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Heteroclinic traveling waves of two-dimensional parabolic Allen–Cahn systems

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Abstract. In this paper we show the existence of traveling waves $w: [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^k$ ($k \geq 2$) for the parabolic Allen–Cahn system $\partial_t w - \Delta w = -\nabla_u V(w)$ in $[0, +\infty) \times \mathbb{R}^2$, satisfying some *heteroclinic* conditions at infinity. The potential V is a nonnegative and smooth multi-well potential, which means that its null set is finite and contains at least two elements. The traveling wave w propagates along the horizontal axis according to a speed $c^\star > 0$ and a profile $\mathfrak{U}$. The profile $\mathfrak{U}$ joins as $x_1 \to \pm\infty$ (in a suitable sense) two locally minimizing one-dimensional heteroclinics which have *different* energies, and the speed $c^\star$ satisfies certain uniqueness properties. The proof is variational and, in particular, it requires the assumption of an upper bound, depending on V, on the difference between the energies of the one-dimensional heteroclinics.

Contents

1. Introduction

Consider the parabolic system of equations

$$\partial_t w - \Delta w = -\nabla_u V(w) \quad \text{in } [0, +\infty) \times \mathbb{R}^2, \tag{1.1}$$

where $V: \mathbb{R}^k \to \mathbb{R}$ is a smooth, nonnegative, multi-well potential (see assumptions (H1), (H2), (H3) later) and $w: [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^k$, with $k \geq 2$. We seek a *traveling wave* solution to (1.1). That is, we impose on w,

$$\forall (t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^2, \quad w(t, x_1, x_2) = \mathfrak{U}(x_1 - c^\star t, x_2),$$

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where $\mathfrak{U}\colon \mathbb{R}^2 \to \mathbb{R}^k$ is the *profile* of the wave and $c^\star > 0$ is the *speed* of propagation of the wave, which occurs in the x_1-direction. The profile and the speed are the unknowns of the problem. Replacing in (1.1), we find that the profile $\mathfrak{U}$ and $c^\star$ must satisfy the elliptic system

$$-c^\star \partial_{x_1} \mathfrak{U} - \Delta \mathfrak{U} = -\nabla_u V(\mathfrak{U}) \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

The system (1.1) can be seen as a reaction–diffusion system. The early works, motivated by questions from population dynamics, of Fisher [23] and Kolmogorov, Petrovsky and Piskunov [28], were devoted to a scalar reaction–diffusion equation in one space dimension known today as the Fisher–KPP equation. Traveling and stationary waves are now known to play a major role in the dynamics of reaction–diffusion problems: for instance, Fife and McLeod [21, 22] proved stability results for the equations considered in [23, 28]. Regarding higher-dimensional problems (but always in the scalar case), existence results for traveling waves were obtained by Aronson and Weinberger [8] for equations with $\mathbb{R}^N$ as space domain and by Berestycki, Larrouturou and Lions [9], Berestycki and Nirenberg [10] for unbounded cylinders of the type $\mathbb{R} \times \omega$, with $\omega \subset \mathbb{R}^{N-1}$ a bounded domain. We also mention that asymptotic stability results (for a suitable class of perturbations) for traveling waves in the scalar Allen–Cahn equation in $\mathbb{R}^N$ were obtained by Matano, Nara and Taniguchi [31].

All the papers mentioned above are devoted to scalar equations and they rely on the application of the maximum principle and its related tools. As is well known, the maximum principle does not apply in general to systems of equations, meaning that other techniques are needed in order to study the existence of traveling waves (and their properties in case they exist) for systems. Different, more general, approaches had been taken in order to circumvent the lack of the maximum principle when dealing with parabolic systems. We refer to the books by Smoller [47] and Volpert, Volpert and Volpert [49]. One of these approaches consists of the use of variational methods. In the context of reaction–diffusion equations, this approach seems to appear for the first time in Heinze's Ph.D. thesis [26] (even though the existence of a variational framework for some classes of reaction diffusion problems has been known since [21, 22]) and subsequently carried on by Muratov [35], Lucia, Muratov and Novaga [30], Alikakos and Katzourakis [6] (see also Alikakos, Fusco and Smyrnelis [5]), Risler [42–44] and, more recently, by Chen, Chien and Huang [20]. In the latter, the authors consider a parabolic Allen–Cahn system in a two-dimensional strip $\mathbb{R} \times (-l, l)$ and find traveling waves which join a well and an approximation of a heteroclinic orbit in $(-l, l)$, for a class of symmetric triple-well potentials. In a spirit which is related to this paper, traveling waves which connect lower-dimensional equilibria have been found (also by variational methods) by Bertsch, Muratov and Primi [12–14] in the context of the three-dimensional harmonic heat flow on an infinite cylinder, as well as Muratov and Shvartsman [36] in another setting connected with the modeling of cellular dynamics. Lastly, we mention that variational methods have also been applied to scalar reaction–diffusion equations; see for instance Bouhours and Nadin [18] for the case of heterogeneous equations, as well as Lucia, Muratov and Novaga [29], Muratov and Shvartsman [36]. These methods have also been applied to the damped wave equation (a type of hyperbolic equation) by Gallay and Joly [25], Luo [19] and Risler [45]. In this paper we will also take a variational approach for dealing with the following question:

> *Question:* Assuming that there exist two *heteroclinic orbits*, joining two fixed wells, with *different* energy (defined in (2.1)) levels, does there exist a solution $(c, \mathfrak{U})$ to (1.2) such that $\mathfrak{U}$ *joins* the two heteroclinic orbits at infinity, in the x_1-direction and uniformly in x_2?

Heteroclinic orbits are curves $\mathfrak{q}\colon \mathbb{R} \to \mathbb{R}^k$ which solve the equation

$$\mathfrak{q}'' = \nabla_u V(\mathfrak{q}) \quad \text{in } \mathbb{R},$$

and join two *different* wells of Σ at $\pm\infty$. Moreover, one asks that the *one-dimensional energy* (i.e. the functional associated with the previous equation; see (2.1)) is finite. We show that, under suitable assumptions, the question we posed has an affirmative answer. Our motivation comes from two different sides:

(1) Stationary heteroclinic-type solutions of (1.1) have been known to exist in several situations for a long time. Indeed, for a class of symmetric potentials, Alama, Bronsard and Gui [2] showed the existence of a stationary wave (that is, a solution to (1.2) with $c = 0$) in the situation such that two heteroclinics with *equal* energy levels exist and are global minimizers of the one-dimensional energy. Their analysis was later extended to potentials without symmetry in several papers, which in some cases obtained similar results by means of different techniques. See Fusco [24], Monteil and Santambrogio [34] (an extension of the previous work by the same authors [33] for the finite-dimensional problem), Schatzman [46], Smyrnelis [48]. A key observation is that this problem can be seen as a heteroclinic orbit problem for a potential (the one-dimensional energy, see (2.1)) defined in the infinite-dimensional space $L^2(\mathbb{R}, \mathbb{R}^k)$. Therefore, it is natural to aim at solving a connecting orbit problem for potentials defined in, say, Hilbert spaces and then deduce the original problem as a particular case. This is the approach taken in [34] (in the metric space setting) and in [48] (in the Hilbert space setting).

(2) Alikakos and Katzourakis [6] showed the existence of traveling waves for a class of one-dimensional parabolic systems of gradient type. Essentially, they assume that the potential possesses two local minima (one of them global) at *different* levels. Hence, their potential is not of multi-well type in general. The profile of the traveling waves connects the two local minima at infinity and the determination of the speed becomes part of the problem.

The results of this paper follow by suitably merging the ideas of the previous items. More precisely, we formulate and provide solutions for a *heteroclinic traveling problem* as that in [6] for potentials defined in an abstract Hilbert space. Then, as a particular case, we recover the existence of a traveling wave solution for (1.1) with heteroclinic behavior at infinity.

2. The main results: Statements and discussions

We now state the results of this paper. In Theorem 1, which is the main result, existence of a traveling wave solution with speed $c^\star$ and profile $\mathfrak{U}$ is established, as well as the uniqueness (in some sense) of $c^\star$ and the L^2 exponential convergence of $\mathfrak{U}$ at the limits $x_1 \to \pm\infty$. To prove such a result, the key assumption is (H6). With the assumptions we make in Theorem 1 we are not able to prove that the analogous exponential convergence for $\mathfrak{U}$ holds as $x_1 \to -\infty$. In Theorem 2, we show that under the previous assumptions we also have uniform convergence of the solution in the x_1- and the x_2-directions. In Theorem 3 we give some properties regarding the speed parameter $c^\star$. We conclude this section by describing the outline and main ideas of our proofs (Section 2.6), which are located in Sections 4 and 5.

2.1. Basic assumptions and definitions

Before stating the results, we recall some standard assumptions, definitions and results and we introduce some notation. The multi-well potentials V considered in this paper satisfy the following:

(H1) Let $V \in \mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^k)$ and $V \geq 0$ in $\mathbb{R}^k$. Moreover, $V(u) = 0$ if and only if $u \in \Sigma$, where, for some $l \geq 2$,

$$\Sigma := \{\sigma_1, \dots, \sigma_l\}.$$

(H2) There exist $\alpha_0, R_0 > 0$ such that for all $u \in \mathbb{R}^k$ with $|u| \geq R_0$ it holds that $\langle \nabla_u V(u), u\rangle \geq \alpha_0 |u|^2$ and, as a consequence, there exists $\beta_0 > 0$ such that $V(u) \geq \beta_0$ for all such u.

(H3) For all $\sigma \in \Sigma$, the matrix $D^2 V(\sigma)$ is positive definite.

One considers the one-dimensional energy functional

$$E(q) := \int_{\mathbb{R}} e(q)(t)\,dt := \int_{\mathbb{R}} \left[\frac{1}{2}|q'(t)|^2 + V(q(t))\right] dt, \quad q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k). \tag{2.1}$$

Given a pair of wells $(\sigma^-, \sigma^+) \in \Sigma^2$, as done for instance in Rabinowitz [41], we define

$$X(\sigma^-, \sigma^+) := \{q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty \text{ and } \lim_{t\to\pm\infty} q(t) = \sigma^{\pm}\},$$

the set of curves in $\mathbb{R}^k$ connecting σ^- and σ^+. The space $X(\sigma^-, \sigma^+)$ is a metric space when it is endowed with the L^2 and the H^1 distances. Indeed, by assumption (H3) it readily follows that if q belongs to $X(\sigma^-, \sigma^+)$ then $q - \sigma^+$ belongs to $L^2([0, +\infty), \mathbb{R}^k)$

and $q-\sigma^-$ belongs to $L^2((-\infty,0],\mathbb{R}^k)$. As a consequence, $q-\tilde{q}\in H^1(\mathbb{R},\mathbb{R}^k)$ whenever q and $\tilde{q}$ belong to $X(\sigma^-,\sigma^+)$. If $\mathfrak{q}$ is a critical point of the energy E in $X(\sigma^-,\sigma^+)$, we say that $\mathfrak{q}$ is a *homoclinic orbit* when $\sigma=\sigma'$ and that $\mathfrak{q}$ is a *heteroclinic orbit* when $\sigma^-\neq\sigma^+$. Define as well the corresponding infimum value

$$\mathfrak{m}_{\sigma^-\sigma^+} := \inf\{E(q) : q\in X(\sigma^-,\sigma^+)\}.$$

If σ^- and σ^+ are two distinct wells in Σ, it turns out that $\mathfrak{m}_{\sigma^-\sigma^+}$ is not attained in general. We need to add the following assumption:

(H4) We have

$$\forall\sigma\in\Sigma\setminus\{\sigma^-,\sigma^+\},\quad \mathfrak{m}_{\sigma^-\sigma^+}<\mathfrak{m}_{\sigma^-\sigma}+\mathfrak{m}_{\sigma\sigma^+}.$$

Notice that one can always find a pair $(\sigma^-,\sigma^+)\in\Sigma^2$ such that (H4) holds. Assuming that (H1), (H2), (H3) and (H4) hold, it is well known that there exists a minimizer of E in $X(\sigma^-,\sigma^+)$. Moreover, we have the compactness of minimizing sequences as follows: for any $(q_n)_{n\in\mathbb{N}}$ in $X(\sigma^-,\sigma^+)$ such that $E(q_n)\to\mathfrak{m}_{\sigma^-\sigma^+}$, there exists $(\tau_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $\mathfrak{q}\in X(\sigma^-,\sigma^+)$ such that $E(\mathfrak{q})=\mathfrak{m}_{\sigma^-\sigma^+}$ and, up to subsequences,

$$\|q_n(\cdot+\tau_n)-\mathfrak{q}\|_{H^1(\mathbb{R},\mathbb{R}^k)}\to 0 \quad \text{as } n\to+\infty. \tag{2.2}$$

This result is well known. The earlier references are Bolotin [16], Bolotin and Kozlov [17], Bertotti and Montecchiari [11] and Rabinowitz [39, 40], sometimes in a slightly different setting. Proofs and applications of the compactness property (2.2) are also given in Alama, Bronsard and Gui [2], Alama et al. [1] and Schatzman [46].

We fix the two wells σ^- and σ^+ for the rest of the paper, as well as $\mathfrak{m}:=\mathfrak{m}_{\sigma^-\sigma^+}$. According to the previous discussion, we have that the set

$$\mathcal{F} := \{\mathfrak{q} : \mathfrak{q}\in X(\sigma^-,\sigma^+) \text{ and } E(\mathfrak{q})=\mathfrak{m}\}, \tag{2.3}$$

is not empty. We term the elements of $\mathcal{F}$ as *globally minimizing heteroclinics* between σ^- and σ^+. The term *heteroclinics* comes from the fact that σ^- and σ^+ are different. An important fact is that, due to the translation invariance of E and $X(\sigma^-,\sigma^+)$, we have that if $\mathfrak{q}\in\mathcal{F}$, then for all $\tau\in\mathbb{R}$ it holds that $\mathfrak{q}(\cdot+\tau)\in\mathcal{F}$.

2.2. Existence

Assumptions (H1), (H2), (H3) and (H4) stated before are classical. In order to obtain our results, we will supplement them with the following one, which is more specific to the setting of this paper:

(H5) Assume that (H1), (H2), (H3) and (H4) hold for the potential V. We keep the previous notation. We assume the following:

(1) It holds that $\mathcal{F}=\{\mathfrak{q}^-(\cdot+\tau):\tau\in\mathbb{R}\}$ for some $\mathfrak{q}^-\in X(\sigma^-,\sigma^+)$, where $\mathcal{F}$ was defined in (2.3). We set $\mathcal{F}^-:=\mathcal{F}$ and $\mathfrak{m}^-:=\mathfrak{m}$.

(2) There exist $\mathfrak{m}^+ > \mathfrak{m}^-$ and $\mathfrak{q}^+ \in X(\sigma^-, \sigma^+)$ such that $E(\mathfrak{q}^+) = \mathfrak{m}^+$ and $\mathfrak{q}^+$ is a local minimizer of E with respect to the H^1-norm. We denote $\mathcal{F}^+ := \{\mathfrak{q}^+(\cdot + \tau) : \tau \in \mathbb{R}\}$.

(3) We have the spectral nondegeneracy assumption as introduced in [46]: For all $q \in X(\sigma^-, \sigma^+)$, let $A(q)$ be the unbounded linear operator in $L^2(\mathbb{R}, \mathbb{R}^k)$ with domain $H^2(\mathbb{R}, \mathbb{R}^k)$ defined as

$$A(q) \colon v \to -v'' + D^2 V(q) v.$$

Then it holds that for any $\mathfrak{q} \in \mathcal{F}^- \cup \mathcal{F}^+$ we have $\mathrm{Ker}(A(\mathfrak{q}))$ is generated by $\mathfrak{q}'$. The fact that $\mathfrak{q}' \in H^2(\mathbb{R}, \mathbb{R}^k)$ follows from the identity $\mathfrak{q}''' = D^2 V(\mathfrak{q})\mathfrak{q}'$.

Notice that if we had $\mathfrak{m}^+ = \mathfrak{m}^-$ we would be in the framework of Alama, Bronsard and Gui [2], for which the two-dimensional solution connecting $\mathfrak{q}^-$ and $\mathfrak{q}^+$ is stationary. Essentially, conditions (1) and (2) in (H5) imply that $\mathfrak{q}^-$ is a globally minimizing heteroclinic and $\mathfrak{q}^+$ is a locally (but not globally) minimizing heteroclinic.

Regarding assumption (3), notice that if $q \in X(\sigma^-, \sigma^+)$ is a critical point of E, then we have for all $h \in \mathbb{R}$ and $v \in H^1(\mathbb{R}, \mathbb{R}^k)$,

$$E(q + hv) = E(q) + \frac{h^2}{2} D^2 E(q)(v, v) + o_{h \to 0}(h^2),$$

where

$$D^2 E(q)(v, v) := \int_{\mathbb{R}} \Big[\frac{|v'(t)|^2}{2} + \langle D^2 V(q(t)) v(t), v(t) \rangle \Big] \, dt = \int_{\mathbb{R}} \langle A(q) v(t), v(t) \rangle \, dt.$$

Hence, $A(q)$ is the self-adjoint operator associated to the second variation of E at q. Moreover, if q is a local minimizer of E (as $\mathfrak{q}^-$ and $\mathfrak{q}^+$ are), then $A(q)$ is also nonnegative. One readily checks that q' always belongs to $\mathrm{Ker}(A(q))$. This is an intrinsic degeneracy due to the invariance by translations of the functional E. Assumption (3) is made so that this is the only source of degeneracy or, in order words, so that $\mathfrak{q}^-$ and $\mathfrak{q}^+$ are nondegenerate critical points of E *up to translations*. In [46, Theorem 4.3] it is shown that it is a *generic* assumption in the following sense: given a potential satisfying (H1), (H2), (H3), (H4) and (1) and (2) in (H5), it is possible to find an arbitrarily small perturbation of it so that (3) also holds, and that without modifying the data Σ, σ^-, σ^+, $\mathfrak{q}^-$ and $\mathfrak{q}^+$. Actually, [46, Theorem 4.3] is stated only for global minimizers of the energy, but an inspection of the proof reveals that only the fact that $A(\mathfrak{q}^\pm)$ is nonnegative is used.

The most important consequence of (H5), as proven in [46], is the existence of two constants $\rho_0^+ > 0$ and $\rho_0^- > 0$ such that for all $q \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$,

$$\begin{aligned} &\mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm \\ &\quad \Rightarrow \exists ! \tau^\pm(q) \in \mathbb{R} \colon \|q - \mathfrak{q}^\pm(\cdot + \tau^\pm(q))\|_{L^2(\mathbb{R},\mathbb{R}^k)} = \mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm), \end{aligned} \tag{2.4}$$

and for some constant $\beta^\pm$ we have, for all $q \in X(\sigma^-, \sigma^+)$,

$$\mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm \Rightarrow \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm)^2 \leq \beta^\pm (E(q) - \mathfrak{m}^\pm). \tag{2.5}$$

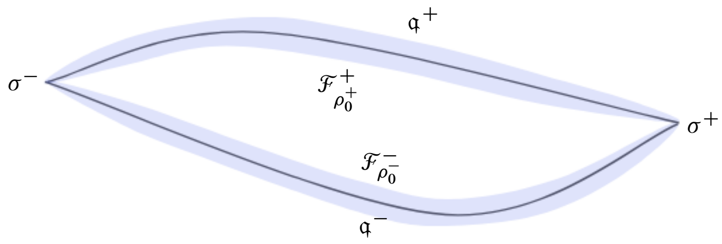

Figure 1. The situation described by (H5). The curves correspond to the traces of $\mathfrak{q}^-$ and $\mathfrak{q}^+$ as indicated. The shadowed regions correspond to the traces of the functions in $\mathcal{F}^-_{\rho_0^-}$ and $\mathcal{F}^+_{\rho_0^+}$, which are a neighborhood of $\mathcal{F}^-$ and $\mathcal{F}^+$ respectively.

Notice that (2.4) and (2.5) state that the energy is *quadratic* around $\mathcal{F}^-$ and $\mathcal{F}^+$, up to the degeneracy associated to the invariance by translations. For the global minimizer $\mathfrak{q}^-$, the identities (2.4) and 2.5 are proven in [46]; more precisely (2.4) is a particular case of [46, Lemma 2.1] and (2.5) is a consequence of [46, Lemma 4.5] (which uses the spectral assumption (3) in (H5)), which states that, for all $q \in X(\sigma^-, \sigma^+)$,

$$\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^-) \le \tilde{\rho}_0^- \Rightarrow \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^-)^2 \le \tilde{\beta}^-(E(q) - \mathfrak{m}^\pm) \tag{2.6}$$

for some $\tilde{\beta}^- > 0$ and $\tilde{\rho}_0^- > 0$. By a contradiction argument, one proves that (2.5) follows by (2.6), also using compactness of minimizing sequences. Actually, property (2.4), proven in [46, Lemma 2.1], does not require the curve to be a global minimizer and it works for any curve q_0 with second derivative in $L^2(\mathbb{R}, \mathbb{R}^k)$, as this allows us to compute the second variation of

$$\tau \in \mathbb{R} \to \|q - q_0(\cdot + \tau)\|^2_{L^2(\mathbb{R},\mathbb{R}^k)} \in \mathbb{R}.$$

This shows that one can find $\rho_0^+ > 0$ so that (2.4) holds for $\mathfrak{q}^+$. Moreover, one can obtain an analogous property for the $H^1(\mathbb{R}, \mathbb{R}^k)$, as done in [46, Lemma 2.1], using that $(\mathfrak{q}^+)''' = D^2V(\mathfrak{q}^+)[(\mathfrak{q}^+)'] \in L^2(\mathbb{R}, \mathbb{R}^k)$, and then deduce suitable upper bounds in the optimal parameter. Lemma 4.5 in [46] combines the previous fact with the spectral assumption (H5) (3), and the fact that the curve is a global minimizer is not used. Hence, up to decreasing the parameter ρ_0^+, one obtains (2.5) for $\mathfrak{q}^+$.

We will define for $r > 0$ the sets

$$\mathcal{F}_r^\pm := \{q \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k) : \mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \le r\}. \tag{2.7}$$

Notice that $\mathcal{F}^-$ and $\mathcal{F}^+$ are at positive L^2 distance. Indeed, we have that $\|\mathfrak{q}^-(\cdot + \tau) - \mathfrak{q}^+\|_{L^2(\mathbb{R},\mathbb{R}^k)}$ tends to $+\infty$ as $|\tau| \to +\infty$, which by continuity means that the infimum is attained, which is hence positive as $\mathfrak{q}^+$ and $\mathfrak{q}^-$ are different up to translations. Therefore, up to decreasing ρ_0^+ and ρ_0^- we can (and will) assume that

$$\mathcal{F}^+_{\rho_0^+} \cap \mathcal{F}^-_{\rho_0^-} = \emptyset.$$

See Figure 1 for an explanatory drawing of (H5). Let us now assume that $\mathfrak{m}^+ - \mathfrak{m}^-$ is bounded above as follows:

(H6) Assume that (H5) holds and, moreover, for the constants $\rho_0{}^\pm$ satisfying (2.4) and (2.5) it holds that

$$0 < \mathfrak{m}^+ - \mathfrak{m}^- < E_{\max},$$

where $E_{\max}$ will be defined later in (2.21). Moreover, assume that

$$\left\{q \in X(\sigma^-, \sigma^+) : E(q) < \mathfrak{m}^+\right\} \subset \mathcal{F}_{\rho_0^-/2}, \tag{2.8}$$

with $\mathcal{F}_{\rho_0^-/2}$ as in (2.7). Furthermore,

$$\mathfrak{m}^+ - \mathfrak{m}^- < \frac{(\mu^- \mathfrak{d}_0)^2}{2},$$

where the constants $\mathfrak{d}_0$ and μ^- are defined later in (2.19) and (2.20) respectively.

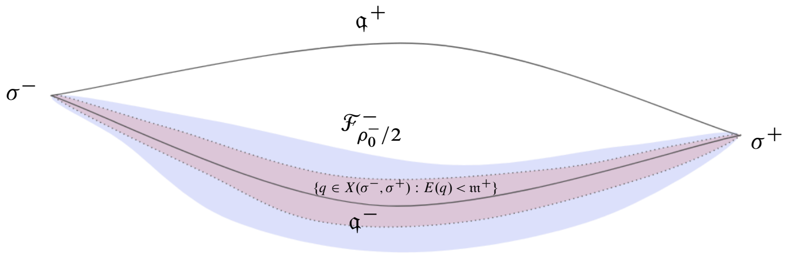

Figure 2. Representation of (H6). While the larger shadowed region corresponds to $\mathcal{F}_{\rho_0^-/2}$, the smaller one that is contained inside represents the set $\{q \in X(\sigma^-, \sigma^+) : E(q) < \mathfrak{m}^+\}$. Moreover, the value $\mathfrak{m}^+ - \mathfrak{m}^-$ must be smaller than $E_{\max}$, defined in (2.21).

