J. Comb. Algebra (Online first)
DOI 10.4171/JCA/90

Stable pseudo-quadratic modules

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Abstract. We give a necessary and sufficient condition for the existence of pseudo-quadratic Tits quadrangles in terms of a stability condition for pseudo-quadratic modules. We also give a sufficient condition for the existence of a natural projective embedding for these Tits quadrangles and indicate connections to results in unitary K-theory of Bak, Vaserstein and others.

1. Introduction

An *opposition relation* is simply a relation that is anti-reflexive and symmetric. An opposition relation $\equiv$ on a set Ω is *k-plump* for some $k \geq 2$ if $\Omega \neq \emptyset$ and for all subsets $S \subset \Omega$ of cardinality at most k, there exists $x \in \Omega$ such that $x \equiv y$ for all $y \in S$.

Tits polygons, which were introduced in [11], are certain bipartite graphs. Their key feature is that for each vertex v, the set of vertices adjacent to v is endowed with a 2-plump opposition relation. We call these relations, one for each vertex v, the *local opposition relations* of the Tits polygon. A Tits polygon is called k-plump for some $k \geq 3$ if all its local opposition relations are k-plump.

We call the opposition relation $\neq$ on a set Ω the *trivial* opposition relation. The trivial opposition relation on a set Ω is k-plump for some $k \geq 2$ if and only if $|\Omega| \geq k + 1$. A Tits polygon in which all the local opposition relations are trivial is the same thing as a Moufang polygon, i.e. an irreducible spherical building of rank 2 satisfying the Moufang condition. Moufang polygons were classified in [16].

Let Δ be an arbitrary irreducible spherical building of rank $r \geq 2$ (assumed to be Moufang when $r = 2$) with corresponding Coxeter system (W, S). Given two subsets J_1 and J_2 of S, the corresponding bipartite graph on the set of residues of type J_1 or J_2, under certain conditions on J_1 and J_2, has in a natural way the structure of a Tits polygon. We call the Tits polygons that arise in this way the Tits polygons *of index type*. Their local opposition relations are, in fact, not just 2-plump, but k-plump as long as every panel of Δ contains at least $k + 1$ chambers.

Mathematics Subject Classification 2020: 20E42 (primary); 51E12, 51E24 (secondary).
Keywords: building, Tits polygon, classical group, stable range.

Like Moufang polygons, Tits polygons have root groups, apartments and commutator relations. A Tits polygon X is called *sharp* if the action of the pointwise stabilizer in $\mathrm{Aut}(X)$ of an apartment on the corresponding root groups fulfills a certain condition. With very few exceptions, the Tits polygons of index type are sharp.

In [13], we completed the classification of Tits polygons satisfying suitable versions of the plump and sharp conditions. The Tits polygons classified in [13] were the pseudo-quadratic Tits quadrangles. These are the Tits polygons that can be parametrized by a pseudo-quadratic module. Pseudo-quadratic modules are essentially the same thing as the pseudo-quadratic forms introduced by Tits (see [14, Sec. 8.2]) in the classification of buildings of type C_n except that the underlying division ring is now replaced by an arbitrary unitary associative ring.

In [13], we did not address the existence question for the pseudo-quadratic Tits quadrangles. It is relatively easy to prove the existence of pseudo-quadratic Tits quadrangles when the underlying associative ring is division.For the existence of pseudo-quadratic Tits quadrangles defined over an arbitrary associative ring, however, quite different methods are required.

Our goal here is to give a necessary and sufficient condition for the existence of pseudo-quadratic Tits quadrangles. This is accomplished in Theorem 8.1, the main result of this paper, where we show that for a given standard pseudo-quadratic module Λ, there exists a corresponding pseudo-quadratic Tits quadrangle if and only if Λ is *strongly stable* as defined in Definition 5.9.

Strong stability is a natural extension of the notion of a ring of *stable range* 1 to pseudo-quadratic modules. The stable range of a ring was introduced by Bass in [2]. It is one of the central notions of algebraic K-theory. See [7] for an overview of the remarkable role of this notion in algebra. Stable range in the context of unitary groups was investigated by Bak, Vaserstein and others; see, for example, [1] and [8] as well as [5, Chaps. 5 and 9].

That there is a connection between stable range and geometry was discovered by Veldkamp. He showed that rings of stable range 1 provide the right setting for developing a theory of projective planes over rings (see [18] and [17]). This connection was investigated further by Faulkner in [4] and [3]. In [10], it was shown that the projective planes investigated by Veldkamp and Faulkner are, in fact, Tits triangles (and vice versa); see also [12]. Our investigations here should be considered as a natural extension of this work to Tits quadrangles.

In Section 9, we investigate some consequences of Theorem 8.1. Let Λ be a pseudo-quadratic module over a ring K. We denote by $E(\Lambda)$ the pseudo-quadratic module obtained by forming the direct sum with the hyperbolic plane over K. If K is a division ring, then Λ is automatically strongly stable and the corresponding Tits quadrangle can be described in terms of the 1- and 2-dimensional totally isotropic subspaces of $E^2(\Lambda) := E(E(\Lambda))$. Suppose that we assume only that Λ is strongly stable but not that K is division. We examine the question to what extent there is an analogous description of the corresponding Tits quadrangle in this more general setting. To do this, we need to replace the notion of a 1-dimensional totally isotropic subspace by the notion of a *hyperbolic* cyclic submodule as defined in Definition 9.12 and the notion of a 2-dimensional totally isotropic subspace by the notion of a *special* submodule as defined in Notation 9.24. In Definition 9.4, we introduce a condition on Λ we call *secure stability* which implies (and is perhaps equivalent to) strong stability (see Remark 9.6). In Theorem 9.27, the main result of Section 9, we show that if Λ, $E(\Lambda)$ and $E^2(\Lambda)$ are all securely stable, then the Tits quadrangle corresponding to Λ has, in fact, a natural description in terms of the hyperbolic cyclic and special submodules of $E^2(\Lambda)$.

In the course of proving Theorem 9.27 (and as a consequence of Theorem 9.27), we prove a result about transitivity on hyperbolic pairs in Proposition 9.9, a result about subgroups generated by certain Eichler transformations in Proposition 9.29 and a cancellation result in Proposition 9.31. In each case, our hypotheses involve some variation of the notion of secure stability. Analogous conclusions, but with very different hypotheses, can be found in [5, Thm. 9.1.3] and [6, Chap. VI, Prop. 4.1.7] for hyperbolic pairs, in [5, Thm. 9.1.1] and [6, Chap. VI, Sec. 4.6 and Thm. 4.7.1] for Eichler transformations and in [8, Cor. 8.3] for cancellation. This suggests that the notion of secure stability might well have a larger role to play in unitary K-theory.

In light of Theorem 9.27 and Proposition 9.31, it is natural to ask whether the secure stability of a pseudo-quadratic module Λ implies the secure stability of $E(\Lambda)$. We succeeded in proving this (in Proposition 9.33), but only for strong stability in place of secure stability. As mentioned above, it remains an open question whether the notions of secure stability and strong stability are equivalent.

The proof of Proposition 9.33 requires a basic result about arbitrary Tits polygons which we prove in Proposition A.10, but which should have been included in [11]. This result says that if X is an arbitrary Tits polygon, then the group G of all its type-preserving automorphisms carries the structure of a Tits set in its action on the edge set of X and in its action on each of the two G-orbits in the vertex set. The notion of a Tits set was introduced in [9, Def. 3.1]. It is, roughly speaking, the isotropic version of the notion of a Moufang set introduced by Tits in [15, Sec. 4.4].

As indicated in [9, Thm. 6.2], Tits sets are ubiquitous in the theory of buildings. Let

$$\mathcal{X} = \left(G, \Omega, \equiv, \{U_x\}_{x\in\Omega}\right)$$

be one of them, let U_0 and U_1 be two opposite root groups of G and let H be the corresponding 2-point stabilizer. By [9, Def. 3.1 (iii)], there is a 2-plump opposition relation on Ω preserved by G. This axiom yields the conclusion (see [9, Prop. 3.6]) that $G = U_1U_0U_1H$. Thus, Definition 5.9 and the result in Proposition 5.16 below point to a connection between any one of the many 2-plump opposition relations in a building Δ and the stability of some algebraic structure underlying Δ. In our opinion, this connection is important and needs to be investigated further.

This paper is organized as follows. In Sections 2 and 3, we review the definitions of the stable range of a ring and a Tits polygon. In Section 4, we introduce pseudo-quadratic modules and in Sections 5 and 6 we prove the main properties of these modules that play a role here. In Section 7 we describe a connection to the theory of root graded groups and we use this connection in Section 8 to prove our main result. In Section 9, we examine more closely the "projective space" associated with a pseudo-quadratic module and use our observations to prove results about the structure of the isometry group of a pseudo-quadratic module reminiscent of results in unitary K-theory. In Section A, we prove the result about Tits sets mentioned above.

Conventions 1.1. Let G be a group. As in [16], we set $a^b = b^{-1}ab$ and

$$[a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$ and if G acts on a set Ω, we assume that $x^{ab} = (x^a)^b$ for all $x \in \Omega$ and all $a, b \in G$.

2. Stability

Let R be an arbitrary associative ring with identity.

Definition 2.1. The ring R has *stable range* 1 if for all $(\alpha, \beta), (s, t) \in R \times R$ such that $s\alpha + t\beta = 1$, there exists $r \in R$ such that $\alpha + r\beta$ is left-invertible. (In [17] and elsewhere, the term 2-*stable* is used in place of "stable range 1".) Thus, R has stable range 1 if and only if for all $(\alpha, \beta) \in R \times R$ and $s \in R$ such that $s\alpha + \beta$ is left-invertible, there exists $t \in R$ such that $\alpha + t\beta$ is left-invertible.

Notation 2.2. We denote the group of units in R by $R^\times$.

Proposition 2.3 ([7, Lem. 1.7]). *Suppose that R has stable range* 1. *Then every left-invertible element is a unit.*

Proof. Suppose that $sr = 1$ for some $r, s \in R$. Then $r \cdot s + 1 \cdot (1 - rs) = 1$. Since R has stable range 1, it follows that $z := s + t \cdot (1 - rs)$ is left-invertible for some $t \in R$. Since

$$zr = sr + t(1 - rs)r = sr + tr - tr \cdot sr = 1,$$

we conclude that $z \in R^\times$, and hence $r = z^{-1} \in R^\times$. ∎

Remark 2.4. A ring in which every left-invertible element is invertible is called *Dedekind-finite*.

3. Tits polygons

Let $\Gamma = (V, E)$ be a graph in which for each $v \in V$, the set Γ_v of vertices adjacent to v is endowed with an opposition relation $\equiv_v$.

Definition 3.1. We say that $u, w \in \Gamma_v$ are *opposite at* v if $u \equiv_v w$. We say that a path of (u, v, w) length 2 is *straight* if u and w are opposite at v and we say that a path or a circuit of arbitrary length is *straight* if every 2-path it contains is straight.

Definition 3.2. A *Veldkamp n-gon* for some $n \geq 3$ is a connected bipartite graph $\Gamma = (V, E)$ in which for each $v \in V$, the set Γ_v is endowed with a 2-plump opposition relation $\equiv_v$ (as defined in Section 1) subject to the following axioms:

(i) If $\alpha = (v_0, \dots, v_k)$ is a straight path of length $k < n$, then α is the unique straight path from v_0 to v_k of length at most k.

(ii) Every straight $(n + 1)$-path is contained in a straight circuit of length $2n$.

Notation 3.3. Let $X = (\Gamma, \{\equiv_v\}_{v \in V})$ be a Veldkamp n-gon. An *apartment* of X is a straight circuit of length $2n$. A *root* of X is a straight path of length n in Γ. The *root group* U_α associated with a root $\alpha = (v_0, v_1, v_2, \dots, v_{n-1}, v_n)$ is the pointwise stabilizer in $\mathrm{Aut}(X)$ of the set

$$\Gamma_{v_1} \cup \Gamma_{v_2} \cup \cdots \cup \Gamma_{v_{n-1}}.$$

We sometimes denote a Tits n-gon $(\Gamma, \{\equiv_v\}_{v \in V})$ by the triple $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$, where $\mathcal{A}$ denotes the set of all apartments; see [11, Rem. 1.3.12].

Definition 3.4. A *Tits n-gon* for some $n \geq 3$ is a Veldkamp n-gon such that for each root α, the root group U_α acts transitively on the set of apartments containing α.

A Tits polygon is a Tits n-gon for some $n \geq 3$. When $n = 3, 4$, etc., we refer to Tits triangles, quadrangles, etc.

See [11, Sec. 1.2] for a description of the family of examples which provided the original motivation for Definition 3.4. For all the basic properties of Tits polygons we will require here we will cite specific results and definitions in either [9] or [11] rather than restate them here with two exceptions. The first exception is the following result, which we will apply it in the proof of Theorem 8.1 below. The second exception is Proposition A.10. This result (actually, the special case formulated in Corollary A.11) will be applied in the proof of Proposition 9.33. Both Propositions 3.5 and A.10 ought to have been included in [11].

Proposition 3.5. *Let X be a Tits polygon and let $(\gamma, i \mapsto w_i)$ be a coordinate system of X as defined in* [9, Def. 4.5]. *Let $i \mapsto U_i$ and H be as in* [9, Not. 4.6 and Prop. 4.8] *and let $D_i = \langle U_i, U_{i+n}\rangle$ for each i. Then*

$$N_{D_i}(U_{i+1}) \cap N_{D_i}(U_{i+n-1}) = D_i \cap H$$

for each i.

Proof. Choose i, let $D = D_i$ and choose $g \in E := N_D(U_{i+1}) \cap N_D(U_{i+n-1})$. Since $g \in D$, g fixes w_i and w_{i+n}. Let $y = w_{i+1}^g$ and $z \in w_{i-1}^g$. Then $y, z \in \Gamma_{w_i}$. Since $g \in N_D(U_{i+1})$ and U_{i+1} fixes w_{i+1}, U_{i+1} fixes y. Since $g \in N_D(U_{i+n-1})$ and U_{i+n-1} fixes w_{i-1}, U_{i+n-1} fixes z. By [11, Lem. 1.4.24], it follows that

$$y = w_{i+1} \quad \text{and} \quad z = w_{i-1}.$$

Since g fixes w_{i+n}, it follows from Definition 3.2 (i) that g fixes w_j for all j. Hence, $g \in H$. Thus, $E \subset D \cap H$. Since H normalizes U_j for all j (by [9, Prop. 4.8]), it follows that $E = D \cap H$. ∎

4. Pseudo-quadratic modules

Definition 4.1. An *involutory ring* is an associative ring K endowed with an involution σ (i.e. an anti-automorphism whose square is trivial) and an additive subgroup K_0 containing 1 such that

(i) $K_\sigma \subset K_0 \subset K^\sigma$, where $K_\sigma = \{a + a^\sigma \mid a \in K\}$ and $K^\sigma = \{a \in K \mid a^\sigma = a\}$;

(ii) $a^\sigma K_0 a \subset K_0$ for all $a \in K$.

We denote an involutory ring by the triple (K, K_0, σ). Thus an involutory ring (K, K_0, σ) is an *involutory set* as defined in [16, Def. 11.1] in the special case that K is a skew-field. In [1] and [5], an involutory ring (K, K_0, σ) is called a *form ring* and the additive subgroup K_0 is referred to as a *form parameter*.

