisfies

$$\pi_1\big(\mathrm{Sk}^2(\mathcal{N}(\mathcal{U}))\big) = G \quad \text{and} \quad s_0\big(\mathrm{Sk}^2(\mathcal{N}(\mathcal{U}))\big) = \mathrm{KW}(G).$$

This implies that $s_0(G) \leq \mathrm{KW}(G)$.

To obtain the second inequality, we take a 2-simplicial complex K such that $\pi_1(K) = G$ and $s_0(K) = s_0(G)$. The open stars of vertices of K form a good cover of K with $s_0(G)$ open subsets. ∎

Let compare the KW-complexity of a group G with the simplicial complexity $\kappa(G)$ introduced formerly in [3], recall the original definition.

Let G be a finitely presented group. Its simplicial complexity $\kappa(G)$ is defined by the following formula:

$$\kappa(G) := \inf_{\pi_1(X)=G} s_2(X);$$

the infimum being taken over all 2-simplicial complexes X with fundamental group G. Then a 2-complex X is said minimal for G if $\pi_1(X) = G$, $s_2(X) = \kappa(G)$, and each vertex is incident to at least 2 edges. If G is of free index zero, the last condition is equivalent to the one that each vertex is incident to a face.

It is obvious from the definition that $\kappa(G) = 0$ if and only if G is free. From a combinatorial point of view, $\kappa(G)$ can be considered as a *discrete area* of the group G. The following simple proposition establishes links between the two kinds of complexity, $\kappa(G)$ and $\mathrm{KW}(G)$.

Proposition 2.1. *Every finitely presented group G satisfies:*

$$\kappa(G) \leq \frac{(\mathrm{KW}(G))^3}{6}.$$

If in addition G has zero free index, then

$$\frac{4}{3}\,\mathrm{KW}(G) \leq \kappa(G).$$

Proof. Let G be a finitely presented group and $\mathrm{KW}(G) = k$. To prove the first inequality consider a minimal complex X for the KW-complexity such that, $\pi_1(X) = G$ and $s_0(X) = k$. This complex has at most

$$\frac{k(k-1)(k-2)}{6} < \frac{k^3}{6}$$

2-simplices.

Let suppose that G has zero free index and Y is a 2-complex such that $\pi_1(Y) = G$ and minimal for $\kappa(G)$, i.e., $s_2(Y) = \kappa(G)$. In this case, any vertex of Y is incident to at least four 2-simplices; see [3]. This implies the second inequality. ■

Proof of Corollary 1.3. Note by $\mathcal{G}_\kappa(T)$ the set of isomorphism classes of groups with free index zero and of simplicial complexity bounded by T with $T \geq 2$. According to [3, Theorem 1.3], we have

$$|\mathcal{G}_\kappa(T)| \leq 2^{6T\log_2 T},$$

where $|\mathcal{G}_\kappa(T)|$ is the number of groups in $\mathcal{G}_\kappa(T)$. If $G \in \mathcal{G}_{\mathrm{KW}}(T)$, i.e., $\mathrm{KW}(G) \leq T$, by proposition 2.1, we obtain

$$\mathcal{G}_{\mathrm{KW}}(T) \subseteq \mathcal{G}_\kappa\Big(\frac{T^3}{6}\Big).$$

Therefore,

$$|\mathcal{G}_{\mathrm{KW}}(T)| \le \left|\mathcal{G}_{\kappa}\left(\frac{T^3}{6}\right)\right| \le 2^{T^3 \log_2(T^3/6)},$$

which gives $|\mathcal{G}_{\mathrm{KW}}(T)| \le 2^{3T^3 \log_2(T)}$. ∎

We use the following technical lemma in several estimates of the KW-complexity.

Definition 2.2. Consider a simplicial complex X and a simplicial subcomplex $Z \subset X$. We say that the embedding $Z \hookrightarrow X$ is maximal if for all simplex $\Delta^m \subset X$ such that its 1-skeleton is included in Z then the simplex $\Delta^m \subset Z$.

Let $\{X, Y\}$ be a pair of simplicial complexes. Suppose that a simplicial complex Z embeds like a simplicial subcomplex in each complex X and Y,

$$X \overset{i}{\hookleftarrow} Z \overset{j}{\hookrightarrow} Y.$$

Consider a pseudo-simplicial complex $W = X \cup_Z Y$ obtained by the gluing together of X and Y along Z. The following Lemma gives us the sufficient conditions for W to be a simplicial complex.

Lemma 2.3. *Assume that the following conditions are satisfied:*

(1) *For all two vertices v_1 and v_2 in Z, there is at most one edge $[v_1, v_2]$ in W that connects them.*

(2) *At least one of the two embeddings i or j is maximal.*

Then W is a simplicial complex.

Proof. Suppose that W is not a simplicial complex, this means that there exist two simplexes Δ_1 and Δ_2 in X and Y respectively, such that $\Delta_1 \cap \Delta_2 = M \subset Z$ where $M \neq \emptyset$ is not a simplex.

(1) If $\mathrm{Sk}^1(M)$ is not a complete graph then there exist in M two vertices v_1 and v_2 which are not related by an edge in M, but there exist two edges $[v_1, v_2]^{(1)} \in \Delta_1$ and $[v_1, v_2]^{(2)} \in \Delta_2$ connecting v_1 and v_2, this contradicts the condition (1).

(2) If $\mathrm{Sk}^1(M)$ is a complete graph with $m+1$ vertices, then there are two simplexes $\Delta_1^m \subset \Delta_1$ and $\Delta_2^m \subset \Delta_2$ such that

$$\mathrm{Sk}^1(M) = \mathrm{Sk}^1(\Delta_1^m) = \mathrm{Sk}^1(\Delta_2^m).$$

If Z is maximal in X, we get $\Delta_1^m \subset Z \subset X$ this implies that $\Delta_1^m = \Delta_2^m$, and then M is a simplex.

Which completes the proof. ∎

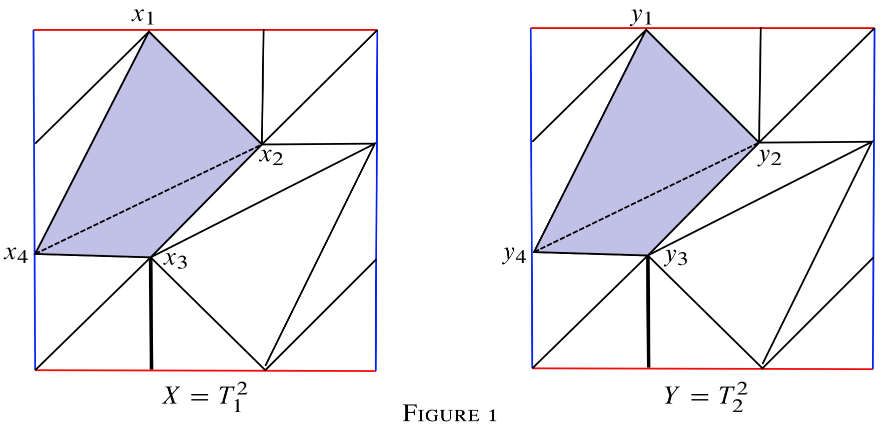

FIGURE 1
Minimal triangulation of a torus T^2.

Remark 2.4. (1) Condition (2) of Lemma 2.3 is not necessary. Fix Δ^3 a 3-dimensional simplex and let $W = \partial\Delta^3$ be the boundary of this simplex and $Z = \mathrm{Sk}^1(\Delta^3)$ its 1-skeleton. We take X as the union of Z and two 2-simplexes of W, and also Y as the union of Z and two remaining 2-simplexes of W. We can see that the embeddings of Z into X and Y are not maximal.

(2) Condition (1) of Lemma 2.3 is however necessary. We illustrate this condition by the following example. Take the 2-dimensional torus $T^2 = S^1 \times S^1$ and consider its minimal triangulation (see Figure 1), which contains 7 vertices. This triangulation is unique and its 1-skeleton is a complete graph K_7. X and Y are both tori, the vertices of each of them are denoted by x_i and y_i, respectively. We choose two adjacent triangles $[x_1, x_2, x_4]$ and $[x_2, x_3, x_4]$ in X ($[y_1, y_2, y_4]$ and $[y_2, y_3, y_4]$, respectively, in Y) as in Figure 1 then we remove them, so we obtain two tori with holes. Take the subcomplex

$$Z = \partial\big(X \setminus \{[x_1, x_2, x_4] \cup [x_2, x_3, x_4]\}\big) \quad \text{in } X$$

and

$$Z = \partial\big(Y \setminus \{[y_1, y_2, y_4] \cup [y_2, y_3, y_4]\}\big) \quad \text{in } Y.$$

There are two ways to glue X and Y along Z to get a surface of genus 2. The first consists to identify the vertices x_i with y_i for $i \in \{1, 2, 3, 4\}$. The second is to identify the vertices x_i with y_{i+1}, $i \in \{1, 2, 3\}$ and x_4 with y_1. Each of the two gluing gives a semi-triangulation of the surface W. The first gluing is not a triangulation because there are in W two edges (which are dark in Figure 1) such that their intersection is two distinct vertices $x_1 = y_1$ and $x_3 = y_3$. But in the second case, condition (1) of Lemma 2.3 is satisfied and we get a triangulation of the surface W. This triangulation contains 10 vertices and it provides a minimal triangulation of the surface of genus 2.