See also Figure 2. Essentially, (H6) requires that $\mathfrak{m}^- - \mathfrak{m}^+$ is not too large and the bound is given by a constant $E_{\max}$ that can be computed through the constants produced in (2.4) and (2.5) as a consequence of (H5). Moreover, it also requires that whenever $q \in X(\sigma^-, \sigma^+)$ is such that $E(q) < \mathfrak{m}^+$, then q is close to $\mathfrak{q}^-$ (more precisely, $\mathfrak{q}^- \in \mathcal{F}_{\rho^-/2}$). One should notice that while the constants $\rho_0^\pm$ could be taken arbitrarily small in order to fulfill (2.4) and (2.5), assumption (H6) imposes an upper bound on these values as the inclusion (2.8) must hold. Therefore, the assumption can be not so obvious to check in applications, but it is possible to obtain examples by performing suitable perturbations on multi-well potentials with several globally minimizing heteroclinics; see Section 6 for more details. If (H6) holds, then we are able to answer the question that we posed at the beginning of the paper in a positive way. More precisely, recall the equation of the profile,

$$-c\partial_{x_1}\mathfrak{U} - \Delta\mathfrak{U} = -\nabla_u V(\mathfrak{U}) \quad \text{in } \mathbb{R}^2, \tag{2.9}$$

and consider the conditions at infinity,

$$\exists L^- \in \mathbb{R},\ \forall x_1 \leq L^-, \quad \mathfrak{U}(x_1,\cdot) \in \mathcal{F}^-_{\rho_0^-/2}, \tag{2.10}$$

$$\exists L^+ \in \mathbb{R},\ \forall x_1 \geq L^+, \quad \mathfrak{U}(x_1,\cdot) \in \mathcal{F}^+_{\rho_0^+/2}. \tag{2.11}$$

Our proof is variational, which implies that the profile $\mathfrak{U}$ can be characterized as a critical point of a functional. The variational framework is as follows: assume that (H6) holds and set

$$S := \{U \in H^1_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R},\mathbb{R}^k)) : \exists L \geq 1,\ \forall x_1 \geq L,\ U(x_1,\cdot) \in \mathcal{F}^+_{\rho_0^+/2}$$
$$\forall x_1 \leq -L,\ U(x_1,\cdot) \in \mathcal{F}^-_{\rho_0^-/2}\}.$$

For $U \in S$ and $c > 0$ we define the energy

$$E_{2,c}(U) := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\partial_{x_1} U(x_1,x_2)|^2}{2}\, dx_2 + \left(E(U(x_1,\cdot)) - \mathfrak{m}^+\right)\right) e^{cx_1}\, dx_1.$$

Formally, critical points of $E_{2,c}$ give rise to solutions of (2.9). If $U \in S$, we can define the translated function $U^\tau := U(\cdot + \tau, \cdot)$ for $\tau \in \mathbb{R}$. Then, for all $c > 0$, we have

$$E_{2,c}(U^\tau) = e^{-c\tau} E_{2,c}(U),$$

which implies that

$$\forall c > 0, \quad \inf_{U \in S} E_{2,c}(U) \in \{-\infty, 0\}.$$

We have now introduced the notation which allows us to state the main result of this paper:

Theorem 1 (Main theorem). *Assume that (H1), (H2), (H3) (H4), (H5) and (H6) hold. Then we have the following properties:*

(1) Existence. *There exist $c^\star > 0$ and $\mathfrak{U} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2,\mathbb{R}^k) \cap S$, $\alpha \in (0,1)$, which fulfill (2.9). The profile $\mathfrak{U}$ satisfies the conditions at infinity (2.10) and (2.11) as well as the variational characterization*

$$E_{2,c^\star}(\mathfrak{U}) = 0 = \inf_{U \in S} E_{2,c^\star}(U). \tag{2.12}$$

(2) Uniqueness of the speed. *The speed $c^\star$ is unique in the following sense: Assume that $\overline{c^\star} > 0$ is such that*

$$\inf_{U \in S} E_{2,\overline{c^\star}}(U) = 0$$

and that $\overline{\mathfrak{U}} \in S$ is such that $(\overline{c^\star}, \overline{\mathfrak{U}})$ solves (2.9) and $E_{2,\overline{c^\star}}(\overline{\mathfrak{U}}) < +\infty$. Then $\overline{c^\star} = c^\star$.

(3) Exponential convergence. *The convergence of $\mathfrak{U}$ at $+\infty$ is exponential with respect to the L^2-norm. More precisely, there exist $\mathfrak{M}^+ > 0$ and $\tau^+ \in \mathbb{R}$ such that for all $x_1 \in \mathbb{R}$,*

$$\|\mathfrak{U}(x_1,\cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{L^2(\mathbb{R},\mathbb{R}^k)} \leq \mathfrak{M}^+ e^{-c^\star x_1}. \tag{2.13}$$

Moreover, it holds that $c^\star < \mu^-$, μ^- *to be defined later in* (2.20)*, and there exists* $\mathfrak{M}^- > 0$ *such that for all* $x_1 \in \mathbb{R}$*,*

$$\|\mathfrak{U}(x_1,\cdot) - \mathfrak{q}^-(\cdot + \tau^-)\|_{L^2(\mathbb{R},\mathbb{R}^k)} \leq \mathfrak{M}^- e^{(\mu^- - c^\star)x_1}. \tag{2.14}$$

Remark 2.1. The existence part of Theorem 1 states that there exists a solution $(c^\star, \mathfrak{U})$ such that $\mathfrak{U}$ is a global minimizer of $E_{c^\star}$ in S. We also have that the speed $c^\star$ is unique for some class of solutions, namely for finite energy solutions and speeds for which the corresponding energy is bounded below in S. In particular, $c^\star$ is unique among the class of globally minimizing profiles. In other words, if $c > 0$ is such that the infimum of E_c in S is attained, then $c = c^\star$. This is analogous to what was shown in Alikakos and Katzourakis [6]. As explained in the introduction, the main drawback of our approach is the existence assumption (H6). In particular, the definition of the upper bound $E_{\max}$ is technical and it is possible that in several situations it could be small. Nevertheless, in Section 6 we show that there exist examples of potentials for which (H6) holds.

2.3. Improving the convergence at infinity

The natural question is whether the convergence properties (2.13) and (2.14) in Theorem 1 can be improved, and in particular, whether the L^2-norm can be replaced by the H^1-norm. We believe that the answer to this question is positive, but we do not have a proof of this fact. However, as one can check in Smyrnelis [48] and Fusco [24], such a fact holds for the balanced two-dimensional heteroclinic solution. They obtain these properties by combining standard elliptic estimates with some properties which are intrinsic to minimal solutions of the elliptic system (1.2) with $c^\star = 0$. See the results in Alikakos, Fusco and Smyrnelis [5, Section 4], mainly based on Alikakos and Fusco [4]. The main obstacle is that even if one was able to extend their analysis to the case $c^\star > 0$, a crucial hypothesis of their results is that solutions are minimal with respect to compactly supported perturbations, a property which (at least when one does not limit the size of these perturbations) does not hold for solutions given by Theorem 1 due to the fact that $\mathfrak{q}^+$ is not a global minimizer of E. Therefore, we leave this question open. Nevertheless, besides the L^2-convergence rates (2.13) and (2.14), we can prove uniform convergence in both the x_1- and the x_2-directions:

Theorem 2. *Assume that* (H1)*,* (H2)*,* (H3) (H4)*,* (H5) *and* (H6) *hold. Let* $(c^\star, \mathfrak{U})$ *be a solution given by Theorem* 1*. Then we have*

$$\lim_{x_1 \to \pm\infty} \|\mathfrak{U}(x_1,\cdot) - \mathfrak{q}^\pm(\cdot + \tau^\pm)\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} = 0 \tag{2.15}$$

and

$$\lim_{x_2 \to \pm\infty} \|\mathfrak{U}(\cdot, x_2) - \sigma^\pm\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} = 0. \tag{2.16}$$

2.4. Min-max characterization of the speed

We provide here a min-max characterization of the speed $c^\star$ and other related properties which are summarized in Theorem 3. The idea of providing a variational characterization for the speed of traveling waves in reaction–diffusion systems can be traced back to Heinze [26] and Heinze, Papanicolau and Stevens [27], and it was used later in several other papers [6, 18, 29, 30, 35].

Theorem 3. *Assume that (H1), (H2), (H3) (H4), (H5) and (H6) hold. Let* $(c^\star, \mathfrak{U})$ *be a solution given by Theorem 1. Then for any* $\widetilde{\mathfrak{U}} \in S$ *such that*

$$E_{2,c^\star}(\widetilde{\mathfrak{U}}) = 0,$$

we have that $(c^\star, \widetilde{\mathfrak{U}})$ *solves (2.9) and*

$$c^\star = \frac{\mathfrak{m}^+ - \mathfrak{m}^-}{\int_{\mathbb{R}^2} |\partial_{x_1} \widetilde{\mathfrak{U}}(x_1, x_2)|^2 \, dx_2 \, dx_1}. \tag{2.17}$$

In particular, the quantity $\int_{\mathbb{R}^2} |\partial_{x_1} \widetilde{\mathfrak{U}}(x_1, x_2)|^2 \, dx_2 \, dx_1$ *is well defined and constant among the set of minimizers of* $E_{2,c}$ *in* S*. Moreover, it holds that*

$$c^\star = \sup\left\{c > 0 : \inf_{U \in S} E_{2,c}(U) = -\infty\right\} = \inf\left\{c > 0 : \inf_{U \in S} E_{2,c}(U) = 0\right\} \tag{2.18}$$

and we have the bound

$$c^\star \le \frac{\sqrt{2(\mathfrak{m}^+ - \mathfrak{m}^-)}}{\mathfrak{d}_0} < \min\left\{\frac{\sqrt{2E_{\max}}}{\mathfrak{d}_0}, \mu^-\right\},$$

where $\mathfrak{d}_0$, μ^- *and* $E_{\max}$ *will be defined later in (2.19), (2.20) and (2.21) respectively, and the second inequality follows from the bound on* $\mathfrak{m}^+ - \mathfrak{m}^-$ *given by (H6).*

Remark 2.2. Notice that the conditions at infinity imply that any $U \in S$ is such that

$$\int_{\mathbb{R}^2} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} \, dx_2 \, dx_1 > 0.$$

As can be seen, Theorem 3 shows that the speed $c^\star$ is characterized by the explicit formula (2.17), which nevertheless requires knowledge about a profile $\widetilde{\mathfrak{U}}$. However, one also has the variational characterization (2.18), which does not involve any information on the profiles. Indeed, one only needs to be able to compute the infimum of the energies with $c > 0$ as a parameter. Moreover, notice that combining (2.18) with the uniqueness part of Theorem 1, we obtain a stronger assertion: if $\bar{c} > c^\star$ and $(\bar{c}, \overline{\mathfrak{U}})$, with $\overline{\mathfrak{U}} \in S$, solves (2.9), then $E_{2,\bar{c}}(\overline{\mathfrak{U}}) = +\infty$. Hence, variational solutions can only exist for $\bar{c} \le c^\star$ and these cannot be global minimizers when $\bar{c} < c^\star$.

2.5. Definition of the upper bounds

We will now define some important numerical constants which are necessary in order to formulate assumption (H6). Assume first that (H5) holds. Let $\rho_0^\pm$ be as in (2.4) and (2.5). Recall that we chose ρ_0^+ and ρ_0^- such that

$$\mathcal{F}^+_{\rho_0^+} \cap \mathcal{F}^-_{\rho_0^-} = \emptyset,$$

which implies that the quantity

$$\mathfrak{d}_0 := \mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(\mathcal{F}^+_{\rho_0^+/2}, \mathcal{F}^-_{\rho_0^-/2}) \tag{2.19}$$

is positive. Therefore, as we argued before, one can see that the constant $\mathfrak{d}_0$ depends only on the distance between the two families of minimizing heteroclinics. Next, under (H5), recall the constants $\beta^\pm$ from (2.5). Set

$$\overline{\beta^\pm} := \frac{1}{2}(\beta^\pm)^2((\beta^\pm)^2 + (\beta^\pm + 1)^2) > 0$$

and, subsequently,

$$\mu^- := \frac{1}{\beta^- + \overline{\beta^-}} > 0, \tag{2.20}$$

which is the constant appearing in (H6). Of course, the nature of the definition given in (2.20) obeys technical considerations. But μ^- should be thought of as a constant depending only on the local behavior of the energy around $\mathcal{F}^-$ and, in particular, independent of the behavior of the energy near $\mathcal{F}^+$. Now, for $r \in (0, \rho_0^\pm]$, let

$$\mathfrak{e}_r^\pm := \inf\{E(q) : q \in X(\sigma^-, \sigma^+),\ \mathrm{dist}_{L^2(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \in [r, \rho_0^\pm]\}.$$

Inequality (2.5) implies that $\mathfrak{e}_r^\pm > \mathfrak{m}^\pm$. Moreover, we also have that for $r \in (0, \rho_0^\pm]$ there exists $\nu^\pm(r) > 0$ such that

$$\forall q \in \mathcal{F}^\pm_{\rho_0^\pm/2}, \quad E(q) - \mathfrak{m}^\pm \le \nu^\pm(r) \Rightarrow \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \le r.$$

This leads us to define the constants

$$\delta_0^- := \min\left\{\sqrt{e^{-1}\frac{\rho_0^-}{4}\sqrt{2(\mathfrak{e}^-_{\rho_0^-/4} - \mathfrak{m}^-)}}, \frac{\rho_0^-}{4}\right\} > 0,$$

$$\mathrm{r}^- := \frac{\rho_0^-}{\beta^- + 1} > 0$$

and

$$E_{\max} := \frac{1}{(\beta^-)^2(\beta^- + 1)} \min\left\{\frac{(\delta_0^-)^2}{4}, \mathfrak{e}^-_{\delta_0^-} - \mathfrak{m}^-, \nu^-(\mathrm{r}^-), \nu^-(\delta_0^-)\right\} > 0, \tag{2.21}$$

which is the constant appearing in (H6). Again, the definition of $E_{\max}$ is essentially due to technical reasons, but it must be thought of as a constant which only depends on local information around $\mathcal{F}^-$.

2.6. Methods and ideas of the proofs

The main result of this paper is Theorem 1, which establishes the existence of a solution $(c^\star, \mathfrak{U})$, with the profile $\mathfrak{U}$ satisfying the heteroclinic asymptotic conditions (2.10), (2.11). We also prove an exponential rate of convergence for the profile at $\pm\infty$ (with respect to the L^2-norm). We finally show that the speed $c^\star$ has some uniqueness properties. Important properties of the profile and the speed, as well as improvements to the results under additional assumptions, are also established in Theorems 2 and 3.

The proof of our results follows by bringing together two different lines of research; see items (1) and (2) in the introduction. More precisely, in the spirit of [34, 46], we adapt the result of Alikakos and Katzourakis [6] (based on the previous work by Alikakos and Fusco [3] for the equal-depth case) to potentials defined in an abstract, possibly infinite-dimensional, Hilbert space and possessing two local minima at different levels. This abstract setting is established in Section 3 and the main abstract results are Theorems 4, 5 and 6. The proofs of these results are found in Section 4. Assumption (H5) guarantees that our main results (Theorems 1–3) are a particular case of the abstract results. Naturally, the advantage of proving the results in an abstract framework is that one can apply them to several problems other than the original one. In our case, the results in this paper apply to the system

$$\partial_t w - \partial_x^2 w = -\nabla_u W(w) \quad \text{in } [0, +\infty) \times \mathbb{R},$$

where W is a smooth potential bounded below, possessing two local and nondegenerate minima at different levels. This system, which is one-dimensional in space, is the one considered in [6], but the results of this paper allow us to somewhat relax the nondegeneracy assumption used in [6] and treat the case of nonisolated minima.

Generalizing the result from [6] to curves taking values in a more general, possibly infinite-dimensional, Hilbert space raises several additional difficulties. A detailed outline of our proof as well as the difficulties is given in Section 4.1. Essentially, the approach in [6] requires the potential W to be quadratic in $W^{-1}((-\infty, \alpha])$ for some $\alpha > 0$. It is not hard to find potentials defined on $\mathbb{R}^N$ which satisfy such a property. However, in our setting, the role of the potential is played by the one-dimensional energy E, defined in the infinite-dimensional space of curves $X(\sigma^-, \sigma^+)$. Moreover, one can only modify the energy E by modifying the potential V, which has to be of Allen–Cahn type. Therefore, it seems that the only natural way of obtaining a functional E satisfying an assumption analogous to that in [6] is to impose (H6). As a consequence, E has quadratic behavior in its “negative region” because it is included in a suitable neighborhood of $\mathfrak{q}^-$, which is selected to be nondegenerate up to translations. This enables us to apply the scheme of [6] in the present setting.

In order to conclude this section, it is worth mentioning that, following our approach, one should be able to obtain a generalization of the result in [3] to curves taking values in a Hilbert space. This would yield yet a new proof for the existence result in the stationary case. Nevertheless, such a result is not a special case of ours, as the fact that the minima are at different levels (or, equivalently, that the speed parameter is nonzero) is used at several points during the proof. Hence, one would need to perform some adjustments in the proofs, which goes beyond the scope of this paper.

3. The abstract setting

Instead of proving Theorems 1 and 3 directly, we introduce an abstract setting similar to those considered in [34] and especially [48], which will allow us to deduce the original ones as particular cases. The proofs of the main abstract results, Theorems 4, 5 and 6 below, are thus the core of the paper. The passage between the abstract and the original setting is established in Section 5, which in turn proves Theorems 1 and 3.

3.1. Main definitions and notation

Our approach will consist of establishing the existence of a pair $(c, \mathbf{U})$ in $(0, +\infty) \times X$ (where X is a suitable space of curves; see (3.30) below) which fulfills

$$\mathbf{U}'' - \mathcal{B}(\mathbf{U}) = -c\mathbf{U}' \quad \text{in } \mathbb{R}, \tag{3.1}$$

where $\mathcal{B}$ is (at the least formally) the gradient of a potential. Moreover, at infinity, $\mathbf{U}$ satisfies the conditions

$$\exists T^- \in \mathbb{R}, \forall t \leq T^-, \quad \mathbf{U}(t) \in \mathcal{F}^-_{r_0^-/2}, \tag{3.2}$$

$$\exists T^+ \in \mathbb{R}, \forall t \geq T^+, \quad \mathbf{U}(t) \in \mathcal{F}^+_{r_0^+/2}. \tag{3.3}$$

Notice that this problem can also be thought of as a heteroclinic connection problem on Hilbert spaces for a second-order potential system with friction term. Such a problem could have its own interest besides the main application to the existence of traveling waves that we give here. Of course, analogous considerations can also be applied to the results in [6], as well as our companion paper [38].

The nature of the objects introduced above will be made precise through this paragraph. Let $\mathcal{L}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and induced norm $\|\cdot\|_{\mathcal{L}}$. Let $\mathcal{H} \subset \mathcal{L}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. In the original setting, $\mathcal{L}$ is $L^2(\mathbb{R}, \mathbb{R}^k)$ and $\mathcal{H}$ is $H^1(\mathbb{R}, \mathbb{R}^k)$, both endowed with their natural inner products. We will take $\mathcal{E}\colon \mathcal{L} \to (-\infty, +\infty]$, a potential bounded below and possessing a local minimum which is not a global one. In the setting of Theorem 1, $\mathcal{E}$ will *essentially* coincide with $E - \mathfrak{m}^+$ in $H^1(\mathbb{R}, \mathbb{R}^k)$ and with $+\infty$ elsewhere.[1] Here we just impose a set of abstract assumptions

[1]This statement is not exact, as the energy E is not defined in $H^1(\mathbb{R}, \mathbb{R}^k)$, but on an affine space based on $H^1(\mathbb{R}, \mathbb{R}^k)$. However, we can trivially obtain a functional defined on $H^1(\mathbb{R}, \mathbb{R}^k)$ from E. See Section 5.1.

on $\mathcal{E}$. Most of those assumptions follow by combining ideas in [6] with ideas in Schatzman [46] and Smyrnelis [48]. We will begin by fixing two sets $\mathcal{F}^-$ and $\mathcal{F}^+$ in $\mathcal{L}$. For $r > 0$, we define

$$\mathcal{F}_r^{\pm} := \{v \in \mathcal{L} : \inf_{\mathbf{v}\in\mathcal{F}^{\pm}} \|v - \mathbf{v}\|_{\mathcal{L}} \leq r\} \tag{3.4}$$

and

$$\mathcal{F}_{\mathcal{H},r}^{\pm} := \{v \in \mathcal{H} : \inf_{\mathbf{v}\in\mathcal{F}^{\pm}} \|v - \mathbf{v}\|_{\mathcal{H}} \leq r\}, \tag{3.5}$$

that is, the closed balls in $\mathcal{L}$ and $\mathcal{H}$ respectively, with radius $r > 0$ and center $\mathcal{F}^{\pm}$. The main assumption reads as follows:

(H1′) The potential $\mathcal{E}$ is weakly lower semicontinuous in $\mathcal{L}$. The sets $\mathcal{F}^-$ and $\mathcal{F}^+$ are closed in $\mathcal{L}$. There exists a constant $a > 0$ such that

$$\forall v \in \mathcal{L},\ \forall \mathbf{v}^- \in \mathcal{F}^-, \quad \mathcal{E}(v) \geq \mathcal{E}(\mathbf{v}^-) = -a$$

and each $\mathbf{v}^+ \in \mathcal{F}^+$ is a local minimizer (see (3.6) below) satisfying $\mathcal{E}(\mathbf{v}^+) = 0$. Moreover, there exist two positive constants r_0^-, r_0^+ such that $\mathcal{F}_{r_0^-}^+ \cap \mathcal{F}_{r_0^+}^- = \emptyset$ (see (3.4)). There also exist $C^{\pm} > 1$ such that

$$\begin{cases} \forall v \in \mathcal{F}_{r_0^+}^+, & (C^+)^{-1}\mathrm{dist}_{\mathcal{L}}(v, \mathcal{F}^+)^2 \leq \mathcal{E}(v), \\ \forall v \in \mathcal{F}_{r_0^-}^-, & (C^-)^{-1}\mathrm{dist}_{\mathcal{L}}(v, \mathcal{F}^-)^2 \leq \mathcal{E}(v) + a. \end{cases} \tag{3.6}$$

Moreover, for any $v \in \mathcal{F}_{r_0^{\pm}}^{\pm}$, there exists a unique $\mathbf{v}^{\pm}(v) \in \mathcal{F}^{\pm}$ such that

$$\|v - \mathbf{v}^{\pm}(v)\|_{\mathcal{L}} = \inf_{\mathbf{v}^{\pm}\in\mathcal{F}^{\pm}} \|v - \mathbf{v}^{\pm}\|_{\mathcal{L}}.$$

Moreover, the projection maps

$$P^{\pm}: v \in \mathcal{F}_{r_0^{\pm}}^{\pm} \to \mathbf{v}^{\pm}(v) \in \mathcal{F}^{\pm}$$

are C^2 with respect to the $\mathcal{L}$-norm.

Hypothesis (H1′) defines $\mathcal{E}$ as an unbalanced double-well potential with respect to $\mathcal{F}^-$ and $\mathcal{F}^+$ and gives local information on the minimizing sets. Compare with (H5) and the remarks that follow. In the concrete setting, $\mathcal{F}^-$ and $\mathcal{F}^+$ are essentially the sets $\mathscr{F}^-$ and $\mathscr{F}^+$ respectively, but perturbed so that they are contained in $H^1(\mathbb{R}, \mathbb{R}^k)$. We have the following immediate consequence, which will be useful in the sequel:

Lemma 3.1. *Assume that (H1′) holds. For $r \in (0, r_0^{\pm}]$ we define*

$$\kappa_r^{\pm} := \inf\{\mathcal{E}(v) : \mathrm{dist}_{\mathcal{L}}(v, \mathcal{F}^{\pm}) \in [r, r_0^{\pm}]\}; \tag{3.7}$$

then we have $\kappa_r^+ > 0$ and $\kappa_r^- > -a$. Moreover,

$$\forall v \in \mathcal{F}_{r_0^+/2}^+, \quad \mathcal{E}(v) \geq 0. \tag{3.8}$$

Proof. It follows directly from (3.6) in (H1′). ∎

We now impose the following regarding the relationship between $\mathcal{L}$ and $\mathcal{H}$:

(H2′) We have $\mathcal{H} = \{v \in \mathcal{L} : \mathcal{E}(v) < +\infty\}$ and $\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{\mathcal{H}}$. In particular, $\mathfrak{F}^{\pm} \subset \mathcal{H}$. Moreover, $\mathcal{E}$ is a C^1 functional on $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ with differential $D\mathcal{E} : v \in \mathcal{H} \to D\mathcal{E}(v) \in \mathcal{H}'$, where $\mathcal{H}'$ is the dual of $\mathcal{H}$. Furthermore, there exists an even smaller space $\widetilde{\mathcal{H}}$ with an inner product $\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}}$ and associated norm $\|\cdot\|_{\widetilde{\mathcal{H}}} \geq \|\cdot\|_{\mathcal{H}}$ such that there exists a bounded linear operator $\mathcal{B}$ in $\mathcal{L}$ with domain $\widetilde{\mathcal{H}}$ such that

$$\forall v \in \widetilde{\mathcal{H}},\ \forall w \in \mathcal{H}, \quad \mathcal{B}(v)(w) = D\mathcal{E}(v)(w). \tag{3.9}$$

Notice that in the context of Theorem 1, assumption (H2′) is easily verified. The space $\widetilde{\mathcal{H}}$ will be chosen as $H^2(\mathbb{R}, \mathbb{R}^k)$, and (3.9) is nothing other that integration by parts. We now continue by imposing a compactness assumption on $\mathfrak{F}^{\pm}$:

(H3′) We have that $\mathfrak{F}^{\pm} \subset \mathcal{H}$ and $\mathcal{L}$-bounded subsets of $\mathfrak{F}^{\pm}$ are relatively compact with respect to $\mathcal{H}$-convergence.[2]

Assumption (H3′) readily implies the following:

Lemma 3.2. *Assume that (H1′) and (H3′) hold. Then the sets $\mathfrak{F}^{\pm}_{r_0^{\pm}/2}$ defined in (3.4) are closed in $\mathcal{L}$.*

Assumption (H3′) is necessary in order to establish the conditions at infinity. In the main context, it follows from the straightforward fact that a bounded sequence of translations (which are real numbers) is relatively compact. Subsequently, we impose the following:

(H4′) Assume that (H1′) holds. For $\mathfrak{F}^{\pm}$, one of the two following alternatives holds:

(1) $\mathfrak{F}^{\pm}$ is $\mathcal{L}$-bounded.

(2) For all $(v, \mathbf{v}^{\pm}) \in \mathfrak{F}^{\pm}_{r_0^{\pm}} \times \mathfrak{F}^{\pm}$, there exists an associated map $\widehat{P}^{\pm}_{(v,\mathbf{v}^{\pm})} : \mathcal{L} \to \mathcal{L}$ such that

$$P^{\pm}(\widehat{P}^{\pm}_{(v,\mathbf{v}^{\pm})}(v)) = \mathbf{v}^{\pm} \tag{3.10}$$

and

$$\operatorname{dist}_{\mathcal{L}}(\widehat{P}^{\pm}_{(v,\mathbf{v}^{\pm})}(v), \mathfrak{F}^{\pm}) = \operatorname{dist}_{\mathcal{L}}(v, \mathfrak{F}^{\pm}). \tag{3.11}$$

Moreover, $\widehat{P}^{\pm}_{(v,\mathbf{v}^{\pm})} : \mathcal{L} \to \mathcal{L}$ is differentiable and

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad \|D(\widehat{P}^{\pm}_{(v,\mathbf{v}^{\pm})})(w_1, w_2)\|_{\mathcal{L}} = \|w_2\|_{\mathcal{L}} \tag{3.12}$$

and

$$\mathcal{E}(\widehat{P}_{(v,\mathbf{v}^{\pm})}(\tilde{v})) = \mathcal{E}(\tilde{v}) \tag{3.13}$$

for all $\tilde{v} \in \mathcal{L}$.

[2]Hence, they are in particular relatively compact with respect to $\mathcal{L}$-convergence.