We are grateful to J.-P. Tignol for the proofs of the next two results.

Proposition 4.2. *Let (K, K_0, σ) be an involutory ring and suppose that $e \in K$ is left-invertible. Suppose, too, that $x \in K^\sigma$ for all $x \in K$ such that $xe^2 = 0$. Then e is a unit.*

Proof. Choose $f \in K$ such that $fe = 1$ and let $\varepsilon = 1 - ef$ and $\varepsilon' = 1 - e^2f^2$. Since $fe = f^2e^2 = 1$, computation shows that ε and ε' are idempotents. Further computations yield

$$\varepsilon' e \varepsilon' = e\varepsilon \quad \text{and} \quad \varepsilon' f \varepsilon' = \varepsilon f. \tag{4.3}$$

We have $\varepsilon' e^2 = 0$, and hence $x\varepsilon' \cdot e^2 = 0$ for all $x \in K$. By hypothesis, therefore, $x\varepsilon' \in K^\sigma$ for all $x \in K$. Thus,

$$\varepsilon' f \varepsilon' \cdot \varepsilon' e \varepsilon' = (\varepsilon' f \varepsilon')^\sigma \cdot (\varepsilon' e \varepsilon')^\sigma = (\varepsilon' e \varepsilon' \cdot \varepsilon' f \varepsilon')^\sigma = \varepsilon' e \varepsilon' \cdot \varepsilon' f \varepsilon'. \tag{4.4}$$

Since $fe = 1$, we have $\varepsilon = \varepsilon f \cdot e\varepsilon$. Using (4.3) and (4.4), we thus obtain

$$\varepsilon = \varepsilon' f \varepsilon' \cdot \varepsilon' e \varepsilon' = \varepsilon' e \varepsilon' \cdot \varepsilon' f \varepsilon' = e\varepsilon \cdot \varepsilon f = e\varepsilon f.$$

Hence, $\varepsilon = fe \cdot \varepsilon \cdot fe = f \cdot e\varepsilon f \cdot e = f \cdot \varepsilon e = f \cdot 0 = 0$. Therefore, $ef = 1$. Thus, e is a unit. ■

Proposition 4.5. *Let (K, K_0, σ) be an involutory ring. Then K is Dedekind-finite if and only if for all left-invertible elements e, the following holds:*

$$x \in K_0 \textit{ for all } x \in K \textit{ such that } e^\sigma x e \in K_0. \tag{4.6}$$

Proof. Suppose that (4.6) holds for all left-invertible elements e. Choose a left-invertible element $e \in K$. Then e^2 is also left-invertible. Suppose that $xe^2 = 0$ for some $x \in K$. Then $(e^2)^\sigma x e^2 = 0 \in K_0$, so $x \in K_0$ by (4.6), and thus $x \in K^\sigma$ by Definition 4.1 (i). By Proposition 4.2, it follows that e is a unit. Thus, if (4.6) holds for all left-invertible elements e, then K is Dedekind-finite. The converse is clear. ■

Notation 4.7. Suppose that an involutory ring (K, K_0, σ) is given and let L be a right K-module. A *skew-hermitian form* on L is a bi-additive map f from $L \times L$ to K such that

(i) $f(a, bt) = f(a, b)t$, and

(ii) $f(a, b)^\sigma = -f(b, a)$, and hence

(iii) $f(at, b) = t^\sigma f(a, b)$

for all $a, b \in L$ and all $t \in K$.

Notation 4.8. A *pseudo-quadratic module* is a 6-tuple

$$\Lambda = (K, K_0, \sigma, L, q, f),$$

where (K, K_0, σ) is an involutory ring as defined in Definition 4.1, L is a right K-module, f is a skew-hermitian form on L and q is a map from L to K such that

(i) $q(a + b) \equiv q(a) + q(b) + f(a, b) \pmod{K_0}$, and

(ii) $q(at) \equiv t^\sigma q(a)t \pmod{K_0}$

for all $a \in L$ and all $t \in K$. We will say that Λ is *standard* if

$$f(a, a) = q(a) - q(a)^\sigma$$

for all $a \in L$.

Remark 4.9. A *sesquilinear form* on L is a bi-additive map satisfying Notation 4.7 (i) and (iii). A pseudo-quadratic module $\Lambda = (K, K_0, \sigma, L, q, f)$ is *trace-valued* if there exists a sesquilinear form g on L such that

$$f(a,b) = g(a,b) - g(b,a)^\sigma \quad \text{and} \quad q(a) \equiv g(a,a) \pmod{K_0}$$

for all $a, b \in L$. If a pseudo-quadratic module is trace-valued, then it is standard as defined in Notation 4.8 (since $K_0 \subset K^\sigma$). The converse holds by [14, Sec. 8.1.4] under the assumption that L has a basis but is probably not true in general. See also [13, Rem. 9.41 and Prop. 9.42].

Notation 4.10. For each pseudo-quadratic module $\Lambda = (K, K_0, \sigma, L, q, f)$, let T_Λ denote the group with underlying set

$$\{(a,t) \in L \times K \mid q(a) - t \in K_0\}$$

and multiplication given by

$$(a,t) \cdot (b,s) = \big(a + b, t + s + f(b,a)\big)$$

for all $(a,t), (b,s) \in T_\Lambda$.

In Definition 4.11, we refer to coordinate systems. See [9, Def. 4.5] for the definition. The associated root group labeling is the map $i \mapsto U_i$, where U_i is as in [9, Not. 4.6]. See [9, Prop. 4.11] for the definition of the subsets $U_1^\sharp$ and $U_4^\sharp$ of U_1 and U_4 that also appear in Definition 4.11.

Definition 4.11. Let X be a Tits quadrangle, let $\Lambda = (K, K_0, \sigma, L, q, f)$ be a pseudo-quadratic module and let T_Λ be as in Notations 4.8 and 4.10. As in [13, Not. 3.8], we say that X is *pseudo-quadratic of type* Λ if there exists a coordinate system $(\gamma, i \mapsto w_i)$ of X with associated root group labeling $i \mapsto U_i$, isomorphisms x_i from T_Λ to U_i for $i = 1, 3$ and isomorphisms x_i from the additive group of K to U_i for $i = 2, 4$ such that

$$\begin{aligned}
[x_1(a,t), x_4(r)^{-1}] &= x_2(tr)x_3(ar, r^\sigma tr),\\
[x_1(a,t), x_3(b,s)^{-1}] &= x_2(f(a,b)),\\
[x_2(r), x_4(z)^{-1}] &= x_3(0, r^\sigma z + z^\sigma r)
\end{aligned} \tag{4.12}$$

for all $(a,t), (b,s) \in T_\Lambda$ and all $r, z \in K$. We will say that X is *unitary* if $(\gamma, i \mapsto w_i)$ and $x_1, \dots, x_4$ can be chosen so that, in addition, $x_1(0,1) \in U_1^\sharp$ and $x_4(1) \in U_4^\sharp$.

Proposition 4.13. *Let X be a Tits quadrangle of type Λ for some pseudo-quadratic space Λ and let $x_1, \dots, x_4$ are as in Definition 4.11. Then Λ is standard as defined in Notation 4.8.*

Proof. Let L be as in Definition 4.11 and choose $a \in L$. Applying (4.12), we obtain

$$\begin{aligned} x_3(a,q(a))x_3(-a,q(a)) &= x_3(0,2q(a)+f(-a,a)) \\ &= x_3(0,2q(a)-f(a,a)), \end{aligned} \tag{4.14}$$

as well as

$$\begin{aligned} [x_1(a,q(a)),x_4(1)x_4(-1)] &= [x_1(a,q(a)),x_4(-1)] \cdot [x_1(a,q(a)),x_4(1)]^{x_4(-1)} \\ &= [x_1(a,q(a)),x_4(1)^{-1}] \cdot [x_1(a,q(a)),x_4(-1)^{-1}]^{x_4(-1)} \\ &= x_2(q(a))x_3(a,q(a)) \cdot x_2(-q(a))x_3(-a,q(a)) \\ &\qquad \cdot [x_2(-q(a)),x_4(1)^{-1}] \\ &= x_3(a,q(a))x_3(-a,q(a))x_3(0,-q(a)-q(a)^\sigma), \end{aligned}$$

and so

$$x_3(a,q(a))x_3(-a,q(a)) = x_3(0,q(a)+q(a)^\sigma)$$

since $x_4(1)x_4(-1)=1$. Hence, $f(a,a)=q(a)-q(a)^\sigma$ by (4.14). ■

Hypothesis 4.15. From now on we fix a standard pseudo-quadratic module

$$\Lambda = (K,K_0,\sigma,L,q,f).$$

We allow that $L=0$, in which case Λ is just the involutory ring (K,K_0,σ).

Proposition 4.16. *$f(a,a)=t-t^\sigma$ and $(a,t)^{-1}=(-a,-t^\sigma)$ for all $(a,t)\in T_\Lambda$.*

Proof. Choose $(a,t)\in T_\Lambda$. Since Λ is standard and $K_0 \subset K^\sigma$, we have

$$f(a,a) = t-t^\sigma.$$

Since $q(-a)-q(a)\in K_0$ by Notation 4.8 (ii) and $t+t^\sigma \in K_\sigma \subset K_0$, we have $(-a,-t^\sigma)\in T_\Lambda$ and

$$(a,t)\cdot(-a,-t^\sigma) = (0,0).$$ ■

Notation 4.17. Let M denote the right K-module $K\oplus K\oplus K\oplus K\oplus L$ (with $L=0$ allowed). Let Q be the map from M to K given by

$$Q(\alpha_1,\beta_1,\alpha_2,\beta_2,d) = \alpha_1^\sigma\beta_1+\alpha_2^\sigma\beta_2+q(d)$$

for all $(\alpha_1,\beta_1,\alpha_2,\beta_2,d)\in M$ and let F be the map from $M\times M$ to K given by

$$\begin{aligned} &F\big((\alpha_1,\beta_1,\alpha_2,\beta_2,d),(\alpha_1',\beta_1',\alpha_2',\beta_2',d')\big) \\ &\qquad = \alpha_1^\sigma\beta_1'-\beta_1^\sigma\alpha_1'+\alpha_2^\sigma\beta_2'-\beta_2^\sigma\alpha_2'+f(d,d') \end{aligned}$$

for all $(\alpha_1,\beta_1,\alpha_2,\beta_2,d),(\alpha_1',\beta_1',\alpha_2',\beta_2',d')\in M$. We observe that, like Λ, the 6-tuple $\Lambda_1 := (K,K_0,\sigma,M,Q,F)$ is also a standard pseudo-quadratic module.

Notation 4.18. Let J denote the group of linear isometries of $(K, K_0, \sigma, M, Q, F)$. In other words, J is the group of the linear automorphisms of M preserving both the pseudo-quadratic form Q modulo K_0 and the skew-hermitian form F. Let

$$u_1 = (1,0,0,0,0), \quad v_1 = (0,1,0,0,0), \quad u_2 = (0,0,1,0,0), \quad v_2 = (0,0,0,1,0),$$

and let H be the pointwise stabilizer in J of the set of submodules

$$\{\langle u_1\rangle, \langle v_1\rangle, \langle u_2\rangle, \langle v_2\rangle\}.$$

We identify L with its image in M under the map $u \mapsto (0,0,0,0,u)$. Thus

$$L = \langle u_1, v_1, u_2, v_2\rangle^{\perp}, \tag{4.19}$$

i.e. $L = \{w \in M \mid F(w, \langle u_1, v_1, u_2, v_2\rangle) = 0\}$.

Notation 4.20. Let $\rho, \tau \in J$ be as follows:

- ρ is the unique element of J that maps u_1 to $-v_2$, u_2 to $-v_1$, v_1 to u_2 and v_2 to u_1 and fixes every element of L.
- τ is the unique element of J that maps u_1 to $-v_1$ and v_1 to u_1, fixes u_2, v_2 and fixes every element of L.

Notation 4.21. Let $x_0, x_1, \ldots, x_5$ be defined as follows:

- For each $s \in K$, let $x_0(s)$ denote the unique element of J that fixes u_1, u_2 and every element of L, maps v_1 to $v_1 + u_2 s$ and maps v_2 to $v_2 + u_1 s^\sigma$.
- For each $(a,t) \in T_\Lambda$, let $x_1(a,t)$ denote the unique element of J that fixes u_1, u_2 and v_2, maps v_1 to $v_1 + u_1 t + a$ and maps an arbitrary element $b \in L$ to $b + u_1 f(a,b)$.
- For each $s \in K$, let $x_2(s)$ denote the unique element of J that fixes u_1, v_2 and every element of L, maps v_1 to $v_1 + v_2 s^\sigma$ and maps u_2 to $u_2 - u_1 s$.
- For each $(a,t) \in T_\Lambda$, let $x_3(a,t)$ denote the unique element of J that fixes u_1, v_1 and v_2, maps u_2 to $u_2 - v_2 t - a$ and maps an arbitrary element $b \in L$ to $b + v_2 f(a,b)$.
- For each $s \in K$, let $x_4(s)$ denote the unique element of J that fixes v_1, v_2 and every element of L, maps u_1 to $u_1 - v_2 s^\sigma$ and maps u_2 to $u_2 - v_1 s$.
- For every $(a,t) \in T_\Lambda$, let $x_5(a,t)$ denote the unique element of J that fixes v_1, u_2 and v_2, maps u_1 to $u_1 - v_1 t + a$ and maps an arbitrary element $b \in L$ to $b - v_1 f(a,b)$.

It follows from Conventions 1.1, Notations 4.10 and 4.21, and Proposition 4.16, and that x_1, x_3 and x_5 are injective homomorphisms from T_Λ to J, x_0, x_2 and x_4

are injective homomorphisms from the additive group of K to J and the following commutator relations hold:

$$\begin{aligned} [x_1(a,t), x_4(r)^{-1}] &= x_2(tr)x_3(ar, r^\sigma tr), \\ [x_1(a,t), x_3(b,s)^{-1}] &= x_2(f(a,b)), \\ [x_2(r), x_4(z)^{-1}] &= x_3(0, r^\sigma z + z^\sigma r), \end{aligned} \tag{4.22}$$

as well as

$$\begin{aligned} [x_0(r), x_2(z)^{-1}] &= x_1(0, zr + r^\sigma z^\sigma), \\ [x_0(r), x_3(a,t)^{-1}] &= x_1(ar, r^\sigma tr)x_2(r^\sigma t) \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} [x_3(b,s), x_5(a,t)^{-1}] &= x_4(f(a,b)), \\ [x_2(r), x_5(a,t)^{-1}] &= x_3(-ar, r^\sigma tr)x_4(t^\sigma r) \end{aligned} \tag{4.24}$$

for all $(a,t), (b,s) \in T_\Lambda$ and all $r, z \in K$.