Proposition 2.5. *Let G_i, $i = \{1,2\}$ be two finitely presented groups. Then*

(1) $\mathrm{KW}(G_1 \times G_2) \leq \mathrm{KW}(G_1) \times \mathrm{KW}(G_2)$*;*

(2) $\mathrm{KW}(G_1 * G_2) \leq \mathrm{KW}(G_1) + \mathrm{KW}(G_2) - 3 + a$*, where*

$$\begin{aligned} a &= 0 \quad \text{if } G_1 \text{ and } G_2 \text{ are both non-free, and} \\ a &= 1 \quad \text{otherwise;} \end{aligned}$$

(3) *if $H < G$ is a subgroup of index k, then* $\mathrm{KW}(H) \leq k\,\mathrm{KW}(G)$*;*

(4) *if* $\mathrm{KW}(G) = n$*, then the Betti numbers of G satisfy*

$$b_k(G) \leq \binom{n-1}{k+1} \quad \text{for } k = 1, 2,$$

where $\binom{n-1}{k+1}$ is the binomial coefficient.

Proof. The first inequality is analogous to the ct-complexity of a product of simplicial complexes given in [17, Remark 7.4].

To prove the second inequality (2), let us consider two simplicial complexes X_1 and X_2 satisfying $\pi_1(X_i) = G_i$ and $\mathrm{KW}(G_i) = \mathrm{ct}(X_i)$ for $i \in \{1,2\}$. We distinguish two cases:

Case 1. If G_1 and G_2 are both non-free groups then $G_1 * G_2$ is a fundamental group of the simplicial complex $X_1 \cup_{\Delta^2} X_2$ obtained by the gluing of X_1 and X_2 along a 2-simplex $\Delta^2 \subset X_i$, $i = \{1,2\}$, so

$$\mathrm{KW}(G_1 * G_2) \leq \mathrm{KW}(G_1) + \mathrm{KW}(G_2) - 3.$$

Case 2. If G_1 or G_2 is free group or both of them then $G_1 * G_2$ is a fundamental group of the simplicial complex $X_1 \cup_Z X_2$ obtained by the gluing of X_1 and X_2 along an edge $Z \subset X_i$, $i = \{1,2\}$, so

$$\mathrm{KW}(G_1 * G_2) \leq \mathrm{KW}(G_1) + \mathrm{KW}(G_2) - 2.$$

The third inequality is immediate. If X is a 2-complex such that $\pi_1(X) = G$ one has $b_1(X) = b_1(G)$ and $b_2(X) \geq b_2(G)$. So the last property is direct from [17, Theorem 3.3]. ■

Remark 2.6. It seems that inequality (1) of Proposition 2.5 is never exact if the two groups are not trivial. A simple example is $G_1 = G_2 = \mathbb{Z}$. In this case

$$\mathrm{KW}(G_1) = \mathrm{KW}(G_2) = 3, \quad \text{but } \mathrm{KW}(G_1 \times G_2) = 7.$$

On the other hand the inequality (2) is optimal as shows the example of $G_1 = \mathbb{F}_n$, where $n = (k-1)(k-2)/2$ with k an integer, and $G_2 = \mathbb{Z}$. Here

$$\mathrm{KW}(G_1) = k, \quad \mathrm{KW}(G_2) = 3, \quad \text{and} \quad \mathrm{KW}(G_1 * G_2) = \mathrm{KW}(\mathbb{F}_{n+1}) = k+1.$$

The behavior of $\mathrm{KW}(G_1 * G_2)$ may be rather different from the behavior of $\kappa(G_1 * G_2)$. For example, let $G_1 = \pi^{\pm}(g)$ be the fundamental group of a non-simply connected surface of genus g (see the introduction) and $G_2 = \mathbb{Z}$. The following remarkable stabilizing property for the simplicial complexity has been proven recently in [7]. For every surface group $\pi^{\pm}(g)$, we have

$$\kappa(\pi^{\pm}(g) * \mathbb{Z}) = \kappa(\pi^{\pm}(g)).$$

Nevertheless, for the KW-complexity we have,

$$\mathrm{KW}(\mathbb{Z}_2 * \mathbb{Z}) = \mathrm{KW}(\pi_1(\mathbb{R}P^2)) + 1 = 7,$$
$$\mathrm{KW}(\pi^+(1) * \mathbb{Z}) = \mathrm{KW}(\pi^+(1)) + 1 = 8,$$

but

$$\mathrm{KW}(\pi^+(2) * \mathbb{Z}) = \mathrm{KW}(\pi^+(2)).$$

It seems that in the general orientable case, for $g \geq 3$, the KW-complexity should be given by

$$\mathrm{KW}(\pi^+(g) * \mathbb{Z}) = \begin{cases} \mathrm{KW}(\pi^+(g)) + 1 & \text{for } g = 12l^2 \pm l \text{ or } g = 12l^2 \pm 7l + 1, \\ \mathrm{KW}(\pi^+(g)) & \text{otherwise.} \end{cases}$$

3. The KW-complexity of certain groups

In this part we study the KW-complexity of certain finitely presented groups, namely Artin and Coxeter groups, abelian, in particular, cyclic groups. For Artin and Coxeter groups, we follow the same notations as in [9].

Let $E = \{a_i\}_{i=1}^n$ be a finite set of elements and F_E a free group generated by this set. Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and taking a_i, a_j two distinct elements of E. For all $m_{ij} \in \overline{\mathbb{N}}$, $< a_i a_j >^{m_{ij}} = a_i a_j a_i \ldots$ denotes an alternating product of a_i and a_j of length m_{ij}, such that

- if $m_{ij} = 2k_{ij}$, $k_{ij} \in \mathbb{N}$, then $< a_i a_j >^{m_{ij}} = (a_i a_j)^{k_{ij}}$, and
- if $m_{ij} = 2k_{ij} + 1$, then $< a_i a_j >^{m_{ij}} = (a_i a_j)^{k_{ij}} a_i$.

Let us note by $M = (m_{ij})$ a symmetric $(n \times n)$-matrix called the Coxeter matrix of elements $m_{ji} = m_{ij} \in \overline{\mathbb{N}}$ such that $m_{ii} = 1$, and if $i \neq j$, then $m_{ij} \geq 2$.

Definition 3.1 (Artin groups). An Artin group generated by the set E is a group with a presentation of the following form:

(3.1) $G_A(M) = \langle a_1, \ldots, a_n \mid < a_i a_j >^{m_{ij}} = < a_j a_i >^{m_{ji}},\ i \neq j,\ i, j \in \{1, \ldots, n\} \rangle.$

Here there is no relation between a_i and a_j if $m_{ij} = \infty$. According to the integers m_{ij}, $i \neq j$, there are three common types of Artin groups:

(1) *right-angled* Artin groups: all elements of M, m_{ij} are equal to 2 or ∞;

(2) an Artin group of *large type* when $m_{ij} \geq 3$;

(3) an Artin group of *extra-large type* when $m_{ij} \geq 4$.

Definition 3.2 (Coxeter groups). We call Coxeter group generated by the set E, a group with a presentation of the form:

$G_C(M) = \langle a_1, \ldots, a_n \mid a_i^2 = e,\ < a_j a_i >^{m_{ji}} = < a_j a_i >^{m_{ji}},\ i \neq j,\ i, j \in \{1, \ldots, n\} \rangle.$

As above,

(1) if $m_{ij} = \{2, \infty\}$, $G_C(M)$ is called a *right-angled* Coxeter group;

(2) if $m_{ij} \geq 3$ for all $i \neq j,\ i, j \in \{1, \ldots, n\}$, then $G_C(M)$ is a Coxeter group of *large type*;

(3) if $m_{ij} \geq 4$ for all $i \neq j,\ i, j \in \{1, \ldots, n\}$, then $G_C(M)$ is a Coxeter group of *extra-large type*.

The chapter is devoted to systematical study the KW-complexity of this two types of groups.

Example 3.3. The simplest example of a Coxeter group is the group of one generator $\mathbb{Z}_2$. Its classifying space is the infinite real projective space $\mathbb{R}P^\infty$ and its 2-skeleton is the real projective plane $\mathbb{R}P^2$. The minimal triangulation of $\mathbb{R}P^2$, given in Figure 2, contains 6 vertices. This number corresponds to the KW-complexity of the group $\mathbb{Z}_2$, i.e., $\mathrm{KW}(\mathbb{Z}_2) = 6$.

