
Short note **On an Erdős inscribed triangle inequality revisited**

Cezar Lupu and Ștefan Spătaru

Abstract. In this note, we give a refinement of an inequality of Torrejon between the area of a triangle and that of an inscribed triangle. Our approach is based on using complex numbers and some elementary facts on geometric inequalities.

Dedicated to the memory of Octavian Ganea, a big enthusiast of complex numbers in geometry

1 Introduction and statement of the main result

Let us consider a triangle ABC . On each of the sides BC , CA and AB , fix arbitrary points A_1 , B_1 and C_1 , respectively. As pointed out in [7, 9], the Erdős–Debrunner inequality

$$\min\{\text{area}(AC_1B_1), \text{area}(C_1BA_1), \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1) \quad (1)$$

is a topic with a long history. Later, Janous [5] generalized inequality (1) by proving

$$\mathcal{M}_{-1}\{\text{area}(AC_1B_1), \text{area}(C_1BA_1), \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1),$$

where $\mathcal{M}_{-1}$ denotes the harmonic mean of three positive numbers. Moreover, Janous formulated a more general question which was extended and solved by Mascioni [7, 8]. Using a different method, Frenzen, Ionașcu and Stănică [4] proved Janous’ conjecture independently of Mascioni. Yet a different method was used by Blatter in [2] for his proof of Janous’ conjecture.

The purpose of this note is to extend the result of Torrejon [9] regarding the areas of triangles $A_1B_1C_1$ and ABC when the points A_1 , B_1 , C_1 satisfy a certain metric property.

Our main result is given by the following.

Theorem 1.1. *Let ABC be a triangle and let A_1 , B_1 and C_1 be on BC , CA and AB , respectively, with none of A_1 , B_1 and C_1 coinciding with a vertex of ABC . If*

$$\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,$$

then

$$\begin{aligned} & \text{area}(A_1 B_1 C_1) \\ & \leq \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)} \left(\text{area}(ABC) + s^4 \left(\frac{\alpha-1}{\alpha+1} \right)^2 \text{area}(ABC)^{-1} \right), \end{aligned}$$

where s is the semi-perimeter of triangle ABC .

For $\alpha = 1$, we obtain ([6])

$$\frac{\text{area}(A_1 B_1 C_1)}{\text{area}(ABC)} \leq \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)}.$$

As the arithmetic-geometric mean inequality shows $(a+b+c)(a^2+b^2+c^2) \geq 9abc$, Theorem 1.1 implies the result ([9])

$$4 \text{ area}(A_1 B_1 C_1) \leq \text{area}(ABC) + s^4 \left(\frac{\alpha-1}{\alpha+1} \right)^2 \text{area}(ABC)^{-1}.$$

For $\alpha = 1$, this inequality reduces to ([1])

$$4 \text{ area}(A_1 B_1 C_1) \leq \text{area}(ABC).$$

Our approach in computing the area of the triangle $A_1 B_1 C_1$ will be different from the one used by Torrejon in [9]. It is based on complex numbers in the plane.

Recall that the area of a positively oriented triangle ABC whose vertices have affixes z_A, z_B and z_C is given by

$$\text{area}(ABC) = \frac{1}{2} \text{Im}(\overline{z_A} z_B + \overline{z_B} z_C + \overline{z_C} z_A).$$

2 Proof of Theorem 1.1

First of all, we prove the following equality:

$$\text{area}(A_1 B_1 C_1) = 2F \cdot \frac{(s-a)(s-b)(s-c) + s^3 \cdot \left(\frac{\alpha-1}{\alpha+1} \right)^2}{abc},$$

where a, b, c denote the side lengths of triangle ABC and F its area. Let furthermore z_P be the (complex) affix of an arbitrary point P .