Notation 4.25. Let

$$\begin{aligned} &\Sigma_0 = \langle u_1, u_2, L\rangle, \quad \Sigma_1 = \langle u_1, u_2, v_2\rangle, \quad \Sigma_2 = \langle u_1, v_2, L\rangle, \\ &\Sigma_3 = \langle u_1, v_1, v_2\rangle, \quad \Sigma_4 = \langle v_1, v_2, L\rangle, \quad \Sigma_5 = \langle v_1, u_2, v_2\rangle, \\ &\Sigma_6 = \langle v_1, u_2, L\rangle, \quad \Sigma_7 = \langle u_1, v_1, u_2\rangle, \quad \Sigma_i = \Sigma_j \end{aligned}$$

for all $i \in \mathbb{Z}\backslash[0,7]$, where j is the residue of i modulo 8. Let

$$U_i = \{g \in J \mid w^g = w \text{ for all } w \in \Sigma_i\}$$

for all i. By Notation 4.20, we have $\Sigma_i^\rho = \Sigma_{4-i}$ and $\Sigma_i^\tau = \Sigma_{6-i}$, and therefore

$$U_i^\rho = U_{4-i} \quad \text{and} \quad U_i^\tau = U_{6-i} \tag{4.26}$$

for all i.

Remark 4.27. Since H stabilizes Σ_i, it normalizes U_i for all i.

Proposition 4.28. *$U_i = x_i(T_\Lambda)$ for $i = 1, 3, 5$ and $U_i = x_i(K)$ for $i = 0, 2, 4$.*

Proof. We have $x_1(T_\Lambda) \subset U_1$ and $x_4(K) \subset U_4$. If $g \in U_1$, then g stabilizes

$$\langle u_2, v_2\rangle^\perp = \langle u_1, v_1, L\rangle \quad \text{and} \quad f(u_1^g, v_1) = f(u_1^g, v_1^g) = 1,$$

and therefore $g \in x_1(T_\Lambda)$. If $h \in U_4$, then h stabilizes

$$L^\perp = \langle u_1, v_1, u_2, v_2\rangle \quad \text{and} \quad f(u_i^h, v_i) = f(u_i^h, v_i^h) = 1,$$

and therefore $h \in x_4(K)$. Thus $U_i = x_1(T_\Lambda)$ for $i = 1$ and $U_i = x_4(K)$ for $i = 4$. By similar arguments, these equations hold for the remaining values of i as well. ■

Proposition 4.29. *The following hold:*

(i) $\tau = x_5(0,1)x_1(0,1)x_5(0,1)$ *and* $\rho = x_0(1)x_4(1)x_0(1)$.

(ii) $(\rho\tau)^2 = (\tau\rho)^2$.

Proof. These relations follow from the definitions in Notations 4.20 and 4.21. ∎

Notation 4.30. Let $J^\dagger = \langle U_i \mid i \in \mathbb{Z}\rangle$.

Proposition 4.31. *Let H be as in Notation 4.18 and let $H^\dagger = H \cap J^\dagger$. Then the following hold:*

(i) *Both H and $H^\dagger$ are normalized by $\langle\rho, \tau\rangle$.*

(ii) ρ^2 *and* τ^2 *both lie in* $H^\dagger$.

Proof. Since $\langle\rho, \tau\rangle$ stabilizes the set $\{\langle u_1\rangle, \langle v_1\rangle, \langle u_2\rangle, \langle v_2\rangle\}$, it normalizes H. By Proposition 4.29 (i), $\langle\rho, \tau\rangle$ is contained in $J^\dagger$. Thus, (i) holds. Since ρ^2 and τ^2 lie in H, (ii) also holds. ∎

Proposition 4.32. $[U_i, U_{i+1}] = 1$ *for all* i.

Proof. By (4.26), for each i there exists $g \in \langle\rho, \tau\rangle$ such that

$$\{U_i, U_{i+1}\}^g = \{U_1, U_2\}.$$

It thus suffices to check, using Proposition 4.28, that $[U_1, U_2] = 1$. ∎

Notation 4.33. Let $U_{[i,j]} = \langle U_i, U_{i+1}, \ldots, U_j\rangle$ for all $i, j \in \mathbb{Z}$ such that $i \le j \le i+3$. It follows from (4.22) and Proposition 4.32 that $U_{[i,j]} = U_iU_{i+1}\cdots U_j$ for all i, j such that $i \le j \le i+3$.

Proposition 4.34. $HU_{[i,i+3]} \cap U_{[i+4,i+7]} = 1$ *for all* i.

Proof. By (4.26) and Remark 4.27, the groups $HU_{[i,i+3]} \cap U_{[i+4,i+7]}$ for $i \in \mathbb{Z}$ are all conjugate to each other under the action of $\langle\rho, \tau\rangle$. It thus suffices to show that the claim holds for $i = 1$. By Notation 4.21, $U_{[1,4]}$ stabilizes both $\langle v_2\rangle$ and $\langle u_1, v_2\rangle$. By (4.26), we have

$$U_{[5,8]} = U_{[1,4]}^{\rho\tau\rho\tau}. \tag{4.35}$$

It follows that $U_{[5,8]}$ stabilizes both $\langle u_2\rangle$ and $\langle v_1, u_2\rangle$. By Notation 4.21, the stabilizer of $\langle u_2\rangle$ in $U_{[1,4]}$ is U_1, and the stabilizer of $\langle v_1, u_2\rangle$ in U_1 is trivial. Thus,

$$HU_{[1,4]} \cap U_{[5,8]} \subset H \cap U_{[5,8]} \quad \text{and} \quad H \cap U_{[1,4]} = 1.$$

By Remark 4.27 and (4.35), it follows that $H \cap U_{[5,8]} = 1$. Thus,

$$HU_{[1,4]} \cap U_{[5,8]} = 1.$$ ∎

5. The subgroups G_1 and G_4

We continue with all the notation and assumptions of the previous section. In particular, M, Q and F are as in Notation 4.17. The main result of this section is Proposition 5.16.

Definition 5.1. An element $z \in M$ is *isotropic* if $Q(z) \equiv 0 \pmod{K_0}$. A submodule is isotropic if all its elements are isotropic.

Notation 5.2. Let K_ℓ denote the set of left-invertible elements of K. Thus, K is Dedekind-finite if and only if $K_\ell = K^\times$.

Proposition 5.3. *Let $z \in \{u_1, v_1, u_2, v_2\}$. Then the set zK_ℓ is H-invariant.*

Proof. Let $h \in H$. Then $z^h = zt$ for some $t \in K^\times$. The claim holds, therefore, since $K^\times K_\ell = K_\ell$. ∎

Proposition 5.4. *Let*

$$A_1 = \{(\alpha, \beta) \in K \oplus K \mid s\alpha + \beta \in K_\ell \textit{ for some } s \in K\},$$
$$A_2 = \{(\alpha, \beta) \in K \oplus K \mid \alpha + t\beta \in K_\ell \textit{ for some } t \in K\}.$$

Then the following hold:

(i) $(u_2 K_\ell)^{U_4 U_0} = \{v_1\alpha + u_2\beta \mid (\alpha, \beta) \in A_1\}$.

(ii) $(v_1 K_\ell)^{U_0 U_4} = \{v_1\alpha + u_2\beta \mid (\alpha, \beta) \in A_2\}$.

Proof. Let $s, t \in K$ and $e \in K_\ell$. Then

$$\begin{aligned}(u_2 e)^{x_4(t)x_0(s)} &= (u_2 e - v_1 t e)^{x_0(s)} \\ &= u_2 e - (v_1 - u_2 s)te \\ &= -v_1 te + u_2(e + ste) = v_1\alpha + u_2\beta \end{aligned} \tag{5.5}$$

with $\alpha = -te$ and $\beta = e - s\alpha$. Thus, $(\alpha, \beta) \in A_1$.

Suppose, conversely, that $(\alpha, \beta) \in A_1$, so $s\alpha + \beta = e$ for some $s \in K$ and some $e \in K_\ell$. Let j be a left-inverse of e. Then by (5.5), we have

$$v_1\alpha + u_2\beta = (u_2 e)^{x_4(-\alpha j)x_0(s)}.$$

Thus, (i) holds. Conjugating by ρ and applying (4.26), we obtain

$$(v_1 K_\ell)^{U_0 U_4} = \{-v_1\beta + u_2\alpha \mid (\alpha, \beta) \in A_1\}.$$

Since $s\alpha + \beta \in K_\ell$ if and only if $-\beta - s\alpha \in K_\ell$, we have $(\alpha, \beta) \in A_1$ if and only if $(-\beta, \alpha) \in A_2$. Thus, (ii) holds. ∎

Remark 5.6. Let A_1 and A_2 be as in Proposition 5.4. By the final remark in Definition 2.1, $A_1 = A_2$ if and only if K has stable range 1.

Proposition 5.7. *Let*

$$\Omega = \{(\alpha, \beta, d) \in K \oplus K \oplus L \mid \alpha^\sigma \beta + q(d) \in K_0\},$$

and let

$$B_1 = \{(\alpha, \beta, d) \in \Omega \mid s^\sigma \alpha + \beta + f(b, d) \in K_\ell \text{ for some } (b, s) \in T_\Lambda\},$$
$$B_2 = \{(\alpha, \beta, d) \in \Omega \mid \alpha - t^\sigma \beta + f(a, d) \in K_\ell \text{ for some } (a, t) \in T_\Lambda\}.$$

Suppose that K is Dedekind-finite. Then the following hold:

(i) $(v_1 K^\times)^{U_1 U_5} = \{u_1\alpha + v_1\beta + d \mid (\alpha, \beta, d) \in B_1\}$.

(ii) $(u_1 K^\times)^{U_5 U_1} = \{u_1\alpha + v_1\beta + d \mid (\alpha, \beta, d) \in B_2\}$.

Proof. Choose $(a, t), (b, s) \in T_\Lambda$ and $e \in K^\times$. Then

$$\begin{aligned}(v_1 e)^{x_1(a,t)x_5(b,s)} &= (v_1 e + u_1 te + ae)^{x_5(b,s)}\\ &= (v_1 e + (u_1 - v_1 s + b)te + ae - v_1 f(b, ae)\\ &= u_1 te + v_1(e - ste - f(b, ae)) + ae + bte\\ &= u_1\alpha + v_1\beta + d,\end{aligned} \tag{5.8}$$

where $\alpha = te$, $\beta = e - s\alpha - f(b, ae)$, and $d = ae + b\alpha$. Hence,

$$f(b, ae) = f(b, d) - f(b, b)\alpha = f(b, d) - s\alpha + s^\sigma \alpha$$

by Proposition 4.16. Therefore,

$$e = s^\sigma \alpha + \beta + f(b, d).$$

Since $v_1 e$ is isotropic, so is $u_1\alpha + v_1\beta + d$. Hence, $(\alpha, \beta, d) \in \Omega$. Thus, $(\alpha, \beta, d) \in B_1$.

Suppose, conversely, that $(\alpha, \beta, d) \in B_1$, so $s^\sigma \alpha + \beta + f(b, d) = e$ for some $(b, s) \in T_\Lambda$ and some $e \in K_\ell$, and $u_1\alpha + v_1\beta + d$ is isotropic. Let j be a left-inverse of e, let $a = (d - b\alpha)j$ and $t = \alpha j$. Then by (5.8), we have

$$(v_1 e + u_1 te + ae)^{x_5(b,s)} = u_1\alpha + v_1\beta + d.$$

In particular, $v_1 e + u_1 te + ae$ is isotropic. Thus, $e^\sigma(q(a) - t)e \in K_0$. We apply now the hypothesis that K is Dedekind-finite to conclude that $(a, t) \in T_\Lambda$ (see Proposition 4.5). Hence,

$$(v_1 e)^{x_1(a,t)} = v_1 e + u_1 te + a.$$

Thus, (i) holds.

Since $K_\sigma \subset K_0$, we have $(\alpha, \beta, d) \in \Omega$ if and only if $(\beta, -\alpha, d) \in \Omega$. Thus, $(\alpha, \beta, d) \in B_1$ if and only if $(\beta, -\alpha, d) \in B_2$. Conjugating (i) by τ and applying (4.26), we thus obtain (ii). ∎

Definition 5.9. We will say that Λ is *weakly stable* if K is Dedekind-finite and $B_1 = B_2$, where B_1 and B_2 are as in Proposition 5.7. We will say that Λ is *strongly stable* if $B_1 = B_2$ and K has stable range 1. Thus by Proposition 2.3, strongly stable implies weakly stable.

A unitary stability condition for involutory rings appears in [5, bottom of p. 526]. It does not appear to be related to the definitions in Definition 5.9.

Proposition 5.10. *Let $G_1 = \langle U_1, U_5 \rangle$ and $G_4 = \langle U_0, U_4 \rangle$, and let $H_1 = G_1 \cap H$ and $H_4 = G_4 \cap H$. Then an element of G_4 that stabilizes both $\langle v_1 \rangle$ and $\langle u_2 \rangle$ is in H_4 and an element of G_1 that stabilizes both $\langle u_1 \rangle$ and $\langle v_1 \rangle$ is in H_1.*

Proof. Let h be an element of G_4 that stabilizes both $\langle v_1 \rangle$ and $\langle u_2 \rangle$. The submodule $W := \langle u_1, v_2 \rangle$ is also stabilized by U_0 and U_4. It follows that h also stabilizes

$$\langle u_1 \rangle = W \cap \langle u_2 \rangle^\perp \quad \text{and} \quad \langle v_2 \rangle = W \cap \langle v_1 \rangle^\perp.$$

Thus, $h \in H_4$. Every element of G_1 stabilizes u_2 and v_2. Hence, if an element of G_1 stabilizes both $\langle u_1 \rangle$ and $\langle v_1 \rangle$, then it lies in H_1. ∎

Proposition 5.11. *$U_0 U_4 U_0 H_4 = U_4 U_0 U_4 H_4$ and $U_1 U_5 U_1 H_1 = U_5 U_1 U_5 H_1$, where H_1 and H_4 are as in Proposition 5.10.*

Proof. By Proposition 4.29, $\rho \in U_0 U_4 U_0 H_4$ and $\tau \in U_5 U_1 U_5 H_1$, and by Proposition 4.31 (i), H_4 is normalized by ρ and H_1 is normalized by τ. Thus,

$$\begin{aligned} U_0 U_4 U_0 H_4 &= (U_0 U_4 U_0 H_4)^\rho = U_4 U_0 U_4 H_4, \\ U_5 U_1 U_5 H_1 &= (U_5 U_1 U_5 H_1)^\tau = U_1 U_5 U_1 H_1 \end{aligned}$$

by (4.26). ∎

Proposition 5.12. *Let G_1, G_4, H_1 and H_4 be as in Proposition 5.10. Then the following hold:*

(i) *If K is Dedekind-finite, then the stabilizer of $\langle v_1 \rangle$ in G_4 is $U_4 H_4$ and the stabilizer of $\langle u_2 \rangle$ in G_4 is $U_0 H_4$.*

(ii) *The stabilizer of $\langle u_1 \rangle$ in G_1 is $U_1 H_1$ and the stabilizer of $\langle v_1 \rangle$ in G_1 is $U_5 H_1$.*

Proof. Suppose that K is Dedekind-finite. Let g be an element of G_4 that stabilizes $\langle u_2 \rangle$. The groups U_0 and U_4 stabilize the submodule $\langle v_1, u_2 \rangle$. Hence,

$$v_1^g = v_1 s + u_2 t$$

for some $s,t \in K$ and $v_1 \in \langle u_2^g, v_1s + u_2t\rangle$. It follows that $sr = 1$ for some $r \in K$. Thus, $rs = 1$ since K is Dedekind-finite. Therefore,

$$v_1^{gx_0(-tr)} = (v_1s + u_2t)^{x_0(-tr)} = (v_1 - u_2tr)s + u_2t = v_1s.$$

Hence, $gx_0(-tr)$ stabilizes both $\langle u_2\rangle$ and $\langle v_1\rangle$. By Proposition 5.10, it follows that $gx_0(-tr) \in H_4$, and hence $g \in H_4U_0$. By Remark 4.27,

$$H_4U_0 = U_0H_4.$$

Thus, an element of G_4 that stabilizes $\langle u_2\rangle$ lies in U_0H_4. To see that an element of G_4 that stabilizes $\langle v_1\rangle$ lies in U_4H_4, it suffices to conjugate by ρ. Thus, (i) holds.