Let M contains $2m$ finite entries and let us consider a right-angled Artin group $G_A(M)$, so $G_A(M)$ has n generators and m relations. The following result gives an estimate of the KW-complexity of the right-angled Artin groups.

Theorem 3.4. *Let $G_A(M)$ be a right-angled Artin group with n generators and m relations. Then*

$$k_A(n, m) \leq \mathrm{KW}(G_A(M)) \leq 2(n + m) + 1,$$

where $k_A(n, m)$ is the function of two variables, defined as follows:

$$k_A(n, m) = \begin{cases} (\sqrt{8n + 1} + 3)/2 & \text{if } m \leq \frac{n}{6}(\sqrt{8n + 1} - 3), \\ \sqrt[3]{6m} + 2 & \text{otherwise}. \end{cases}$$

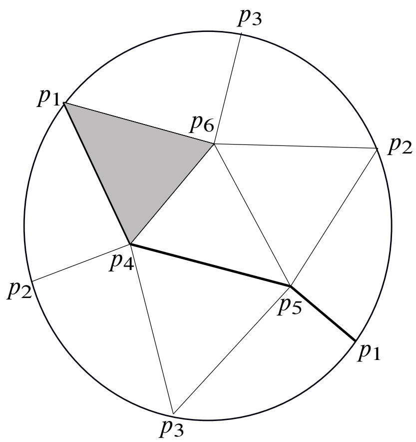

FIGURE 2
Minimal triangulation of $\mathbb{R}P^2$.

Proof. Let us start with the upper bound. For right-angled Artin group, the relations in (3.1) are the commutativity relations, so the 2-complex $K(n,m)$ obtained by the presentation of $G_A(M)$ is naturally a subcomplex of the 2-skeleton of a n-dimensional torus, T^n.

Consider the canonical cellular decomposition of torus T^n that given by the product of n circles. Then

$$K(n,m) = \Big(\bigvee_{i=1}^{n} S_i^1\Big) \bigcup \Big(\bigcup_{\substack{i<j,\\ m_{ij}<\infty}} T^2_{\{ij\}}\Big) \subset \mathrm{Sk}^2(T^n), \tag{3.2}$$

where the circles S_i^1 of the wedge sum correspond to the generators a_i of $G_A(M)$. Each 2-dimensional torus $T^2_{\{ij\}}$ corresponds to the commutativity relation $a_i a_j = a_j a_i$ of (3.1). We can see that this complex coincides with the 2-skeleton of the classifying space of $G_A(M)$; see, for example, [10].

Now we give the triangulation of $K(n,m)$ as follows, on each circle of (3.2) we consider the triangulation of the complete graph K_3 such that one vertex of this graph coincides with the base vertex of the wedge sum. For each 2-cell $T^2_{\{ij\}}$, we take a minimal triangulation of the torus T^2 given in Figure 1. The three vertical edges of the square coincide with those of the circle S_i^1, and the horizontal edges correspond to those of the circle S_j^1.

To prove that the pseudo-triangulation given above is a triangulation, we proceed by induction.

If $m = 0$, then $K(n,0) = (\bigvee_{i=1}^{n} S_i^1)$, and the conclusion is obvious. Assume that by adding m tori one obtains the complex (3.2) whose triangulation is supposed to be convenient. We now add another torus $T^2_{\{kl\}}$:

$$K(n,m+1) = K(n,m)\bigcup_Z T^2_{\{kl\}},$$

where Z is the 1-dimensional subcomplex belonging to both $K(n,m)$ and $T^2_{\{kl\}}$.

To verify that $K(n,m+1)$ is a simplicial complex, we check the conditions of Lemma 2.3. From the geometry of cellular decomposition of $K(n,m)$ we have three possibilities for Z. The first one is $Z = K_3 \vee K_3$. Thus this subcomplex corresponds to the wedge sum $S^1_k \vee S^1_l \subset T^2_{\{kl\}}$. The circles are not contractible so the embedding of $Z \hookrightarrow T^2_{\{kl\}}$ is maximal. To satisfy condition (1) of the same lemma, note that from the triangulation of $T^2_{\{kl\}}$, two vertices of Z are connected by one edge and which does not belong to Z if these vertices are different from the base point p and each of them belongs to the circles S^1_k and S^1_l. In this case, these vertices do not belong to the same torus $T^2_{\{ij\}} \subset K(n,m)$ then they are not connected by an edge in $K(n,m)$. The second and third possibilities are respectively $Z = K_3$ and $Z = \{pt\}$. In these two cases Lemma 2.3 applies obviously.

To complete the first part of the proof, we calculate the number of vertices obtained from the triangulation of $K(n,m)$. The wedge sum in (3.2) gives $2n+1$ vertices. Each torus $T^2_{\{ij\}}$ of (3.2) contains in addition 2 vertices which belong to $T^2_{\{ij\}} \setminus \{S^2_i \vee S^2_j\}$. Then the triangulation of $K(n,m)$ contains $2n+2m+1$ vertices.

For the lower bound, let us take $k = \mathrm{KW}(G_A(M))$ so the number n of generators and the number m of relations of $G_A(M)$ satisfy:

$$n \leq \frac{(k-1)(k-2)}{2} \tag{3.3}$$

and

$$m \leq \frac{(k-1)(k-2)(k-3)}{6}. \tag{3.4}$$

Consider the positive part of Euclidean plan $(x,y) \in \mathbb{R}^2_+$ and define a parametric curve

$$\left(x = \frac{(t-1)(t-2)}{2}; y = \frac{(t-1)(t-2)(t-3)}{6}\right), \quad t \geq 3,$$

whose Cartesian equation is $y = \frac{x}{6}(\sqrt{8x+1}-3)$. This curve divides the first quadrant of the plane into two parts: the lower part

$$\left\{m \leq \frac{n}{6}(\sqrt{8n+1}-3)\right\},$$

and the upper part

$$\left\{m \geq \frac{n}{6}(\sqrt{8n+1}-3)\right\}.$$

If (n, m) is in the lower part then the right-hand side of (3.3) is higher than the one in (3.4), hence $k \geq (\sqrt{8n+1}+3)/2$. In the opposite case, i.e., (n, m) is in the upper part of the quadrant and the inequality (3.3) becomes more powerful than inequality (3.4). This implies that $k \geq t_*(m)$ where $t_*(m)$ is the real root of the equation $(t-1)(t-2)(t-3) = 6m$. Therefore, $k \geq \sqrt[3]{6m} + 2$, hence the result. ∎

For Artin groups, the number of relations is bounded by $n(n-1)/2$, where n is the number of generators. When $m = n(n-1)/2$, the right-angled Artin group corresponds to the free abelian group of rank n. From Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *The* KW*-complexity of free abelian group $\mathcal{A}_n$ of rank n satisfies*

$$\lceil (3n(n-1))^{1/3} \rceil + 2 \leq \mathrm{KW}(\mathcal{A}_n) \leq n^2 + n + 1.$$

Example 3.6. For the abelian group $\mathcal{A}_2$ of rank 2, the Corollary 3.5 implies

$$4 \leq \mathrm{KW}(\mathcal{A}_2) \leq 7.$$

On the other hand, this group is a fundamental group of torus T^2 and we know from [6] that the exact value of its KW-complexity is equal to 7.

If the rank is equal to 3, from the same Corollary 3.5 we get $5 \leq \mathrm{KW}(\mathcal{A}_3) \leq 13$. The exact value of its KW-complexity is not yet known.

Theorem 3.7. *Let $G_C(M)$ be a right-angled Coxeter group with n generators and m commutation relations. Then*

$$k_C(n, m) \leq \mathrm{KW}(G_C(M)) \leq 5n + 2m + 1,$$

where $k_C(n, m)$ is a function of two variables defined in the following form:

$$k_C(n, m) = \begin{cases} (\sqrt{8n+1}+3)/2 & \text{if } m \leq \frac{n}{6}(\sqrt{8n+1}-3), \\ \sqrt[3]{6(m+n)} + 2 & \text{otherwise.} \end{cases}$$

Proof. We apply the same reasoning as above. The relations $a_i^2 = e$, $i \in \{1, \ldots, n\}$ of $G_C(M)$ correspond in the classifying space of the group to the infinite real projective spaces $\mathbb{R}P^\infty$. We define the 2-complex $P(n, m)$ such that $\pi_1(P(n, m)) = G_C(M)$ by its cellular decomposition such that:

$$P(n, m) = \left(\bigvee_{i=1}^{n} \mathbb{R}P_i^2 \right) \bigcup \left(\bigcup_{\substack{i<j, \\ m_{ij}<\infty}} T_{\{ij\}}^2 \right) \subset \mathrm{Sk}^2 \left(\prod_{i=1}^{n} \mathbb{R}P_i^\infty \right). \tag{3.5}$$

We can see that this decomposition is the corresponding decomposition of the 2-skeleton of the classifying space of $G_C(M)$.