Firstly, $2s = a + b + c = (AB + AB_1) + (BC + CB_1) = (\alpha + 1)(c + AB_1)$, and consequently, $AB_1 = \frac{2s}{\alpha+1} - c$ and

$$CB_1 = CA - AB_1 = b - \frac{2s}{\alpha+1} + c = 2s - a - \frac{2s}{\alpha+1} = \frac{2s\alpha}{\alpha+1} - a.$$

Analogously, we have

$$BC_1 = \frac{2s}{\alpha+1} - a, \quad CA_1 = \frac{2s}{\alpha+1} - b, \quad BA_1 = \frac{2s\alpha}{\alpha+1} - c \quad \text{and} \quad AC_1 = \frac{2s\alpha}{\alpha+1} - b.$$

Therefore, the affixes of A_1 , B_1 , and C_1 are given by

$$\begin{aligned} z_{A_1} &= \frac{\left(\frac{2s}{\alpha+1} - b\right)z_B + \left(\frac{2s\alpha}{\alpha+1} - c\right)z_C}{a}, \\ z_{B_1} &= \frac{\left(\frac{2s}{\alpha+1} - c\right)z_C + \left(\frac{2s\alpha}{\alpha+1} - a\right)z_A}{b}, \\ z_{C_1} &= \frac{\left(\frac{2s}{\alpha+1} - a\right)z_A + \left(\frac{2s\alpha}{\alpha+1} - b\right)z_B}{c}, \end{aligned}$$

respectively. Before computing the area of triangle $A_1B_1C_1$, we note that the expressions $z_P \overline{z_P}$ and $z_P \overline{z_Q} + \overline{z_P} z_Q$ are real numbers for arbitrary points P and Q . Therefore,

$$\begin{aligned} 2 \text{area}(A_1B_1C_1) &= \text{Im}\left(\sum_{\text{cyc}} \overline{z_{A_1}} z_{B_1}\right) \\ &= \text{Im}\left(\sum_{\text{cyc}} \frac{\left(\frac{2s}{\alpha+1} - b\right)\overline{z_B} + \left(\frac{2s\alpha}{\alpha+1} - c\right)\overline{z_C}}{a} \cdot \frac{\left(\frac{2s}{\alpha+1} - c\right)z_C + \left(\frac{2s\alpha}{\alpha+1} - a\right)z_A}{b}\right) \\ &= \frac{1}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_B} z_C \left[c\left(\frac{2s}{\alpha+1} - b\right)\left(\frac{2s}{\alpha+1} - c\right) \right. \right. \\ &\quad \left. \left. + b\left(\frac{2s\alpha}{\alpha+1} - c\right)\left(\frac{2s\alpha}{\alpha+1} - b\right) \right] \right) \\ &\quad + \frac{1}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_C} z_B a\left(\frac{2s\alpha}{\alpha+1} - b\right)\left(\frac{2s}{\alpha+1} - c\right)\right) \\ &= \frac{1}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_B} z_C \left[c\left(\frac{2s}{\alpha+1} - b\right)\left(\frac{2s}{\alpha+1} - c\right) \right. \right. \\ &\quad \left. \left. + b\left(\frac{2s\alpha}{\alpha+1} - c\right)\left(\frac{2s\alpha}{\alpha+1} - b\right) \right. \right. \\ &\quad \left. \left. - a\left(\frac{2s\alpha}{\alpha+1} - b\right)\left(\frac{2s}{\alpha+1} - c\right) \right] \right) \\ &= \frac{1}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_B} z_C \left[b\left(\frac{s(1-\alpha)}{\alpha} + s - b\right)\left(\frac{s(1-\alpha)}{\alpha} + s - c\right) \right. \right. \\ &\quad \left. \left. + c\left(\frac{s(\alpha-1)}{\alpha} + s - b\right)\left(\frac{s(\alpha-1)}{\alpha} + s - c\right) \right. \right. \\ &\quad \left. \left. - a\left(\frac{s(\alpha-1)}{\alpha} + s - b\right)\left(\frac{s(1-\alpha)}{\alpha} + s - c\right) \right] \right) \\ &= \frac{1}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_B} z_C \left[2(s-a)(s-b)(s-c) \right. \right. \\ &\quad \left. \left. + s \cdot \left(\frac{1-\alpha}{1+\alpha}\right)(ab-ac+ac-ab) \right. \right. \\ &\quad \left. \left. + s^2 \cdot \left(\frac{\alpha-1}{\alpha+1}\right)^2 (a+b+c) \right] \right). \end{aligned}$$

Now it can be checked without much difficulty that the coefficient of $\overline{z_B} z_C$ equals

$$2(s-a)(s-b)(s-c) + 2s^3 \cdot \left(\frac{\alpha-1}{\alpha+1}\right)^2.$$

This results in

$$\text{area}(A_1 B_1 C_1) = \frac{(s-a)(s-b)(s-c) + s^3 \cdot \left(\frac{\alpha-1}{\alpha+1}\right)^2}{abc} \cdot \text{Im}\left(\sum_{\text{cyc}} \overline{z_B} z_C\right).$$