Essentially, in (2) we impose that the projections $P^{\pm}$ from (H1′) can be transported in a direction parallel to the sets $\mathcal{F}^{\pm}$. Again, this is straightforward in the concrete setting, as the projections $P^{\pm}$ consist of performing a translation. We now impose an assumption for the sets $\mathcal{F}^{\pm}_{\mathcal{H},r_0}$:

(H5′) For any $v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}}$, as defined in (3.5), there exists a unique $\mathbf{v}^{\pm}_{\mathcal{H}}(v) \in \mathcal{F}^{\pm}$ such that

$$\|v - \mathbf{v}^{\pm}_{\mathcal{H}}(v)\|_{\mathcal{L}} = \inf_{\mathbf{v}^{\pm} \in \mathcal{F}^{\pm}} \|v - \mathbf{v}^{\pm}\|_{\mathcal{L}}.$$

Moreover, the projection maps

$$P^{\pm}_{\mathcal{H}} : v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}} \to \mathbf{v}^{\pm}_{\mathcal{H}}(v) \in \mathcal{F}^{\pm}$$

are C^1 with respect to the $\mathcal{H}$-norm. Moreover, if $C^{\pm} > 1$ is the constant from (H1′), we have

$$\forall v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}}, \quad \|P^{\pm}(v) - P^{\pm}_{\mathcal{H}}(v)\|_{\mathcal{H}} \le C^{\pm}\|v - P^{\pm}_{\mathcal{H}}(v)\|_{\mathcal{H}}. \tag{3.14}$$

Furthermore, for each $r^{\pm} \in (0, r_0^{\pm}]$ there exist constants $\beta^{\pm}(r^{\pm}) > 0$ such that in the case that $v \in \mathcal{F}^{\pm}_{r_0^{\pm}}$ satisfies

$$\mathcal{E}(v) \le \min\{\pm a, 0\} + \beta^{\pm}(r^{\pm}), \tag{3.15}$$

then $v \in \mathcal{F}^{\pm}_{\mathcal{H},r}$. Finally, we have

$$\begin{aligned} \forall v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}}, \quad (C^{\pm})^{-2}\|v - P^{\pm}_{\mathcal{H}}(v)\|^2_{\mathcal{H}} &\le \mathcal{E}(v) - \min\{\pm a, 0\} \\ &\le (C^{\pm})^2\|v - P^{\pm}_{\mathcal{H}}(v)\|^2_{\mathcal{H}}. \end{aligned} \tag{3.16}$$

Assumption (H5′) is made in order to ensure suitable local properties around $\mathcal{F}^{\pm}$ in $\mathcal{H}$. In the main setting, those are known results which follow essentially from the spectral assumption by Schatzman [46]. Before introducing the last assumptions, we need some additional notation. For $U \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L})$ and $c > 0$, we (formally) define

$$\mathbf{E}_c(U) := \int_{\mathbb{R}} \mathbf{e}_c(U)(t)\, dt := \int_{\mathbb{R}} \Big[\frac{\|U'(t)\|^2_{\mathcal{L}}}{2} + \mathcal{E}(U(t))\Big] e^{ct}\, dt. \tag{3.17}$$

More generally, for $I \subset \mathbb{R}$ a nonempty interval and $U \in H^1_{\mathrm{loc}}(I, \mathcal{L})$, put

$$\mathbf{E}_c(U; I) := \int_I \mathbf{e}_c(U)(t)\, dt. \tag{3.18}$$

Notice that the integrals defined in (3.17) and (3.18) might not even make sense in general due to the fact that $\mathcal{E}$ has a sign. Nevertheless, we can define the notion of a *local minimizer* of $\mathbf{E}_c(\cdot; I)$ as follows:

Definition 3.1. Assume that (H1′) and (H2′) hold. Let $I \subset \mathbb{R}$ be a bounded, nonempty interval. Assume that $U \in H^1_{\mathrm{loc}}(I, \mathcal{L})$ is such that $E_c(U; I)$ is well defined and finite. Assume also that there exists $C > 0$ such that for any $\phi \in \mathcal{C}^1_c(\mathrm{int}(I), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ such that

$$\max_{t \in I} \|\phi(t)\|_{\mathcal{H}} < C,$$

the quantity $\mathbf{E}_c(U + \phi; I)$ is well defined and larger than $\mathbf{E}_c(U; I)$. Then we say that U is a *local minimizer* of $\mathbf{E}_c(\cdot; I)$.

We assume the following property for local minimizers:

(H6′) Assume that (H1′) and (H2′) hold. There exists a map $\mathfrak{P}: \mathcal{L} \to \mathcal{L}$ such that

$$\forall v \in \mathcal{L}, \qquad \mathcal{E}(\mathfrak{P}(v)) \leq \mathcal{E}(v) \text{ and } \mathcal{E}(\mathfrak{P}(v)) = \mathcal{E}(v) \Leftrightarrow \mathfrak{P}(v) = v, \tag{3.19}$$

$$\forall (v_1, v_2) \in \mathcal{L}^2, \quad \|\mathfrak{P}(v_1) - \mathfrak{P}(v_2)\|_{\mathcal{L}} \leq \|v_1 - v_2\|_{\mathcal{L}}, \tag{3.20}$$

and

$$\mathfrak{P}|_{\mathcal{F}^{\pm}} = \mathrm{Id}|_{\mathcal{F}^{\pm}}. \tag{3.21}$$

In the main setting, $\mathfrak{P}$ is the projection on a fixed ball of $\mathbb{R}^k$, which, when applied to curves, can be shown to reduce their energy while keeping a uniform bound in the L^∞-norm. We also assume the following:

(H7′) Let $I \subset \mathbb{R}$, possibly unbounded and nonempty. Let $c > 0$. If $\mathbf{W} \in H^1_{\mathrm{loc}}(I, \mathcal{L})$ is a local minimizer of $\mathbf{E}_c(\cdot; I)$ in the sense of Definition 3.1, which, additionally, is such that for all $t \in I$, $\mathbf{W}(t) = \mathfrak{P}(\mathbf{W}(t))$, then $\mathbf{W} \in \mathcal{A}(I)$ where for any open set $O \subset \mathbb{R}$, $\mathcal{A}(O)$ is defined as

$$\mathcal{A}(O) := \mathcal{C}^2_{\mathrm{loc}}(O, \mathcal{L}) \cap \mathcal{C}^1_{\mathrm{loc}}(O, (\mathcal{H}, \|\cdot\|_{\mathcal{H}})) \cap \mathcal{C}^0_{\mathrm{loc}}(O, (\tilde{\mathcal{H}}, \|\cdot\|_{\tilde{\mathcal{H}}})) \tag{3.22}$$

and $\mathbf{W}$ solves

$$\mathbf{W}'' - \mathcal{B}(\mathbf{W}) = -c\mathbf{W}' \quad \text{in } I,$$

where $\mathcal{B}$ was introduced in (H2′).

In the context of Theorem 1, (H7′) is a consequence of classical elliptic regularity results as well as properties of the energy functional. Before stating the abstract result, we introduce the following constants (assuming that all the previous assumptions hold) which are obviously analogous to those introduced in Section 2.5:

$$\eta_0^- := \min\left\{\sqrt{e^{-1}\frac{r_0^-}{4}\sqrt{2(\kappa_{r_0^-/4}^- + a)}}, \frac{r_0^-}{4}\right\} > 0, \tag{3.23}$$

$$\hat{r}^- := \frac{r_0^-}{C^- + 1} > 0, \tag{3.24}$$

$$\mathcal{E}_{\max}^- := \frac{1}{(C^-)^2(C^- + 1)} \min\left\{\frac{(\eta_0^-)^2}{4}, \kappa_{\eta_0^-}^- + a, \beta^-(\hat{r}^-), \beta^-(\eta_0^-)\right\} > 0, \tag{3.25}$$

$$\mathsf{c}^{\pm} := \frac{1}{2}(C^{\pm})^2((C^{\pm})^2 + (C^{\pm}+1)^2) > 0, \tag{3.26}$$

$$\gamma^{-} := \frac{1}{C^{-} + \mathsf{c}^{-}} > 0 \tag{3.27}$$

and

$$d_0 := \mathrm{dist}_{\mathcal{L}}(\mathcal{F}^{+}_{r_0^+/2}, \mathcal{F}^{-}_{r_0^-/2}) > 0, \tag{3.28}$$

where the constants C^{-}, $\beta^{-}(\hat{r}^{-})$, $\beta^{-}(\eta_0^{-})$ are those from (H5′) and $\kappa_r^{\pm}$ for $r > 0$ are defined in (3.7). The fact that $d_0 > 0$ follows from Lemma 3.2 and (H1′). We can finally state the following assumption:

(H8′) Assume that (H1′) and (H2′) hold. Moreover, assume that

$$a < \mathcal{E}^{-}_{\max}$$

and

$$\{v \in \mathcal{H} : \mathcal{E}(v) < 0\} \subset \mathcal{F}^{-}_{r_0^-/2}. \tag{3.29}$$

Assumption (H8′) is essentially the abstract version of (H6).

3.2. Statement of the abstract results

Let us define the space

$$X := \Big\{U \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}) : \exists T \geq 1, \forall t \geq T, \quad \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^{+}) \leq \tfrac{r_0^+}{2}, \\ \forall t \leq -T, \quad \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^{-}) \leq \tfrac{r_0^-}{2}\Big\}. \tag{3.30}$$

The statement of the main abstract result is as follows:

Theorem 4 (Main abstract result). *Assume that (H1′), (H2′), (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Then the following holds:*

(1) Existence. *There exist $c^{\star} > 0$ and $\mathbf{U} \in \mathcal{A}(\mathbb{R}) \cap X$, $\mathcal{A}(\mathbb{R})$ as in (3.22) and X as in (3.30), such that $(c^{\star}, \mathbf{U})$ solves (3.1) with conditions at infinity (3.2), (3.3) and $\mathbf{U}$ is a global minimizer of $\mathbf{E}_c$ in X, that is, $\mathbf{E}_c(\mathbf{U}) = 0$. Moreover, for all $t \in \mathbb{R}$, $\mathbf{U}(t) = \mathfrak{P}(\mathbf{U}(t))$, where $\mathfrak{P}$ is as in (H6′).*

(2) Uniqueness of the speed. *The speed $c^{\star}$ is unique in the following sense: if $\overline{c^{\star}} > 0$ is such that*

$$\inf_{U \in X} \mathbf{E}_{\overline{c^{\star}}}(U) = 0$$

and there exists $\bar{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap X$ such that $(\overline{c^{\star}}, \bar{\mathbf{U}})$ solves (3.1) and $\mathbf{E}_{\overline{c^{\star}}}(\bar{\mathbf{U}}) < +\infty$, then $\overline{c^{\star}} = c^{\star}$.

(3) Exponential convergence. *There exists a constant $M^{+} > 0$ such that for all $t \in \mathbb{R}$ we have*

$$\|\mathbf{U}(t) - \mathbf{v}^{+}(\mathbf{U})\|_{\mathcal{L}} \leq M^{+} e^{-ct}, \tag{3.31}$$

for some $\mathbf{v}^{+}(\mathbf{U}) \in \mathcal{F}^{+}$.

Remark 3.1. Given the definition of X in (3.30), we have that for any $U \in X$ and $\tau \in \mathbb{R}$ it holds that $U(\cdot + \tau) \in X$ and for any $c > 0$ it holds that $\mathbf{E}_c(U(\cdot + \tau)) = e^{-c\tau}\mathbf{E}_c(U)$. Such statements imply

$$\forall c > 0, \quad \inf_{U \in X} \mathbf{E}_c(U) \in \{-\infty, 0\}.$$

Moreover, we see that in the case that $c > 0$ is such that $\inf_{U \in X} \mathbf{E}_c(U) = 0$, one can find plenty of examples of minimizing sequences in X which cannot ever reasonably produce a global minimizer. Indeed, consider any function $\tilde{U} \in X$ such that $E_c(\tilde{U}) > 0$ and then take the minimizing sequence $(\tilde{U}(\cdot + n))_{n \in \mathbb{N}}$.

Remark 3.2. A more general statement can be given about the uniqueness of the speed, which in particular works for eventual nonminimizing solutions. See Proposition 4.3.

Theorem 4 will be shown to contain Theorem 1 in Section 5. Notice that, as before, the conditions at infinity (3.2) are rather weak (and not really of heteroclinic type), since we do not have convergence to an element of $\mathcal{F}^-$ as $t \to -\infty$. It is however clear that the conditions at infinity (3.2), (3.3) are enough to ensure that the solution given by Theorem 4 is not constant. In any case, we can impose an additional assumption in order to obtain stronger conditions at $-\infty$ on the solution:

(H9′) Hypothesis (H8′) is fulfilled and, additionally,

$$a < \frac{(d_0\gamma^-)^2}{2}, \tag{3.32}$$

where d_0 and γ^- were defined in (3.28) and (3.27) respectively.

Then we can show the following exponential convergence result:

Theorem 5. *Assume that (H1′), (H2′), (H3′), (H4′), (H5′), (H6′), (H7′), (H8′) and (H9′) hold. Then, if $(c^\star, \mathbf{U})$ is the solution given by Theorem 4, it holds that $\gamma^- > c^\star$ (γ^- as in (3.27)) and there exists $\mathfrak{M}^- > 0$ such that for all $t \in \mathbb{R}$,*

$$\|\mathbf{U}(t) - \mathbf{v}^-(\mathbf{U})\|_{\mathcal{L}} \leq M^- e^{(\gamma^- - c^\star)t} \tag{3.33}$$

for some $\mathbf{v}^-(U) \in \mathcal{F}^-$.

Theorem 5 corresponds to the second part of Theorem 1. Finally, we will prove the following result:

Theorem 6. *Assume that (H1′), (H2′), (H3′), (H4′), (H5′), (H6′), (H7′), (H8′) and (H9′) hold. Let $(c^\star, \mathbf{U})$ be the solution given by Theorem 4. Then, if $\tilde{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap X$ is such that*

$$\mathbf{E}_{c^\star}(\tilde{\mathbf{U}}) = 0,$$

we have that $(c^\star, \tilde{\mathbf{U}})$ solves (3.1) and

$$c^\star = \frac{a}{\int_{\mathbb{R}} \|\tilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 \, dt}. \tag{3.34}$$

In particular, the quantity $\int_{\mathbb{R}} \|\widetilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 \, dt$ *is finite. Moreover, we have*

$$c^\star = \sup\{c > 0 : \inf_{U\in X} \mathbf{E}_c(U) = -\infty\} = \inf\{c > 0 : \inf_{U\in X} \mathbf{E}_c(U) = 0\}, \quad (3.35)$$

as well as the bound

$$c^\star \le \frac{\sqrt{2a}}{d_0} < \min\left\{\frac{\sqrt{2\mathcal{E}^-_{\max}}}{d_0}, \gamma^-\right\}, \quad (3.36)$$

with $\mathcal{E}^-_{\max}$ *as in* (3.25)*,* d_0 *as in* (3.28) *and* γ^- *as in* (3.27)*. The second inequality follows from the bounds on* a *given by* (H8′) *and* (H9′)*.*

Theorem 6 corresponds to Theorem 3.

4. Proofs of the abstract results

4.1. Scheme of the proofs

As pointed out several times, the structure of the proofs of our abstract results, Theorems 4, 5 and 6, is analogous to that in Alikakos and Katzourakis [6], which has its roots in Alikakos and Fusco [3]. In fact, most of their results also carry into the abstract setting with the suitable modifications. In fact, the structure of our proofs should rather be compared with Alikakos, Fusco and Smyrnelis [5, Section 2.6], which slightly modifies and simplifies the argument in [6]. We will also rely on some arguments provided in Smyrnelis [48], when an analogous abstract approach is taken for the stationary problem. As usual, most of the intermediate results we prove hold under smaller subsets of assumptions (with respect to the set of all assumptions that we dropped in the previous section). Therefore, for the sake of clarity and generality, the necessary assumptions (and only these) that we use to prove a result are specified in its statement.

Despite the previous facts, and as pointed before, several important difficulties not present in [6] arise when one tries to tackle the same problem in the abstract setting we introduced in the previous section. One of those extra difficulties is due to the fact that, in our setting, we need deal with two different norms in the configuration space of the curves, $\mathcal{L}$ and $\mathcal{H}$ (to be thought of as L^2 and H^1 respectively, for simplification) and that the potential $\mathcal{E}$ is only lower semicontinuous with respect to $\mathcal{L}$-convergence. An additional difficulty comes from the fact that, due to the requirements of our original problem, we are not looking at curves that join two *isolated minimum points*, but rather two *isolated minimum sets*. This turns out to be an obstacle when one tries to adapt arguments in [6], even if one were to restrict to finite-dimensional configuration spaces. However, this difficulty is successfully dealt with using the precise knowledge about the *projection mappings* (namely assumptions (H1′), (H4′) and (H5′)) that is available. That is, one uses that, for a suitable neighborhood of the minimum sets, the projection onto the sets (with both the $\mathcal{L}$- and $\mathcal{H}$-norms) is well defined and enjoys some type of continuity and differentiability properties. This idea, in the Allen–Cahn systems setting, has to be traced back to Schatzman [46].

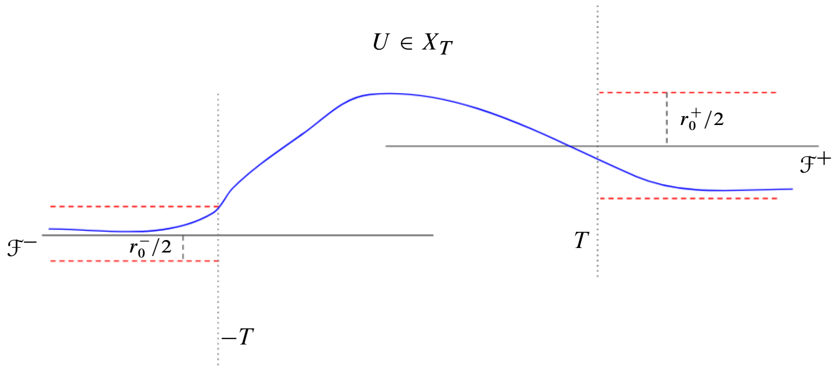

Figure 3. One-dimensional representation of X_T. The blue line represents a function U belonging to X_T. The red lines contain the points which are at $\mathcal{L}$-distance smaller than $r_0^{\pm}/2$ from $\mathcal{F}^{\pm}$.

We will now briefly sketch the scheme of the proof of Theorem 4. Recall that, according to Remark 3.1, direct minimization of $\mathbf{E}_c$ in X cannot yield solutions to the problem, the reason being the action of the group of translations. The spaces X_T, which were introduced in [6] (also in [3] for the equal-depth case) and will be precisely presented in (4.1), are defined in order to overcome this source of degeneracy, as they are no longer invariant by the action of the group of translations. See the design in Figure 3. As a consequence, compactness is restored and the corresponding minimization problem has a solution for all $c > 0$ and $T \geq 1$. See Lemma 4.7 later on. In general, minimizers in X_T solve the profile equation on a (possibly proper) subset of $\mathbb{R}$ (see Lemma 4.8), meaning that they are in general not solutions of (3.1). However, such constrained minimizers are in fact solutions of (3.1) in the case they do not saturate the constraints. Therefore, the goal will be to show the existence of the speed $c^\star$ such that, for some $T \geq 1$, there exists a constrained minimizer in X_T which does not saturate the constraints. For that purpose, a careful analysis of the behavior of the constrained minimizers is needed. Indeed, one needs a uniform bound (independent of T and continuous on c) on the distance between the *entry times*, i.e. the times in which the constrained minimizers enter $\mathcal{F}^{\pm}_{r_0^{\pm}/2}$. In the balanced case this follows from the fact that the energy density is bounded below by a positive constant outside $\mathcal{F}^-_{r_0^-/2} \cup \mathcal{F}^+_{r_0^+/2}$ (see for instance Smyrnelis [48]). However, this is no longer true for our unbalanced problem, which makes it more involved: if one does not have the positivity of the energy density, the constrained solutions can oscillate between the regions $\mathcal{F}^{\pm}_{r_0^{\pm}/2}$ (producing energy compensations) in larger and larger intervals as $T \to \infty$, so that no T-independent bound can be found. This is the main new difficulty with respect to the balanced setting, as one needs new ideas in order to obtain a uniform bound on the distance between the entry times. Our assumption (H8′) provides this control because the energy density of the constrained minimizers is bounded below by a positive constant in the interval given by the two entry times mentioned before, meaning that we can argue as in the balanced case. The precise result is Corollary 4.1. This is the main step in which our proof differs with that in [6].

The natural question is what happens if we remove (H8′). A natural approach is to replace (H8′) by an assumption more closely related to the one used in [6] and [5]. This would lead us to introduce a convexity assumption on the level sets of $\mathcal{E}$, as well as some sort of strict monotonicity on well-chosen segments. While this assumption can be worked out in the abstract setting and is applicable for the finite-dimensional situation considered in [6] (as we show in the companion paper [38]), we believe it to be too restrictive to be applied to our original problem.

In any case, after the uniform bound on the entry times of constrained minimizers is obtained, one needs to find the speed $c^\star$ as, until this point, the speed $c > 0$ has only been considered as a parameter of the problem without any special role. Our argument adapts without major difficulty from [5] and it goes as follows: One introduces a set which classifies the speeds according to the value of the infimum of the corresponding energy on X (which, due to the weight and the invariance by translations, is either $-\infty$ or 0). Such a set is $\mathcal{C}$, defined in (4.87). Subsequently, one shows (Lemma 4.11) that $\mathcal{C}$ is open, bounded, nonempty and that its positive limit points give rise to entire minimizing solutions of the equations (since for those points one can find corresponding constrained minimizers which do not saturate the constraints). The speed $c^\star$ is then defined as the supremum of $\mathcal{C}$, which is in fact the unique positive limit point of the set, as shown in Corollary 4.3. At this point, the process of the proof of Theorem 4 is completed. Later on, we show that the asymptotic behavior of the constrained solutions can be improved under an additional assumption, namely an upper bound on the speed. This is Proposition 4.4. Theorems 5 and 6 can then be proven.

4.2. Preliminaries

Let r_0^- and r_0^+ be the constants introduced in Section 3 and $\mathfrak{F}^{\pm}_{r_0^{\pm}/2}$ be the corresponding closed balls as in (3.4). Assume that (H1′) holds. For $T \geq 1$, we define the sets

$$X_T^- := \left\{ U \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}) : \forall t \leq -T,\ U(t) \in \mathfrak{F}^-_{r_0^-/2} \right\},$$
$$X_T^+ := \left\{ U \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}) : \forall t \geq T,\ U(t) \in \mathfrak{F}^+_{r_0^+/2} \right\}.$$

Subsequently, we set

$$X_T := X_T^- \cap X_T^+. \tag{4.1}$$

Recall the space X introduced in (3.30). Notice that

$$X = \bigcup_{T \geq 1} X_T.$$

We have the following preliminary properties on the spaces X_T:

Lemma 4.1. *Assume that (H1′) holds. Let $c > 0$ and $T \geq 1$. For any $U \in X_T$, we have*

$$\forall t \geq T, \quad \mathcal{E}(U(t)) \geq 0. \tag{4.2}$$

Moreover, the quantity $\mathbf{E}_c(U)$ as introduced in (3.17) belongs to $(-\infty, +\infty]$.

Proof. Let $U \in X_T$. Notice that for $t \geq T$, we have $U(t) \in \mathcal{F}^+_{r_0/2}$. Therefore, (4.2) follows directly from (3.8) in Lemma 3.1.

Now let $\mathcal{E}^+(U) \geq 0$ and $\mathcal{E}^-(U) \geq 0$ be, respectively, the nonnegative and the nonpositive parts of $\mathcal{E}(U)$, so that $\mathcal{E}(U) = \mathcal{E}^+(U) - \mathcal{E}^-(U)$. We have that $\mathcal{E}^-(U)$ is null on $[T, +\infty)$, that is,

$$\int_{-\infty}^{+\infty} \mathcal{E}^-(U(t))e^{ct}\, dt = \int_{-\infty}^{T} \mathcal{E}^-(U(t))e^{ct}\, dt \leq \frac{a}{c}e^{cT} < +\infty,$$

where a is the minimum value from (H1′). Therefore, the negative part of the energy density $\mathbf{e}_c(U)$ (see (3.17)) belongs to $L^1(\mathbb{R})$, which establishes the result. ■

Lemma 4.1 shows that for any $T \geq 1$ and $c > 0$, $\mathbf{E}_c$ is well defined as an extended functional on X_T, at least if sufficient hypotheses are made. Moreover, it gives the following useful inequalities:

Lemma 4.2. *Assume that (H1′) holds. Let $c > 0$ and $T \geq 1$. For any $U \in X_T$, we have*

$$\int_{\mathbb{R}} \frac{\|U'(t)\|^2_{\mathcal{L}}}{2} e^{ct}\, dt \leq \mathbf{E}_c(U) + \frac{a}{c}e^{cT} \tag{4.3}$$

and

$$\int_{\mathbb{R}} |\mathcal{E}(U(t))| e^{ct}\, dt \leq \mathbf{E}_c(U) + \frac{a}{c}e^{cT}. \tag{4.4}$$

Finally, we have that for all $t \in \mathbb{R}$,

$$\int_{t}^{+\infty} \|U'(s)\|_{\mathcal{L}}\, ds \leq \left(\left(\mathbf{E}_c(U) + \frac{a}{c}e^{cT}\right)\frac{e^{-ct}}{c}\right)^{\frac{1}{2}}. \tag{4.5}$$

Proof. Using (4.2) in Lemma 4.1, we get

$$\int_{\mathbb{R}} \frac{\|U'(t)\|^2_{\mathcal{L}}}{2} e^{ct}\, dt \leq \mathbf{E}_c(U) - \int_{-\infty}^{T} \mathcal{E}(U(t))e^{ct},$$

which, by (H1′), implies that (4.3) holds. Inequality (4.4) is obtained in the same fashion. Finally, we have that (4.5) follows by combining (4.3) with the Cauchy–Schwarz inequality. ■

The previous results allow us to prove the following convergence properties at $+\infty$ for finite energy functions in X_T:

Lemma 4.3. *Assume that (H1′) and (H5′) hold. Let $c > 0$ and $T \geq 1$. Take $U \in X_T$ such that $\mathbf{E}_c(U) < +\infty$. Then we have that there exists a subsequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $t_n \to +\infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \mathcal{E}(U(t_n))e^{ct_n} = 0. \tag{4.6}$$

Moreover, there exists $\mathbf{v}^+(U) \in \mathcal{F}^+$ such that for all $t \in \mathbb{R}$ it holds that

$$\|U(t) - \mathbf{v}^+(U)\|_{\mathcal{L}}^2 \leq \Big(\frac{\mathbf{E}_c(U) + \frac{a}{c}e^{cT}}{c}\Big)e^{-ct}. \tag{4.7}$$

That is, U tends to $\mathbf{v}^+(U)$ at $+\infty$ with an exponential rate of convergence and with respect to the $\mathcal{L}$-norm.