To triangulate properly this cellular complex (3.5), we shall use the minimal triangulation of the torus T^2 and of the real projective plane $\mathbb{R}P^2$; see, respectively, Figures 1 and 2. For the real projective planes the vertex $p_1(i)$ is associated to the base vertex of wedge sum (3.5). The generators a_i for all $i \in \{1, \dots, n\}$ correspond to the concatenation of three edges

$$[p_1(i), p_4(i)], \quad [p_4(i), p_5(i)], \quad [p_5(i), p_1(i)]$$

of this triangulation.

Assume that the 2-cells of $P(n,m)$ are triangulated. By applying Lemma 2.3 in the same way as in the proof of Theorem 3.4, we see that the triangulations of all 2-cells of $P(n,m)$ are coherent and so we obtain a global triangulation of $P(n,m)$. Now we calculate the number of vertices derived from this triangulation. Then

$$s_0\big(\mathrm{Sk}^2(BG_C(M))\big) \leq 2(n+m) + 1 + 3n = 5n + 2m + 1,$$

where $BG_C(M)$ is the classifying space of $G_C(M)$. This implies

$$\mathrm{KW}(G_C(M)) \leq 5n + 2m + 1.$$

For the lower bound, we proceed as in the previous proof of Theorem 3.4, the difference is that in this case the number of relations is $r = m + n$, and this ends the proof. ∎

If all generators of $G_C(M)$ commute two-by-two so $m = n(n+1)/2$ and the corresponding group is the direct sum of m cyclic groups $\mathbb{Z}_2$,

$$G_C(M) = \bigoplus_{i=1}^{n} (\mathbb{Z}_2)_i.$$

We then have the following result.

Corollary 3.8. $\frac{4}{3}n^{2/3} + 1 \leq \mathrm{KW}(\bigoplus_{i=1}^{n}(\mathbb{Z}_2)_i) \leq n^2 + 4n + 1.$

3.1. Artin/Coxeter groups of large and extra-large type. Throughout this section, we focus on the study of Artin/Coxeter groups of large type. The construction used below is universal and it can be applied to all types of Artin/Coxeter groups. It becomes really interesting if groups are of the uniformly extra-large type. The last means that $m_{ij} \gg 1$ for all $i \neq j$. We begin with the following proposition.

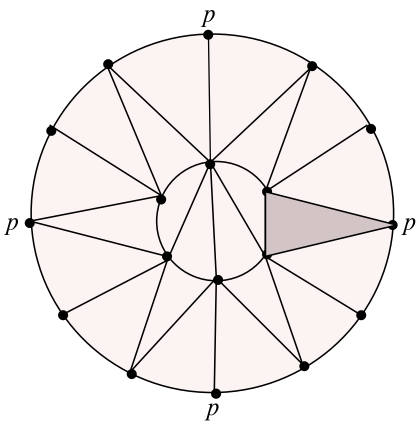

FIGURE 3
Triangulation of the disk P_l for even l ($l = 4$).

Proposition 3.9. *Let G be a finitely presented group with n generators and one relation*

$$G = \langle a_1, a_2, \dots, a_n \mid w^m = v^m \rangle,$$

where $w = a_{t_1}^{\varepsilon_1} a_{t_2}^{\varepsilon_2} \dots a_{t_l}^{\varepsilon_l}$ and $v = a_{t'_1}^{\varepsilon'_1} a_{t'_2}^{\varepsilon'_2} \dots a_{t'_{l'}}^{\varepsilon'_{l'}}$ are two cyclically reduced words such that

$$\varepsilon_i, \varepsilon'_j \in \{-1, +1\}, \quad t_i, t'_j \in \{1, \dots, n\}$$

for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, l'\}$, where l and l' are the lengths of w and v, respectively. Then there exists a 2-simplicial complex K, such that $\pi_1(K) = G$ and

$$s_0(K) \leq 8 \log_2 m + 2n + \frac{3}{2}(l + l') + 5.$$

Proof. To construct such a complex, we apply the same telescopic process as that of the proof of [3, Lemma 4.1].

We start by defining two simplicial complexes K_v and K_w. Let $S = \bigvee_{i=1}^n S_i^1$ be a wedge sum of n circles. Each circle S_i^1 is associated to a generator a_i, $i \in \{1, \dots, n\}$. We triangulate S, by taking each circle as a concatenation of three edges and denote by p the base vertex.

Take P_l and $P_{l'}$ two triangulated disks as follows, the boundaries of P_l and $P_{l'}$ are triangulated into $3l$ and $3l'$ edges, respectively. The inside of each of the two disks is decomposed into sections, each of them containing 4 or 5 2-simplexes placed in an alternate manner (see Figures 3 and 4). The central section of P_l contains l or $l + 1$ 2-simplexes, and that of $P_{l'}$ contains l' or $l' + 1$ 2-simplexes depending on the parity of l and l', respectively. In total, we get $l + 1$ sections for P_l and $l' + 1$ for $P_{l'}$.

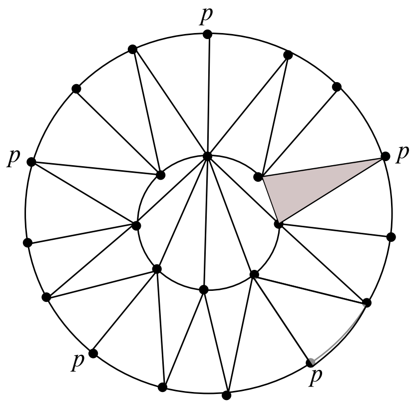

FIGURE 4
Triangulation of the disk P_l for odd l ($l = 5$).

The complexes P_w and P_v are obtained by the gluing of P_l and $P_{l'}$ with S along the words w and v, respectively. It is easy to verify that these complexes provide two triangulations.

Fix two 2-simplices Δ^2_p and Δ'^2_p of P_w and P_v, respectively. Each of them contains the vertex p, e.g., the dark triangles of Figures 3 and 4, and remove them. We get two new complexes P'_w and P'_v whose fundamental groups are free groups of rank n, and homotopy classes of curves $\partial\Delta^2_p$ and $\partial\Delta'^2_p$ correspond to the words w and v, respectively.

Now, starting with $\partial\Delta^2_p$ and $\partial\Delta'^2_p$, we construct two Möbius telescopes $\mathcal{J}_k$ and $\mathcal{L}_k$ of length k. We are going to use the construction of the proof of [3, Lemma 4.1].

Let $\{M_i\}_{i\in\mathbb{N}}$ and $\{M'_i\}_{i\in\mathbb{N}}$ be two sequences of Möbius bands, satisfying:

(1) p_i is a point in ∂M_i, and p'_i is a point in $\partial M'_i$;

(2) γ_i is a simple loop based at p_i, such that $\gamma_i \setminus \{p_i\}$ lies in the interior of M_i and $\{\partial M_i\} = 2\{\gamma_i\} \in \pi_1(M_i) = \langle[\gamma_i]\rangle$ (likewise for γ'_i in M'_i, $i \in \{1, \dots, k\}$).

The triangulation of the Möbius band given in Figure 5, which is not minimal, is obtained from that of the projective plane (Figure 2) from which a 2-simplex e.g., $\Delta(p_1, p_4, p_6)$ has been removed. The curve γ_i of the left-hand side of Figure 5 corresponds to the concatenation

$$[p_1(i), p_3(i)] \cup [p_3(i), p_2(i)] \cup [p_2(i), p_1(i)]$$

in the triangulation. Also the border ∂M_i corresponds to the concatenation of three edges

$$[p_1(i), p_4(i)] \cup [p_4(i), p_6(i)] \cup [p_6(i), p_1(i)]$$

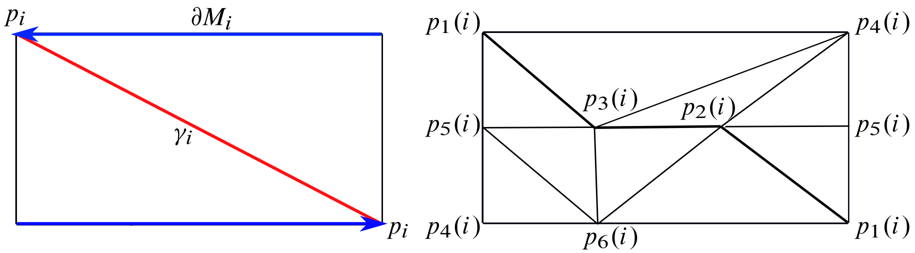

Figure 5
The triangulation of the Möbius band.

in the triangulation. The advantage of choosing this triangulation is that both curves, γ_i and ∂M_i pass through the vertex $p_i = p_1(i)$ and they have the same simplicial length, i.e., 3 edges.