This and $\text{Im}(\sum_{\text{cyc}} \overline{z_B} z_C) = 2F$ readily lead to the stated formula for $\text{area}(A_1 B_1 C_1)$.
Therefore, we get

$$\frac{abc \cdot s}{2} \cdot \text{area}(A_1 B_1 C_1) = F \cdot \left(s(s-a)(s-b)(s-c) + s^4 \cdot \left(\frac{\alpha-1}{\alpha+1}\right)^2\right),$$

that is (via Heron's formula),

$$\frac{abc \cdot s}{2} \cdot \text{area}(A_1 B_1 C_1) = F^3 + s^4 \cdot \left(\frac{\alpha-1}{\alpha+1}\right)^2 \cdot F.$$

At this point, we are only left to prove the inequality

$$\frac{sabc}{2} \geq 4F^2 \cdot \frac{(a+b+c)(a^2+b^2+c^2)}{9abc}, \quad (2)$$

that is,

$$a^2 b^2 c^2 \geq 16F^2 \frac{a^2 + b^2 + c^2}{9}.$$

Using $abc = 4RF$, we get the equivalent inequality

$$9R^2 \geq a^2 + b^2 + c^2.$$

However, this inequality is evident since the distance between the circumcenter O and the centroid G is given by the Leibniz identity $OG^2 = 9R^2 - (a^2 + b^2 + c^2)$.

Finally, we conclude

$$F^3 + s^4 \left(\frac{\alpha-1}{\alpha+1}\right)^2 F \geq 4F^2 \cdot \frac{(a+b+c)(a^2+b^2+c^2)}{9abc} \text{area}(A_1 B_1 C_1),$$

which is equivalent to the inequality of our theorem. ■

Remark. It is worth noting that inequality (2) is equivalent to

$$F \leq \frac{3}{4} \cdot \frac{abc}{\sqrt{a^2 + b^2 + c^2}}$$

and thus improves the result ([3, Item 4.13])

$$F \leq \frac{3\sqrt{3}}{4} \cdot \frac{abc}{a+b+c}.$$

References

- [1] M. Aassila, Problem 1717. *Math. Mag.* **78** (2005), 158
- [2] C. Blatter, In Dreiecken einbeschriebene Dreiecke. *Elem. Math.* **63** (2008), no. 1, 18–22 Zbl [1176.51006](#) MR [2369033](#)
- [3] O. Bottema, R. Ž. Djordjevič, R. R. Janič, D. S. Mitrinović, and P. M. Vasic, Geometric Inequalities. Wolters-Noordhoff Publishing, Groningen, 1969 Zbl [0174.52401](#)
- [4] C. L. Frenzen, E. J. Ionascu, and P. Stănică, A proof of two conjectures related to the Erdős–Debrunner inequality. *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 3, Article 68, 13 Zbl [1198.51009](#) MR [2345923](#)
- [5] W. Janous, A short note on the Erdős–Debrunner inequality. *Elem. Math.* **61** (2006), no. 1, 32–35 Zbl [1135.51017](#) MR [2209146](#)
- [6] C. Lupu and T. Lupu, Problem 1857. *Math. Mag.* **83** (2010), 391
- [7] V. Mascioni, On the Erdős–Debrunner inequality. *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 32, 5 Zbl [1048.26014](#) MR [2320605](#)
- [8] V. Mascioni, An extension of the Erdős–Debrunner inequality to general power means. *JIPAM. J. Inequal. Pure Appl. Math.* **9** (2008), no. 3, Article 67, 11 Zbl [1163.26335](#) MR [2476646](#)
- [9] R. M. Torrejón, On an Erdős inscribed triangle inequality. *Forum Geom.* **5** (2005), 137–141 Zbl [1119.51018](#) MR [2195743](#)

Cezar Lupu

Beijing Institute of Mathematical Sciences and Applications (BIMSA)

101408 Beijing

Yau Mathematical Sciences Center (YMSC)

Tsinghua University

100084 Beijing, P. R. China

lupucezar@gmail.com, lupucezar@bimsa.cn

Ștefan Spătaru

Harvard University

Cambridge, MA-02138, USA

sspataru96@gmail.com