Proof. We have by (4.4) in Lemma 4.2 that $t \in \mathbb{R} \to \mathcal{E}(U(t))e^{ct} \in \mathbb{R}$ belongs to $L^1(\mathbb{R})$ because $\mathbf{E}_c(U) < +\infty$. Therefore, combining with (4.2) in Lemma 3.1, we obtain (4.6).

Subsequently, notice that (4.5) in Lemma 4.2 and the fact that $\mathbf{E}_c(U) < +\infty$ give the existence of $\mathbf{v}^+(U) \in \mathcal{F}^+$ such that $\lim_{t\to+\infty}\|U(t) - \mathbf{v}^+(U)\|_{\mathcal{L}} = 0$. Therefore, fix $t \in \mathbb{R}$ and notice that for any $\tilde{t} > t$ we have

$$\|U(\tilde{t}) - U(t)\|_{\mathcal{L}} \leq \int_t^{\tilde{t}} \|U'(s)\|_{\mathcal{L}}\, ds \leq \int_t^{+\infty} \|U'(s)\|_{\mathcal{L}},$$

which by (4.5) in Lemma 4.2 means that

$$\|U(\tilde{t}) - U(t)\|_{\mathcal{L}}^2 \leq \Big(\frac{\mathbf{E}_c(U) + \frac{a}{c}e^{cT}}{c}\Big)e^{-ct}.$$

Therefore, passing to the limit $\tilde{t} \to +\infty$ we obtain (4.7), also due to the fact that U is continuous with respect to the $\mathcal{L}$-norm. ∎

Remark 4.1. Notice that (4.7) in Lemma 4.3 does not imply convergence of $\mathcal{E}(U)$ towards 0 at $+\infty$, due to the fact that $\mathcal{E}$ is not continuous with respect to the $\mathcal{L}$-norm.

Remark 4.2. Regarding the behavior at $-\infty$, notice that we can only say that if $U \in X_T$ is such that $\mathbf{E}_c(U) < +\infty$, then $\mathcal{E}(U)$ does not go to $+\infty$ faster than e^{ct} at the limit $t \to -\infty$. That is, almost nothing can be said for generic finite energy solutions regarding their behavior at $-\infty$.

4.3. The infima of $\mathbf{E}_c$ in X_T are well defined

Once we have defined the spaces X_T, we show that the corresponding infimum of $\mathbf{E}_c$ is well defined as a real number for all $c > 0$. Set

$$\mathbf{m}_{c,T} := \inf_{U\in X_T} \mathbf{E}_c(U) \in [-\infty, +\infty]. \tag{4.8}$$

We have the following:

Lemma 4.4. *Assume that (H1′) and (H2′) hold. Fix* $\hat{\mathbf{v}}^{\pm} \in \mathcal{F}^{\pm}$. *Let* $c > 0$ *and* $T \geq 1$. *For all* $T \geq 1$, *the function*

$$\Psi(t) := \begin{cases} \hat{\mathbf{v}}^- & \text{if } t \leq -1, \\ \frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+ & \text{if } -1 \leq t \leq 1, \\ \hat{\mathbf{v}}^+ & \text{if } t \geq 1, \end{cases} \tag{4.9}$$

belongs to X_T. *Moreover, for all* $c > 0$,

$$\mathbf{E}_c(\Psi) < +\infty. \tag{4.10}$$

Furthermore, we have

$$-\infty < \mathbf{m}_{c,T} < +\infty. \tag{4.11}$$

Proof. It is clear that $\Psi \in X_T$. We now show that (4.10) holds. Notice first that

$$\int_{-\infty}^{1} \mathbf{e}_c(\Psi) = \int_{-\infty}^{1} -ae^{ct}\,dt = -\frac{a}{c}e^c,$$

where $-a$ is the minimum value from (H1′). Subsequently, we have

$$\int_{1}^{+\infty} \mathbf{e}_c(\Psi) = 0$$

and

$$\begin{aligned} \int_{-1}^{1} \mathbf{e}_c(\Psi) &= \int_{-1}^{1} \Big[\frac{\|\hat{\mathbf{v}}^+ - \hat{\mathbf{v}}^-\|_{\mathcal{L}}^2}{8} + \mathcal{E}\Big(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\Big)\Big]e^{ct}\,dt \\ &\leq \Big[\frac{\|\hat{\mathbf{v}}^+ - \hat{\mathbf{v}}^-\|_{\mathcal{L}}^2}{4} + 2\max_{t\in[-1,1]} \mathcal{E}\Big(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\Big)\Big]\frac{e^c - e^{-c}}{c}, \end{aligned}$$

and we have

$$\max_{t\in[-1,1]} \mathcal{E}\Big(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\Big) < +\infty,$$

by (H2′). Therefore, we have obtained $E_c(\Psi) < +\infty$, which readily implies that $\mathbf{m}_{c,T} < +\infty$. In order to establish (4.11), we still need to show that $\mathbf{m}_{c,T} > -\infty$. For that purpose, let $U \in X_T$. By (4.2) in Lemma 4.1, we have

$$\int_{T}^{+\infty} \mathbf{e}_c(U) \geq 0.$$

We also have

$$\int_{-\infty}^{T} \mathbf{e}_c(U) \geq \int_{-\infty}^{T} -a\mathbf{e}^{ct}\,dt = -\frac{a}{c}e^{cT}.$$

That is,

$$\forall U \in X_T, \quad \mathbf{E}_c(U) \geq -\frac{a}{c}e^{cT} > -\infty,$$

which means that $\mathbf{m}_{c,T} > -\infty$. ∎

The next goal will be to show that, under the proper assumptions, we have that for any $c > 0$ and $T \geq 1$, the infimum values defined in (4.8) are attained. Such a fact is not hard to prove since the constraints that define the spaces X_T allow us to restore compactness. It relies on some properties that will be proven in the next section.

4.4. General continuity and semicontinuity results

We now provide some results which address continuity and semicontinuity properties of the energies $\mathbf{E}_c$ in the spaces X_T. Such properties will allow us to show that the infimum values defined in (4.8) are attained under the proper assumptions. They will also be useful in a more advanced stage of the proof, when the constraints will be removed. For now, we essentially adapt some results from [6] to our setting.

Our first result resembles [6, Lemma 26]:

Lemma 4.5. *Assume that (H1′) holds. Fix $T \geq 1$ and $U \in X_T$. Consider the set*

$$A_{T,U} := \{c > 0 : \mathbf{E}_c(U) < +\infty\}.$$

Then the correspondence

$$c \in A_{T,U} \to \mathbf{E}_c(U) \in \mathbb{R}$$

is continuous.

Proof. Consider a sequence $(c_n)_{n\in\mathbb{N}}$ in $A_{T,U}$ such that $c_n \to c_\infty \in A_{T,U}$. One checks that for all $n \in \mathbb{N}$,

$$|\mathbf{e}_{c_n}(U(\cdot))| \leq |\mathbf{e}_{c_{\max}}(U(\cdot))| \quad \text{in } [0, +\infty) \tag{4.12}$$

and

$$|\mathbf{e}_{c_n}(U(\cdot))| \leq |\mathbf{e}_{c_{\min}}(U(\cdot))| \quad \text{in } (-\infty, 0], \tag{4.13}$$

where $c_{\max} := \sup_{n\in\mathbb{N}} c_n \in A_{T,U}$ and $c_{\min} := \inf_{n\in\mathbb{N}} c_n \in A_{T,U}$. By Lemma 4.2, the right-hand side in both (4.12) and (4.13) is integrable, which allows us to conclude by the dominated convergence theorem. ∎

We now show a lower semicontinuity result, which in particular will imply the existence of the constrained solutions:

Lemma 4.6. *Assume that (H1′), (H3′) and (H4′) hold. Let $T \geq 1$ be fixed. Let $(U_n^i)_{n\in\mathbb{N}}$ be a sequence in X_T and $(c_n)_{n\in\mathbb{N}}$ a convergent sequence of positive real numbers such that*

$$\sup_{n\in\mathbb{N}} \mathbf{E}_{c_n}(U_n^i) < +\infty. \tag{4.14}$$

Then there exist a sequence $(U_n)_{n\in\mathbb{N}}$ in X_T and $U_\infty \in X_T$ such that up to extracting a subsequence in $(U_n, c_n)_{n\in\mathbb{N}}$ it holds that

$$\forall n \in \mathbb{N}, \quad \mathbf{E}_{c_n}(U_n) = \mathbf{E}_{c_n}(U_n^i), \tag{4.15}$$

$$\forall t \in \mathbb{R}, \quad U_n(t) \rightharpoonup U_\infty(t) \quad \textit{weakly in } \mathcal{L} \tag{4.16}$$

and

$$U_n' h_{c_n} \rightharpoonup U_\infty' h_{c_\infty} \quad \textit{weakly in } L^2(\mathbb{R}, \mathcal{L}), \tag{4.17}$$

where, for $k \in \mathbb{R}$, $h_k : t \in \mathbb{R} \to e^{kt/2} \in \mathbb{R}$ *and* $c_\infty := \lim_{n\to\infty} c_n$. *Moreover,*

$$\mathbf{E}_{c_\infty}(U_\infty) \leq \liminf_{n\to\infty} \mathbf{E}_{c_n}(U_n). \tag{4.18}$$

Proof. Denote $M := \sup_{n\in\mathbb{N}} E_{c_n}(U_n^i)$, which is finite by (4.14). We will now use (H4′). We assume that (2) holds, the argument when (1) holds being similar and easier. Fix any $\mathbf{v}^+ \in \mathcal{F}^\pm$ and for all $n \in \mathbb{N}$, set $v_n := U_n^i(T) \in \mathcal{F}^+$. Define

$$U_n : t \in \mathbb{R} \to \widehat{P}_{(v_n, \mathbf{v}^+)}(U_n^i(t)),$$

where $\widehat{P}_{\mathbf{v}^+}$ is the differentiable operator introduced in (H4′). We apply the properties summarized in (2) of (H4′). Notice that for all $n \in \mathbb{N}$ we have $U_n \in X_T$ due to (3.11). The energy equality (4.15) follows from (3.12) and (3.13). Moreover, (3.10) implies that for all $n \in \mathbb{N}$,

$$P^+(U_n(T)) = P^+\big(\widehat{P}_{(v_n,\mathbf{v}^+)}(U_n^i(T))\big) = P^+(\widehat{P}_{(v_n,\mathbf{v}^+)}(v_n)) = \mathbf{v}^+,$$

which in particular means

$$\|U_n(T) - \mathbf{v}^+\|_{\mathcal{L}} \leq \frac{r_0^+}{2}. \tag{4.19}$$

Notice now that $\mathcal{E}(U(\cdot))$ is nonnegative in $[T, +\infty)$ as $U \in X_T$ by (4.2) in Lemma 4.1; therefore

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad \frac{1}{2}\int_{\mathbb{R}} \|U_n'(t)\|_{\mathcal{L}}^2 e^{c_n t}\, dt &\leq M - \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t}\, dt \\
&\leq M - \int_{-\infty}^{T} \mathcal{E}(U_n(t)) e^{c_n t}\, dt \\
&\leq \sup_{n\in\mathbb{N}} \left\{ M + \frac{a}{c_n} e^{c_n T} \right\} < +\infty.
\end{aligned} \tag{4.20}$$

That is, we have that $(U_n' h_{c_n})_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{R}, \mathcal{L})$. Therefore, there exists $\widetilde{U} \in L^2(\mathbb{R}, \mathcal{L})$ such that

$$U_n' h_{c_n} \rightharpoonup \widetilde{U} \quad \text{weakly in } L^2(\mathbb{R}, \mathcal{L}) \tag{4.21}$$

up to subsequences. Such a condition implies

$$\int_{\mathbb{R}} \|\widetilde{U}(t)\|_{\mathcal{L}}^2\, dt \leq \liminf_{n\to\infty} \int_{\mathbb{R}} \|U_n'(t)\|_{\mathcal{L}}^2 e^{c_n t}\, dt. \tag{4.22}$$

Now, notice that by (4.19) we have that $(U_n(T))_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}$. Therefore, up to an extraction there exists $v_\infty \in \mathcal{L}$ such that

$$U_n(T) \rightharpoonup v_\infty \quad \text{in } \mathcal{L}. \tag{4.23}$$

As in [48], we point out that

$$\forall t \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad U_n(t) = U_n(T) + \int_T^t U_n'(s)\,ds.$$

Now, notice that for all $t \in \mathbb{R}$ we have $\mathbf{1}_{(0,t)} h_{-c_n} \to \mathbf{1}_{(0,t)} h_{-c_\infty}$ in $L^\infty(\mathbb{R})$, where $\mathbf{1}$ stands for the indicator function of a set. Therefore, we obtain by (4.21) and (4.23),

$$\forall t \in \mathbb{R}, \quad U_n(t) \rightharpoonup U_\infty(t) := v_\infty + \int_T^t \widetilde{U}(s) e^{-c_\infty s/2}\,ds,$$

which gives (4.16). Moreover, we have $U_\infty \in H^1_{\text{loc}}(\mathbb{R}, \mathcal{L})$ and $U_\infty' = \widetilde{U} h_{-c_\infty}$, meaning by (4.21) that (4.17) also holds.

Recall now that $\mathcal{E}$ is lower semicontinuous on $\mathcal{L}$ by (H1′), so that (4.16) gives

$$\forall t \in \mathbb{R}, \quad \mathcal{E}(U_\infty(t)) \le \liminf_{n\to\infty} \mathcal{E}(U_n(t)). \tag{4.24}$$

We need to show that $U_\infty \in X_T$ and to establish the inequality (4.18).

- We begin by showing that $U_\infty \in X_T$. We need to show that for all $t \in [T, +\infty)$, it holds that $U_\infty(t) \in \mathcal{F}^+_{r_0^+/2}$ and similarly for $(-\infty, -T]$. Fix $t \in [T, +\infty)$. We have $U_n(t) \in \mathcal{F}^+_{r_0^+/2}$, so we can define the sequence $(\mathbf{v}_n^+(t))_{n\in\mathbb{N}}$ in $\mathcal{F}^+$ as $\mathbf{v}_n^+(t) := P^+(U_n(t))$. We show that such a sequence is bounded. Indeed, we have

$$\forall n \in \mathbb{N}, \quad \|\mathbf{v}_n^+(t)\|_{\mathcal{L}} \le \frac{r_0^+}{2} + \|U_n(t)\|_{\mathcal{L}}$$

and $(U_n(t))_{\mathcal{L}}$ converges weakly in $\mathcal{L}$, so in particular it is bounded. Therefore, up to an extraction we can assume that $\mathbf{v}_n^+(t) \rightharpoonup \mathbf{v}_\infty^+(t) \in \mathcal{L}$ and by (H3′) we have $\mathbf{v}_\infty^+(t) \in \mathcal{F}^\pm$. Now using the convergence properties we get the inequality

$$\|U_\infty(t) - \mathbf{v}_\infty^+(t)\|_{\mathcal{L}} \le \liminf_{n\to\infty} \|U_n(t) - \mathbf{v}_n^+(t)\|_{\mathcal{L}} \le \frac{r_0^+}{2},$$

so that $U_n(t) \in \mathcal{F}^+_{r_0^+/2}$. An identical argument shows that for all $t \in (-\infty, -T]$ we have $U_n(t) \in \mathcal{F}^-_{r_0^-/2}$. Therefore, we have shown that $U_\infty \in X_T$.

- Next we prove (4.18). We have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t}\,dt \le M - \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} \frac{\|U_n'(t)\|_{\mathcal{L}}^2}{2} e^{ct}\,dt < +\infty,$$

by (4.20). Hence, we can apply Fatou's lemma to $(t \in \mathbb{R} \to \mathcal{E}(U_n(t)) e^{c_n t})_{n\in\mathbb{N}}$ (a sequence of functions uniformly bounded below by $-a$) to show

$$\int_{\mathbb{R}} \liminf_{n\to+\infty} \mathcal{E}(U_n(t)) e^{c_n t}\,dt \le \liminf_{n\to\infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t}\,dt,$$

which, combined with (4.24) implies

$$\int_{\mathbb{R}} \mathcal{E}(U_\infty(t))e^{ct}\,dt \leq \liminf_{n\to\infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t))e^{c_n t}\,dt. \tag{4.25}$$

Combining (4.22) and (4.25) we get

$$\mathbf{E}_c(U_\infty) \leq \liminf_{n\to\infty} \int_{\mathbb{R}} \frac{\|U_n'(t)\|_{\mathcal{L}}^2}{2} e^{c_n t}\,dt + \liminf_{n\to\infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t))e^{c_n t}\,dt,$$

which, by superadditivity of the limit inferior gives (4.18). ∎

4.5. Existence of an infimum for $\mathbf{E}_c$ in X_T

The goal now is to show that, for $T \geq 1$ and $c > 0$ fixed, the infimum $\mathbf{m}_{c,T}$ as defined in (4.8) is attained by a function in X_T. This will actually follow easily from Lemma 4.6.

Lemma 4.7. *Assume that (H1′), (H2′), (H3′) and (H4′) hold. Let $c > 0$, $T \geq 1$ and $\mathbf{m}_{c,T}$ be as in (4.8). Then $\mathbf{m}_{c,T}$ is attained for some $\mathbf{U}_{c,T} \in X_T$.*

Proof. By (4.11) in Lemma 4.4, we have that there exists a minimizing sequence $(U_n)_{n\in\mathbb{N}}$ for $\mathbf{E}_c$ in X_T. We apply Lemma 4.6 to $(U_n)_{n\in\mathbb{N}}$ and the sequence of speeds constantly equal to c. We obtain a function $\mathbf{U}_{c,T} \in X_T$ such that

$$\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \liminf_{n\to\infty} \mathbf{E}_c(U_n) = \mathbf{m}_{c,T},$$

due to (4.18). Therefore, $\mathbf{m}_{c,T}$ is attained by $\mathbf{U}_{c,T}$ in X_T. ∎

Subsequently, we show that assumption (H7′) implies that the constrained minimizers are solutions of the equation in a certain set containing $(-T, T)$, with the proper regularity.

Lemma 4.8. *Assume that (H6′) and (H7′) hold. Let $c > 0$, $T \geq 1$ and $\mathbf{m}_{c,T}$ be as in (4.8). Let $\mathbf{U}_{c,T} \in X_T$ be such that $\mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T}$. Then $\mathbf{U}_{c,T} \in \mathcal{A}((-T,T))$, $\mathcal{A}((-T,T))$ as in (3.22) and*

$$\mathbf{U}_{c,T}'' - \mathcal{B}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \quad \text{in } (-T,T). \tag{4.26}$$

Moreover, if $t \geq T$ is such that

$$\operatorname{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \tag{4.27}$$

then there exists $\delta^+(t) > 0$ such that $\mathbf{U}_{c,T} \in \mathcal{A}((t-\delta^+(t), t+\delta^+(t)))$ and

$$\mathbf{U}_{c,T}'' - \mathcal{B}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \quad \text{in } (t-\delta^+(t), t+\delta^+(t)). \tag{4.28}$$

Similarly, if $t \leq -T$ is such that

$$\operatorname{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) < \frac{r_0^-}{2}, \tag{4.29}$$

then there exists $\delta^-(t) > 0$ such that $\mathbf{U}_{c,T} \in \mathcal{A}((t-\delta^-(t), t+\delta^-(t)))$ and

$$\mathbf{U}_{c,T}'' - \mathcal{B}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \quad \text{in } (t-\delta^-(t), t+\delta^-(t)).$$

Proof. We first show that

$$\forall t \in \mathbb{R}, \quad \mathfrak{P}(\mathbf{U}_{c,T}(t)) = \mathbf{U}_{c,T}(t), \tag{4.30}$$

where $\mathfrak{P}$ is the map from (H6′). We claim that the function

$$\mathbf{U}^{\mathfrak{P}}_{c,T} : t \in \mathbb{R} \to \mathfrak{P}(\mathbf{U}_{c,T}(t))$$

belongs to X_T. Indeed, this follows from (3.20) and (3.21). Property (3.19) implies that

$$\forall t \in \mathbb{R}, \quad \mathcal{E}(\mathbf{U}^{\mathfrak{P}}_{c,T}(t)) \leq \mathcal{E}(\mathbf{U}_{c,T}(t)). \tag{4.31}$$

Now take $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{t\}$. Property (3.20) implies that

$$\left\| \frac{\mathbf{U}^{\mathfrak{P}}_{c,T}(t) - \mathbf{U}^{\mathfrak{P}}_{c,T}(s)}{t-s} \right\|_{\mathcal{L}} \leq \left\| \frac{\mathbf{U}_{c,T}(t) - \mathbf{U}_{c,T}(s)}{t-s} \right\|_{\mathcal{L}},$$

which implies that

$$\text{for a.e. } t \in \mathbb{R}, \quad \|(\mathbf{U}^{\mathfrak{P}}_{c,T})'(t)\|_{\mathcal{L}} \leq \|\mathbf{U}'_{c,T}(t)\|_{\mathcal{L}}, \tag{4.32}$$

as the metric derivative coincides with the distributional derivative. By contradiction, assume now that there exists $t \in \mathbb{R}$ such that $\mathbf{U}_{c,T}(t) \neq \mathfrak{P}(\mathbf{U}_{c,T}(t)) = \mathbf{U}^{\mathfrak{P}}_{c,T}(t)$. Property (3.20) implies that $\mathfrak{P}$ is continuous into $\mathcal{L}$. Therefore, since $\mathbf{U}_{c,T}$ is continuous into $\mathcal{L}$, we must have that for some nonempty interval $I_t \ni t$, it holds that

$$\forall s \in I_t, \quad \mathbf{U}_{c,T}(s) \neq \mathfrak{P}(\mathbf{U}_{c,T}(s)) = \mathbf{U}^{\mathfrak{P}}_{c,T}(s),$$

so that, using (3.19) we get

$$\forall s \in I_t, \quad \mathcal{E}(\mathbf{U}^{\mathfrak{P}}_{c,T}(s)) < \mathcal{E}(\mathbf{U}_{c,T}(s)),$$

so that, combining with (4.31) and (4.32) we obtain

$$\mathbf{E}_c(\mathbf{U}^{\mathfrak{P}}_{c,T}) < \mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T},$$

which contradicts the definition of $\mathbf{m}_{c,T}$ (4.8) since $\mathbf{U}^{\mathfrak{P}}_{c,T} \in X_T$. Therefore, we have shown that (4.30) holds. Next, notice that

$$\mathbf{E}_c(\mathbf{U}_{c,T}; [-T, T]) \leq \mathbf{m}_{c,T} + \frac{a}{c} e^{-cT} < +\infty,$$

and for any $\phi \in \mathcal{C}^1_c((-T,T), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ we have $\mathbf{U}_{c,T} + \phi \in X_T$, so that

$$\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \mathbf{E}_c(\mathbf{U}_{c,T} + \phi).$$

Therefore, the restriction of $\mathbf{U}_{c,T}$ in $(-T, T)$ is a local minimizer of $\mathbf{E}_c(\cdot, [-T, T])$ in the sense of Definition 3.1. Since $\mathbf{U}_{c,T}$ also verifies (4.30), we can apply the regularity assumption (H7′). Therefore, $\mathbf{U}_{c,T} \in \mathcal{A}((-T,T))$ and (4.26) holds. Assume now that there exists $t \geq T$ such that (4.27) holds. Then there exists $\mathbf{v}^+(t) \in \mathcal{F}^+$ such that

$$\|\mathbf{U}_{c,T}(t) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2}$$

which, since $\mathbf{U}_{c,T}$ is continuous into $\mathcal{L}$, implies that there exists $\delta^+(t) > 0$ such that

$$\forall s \in (t - \delta^+(t), t + \delta^+(t)), \quad \|\mathbf{U}_{c,T}(s) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2} - d^+(t),$$

where

$$d^+(t) := \frac{1}{2}\Big(\frac{r_0^+}{2} - \|\mathbf{U}_{c,T}(t) - \mathbf{v}^+(t)\|_{\mathcal{L}}\Big) > 0.$$

Therefore, if $\phi \in \mathcal{C}_c^1((t - \delta^+(t), t + \delta^+(t)), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ is such that

$$\max_{t \in [t-\delta^+(t), t+\delta^+(t)]} \|\phi(t)\|_{\mathcal{H}} \leq \frac{d^+(t)}{2},$$

we have

$$\forall s \in (t - \delta^+(t), t + \delta^+(t)), \quad \|\mathbf{U}_{c,T}(s) + \phi(s) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2} - \frac{d^+(t)}{2},$$

so that $\mathbf{U}_{c,T} + \phi \in X_T$. Therefore,

$$\mathbf{E}_{c,T}(\mathbf{U}_{c,T}) \leq \mathbf{E}_{c,T}(\mathbf{U}_{c,T} + \phi).$$

Since ϕ is supported on $[t - \delta^+(t), t + \delta^+(t)]$, the previous argument implies that

$$\mathbf{E}_{c,T}(\mathbf{U}_{c,T}; [t - \delta^+(t), t + \delta^+(t)]) \leq \mathbf{E}_{c,T}(\mathbf{U}_{c,T} + \phi; [t - \delta^+(t), t + \delta^+(t)]),$$

so that $\mathbf{U}_{c,T}$ is a local minimizer of $\mathbf{E}_c(\cdot; [t - \delta^+(t), t + \delta^+(t)])$ in the sense of Definition 3.1. Since (4.30) also holds, we can apply (H7′) and obtain that $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^+(t), t + \delta^+(t)))$ and equation (4.28) holds. If $t \leq -T$ is such that (4.29) holds, the same reasoning shows that for some $\delta^-(t) > 0$, $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^-(t), t + \delta^-(t)))$ and (4.27) holds, which concludes the proof of the result. ∎

4.6. The comparison result

The goal of this section is to obtain relevant information on the behavior of the constrained minimizers. Such information is contained in Corollary 4.1 and it will allow us to remove the constraints later on. In order to carry on these arguments, assumption (H8′) will become necessary since it will show that our problem can be somehow dealt with as in the balanced one, which will allow us to argue in a fashion similar to Smyrnelis

[48]. We begin by introducing some constants. For $0 < r \leq r_0^{\pm}$, recall the definition of $\kappa_r^{\pm}$ introduced in (3.7), Lemma 3.1. We define

$$\eta_0^+ := \min\left\{ \sqrt{e^{-1}\frac{r_0^+}{4}\sqrt{2\kappa^+_{r_0^+/4}}}, \frac{r_0^+}{4} \right\} > 0, \tag{4.33}$$

$$\hat{r}^+ := \frac{r_0^+}{C^+ + 1} > 0, \tag{4.34}$$

$$\mathcal{E}^+_{\max} := \frac{1}{(C^+)^2(C^+ + 1)} \min\left\{ \frac{(\eta_0^+)^2}{4}, \kappa^+_{\eta_0^+}, \beta^+(\hat{r}^+), \beta^+(\eta_0^+) \right\} > 0, \tag{4.35}$$

where the constants $C^{\pm}$, $\beta^{\pm}(\hat{r}^{\pm})$, $\beta(\eta_0^{\pm})$ were introduced in (H5′). Recall that in (3.23), (3.24), (3.25) we introduced the analogous constants

$$\eta_0^- := \min\left\{ \sqrt{e^{-1}\frac{r_0^-}{4}\sqrt{2(\kappa^-_{r_0^-/4} + a)}}, \frac{r_0^-}{4} \right\} > 0, \tag{4.36}$$

$$\hat{r}^- := \frac{r_0^-}{C^- + 1} > 0 \tag{4.37}$$

and

$$\mathcal{E}^-_{\max} := \frac{1}{(C^-)^2(C^- + 1)} \min\left\{ \frac{(\eta_0^-)^2}{4}, \kappa^-_{\eta_0^-} + a, \beta^-(\hat{r}^-), \beta^-(\eta_0^-) \right\} > 0. \tag{4.38}$$

For any $U \in X_T$, define

$$t^-(U, \mathcal{E}^-_{\max}) := \sup\left\{ t \in \mathbb{R} : \mathcal{E}(U(t)) \leq -a + \mathcal{E}^-_{\max} \text{ and } \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \leq \tfrac{r_0^-}{2} \right\} \tag{4.39}$$

and

$$t^+(U, \mathcal{E}^+_{\max}) := \inf\left\{ t \in \mathbb{R} : \mathcal{E}(U(t)) \leq \mathcal{E}^+_{\max} \text{ and } \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \leq \tfrac{r_0^+}{2} \right\}. \tag{4.40}$$

We have the following technical property:

Lemma 4.9. *Assume that (H1′) and (H5′) hold. Let $\hat{r}^{\pm} > 0$ be as in (4.34), (4.37) and $\mathcal{E}^{\pm}_{\max}$ be as in (4.35), (4.38). Then, if $v \in \mathcal{F}^{\pm}_{r_0^{\pm}}$ is such that*

$$\mathcal{E}(v) \leq \min\{\pm a, 0\} + \beta^{\pm}(\hat{r}^{\pm}), \tag{4.41}$$

then

$$\forall \lambda \in [0,1], \quad \mathcal{E}(\lambda v + (1-\lambda)P^{\pm}(v)) \leq \min\{\pm a, 0\} + \mathsf{C}^{\pm}(\mathcal{E}(v) - \min\{\pm a, 0\}). \tag{4.42}$$

In particular, if

$$\mathcal{E}(v) \leq \min\{\pm a, 0\} + \mathcal{E}^{\pm}_{\max}, \tag{4.43}$$

then

$$\forall \lambda \in [0,1], \quad \mathcal{E}(\lambda v + (1-\lambda)P^{\pm}(v)) \leq \min\{\pm a, 0\} + \mathsf{C}^{\pm}\mathcal{E}^{\pm}_{\max}. \tag{4.44}$$

The constants $\mathcal{E}^{\pm}_{\max}$, $\mathsf{C}^{\pm}$ were defined in (4.38), (4.35) and (3.26) respectively, and $P^{\pm}(u)$ is the projection introduced in (H1′).