One defines two similar telescopic towers $\mathcal{J}_k$ and $\mathcal{L}_k$ as follows:

$$\begin{cases} \gamma_0 = \partial \Delta_p^2, \\ \mathcal{J}_1 = M_0, \\ \mathcal{J}_{k+1} = \mathcal{J}_k \cup_{\varphi_k} M_k \end{cases}$$

and

$$\begin{cases} \gamma_0' = \partial \Delta_p'^2, \\ \mathcal{L}_1 = M_0', \\ \mathcal{L}_{k+1} = \mathcal{L}_k \cup_{\psi_k} M_k', \end{cases}$$

where φ_k and ψ_k are gluing PL-homeomorphisms satisfying:

(1) $\varphi_k(\gamma_k) = \partial M_{k-1}, \varphi_k(p_k) = p_{k-1}$, we require that all vertices p_i are glued on the same vertex p for all $i \in \{1, \ldots, k\}$,

(2) $\psi_k(\gamma_k') = \partial M_{k-1}', \psi_k(p_k') = p_{k-1}'$, the same requirement for the vertices p_i', they are all glued on the same vertex p.

Every gluing homeomorphism will be chosen to be piecewise linear in the sequel.

Following the recursive construction the straightforward verification as in [3] and Lemma 2.3 ensure that both telescopes $\mathcal{J}_k$ and $\mathcal{L}_k$ are simplicial complexes. Observe that

- $\mathcal{J}_1 \subset \mathcal{J}_2 \subset \ldots \subset \mathcal{J}_{k-1} \subset \mathcal{J}_k$;
- γ_0 is a deformation retract of $\mathcal{J}_k$, and thus $\pi_1(\mathcal{J}_k) = \mathbb{Z}$;
- $\{\gamma_i\} = 2^i\{\gamma_0\}$ for $i \in \{0, \ldots, k-1\}$.

Also,

- $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots \subset \mathcal{L}_{k-1} \subset \mathcal{L}_k$;
- γ'_0 is a deformation retract of $\mathcal{L}_k$, and thus $\pi_1(\mathcal{L}_k) = \mathbb{Z}$;
- $\{\gamma'_i\} = 2^i\{\gamma'_0\}$ for $i \in \{0, \ldots, k-1\}$.

Let k be the smallest integer such that: $m < 2^{k+1}$. The dyadic decomposition of m is written as

$$m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s},$$

where $(k_j)_{j\in\{0,\ldots,k-1\}}$ are integers, satisfying $0 \le k_1 < k_2 < \cdots < k_s = k$ and $s \le k+1$.

Let $\xi(m)$ and $\xi'(m)$ be two closed curves based on p such that

$$\begin{aligned} \xi(m) &= \gamma_{k_1} \star \gamma_{k_2} \star \cdots \star \gamma_{k_{s-1}} \star \partial M_{k-1} \in \mathcal{J}_k, \\ \xi'(m) &= \gamma'_{k_1} \star \gamma'_{k_2} \star \cdots \star \gamma'_{k_{s-1}} \star \partial M'_{k-1} \in \mathcal{L}_k, \end{aligned}$$

we get $\{\xi(m)\} = m\{\gamma_0\}$ and $\{\xi'(m)\} = m\{\gamma'_0\}$. The symbol $\star$ means the concatenation of loops γ_{k_j} (and γ'_{k_j}) based on p for all $j \in \{1, \ldots, s-1\}$. Thus, we get two simplicial complexes

$$K_w = \left(P'_w \cup_{\gamma_0} \mathcal{J}_k\right) \bigcup_{\xi(m)\sim\partial D_1^2} D_1^2, \quad K_v = \left(P'_v \cup_{\gamma'_0} \mathcal{L}_k\right) \bigcup_{\xi'(m)\sim\partial D_2^2} D_2^2.$$

Now fix two 2-simplexes $T_1 \subset D_1 \subset K_w$ and $T_2 \subset D_2 \subset K_v$. They both contain the vertex p (see the dark triangle of Figures 3), and we remove them. Let K'_w and K'_v be two new obtained complexes. Note that the both complexes contain the initial bouquet of circles:

$$K'_w \hookleftarrow S \hookrightarrow K'_w,$$

and let $K' = K'_w \cup_S K'_v$.

Remark that $S \subset K'$ is a deformation retract of K'. The curve ∂T_1 is homotopic to the element of $\pi_1(S)$ given by the word w^m in generators $\{a_i\}_{i=1}^n$. In the same way the curve ∂T_2 is homotopic to the element of $\pi_1(S)$ given by the word v^m in the same system of generators.

Let $\phi\colon \partial T_1 \to \partial T_2$ be a usual PL-homeomorphism preserving p. The complex

$$K = K'/\phi,$$

which consists of the identification of ∂T_i, $i = 1, 2$ by ϕ, is naturally pseudo-simplicial. It is easy to see by construction that

$$\pi_1(K) = \langle a_1, a_2, \ldots, a_n \mid w^m = v^m \rangle.$$

Remark now that the pseudo-triangulation obtained on K is in fact a triangulation. Let $\{u_i, v_i, p\}$ be the vertices of T_i, $i = 1, 2$. The shortest combinatorial way from u_1 to u_2 or v_2 is $[u_1, p] \cup [p, u_2]$ (respectively, $[u_1, p] \cup [p, v_2]$) and it has the length 2, idem for v_1. It is easy to see from construction that any other combinatorial way joining u_1 or v_1 with u_2 or v_2 is at least of length 4. So the gluing by ϕ does not degenerate the triangulation.

Let estimate the number of vertices of the constructed triangulation of K. We have

$$s_0(K) = s_0(K'_w) + s_0(K'_v) - [s_0(\partial T_1) + 2n].$$

We start with calculation of $s_0(K'_w)$. The triangulation of the Möbius band given by the Figure 5, of generator curve

$$\gamma_i = [p_1(i), p_3(i)] \cup [p_3(i), p_2(i)] \cup [p_2(i), p_1(i)],$$

which is homotopically equivalent to $[p_1(i), p_4(i)] \cup [p_4(i), p_5(i)] \cup [p_5(i), p_1(i)]$, contains 6 vertices, of which one is the vertex p.

By construction of $\mathcal{J}_k$, the number of vertices that it contains is

$$s_0(\mathcal{J}_k) = s_0(\gamma_0) + 3k,$$

where k is the number of Möbius bands and $s_0(\gamma_0) = 3$, then in total we get $s_0(\mathcal{J}_k) = 3k + 3$.

The triangulation of the complex P'_w gives

$$s_0(P'_w) = \begin{cases} 1 + 2t + l + l/2 & \text{if } l \text{ is even,} \\ 1 + 2t + l + (l+1)/2 & \text{otherwise.} \end{cases}$$

We can write $s_0(P'_w) \leq 1 + 2t + (3l + 1)/2$ for all l, where 1 corresponds to the vertex p, $t \leq n$ is the number of generators that form the word w, so

$$s_0(P'_w) \leq 1 + 2n + \frac{3l + 1}{2}.$$

The triangulation of the disk D_1^2 is different from that of P_w, because its boundary $\partial D_1^2 \sim \xi(m)$ is composed of s sections, all different from each other. Each section contains 2 vertices and the base vertex p, so $s_0(\partial D_1^2) = 2s + 1$, we obtain $s_0(D_1^2) = 3s + 1$. Figure 6 illustrates this triangulation for $s = 6$. Therefore,

$$\begin{aligned} s_0(K'_w) &= s_0(P'_w) + s_0(\mathcal{J}_k) - s_0(\gamma_0) + s_0(D_1^2) - s_0(\partial D_1^2) \\ &\leq \left(1 + 2n + \frac{3l+1}{2}\right) + (3k + 3) - 3 + (3s + 1) - (2s + 1) \\ &\leq 4k + 2n + \frac{3l+1}{2} + 2 \quad \text{because } s \leq k + 1. \end{aligned}$$

So, $s_0(K'_w) \leq 4 \log_2 m + 2n + \frac{3}{2}l + \frac{5}{2}$.

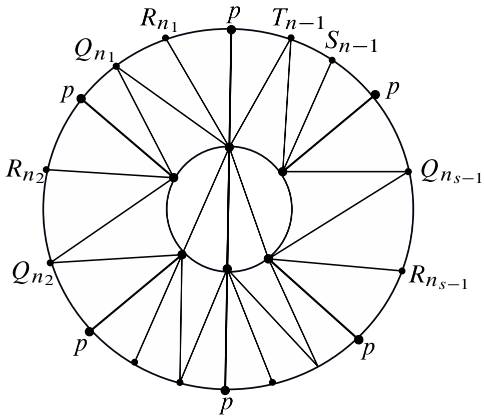

FIGURE 6
Triangulation of disk D_1^2 for $s = 6$.