To prove (ii), we do not need the assumption that K is Dedekind-finite. Let g be an element of G_1 that stabilizes $\langle u_1\rangle$. Hence, $u_1^g = u_1e$ for some $e \in K^\times$. The groups U_1 and U_5 stabilize the submodule $\langle u_1, v_1, L\rangle$. Hence,

$$v_1^g = u_1t + v_1r + a$$

for some $t, r \in K$ and some $a \in L$. From $F(u_1^g, v_1^g) = F(u_1, v_1) = 1$, it follows that

$$r = e^{-\sigma}. \tag{5.13}$$

Since v_1 is isotropic, so is v_1^g, and thus

$$t^\sigma r + q(a) \in K_0. \tag{5.14}$$

Let

$$b = -ae^\sigma \quad \text{and} \quad s = -et^\sigma. \tag{5.15}$$

Then $q(b) \equiv eq(a)e^\sigma \pmod{K_0}$. By (5.13) and (5.14), therefore,

$$et^\sigma + q(b) = e(t^\sigma r + q(a))e^\sigma \equiv 0 \pmod{K_0}.$$

Hence, by (5.15), $q(b) - s = q(b) + et^\sigma \in K_0$. Thus, $(b,s) \in T_\Lambda$ and

$$\begin{aligned}(u_1t + v_1r + a)^{x_1(b,s)} &= u_1t + (v_1 + u_1s + b)r + a + u_1f(b,a)\\ &= u_1(t + sr + f(b,a)) + v_1r + br + a.\end{aligned}$$

By (5.13) and (5.15), $br + a = 0$ and $s^\sigma = -te^\sigma$, hence

$$f(b,a) = -f(b,br) = (s^\sigma - s)r$$

by Proposition 4.16, and thus

$$t + sr + f(b,a) = t + s^\sigma r = t + (-te^\sigma)r = 0.$$

Thus, $h := gx_1(b,s)$ stabilizes both $\langle u_1\rangle$ and $\langle v_1\rangle$. By Proposition 5.10, therefore, $h \in H_1$, and hence

$$g \in H_1U_1 = U_1H_1.$$

To see that an element of G_1 that stabilizes $\langle v_1\rangle$ lies in U_5H_1, it suffices to conjugate by τ. Thus, (ii) holds. ∎

Proposition 5.16. *Let G_1, G_4, H_1 and H_4 be as in Proposition 5.10, and let A_1, A_2, B_1 and B_2 be as in Propositions 5.4 and 5.7. Then the following hold:*

(i) $G_4 = U_0U_4U_0H_4$ *if and only if* $A_1 = A_2$.

(ii) *Suppose that K is Dedekind-finite. Then $G_1 = U_5U_1U_5H_1$ if and only if $B_1 = B_2$.*

Proof. Suppose that $A_1 = A_2$ and choose $g \in G_4$. By Proposition 2.3 and Remark 5.6, K is Dedekind-finite. By Proposition 5.4, therefore, we have

$$(u_2K^\times)^{U_4U_0} = (v_1K^\times)^{U_0U_4}.$$

Thus, $(u_2K^\times)^{U_4U_0}$ is U_0- and U_4-invariant, and hence also G_4-invariant. Therefore, the coset gU_0U_4 contains an element $\hat{g}$ stabilizing $\langle u_2\rangle$. By Proposition 5.12 (i), therefore, $\hat{g} \in H_4U_0$ and hence

$$g \in U_0U_4U_0H_4 = H_4U_0U_4U_0$$

(by Remark 4.27). Therefore, $G_4 = U_0U_4U_0H_4$.

Suppose, conversely, that $G_4 = U_0U_4U_0H_4$, and hence

$$G_4 = H_4U_4U_0U_4$$

by Proposition 5.11. The element ρ interchanges u_2K_ℓ and v_1K_ℓ. It follows by Proposition 4.29 (i) that $\rho \in G_4$. Thus, $(u_2K_\ell)^{G_4} = (v_1K_\ell)^{G_4}$. Hence,

$$(u_2K_\ell)^{G_4} = (v_1K_\ell)^{H_4U_4U_0U_4} = (v_1K_\ell)^{U_0U_4}$$

and

$$(v_1K_\ell)^{G_4} = (u_2K_\ell)^{H_4U_0U_4U_0} = (u_2K_\ell)^{U_4U_0}$$

are equal. By Proposition 5.4, therefore, $A_1 = A_2$. Thus, (i) holds.

We assume now K is Dedekind-finite. Suppose that $B_1 = B_2$ and choose $g \in G_1$. By Proposition 5.7, we have

$$(u_1K^\times)^{U_5U_1} = (v_1K^\times)^{U_1U_5}.$$

Thus, $(v_1 K^\times)^{U_1 U_5}$ is U_1- and U_5-invariant and hence also G_1-invariant. Therefore, the coset gU_5U_1 contains an element $\hat{g}$ stabilizing $\langle v_1 \rangle$. By Proposition 5.12 (ii), therefore, $\hat{g} \in H_1U_5$, and hence $g \in U_5U_1U_5H_1$. Therefore,

$$G_1 = U_5U_1U_5H_1.$$

Suppose, conversely, that $G_1 = U_5U_1U_5H_1$. Thus, $G_1 = H_1U_1U_5U_1$ by Proposition 5.11. The element τ interchanges $u_1K^\times$ and $v_1K^\times$. By Proposition 4.29 (i), $\tau \in G_1$. Thus, $(u_1K^\times)^{G_1} = (v_1K^\times)^{G_1}$. Hence,

$$(u_1K^\times)^{G_1} = (u_1K^\times)^{H_1U_1U_5U_1} = (u_1K^\times)^{U_5U_1}$$

and

$$(v_1K^\times)^{G_1} = (v_1K^\times)^{H_1U_5U_1U_1} = (v_1K^\times)^{U_1U_5}$$

are equal. By Proposition 5.7, therefore, $B_1 = B_2$. Thus, (ii) holds. ∎

Remark 5.17. Does perhaps $B_1 = B_2$ imply that K is Dedekind-finite?

6. The subgroups P_1 and P_4

We continue with all the notation and assumptions of the previous sections. In particular, $\Lambda = (K, K_0, \sigma, L, q, f)$ is an arbitrary standard pseudo-quadratic module. The main result of this section is Proposition 6.8.

Notation 6.1. Let $Z_0 = \{\varepsilon \in Z(K) \mid \varepsilon \cdot \varepsilon^\sigma = 1\}$, let ι_ε denote the map $z \mapsto z\varepsilon$ from M to itself for each $\varepsilon \in Z_0$, and let $I = \langle \iota_\varepsilon \mid \varepsilon \in Z_0 \rangle$. We observe that $I \subset Z(J) \cap H$, where J is as in Notation 4.18.

Proposition 6.2. *$C_H(U_{[2,3]}) = I$, where $U_{[2,3]}$ is as in Notation 4.33.*

Proof. Let $h \in C_H(U_{[2,3]})$. Since $h \in H$, there exist $\varepsilon, \varepsilon', \alpha, \alpha' \in K^\times$ such that

$$u_1^h = u_1\varepsilon, \quad u_2^h = u_2\alpha, \quad v_1^h = v_1\varepsilon', \quad v_2^h = v_2\alpha'.$$

Since h leaves the skew-symmetric form F invariant, we have

$$\varepsilon^\sigma \cdot \varepsilon' = \alpha^\sigma \cdot \alpha' = 1.$$

Since $[U_2, h]$ fixes u_2, we have $\varepsilon s = s\alpha$ for all $s \in K$ (by Notation 4.21). Hence, $\alpha = \varepsilon \in Z(K)$. By (4.19), there exists an isometry φ of Λ such that $a^h = \varphi(a)$ for all $a \in L$. By Definition 4.1, $1 \in K_0$, and hence $(0, 1) \in T_\Lambda$ by Notation 4.10. Since $[U_3, h]$ fixes u_2, we have

$$\varphi(a) = a\alpha \quad \text{and} \quad t\alpha = \alpha' t$$

for all $(a,t) \in T_\Lambda$. Since $(0,1) \in T_\Lambda$, it follows that $\alpha = \alpha'$, and hence

$$\varepsilon = \varepsilon'.$$

Since $(a, q(a)) \in T_\Lambda$ for all $a \in L$, we have $\varphi(a) = a\alpha$ for all $a \in L$. Thus, $h \in I$. ∎

Proposition 6.3. *The following hold:*

(i) $H_4 = N_{\langle U_0, U_4\rangle}(U_1) \cap N_{\langle U_0, U_4\rangle}(U_3)$.

(ii) $H_1 = N_{\langle U_1, U_5\rangle}(U_2) \cap N_{\langle U_1, U_5\rangle}(U_4)$.

Proof. Suppose that g is an element of $\langle U_0, U_4\rangle$ normalizing both U_1 and U_3. The subgroup $\langle U_0, U_4\rangle$ stabilizes $\langle u_2, v_1\rangle$. Thus, g stabilizes both

$$C_{\langle u_2, v_1\rangle}(U_1) = \langle u_2\rangle \quad \text{and} \quad C_{\langle u_2, v_1\rangle}(U_3) = \langle v_1\rangle.$$

By Proposition 5.10, therefore, $g \in H_4$. Thus, (i) holds

Suppose now that g is an element of $\langle U_1, U_5\rangle$ normalizing both U_2 and U_4. The group $\langle U_1, U_5\rangle$ stabilizes $\langle u_1, v_1\rangle$. Thus g stabilizes both

$$C_{\langle u_1, v_1\rangle}(U_2) = \langle u_1\rangle \quad \text{and} \quad C_{\langle u_1, v_1\rangle}(U_4) = \langle v_1\rangle.$$

By Proposition 5.10, therefore, $g \in H_1$. Thus, (ii) holds. ∎

Proposition 6.4. *The images of $C_{\langle U_0, U_4, H\rangle}(U_{[1,3]})$ and $C_{\langle U_1, U_5, H\rangle}(U_{[2,4]})$ in J/I are both trivial.*

Proof. By Remark 4.27,

$$\langle U_0, U_4, H\rangle = H \cdot \langle U_0, U_4\rangle \quad \text{and} \quad \langle U_1, U_5, H\rangle = H \cdot \langle U_1, U_5\rangle.$$

By Proposition 6.3, therefore,

$$C_{\langle U_0, U_4, H\rangle}(U_{[1,3]}) \subset C_H(U_{[1,3]}) \quad \text{and} \quad C_{\langle U_1, U_5, H\rangle}(U_{[2,4]}) \subset C_H(U_{[2,4]}).$$

The claims hold, therefore, by Proposition 6.2. ∎

Proposition 6.5. *$\langle U_0, U_4, H\rangle \cap U_{[1,3]}$, $\langle U_1, U_5, H\rangle \cap U_{[2,4]}$ and $H \cap U_{[1,4]}$ are all trivial.*

Proof. It suffices to observe that by Notation 4.21 and some calculation, $\langle U_0, U_4, H\rangle$ stabilizes $\langle v_1, u_2\rangle$ and $\langle U_1, U_5, H\rangle$ stabilizes $\langle u_2\rangle$, but no non-trivial element of $U_{[1,3]}$ stabilizes $\langle v_1, u_2\rangle$, no non-trivial element of $U_{[2,4]}$ stabilizes $\langle u_2\rangle$ and no non-trivial element of $U_{[1,4]}$ stabilizes both these submodules. ∎

It follows from Remark 4.27 that the groups H_1 and H_4 both normalize both G_1 and G_4 (defined in Proposition 5.10). Hence, H_1 and H_4 normalize each other. Thus, in particular, H_1H_4 is a subgroup of H.

Notation 6.6. Let P_1 denote the image of $U_{[2,4]} \cdot \langle U_1, U_5, H_4\rangle$ in J/I, let P_4 denote the image of $U_{[1,3]} \cdot \langle U_0, U_4, H_1\rangle$ in J/I, and let B denote the image of $U_{[1,4]} \cdot H_1H_4$ in J/I. Then B is a subgroup of both P_1 and P_4. Let $\hat{J}$ denote the free amalgamated product $P_1 *_B P_4$.

Remark 6.7. Let N_1 denote the subgroup of $\mathrm{Aut}(U_{[2,4]})$ induced by $\langle U_1, U_5, H_4\rangle$, let N_4 denote the subgroup of $\mathrm{Aut}(U_{[1,3]})$ induced by $\langle U_0, U_4, H_1\rangle$ and let A denote the subgroup of $\mathrm{Aut}(U_{[1,4]})$ induced by H_1H_4. By Propositions 6.4 and 6.5, P_1 is isomorphic to the semi-direct product $U_{[2,4]}N_1$, P_4 to the semi-direct product $U_{[1,3]}N_4$ and B to the semi-direct product $U_{[1,4]}A$.

Proposition 6.8. *The free amalgamated product $\hat{J}$ defined in Notation 6.6 is uniquely determined by the commutator relations in (4.22), (4.23), (4.24) and Proposition 4.32, and the observations in Propositions 6.3, 6.4 and 6.5.*

Proof. Let N_1, N_4 and A be as in Remark 6.7. Let Q_1 denote the subgroup of N_1 induced by $\langle U_1, U_5\rangle$ and let Q_4 denote the subgroup of N_4 induced by $\langle U_0, U_4\rangle$. The semi-direct products $U_{[2,4]}Q_1$ and $U_{[1,3]}Q_4$ are uniquely determined by the commutator relations in (4.22), (4.23) and (4.24), and Proposition 4.32. By Proposition 6.3, the subgroup of Q_1 induced by H_1 is the intersection of the normalizers of U_2 and U_4 and the subgroup of Q_4 induced by H_4 is the intersection of the normalizers of U_1 and U_3. Thus the action of H_1 on $U_{[1,3]}$ is uniquely determined by $U_{[2,4]}Q_1$ and the action of H_4 on $U_{[2,4]}$ is uniquely determined by $U_{[1,3]}Q_4$. Thus, the semi-direct products $U_{[1,3]}N_4$ and $U_{[2,4]}N_1$ are uniquely determined by $U_{[2,4]}Q_1$ and $U_{[1,3]}Q_4$. Let $\hat{J}_0$ denote the free amalgamated product

$$U_{[2,4]}N_1 *_{U_{[1,4]}A} U_{[1,3]}N_4$$

formed with respect to the natural inclusions from the semi-direct product $U_{[1,4]}A$ into $U_{[2,4]}Q_1$ and into $U_{[1,3]}Q_4$. By Remark 6.7, where we applied Propositions 6.4 and 6.5, we conclude that there is an isomorphism from $\hat{J}_0$ to $\hat{J}$ mapping $U_{[2,4]}N_1$ to P_1, $U_{[1,3]}N_4$ to P_4 and B to $U_{[1,4]}A$. ∎

We conclude this section with one more observation.