Proof. Assume that (4.41) holds for $v \in \mathcal{F}^{\pm}_{r_0^{\pm}}$. Then, invoking (H5′), we have $v \in \mathcal{F}_{\mathcal{H},\hat{r}^{\pm}}$, so in particular the projection $P^{\pm}_{\mathcal{H}}(u)$ is well defined. Fix $\lambda \in [0,1]$. Since $v \in \mathcal{F}^{\pm}_{r_0^{\pm}}$, the projection $P^{\pm}(v)$ is well defined by (H1′). Using (3.14) we obtain

$$\begin{aligned}\|\lambda v + (1-\lambda)P^{\pm}(v) - P^{\pm}(v)\|_{\mathcal{H}} &= \lambda\|v - P^{\pm}(v)\|_{\mathcal{H}} \\ &\le (C^{\pm}+1)\|v - P^{\pm}_{\mathcal{H}}(v)\|_{\mathcal{H}} \le (C^{\pm}+1)\hat{r}^{\pm} \end{aligned} \tag{4.45}$$

so that $\lambda v + (1-\lambda)P^{\pm}(v) \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}}$ by the definition of $\hat{r}^{\pm}$ in (4.34), (4.37). Now using (3.14) again, along with the estimate (3.16) in (H5′), we get

$$\|P^{\pm}(v) - P^{\pm}_{\mathcal{H}}(v)\|^2_{\mathcal{H}} \le (C^{\pm})^2(\mathcal{E}(v) - \min\{\pm a, 0\})$$

which, plugging into (4.45), gives

$$\|\lambda v + (1-\lambda)P^{\pm}(v) - P^{\pm}_{\mathcal{H}}(v)\|^2_{\mathcal{H}} \le \frac{1}{2}((C^{\pm})^2 + (C^{\pm}+1)^2)(\mathcal{E}(v) - \min\{\pm a, 0\}),$$

that, using (3.16) again, implies exactly (4.42). Assuming now that (4.43) holds, we have by (4.38), (4.35) that in particular (4.41) holds. Therefore, (4.44) follows from (4.42). ■

Next we have the following property:

Lemma 4.10. *Assume that (H5′) and (H8′) hold. Let $c > 0$ and $T \ge 1$. Assume that $U \in X_T$ is such that $\mathbf{E}_c(U) \le 0$. Then the quantities $t^-(U, \mathcal{E}^-_{\max})$ and $t^+(U, \mathcal{E}^+_{\max})$ defined in (4.39) and (4.40), respectively, are well defined as real numbers. Moreover, it holds that*

$$\mathcal{E}\big(U(t^-(U,\mathcal{E}^-_{\max}))\big) \le -a + \mathcal{E}^-_{\max}, \quad \mathrm{dist}_{\mathcal{L}}(U(t^-(U,\mathcal{E}^-_{\max})), \mathcal{F}^-) \le \frac{r_0^-}{2} \tag{4.46}$$

and

$$\mathcal{E}\big(U(t^+(U,\mathcal{E}^+_{\max}))\big) \le \mathcal{E}^+_{\max}, \quad \mathrm{dist}_{\mathcal{L}}(U(t^+(U,\mathcal{E}^+_{\max})), \mathcal{F}^+) \le \frac{r_0^+}{2}. \tag{4.47}$$

Proof. Using that $\mathbf{E}_c(U) \le 0$ and the fact that $\{t \in \mathbb{R} : \mathcal{E}(U(t)) > 0\}$ is nonempty since $U \in X_T$, we must have

$$\{t \in \mathbb{R} : \mathcal{E}(U(t)) < 0\} \neq \emptyset,$$

and if $v \in \mathcal{L}$ is such that $\mathcal{E}(v) < 0$, then we have $\mathrm{dist}_{\mathcal{L}}(v, \mathcal{F}^-) \le r_0^-/2$ by (3.29) in (H8′) and $\mathcal{E}(v) < -a + \mathcal{E}^-_{\max}$ since we assume $-a + \mathcal{E}^-_{\max} > 0$. Therefore, $t^-(U, \mathcal{E}^-_{\max})$ is well defined, as we have shown that

$$\Big\{t \in \mathbb{R} : \mathcal{E}(U(t)) \le -a + \mathcal{E}^-_{\max} \text{ and } \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \le \tfrac{r_0^-}{2}\Big\} \neq \emptyset$$

and such a set is bounded above by T, because $\mathcal{F}^-_{r_0^-/2} \cap \mathcal{F}^+_{r_0^+/2} = \emptyset$. Using Lemma 4.3, we have that $t^+(U, \mathcal{E}^+_{\max})$ is well defined. Finally, inequalities (4.46) and (4.47) follow because $t \in \mathbb{R} \to \mathcal{E}(U(t)) \in \mathbb{R}$ is lower semicontinuous by (H1′) and $t \to \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^{\pm})$ is continuous whenever $U(t) \in \mathcal{F}^{\pm}_{r_0^{\pm}/2}$ by (H1′) (recall that $t \in \mathbb{R} \to U(t) \in \mathcal{L}$ is continuous because $U \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L})$). ■

The main work is done by the following result:

Proposition 4.1. *Assume that (H5′) and (H8′) hold. Let $c > 0$ and $T \geq 1$. Consider $U \in X_T$ with $\mathbf{E}_c(U) \leq 0$. Let $t^{\pm} := t^{\pm}(U, \mathcal{E}^{\pm}_{\max}, \eta^{\pm}_0)$ be as in (4.39) and (4.40). Then $t^{\pm}$ are well defined by Lemma 4.10. Moreover, if there exists $\tilde{t}^- < t^-$ such that*

$$r_0^- \geq \operatorname{dist}_{\mathcal{L}}(U(\tilde{t}^-), \mathcal{F}^-) \geq \frac{r_0^-}{2}, \tag{4.48}$$

then we can find $\widetilde{U}^- \in X_T$ such that

$$\forall t \leq t^-, \quad \operatorname{dist}_{\mathcal{L}}(\widetilde{U}^-(t), \mathcal{F}^-) < \frac{r_0^-}{2} \tag{4.49}$$

and

$$\mathbf{E}_c(\widetilde{U}^-) < \mathbf{E}_c(U). \tag{4.50}$$

Analogously, if there exists $\tilde{t}^+ > t^+$ such that

$$r_0^+ \geq \operatorname{dist}_{\mathcal{L}}(U(\tilde{t}^+), \mathcal{F}^+) \geq \frac{r_0^+}{2}, \tag{4.51}$$

then we can find $\widetilde{U}^+ \in X_T$ such that

$$\forall t \geq t^+, \quad \operatorname{dist}_{\mathcal{L}}(\widetilde{U}^+(t), \mathcal{F}^+) < \frac{r_0^+}{2} \tag{4.52}$$

and

$$\mathbf{E}_c(\widetilde{U}^+) < \mathbf{E}_c(U). \tag{4.53}$$

Furthermore, we have

$$0 < t^+ - t^- \leq \mathsf{T}_\star(c), \tag{4.54}$$

where

$$\mathsf{T}_\star(c) := \frac{1}{c} \ln\left(\frac{a}{\alpha_\star} + 1\right), \tag{4.55}$$

with $\alpha_\star > 0$ a constant which is independent from c, T and U.

The idea of the proof of Proposition 4.1 is pictured in Figure 4.

Proof of Proposition 4.1. We begin by proving the first part of the result for $\mathcal{F}^-$. Recall that Lemma 4.10 gives

$$\mathcal{E}(U(t^-)) \leq -a + \mathcal{E}^-_{\max}, \tag{4.56}$$

and $U(t^-) \in \mathcal{F}^-_{r_0^-/2}$. Since $\mathcal{E}^-_{\max} \leq \frac{(\eta_0^-)^2}{4(C^-)^2}$ by the definition of $\mathcal{E}^-_{\max}$, (4.38), we have by (3.6) in (H1′) and (H8′) that

$$\operatorname{dist}_{\mathcal{L}}(U(t^-), \mathcal{F}^-) \leq \eta_0^-. \tag{4.57}$$

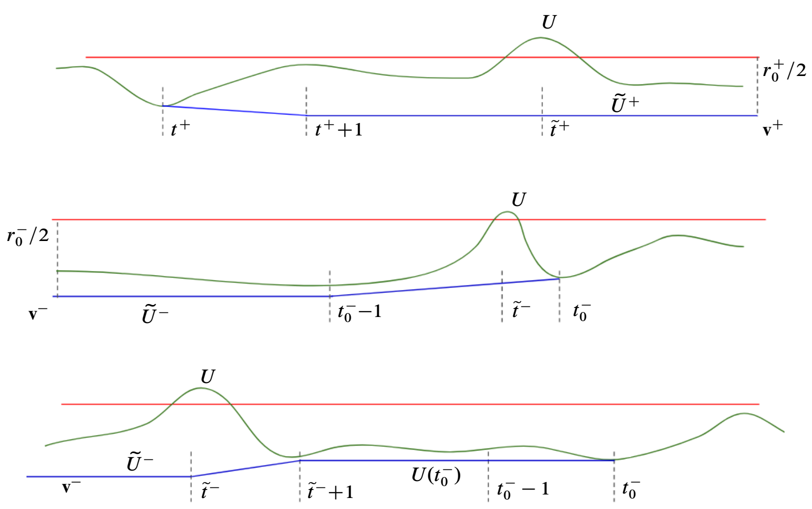

Figure 4. As has been shown, the proof of Proposition 4.1 consists of showing that if the function U gets too far from $\mathcal{F}^{\pm}$ after getting too close, then we can find a suitable competitor with strictly less energy. In the figure, we see a design for the positive case (the competitor $\widetilde{U}^+$ is represented in blue). The second and third pictures correspond to the two possible scenarios for the negative case (the competitor $\widetilde{U}^-$ is represented in blue).

Assume that there exists $\tilde{t}^- < t^-$ such that (4.48) is satisfied. Moreover, we assume, as we can, that

$$\tilde{t}^- := \max\{t \le t^- : \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \ge \tfrac{r_0^-}{2}\} \tag{4.58}$$

(the sup can be replaced by a max by continuity). Define

$$t_0^- := \inf\{t \in [\tilde{t}^-, t^-] : \mathcal{E}(U(t)) \le -a + \mathcal{E}_{\max}^- \text{ and } \mathrm{dist}_{\mathcal{L}}(U(t)) \le \eta_0^-\}. \tag{4.59}$$

Let $\mathbf{v}^- := P^-(U(t_0^-)) \in \mathcal{F}^-$, with P^- as in (H1′). Notice that due to (4.58), we have

$$\forall t \in [t_0^-, t^-], \quad \mathrm{dist}(U(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \tag{4.60}$$

The proof now bifurcates according to two possible cases:

Case 1*:* $t_0^- \le \tilde{t}^- + 1$. In this case, set

$$\widetilde{U}^-(t) := \begin{cases} \mathbf{v}^- & \text{if } t \le t_0^- - 1, \\ (t_0^- - t)\mathbf{v}^- + (t - t_0^- + 1)U(t_0^-) & \text{if } t_0^- - 1 \le t \le t_0^-, \\ U(t) & \text{if } t \ge t_0^-, \end{cases}$$

which belongs to X_T. Due to the definition of $\widetilde{U}^-$ and (4.60), we have that $\widetilde{U}^-$ satisfies (4.49). It remains to check (4.50). We have

$$\int_{t_0^- -1}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt \leq \int_{t_0^- -1}^{t_0^-} \Big[\frac{\|U(t_0^-) - \mathbf{v}^-\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\widetilde{U}^-(t))\Big] e^{ct}\,dt. \tag{4.61}$$

Fix $t \in [t_0^- - 1, t_0^-]$. Choosing $\lambda = t - t_0^- + 1 \in [0,1]$ and applying (4.44) in Lemma 4.9 and (4.56), we have

$$\mathcal{E}(\widetilde{U}^-(t)) \leq -a + \mathsf{C}^- \mathcal{E}_{\max}^-.$$

The previous fact combined with (4.57), (4.56) and (4.61) gives

$$\int_{t_0^- -1}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt \leq \Big(\frac{(\eta_0^-)^2}{2} + \mathsf{C}^- \mathcal{E}_{\max}^-\Big) e^{ct_0^-} + \frac{-a(e^{ct_0^-} - e^{c(t_0^- -1)})}{c}. \tag{4.62}$$

The continuity of U and (4.48) implies that there exists $\tilde{t}_2^- \in (\tilde{t}^-, t_0^-)$ such that

$$\operatorname{dist}(U(\tilde{t}_2^-), \mathcal{F}^-) = \frac{r_0^-}{4} \quad \text{and} \quad \forall t \in [\tilde{t}^-, \tilde{t}_2^-],\ \operatorname{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \geq \frac{r_0^-}{4}. \tag{4.63}$$

Using (4.63), we get

$$\int_{\tilde{t}^-}^{\tilde{t}_2^-} \|U'(t)\|_{\mathcal{L}} e^{ct}\,dt \geq \frac{r_0^- e^{c\tilde{t}^-}}{4} \tag{4.64}$$

and (4.63) also implies

$$\forall t \in [\tilde{t}^-, \tilde{t}_2^-], \quad \mathcal{E}(U(t)) \geq \kappa_{r_0^-/4}^-. \tag{4.65}$$

Inequalities (4.64) and (4.65) along with the definition of η_0^- in (4.36) and Young's inequality give

$$\begin{aligned}\int_{\tilde{t}^-}^{\tilde{t}_2^-} \mathbf{e}_c(U(t))\,dt &\geq \frac{r_0^- e^{c\tilde{t}_0^-}}{4}\sqrt{2(\kappa_{r_0^-/4}^- + a)} - a\frac{e^{c\tilde{t}_2^-} - e^{c\tilde{t}^-}}{c} \\ &= e(\eta_0^-)^2 e^{c\tilde{t}^-} - a\frac{e^{c\tilde{t}_2^-} - e^{c\tilde{t}^-}}{c},\end{aligned}$$

which, also using that $\tilde{t}^- \geq t_0^- - 1$, gets to

$$\begin{aligned}\int_{-\infty}^{t_0^-} \mathbf{e}_c(U(t))\,dt &= \int_{(-\infty, t_0^-]\setminus[\tilde{t}^-, \tilde{t}_2^-]} \mathbf{e}_c(U(t))\,dt + \int_{\tilde{t}^-}^{\tilde{t}_2^-} \mathbf{e}_c(U(t))\,dt \\ &\geq e(\eta_0^-)^2 e^{c\tilde{t}^-} - a\frac{e^{ct_0^-}}{c} \geq (\eta_0^-)^2 e^{ct_0^-} - a\frac{e^{ct_0^-}}{c}.\end{aligned} \tag{4.66}$$

Now using (4.62) we get

$$\begin{aligned}\int_{-\infty}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt &= \int_{-\infty}^{t_0^- -1} \mathbf{e}_c(\widetilde{U}^-(t))\,dt + \int_{t_0^- -1}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt \\ &\leq \Big(\frac{(\eta_0^-)^2}{2} + \mathsf{C}^- \mathcal{E}_{\max}^-\Big) e^{ct_0^-} - \frac{ae^{ct_0^-}}{c}.\end{aligned} \tag{4.67}$$

Therefore, subtracting (4.67) from (4.66), we get

$$\int_{-\infty}^{t_0^-} \mathbf{e}_c(U(t))\,dt - \int_{-\infty}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt \geq \Big(\frac{(\eta_0^-)^2}{2} + \mathsf{C}^-\mathcal{E}_{\max}^-\Big)e^{ct_0^-},$$

which is positive because (4.38) implies

$$\mathsf{C}^-\mathcal{E}_{\max}^- \leq \frac{(\eta_0^-)^2}{4}.$$

Since $\widetilde{U}^-$ and U coincide in $[t_0^-, +\infty)$, the proof of the first case is concluded.

Case 2*:* $t_0^- > \tilde{t} + 1$. In such a case, set

$$\widetilde{U}^-(t) := \begin{cases} \mathbf{v}^- & \text{if } t \leq \tilde{t}^-, \\ (t-\tilde{t}^-)U(t_0^-) + (\tilde{t}^- + 1 - t)\mathbf{v}^- & \text{if } \tilde{t}^- \leq t \leq \tilde{t}^- + 1, \\ U(t_0^-) & \text{if } \tilde{t}^- + 1 \leq t \leq t_0^-, \\ U(t) & \text{if } t_0^- \leq t, \end{cases}$$

which clearly belongs to X_T and for all $t \leq t^-$, $U(t) \in \mathcal{F}^-_{r_0^-/2}$ by (4.60). We have that $\widetilde{U}^-$ is constant in $[\tilde{t}^- + 1, t_0^-]$, and therefore

$$\int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt \leq (-a + \mathcal{E}_{\max}^-)\frac{e^{ct_0^-} - e^{c\tilde{t}^-+1}}{c}$$

and, due to the definitions of $\mathcal{E}_{\max}^-$ in (4.38) and t_0^- in (4.59),

$$\int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(U(t))\,dt \geq \min\{-a + \mathcal{E}_{\max}^-, \kappa_{\eta_0^-}\}\frac{e^{ct_0^-} - e^{c\tilde{t}^-+1}}{c} \geq \int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(\widetilde{U}^-(t))\,dt,$$

because $\mathcal{E}_{\max}^- + a > 0$ by (H8′) and $t_0 \geq \tilde{t}^- + 1$ by assumption. Hence

$$\int_{\tilde{t}^-+1}^{+\infty} \mathbf{e}_c(\widetilde{U}(t))\,dt \leq \int_{\tilde{t}^-+1}^{\infty} \mathbf{e}_c(U(t))\,dt.$$

Arguing as in the first case scenario, we can prove that

$$\int_{-\infty}^{\tilde{t}^-+1} \mathbf{e}_c(\widetilde{U}(t))\,dt < \int_{-\infty}^{\tilde{t}^-+1} \mathbf{e}_c(U(t))\,dt,$$

which concludes the proof of the second case.

To sum up, we have shown that if (4.48) is satisfied, then there exists $\widetilde{U}^-$ such that (4.49) and (4.50) hold, as we wanted.

Assume now that there exists $\tilde{t}^+ > t^+$ such that (4.51) holds. As before, Lemma 4.10 and the definition of $\mathcal{E}_{\max}^+$, (4.35), imply that $t^+ := t^+(U, \mathcal{E}_{\max}^+)$ is such that

$$\operatorname{dist}_{\mathcal{L}}(U(t^+), \mathcal{F}^+) \leq \eta_0^+ \tag{4.68}$$

and

$$\mathcal{E}(U(t^+)) \le \mathcal{E}^+_{\max}. \tag{4.69}$$

We claim that we can assume without loss of generality that

$$\forall t \in [t^+, +\infty), \quad \mathcal{E}(U(t)) \ge 0. \tag{4.70}$$

Indeed, if we can find $t_0 \in (t^+, +\infty)$ such that $\mathcal{E}(U(t)) < 0$, then by (H8′) we have $\mathcal{E}(U(t_0)) \le -a + \mathcal{E}^-_{\max}$ and by (3.29) in (H8′) we also have $\mathrm{dist}_{\mathcal{L}}(U(t_0), \mathcal{F}^-) \le r_0^-/2$. Therefore, we have by the definitions (4.39) and (4.40) that $t^- \ge t_0$ and $t^+ > t^-$, a contradiction since we assume $t_0 > t^+$.

For the positive case, the proof is simpler as it suffices to define $\mathbf{v}^+ := P^+(U(t^+))$ and

$$\tilde{U}^+(t) := \begin{cases} \mathbf{v}^+ & \text{if } t \ge t^+ + 1, \\ (t - t^+)\mathbf{v}^+ + (t^+ + 1 - t)U(t^+) & \text{if } t^+ + 1 \ge t \ge t^+, \\ U(t) & \text{if } t^+ \ge t, \end{cases}$$

which is such that $U \in X_T$. Moreover, it holds that for all $t \ge t^+$, we have $\tilde{U}^+(t) \in \mathcal{F}^+_{r_0^+/2}$. Therefore, the requirements (4.52) and (4.53) hold for $\tilde{U}^+$. It remains to check that (4.53) is also fulfilled. We have

$$\int_{t^+}^{t^+ + 1} \mathbf{e}_c(\tilde{U}^+(t))\, dt = \int_{t^+}^{t^+ + 1} \Big[\frac{\|U(t^+) - \mathbf{v}^+\|^2_{\mathcal{L}}}{2} + \mathcal{E}(\tilde{U}^+(t)) \Big] e^{ct}\, dt. \tag{4.71}$$

Using (4.44) in Lemma 4.9 and (4.69), we get

$$\int_{t^+}^{t^+ + 1} \mathcal{E}(\tilde{U}^+(t)) e^{ct}\, dt \le \mathsf{C}^+ \mathcal{E}^+_{\max} e^{c(t^+ + 1)}. \tag{4.72}$$

Now using (4.68), we get

$$\int_{t^+}^{t^+ + 1} \frac{\|U(t^+) - \mathbf{v}^+\|^2_{\mathcal{L}}}{2} e^{ct}\, dt \le \frac{(\eta_0^+)^2}{2} e^{c(t^+ + 1)}. \tag{4.73}$$

Plugging (4.72) and (4.73) into (4.71), we get

$$\int_{t^+}^{t^+ + 1} \mathbf{e}_c(\tilde{U}^+(t))\, dt \le \Big(\frac{(\eta_0^+)^2}{2} + \mathsf{C}^+ \mathcal{E}^+_{\max} \Big) e^{ct^+ + 1}. \tag{4.74}$$

Since for all $t \ge t^+ + 1$ we have $\tilde{U}^+(t) = \mathbf{v}^+$, we obtain from (4.74),

$$\int_{t^+}^{+\infty} \mathbf{e}_c(\tilde{U}^+(t))\, dt \le \Big(\frac{(\eta_0^+)^2}{2} + \mathsf{C}^+ \mathcal{E}^+_{\max} \Big) e^{ct^+ + 1}. \tag{4.75}$$

Next, notice that by continuity we can find $\tilde{t}_2^+ \in (t^+, \tilde{t}^+)$ such that

$$\mathrm{dist}(U(\tilde{t}_2^+), \mathcal{F}^+) = \frac{r_0^+}{4} \quad \text{and} \quad \forall t \in [\tilde{t}^+, \tilde{t}_2^+], \; \frac{r_0^+}{2} \ge \mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \ge \frac{r_0^+}{4}. \tag{4.76}$$

Therefore, using (4.51) and (4.76), we obtain

$$\int_{\tilde{t}_2^+}^{\tilde{t}^+} \|U'(t)\|_{\mathcal{L}} e^{ct}\, dt \geq \frac{r_0^+}{4} e^{ct^+ + 1} e^{-1} \tag{4.77}$$

and (4.76), (3.6) in (H1′) imply

$$\forall t \in [\tilde{t}^+, \tilde{t}_2^+], \quad \mathcal{E}(U(t)) \geq \kappa^+_{r_0^+/4}. \tag{4.78}$$

Inequalities (4.77), (4.78) yield, by Young's inequality,

$$\int_{\tilde{t}_2^+}^{\tilde{t}^+} \mathbf{e}_c(U(t))\, dt \geq \frac{r_0^+}{4} e^{ct^+ + 1} e^{-1} \sqrt{2\kappa^+_{r_0^+/4}} = (\eta_0^+)^2 e^{t^+ + 1},$$

where the last equality is due to the definition of η_0^+ in (4.33). Combining with (4.70), we get

$$\int_{t^+}^{+\infty} \mathbf{e}_c(U(t))\, dt \geq (\eta_0^+)^2 e^{t^+ + 1}.$$

The definition of $\mathcal{E}^+_{\max}$ in (4.35) together with (4.75) implies then that

$$\int_{t^+}^{+\infty} \mathbf{e}_c(U(t))\, dt > \int_{t^+}^{+\infty} \mathbf{e}_c(\tilde{U}^-(t))\, dt,$$

which establishes (4.53).