We repeat the same calculations to estimate $s_0(K'_v)$, and we obtain

$$s_0(K'_v) \leq 4\log_2 m + 2n + \frac{3}{2}l' + \frac{5}{2}.$$

We finally conclude that $s_0(K)$ satisfies

$$\begin{aligned} s_0(K) &\leq \Big(4\log_2 m + 2n + \frac{3}{2}l + \frac{5}{2}\Big) + \Big(4\log_2 m + 2n + \frac{3}{2}l' + \frac{5}{2}\Big) - (3 + 2n) \\ &\leq 8\log_2 m + 2n + \frac{3}{2}(l + l') + 2. \end{aligned}$$

This implies that $\mathrm{KW}(G) \leq 8\log_2 m + 2n + \frac{3}{2}(l + l') + 2$. ∎

Remark 3.10. If in addition, the word w satisfies $a_{t_i} \neq a_{t_{i+1}}$ for $i \in \{1, \ldots, l-1\}$, then the triangulation of the disk P_l will have fewer vertices than the one given in Figures 3 and 4, as shown in the Figure 7 for $w = a_1 a_2 a_1^{-1} a_2$, where a_1 and a_2 are the generators of the group.

Corollary 3.11. *Let G be a finitely presented group defined by*

$$G = \langle a_1, \ldots, a_n \mid w_1^{m_1} = v_1^{m_1}, \ldots, w_r^{m_r} = v_r^{m_r} \rangle,$$

where w_i and v_i are two reduced words (they satisfy the properties of Proposition 3.9) of lengths l_i and l'_i, respectively, m_i are non-zero integers for all $i \in \{1, \ldots, r\}$. Then,

$$\mathrm{KW}(G) \leq \Big[\sum_{i=1}^{r}\Big(8\log_2 m_i + \frac{3}{2}(l_i + l'_i)\Big)\Big] + 2n + r + 1.$$

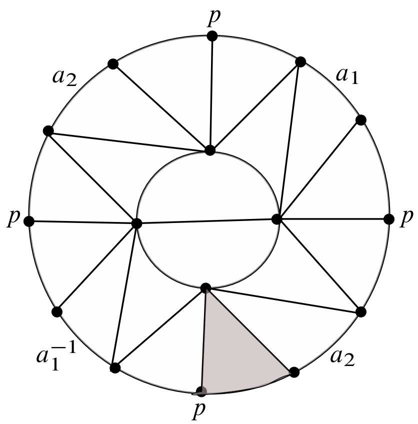

FIGURE 7
An associated disk P to the word $w = a_1 a_2 a_1^{-1} a_2$.

Proof. As above, we construct r simplicial 2-complexes K_i, where $i \in \{1, \ldots, r\}$, such that each of them is associated to the relation $w_i^{m_i} = v_i^{m_i}$. The union of these simplicial 2-complexes induces a new simplicial 2-complex K such that $K = \bigcup_{i=1}^{r} K_i$ and $\pi_1(K) = G$. Therefore,

$$s_0(K) \leq \left[\sum_{i=1}^{r} s_0(K_i) \right] - [(r-1)(2n+1)],$$

where $2n+1$ is the number of vertices which are on S. Then

$$\begin{aligned} s_0(K) &\leq \left[\sum_{i=1}^{r} \Big(8 \log_2 m_i + 2n + \frac{3}{2}(l_i + l_i') + 2 \Big) \right] - (2rn - 2n + r - 1) \\ &\leq \left[\sum_{i=1}^{r} \Big(8 \log_2 m_i + \frac{3}{2}(l_i + l_i') \Big) \right] + 2n + r + 1, \end{aligned}$$

and this completes the proof. ∎

Consider now an Artin group $G_A(M)$ of large or extra-large size. Its relations are divided into two natural families:

$$\mathcal{E} = \{m_{ij} \mid m_{ij} = 2k_{ij} < \infty,\ i < j\}, \quad \mathcal{O} = \{m_{ij} \mid m_{ij} = 2k_{ij} + 1 < \infty,\ i < j\}.$$

We denote $r = |\mathcal{E}| + |\mathcal{O}|$ the total number of relations, and also

$$\mu = \prod_{m_{ij} \in \mathcal{E}} m_{ij} \prod_{m_{ij} \in \mathcal{O}} (m_{ij} - 1). \tag{3.6}$$

Corollary 3.12. *Let $G_A(M)$ be an Artin group of large or extra-large size:*

$$G_A(M) = \langle a_1, \dots, a_n \mid < a_i a_j >^{m_{ij}} = < a_j a_i >^{m_{ij}},\ i \neq j,\ i, j \in \{1, \dots, n\} \rangle.$$

Then

$$\mathrm{KW}(G_A(M)) \leq 8\Big(\sum_{m_{ij} \in \mathcal{E}} \log_2 m_{ij} + \sum_{m_{ij} \in \mathcal{O}} \log_2 (m_{ij} - 1) \Big) + 2n - r + 1.$$

Note that this directly implies $\mathrm{KW}(G_A(M)) \leq 8 \log_2 \mu + 2n - r + 1$.

Proof. Let K_{ij} be a 2-simplicial complex associated to the relations

$$< a_i a_j >^{m_{ij}} = < a_j a_i >^{m_{ij}}, \quad m_{ij} \in \mathcal{E},$$

and constructed in the proof of Proposition 3.9. We consider $K_1 = \bigcup_{m_{ij} \in \mathcal{E}} K_{ij}$ and apply Corollary 3.11, where $w_{ij} = a_i a_j$, $v_{ij} = a_j a_i$, $|w_{ij}| = l_{ij} = 2$ and $|v_{ij}| = l'_{ij} = 2$ and $k_{ij} = m_{ij}/2$, so

$$s_0(K_1) \leq \Big[\sum_{m_{ij} \in \mathcal{E}} \Big(8 \log_2 k_{ij} + \frac{3}{2}(l_{ij} + l'_{ij}) \Big) \Big] + 2n + |\mathcal{E}| + 1.$$

Thus we obtain the following upper estimate

$$s_0(K_1) = 8\Big(\sum_{m_{ij}} \log_2 m_{ij} \Big) + 2n - |\mathcal{E}| + 1. \tag{3.7}$$

We proceed now with the relations

$$< a_i a_j >^{m_{ij}} = < a_j a_i >^{m_{ij}}, \quad m_{ij} \in \mathcal{O}.$$

For a given $m_{ij} \in \mathcal{O}$ where $m_{ij} = 2k_{ij} + 1$ and $k_{ij} \in \mathbb{N} \setminus \{0\}$, we have

$$< a_i a_j >^{m_{ij}} = (a_i a_j)^{k_{ij}} a_i \quad \text{and} \quad < a_j a_i >^{m_{ij}} = (a_j a_i)^{k_{ij}} a_j.$$

Taking $w_{ij} = a_i a_j$, $v_{ij} = a_j a_i$, where $|w_{ij}| = l_{ij} = 2$, $|v_{ij}| = l'_{ij} = 2$ we build, as in the proof of Proposition 3.9, two complexes

$$K_{w_{ij}} = \Big(P'_{w_{ij}} \bigcup_{\gamma_0(m_{ij})} \mathcal{J}_{k'_{ij}} \Big) \bigcup_{(\xi(k_{ij}) \vee S^1_{a_i}) \sim \partial D^2_1} D^2_1,$$

$$K_{v_{ij}} = \Big(P'_{v_{ij}} \bigcup_{\gamma'_0(m_{ij})} \mathcal{L}_{k'_{ij}} \Big) \bigcup_{(\xi'(k_{ij}) \vee S^1_{a_j}) \sim \partial D^2_2} D^2_2,$$

where $\mathcal{J}_{k'_{ij}}$ and $\mathcal{L}_{k'_{ij}}$ are two telescopic constructions, both of length k'_{ij}, such that k'_{ij} is the smallest integer that satisfies $k_{ij} \le 2^{k'_{ij}+1}$. These two complexes are similar to K_w and K_v from the proof of Proposition 3.9 with only slight modification in the second gluing which takes into account generators a_i and a_j, respectively.

Like in the proof of Proposition 3.9, we deduce by the straightforward calculation

$$\begin{aligned} s_0(K_{w_{ij}}) \le s_0(P'_{w_{ij}}) + s_0(\mathcal{J}_{k'_{ij}}) - s_0(\gamma_0(m_{ij})) \\ + s_0(D_1^2) - s_0\big((\xi(k_{m_{ij}}) \vee S^1_{a_i}) \sim \partial D_1^2\big). \end{aligned}$$

So,

$$s_0(K_{w_{ij}}) \le 4\log_2 k_{ij} + 2n + \frac{3}{2} l_{ij} + \frac{7}{2}, \tag{3.8}$$

and similarly

$$s_0(K_{v_{ij}}) \le 4\log_2 k_{ij} + 2n + \frac{3}{2} l'_{ij} + \frac{7}{2}. \tag{3.9}$$

Then $K_{ij} = K_{w_{ij}} \cup_S K_{v_{ij}}$, where $S = \bigvee_{s=1}^n S^1_{a_s}$.