Proposition 6.9. *Suppose that Λ is non-degenerate, i.e. that*

$$\{a \in L \mid f(a, L) = 0 \text{ and } q(a) \equiv 0 \ (\mathrm{mod}\, K_0)\} = 0.$$

Then $C_J(J^\dagger) = I$.

Proof. Let $g \in C_J(J^\dagger)$. Since g centralizes U_i, it stabilizes the set of elements of M fixed by U_i for all i, in particular, for all odd i. It follows that for all $z \in Z := \{u_1, v_1, u_2, v_2\}$, we have $z^g \in \langle z, L_0 \rangle$, where

$$L_0 = \{a \in L \mid f(a, L) = 0\}.$$

By hypothesis, the restriction of q to L_0 is anisotropic. Thus, for each $z \in Z$, if $z_1 \in \langle z, L_0 \rangle$ is isotropic, then $z_1 \in \langle z \rangle$. Hence, $g \in H$. By Proposition 6.2, therefore, $g \in I$. ∎

7. Root graded groups

We continue with all the notation and assumptions in the previous sections. In particular, $J^\dagger$ is as in Notation 4.30 and H_1 and H_4 are as in Proposition 5.10.

Notation 7.1. Let $\zeta = e^{\pi\sqrt{-1}/4}$, let $\alpha_i = \zeta^i$ for all $i \in \mathbb{Z}$, and let $\Phi_4 = \{\alpha_i \mid i \in \mathbb{Z}\}$. We view Φ_4 as a set of eight unit vectors in the real plane.

In Proposition 7.2, we refer to Φ_4-gradings of a group and the torus of a Φ_4-grading. These notions are defined in [9, Def. 2.3].

Proposition 7.2. *Let $\{U_i \mid i \in \mathbb{Z}\}$ be as in Notation 4.25 and let M_{α_i} for all $\alpha_i \in \Phi_n$ be as in* [9, Def. 2.3 (iii)]. *The map $\alpha_i \mapsto U_i$ is a Φ_4-grading of $J^\dagger$ with torus $H_1 H_4$, $M_{\alpha_1} = \{\tau\}$ and $M_{\alpha_4} = \{\rho\}$. It is stable as defined in* [9, (2.4)] *if and only if Λ is strongly stable as defined in Definition 5.9.*

Proof. By Remark 4.27, the condition in [9, Def. 2.3 (i)] holds. By (4.22) and Proposition 4.32, [9, Def. 2.3 (ii)] holds for the subgroups U_i with subscripts i in the interval $[1, 4]$.

If $i \in \mathbb{Z}$ is arbitrary, then by (4.26), there exists $\omega \in \langle \rho, \tau \rangle$ such that either

$$U_k^\omega = U_{k-i+1} \subset U_{[1,4]}$$

for all $k \in [i, i+3]$, or

$$U_k^\omega = U_{4-k+i} \subset U_{[1,4]}$$

for all $k \in [i, i+3]$, where $U_{[1,4]}$ is as in Notation 4.33. Thus, [9, Def. 2.3 (ii)] holds.

By (4.26), every U_i is conjugate either to U_1 or U_4 under the action of $\langle \rho, \tau \rangle$ in case (iii). It follows from Propositions 4.29 (i) and 4.31 that the condition in [9, Def. 2.3 (iii)] holds. By Proposition 4.34, the condition in [9, Def. 2.3 (iv)] holds. In light of Remark 5.6 and Definition 5.9, it remains only to apply Proposition 5.16. ∎

8. Pseudo-quadratic Tits quadrangles

The main result of this section is Theorem 8.1. We continue with all the notation and assumptions in the previous sections. In particular, $\Lambda = (K, K_0, \sigma, L, q, f)$ continues to denote an arbitrary standard pseudo-quadratic module as defined in Notation 4.8 and T_Λ is the group defined in Notation 4.10.

Theorem 8.1. *Let Λ be a standard pseudo-quadratic module as defined in Notation 4.8. Then there exists a unitary pseudo-quadratic Tits quadrangle of type Λ as defined in Definition 4.11 if and only if Λ is strongly stable as defined in Definition 5.9.*

Proof. Suppose that Λ is strongly stable and let α_i for each i be as in Notation 7.1. By Proposition 7.2, $\alpha_i \mapsto U_i$ is a stable Φ_4-grading of $J^\dagger$ with torus H_1H_4. Let

$$X = \big(\Gamma, \mathcal{A}, \{\equiv\}_{v\in V}\big)$$

be the Tits quadrangle obtained by applying [9, Thm. 5.3] to this Φ_4-grading with $m_4 = \rho$ and $m_1 = \tau$ in [9, Con. 5.2 (b)] and let $(\gamma, i \mapsto w_i)$ be the coordinate system of X defined in [9, Not. 5.7]. Then $J^\dagger$ acts on X and H_1H_4 is the pointwise stabilizer of γ in $J^\dagger$. By the intermediate steps [9, Props. 5.17 and 5.18] in the proof of [9, Thm. 5.3], we can identify U_i with the root group of X associated with the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ for each i. By (4.22), it follows that isomorphisms $x_1, \ldots, x_4$ can be chosen so that (4.12) holds. Thus, X is pseudo-quadratic of type Λ. By (4.26), Proposition 4.29 (i) and [9, Prop. 4.15], $x_1(0,1) \in U_1^\sharp$ and $x_4(1) \in U_4^\sharp$, so X is, in fact, unitary as defined in Definition 4.11.

Suppose, conversely, that there exists a unitary pseudo-quadratic Tits quadrangle $X = (\Gamma, \mathcal{A}, \{\equiv\}_{v\in V})$ of type Λ. Our goal is to show that Λ is strongly stable. Let $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$ and x_1, x_2, x_3, x_4 be as in Definition 4.11, let $G = \mathrm{Aut}(X)$, let $G^\dagger$ denote the subgroup of G generated by all the root groups, let H be the pointwise stabilizer of γ in G, and let $H^\dagger = H \cap G^\dagger$. By [9, Prop. 4.8], H normalizes U_i for all i. By Proposition 3.5, we have

$$H_1 = N_{\langle U_1,U_5\rangle}(U_2) \cap N_{\langle U_1,U_5\rangle}(U_4), \quad H_4 = N_{\langle U_0,U_4\rangle}(U_1) \cap N_{\langle U_0,U_4\rangle}(U_3), \tag{8.2}$$

where $H_1 = \langle U_1, U_5\rangle \cap H$ and $H_4 = \langle U_0, U_4\rangle \cap H$. Since X is unitary, we can set

$$\rho = \mu_\gamma(x_4(1)) \quad \text{and} \quad \tau = \mu_\gamma(x_1(0,1)),$$

where μ_γ is as in [9, Prop. 4.11]. Thus, $U_4^\rho = U_0$ and $U_1^\tau = U_5$. We set

$$x_0(s) = x_4(s)^\rho \quad \text{and} \quad x_5(a,t) = x_1(a,t)^\tau \tag{8.3}$$

for all $s \in K$ and all $(a,t) \in T_\Lambda$.

Let $\mathcal{S}$ denote the subset of $G^\dagger$ containing all the elements in $U_0, U_1, \ldots, U_5$ and let $\mathcal{R}$ denote the set of relations in (4.12) together with

$$\begin{aligned} [x_0(r), x_2(z)^{-1}] &= x_1(0, zr + r^\sigma z^\sigma), \\ [x_0(r), x_3(a,t)^{-1}] &= x_1(ar, r^\sigma tr)x_2(r^\sigma t), \end{aligned} \tag{8.4}$$

and

$$\begin{aligned} [x_3(b,s), x_5(a,t)^{-1}] &= x_4(f(a,b)), \\ [x_2(r), x_5(a,t)^{-1}] &= x_3(-ar, r^\sigma tr)x_4(t^\sigma r) \end{aligned} \tag{8.5}$$

for all $(a,t), (b,s) \in T_\Lambda$ and all $r, z \in K$ (exactly as in (4.23) and (4.24)) as well as $[U_i, U_{i+1}] = 1$ for $i \in [1,4]$.

We now consider the following assertion:

$$\text{The relations in } \mathcal{R} \text{ hold in } G^\dagger. \tag{8.6}$$

We will show that if (8.6) holds, then Λ is strongly stable and then that (8.6) must, in fact, hold.

Suppose that (8.6) holds. By [11, Prop. 1.5.16],

$$C_{\langle U_0, U_4, H\rangle}(U_{[1,3]}) = 1 \quad \text{and} \quad C_{\langle U_1, U_5, H\rangle}(U_{[2,4]}) = 1, \tag{8.7}$$

where $U_{[2,4]}$ and $U_{[1,3]}$ are as in [9, Prop. 4.9 and Cor. 4.10]. By [11, Thm. 1.5.19],

$$G^\dagger \cap H = H_1 H_4.$$

By [11, Prop. 1.5.18], the stabilizer of w_4 in $G^\dagger$ is $P_4 := \langle U_0, U_4, H_1\rangle \cdot U_{[1,3]}$, the stabilizer of w_5 is $P_1 = \langle U_1, U_5, H_4\rangle \cdot U_{[2,4]}$ and

$$B := P_1 \cap P_2 = U_{[1,4]} H_1 H_4.$$

The vertices w_8 and w_4 are opposite as defined in [11, Not. 1.3.16] as are the vertices w_1 and w_5 and the edges $\{w_4, w_5\}$ and $\{w_1, w_8\}$. Furthermore, $\langle U_0, U_4, H\rangle$ stabilizes w_8, $\langle U_1, U_5, H\rangle$ stabilizes w_1 and H stabilizes $\{w_4, w_5\}$. It follows from [11, Prop. 1.3.38] and [11, Thm. 1.3.36 (i) and Prop. 1.3.37], therefore, that

$$\langle U_0, U_4, H\rangle \cap U_{[1,3]} = 1, \quad \langle U_1, U_5, H\rangle \cap U_{[2,4]} = 1, \quad H \cap U_{[1,4]} = 1. \tag{8.8}$$

We conclude by Proposition 6.8, (8.2), (8.6), (8.7) and (8.8) that the map from $\mathcal{S}$ to $J^\dagger$ sending each element to the element of $J^\dagger$ having the same name extends to an isomorphism π from $P_1 *_B P_4$ to the free amalgamated product $\widehat{J}$ defined in Notation 6.6.

By [11, Prop. 1.4.17], $(\rho\tau)^2 = (\tau\rho)^2$ in $G^\dagger$ and by [11, Thm. 1.5.19], $G^\dagger$ is isomorphic to the quotient of $P_1 *_B P_4$ by the normal subgroup generated by this relation. By Proposition 4.29 (ii), it follows that π induces a surjective homomorphism from $G^\dagger$ to the image of $J^\dagger$ in J/I, where I is as in Notation 6.1. By [9, Props. 4.8 and 4.13], we have

$$\langle U_0, U_4\rangle = U_0U_4U_0H_4 \quad \text{and} \quad \langle U_1, U_5\rangle = U_5U_1U_5H_1$$

in $G^\dagger$. For each $g \in \langle U_0, U_4\rangle$ in $J^\dagger$, therefore, there exists $g_1 \in U_0U_4U_0$ such that $gg_1^{-1} \in H_4I \subset H$, and hence

$$gg_1^{-1} \in \langle U_0, U_4\rangle \cap H = H_4.$$

Hence, $\langle U_0, U_4\rangle = U_0U_4U_0H_4$ holds in $J^\dagger$. Similarly, $\langle U_1, U_5\rangle = U_5U_1U_5H_1$ holds in $J^\dagger$. By Propositions 2.3 and 5.16, we conclude that Λ is strongly stable.

It thus suffices to show that (8.6) does, in fact, hold. We have $[U_i, U_{i+1}] = 1$ for all i by [9, Prop. 4.9]. The relations (4.12) hold by hypothesis. In particular,

$$[x_1(0,\varepsilon), x_4(\varepsilon s)^{-1}]_2 = x_2(s) \quad \text{and} \quad [x_1(a,t), x_4(\varepsilon)^{-1}]_3 = x_3(a\varepsilon, t) \tag{8.9}$$

for all $(a,t) \in T_\Lambda$, all $s \in K$ and $\varepsilon = 1$ and -1. By [11, Prop. 1.4.13 (i)], we have

$$\rho^{-1} = \mu_\gamma(x_4(-1)) \quad \text{and} \quad \tau^{-1} = \mu_\gamma(x_1(0,-1)).$$

By [11, Prop. 1.4.16] applied to the equations in (8.9), therefore, we obtain

$$x_4(s)^\tau = x_2(s) \quad \text{and} \quad x_3(a,t)^\rho = x_1(a,t) \tag{8.10}$$

if $\varepsilon = 1$, and

$$x_4(-s)^{\tau^{-1}} = x_2(s) \quad \text{and} \quad x_3(-a,t)^{\rho^{-1}} = x_1(a,t)$$

for all $(a,t) \in T_\Lambda$ and all $s \in K$ if $\varepsilon = -1$. Hence,

$$x_1(a,t)^\rho = x_3(-a,t) \quad \text{and} \quad x_2(s)^\tau = x_4(-s) \tag{8.11}$$

for all $(a,t) \in T_\Lambda$ and all $s \in K$.

We claim now that

$$[\tau, U_3] = 1. \tag{8.12}$$

By [9, Prop. 4.11],

$$U_3^\tau = U_3 \tag{8.13}$$

and $\tau = a_5x_1(0,1)b_5$ for some $a_5, b_5 \in U_5$. Choose $a_3 \in U_3$. By [9, Prop. 4.9], $a_3^{a_5} = a_3a_4$ for some $a_4 \in U_4$. By [11, Prop. 1.4.16 (i)],

$$[x_1(0,1), a_4]_2 \neq 1$$

if $a_4 \neq 1$. By (4.12), therefore,

$$(a_3a_4)^{x_1(0,1)} = a_3a_4^{x_1(0,1)} = a_2b_3a_4$$

for some $a_2 \in U_2$, which is non-trivial if $a_4 \neq 1$, and some $b_3 \in U_3$, which equals a_3 if $a_4 = 1$. Hence, $a_3^\tau = (a_2b_3a_4)^{b_5} \in a_2U_{[3,4]}$. By (8.13) and [11, Thm. 1.3.36 (iii)], it follows that $a_2 = 1$. Thus, $a_4 = 1$, and hence $b_3 = a_3$. Therefore, $[\tau, a_3] = 1$, and thus (8.12) holds as claimed.