We now show the last part of the proof: we show that (4.54) holds with the constant $T_\star(c)$ defined in (4.55). The argument is the same as in [5, Lemma 2.10]. Assume by contradiction that there exists $t \in (t^-, +\infty)$ such that $\mathcal{E}(U(t)) < 0$. Then, arguing as above, we must have $t < t^-$ by the definition of t^- in (4.39), a contradiction. Therefore, we can write

$$\mathbf{E}_c(U) = \frac{1}{2} \int_{\mathbb{R}} \|U'(t)\|^2_{\mathcal{L}} e^{ct}\, dt - \int_{-\infty}^{t^-} \mathcal{E}^-(U(t)) e^{ct}\, dt + \int_{\mathbb{R}} \mathcal{E}^+(U(t)) e^{ct}\, dt, \tag{4.79}$$

where $\mathcal{E}^-$ and $\mathcal{E}^+$ stand for the positive and the negative parts of $\mathcal{E}$, respectively. We have

$$\int_{-\infty}^{t^-} \mathcal{E}^-(U(t)) e^{ct}\, dt \leq \frac{a}{c} e^{ct^-}. \tag{4.80}$$

Set $\alpha_\star := \min\{\mathcal{E}^+_{\max}, \mathcal{E}^-_{\max} - a\} > 0$, which is independent of U, c and T. Notice that for all $t \in (t^-, t^+)$ we have $\mathcal{E}(U(t)) \geq \alpha_\star$. Indeed, if $\mathcal{E}(U(t)) < \alpha_\star$, then by the definition of t^- and t^+ in (4.39) and (4.40) we get

$$\mathrm{dist}_{\mathcal{L}}(U(t), \mathcal{F}^\pm) \geq \frac{r_0^\pm}{2},$$

which implies that

$$\mathcal{E}(U(t)) \geq \min\{\kappa^+_{r_0^+/2}, \kappa^-_{r_0^-/2} - a\} \geq \alpha_\star,$$

by (4.38) and (4.35), a contradiction. Therefore,

$$\int_{\mathbb{R}} \mathcal{E}^+(U(t))e^{ct}\,dt \geq \int_{t^-}^{t^+} \mathcal{E}^+(U(t))e^{ct}\,dt \geq \frac{\alpha_\star}{c}(e^{ct^+} - e^{ct^-}). \tag{4.81}$$

Plugging (4.80) and (4.81) into (4.79) and using that $\mathbf{E}_c(U) \leq 0$, we obtain

$$0 \geq -\frac{a}{c}e^{ct^-} + \frac{\alpha_\star}{c}(e^{ct^+} - e^{ct^-}) \geq \Big(-\frac{a}{c} + \frac{\alpha_\star}{c}(e^{c(t^+-t^-)} - 1)\Big)e^{ct^-},$$

that is,

$$0 \geq -\Big(\frac{a}{\alpha_\star} + 1\Big) + e^{c(t^+-t^-)},$$

which implies

$$0 < t^+ - t^- \leq \frac{1}{c}\ln\Big(\frac{a}{\alpha_\star} + 1\Big) = \mathsf{T}_\star(c),$$

which is exactly (4.54) according to the definition (4.55). ∎

The importance of Proposition 4.1 is summarized by the following result, which gives important information on the behavior of the constrained minimizers:

Corollary 4.1. *Assume that (H3′), (H4′), (H5′) and (H8′) hold. Let $c > 0$ and $T \geq 1$. Let $\mathbf{U}_{c,T}$ be an associated minimizer of $\mathbf{E}_c$ in X_T given by Lemma 4.7. Then, if $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}^\pm_{\max})$ are as in (4.39), (4.40), it holds that*

$$\forall t \leq t^-, \quad \mathrm{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) < \frac{r_0^-}{2} \tag{4.82}$$

and

$$\forall t \geq t^+, \quad \mathrm{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}. \tag{4.83}$$

Moreover, we have

$$\forall t \geq t^-, \quad \mathcal{E}(\mathbf{U}_{c,T}(t)) \geq 0. \tag{4.84}$$

Finally, we have that if $\mathbf{E}_c(\mathbf{U}_{c,T}) \leq 0$, then

$$0 < t^+ - t^- \leq \mathsf{T}_\star(c), \tag{4.85}$$

where $\mathsf{T}_\star(c)$ is as in (4.55). In particular, the function

$$c \in (0, +\infty) \to \mathsf{T}_\star(c)$$

is continuous.

Proof. If we assume by contradiction that (4.83) does not hold, then we necessarily have that there exists $\tilde{t}^- < t^-$ such that (4.48) holds. Proposition 4.1 implies then the existence of $\widetilde{U} \in X_T$ such that $\mathbf{E}_c(\widetilde{U}) < \mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T}$, a contradiction. Therefore, (4.83) holds. Similarly, we can show that (4.82) also holds. Finally, in order to establish (4.84), we argue as in the proof Proposition 4.1. Indeed, due to the definition of t^-, we have that for $t \geq t^-$ it holds that either

$$\mathcal{E}(\mathbf{U}_{c,T}(t)) \geq -a + \mathcal{E}^-_{\max} > 0$$

(which is (H8′)) or

$$\mathrm{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) \geq \frac{r_0^-}{2}$$

which by (3.29) in (H8′) implies that $\mathcal{E}(\mathbf{U}_{c,T}(t)) \geq 0$. Therefore, (4.84) holds and the proof is concluded. ∎

Moreover, Lemma 4.8 applies to $\mathbf{V}_{c,T}$ as follows:

Corollary 4.2. *Assume that (H3′), (H4′), (H5′) and (H8′) hold. Let $c > 0$ and $T \geq 1$. Let $\mathbf{U}_{c,T}$ be an associated minimizer of $\mathbf{E}_c$ in X_T given by Lemma 4.7. Then, if $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}^\pm_{\max})$ are as in (4.39), (4.40), it holds that there exists $\delta_{c,T} > 0$ such that the set*

$$S_{c,T} := (-\infty, t^- + \delta_{c,T}) \cup (-T, T) \cup (t^+ - \delta_{c,T}, +\infty)$$

is such that $\mathbf{U}_{c,T} \in \mathcal{A}(S_{c,T})$ (see (3.22)) and

$$\mathbf{U}''_{c,T} - \mathcal{B}(\mathbf{U}_{c,T}) = -c\mathbf{U}'_{c,T} \quad \text{in } S_{c,T}. \tag{4.86}$$

The proof of 4.2 is obtained in a straightforward manner by combining Lemma 4.8 with the information given by Corollary 4.1. Notice that Corollary 4.2 implies that constrained solutions are piecewise solutions and, in particular, they solve the equation for times with large absolute value.

4.7. Existence of the unconstrained solutions

We now establish the existence of the unconstrained solutions making use of the previous comparison results. As in [5,6], we define the set

$$\mathcal{C} := \left\{c > 0 : \exists T \geq 1 \text{ and } U \in X_T \text{ such that } \mathbf{E}_c(U) < 0\right\}. \tag{4.87}$$

We first prove some important properties for $\mathcal{C}$ which are the same as [5, Lemma 2.12] and [6, Lemma 27]:

Lemma 4.11. *Assume that (H3′), (H4′) and (H5′) hold. Let $\mathcal{C}$ be the set defined in (4.87). Then $\mathcal{C}$ is open and nonempty. Moreover, if we assume that (H8′) holds, then $\mathcal{C}$ is also bounded with*

$$\sup \mathcal{C} \leq \frac{\sqrt{2a}}{d_0}, \tag{4.88}$$

where d_0 was defined in (3.28).

Proof. First, we show that $\mathcal{C} \neq \emptyset$. For that purpose, consider the function Ψ introduced in (4.9). Consider the function

$$f : c \in (0, +\infty) \to e^{-c}\left(-\frac{a}{c} + e^{2c}\int_{-1}^{1}\left(\frac{\|\Psi'(t)\|^2_{\mathcal{L}}}{2} + \mathcal{E}(\Psi(t))\right)dt\right) \in \mathbb{R},$$

which is well defined by Lemma 4.4. We have that for all $c > 0$,

$$\mathbf{E}_c(\Psi) = \frac{a}{c}e^{-c} + \int_{-1}^{1}\Big(\frac{\|\Psi'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\Psi(t))\Big)e^{ct}\,dt \leq f(c) \tag{4.89}$$

and f is a continuous function such that $\lim_{c\to 0} f(c) = -\infty$ because $a < 0$. Moreover, we have that for all $c > 0$,

$$f'(c) = e^{-c}a + ce^{2c}\int_{-1}^{1}\Big(\frac{\|\Psi'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\Psi(t))\Big)\,dt > 0$$

and $\lim_{c\to+\infty} f(c) = +\infty$. Therefore, there exists a unique $c_\Psi > 0$ such that $f(c_\Psi) = 0$ and for all $c < c_\Psi$ we have $\mathbf{E}_c(\Psi) < 0$ by (4.89). Therefore, $(0, c_\Psi) \subset \mathcal{C}$, meaning that $\mathcal{C} \neq \emptyset$ as we wanted to show.

We next prove that $\mathcal{C}$ is open. Let $c \in \mathcal{C}$; we have $\mathbf{E}_c(\mathbf{U}_{c,T}) < 0$, where $\mathbf{U}_{c,T}$ is a minimizer of $\mathbf{E}_c$ in X_T given by Lemma 4.7. By (4.6) in Lemma 4.3, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[T, +\infty)$ such that $t_n \to +\infty$ and

$$\lim_{n\to\infty} \mathcal{E}(\mathbf{U}_{c,T}(t_n))e^{ct_n} = 0. \tag{4.90}$$

Up to subsequences, we have that for all $n \in \mathbb{N}$, $\mathbf{U}_{c,T}(t_n) \in \mathcal{F}^+_{r_0^+}$. Hence, we can define

$$\mathbf{U}^n_{c,T}(s) := \begin{cases} \mathbf{U}_{c,T}(s) & \text{if } s \leq t, \\ (1 + t_n - s)\mathbf{U}_{c,T}(t_n) + (s - t_n)P^+(\mathbf{U}_{c,T}(t_n)) & \text{if } t_n \leq s \leq t_n + 1, \\ P^+(\mathbf{U}_{c,T}(t_n)) & \text{if } t_n + 1 \leq s. \end{cases}$$

We have that for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{E}_c(\mathbf{U}^n_{c,T}(s)) &= \int_{-\infty}^{t_n} \mathbf{e}_c(\mathbf{U}_{c,T}(s))\,ds + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2}{2}e^{ct_n} \\ &\quad + \int_{t_n}^{t_n+1} \mathcal{E}(\mathbf{U}^n_{c,T}(s))e^{cs}\,ds \\ &\leq \mathbf{E}_c(\mathbf{U}_{c,T}) + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2}{2}e^{ct_n} \\ &\quad + \int_{t_n}^{t_n+1} \mathcal{E}((1+t_n-s)\mathbf{U}_{c,T}(t_n) + (s-t_n)P^+(\mathbf{U}_{c,T}(t_n)))e^{ct_n}\,ds, \end{aligned} \tag{4.91}$$

where we have used that $t_n \geq T$ in order to obtain the inequality. Let $\beta^+(\hat{r}^+)$ be as in Lemma 4.9. Up to a subsequence, we have that for all $n \in \mathbb{N}$ it holds that $\mathcal{E}(\mathbf{U}_{c,T}(t_n)) \leq \beta^+(\hat{r}^+)$. Therefore, by Lemma 4.9 we have that for all $\lambda \in [0,1]$ and $n \in \mathbb{N}$ it holds that

$$\mathcal{E}\big(\lambda\mathbf{U}_{c,T}(t_n) + (1-\lambda)P^+(\mathbf{U}_{c,T}(t_n))\big) \leq \mathsf{C}^+\mathcal{E}(\mathbf{U}_{c,T}(t_n)),$$

where $\mathtt{C}^+ > 0$ is independent of n (see (3.26)). Plugging into (4.91), we obtain that for all $n \in \mathbb{N}$ it holds that

$$\mathbf{E}_c(\mathbf{U}^n_{c,T}(s)) \leq \mathbf{E}_c(\mathbf{U}_{c,T}) + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|^2_{\mathcal{L}}}{2} e^{ct_n} + \mathtt{C}^+ \mathcal{E}(\mathbf{U}_{c,T}(t_n)) e^{ct_n}. \tag{4.92}$$

Notice also that (4.90) implies in particular that

$$\lim_{n \to +\infty} \|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|^2_{\mathcal{L}} e^{ct_n} = 0,$$

which, in combination with inequalities (4.90) and (4.92), together with the fact that

$$\mathbf{E}_c(\mathbf{U}_{c,T}) < 0,$$

gives that there exists $N \in \mathbb{N}$ such that $\mathbf{E}_c(\mathbf{U}^N_{c,T}) < 0$. Since $\mathbf{U}^N_{c,T}$ is constant in $[t_N + 1, +\infty)$, we have that for all $\tilde{c} > 0$, $\mathbf{E}_{\tilde{c}}(\mathbf{U}^N_{c,T}) < +\infty$. Therefore, by Lemma 4.5 we have that

$$\tilde{c} \in (0, +\infty) \to \mathbf{E}_{\tilde{c}}(\mathbf{U}^N_{c,T}) \in \mathbb{R}$$

is well defined and continuous. Therefore, we can find some $\delta > 0$ such that for all $\tilde{c} \in (c - \delta, c + \delta)$, $\mathbf{E}_{\tilde{c}}(\mathbf{U}^N_{c,T}) < 0$. As a consequence, we have $(c - \delta, c + \delta) \subset \mathcal{C}$, which shows that $\mathcal{C}$ is open.

We now assume that (H8′) holds and we use it to establish the bound (4.88). In particular, we can apply Proposition 4.1. Let $c > 0$ and $T \geq 1$ be such that $\mathbf{E}_c(\mathbf{U}_{c,T}) < 0$ with $\mathbf{U}_{c,T} \in X_T$ a minimizing solution given by Lemma 4.7. Let $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}^\pm_{\max})$ be as in (4.39), (4.40). Inequality (4.85) in Proposition 4.1 implies that $t^- < t^+$. Recall the definition of d_0 in (3.28) and the fact that $\mathbf{U}_{c,T}(t^\pm) \in \mathcal{F}^\pm_{r_0^\pm/2}$. These facts imply

$$d_0 \leq \|\mathbf{U}_{c,T}(t^+) - \mathbf{U}_{c,T}(t^-)\|_{\mathcal{L}}. \tag{4.93}$$

Since (H8′) holds, we can use (4.84) in Corollary 4.1 to obtain

$$\begin{aligned}\|\mathbf{U}_{c,T}(t^+) - \mathbf{U}_{c,T}(t^-)\|^2_{\mathcal{L}} &\leq 2 \int_{\mathbb{R}} \frac{\|\mathbf{U}'_{c,T}(t)\|^2}{2} e^{ct}\, dt \Big(\frac{e^{-ct^-} - e^{-ct^+}}{c} \Big) \\ &\leq 2 \Big(\mathbf{E}_c(\mathbf{U}_{c,T}) + \frac{a}{c} e^{ct^-} \Big) \Big(\frac{e^{-ct^-} - e^{-ct^+}}{c} \Big).\end{aligned}$$

Now using that $\mathbf{E}_c(\mathbf{U}_{c,T}) \leq 0$, the fact that $t^- < t^+$ and (4.93), the inequality above becomes

$$d_0^2 \leq 2a \frac{1 - e^{c(t^- - t^+)}}{c^2} \leq \frac{2a}{c^2},$$

so that (4.88) follows. ∎

We now have all the ingredients for establishing the existence of the unconstrained solutions:

Proposition 4.2. *Assume that (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Let* $\bar{c} \in \partial(\mathcal{C}) \cap (0, +\infty)$*, where* $\partial(\mathcal{C})$ *stands for the boundary of the set* $\mathcal{C}$ *defined in (4.87). Then there exist* $\bar{T} \geq 1$ *such that* $\mathbf{m}_{\bar{c},\bar{T}} = 0$ *(*$\mathbf{m}_{\bar{c},T}$ *as in (4.8)) and* $\bar{\mathbf{U}} \in X_{\bar{T}}$ *an associated minimizer of* $\mathbf{E}_{\bar{c}}$ *in* $X_{\bar{T}}$ *which does not saturate the constraints, i.e.*

$$\forall t \geq \bar{T}, \quad \operatorname{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \tag{4.94}$$

and

$$\forall t \leq -\bar{T}, \quad \operatorname{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \tag{4.95}$$

Moreover, $\bar{\mathbf{U}} \in \mathcal{A}(\mathbb{R})$ *and the pair* $(\bar{c}, \bar{\mathbf{U}})$ *solves (3.1).*

Remark 4.3. Notice that Lemma 4.11 implies that (under the necessary assumptions) the set $\mathcal{C}$ is bounded, meaning that $\partial(\mathcal{C}) \cap (0, +\infty) \neq \emptyset$. Such a fact, in combination with Proposition 4.2, shows the existence of the unconstrained solutions.

Proof of Proposition 4.2. By Lemma 4.11, we have that $\mathcal{C} \neq \emptyset$ is open, which implies that $\partial(\mathcal{C}) \subset \mathbb{R} \setminus \mathcal{C}$. Therefore, we have $\bar{c} \notin \mathcal{C}$. Recall that due to the definition of $\mathcal{C}$ in (4.87), we have

$$\forall T \geq 1, \quad \mathbf{m}_{\bar{c},T} \geq 0. \tag{4.96}$$

The definition of the boundary allows us to consider a sequence $(c_n)_{n\in\mathbb{N}}$ contained in $\mathcal{C}$ such that $c_n \to \bar{c}$. Then, for each $n \in \mathbb{N}$, there exists $T_n \geq 1$ such that $\mathbf{E}_{c_n}(\mathbf{U}_{c_n,T_n}) < 0$, where, for each $n \in \mathbb{N}$, $\mathbf{U}_{c_n,T_n}$ is a minimizer of $\mathbf{E}_{c_n}$ in X_{T_n}. For each $n \in \mathbb{N}$, set $t_n^{\pm} := t^{+}(\mathbf{U}_{c_n,T_n}, \mathcal{E}_{\max}^{\pm})$ as in (4.39), (4.40). Using (4.85) in Corollary 4.1 we have

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_n^- \leq \mathsf{T}_\star(c_n),$$

and the function

$$c \in (0, +\infty) \to \mathsf{T}_\star(c) \in (0, +\infty)$$

is continuous. Since the sequence $(c_n)_{n\in\mathbb{N}}$ is bounded, we have

$$\mathsf{T}_\star := \max\{1, \sup_{n\in\mathbb{N}} \mathsf{T}_\star(c_n)\} < +\infty$$

and

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_n^- \leq \mathsf{T}_\star, \tag{4.97}$$

so that we have a bound on $(t_n^+ - t_n^-)_{n\in\mathbb{N}}$. Moreover, (4.82) and (4.83) in Corollary 4.1 imply

$$\forall n \in \mathbb{N}, \ \forall t \geq t_n^+, \quad \operatorname{dist}_{\mathcal{L}}(\mathbf{U}_{c_n,T_n}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \tag{4.98}$$

and

$$\forall n \in \mathbb{N}, \ \forall t \leq t_n^-, \quad \operatorname{dist}_{\mathcal{L}}(\mathbf{U}_{c_n,T_n}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \tag{4.99}$$

For each $n \in \mathbb{N}$, define the function $\mathbf{U}^{t_n^+}_{c_n,T_n} := \mathbf{U}_{c_n,T_n}(\cdot + t_n^+)$. Then (4.97) implies that (4.98) and (4.99) can be written as

$$\forall n \in \mathbb{N},\ \forall t \geq 0, \qquad \mathrm{dist}_{\mathcal{L}}(\mathbf{U}^{t_n^+}_{c_n,T_n}(t), \mathcal{F}^-) < \frac{r_0^-}{2}$$

and

$$\forall n \in \mathbb{N},\ \forall t \leq -\mathsf{T}_\star, \quad \mathrm{dist}_{\mathcal{L}}(\mathbf{U}_{c_n,T_n}(t), \mathcal{F}^-) < \frac{r_0^-}{2},$$

so that for all $n \in \mathbb{N}$ we have $\mathbf{U}^{t_n^+}_{c_n,T_n} \in X_{\mathsf{T}_\star}$. Moreover, a computation shows

$$\forall n \in \mathbb{N}, \quad \mathbf{E}_{c_n}(\mathbf{U}^{t_n^+}_{c_n,T_n}) = e^{-c_n t_n^+}\mathbf{E}_{c_n}(\mathbf{U}_{c_n,T_n}) < 0.$$

Therefore, if we apply Lemma 4.6 with sequence of speeds $(c_n)_{n\in\mathbb{N}}$ and the sequence $(\mathbf{U}^{t_n^+}_{c_n,T_n})_{n\in\mathbb{N}}$ in $X_{\mathsf{T}_\star}$, we obtain $\bar{\mathbf{U}} \in X_{\mathsf{T}_\star}$ such that

$$\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) \leq \liminf_{n\to\infty} \mathbf{E}_{c_n}(\mathbf{U}^{t_n^+}_{c_n,T_n}) \leq 0,$$

which in combination with (4.96) implies that $\mathbf{m}_{\bar{c},\mathsf{T}_\star} = 0$. Therefore, we have $\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) = 0$, so that $\bar{\mathbf{U}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in $X_{\mathsf{T}_\star}$. Set $t_\star^\pm := t^\pm(\bar{\mathbf{U}}, \mathcal{E}^\pm_{\max})$ as in (4.39), (4.40). Invoking (4.96) and Corollary 4.2, we obtain that for all $T \geq 1$ such that $\mathbf{m}_{\bar{c},T} = 0$ and $\bar{\mathbf{U}} \in X_T$, we have $\bar{\mathbf{U}} \in \mathcal{A}(S_{\bar{c},T})$ with

$$S_{\bar{c},T} := (-\infty, t_\star^- + \delta_\star(T)) \cup (-T, T) \cup (t_\star^+ - \delta_\star(T), +\infty) \tag{4.100}$$

for some $\delta_\star(T) > 0$ and

$$\bar{\mathbf{U}}'' - \mathcal{B}(\bar{\mathbf{U}}) = -\bar{c}\bar{\mathbf{U}}' \quad \text{in } S_{\bar{c},T}.$$

Moreover, using (4.82) and (4.83) in Corollary 4.1, we obtain as before that

$$\forall t \geq t_\star^+, \quad \mathrm{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \tag{4.101}$$

and

$$\forall t \leq t_\star^-, \quad \mathrm{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \tag{4.102}$$

Therefore, if we set $\bar{T} = \max\{1, t_\star^+, -t_\star^-\}$, then (4.101) and (4.102) imply that $\bar{\mathbf{U}} \in X_{\bar{T}}$ and that (4.94), (4.95) hold. Moreover, we have $\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) = 0$, so that $\bar{\mathbf{U}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in $X_{\bar{T}}$ by (4.96). Therefore, we obtain that $\bar{\mathbf{U}} \in \mathcal{A}(S_{\bar{c},\bar{T}})$ and

$$\bar{\mathbf{U}}'' - \mathcal{B}(\bar{\mathbf{U}}) = -\bar{c}\bar{\mathbf{U}}' \quad \text{in } S_{\bar{c},\bar{T}},$$

with $S_{\bar{c},\bar{T}}$ as in (4.100). The choice of $\bar{T}$ implies that $S_{\bar{c},\bar{T}} = \mathbb{R}$. Therefore, $\bar{\mathbf{U}} \in \mathcal{A}(\mathbb{R})$ and $(\bar{c}, \bar{\mathbf{U}})$ solves (3.1), which finishes the proof. ∎

Notice that our Proposition 4.2 follows very similar lines to the analogous results in [5, 6].

4.8. Uniqueness of the speed

The precise statement of the uniqueness result is as follows:

Proposition 4.3. *Assume that (H6′) and (H7′) hold. Let X be the set defined in (3.30). Let $(c_1, c_2) \in (0, +\infty)^2$ be such that there exist $\mathbf{U}_1$ and $\mathbf{U}_2$ in $X \cap \mathcal{A}(\mathbb{R})$ such that $(c_1, \mathbf{U}_1)$ and $(c_2, \mathbf{U}_2)$ solve (3.1) and for each $i \in \{1, 2\}$, $\mathbf{E}_{c_i}(\mathbf{U}_i) < +\infty$. Assume moreover that*

$$\forall i \in \{1,2\}, \ \forall j \in \{1,2\} \setminus \{i\}, \quad \mathbf{E}_{c_i}(\mathbf{U}_j) \geq 0. \tag{4.103}$$

Then we have $c_1 = c_2$.

