Elemente der Mathematik (Online first)
DOI 10.4171/EM/533

Published by EMS Press

Short note

A generalization of Boole's formula derived from a system of linear equations

Haoran Zhu

Abstract. We analyze a system of linear algebraic equations whose solutions lead to a proof of a generalization of Boole's formula. In particular, our approach provides an elementary and short alternative to Katsuura's proof of this generalization.

1 Introduction

Due to their numerous relations, binomial coefficients play an important role in various mathematical fields, including enumerative combinatorics, statistics and number theory. In Boole's classical book "Calculus of Finite Differences" [5], the following beautiful formula is given, which holds for $1 \leq m \leq n \in \mathbb{N}$:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m = \begin{cases} n! & \text{if } m = n, \\ 0 & \text{if } m < n. \end{cases} \tag{1}$$

It is known that the formula is related to Stirling's partition numbers $S(m,n)$ (see, e.g., [7]), which is given by the following equation:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m = n! \cdot S(m,n) \quad (m,n \in \mathbb{N}).$$

The relations between the formulas have implications for the study of the degrees of normed null-polynomials and the derivation of inequalities associated with the Smarandache function (see [10, 11]).

The enduring interest in Boole's formula has led to a variety of proof techniques being developed over the years. Gould [6] discussed its properties and called it *Euler's formula*. In 2005, Anglani and Barile [2] introduced two proofs via methods from real analysis and combinatorics. Subsequently, Phoata [9] and Katsuura [8] provided new proofs and gave a generalization of Boole's formula. In this note, we give a short and elementary proof of this formula which is based on a system of linear equations. More recently, Alzey and Chapman [1] have presented a novel proof, while Batır and Atpınar [3, 4] have independently developed two entirely new approaches to validating Boole's formula.

2 Solutions of linear algebraic equations

Let us consider the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a & a+b & \cdots & a+nb \\ a^2 & (a+b)^2 & \cdots & (a+nb)^2 \\ \vdots & \vdots & \ddots & \vdots \\ a^n & (a+b)^n & \cdots & (a+nb)^n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b^n \cdot n! \end{bmatrix}. \tag{2}$$

Here a, b can be any real numbers and the coefficient matrix V is a Vandermonde matrix.

To solve this system, we first calculate the determinant of V. Since V is a Vandermonde matrix, its determinant can be computed as follows:

$$\det(V) = \prod_{0 \le j < i \le n} \big((a+ib) - (a+jb)\big) = n! \cdot (n-1)! \cdots 1! \cdot b^{\frac{n(n+1)}{2}}.$$

We then proceed to define the matrix V_k as

$$V_k = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ a & a+b & \cdots & a+(k-1)b & 0 & a+(k+1)b & \cdots & a+nb \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^n & (a+b)^n & \cdots & (a+(k-1)b)^n & b^n \cdot n! & (a+(k+1)b)^n & \cdots & (a+nb)^n \end{bmatrix},$$

and we denote the matrix obtained by removing the $(n+1)$-th row and $(k+1)$-th column from V_k as V_k'. The determinant of V_k is computed by applying Laplace's expansion along the $(k+1)$-th column, which yields the determinant of the submatrix V_k'. This submatrix is also a Vandermonde matrix, and its determinant can be computed using the way previously showed:

$$\begin{aligned} \det(V_k) &= (-1)^{n-k} \cdot b^n \cdot n! \cdot \det(V_k') \\ &= (-1)^{n-k} \cdot b^{\frac{n(n+1)}{2}} \cdot n! \cdot \frac{n! \cdots (k+1)! \cdot (k-1)! \cdots 1!}{(n-k)!}. \end{aligned}$$

Then, by Cramer's rule, we obtain the following explicit expression for x_k:

$$x_k = \frac{\det(V_k)}{\det(V)} = \frac{(-1)^{n-k} \cdot n!}{k! \cdot (n-k)!} = (-1)^{n-k} \binom{n}{k}.$$

Reading the equations in (2) one by one, we obtain the following result.

Theorem (Generalization of Boole's formula). *For any real numbers a and b, and for $1 \le m \le n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+bk)^m = \begin{cases} 0 & \text{if } m < n, \\ (-1)^n \cdot b^n \cdot n! & \text{if } m = n. \end{cases}$$

For the special case $a = 0$, $b = 1$, the result in the theorem implies Boole's formula (1). Perhaps, similar approaches from linear algebra can also be used to generalize other combinatorial identities.

Acknowledgments. The author thanks the referees for their constructive feedback, which enhanced the paper's organization.

References

[1] H. Alzer and R. Chapman, On Boole's formula for factorials. *Australas. J. Combin.* **59** (2014), 333–336 Zbl 1296.05013 MR 3245408

[2] R. Anglani and M. Barile, Two very short proofs of a combinatorial identity. *Integers* **5** (2005), no. 1, A18, 3 MR 2192237

[3] N. Batir, On some combinatorial identities and harmonic sums. *Int. J. Number Theory* **13** (2017), no. 7, 1695–1709 Zbl 1074.74619 MR 3667490

[4] N. Batır and S. Atpınar, A new proof of Boole's additive combinatorics formula. *Discrete Math. Lett.* **10** (2022), 112–114 Zbl 1513.05012 MR 4480516

[5] G. Boole, *Calculus of finite differences*. Chelsea Publishing Co., New York, 1957 Zbl 0084.07701 MR 115025

[6] H. W. Gould, Euler's formula for nth differences of powers. *Amer. Math. Monthly* **85** (1978), no. 6, 450–467 Zbl 0397.10055 MR 480057

[7] L. Halbeisen, N. Hungerbühler, and H. Läuchli, Powers and polynomials in Z_m. *Elem. Math.* **54** (1999), no. 3, 118–129 Zbl 1007.11002 MR 1713222

[8] H. Katsuura, Summations involving binomial coefficients. *College Math. J.* **40** (2009), no. 4, 275–278 MR 2548966

[9] C. Pohoata, Boole's formula as a consequence of Lagrange's interpolating polynomial theorem. *Integers* **8** (2008), A23, 2 MR 2425621

[10] E. Specker, N. Hungerbühler, and M. Wasem, The ring of polyfunctions over $\mathbb{Z}/n\mathbb{Z}$. *Comm. Algebra* **51** (2023), no. 1, 116–134 Zbl 1502.13061 MR 4525284

[11] E. Specker, N. Hungerbühler, and M. Wasem, Polyfunctions over commutative rings. *J. Algebra Appl.* **23** (2024), no. 1, Paper No. 2450014, 10 MR 4688780

Haoran Zhu
College of Sciences
Northeastern University
Liaoning 110004 Shenyang, P. R. China
whrzhu@outlook.com