From the upper bounds (3.8) and (3.9), we obtain

$$\begin{aligned} s_0(K_{ij}) &\le s_0(K_{w_{ij}}) + s_0(K_{v_{ij}}) - (5 + 2n) \\ &\le 8\log_2 k_{ij} + 2n + \frac{3}{2}(l_{ij} + l'_{ij}) + 2 \\ &= 8\log_2(m_{ij} - 1) + 2n. \end{aligned}$$

Now, let $K_2 = \bigcup_{m_{ij} \in \mathcal{O}} K_{ij}$ be the 2-simplicial complex analogous to K_1, but corresponding to relations of $\mathcal{O}$. The last upper bound implies

$$s_0(K_2) \le \sum_{m_{ij} \in \mathcal{O}} s_0(K_{ij}) - \big(|\mathcal{O}| - 1\big)(2n + 1),$$

so

$$s_0(K_2) \le 8\Bigg(\sum_{m_{ij} \in \mathcal{O}} \log_2(m_{ij} - 1)\Bigg) + 2n - |\mathcal{O}| + 1. \tag{3.10}$$

Finally, we denote by K_A the simplicial complex obtained by the union of K_1 and K_2 along the wedge sum S. It satisfies $\pi_1(K_A) = G_A(M)$. The equations (3.7) and (3.10) imply

$$s_0(K_A) \le s_0(K_1) + s_0(K_2) - (2n + 1),$$

then

$$s_0(K_A) \le 8\Bigg(\sum_{m_{ij} \in \mathcal{E}} \log_2 m_{ij} + \sum_{m_{ij} \in \mathcal{O}} \log_2(m_{ij} - 1)\Bigg) + 2n - \big(|\mathcal{E}| + |\mathcal{O}|\big) + 1.$$

This completes the proof. ∎

For the following result, we use the same notations as in Corollary 3.12.

Corollary 3.13. *Let $G_C(M)$ be a Coxeter group of large or extra-large size. As*

$$G_C(M) = \langle a_1, \ldots, a_n \mid a_i^2 = e, < a_i a_j >^{m_{ij}} = < a_j a_i >^{m_{ji}}, i \neq j, i, j \in \{1, \ldots, n\} \rangle,$$

then

$$\mathrm{KW}(G_C(M)) \leq 8 \log_2 \mu + 5n - r + 1.$$

In other words

$$\mathrm{KW}(G_C(M)) \leq \mathrm{KW}(G_A(M)) + 3n.$$

Proof. Same as the proof of Corollary 3.12, except that here the 2-simplicial complex K_C, which satisfies $\pi_1(K) = G_C(M)$, contains n real projective planes associated to the relations $a_i^2 = e$ for all $i \in \{1, \ldots, n\}$. Each of $\mathbb{R}P_i^2$ lies on the circle S_i^1 given by

$$[p_1(i) p_4(i)] \cup [p_4(i) p_5(i)] \cup [p_5(i) p_1(i)],$$

cf. Figure 2. Let K_A be the 2-simplicial complex constructed in the proof of Corollary 3.12, then

$$s_0(K_C) = s_0(K_A) + 3n,$$

and this concludes the proof. ■

Remark 3.14. Let $G(n, \mu)$ be an Artin or Coxeter group with fixed number n of generators and μ be the total weight of the relations; see (3.6). It is evident that $\mathrm{KW}(G(n, \mu))$ is not a bounded function of μ. An interesting and relevant problem is to find some explicit lower bound of $\mathrm{KW}(G(n, \mu))$ as a function of μ. This problem remains open even for so-called 2-dimensional Artin groups. This latter means that the corresponding classifying space $K(G(n, \mu), 1)$ can be chosen as a 2-dimensional cellular complex. Note that 2-dimensional Artin groups provide an interesting geometric examples. Right-angled 2-dimensional Artin groups of positive volume entropy was recently founded in [8]. It seems likely some effective lower bound of $\mathrm{KW}(G(n, \mu))$ as a function of μ, can eventually provide an effective lower bound of the volume entropy for such groups; see also Remark 1.6.

3.2. The cyclic and abelian groups. In this section we obtain several effective lower and upper bounds for KW-complexity in the case of finite Abelian groups.

Corollary 3.15. *Let $\mathbb{Z}_m = \langle a \mid a^m = e \rangle$ be a cyclic group of order m, so*

$$\sqrt[3]{12 \log_3 m} \leq \mathrm{KW}(\mathbb{Z}_m) \leq 4 \log_2 m + 4.$$

Proof. We consider the upper and lower bounds separately.

(1) For the upper bound, we take the first part of the proof of Proposition 3.9, except that here the word $w = a$ is of length one, i.e., $|w| = 1$, so no polygon is associated and the telescopic construction begins directly from the copies of Möbius bands $\{M_i\}_{i\in\mathbb{N}}$. We get a 2-simplicial complex $X_m = \mathcal{J}_k \cup_{\xi(m)} \{D^2/\sim\}$ satisfying $\pi_1(K) = \mathbb{Z}_m$, and

$$s_0(X_m) = s_0(\mathcal{J}_k) + s_0(D^2/\sim) - s_0(\xi(m)),$$

then $\mathrm{KW}(\mathbb{Z}_m) \leq 4\log_2 m + 4$.

(2) For the lower bound, we take $\mathrm{KW}(\mathbb{Z}_m) = s_0$. Applying Proposition 2.1 and according to [3, Theorem 4.1], we have

$$2\log_3 m \leq \kappa(G) \leq \frac{s_0^3}{6},$$

then

$$s_0 \geq (12\log_3 m)^{1/3},$$

hence the conclusion. ■

Remark 3.16. Corollary 3.15 implies that $3 \leq \mathrm{KW}(\mathbb{Z}_4) \leq 12$. By using the direct disk attaching construction for $\mathbb{Z}_4$, we obtain $\mathrm{KW}(\mathbb{Z}_4) \leq 11$. For the case of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, Corollary 3.8 implies $4 \leq \mathrm{KW}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \leq 13$. We do not know the exact values for either of $\mathrm{KW}(\mathbb{Z}_4)$ or $\mathrm{KW}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, and we also do not know if they coincide.

Corollary 3.17. *Let G be a finite abelian group decomposed into the direct sum of its invariant factors,*

$$G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s},$$

such that $n_i | n_{i+1}$, $i = \{1, \ldots, s-1\}$. Then

$$\big(12\log_3 \mid G \mid\big)^{1/3} \leq \mathrm{KW}(G) \leq \Big(\frac{4}{s}\log_2 \mid G \mid + 4\Big)^s.$$

Proof. For the lower bound, the reasoning is the same as in the proof of Corollary 3.15. The number of elements of G satisfies $|G| = n_1 n_2 \ldots n_s$, so

$$\mathrm{KW}(G) \geq \big(12\log_3 |G|\big)^{1/3}.$$

On the other hand, we have

$$\mathrm{KW}(G) \leq \prod_{i=1}^{s} \mathrm{KW}(\mathbb{Z}_{n_i}) \leq \prod_{i=1}^{s}(4\log_2 n_i + 4).$$

By using the inequality between the arithmetic and the geometric means, we obtain

$$\mathrm{KW}(G) \leq \Big(\frac{1}{s}\sum_{i=1}^{s}(4\log_2 n_i + 4)\Big)^s \leq \Big(\frac{4}{s}\log_2 \mid G \mid + 4\Big)^s. \qquad ■$$

4. Complexity and geometric invariants for groups

Throughout this part, we shall give the proofs of two main theorems announced in the introduction, Theorems 1.4 and 1.5

4.1. Proof of Theorem 1.4. Consider first inequality, let $k = \mathrm{KW}(G)$ and X be a 2-simplicial complex verifying $\pi_1(X) = G$ and $s_0(X) = k$. Assume that the number of 2-simplexes in X is minimal possible under considered conditions. The last means that there is no edge which is adjacent to only one 2-simplex. Endow X with the special metric h such that:

(1) the length of each edge is equal to 1;

(2) each 2-simplex is a spherical equilateral triangle of angle $\pi/2$ and of spherical radius $2/\pi$.

This means that each edge of triangle is equal to 1 and its area $S = 2/\pi$.

First, we prove that $\mathrm{sys}(X, h) \geq 3$. Let $\gamma(t)$ be a systolic geodesic, so it is a simple non-contractible closed curve of length $\mathrm{sys}(X, h)$. Let us suppose that $\gamma(t)$ does not meet vertices of X. Because $\gamma(t)$ is a locally minimizing geodesic it is a broken spherical line, this means that $\gamma(t)$ intersects any 2-simplex by some spherical geodesic arc (of radius $2/\pi$).