Proof. We prove the result by contradiction. Hence, we can assume without loss of generality that $c_1 < c_2$. A direct computation shows that for every $(c, U) \in (0, +\infty) \times (X \cap \mathcal{A}(\mathbb{R}))$ a solution to (3.1), we have

$$\forall t \in \mathbb{R}, \quad \frac{\|U'(t)\|^2_{\mathcal{L}}}{2} + \mathcal{E}(U(t)) = e^{-ct}\Big(\frac{e^{ct}}{c}\Big(\mathcal{E}(U(t)) - \frac{\|U'(t)\|^2_{\mathcal{L}}}{2}\Big)\Big)'. \tag{4.104}$$

Replacing (c_2, U_2) in (4.104) and multiplying for each $t \in \mathbb{R}$ by $e^{c_1 t}$, computations show that

$$\begin{aligned}\forall t_1 < t_2, \quad c_1 \mathbf{E}_{c_1}(\mathbf{U}_2; (t_1, t_2)) = (c_1 - c_2) \int_{t_1}^{t_2} \|\mathbf{U}_2'(t)\|^2_{\mathcal{L}} e^{c_1 t}\, dt \\ + \Big[e^{c_1 t}\Big(\mathcal{E}(\mathbf{U}_2(t)) - \frac{\|\mathbf{U}_2'(t)\|^2_{\mathcal{L}}}{2}\Big)\Big]_{t_1}^{t_2}.\end{aligned} \tag{4.105}$$

Notice now that the definition of X in (3.30) implies that

$$X = \bigcup_{T \geq 1} X_T,$$

which means that there exists $T \geq 1$ such that $\mathbf{U}_2 \in X_T$. Then, combining Lemma 4.1 and the fact that $\mathbf{E}_{c_2}(\mathbf{U}_2) < +\infty$, we get that $\mathbf{e}_{c_2}(\mathbf{U}_2(\cdot)) \in L^1(\mathbb{R})$. Therefore, we can find two sequences $(t_n^+)_{n \in \mathbb{N}}$ and $(t_n^-)_{n \in \mathbb{N}}$ such that $t_n^{\pm} \to \pm\infty$ and

$$\lim_{n \to \infty} \mathbf{e}_{c_2}(\mathbf{U}_2(t_n^{\pm})) = 0. \tag{4.106}$$

Since we have $c_1 < c_2$, it holds that

$$\forall t \in \mathbb{R}, \quad e^{c_1 t}\Big|\mathcal{E}(\mathbf{U}_2(t)) - \frac{\|\mathbf{U}_2'(t)\|^2_{\mathcal{L}}}{2}\Big| \leq |\mathbf{e}_{c_2}(\mathbf{U}_2(t))|,$$

which in combination with (4.106) implies

$$\lim_{n \to \infty} e^{c_1 t_n^{\pm}}\Big(\mathcal{E}(\mathbf{U}_2(t_n^{\pm})) - \frac{\|\mathbf{U}_2'(t_n^{\pm})\|^2_{\mathcal{L}}}{2}\Big) = 0. \tag{4.107}$$

Therefore, if we take $t_1 = t_n^-$ and $t_2 = t_n^+$ in (4.105) and then pass to the limit $n \to \infty$, then by (4.107) it follows that

$$c_1 \mathbf{E}_{c_1}(\mathbf{U}_2) = (c_1 - c_2) \int_{\mathbb{R}} \|\mathbf{U}_2'(t)\|_{\mathcal{L}}^2 e^{c_1 t} \, dt < 0,$$

because we assume $c_1 < c_2$. However, by (4.103) we have $\mathbf{E}_{c_1}(\mathbf{U}_2) \geq 0$, which is a contradiction. ■

Remark 4.4. Again, the proof of Proposition 4.3 is essentially a direct adaptation of that given in [5,6]. Our hypotheses are slightly weaker, since we only assume that the solutions have finite energies and (4.103), while in [5,6] it is assumed that the solutions are global minimizers of the corresponding energy functionals. Notice also that (H8′) is not needed for proving Proposition 4.3, which holds in a more general setting.

Proposition 4.3 along with Proposition 4.2 allows us to show that the set $\mathcal{C}$ defined in (4.87) is in fact an open interval:

Corollary 4.3. *Assume that (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Let*

$$c(\mathcal{C}) := \sup \mathcal{C}.$$

Then we have $\mathcal{C} = (0, c(\mathcal{C}))$.

Proof. The statement of the result is equivalent to showing that

$$\partial(\mathcal{C}) \cap (0, +\infty) = \{c(\mathcal{C})\}.$$

The quantity $c(\mathcal{C})$ belongs to $(0, +\infty)$ because $\mathcal{C}$ is nonempty and bounded by Lemma 4.11. Therefore, we have $c(\mathcal{C}) \in \partial(\mathcal{C}) \cap (0, +\infty)$ because $\mathcal{C}$ is open, so it does not contain its limit points. By Proposition 4.2, we find $\mathbf{U}^{\mathcal{C}} \in X$ such that $(c(\mathcal{C}), \mathbf{U}^{\mathcal{C}})$ solves (3.1). Now let $\bar{c} \in \partial(\mathcal{C}) \cap (0, +\infty)$. If we show that $\bar{c} = c(\mathcal{C})$, the proof will be finished. Applying Proposition 4.2 with $\bar{c}$, we find $\bar{\mathbf{U}} \in X$ such that $(\bar{c}, \bar{\mathbf{U}})$ solves (3.1). Proposition 4.2, along with the fact that $\bar{c}$ and $c(\mathcal{C})$ do not belong to $\mathcal{C}$, also implies that

$$\inf_{U \in X} \mathbf{E}_{\bar{c}}(U) = \mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) = 0 = \mathbf{E}_{c(\mathcal{C})}(\mathbf{U}^{\mathcal{C}}) = \inf_{U \in X} \mathbf{E}_{c(\mathcal{C})}(U),$$

so that

$$\mathbf{E}_{c(\mathcal{C})}(\bar{\mathbf{U}}) \geq 0 \quad \text{and} \quad \mathbf{E}_{\bar{c}}(\mathbf{U}^{\mathcal{C}}) \geq 0,$$

meaning that we can apply Proposition 4.3 to $(c(\mathcal{C}), \mathbf{U}^{\mathcal{C}})$, $(\bar{c}, \bar{\mathbf{U}})$. As a consequence, we have $\bar{c} = c(\mathcal{C})$, which concludes the proof. ■

4.9. Proof of Theorem 4 completed

All the elements of the proof of Theorem 4 are already present in the previous result. Indeed, Proposition 4.2 along with Corollary 4.3 implies the existence of $(c^\star, \mathbf{U}) \in (0, +\infty) \times X_{T^\star}$, a solution to (3.1) with $c^\star = c(\mathcal{C})$. Conditions (3.2) and (3.3) are satisfied due to the fact that $\mathbf{U} \in X_{T^\star}$. The statement regarding the uniqueness of the speed $c^\star$ follows from Proposition 4.3. Finally, we have that (4.7) is exactly the exponential rate of convergence (3.31), which completes the proof. ■

4.10. Asymptotic behavior of the constrained solutions at $-\infty$

As has been pointed out before, almost nothing can be said about the behavior of an arbitrary function in X_T at $-\infty$. However, it turns out that constrained minimizers converge exponentially at $-\infty$ with respect to the $\mathcal{L}$-norm, provided that the speed fulfills an explicit upper bound; see Proposition 4.4. Such an upper bound also allows us to establish some other properties. Once Proposition 4.4 has been established, we will be able to complete the proofs of Theorems 5 and 6. The results of this section are obtained by combining ideas from Smyrnelis [48], Alikakos and Katzourakis [6] and Alikakos, Fusco and Smyrnelis [5]. It is worth pointing out that the arguments we present here strongly rely on the fact that the solutions considered are minimizers and that we do not expect them to hold for more general critical points.

We begin by showing a preliminary result, which follows by a direct computation:

Lemma 4.12. *Assume that (H6′) and (H7′) hold. Let $c > 0$, $t_1 < t_2$ and $U \in \mathcal{A}((t_1, t_2))$ be such that*

$$U'' - \mathcal{B}(U) = -cU' \quad \text{in } (t_1, t_2).$$

Then we have the formula

$$\forall t \in (t_1, t_2), \quad \frac{d}{dt}\Big(\mathcal{E}(U(t)) - \frac{\|U'(t)\|_{\mathcal{L}}^2}{2}\Big) = c\|U'(t)\|_{\mathcal{L}}^2. \tag{4.108}$$

Lemma 4.12 gives the following pointwise bounds for constrained solutions:

Lemma 4.13. *Assume that (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 4.7 and $t^- := t^-(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^-)$ be as in (4.39). Then for all $t < t^-$ we have the inequality*

$$\frac{\|\mathbf{U}_{c,T}'(t)\|_{\mathcal{L}}^2}{2} \leq \mathcal{E}(\mathbf{U}_{c,T}(t)) + a. \tag{4.109}$$

Similarly, it holds that for all $t > t^+$,

$$\mathcal{E}(\mathbf{U}_{c,T}(t)) \leq \frac{\|\mathbf{U}_{c,T}'(t)\|_{\mathcal{L}}^2}{2}, \tag{4.110}$$

where $t^+ := t^+(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^+)$ is as in (4.40).

Proof. Notice that (4.86) in Corollary 4.2 implies that $\mathbf{U}_{c,T}$ solves

$$\mathbf{U}_{c,T}'' - \mathcal{B}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \quad \text{in } (-\infty, t^-).$$

Therefore, the function

$$f_{c,T}\colon t \in (-\infty, t^-] \to e^{ct}\Big(\mathcal{E}(\mathbf{U}_{c,T}(t)) + a - \frac{\|\mathbf{U}_{c,T}'(t)\|_{\mathcal{L}}^2}{2}\Big),$$

is C^1 and we clearly have $f_{c,T} \in L^1((-\infty, t^-])$. By (4.108) in Lemma 4.12, we have

$$\forall t \in (-\infty, t^-), \quad f'_{c,T}(t) = c f_{c,T}(t) + c e^{ct} \|\mathbf{U}'_{c,T}(t)\|^2_{\mathcal{L}} \geq 0, \tag{4.111}$$

and we also have $f'_{c,T} \in L^1((-\infty, t^-))$. Therefore, it holds that

$$\lim_{t \to -\infty} f_{c,T}(t) = 0. \tag{4.112}$$

Fix $t_1 < t_2 \leq t^-$. Integrating (4.111) in $[t_1, t_2]$ we get

$$f_{c,T}(t_2) \geq f_{c,T}(t_1),$$

which in combination with (4.112) gives

$$\forall t < t^-, \quad f_{c,T}(t) \geq 0,$$

which is (4.109). Inequality (4.110) is obtained in an identical fashion. ∎

We conclude this section by proving the exponential convergence result, which is inspired by the ideas in Smyrnelis [48, Proof of (28)].

Proposition 4.4. *Assume that (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Let $c > 0$ and $T \geq 1$. Assume moreover that $c < \gamma^-$, where γ^- is defined in (3.27). Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 4.7. Then there exists $\bar{M}^- > 0$ such that for all $\varepsilon \in (0, \gamma^- - c)$ and $t \in \mathbb{R}$ it holds that*

$$\int_{-\infty}^{t} \big(\mathcal{E}(\mathbf{U}_{c,T}(s)) + a\big) e^{-\varepsilon s}\, ds \leq \bar{M}^- e^{(\gamma^- - c - \varepsilon)t}. \tag{4.113}$$

Furthermore, there exist $M^- > 0$ and $\mathbf{v}^-_{c,T} \in \mathcal{F}^-$ such that for all $t \in \mathbb{R}$,

$$\|\mathbf{U}_{c,T}(t) - \mathbf{v}^-_{c,T}\|^2_{\mathcal{L}} \leq M^- e^{(\gamma^- - c)t}. \tag{4.114}$$

Proof. Let $t^- := t^-(\mathbf{U}_{c,T}, \mathcal{E}^-_{\max})$ be as in (4.39). By applying (4.82) in Corollary 4.1, we obtain that for all $t \leq t^-$, $\mathbf{U}_{c,T}(t) \in \mathcal{F}^-_{r_0^-/2}$. For all $t \leq t^-$, define $\mathbf{v}^-(t) := P^-(\mathbf{U}_{c,T}(t))$. Consider the function

$$\tilde{U}^-_t(s) := \begin{cases} \mathbf{v}^-(t) & \text{if } s \leq t-1, \\ (t-s)\mathbf{v}^-(t) + (s-t+1)\mathbf{U}_{c,T}(t) & \text{if } t-1 \leq s \leq t, \\ \mathbf{U}_{c,T}(s) & \text{if } t \leq s, \end{cases}$$

which belongs to X_T. Therefore,

$$\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \mathbf{E}_c(\tilde{U}^-_t)$$

and, equivalently,

$$\int_{-\infty}^{t} \mathbf{e}_c(\mathbf{U}_{c,T}(s))\,ds \leq -\frac{a}{c}e^{ct} + \int_{t-1}^{t}\Big(\frac{\|\mathbf{U}_{c,T}(t)-\mathbf{v}^-(t)\|_{\mathcal{L}}^2}{2} + \big(\mathcal{E}(\tilde{U}_t^-(s))+a\big)\Big)e^{cs}\,ds. \tag{4.115}$$

Using Lemma 4.9 and (3.6) in (H1′), (4.115) becomes

$$\int_{-\infty}^{t} \mathbf{e}_c(\mathbf{U}_{c,T}(s))\,ds \leq -\frac{a}{c}e^{ct} + (C^- + \mathsf{C}^-)\big(\mathcal{E}(\mathbf{U}_{c,T}(t))+a\big)e^{ct},$$

which gives

$$\int_{-\infty}^{t} \big(\mathcal{E}(\mathbf{U}_{c,T}(s))-a\big)e^{cs}\,ds \leq \frac{1}{\gamma^-}\big(\mathcal{E}(\mathbf{U}_{c,T}(t))+a\big)e^{ct}, \tag{4.116}$$

where γ^- was defined in (3.27). Define the function

$$\theta_{c,T}^- : t \in (-\infty, t^-] \to \int_{-\infty}^{t} \big(\mathcal{E}(\mathbf{U}_{c,T}(s))+a\big)e^{cs}\,ds \in \mathbb{R}. \tag{4.117}$$

By (H7′), the function $\theta_{c,T}^-$ defined in (4.117) verifies that for all $t \in (-\infty, t^-)$,

$$(\theta_{c,T}^-)'(t) = \big(\mathcal{E}(\mathbf{U}_{c,T}(t))+a\big)e^{ct}$$

which, by (4.116), implies

$$\forall t \leq t^-, \quad \gamma^-\theta_{c,T}^-(t) \leq (\theta_{c,T}^-)'(t).$$

Now fix $t \in (-\infty, t^-)$ and assume that $\theta_{c,T}^-(t) > 0$. The previous inequality is equivalent to

$$\gamma^- \leq \big(\ln(\theta_{c,T}^-(t))\big)'.$$

which, by integrating in $[t, t^-]$, becomes

$$\gamma^-(t^- - t) \leq \ln(\theta_{c,T}^-(t^-)) - \ln(\theta_{c,T}^-(t)),$$

hence

$$e^{\gamma^-(t^- - t)} \leq \frac{\theta_{c,T}^-(t^-)}{\theta_{c,T}^-(t)},$$

that is,

$$\theta_{c,T}^-(t)e^{\gamma^-(t^- - t)} \leq \theta_{c,T}^-(t^-),$$

which clearly also holds if we drop the assumption $\theta_{c,T}^-(t) > 0$, as $\theta_{c,T}^-$ is a nonnegative function. Thus, we have shown that

$$\forall t \leq t^-, \quad \theta_{c,T}^-(t) \leq \theta_{c,T}^-(t^-)e^{-\gamma^-(t^- - t)}. \tag{4.118}$$

Now we have that using (4.118) we get for any fixed $t \leq t^- - 1$, $\varepsilon > 0$ and $i \in \mathbb{N}$,

$$\begin{aligned}\int_{t-i-1}^{t-i} &\big(\mathcal{E}(\mathbf{U}_{c,T}(s)) + a\big)e^{-\varepsilon s}\, ds \\ &\leq e^{-(c+\varepsilon)(t-i-1)} \int_{t-i-1}^{t-i} \big(\mathcal{E}(\mathbf{U}_{c,T}(s)) + a\big)e^{cs}\, ds \\ &\leq e^{-(c+\varepsilon)(t-i-1)} \theta^-_{c,T}(t^-) e^{-\gamma^-(t^- - t + i)} \\ &= e^{(c+\varepsilon)(1-t^-)} \theta^-_{c,T}(t^-) e^{(\gamma^- - c - \varepsilon)(t - t^-)} e^{(c+\varepsilon-\gamma^-)i}. \end{aligned} \tag{4.119}$$

Since we assume that $c < \gamma^-$, by choosing any $\varepsilon \in (0, \gamma^- - c)$ it holds that

$$\sum_{i \in \mathbb{N}} e^{(c+\varepsilon-\gamma^-)i} = \frac{1}{1 - e^{(c+\varepsilon-\gamma^-)}},$$

which, in combination with (4.119), gives (4.113) (notice that the case $t > t^- - 1$ presents no problem, as $e^{(\gamma^- - c - \varepsilon)t}$ is then large). Therefore, by (4.109) in Lemma 4.13 we have that for all $\varepsilon \in (0, \gamma^- - c)$ and $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} \frac{\|U'(s)\|^2_{\mathcal{L}}}{2} e^{-\varepsilon s}\, ds \leq \bar{M}^- e^{(\gamma^- - c - \varepsilon)t},$$

which, by the Cauchy–Schwarz inequality, means that

$$\int_{-\infty}^{t} \|\mathbf{U}'_{c,T}(s)\|_{\mathcal{L}}\, ds \leq \left(\frac{e^{\varepsilon t}}{\varepsilon} \int_{-\infty}^{t} \|\mathbf{U}'_{c,T}(s)\|^2_{\mathcal{L}} e^{-\varepsilon s}\, ds \right)^{\frac{1}{2}} \leq \frac{2\bar{M}^-}{\varepsilon} e^{(\gamma^- - c)t}, \tag{4.120}$$

where we have used that $\lim_{s \to -\infty} e^{\varepsilon s} = 0$, because $\varepsilon > 0$. Since $c < \gamma^-$, in particular, inequality (4.120) implies the existence of some $\tilde{v}^- \in \mathcal{L}$ such that

$$\lim_{t \to -\infty} \|\mathbf{U}_{c,T}(t) - \tilde{v}^-\|_{\mathcal{L}} = 0. \tag{4.121}$$

Inequality (4.120) also implies that for all $\tilde{t} < t \in \mathbb{R}$ we have

$$\|\mathbf{U}_{c,T}(t) - \mathbf{U}_{c,T}(\tilde{t}^-)\|_{\mathcal{L}} \leq \frac{2\bar{M}^-}{\varepsilon} e^{(\gamma^- - c)t},$$

which by taking the limit $\tilde{t} \to -\infty$ and using (4.121) gives (4.114), by choosing for instance $\varepsilon = (\gamma^- - c)/2 \in (0, \gamma^- - c)$ and $M^- = \frac{2\bar{M}^-}{\varepsilon} > 0$. ■

Remark 4.5. Notice that combining (4.113) in Proposition 4.4 with (4.109) in Lemma 4.13, we obtain in particular that $\mathbf{U}'_{c,T} \in L^2(\mathbb{R}, \mathcal{L})$ provided that $c < \gamma^-$ (see the statements of the results for the notation).

4.11. Proof of Theorem 5 completed

Assume first that (H8′) holds. Let $(c^\star, \mathbf{U})$ be the solution to (3.1) with conditions at infinity (3.2) and (3.31) given by Proposition 4.2 and Lemma 4.3. Since we took $c^\star = \sup \mathscr{C}$ with $\mathscr{C}$ as in (4.87), inequality (4.88) in Lemma 4.11 implies that

$$c^\star \leq \frac{\sqrt{2a}}{d_0},$$

which by (3.32) in (H9′) implies that

$$c^\star < \gamma^-,$$

so that we can apply (4.114) in Proposition 4.4 to $\mathbf{U}$, as it is a minimizer of $\mathbf{E}_{c^\star}$ in $X_{T^\star}$ for some $T^\star \geq 1$. Therefore, (3.33) holds for $\mathbf{U}$, which completes the proof. ∎

4.12. Proof of Theorem 6 completed

Since we assume that (H9′) holds and $\widetilde{\mathbf{U}}$ is such that $\widetilde{\mathbf{U}} \in X_T$ for some $T \geq 1$ and $\mathbf{E}_{c^\star}(\widetilde{\mathbf{U}}) = 0$, then by Proposition 4.2, we can apply Proposition 4.4 to $\mathbf{U}$. We recall that by Remark 4.5 we have $\mathbf{U}' \in L^2(\mathbb{R}, \mathcal{L})$ and by (4.113) in Proposition 4.4 we have $\mathscr{E} \circ \mathbf{U} \in L^1((-\infty, t])$ for all $t \in \mathbb{R}$. Therefore, we can find a sequence $(t_n^-)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{n\to\infty} t_n^- = -\infty \tag{4.122}$$

and

$$\lim_{n\to\infty} \left(\mathscr{E}(\mathbf{U}(t_n^-)) - a - \frac{\|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2}{2} \right) = 0. \tag{4.123}$$

Similarly, since $\mathbf{E}_c(\mathbf{U}) = 0 < +\infty$, we have $\mathscr{E} \circ \mathbf{U} \in L^1([t, +\infty))$ for all $t \in \mathbb{R}$, which means that we can find $(t_n^+)_{n\in\mathbb{N}}$, a sequence of real numbers such that

$$\lim_{n\to\infty} t_n^+ = +\infty \tag{4.124}$$

and

$$\lim_{n\to\infty} \left(\frac{\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2}{2} - \mathscr{E}(\mathbf{U}(t_n^+)) \right) = 0. \tag{4.125}$$

Taking the scalar product in $\mathcal{L}$ between equation (3.1) and $\mathbf{U}'$, we obtain

$$\forall t \in \mathbb{R}, \quad \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} - \langle \mathscr{B}(\mathbf{U}(t)), \mathbf{U}'(t) \rangle_{\mathcal{L}} = -c \|\mathbf{U}'(t)\|_{\mathcal{L}}^2$$

so that

$$\forall t \in \mathbb{R}, \quad \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} - \big(\mathscr{E}(\mathbf{U}(t))\big)' = -c^\star \|\mathbf{U}'(t)\|_{\mathcal{L}}^2.$$

Fix $n \in \mathbb{N}$. Integrating the equality above in $[t_n^-, t_n^+]$ (which is nonempty up to an extraction) we obtain

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}}\, dt - \mathscr{E}(\mathbf{U}(t_n^+)) + \mathscr{E}(\mathbf{U}(t_n^-)) = -c^\star \int_{t_n^-}^{t_n^+} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2\, dt. \tag{4.126}$$

Integrating by parts we obtain

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} \, dt = \|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2 - \|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2 - \int_{t_n^-}^{t_n^+} \langle \mathbf{U}'(t), \mathbf{U}''(t) \rangle_{\mathcal{L}} \, dt,$$

which means

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} \, dt = \frac{1}{2}(\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2 - \|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2).$$

Plugging into (4.126) we obtain

$$\begin{aligned} &-a + \left(\mathcal{E}(\mathbf{U}(t_n^-)) + a - \frac{\|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2}{2}\right) + \left(\frac{\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2}{2} - \mathcal{E}(\mathbf{U}(t_n^+))\right) \\ &= -c^\star \int_{t_n^-}^{t_n^+} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2 \, dt. \end{aligned}$$

Using (4.122), (4.123), (4.124) and (4.125), along with the fact that $\mathbf{U}' \in L^2(\mathbb{R}, \mathcal{L})$, we can pass to the limit $n \to \infty$ and we get that

$$-a = -c^\star \int_{\mathbb{R}} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2 \, dt,$$

which shows (3.34). We now show that (3.35) holds. Inspecting the proof of Theorem 4 again, we have that $c^\star$ is equal to $c(\mathscr{C})$ as in Corollary 4.3. Take $c < c^\star$; then by Corollary 4.3 we have $c \in \mathscr{C}$. The definition of $\mathscr{C}$ in (4.87) implies then that

$$\exists T \geq 1, \quad \inf_{U \in X_T} \mathbf{E}_c(U) < 0$$

which, by considering $\widetilde{U} \in X_T$ such that $\mathbf{E}_c(\widetilde{U}) < 0$ and then the sequence $(\widetilde{U}(\cdot + n))_{n \in \mathbb{N}}$ which is contained in X, implies that $\inf_{U \in X} \mathbf{E}_c(U) = -\infty$. If we now take $c > c^\star$, we have again by Corollary 4.3 that

$$\forall T \geq 1, \quad \inf_{U \in X_T} \mathbf{E}_c(U) \geq 0$$

which means

$$\inf_{U \in X} \mathbf{E}_c(U) = 0.$$

Therefore, (3.35) follows. Finally, we have that (3.36) is exactly (4.88) in Lemma 4.11. ∎

5. Proofs of the main results completed

Once we have proven the abstract results, we are ready to prove the main ones. In order to do this, we need to show that the main problem can be put into the abstract framework. This is shown in Lemma 5.1 which is in Section 5.1. The next sections are then devoted to the conclusion of the proofs of the main results, which are Theorems 1 and 3. However, as pointed out before, we do not have a counterpart of Theorem 2 in the abstract setting, which means that we prove it using arguments relative to the main setting.

5.1. Proving the link between the main setting and the abstract setting

The following result establishes the link between the main assumptions and the abstract ones. As a consequence, the main results can be deduced from the abstract framework, which we have already established.

Lemma 5.1. *Assume that (H5) holds. Set*

$$\mathcal{L} := L^2(\mathbb{R}, \mathbb{R}^k), \quad \mathcal{H} := H^1(\mathbb{R}, \mathbb{R}^k), \quad \widetilde{\mathcal{H}} := H^2(\mathbb{R}, \mathbb{R}^k), \tag{5.1}$$

$$r_0^{\pm} := \rho_0^{\pm} \tag{5.2}$$

and

$$\forall v \in \mathcal{L}, \quad \mathcal{E}(v) := \begin{cases} E(\psi + v) - \mathfrak{m}^+ & \text{if } v \in \mathcal{H}, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\mathfrak{m}^+$ was introduced in (H5). The constants $\rho_0^{\pm}$ are those from (2.4) and the function ψ is any smooth function in $X(\sigma^-, \sigma^+)$ converging to $\sigma^{\pm}$ at $\pm\infty$ at an exponential rate and such that $\psi' \in H^2(\mathbb{R}, \mathbb{R}^k)$. Finally, we set $\mathfrak{F}^{\pm} = \mathcal{F}^{\pm} - \psi$. Under this choice, assumptions (H1′), (H2′), (H3′), (H4′), (H5′), (H6′) and (H7′) hold. Moreover, we have that

- *if (H6) holds, then (H8′) and (H9′) hold.*

Proof. The fact that the functional

$$v \in \mathcal{H} \to E(\psi + v)$$

is well defined and, moreover, is a C^1 functional on $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is proven by classical arguments. See, for instance, Bisgard [15], Montecchiari and Rabinowitz [32]. See also [37] for the precise statement in this setting. We now pass to proving that the assumptions are satisfied.

Assumption (H1′) is satisfied: The fact that $\mathcal{E}$ is weakly lower semicontinuous in $\mathcal{L}$ is standard; see [48, Lemma 3.1]. We have already invoked Schatzman [46, Lemma 2.1], so that (2.4) and (2.5) hold. That is, due to (5.2) we have that if

$$\inf_{\tau \in \mathbb{R}} \|v + \psi - \mathfrak{q}^{\pm}(\cdot + \tau)\|_{\mathcal{L}} \le r_0^{\pm}, \tag{5.3}$$

there is a unique $\tau(v) \in \mathbb{R}$ which attains the infimum in (5.3). Moreover, the correspondence $v \to \tau(v)$ defined on the subset of $\mathcal{L}$ composed of functions that verify (5.3) is of class C^2. Therefore, the applications

$$P^{\pm} : v \in \mathfrak{F}^{-}_{r_0^{\pm}/2} \to \mathfrak{q}^{\pm}(\cdot + \tau(v)) - \psi \in \mathfrak{F}^{\pm} \tag{5.4}$$

satisfy the properties required. Finally, we have that estimate (3.6) follows by [34, Lemma 3.2], up to modifying the choice of the constants $\rho_0^{\pm}$, $\beta_0^{\pm}$.