By [9, Prop. 4.11],

$$U_2^\rho = U_2. \tag{8.14}$$

We have $\rho^{-1} = a_0x_4(-1)b_0$ for some $a_0, b_0 \in U_0$. Choose $s \in K$. By [9, Prop. 4.9], $x_2(s)^{a_0} = a_1x_2(s)$ for some $a_1 \in U_1$. By (4.22), we have

$$(a_1x_2(s))^{x_4(-1)} = a_1 \cdot [a_1, x_4(1)^{-1}] \cdot x_2(s)x_3(0, s + s^\sigma). \tag{8.15}$$

By (8.14), $(a_1x_2(s))^{x_4(-1)b_0} \in U_2$. By [9, Prop. 4.9], it follows that

$$[a_1, x_4(1)^{-1}]_3 \cdot x_3(0, s + s^\sigma) = 1.$$

Hence,

$$a_1 = x_3(0, -s - s^\sigma)^\rho = x_1(0, -s - s^\sigma)$$

by (8.10) and [11, Prop. 1.4.16 (i)]. Therefore,

$$[a_1, x_4(1)^{-1}]_2 = x_2(-s - s^\sigma)$$

by (4.22). Hence, by (8.15),

$$x_2(s)^{\rho^{-1}} = x_2(s)^{a_0x_4(-1)b_0} = (a_1x_2(-s^\sigma))^{b_0}.$$

By [9, Prop. 4.9], we have $(a_1x_2(-s^\sigma))^{b_0} \in U_1x_2(-s^\sigma)$, and by (8.14), we have $(a_1x_2(-s^\sigma))^{b_0} \in U_2$. Thus by [11, Thm. 1.3.36 (iii)], $x_2(s)^{\rho^{-1}} = x_2(-s^\sigma)$. Substituting $-s^\sigma$ for s, we conclude that

$$x_2(s)^\rho = x_2(-s^\sigma) \tag{8.16}$$

for all $s \in K$.

By Proposition 4.16, $x_i(a,t)^{-1} = x_i(-a, -t^\sigma)$ for all $(a,t) \in T_\Lambda$. By (8.3), (8.10), (8.11), (8.12) and (8.16), therefore, conjugation of the relations in (4.12) by ρ yields the relations in (8.4) and conjugation of the relations in (4.12) by τ yields the relations in (8.5). Thus (8.6) holds. This completes the proof of Theorem 8.1. ∎

Remark 8.17. We do not know whether or not non-unitary pseudo-quadratic Tits quadrangles exist.

9. Hyperbolic pairs

We continue one last time with all the notation and assumptions in the previous sections. In particular, $\Lambda = (K, K_0, \sigma, L, q, f)$ is the standard pseudo-quadratic module we introduced in Hypothesis 4.15. In the main result of this section (Theorem 9.27), we show that under a suitable condition, the Tits quadrangle X in Notation 9.1 below has a representation in the "projective space" associated with the module Λ_1 defined in Notation 4.17, and we draw some consequences about the structure of J and Witt cancellation from this result in Propositions 9.29 and 9.31.

Notation 9.1. Let X be the Tits quadrangle with coordinate system $(\gamma, i \mapsto w_i)$, root group labeling $i \mapsto U_i$ and torus H_1H_4 constructed by applying [9, Thm. 5.3] to $J^\dagger$ and the Φ_4-grading $\alpha_i \mapsto U_i$ of $J^\dagger$ be as in the first paragraph of the proof of Theorem 8.1.

Notation 9.2. Let M and F be as in Notation 4.17 and let x and y be cyclic submodules of M. We will say that (x, y) is an *hyperbolic pair* if x and y are isotropic and $1 \in F(x, y)$.

Remark 9.3. Let (x, y) be an hyperbolic pair. Then $\langle x, y\rangle$ is a free module and $M = \langle x, y\rangle \oplus \langle x, y\rangle^\perp$.

Definition 9.4. Let Ω be as in Proposition 5.7. We will say that Λ is *securely stable* if the following two conditions hold:

(i) For all $(\alpha, \beta, d), (t, -s, b) \in \Omega$ such that $s^\sigma\alpha + t^\sigma\beta + f(b, d) = 1$, there exists $(r, 1, a) \in \Omega$ such that $\alpha - r^\sigma\beta + f(a, d) \in K_\ell$.

(ii) K has stable range 1.

Proposition 9.5. *Suppose that Λ satisfies Definition 9.4* (i) *and let B_1 and B_2 be as in Proposition 5.7. Then $B_1 = B_2$.*

Proof. Let $(\alpha, \beta, d) \in B_1$. Then there exists $(b, s) \in T_\Lambda$ such that

$$r := s^\sigma\alpha + \beta + f(b, d) \in K_\ell.$$

Let p be a left-inverse of r. Then $(p^\sigma, -sp^\sigma, bp^\sigma) \in \Omega$ and

$$ps^\sigma\alpha + p\beta + f(bp^\sigma, d) = pr = 1.$$

By Definition 9.4 (i), therefore, there exists $(t, 1, a) \in \Omega$ such that

$$\alpha - t^\sigma\beta + f(a, d) \in K_\ell.$$

We have $(a, t) \in T_\Lambda$, and therefore $(\alpha, \beta, d) \in B_2$. Hence, $B_1 \subset B_2$.

Suppose now that $(\alpha, \beta, d) \in B_2$. Then there exists $(a, t) \in T_\Lambda$ such that

$$r := \alpha - t^\sigma \beta + f(a, d) \in K_\ell.$$

Again let p be a left-inverse of r. Then $(-p^\sigma, tp^\sigma, ap^\sigma) \in \Omega$ and

$$p\alpha - pt^\sigma \beta + f(ap^\sigma, d) = pr = 1.$$

Applying Definition 9.4 (i) with $(\beta, -\alpha, d)$ in place of (α, β, d) and $(p^\sigma, -tp^\sigma, ap^\sigma)$ in place of (t, s, b), it follows that there exists an element $(s, 1, b) \in \Omega$ such that

$$\beta + s^\sigma \alpha + f(b, d) \in K_\ell.$$

We have $(b, s) \in T_\Lambda$, and therefore $(\alpha, \beta, d) \in B_1$. Hence, $B_2 \subset B_1$. ■

Remark 9.6. The result in Proposition 9.5 implies that if Λ is securely stable, then it is strongly stable as defined in Definition 5.9. We do not know if the converse holds, but we prove a partial converse in Proposition 9.10 below.

Notation 9.7. Let $M_i = \langle u_i, v_i, L\rangle$ for $i = 1, 2$, where u_i and v_i are as in Notation 4.18.

Remark 9.8. The next result is reminiscent of [5, Thm. 9.1.3]; see also [6, Chap. VI, (4.1.7)]. Note that the group G_1 defined in Proposition 5.10 is the same as the group $EU_{u,w}(M)$ defined on [5, p. 510] with u_1, v_1 and M_1 in place of u, w and M. (See [5, p. 214] for the definition of an Eichler transformation.)

Proposition 9.9. *Suppose that K is Dedekind-finite and that Λ satisfies condition* (i) *of Definition 9.4. Let M_1 be as in Notation 9.7. Then the group G_1 defined in Proposition 5.10 acts transitively on the set of hyperbolic pairs in M_1 as defined in Notations 9.2 and 9.7.*

Proof. Let Ω and B_2 be as in Proposition 5.7 and let (x, y) be an hyperbolic pair. We claim first that $y \in \langle u_1\rangle^{G_1}$. By condition (i) of Definition 9.4, there exist isotropic elements $z = u_1 t - v_1 s + b \in x$ and $w = u_1\alpha + v_1\beta + d \in y$ such that

$$F(z, w) = s^\sigma \alpha + t^\sigma \beta + f(b, d) = 1.$$

Since y and z are isotropic, both (α, β, d) and $(t, -s, b)$ lie in Ω. By condition (i) of Definition 9.4, there exists $(r, 1, a) \in \Omega$ such that $\alpha - r^\sigma \beta + f(a, d) \in K^\times$. Since $(r, 1, a) \in \Omega$, we have $(a, r) \in T_\Lambda$. Thus, $(\alpha, \beta, d) \in B_2$. By Proposition 5.7 (ii), therefore, $y \in \langle u_1\rangle^g$ for some $g \in \langle U_1, U_5\rangle$, as claimed.

We can thus assume that $y = \langle u_1 \rangle$. By Notation 9.2, there exist $t \in K$ and $z \in x$ such that $F(z, u_1 t) = 1$. Thus, $F(z, u_1)t = 1$, so $t \in K_\ell = K^\times$ and $F(z, u_1) = t^{-1}$. Hence, $F(zt^\sigma, u_1) = 1$. We can assume, therefore, that $F(z, u_1) = 1$. Therefore,

$$z = u_1 s + v_1 + b$$

for some $s \in K$ and some $b \in L$. Since z is isotropic, we have $(b, s) \in T_\Lambda$, and thus

$$v_1^{x_1(b,s)} = z.$$

By Proposition 4.29 (i), we conclude that

$$(x, y) = \big(\langle v_1 \rangle, \langle u_1 \rangle\big)^{x_1(b,s)} = \big(\langle u_1 \rangle, \langle v_1 \rangle\big)^{\tau x_1(b,s)} \in \big(\langle u_1 \rangle, \langle v_1 \rangle\big)^{G_1}. \qquad \blacksquare$$

Proposition 9.10. *Let M_1 be as in Notation 9.7. Suppose that Λ is weakly stable as defined in Definition 5.9 and that G_1 acts transitively on the set of hyperbolic pairs in M_1. Then Λ satisfies Definition 9.4* (i).

Proof. Since Λ is weakly stable, we have $B_1 = B_2$ and K is Dedekind-finite. By Remark 4.27 and Propositions 5.11 and 5.16 (ii),

$$G_1 = U_5 U_1 U_4 H_1 = H_1 U_1 U_5 U_1,$$

and hence

$$\langle u_1 K^\times \rangle^{G_1} = \langle u_1 K^\times \rangle^{U_5 U_1}.$$

By Proposition 5.7 (ii), it follows that

$$\langle u_1 K^\times \rangle^{G_1} = \langle u_1 \alpha + v_1 \beta + d \mid (\alpha, \beta, d) \in B_2 \rangle. \tag{9.11}$$

Now suppose that $s^\sigma \alpha + t^\sigma \beta + f(a, d) = 1$ for (α, β, d) and $(t, -s, b) \in \Omega$. Let

$$y_1 = u_1 \alpha + v_1 \beta + d, \quad x_1 = u_1 t - v_1 s + b,$$
$$y = \langle y_1 \rangle \quad \text{and} \quad x = \langle x_1 \rangle.$$

Then (x, y) is a hyperbolic pair. Thus, by hypothesis, $(x, y) \in (\langle v_1 \rangle, \langle u_1 \rangle)^{G_1}$. Hence,

$$(x_1, y_1)^g = (v_1 r, u_1 z)$$

for some $g \in G_1$ and some $r, z \in K$ such that $F(u_1 r, v_1 z) = 1$. Thus, $z \in K^\times$, and so $y_1 \in \langle u_1 K^\times \rangle^{G_1}$. By (9.11), we conclude that $(\alpha, \beta, d) \in B_2$. Hence, there exists $(a, r) \in T_\Lambda$ such that $\alpha - r^\sigma \beta + f(a, d) \in K^\times$. Thus, Definition 9.4 (i) holds. $\blacksquare$

Definition 9.12. We call a cyclic submodule *hyperbolic* if it is the first coordinate of an hyperbolic pair.

Proposition 9.13. *Let J and H be as in Notation 4.18, let $Y_1 = U_{[2,4]} \cdot \langle U_1, U_5, H\rangle$ and let $Y_4 = U_{[1,3]} \cdot \langle U_0, U_4, H\rangle$. Then the following hold:*

(i) *If K has stable range 1, then G_4 acts transitively on the set of all hyperbolic cyclic submodules of M contained in $\langle u_1, v_2\rangle$.*

(ii) *If K has stable range 1, then the stabilizer of $\langle u_1, v_2\rangle$ in J is Y_4.*

(iii) *If K is Dedekind-finite and Λ satisfies Definition 9.4* (i)*, then the stabilizer of $\langle v_2\rangle$ in J is Y_1.*

Proof. Suppose that K has stable range 1 and let $N = \langle u_1, v_2\rangle$. We identify N with $K \oplus K$ via the map $u_1 s + v_2 t \mapsto (s, t)$. Choose $a = u_1\alpha + v_2\beta$ for some $\alpha, \beta \in K$ and suppose that $F(a, b) = 1$ for some $b \in M$, so that $(\langle a\rangle, \langle b\rangle)$ is an hyperbolic pair. Then a is a unimodular element of $N = K \oplus K$ as defined in [17, (1.1)] (where "stable range 1" is called "2-stable"). By [17, (1.4)], the group G_4 acts transitively on the set of unimodular elements of N. Thus, (i) holds.

Choose $g \in J$ stabilizing N. We want to show that $g \in Y_4$. By (i), we can assume that $u_1^g \in u_1 K^\times$. Let $c = v_2^g$. Then $c = u_1 s + v_2 t$ for some $s, t \in K$ and $N = \langle u_1, c\rangle$. Since $v_2 \in \langle u_1, c\rangle$, we have $t \in K^\times$. Hence, $x_0(-(st^{-1})^\sigma)$ fixes u_1 and maps c to $v_2 t$ (by Notation 4.21). We can thus assume that $v_2^g \in v_2 K^\times$. It follows from

$$F(u_1, u_2) = 0 \quad \text{and} \quad F(u_2, v_2) = 1$$

that

$$u_2^g = u_1 r + u_2 s + v_2 t + b$$

for some $r, t \in K$, $s \in K^\times$ and $b \in L$, and $s^\sigma t + q(b) \in K_0$ since u_2 is isotropic. Thus, $q(bs^{-1}) + ts^{-1} \in K_0$, so $(bs^{-1}, s^{-\sigma} t^\sigma) \in T_\Lambda$. Applying Proposition 4.16 (and Notation 4.21), we obtain

$$\begin{aligned} u_2^{g x_3(bs^{-1}, s^{-\sigma}t^\sigma)} &= u_1 r + (u_2 - v_2 s^{-\sigma} t^\sigma - bs^{-1}) s + v_2 t + b + v_2 f(bs^{-1}, b) \\ &= u_1 r + u_2 s + v_2\bigl(-s^{-\sigma} t^\sigma s + t + f(bs^{-1}, bs^{-1}) s\bigr) \\ &= u_1 r + u_2 s + v_2\bigl(-s^{-\sigma} t^\sigma s + t + (s^{-\sigma} t^\sigma - ts^{-1}) s\bigr) \\ &= u_1 r + u_2 s. \end{aligned}$$

We can thus assume that g stabilizes $\langle u_1, u_2\rangle$. Thus, $u_2^g = u_1 s + u_2 t$ for some $s, t \in K$ and since $u_2 \in \langle u_1^g, u_2^g\rangle$, we have $t \in K^\times$. Replacing g by a suitable element of gU_2, we can thus assume that $u_2^g \in u_2 K^\times$. Since

$$F(u_1, v_1) = 1 \quad \text{and} \quad F(v_1, u_2) = F(v_1, v_2) = 0,$$

we have

$$v_1^g = v_1 p + u_1 m + c$$

for some $p \in K^\times, m \in K, c \in L$, and since v_1 is isotropic, we have $m^\sigma p + q(c) \in K_0$.

Therefore, $q(cp^{-1}) - mp^{-1} \in K_0$, so $(cp^{-1}, mp^{-1}) \in T_\Lambda$ and

$$(v_1 p)^{x_1(cp^{-1},mp^{-1})} = v_1^g.$$

We can thus assume that g stabilizes $\langle u_1\rangle$, $\langle v_1\rangle$, $\langle u_2\rangle$, and $\langle v_1\rangle$. Hence, $g \in H \subset Y_4$. Thus, (ii) holds.