Let $\Delta = \Delta(p_1, p_2, p_3)$ be a 2-simplex of vertices p_1, p_2, p_3 meeting $\gamma(t)$. Suppose $a \in \,]p_1, p_2[$ and $b \in \,]p_1, p_3[$ are incoming and outgoing points, respectively. This means that before Δ, $\gamma(t)$ passes through some 2-simplex $\Delta(p_0, p_1, p_2)$ and after Δ, it passes through some 2-simplex $\Delta(p_1, p_3, p_4)$. The union of these three simplices

$$\mathbf{U} = \Delta(p_0, p_1, p_2) \cup \Delta(p_1, p_2, p_3) \cup \Delta(p_1, p_3, p_4)$$

forms a part of the hemisphere of radius $2/\pi$ and the concatenation of edges

$$\mathbf{u} = [p_0, p_2] \cup [p_2, p_3] \cup [p_3, p_4]$$

forms a part of the angle $3\pi/2$ lying on the equatorial circle. So the part $\gamma_{\mathbf{U}}(t) = \gamma(t) \cap \mathbf{U}$ is one half of some great circle and its length is equal to 2. Let $\gamma_{\mathbf{U}}(t)$ meet the arc $\mathbf{u}$ in two points $a' \in \,]p_0, p_2[$ and $b' \in \,]p_3, p_4[$.

The arc $\gamma_{\mathbf{U}}(t)$ is deformable by rotation in $\mathbf{U}$ around a' and b' on the new arc

$$\gamma_1 = [a', p_2] \cup [p_2, p_3] \cup [p_3, b'] \subset \mathbf{U}.$$

By replacing the arc $\gamma_{\mathbf{U}}$ by the arc γ_1 in γ, we obtain a new closed curve $\widetilde{\gamma}$ which is homotopically equivalent to γ and of the same length so it is a new systolic geodesic. But this new broken curve is not locally minimal in the neighborhoods of a' and b'. Then γ is not a systolic geodesic. This contradiction shows that γ has to pass through the vertices.

Lemma 4.1. *Let $\gamma(t)$ be a geodesic curve in (X,h) which passes through a vertex $q=\gamma(t_0)\in X$. Assume that $\gamma(t)$, $t_0<t<t_0+\varepsilon$ does not touch any edge, then $q_2=\gamma(t_0+2)$ is a new vertex and the curve $\gamma(t)$, $t_0\le t\le t_0+2$ is deformable* rel$\{q,q_2\}$ *into a concatenation of two edges and this deformation does not change the length of the curves.*

Proof. Let $\Delta(q,u,v)$ be a triangle which contains the curve $\gamma(t)$ after it has passed through the vertex q. Then $\gamma(t)\cap\Delta(q,u,v)$ is a spherical geodesic arc connecting q and $q_1\in[u,v]$ and the angle of intersection $\gamma(t)\cap[u,v]$ is $\pi/2$. Passing through q_1, $\gamma(t)$ passes in a new triangle $\Delta(u,v,q_2)$. As $\gamma(t)$ is minimal, this implies that $\gamma(t)\cap\Delta(u,v,q_2)$ is a spherical geodesic arc orthogonal to $[u,v]$, then it passes through the vertex q_2. This leads to

$$\gamma_2(t)=\gamma(t)\cap\big(\Delta(q,u,v)\cup\Delta(u,v,q_2)\big),$$

which is a semi-circle of large radius on the sphere of radius $2/\pi$ whose vertices q and q_2 are opposite with respect to the edge $[u,v]$. By rotation around q and q_2, we can, by homotopy, bring the part $\gamma_2(t)$ to the concatenation of two edges, for example, $[q,u]\cup[u,q_2]$. This deformation does not change neither the homotopy class nor the length. In particular, $\mathrm{length}_h(\gamma_2(t))=2$. ∎

By iterating, if necessary, the process of Lemma 4.1, we will see that each closed systolic geodesic can be deformed into a new systolic geodesic that passes only through the edges. It is obvious that a closed non-contractible curve which passes only by edges cannot have a length less than 3. Then $\mathrm{sys}(X,h)\ge 3$.

To complete this first part of the proof, we give the upper bound of the area of (X,h). We have $\mathrm{vol}(X,h)=s_2(X)2/\pi$, so arguing as in the proof of Proposition 2.1, we obtain

$$s_2(X)\le C_k^3\le\frac{k^3}{6}, \tag{4.1}$$

this implies the result.

We consider now the second inequality of Theorem 1.4 giving the lower bound of the systolic area. Choose $0<\varepsilon<1/12$, by [27, Theorem 3.5 and Lemma 4.2], there exists an ε-optimal finite Riemannian 2-complex (Y,h') such that:

(1) $\pi_1(Y)=G$;

(2) $\mathrm{sys}(Y,h')=1$;

(3) $\sigma(Y)<\sigma(G)+\varepsilon$;

(4) for all $\varepsilon < R < 1/2$ and for any $y \in Y$, the ball $B(y, R)$ of radius R centered at y, satisfies

$$|B(y,R)| \geq \frac{1}{4}R^2.$$

Here, $|B(y,R)|$ means the h'-area of the ball. Fix $\varepsilon < R < 1/12$ and consider a maximal filling of Y by closed disjoint balls of radii R and denote by $\{y_i\}_{i=1}^k$ the centers of these balls. By construction, it is obvious that

$$\frac{k}{4}R^2 \leq \mathrm{vol}(Y, h') < \sigma(G) + \varepsilon. \tag{4.2}$$

The balls of radius $2R$ and of the same centers $\{y_i\}_{i=1}^k$ form a cover

$$\mathcal{U} = \{B(y_i, 2R)\}_{i=1}^k$$

of Y.

Consider the nerve $\mathcal{N}(\mathcal{U})$ of this cover and let N be its 2-skeleton. It is a 2-simplicial complex with k vertices. If $R < 1/12$, then its fundamental group is isomorphic to G; see, for example, [3, Lemma 3.2]. From (4.2), this implies that

$$\mathrm{KW}(G) \leq k < \frac{4(\sigma(G)+\varepsilon)}{R^2}.$$

As ε tends to zero and R goes to $1/12$, we deduce $\mathrm{KW}(G) \leq 576\sigma(G)$, and hence the result. ∎

4.2. Proof of Theorem 1.5. Let $\mathrm{KW}(G) = k$ and X be an optimal simplicial complex such that $\pi_1(X) = G$ and $s_0(X) = k$. As G is not free X is obviously 2-dimensional. Endow X with a special metric h given in the proof of Theorem 1.4. Recall that each edge of X is of h-length equal to 1, and the h-area of each 2-simplex of X is $S = 2/\pi$.

Let $\widehat{X}$ be the universal cover of X and $\widehat{h}$ is the lift of h. Choose a vertex $q \in \widehat{X}$ and consider $\widehat{B}_q(R)$ the $\widehat{h}$-geodesical ball of radius R and centered at q. It is obvious that

$$\mathrm{vol}\big(\widehat{B}_q(R); \widehat{h}\big) \leq s_0\big(\widehat{B}_q(R)\big)\,\mathrm{vol}(X; h),$$

where $s_0(\widehat{B}_q(R))$ means the number of vertices of $\widehat{X}$ in the ball. By applying the definition of minimal volume entropy (1.2), we get

$$\mathrm{ent}(X,h) \leq \lim_{R\to\infty} \frac{\log\big(s_0(\widehat{B}_q(R))\big)}{R}. \tag{4.3}$$

Let r be a positive integer and $x \in \widehat{X}$ be a point such that $\mathrm{dist}_{\widehat{h}}(q,x) = r$. Let $\gamma(t)$, $0 \leq t \leq r$ be a minimal geodesic curve joining q and x. The local geometry on $(\widehat{X}, \widehat{h})$

is the same as that on (X, h). Applying systematically Lemma 4.1 to the geodesic curve $\gamma(t)$, $0 \leq t \leq r$, we get that it is deformable into a geodesic curve $\gamma_1(t)$ of the same length which joins q and x and passes through the 1-skeleton of $\widehat{X}$ for at least $0 \leq t \leq r-1$. This implies that the number of vertices in $\widehat{B}_q(r)$ is the same that in the ball of radius r in the graph $\mathrm{Sk}^1(\widehat{X})$. Each vertex of this graph is of valence at most $k-1$, so the number of vertices in $\widehat{B}_q(r)$ is bounded by

$$s_0(\widehat{B}_q(r)) \leq 2\frac{(k-1)^{r+1}}{k-2}.$$

Using (4.3), we deduce that

$$\mathrm{ent}(X, h) \leq \log(k-1). \tag{4.4}$$

The area of (X, h) is $2/\pi s_2(X)$ and with (4.1), we get $\mathrm{vol}(X, h) \leq k^3/9$. This upper bound with (4.4) gives us

$$\omega(G) \leq \omega(X, h) = \mathrm{ent}(X, h)(\mathrm{vol}(X, h))^{1/2} \leq \frac{1}{3}\log(k-1)k^{3/2}.$$

This ends the proof. ■

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