Suppose now that K is Dedekind-finite (but not that K has stable range 1) and that Λ satisfies Definition 9.4 (i). Choose $g \in J$ such that $v_2^g = v_2 p^\sigma$ for some $p \in K^\times$. We want to show that g is in Y_1. Since $F(u_2, v_2) = 1$, it follows that

$$u_2^g = u_2 p^{-1} + v_2 r + u_1 s + v_1 t - a$$

for some $r, s, t \in K$ and some $a \in K$. We have

$$u_2^{g x_2(sp) x_4(tp)} = u_2 p^{-1} + v_2 r' - a$$

with $r' = r + p^\sigma s^\sigma t$. Since u_2 is isotropic, we have $p^{-\sigma} r' + q(a) \in K_0$. Thus, $r'p + q(ap) \in K_0$. Hence, $(ap, -r'p) \in T_\Lambda$ and

$$(u_2 p^{-1})^{x_3(ap,-r'p)} = u_2 p^{-1} + v_2 r' - a.$$

We can assume, therefore, that $u_2^g = u_2 p^{-1}$. Thus, g stabilizes $\langle u_2\rangle$ and $\langle v_2\rangle$. By Proposition 9.9, we can assume that g stabilizes $\langle u_1\rangle$ and $\langle v_1\rangle$ as well. Hence, $g \in H \subset Y_1$. Thus, (iii) holds. ∎

Notation 9.14. Let

$$\begin{aligned}
\hat{w}_1 &= \langle u_2\rangle, & \hat{w}_2 &= \langle u_1, u_2\rangle, & \hat{w}_3 &= \langle u_1\rangle,\\
\hat{w}_4 &= \langle u_1, v_2\rangle, & \hat{w}_5 &= \langle v_2\rangle, & \hat{w}_6 &= \langle v_1, v_2\rangle,\\
\hat{w}_7 &= \langle v_1\rangle, & \hat{w}_8 &= \langle u_2, v_1\rangle, & \hat{w}_i &= \hat{w}_j
\end{aligned}$$

for all $i \in \mathbb{Z}\backslash[1,8]$, where j is the residue of i modulo 8.

Notation 9.15. Let J be as in Notation 4.18, let Υ_1 be the orbit $\langle v_2\rangle^J$, let Υ_4 be the orbit $\langle u_1, v_2\rangle^J$, let $E = (\langle v_2\rangle, \langle u_1, v_2\rangle)^J$ (so $E \subset \Upsilon_1 \times \Upsilon_4$), and let Γ denote the bipartite graph whose vertex set is the disjoint union $V := \Upsilon_1 \cup \Upsilon_4$ and whose edge set is E. If $x \in \Upsilon_1$ and $y, z \in \Gamma_x$, we set $y \equiv_x z$ whenever $\langle y, z\rangle$ contains an hyperbolic pair. If $x \in \Upsilon_4$ and $y, z \in \Gamma_x$, we set $y \equiv_x z$ whenever $x = y \oplus z$. The submodules $\hat{w}_i$ defined in Notation 9.14 are contained in V for all i. Let $\hat{\gamma}$ denote the subgraph of Γ with vertex set $\{\hat{w}_i \mid i \in \mathbb{Z}\}$ and edge set $\{\{\hat{w}_i, \hat{w}_{i+1}\} \mid i \in \mathbb{Z}\}$. Let $\mathcal{A}$ denote the orbit $\hat{\gamma}^J$, and let $\hat{X}$ denote the 3-tuple $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$.

Proposition 9.16. *Suppose that Λ is securely stable as in Definition 9.4, let $\widehat{X}$ and $\widehat{w}_i$ for $i \in \mathbb{Z}$ be as in Notation 9.15, and let X and w_i for $i \in \mathbb{Z}$ be as in Notation 9.1. Then $\widehat{X}$ is a Tits quadrangle and there is an isomorphism from $\widehat{X}$ to X mapping $\widehat{w}_i$ to w_i for all i.*

Proof. It follows from Remark 4.27, and the description of X and the coordinate system (γ, w_i) in [9, Con. 5.2] that there is a natural action of H on X with respect to which an arbitrary $h \in H$ maps the vertex $P_i g$ of X for $i = 1, 4$ for all $g \in J^\dagger$ to $(P_i g)^h = P_i g^h$. By [9, Not. 5.7], we have

$$w_{n+1+2i} = P_1(\rho\tau)^i \quad \text{and} \quad w_{n+2i} = P_n(\tau\rho)^i$$

for all i. By Notation 4.20, both $\rho^h\rho^{-1}$ and $\tau^h\tau^{-1}$ are contained in H for all $h \in H$ and by Proposition 4.29 (i), $\rho \in G_4$ and $\tau \in G_1$. By Remark 4.27, H normalizes both G_1 and G_4. Hence, $\rho^h\rho^{-1} \in H_4$ and $\tau^h\tau^{-1} \in H_1$ for all $h \in H$. By Proposition 4.31 (i), therefore, H fixes w_i for all i. By Notation 9.1 and Remark 9.6, H_1H_4 is the pointwise stabilizer of γ in $J^\dagger$. Therefore,

$$H \cap J^\dagger = H_1H_4.$$

Hence, $Y_1 \cap J^\dagger = \langle U_1, U_5, H_4\rangle$ and $Y_4 \cap J^\dagger = \langle U_0, U_5, H_1\rangle$, where Y_1 and Y_4 are as in Proposition 9.13. By Notation 6.6, we conclude that the image of $Y_i \cap J^\dagger$ in J/I is P_i for $i = 1, 4$.

Let Υ_1, Υ_4, V, E and $\mathcal{A}$ be as in Notation 9.15. By Proposition 9.13 (ii)–(iii) and the conclusion of the previous paragraph, there are bijections φ_4 from Υ_4 to the set of right cosets of P_4 in J and φ_1 from Υ_1 to the set of right cosets of P_1 in J such that

$$\varphi_4\big(\langle u_1, v_2\rangle^g\big) = P_4 g \quad \text{and} \quad \varphi_1\big(\langle v_2\rangle^g\big) = P_1 g$$

for all $g \in J$. Let φ denote the bijection from V to the vertex set of X that restricts to φ_i on Υ_i for $i = 1, 4$. Then φ maps E to the edge set of X and, in particular,

$$\varphi(w_i) = \widehat{w}_i$$

for all i. Since ρ interchanges $\langle v_2\rangle$ with $\langle u_1\rangle$ and τ interchanges $\langle u_1, v_2\rangle$ with $\langle v_1, v_2\rangle$, it follows that φ carries $\{\equiv_v\}_{v\in V}$ to the set of local opposition relations of X and by [9, Con. 5.2 (b)], $\varphi(\mathcal{A})$ is the set of apartments of X. Thus, φ is a bijection from $\widehat{X}$ to X. Since X is a Tits quadrangle, so is $\widehat{X}$. ∎

Notation 9.17. Let $M_0 = K \oplus K \oplus L$, let

$$Q_0(\alpha, \beta, w) = \alpha^\sigma\beta + q(w)$$

for all $(\alpha, \beta, w) \in M_0$, and let

$$F_0\big((\alpha, \beta, w), (\alpha', \beta', w')\big) = \alpha^\sigma \beta' - \beta^\sigma \alpha' + f(w, w')$$

for all $(\alpha, \beta, w), (\alpha', \beta', w') \in M_0$. Then $(K, K_0, \sigma, M_0, Q_0, F_0)$ is a standard pseudo-quadratic module which we denote by Λ_0. We think of Λ as a submodule of Λ_0 via the embedding $a \mapsto (0, 0, a)$ from L to M_0.

Notation 9.18. The construction in Notation 9.17 can be applied to any standard pseudo-quadratic module over (K, K_0, σ). If we apply it to a standard pseudo-quadratic module Δ, we denote the standard pseudo-quadratic module we construct by $E(\Delta)$. Thus, Λ_0 in Notation 9.17 is $E(\Lambda)$ and the standard pseudo-quadratic module Λ_1 constructed in Notation 4.17 is

$$E^2(\Lambda) := E(E(\Lambda)) = E(\Lambda_0).$$

Let $\Lambda_2 = E^3(\Lambda)$. Thus, $\Lambda_2 = E^2(\Lambda_0) = E(\Lambda_1)$. Let J_2 denote the group of all isometries of Λ_2. Note that J can be thought of as the subgroup of J_2 acting trivially on the submodule $E(\Lambda_0)^\perp$.

Notation 9.19. Let Λ_0 be as in Notation 9.17. By Notation 4.10,

$$\begin{aligned} T_{\Lambda_0} &= \big\{(A, T) \in M_0 \times K \mid Q_0(A) - T \in K_0\big\} \\ &= \big\{\big((y, z, a),\ y^\sigma z + t\big) \mid (a, t) \in T_\Lambda,\ y, z \in K\big\} \end{aligned}$$

and multiplication in T_{Λ_0} is given by

$$\begin{aligned} &\big((y, z, a), y^\sigma z + t\big) \cdot \big((\tilde{y}, \tilde{z}, b),\ \tilde{y}^\sigma \tilde{z} + s\big) \\ &\quad = \big((y + \tilde{y}, z + \tilde{z}, a + b), y^\sigma z + \tilde{y}^\sigma \tilde{z} + F_0\big((\tilde{y}, \tilde{z}, b), (y, z, a)\big) + t + s\big) \\ &\quad = \big((y + \tilde{y}, z + \tilde{z}, a + b), (y + \tilde{y})^\sigma (z + \tilde{z}) + f(b, a) + t + s - \tilde{z}^\sigma y - y^\sigma \tilde{z}\big) \end{aligned}$$

for all $(a, t), (b, s) \in T_\Lambda$ and all $y, z, \tilde{y}, \tilde{z} \in K$.

Notation 9.20. Let Λ_0 be as in Notation 9.17 and let

$$\tilde{x}_1\big((y, z, a),\ y^\sigma z + t\big) = x_0(y) x_1(a, t + y^\sigma z + z^\sigma y) x_2(z^\sigma)$$

for all $(a, t) \in T_\Lambda$ and $y, z \in K$. By Notations 4.10 and 9.19, (4.23), and some calculation, $\tilde{x}_1$ is an homomorphism from T_{Λ_0} to $J^\dagger$. It is injective and for given $(a, t), y, z$, the element $\tilde{x}_1((y, z, a),\ y^\sigma z + t)$ fixes u_1, maps v_1 to

$$v_1 + u_1(y^\sigma z + t) + u_2 y + v_2 z + a$$

and maps an arbitrary element $u_2 m + v_2 n + b$ of M_2 to

$$u_2 m + v_2 n + b + u_1\big(y^\sigma n - z^\sigma m + f(a, b)\big).$$

This last expression is equal to

$$u_2m + v_2n + b + u_1F_0(u_2y + v_2z + a, u_2m + v_2n + b).$$

Now let M_2 be as in Notation 9.7. We identify M_0 with M_2 via the map

$$(y, z, a) \mapsto u_2y + v_2z + a.$$

Then for each $(A, T) \in T_{\Lambda_0}$, the element $\tilde{x}_1(A, T)$ fixes u_1, maps v_1 to $v_1 + u_1T + A$ and maps an arbitrary element $B \in M_2$ to $B + u_1F_0(A, B)$.

Notation 9.21. Let Λ_0 and $\tilde{x}_1$ be as in Notation 9.20. Note that τ stabilizes M_2, which we have identified with M_0. Let $\tilde{x}_5(A, T) = \tilde{x}_1(A, T)^\tau$ for all $(A, T) \in T_{\Lambda_0}$. Then for each $(A, T) \in T_{\Lambda_0}$, the element $\tilde{x}_5(A, T)$ fixes v_1, maps u_1 to $u_1 - v_1T + A$ and maps an arbitrary element $B \in M_2$ to $B - v_1F_0(A, B)$.

Notation 9.22. Let $\tilde{U}_i = \tilde{x}_i(T_{\Lambda_0})$ for $i = 1, 5$, and let $\tilde{G}_1 := \langle \tilde{U}_1, \tilde{U}_5 \rangle$. By Notations 9.20 and 9.21, we have $\tilde{U}_1 = U_{[0,2]}$ and $\tilde{U}_5 = U_{[4,6]}$. In particular, we have $\tilde{G}_1 \subset J^\dagger$.

Proposition 9.23. *Let $\tilde{G}_1$ be as in Notation 9.22, let Λ_0 be as in Notation 9.17 and suppose that Λ_0 is securely stable as defined in Definition 9.4. Then $\tilde{G}_1$ acts transitively on the set of all hyperbolic pairs in M.*

Proof. It suffices to observe that by the formulas for the action of the elements of $\tilde{U}_1$ and $\tilde{U}_5$ in Notations 9.20 and 9.21, we can apply Proposition 9.9 with Λ_0 in place of Λ, $M = K \oplus K \oplus M_2$ in place of $M_1 = K \oplus K \oplus L$ and $\tilde{G}_1 = \langle \tilde{U}_1, \tilde{U}_5 \rangle$ in place of $G_1 = \langle U_1, U_5 \rangle$. ∎

Notation 9.24. Two hyperbolic pairs (x, y) and (x', y') are *orthogonal* if $\langle x, y \rangle \subset \langle x', y' \rangle^\perp$. A submodule N of M is *special* if there exist orthogonal hyperbolic pairs (x, y) and (x', y') such that $N = \langle x, x' \rangle$. Note that a special submodule is free.

Definition 9.25. Let $\Lambda_0 = E(\Lambda)$, Λ_1, Λ_2 and the operator E be as in Notations 4.17, 9.17 and 9.18, let $\Lambda_{-1} = \Lambda$ and let $\Lambda_n = E(\Lambda_{n-1})$ for all $n \geq 3$. We will say that Λ is *n-securely stable* for some $n \geq 0$ if Λ_{m-1} is securely stable for all m in the interval $[0, n]$. Thus, Λ is 0-securely stable if and only if it is securely stable, and Λ is 1-securely stable if and only if both Λ and Λ_0 are securely stable.

Proposition 9.26. *Suppose that Λ is 1-securely stable as defined in Definition 9.25. Then $J^\dagger$ acts transitively on the set of all hyperbolic cyclic submodules of M, on the set of all special submodules and on the set of pairs (x, N) such that N is a special submodule, x an hyperbolic cyclic submodule and $x \subset N$.*

Proof. By Propositions 9.9 and 9.23, $J^\dagger$ acts transitively on ordered pairs of orthogonal hyperbolic pairs. Thus, the first two claims hold. The third claim follows from the second by Proposition 9.13 (i). ∎

Theorem 9.27. *Let Υ_1 denote the set of all hyperbolic cyclic submodules of M, let Υ_4 denote the set of all special submodules of M, let E denote the set of pairs $(A, B) \in \Upsilon_1 \times \Upsilon_4$ such that $A \subset B$, and let Γ denote the bipartite graph whose vertex set is the disjoint union $V := \Upsilon_1 \cup \Upsilon_4$ and whose edge set is E. If $x \in \Upsilon_1$ and $y, z \in \Gamma_x$, we set $y \equiv_x z$ whenever $\langle y, z\rangle$ contains an hyperbolic pair. If $x \in \Upsilon_4$ and $y, z \in \Gamma_x$, we set $y \equiv_x z$ whenever $x = y \oplus z$. Let $\mathcal{A}$ denote the set of all 8-circuits γ in Γ such that $y \equiv_x z$ for all 2-paths (y, x, z) contained in γ and let $\hat{X}$ denote the 3-tuple $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$.*

Suppose that Λ is 1-securely stable as defined in Definition 9.25. Then Λ is strongly stable and $\hat{X}$ is a Tits quadrangle of type Λ isomorphic to the Tits quadrangle X constructed from $J^\dagger$ in Notation 9.1.

Proof. By Remark 9.6 and Definition 9.25, Λ is strongly stable. The claims hold, therefore, by Propositions 9.16 and 9.26. ∎

Remark 9.28. Let $\hat{X} = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be as in Notation 9.15, let J_1 denote the kernel of the action of J on $\hat{X}$, and let I be as in Notation 6.1. Then $I \subset J_1 \subset H$ and $U_i \cap J_1 \subset U_i \cap H = 1$, so U_i acts faithfully on the vertex set of Γ for all i. By Remark 4.27, therefore, J_1 centralizes U_i for all i. Hence, $J_1 = I$ by Proposition 6.2.

The following result should be compared with [5, Thm. 9.1.1] and [6, Chap. VI, (4.6) and (4.7.1)].

Proposition 9.29. *Suppose that Λ is 1-securely stable as defined in Definition 9.25. Let J, H and $J^\dagger$ be as in Notations 4.18 and 4.30. Then the following hold:*

(i) $J = J^\dagger H$.

(ii) *$J^\dagger$ is a normal subgroup of J.*

Proof. Let $\hat{X}$ and $\hat{\gamma}$ be as in Notation 9.15. By Proposition 9.16, $\hat{X}$ is a Tits quadrangle and $\{U_i \mid i \in \mathbb{Z}\}$ is the set of root groups of $\hat{X}$ associated with all the roots contained in $\hat{\gamma}$. The kernel of the action of J on $\hat{X}$ lies in H (as was observed in Remark 9.28). By [9, Prop. 4.8], therefore, (i) holds. Since H normalizes $J^\dagger$ (by Remark 4.27), it follows that (ii) holds. ∎

Definition 9.30. A *hyperbolic plane* is a submodule generated by the two cyclic submodules making up a hyperbolic pair.

The following observation should be compared with [8, Cor. 8.3].

Proposition 9.31. *Let Λ, Λ', Λ'' be three pseudo-quadratic modules over (K, K_0, σ). Suppose that Λ is n-securely stable for some $n \geq 1$ as defined in Definition 9.25, that Λ'' is the orthogonal sum of n hyperbolic planes and that $\Lambda \perp \Lambda''$ is isometric to $\Lambda' \perp \Lambda''$. Then Λ is isometric to Λ'.*

Proof. We have $\Lambda \perp \Lambda'' = E(E^{n-1}(\Lambda))$ and $E^{n-1}(\Lambda)$ is securely stable. By Proposition 9.9, therefore, the isometry group of $\Lambda \perp \Lambda''$ acts transitively on the set of hyperbolic pairs in $\Lambda \perp \Lambda''$. It follows by induction that the isometry group of $\Lambda \perp \Lambda''$ acts transitively on the set of submodules isometric to Λ''. Thus, there exists an isometry from $\Lambda \perp \Lambda''$ to $\Lambda' \perp \Lambda''$ that restricts to an isometry from Λ to Λ'. ∎

Notation 9.32. By Notation 9.18, we can think of the underlying module of Λ_2 as

$$K \oplus K \oplus M,$$

where M, as usual, is as in Notation 4.17. Let $u_0 = (1, 0, 0)$ and $v_0 = (0, 1, 0)$ in this underlying module, let J_2 be as in Notation 9.18, and let H_2 denote the pointwise stabilizer in J_2 of the set $\{\langle u_0\rangle, \langle v_0\rangle, \langle u_2\rangle, \langle v_2\rangle\}$. Thus, $\langle H, U_1, U_5\rangle \subset H_2$.

In the proof of the following result, we apply Corollary A.11 which is stated and proved in the next section.

Proposition 9.33. *Suppose that Λ is strongly stable as in Definition 5.9. Then Λ_i is strongly stable for all $i \geq 0$, where Λ_i is as in Definition 9.25.*

Proof. By induction, it suffices to show that Λ_0 is strongly stable. Let

$$X = \big(\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V}\big)$$

and $(\gamma, i \mapsto w_i)$ be the unitary pseudo-quadratic Tits quadrangle and coordinate system in Notation 9.1, and let $\widetilde{U}_1 = U_{[0,2]}$, $\widetilde{U}_5 = U_{[4,6]}$, and $\widetilde{G}_1$ be as in Notation 9.22. Let $H^\dagger = H \cap G^\dagger$ and $\widetilde{H} = \langle U_1, U_5, H^\dagger\rangle$. Thus,

$$\widetilde{H} \subset H_2 \cap \langle \widetilde{U}_1, \widetilde{U}_5\rangle,$$

where H_2 is as in Notation 9.32. By Corollary A.11 below (a special case of Proposition A.10), we have

$$\widetilde{G}_1 = \widetilde{U}_5 \widetilde{U}_1 \widetilde{U}_5 \widetilde{H}.$$

By Proposition 5.16 (ii), it follows that Λ_0 is strongly stable. ∎

A. Appendix: Tits polygons and Tits sets

In this section, we want to prove Proposition A.10 whose Corollary A.11 we applied in the proof of Proposition 9.33. Proposition A.10 is a result about arbitrary Tits polygons and some Tits sets associated with them.

Notation A.1. Suppose that $X = (\Gamma, \mathcal{A}, \{\equiv_v\})$ is a Tits n-gon for some $n \geq 3$. Let $(\gamma, i \mapsto w_i)$ be a coordinate system of X, let α_i denote the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ and let U_i denote the root group U_{α_i} for all i. Let $G = \mathrm{Aut}(X)$ and let $G^\dagger$ denote the subgroup of G generated by all the root groups.

Notation A.2. Let Ψ denote either the set of edges of Γ (case I) or one of the two $G^\dagger$-orbits in the set of vertices of Γ (case II) and let $x \equiv y$ for $x, y \in \Psi$ whenever x and y are opposite as defined in [11, Not. 1.3.16]. Thus, Ψ is an opposition relation as defined in Section 1. In case II, we assume additionally that n is even. Note that in case II, $\equiv$ is empty if n is odd.

Notation A.3. For each vertex x, let $\mathcal{R}_x$ denote the set of roots containing x and two neighbors of x.

Definition A.4. The *unipotent radical* of an edge e is the subgroup generated by the root groups U_α for all roots α containing e and the *unipotent radical of a vertex* x is the subgroup generated by the root groups U_α for all roots α in the set $\mathcal{R}_x$ defined in Notation A.3. Since $gU_\alpha g^{-1} = U_{\alpha^g}$ for all roots α and all $g \in G$, we have

$$g^{-1}U_e g = U_{e^g} \quad \text{and} \quad g^{-1}U_x g = U_{x^g} \tag{A.5}$$

for all $g \in G$, for all edges e and all vertices x.

Proposition A.6. *$U_x = U_{[1,n]}$ if $x = \{w_n, w_{n+1}\}$, $U_x = U_{[1,n-1]}$ if $x = w_n$, and $U_x = U_{[2,n]}$ if $x = w_1$.*

Proof. By [11, Defs. 1.3.28 and 1.3.32, and Thm. 1.3.36 (i)], we have $U_x = U_{[1,n]}$ if $x = \{w_n, w_{n+1}\}$. We have

$$\begin{aligned} U_{[1,n-1]} &\subset U_x \quad \text{if } x = w_n, \text{ and} \\ U_{[2,n]} &\subset U_x \quad \text{if } x = w_1. \end{aligned}$$

By [11, Thm. 1.3.36 (ii)], $U_{[1,n-1]}$ is normalized by $\langle U_0, U_n\rangle$ and $U_{[2,n]}$ is normalized by $\langle U_1, U_{n+1}\rangle$. Let

$$Q_1 = U_{[1,n-1]} \cdot \langle U_0, U_n\rangle \quad \text{and} \quad Q_n = U_{[2,n]} \cdot \langle U_1, U_{n+1}\rangle.$$

By [11, Prop. 1.3.15], $\langle U_0, U_n\rangle$ acts transitively on the set of ordered pairs of vertices in Γ_{w_n} opposite each other at w_n. Let $\alpha \in \mathcal{R}_{w_n}$. Then there exists $g_1 \in \langle U_0, U_n\rangle$ such that the root α^{g_1} contains w_{n-1} and w_{n+1}. By repeated application of [11, Def. 1.1.8 (iii)], there exists $g_2 \in U_{[1,n-1]}$ such that $\alpha^{g_1 g_2} = \alpha_i$ for some $i \in [1, n-1]$. Thus every root in $\mathcal{R}_{w_n}$ is in the same Q_n-orbit as α_i for some $i \in [1, n-1]$. By a similar argument, every root in $\mathcal{R}_{w_1}$ is in the same Q_1-orbit as α_i for some $i \in [2, n]$. Thus, if $\alpha \in \mathcal{R}_{w_n}$, then for some $g \in Q_n$,

$$U_\alpha \subset g^{-1} U_{[1,n-1]} g = U_{[1,n-1]}$$

and if $\alpha \in \mathcal{R}_{w_1}$, then for some $g \in Q_1$,

$$U_\alpha \subset g^{-1} U_{[2,n]} g = U_{[2,n]}.$$

It follows that $U_x = U_{[1,n-1]}$ if $x = w_n$ and $U_x = U_{[2,n]}$ if $x = w_1$. ∎

Proposition A.7. *U_x acts sharply transitively on the set $\{y \in \Psi \mid x \equiv y\}$.*

Proof. By (A.5), we can assume that $x = \{w_n, w_{n+1}\}$ in case I, and that $x = w_n$ or $x = w_1$ in case II. The claim holds, therefore, by Proposition A.6 and [11, Props. 1.3.37 and 1.3.38]. ∎

Proposition A.8. *The opposition relation $\equiv$ on Ψ is 2-plump as defined in the first few lines of Section 1.*

Proof. This holds by [11, Prop. 1.5.1 (i)–(ii)]. ∎

Notation A.9. Let H denote the pointwise stabilizer of γ in G. Let $\widetilde{U} = U_{[1,n]}$, $\widetilde{U}^\circ = U_{[n+1,2n]}$ and $H^\flat = H$ in case I. In case II, let $\widetilde{U} = U_{[1,n-1]}$, $\widetilde{U}^\circ = U_{[n+1,2n-1]}$ and $H^\flat = \langle U_0, U_n, H\rangle$ if $w_n \in \Psi$, and let $\widetilde{U} = U_{[2,n]}$, $\widetilde{U}^\circ = U_{[n+2,2n]}$ and $H^\flat = \langle U_1, U_{n+1}, H\rangle$ if $w_1 \in \Psi$.

Proposition A.10. *Let $G^\dagger$ and $(\gamma, i \mapsto w_i)$ be as in Notation A.1, let Ψ, $\equiv$ and U_x for $x \in \Psi$ be as in Notation A.2 and Definition A.4, and let $\widetilde{U}$, $\widetilde{U}^\circ$ and $H^\flat$ be as Notation A.9. Then the following hold:*

(i) $G^\dagger = \langle \widetilde{U}, \widetilde{U}^\circ\rangle$.

(ii) *$H^\flat$ is the two-point stabilizer of $\{w_0, w_1\}$ and $\{w_n, w_{n+1}\}$, respectively w_i and w_{n+i} for $i = 0$ or 1.*

(iii) *The 4-tuple $(G, \Psi, \{\equiv\}, \{U_x\}_{x\in\Psi})$ is a Tits set as defined in* [9, Def. 3.1].

(iv) $G = \widetilde{U}^\circ \widetilde{U} \widetilde{U}^\circ H^\flat$.

Proof. Both U_i and U_{i+n} lie in $\langle \widetilde{U}, \widetilde{U}^\circ \rangle$ for $i = 2, 3$. By [11, Prop. 1.4.4], therefore, $\mu_\gamma(U_i^\sharp) \subset \langle \widetilde{U}, \widetilde{U}^\circ \rangle$ for $i = 2, 3$. By [11, (1.4.6)], $\langle \mu_\gamma(U_2^\sharp), \mu_\gamma(U_3^\sharp) \rangle$ acts by conjugation transitively on the set $\{U_i \mid i \in \mathbb{Z}\}$. Thus, $U_i \subset \langle \widetilde{U}, \widetilde{U}^\circ \rangle$ for all i. By [11, Prop. 1.3.39], we conclude that

$$G^\dagger = \langle \widetilde{U}, \widetilde{U}^\circ \rangle.$$

Thus, (i) holds.

By [11, Cor. 1.3.14 and Rem. 1.3.40], we have

$$G = G^\dagger H.$$

By [11, Prop. 1.3.17], H is the two-point stabilizer of the edges $\{w_n, w_{n+1}\}$ and $\{w_0, w_1\}$ in G. By [11, Props. 1.3.5 and 1.3.18], $\langle U_0, U_n \rangle$ acts transitively on the set of apartments containing w_0 and w_n and hence $\langle U_0, U_n, H \rangle$ is the two-point stabilizer of the vertices w_0 and w_n in G. Similarly, $\langle U_1, U_{n+1}, H \rangle$ is the two-point stabilizer of the vertices w_1 and w_{n+1} in G. Thus, (ii) holds.

By (A.5), and Propositions A.7 and A.8, (iii) holds. By (ii), (iii) and [9, Prop. 3.6], it follows that (iv) holds. ∎

Corollary A.11. *Let $n = 4$, $\widetilde{U} = U_{[0,2]}$, $\widetilde{U}^\circ = U_{[4,6]}$, $H^\dagger = H \cap G^\dagger$ and $\widetilde{H} = \langle U_1, U_5, H^\dagger \rangle$. Then $G^\dagger = \langle \widetilde{U}, \widetilde{U}^\circ \rangle$ and*

$$G^\dagger = \widetilde{U}^\circ \widetilde{U} \widetilde{U}^\circ \widetilde{H}.$$

Proof. Let Ψ_1 denote the $G^\dagger$-orbit in the vertex set of Γ containing w_1. Setting $n = 4$ and $\Psi = \Psi_1$ in Notation A.9 and Proposition A.10, we obtain

$$G^\dagger = \langle \widetilde{U}, \widetilde{U}^\circ \rangle \quad \text{and} \quad G = \widetilde{U}^\circ \widetilde{U} \widetilde{U}^\circ H^\flat,$$

where $H^\flat = \langle U_1, U_5, H \rangle$. Since $H^\flat \cap G^\dagger = \widetilde{H}$, we conclude that $G^\dagger = \widetilde{U}^\circ \widetilde{U} \widetilde{U}^\circ \widetilde{H}$. ∎

Funding. The work of Bernhard Mühlherr was partially supported by DFG Grant MU1281/7-1, and the work of Richard M. Weiss was partially supported by Simons Foundation Collaboration Grant 516364.

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Received 4 May 2023.

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