Assumption (H2′) is satisfied: By (5.1), we have $\widetilde{\mathcal{H}} \subset \mathcal{H} \subset \mathcal{L}$ and the associated norms verify

$$\|\cdot\|_{\mathcal{L}} \le \|\cdot\|_{\mathcal{H}} \le \|\cdot\|_{\widetilde{\mathcal{H}}}.$$

As we pointed out before, $\mathcal{E}$ restricted to $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a C^1 functional. Moreover, as shown in [15, 32], we have that the differential is given by

$$\forall v \in \mathcal{H}, \quad D\mathcal{E}(v)\colon w \in \mathcal{H} \to \int_{\mathbb{R}} (\langle \psi' + v', w' \rangle + \langle \nabla V(\psi + v), w \rangle) \in \mathbb{R}. \tag{5.5}$$

Now let $v \in \widetilde{\mathcal{H}}$; since ψ is smooth with good behavior at infinity we can integrate by parts to get

$$\begin{aligned} \forall w \in \mathcal{H}, \quad D\mathcal{E}(v)(w) &= \int_{\mathbb{R}} \langle -(\psi'' + v'') + \nabla V(\psi + v), w \rangle \\ &= \langle \mathcal{B}(v), w \rangle_{\mathcal{L}}, \end{aligned} \tag{5.6}$$

where we have set

$$\mathcal{B}\colon v \in (\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}) \to -(\psi'' + v'') + \nabla V(\psi + v) \in (\mathcal{L}, \|\cdot\|_{\mathcal{L}}),$$

which, by standard arguments, can be shown to be continuous. Notice that (3.9) in (H2′) is exactly (5.6) above, which concludes this part of the proof.

Assumption (H3′) is satisfied: Let $(\mathbf{v}_n^-)_{n\in\mathbb{N}}$ be an $\mathcal{L}$-bounded sequence in $\mathfrak{F}^-$. We want to show the existence of a subsequence of $(\mathbf{v}_n^-)_{n\in\mathbb{N}}$ strongly convergent in $\mathcal{H}$. Since

$$\mathfrak{F}^- = \mathcal{F}^- - \psi = \left\{\mathfrak{q}^-(\cdot + \tau) - \psi : \tau \in \mathbb{R}\right\},$$

we have $(\mathbf{v}_n^-)_{n\in\mathbb{N}} = (\mathfrak{q}^-(\cdot + \tau_n) - \psi)_{n\in\mathbb{N}}$ with $(\tau_n)_{n\in\mathbb{N}}$ a bounded sequence of real numbers. Since such a sequence is bounded in $\mathcal{L}$, we know that, up to an extraction, there exists $\tilde{v} \in \mathcal{L}$ such that $\mathfrak{q}^-(\cdot + \tau_n) - \psi \rightharpoonup \tilde{v}$ weakly in $\mathcal{L}$. Due to the weak lower semicontinuity of $\mathcal{E}$, we have

$$\mathcal{E}(\tilde{v}) \le \liminf_{n\to+\infty} \mathcal{E}(\mathfrak{q}^-(\cdot + \tau_n) - \psi) = a,$$

which, by minimality, implies that $\mathcal{E}(v) = 0$, that is, $\tilde{v} \in \mathfrak{F}^-$. We can then write $\tilde{v} = \mathfrak{q}^-(\cdot + \tau) - \psi$ for some $\tau \in \mathbb{R}$. Now, notice that, by the compactness of minimizing sequences (2.2), there exists a sequence $(\tau_n')_{n\in\mathbb{N}}$ of real numbers such that, up to an extraction,

$$\mathfrak{q}^-(\cdot + \tau_n + \tau_n') - \mathfrak{q}^- \to 0 \text{ strongly in } \mathcal{H} \tag{5.7}$$

which necessarily implies that

$$\tau_n + \tau_n' \to 0$$

and, since $(\tau_n)_{n\in\mathbb{N}}$ is bounded, we have that $(\tau_n')_{n\in\mathbb{N}}$ is a bounded sequence as well. Therefore, we can assume, up to an extraction, that $\tau_n' \to \tau$. Combining this information with (5.7), we obtain

$$\mathfrak{q}^-(\cdot + \tau_n) - \mathfrak{q}^-(\cdot - \tau^-) \to 0 \text{ strongly in } \mathcal{H},$$

which establishes the claim.

We need to show the same for $\mathcal{F}^+$. The argument is identical to the one above, except for the fact that the compactness of minimizing sequences is replaced by (3) in assumption (H5), which is in fact stronger, and we use that the elements in $\mathcal{F}^+$ are local minimizers (instead of global ones), which does not require any modification of the reasoning.

Assumption (H4′) is satisfied: More precisely, we show that (2) in (H4′) holds. Notice that since the results are local in nature and (H1′) implies that locally the situation does not change between $\mathcal{F}^-$ and $\mathcal{F}^+$, we can treat both cases together. Let $(v, \mathbf{v}^\pm) \in \mathcal{F}^\pm_{r_0^\pm} \times \mathcal{F}^\pm$. Let $\tau(v)$ be given by the projection map defined in (5.4). We have $\mathbf{v}^\pm = \mathfrak{q}^\pm(\cdot + \tau) - \psi$ for some $\tau \in \mathbb{R}$. Define

$$\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}\colon w \in \mathcal{L} \to w(\cdot - \tau(v) + \tau) - \psi + \psi(\cdot - \tau(v) + \tau) \in \mathcal{L}.$$

Clearly, using the definition of the projection in (5.4) and $\tau(v)$,

$$\begin{aligned}\|\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(v) - \mathbf{v}^\pm\|_{\mathcal{L}} &= \big\|v(\cdot - \tau(v) + \tau) - \big(\mathfrak{q}^\pm(\cdot + \tau) - \psi(\cdot - \tau(v) + \tau)\big)\big\|_{\mathcal{L}} \\ &= \big\|v - \big(\mathfrak{q}^\pm(\cdot + \tau(v)) - \psi\big)\big\|_{\mathcal{L}} = \inf_{\tilde{\tau}\in\mathbb{R}} \|v - (\mathfrak{q}^\pm(\cdot + \tilde{\tau}) - \psi)\|_{\mathcal{L}},\end{aligned}$$

meaning that

$$P^\pm(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(v)) = \mathbf{v}^\pm$$

and

$$\operatorname{dist}_{\mathcal{L}}(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(v), \mathcal{F}^\pm) = \operatorname{dist}_{\mathcal{L}}(v, \mathcal{F}^\pm),$$

which are (3.10) and (3.11) respectively. Next, notice that for $(w_1, w_2) \in \mathcal{L}^2$ and $h \in \mathbb{R}$ we have

$$\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(w_1 + hw_2) = \widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(w_1) + hw_2(\cdot - \tau(v) + \tau)$$

so that $\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}$ is differentiable and

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad D(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)})(w_1, w_2) = w_2(\cdot - \tau(v) + \tau),$$

so that

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad \|D(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)})(w_1, w_2)\|_{\mathcal{L}} = \|w_2\|_{\mathcal{L}},$$

which is (3.12). Finally, notice that $\tilde{v} \in \mathcal{H}$ if and only if $\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(\tilde{v}) \in \mathcal{H}$. Assuming that $\tilde{v} \in \mathcal{H}$ we have

$$\begin{aligned}\mathcal{E}(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(\tilde{v})) &= \mathcal{E}\big(\tilde{v}(\cdot - \tau(v) + \tau) - \psi + \psi(\cdot - \tau(v) + \tau)\big) \\ &= E\big(\tilde{v}(\cdot - \tau(v) + \tau) + \psi(\cdot - \tau(v) + \tau)\big) = E(\psi + \tilde{v}) = \mathcal{E}(\tilde{v})\end{aligned}$$

and if $\tilde{v} \in \mathcal{L} \setminus \mathcal{H}$, we have $\mathcal{E}(\widehat{P}^\pm_{(v,\mathbf{v}^\pm)}(\tilde{v})) = +\infty = \mathcal{E}(\tilde{v})$. Therefore, (3.13) holds. We have then showed that (H4′) holds.

Assumption (H5′) is satisfied: Schatzman [46, Lemma 2.1] states that for $v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}}$ the problem

$$\inf_{\tau \in \mathbb{R}} \|v + \psi - \mathfrak{q}^{\pm}(\cdot + \tau)\|_{\mathcal{H}}$$

has a unique solution $\tau_{\mathcal{H}}(v) \in \mathbb{R}$ and the projection map

$$P^{\pm}_{\mathcal{H}} : v \in \mathcal{F}^{\pm}_{\mathcal{H},r_0^{\pm}} \to \mathfrak{q}^{\pm}(\cdot + \tau_{\mathcal{H}}(v)) \in \mathcal{F}^{\pm}$$

is C^1 with respect to the $\mathcal{H}$-norm. Next we have that (3.14) is [46, Corollary 2.3]. Finally, the fact that (3.15) implies (3.16) for the constants $C^{\pm}$ (up to possibly increasing) is a consequence of the compactness of the minimizing sequences. See, for example, [46, Corollary 3.2].

Assumption (H6′) is satisfied: We show the existence of the map $\mathfrak{P}$. We follow [46, Lemma 3.3]. Let $R_0 > 0$ be the constant from (H2). For $R \geq R_0$, define in $\mathbb{R}^k$,

$$f_R(u) := \begin{cases} u & \text{if } |u| \leq R, \\ R\dfrac{u}{|u|} & \text{otherwise,} \end{cases}$$

where R_0 is the constant from (H2). For $u \in R^k$ such that $|u| \leq R$, we have $f_R(u) = u$. Assume that $u \in \mathbb{R}^k$ is such that $|u| > R$. In that case, there exists $\xi \in (\frac{R}{|u|}, 1)$ such that

$$\begin{aligned} V(u) &= V(f_R(u)) + \langle \nabla_u V(\xi u), u - f_R(u) \rangle \\ &= V(f_R(u)) + \frac{1}{\xi}\Big(1 - \frac{R}{|u|}\Big)\langle \nabla_u V(\xi u), \xi u \rangle \end{aligned}$$

which, by (H2), implies

$$\begin{aligned} \forall R \geq R_0,\ \forall u \in \mathbb{R}^k : |u| > R, \quad V(u) &\geq V(f_R(u)) + \frac{1}{\xi}\Big(1 - \frac{R}{|u|}\Big)\nu_0 |\xi u|^2 \\ &> V(f_R(u)). \end{aligned} \tag{5.8}$$

In particular, we have shown

$$\forall R \geq R_0,\ \forall u \in \mathbb{R}^k, \quad V(u) \geq V(f_R(u)). \tag{5.9}$$

Next, let $J \subset \mathbb{R}$ be a compact interval and $v \in H^1(J, \mathbb{R}^k)$. For $R \geq R_0$, consider the function $v_R := f_R \circ v$. Since we clearly have that for all $u \in \mathbb{R}^k$, $|f_R(u)| \leq |u|$, it holds that $v_R \in L^2(J, \mathbb{R}^k)$. Next we have that f_R is the projection onto the closed ball of center 0 and radius R, so that it is nonexpansive. As a consequence, we have

$$\forall R \geq R_0,\ \forall u \in \mathbb{R}^k, \quad |Df_R(u)| \leq 1. \tag{5.10}$$

Therefore, applying the chain rule we obtain

$$\text{for a.e. } t \in J, \quad |v_R'(t)| \leq |v'(t)|,$$

which means that $v_R \in H^1(J, \mathbb{R}^k)$ and, combining with (5.9), we obtain

$$E(v_R; J) \leq E(v; J), \tag{5.11}$$

and, by (5.8), the equality above holds if and only if $v_R = v$. Now let

$$R_{\max} := 2 \max\{R_0, \|\mathfrak{q}^-\|_{L^\infty(\mathbb{R},\mathbb{R}^k)}, \|\mathfrak{q}^+\|_{L^\infty(\mathbb{R},\mathbb{R}^k)}\}.$$

Now consider the application

$$\mathfrak{P}\colon v \in \mathcal{L} \to f_{R_{\max}} \circ (v + \psi) - \psi \in \mathcal{L}, \tag{5.12}$$

which is well defined due to the previous considerations. Moreover, the choice of $R_{\max}$ implies that $\mathfrak{P}$ equals the identity on $\{\mathfrak{q}^-(\cdot + \tau) - \psi : \tau \in \mathbb{R}\}$ and $\{\mathfrak{q}^+(\cdot + \tau) - \psi : \tau \in \mathbb{R}\}$, which is exactly (3.21). Inequality (5.11) gives (3.19). Finally, using (5.10) we have

$$\begin{aligned}\forall (v_1, v_2) \in \mathcal{L}^2, \quad \|\mathfrak{P}(v_1) - \mathfrak{P}(v_2)\|_{\mathcal{L}}^2 &= \int_{\mathbb{R}} |f_{R_{\max}} \circ (v_1 + \psi) - f_{R_{\max}} \circ (v_2 + \psi)|^2 \\ &\leq \int_{\mathbb{R}} \sup_{u \in \mathbb{R}^k} |Df_{R_{\max}}(u)|^2 |v_1 - v_2|^2 \\ &\leq \int_{\mathbb{R}} |v_1 - v_2|^2 = \|v_1 - v_2\|_{\mathcal{L}}^2,\end{aligned}$$

which is (3.20). Therefore, our map $\mathfrak{P}$ satisfies the required properties.

Assumption (H7′) is satisfied: Let $\mathbf{W}$ be a local minimizer of $\mathbf{E}_c$. We show that $\mathbf{W}$ satisfies the desired regularity properties, that is, $\mathbf{W} \in \mathcal{A}(I)$ with $\mathcal{A}(I)$ as in (3.22). Write $\overline{\mathbf{W}} := \mathbf{W} + \psi$. We assume that for all $t \in I$, $\mathbf{W}(t) = \mathfrak{P}(\mathbf{W})$. The definition of $\mathfrak{P}$ in (5.12) implies that

$$\forall (x_1, x_2) \in I \times \mathbb{R}, \quad \overline{\mathbf{W}}(x_1, x_2) = f_{R_{\max}}(\overline{\mathbf{W}}(x_1, x_2))$$

so that

$$\|\overline{\mathbf{W}}\|_{L^\infty(I\times\mathbb{R},\mathbb{R}^k)} \leq R_{\max}.$$

Therefore, by classical elliptic regularity arguments, we have that, with the obvious identifications, $\overline{\mathbf{W}}$ solves

$$-c\partial_{x_1}\overline{\mathbf{W}} - \Delta\overline{\mathbf{W}} = -\nabla_u V(\overline{\mathbf{W}}) \quad \text{in } I \times \mathbb{R},$$

and for all $\alpha \in (0, 1)$ we have $\overline{\mathbf{W}} \in \mathcal{C}^{3,\alpha}(I_C \times \mathbb{R}, \mathbb{R}^k)$ for any compact $I_C \subset I$. It is then clear that

$$\mathbf{W} \in \mathcal{C}^2(I_C, L^2(\mathbb{R}, \mathbb{R}^k)) \cap \mathcal{C}^1(I_C, H^1(\mathbb{R}, \mathbb{R}^k)) \cap \mathcal{C}^0(I_C, H^2(\mathbb{R}, \mathbb{R}^k))$$

for any $I_C \subset I$ compact, which means that $\mathbf{W} \in \mathcal{A}(I)$.

Assumption (H6) implies (H8′) and (H9′): Immediate. ∎

Once Lemma 5.1 has been established, the main results are easily obtained by rephrasing the abstract ones.

5.2. Proof of Theorem 1 completed

Assume that (H6) holds. Notice that (H6) implies that (H1), (H2), (H3), (H4) and (H5) hold. Therefore, applying Lemma 5.1 we have that, choosing the objects as in its statement, we get that (H3′), (H4′), (H5′), (H6′), (H7′) and (H8′) hold. Those are exactly the assumptions which are needed for Theorem 4 to hold, meaning that we obtain $(c^\star, \mathbf{U})$ with $c^\star > 0$ and $\mathbf{U} \in \mathcal{A}(\mathbb{R}) \cap X$, with $\mathcal{A}(\mathbb{R})$ as in (3.22) and X as in (3.30), which solves

$$\mathbf{U}'' - \mathcal{B}(\mathbf{U}) = -c\mathbf{U}' \quad \text{in } \mathbb{R} \tag{5.13}$$

and satisfies the conditions at infinity

$$\begin{aligned} &\exists T^- \le 0\colon \forall t \le T^-, \quad \mathbf{U}(t) \in \mathcal{F}^-_{r_0^-/2} \quad \text{and} \\ &\qquad\qquad \exists \mathbf{v}^+(\mathbf{U}) \in \mathcal{F}^+\colon \lim_{t\to+\infty} \|\mathbf{U}(t) - \mathbf{v}^+(\mathbf{U})\|_{\mathcal{H}} = 0. \end{aligned} \tag{5.14}$$

We now pass to proving each of the three statements of Theorem 1 separately:

(1) *Existence*. Recall that for all $t \in \mathbb{R}$ we have $\mathbf{U}(t) \in \mathcal{L} = L^2(\mathbb{R}, \mathbb{R}^k)$. Let us then define

$$\mathfrak{U}\colon (x_1, x_2) \in \mathbb{R}^2 \to \mathbf{U}(x_1)(x_2) \in \mathbb{R}^k. \tag{5.15}$$

It is clear then that since $\mathbf{U} \in \mathcal{A}(\mathbb{R})$ we have $\mathfrak{U} \in \mathcal{C}^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^k)$ and, moreover, for all (x_1, x_2) and any pair of indexes $(i, j) \in \{0, 1, 2\}^2$ such that $i + j \le 2$, we have

$$\partial^i_{x_1} \partial^j_{x_2} \mathfrak{U}(x_1, x_2) = (\mathbf{U}^{(i)}(x_1))^{(j)}(x_2), \tag{5.16}$$

where for a curve f taking values in a Hilbert space we denote by $f^{(i)}$ its ith derivative, $i \in \mathbb{N}$. As a consequence of (5.13), (5.16) and the formula for $D\mathcal{E}$, when we make $\mathcal{E} = E - \mathfrak{m}^+$ (see (5.5)) we obtain that

$$-c\partial_{x_1}\mathfrak{U} - \Delta\mathfrak{U} = -\nabla_u V(\mathfrak{U}) \quad \text{in } \mathbb{R}^2,$$

and by (5.14) we obtain that for some $L \in \mathbb{R}$ we have for some $x_1 \le L$ that $\mathfrak{U}(x_1, \cdot) \in \mathcal{F}^-_{\rho^-/2}$, since we choose $r_0^\pm = \rho^\pm$, so that $\mathcal{F}^\pm_{\rho^\pm/2} = \mathcal{F}^\pm_{r_0^\pm/2}$. The variational characterization (2.12) follows directly from Theorem 1, using the fact that we have $X = S$ and $\mathbf{E}_c = E_{2,c}$ for all $c > 0$ (again we implicitly identify $\mathfrak{U}$ with $\mathbf{U}$ via (5.15)). Finally, we have that for all $t \in \mathbb{R}$, $\mathbf{U}(t) = \mathfrak{P}(\mathbf{U}(t))$. According to the choice of $\mathfrak{P}$ made in Lemma 5.1, this implies that $\|\mathfrak{U}\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} < +\infty$, which by classical Schauder theory and the smoothness properties of V, implies that for all $\alpha \in (0, 1)$, $\mathfrak{U} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$. The proof of the existence part of Theorem 1 is hence completed.

(2) *Uniqueness of the speed*. Again, we have $X = S$ and $\mathbf{E}_c = E_{2,c}$ for all $c > 0$, meaning that the proof of this statement follows from the analogous one in Theorem 4.

(3) *Exponential convergence*. Using the exponential rate of convergence of $\mathbf{U}$ given by Theorem 4, which is (3.31), we obtain that for some $b > c^\star/2$ it holds that

$$\lim_{x_1 \to +\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{H^1(\mathbb{R},\mathbb{R}^k)} e^{bx_1} = 0$$

for some $\tau^+ \in \mathbb{R}$. Applying Theorem 5, we obtain the exponential convergence at $-\infty$. This concludes the proof of the statement.

The proof of Theorem 1 is concluded. ∎

5.3. Proof of Theorem 2

We now provide the proof of Theorem 2, which is a consequence of the following results, which are more general than required by Theorem 2 and might be of independent interest:

Lemma 5.2. *Assume that (H1), (H2) and (H3) hold. Let* $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ *(possibly equal) and* $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$*. Assume moreover that there exist* $L^+ \in \mathbb{R}$ *and* $U \in H^1_{\mathrm{loc}}([L^+, +\infty) \times \mathbb{R}, \mathbb{R}^k)$ *uniformly continuous and such that*

$$\int_{L^+}^{+\infty} |E(U(x_1, \cdot)) - E(q)|\, dx_1 < +\infty, \tag{5.17}$$

$$\lim_{x_1 \to +\infty} \|U(x_1, \cdot) - q\|_{L^2(\mathbb{R},\mathbb{R}^k)} = 0. \tag{5.18}$$

Then it holds that

$$\lim_{x_1 \to +\infty} \|U(x_1, \cdot) - q\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} = 0 \tag{5.19}$$

and

$$\lim_{x_2 \to \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([L^+,+\infty),\mathbb{R}^k)} = 0. \tag{5.20}$$

Similarly, we have the following:

Lemma 5.3. *Assume that (H1), (H2) and (H3) hold. Let* $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ *(possibly equal) and* $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$*. Assume moreover that there exist* $L^- \in \mathbb{R}$ *and* $U \in H^1_{\mathrm{loc}}((-\infty, L^-] \times \mathbb{R}, \mathbb{R}^k)$ *uniformly continuous and such that*

$$\int_{-\infty}^{L^-} |E(U(x_1, \cdot)) - E(q)|\, dx_1 < +\infty,$$

$$\lim_{x_1 \to -\infty} \|U(x_1, \cdot) - q\|_{L^2(\mathbb{R},\mathbb{R}^k)} = 0. \tag{5.21}$$

Then it holds that

$$\lim_{x_1 \to -\infty} \|U(x_1, \cdot) - q\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} = 0 \tag{5.22}$$

and

$$\lim_{x_2 \to \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty((-\infty,L^-],\mathbb{R}^k)} = 0. \tag{5.23}$$

In the proofs of Lemmas 5.2 and 5.3, we will need to use the following fact:

Lemma 5.4. *Assume that (H1), (H2) and (H3) hold. Let* $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ *(possibly equal) and* $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$*. Assume that* $(q_n)_{n\in\mathbb{N}}$ *is a sequence in* $X(\hat{\sigma}_-, \hat{\sigma}_+)$ *such that*

$$\lim_{n\to\infty} \|q_n - q\|_{L^2(\mathbb{R},\mathbb{R}^k)} = 0 \tag{5.24}$$

and

$$\lim_{n\to\infty} E(q_n) = E(q). \tag{5.25}$$

Then it holds that

$$\lim_{n\to\infty} \|q_n - q\|_{H^1(\mathbb{R},\mathbb{R}^k)} = 0. \tag{5.26}$$

Proof. First, notice that

$$\sup_{n\in\mathbb{N}} \|q_n\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} < +\infty. \tag{5.27}$$

Indeed, (5.25) implies that $(q_n')_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R},\mathbb{R}^k)$ which, in combination with (5.24) means that $(q_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R},\mathbb{R}^k)$, hence in $L^\infty(\mathbb{R},\mathbb{R}^k)$. We also have

$$\nabla V(q) \in L^2(\mathbb{R},\mathbb{R}^k), \tag{5.28}$$

which follows easily from the fact that V is smooth and quadratic near the wells. For all $n \in \mathbb{N}$, we write the expansion

$$V(q_n) = V(q) + \langle \nabla V(q), q_n - q\rangle + \int_0^1 D^2V(q + \lambda(q_n - q))(q_n - q)(q_n - q)\, d\lambda,$$

which holds pointwise in $\mathbb{R}$. Therefore, the Cauchy–Schwarz inequality implies

$$\left(\int_{\mathbb{R}} |V(q_n) - V(q)|\right)^2 \leq \left(\int_{\mathbb{R}} |\nabla V(q)|^2 + \sup_{\substack{u\in\mathbb{R}^k \\ |u|\leq \|q_n - q\|_{L^\infty}}} |D^2V(u)|\right) \|q_n - q\|^2_{L^2(\mathbb{R},\mathbb{R}^k)},$$

hence, by (5.27) and (5.28) we find a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |V(q_n) - V(q)| \leq C\|q_n - q\|_{L^2(\mathbb{R},\mathbb{R}^k)},$$

which by (5.24) means that $V(q_n) - V(q) \to 0$ in $L^1(\mathbb{R},\mathbb{R}^k)$. As a consequence, (5.25) implies that

$$\lim_{n\to\infty} \|q_n'\|_{L^2(\mathbb{R},\mathbb{R}^k)} = \|q'\|_{L^2(\mathbb{R},\mathbb{R}^k)}. \tag{5.29}$$

Now suppose, by contradiction, that (5.26) does not hold. Then we can find a subsequence $(q_{n_m})_{m\in\mathbb{N}}$ and $\hat{\delta} > 0$ such that for all $m \in \mathbb{N}$,

$$\|q_{n_m} - q\|_{H^1(\mathbb{R},\mathbb{R}^k)} \geq \hat{\delta}. \tag{5.30}$$

Since $(q'_{n_m})_{m\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R},\mathbb{R}^k)$, it converges weakly in L^2 up to an extraction, and the limit is q' by uniqueness of the limit in the sense of distributions. By (5.29), we have that such a subsequence also converges strongly in $L^2(\mathbb{R},\mathbb{R}^k)$, which combining with (5.24) contradicts (5.30). ∎

We now prove Lemma 5.2. The proof of Lemma 5.3 being analogous, we skip it.

Proof of Lemma 5.2. Assume by contradiction that (5.19) does not hold. Then we can find a sequence $(x_{1,n})_{n\in\mathbb{N}}$ in $[L^+,+\infty)\times\mathbb{R}$ such that $x_{1,n}\to+\infty$ as $n\to\infty$ as well as $\hat{\delta}>0$ such that for all $n\in\mathbb{N}$,

$$\|U(x_{1,n},\cdot)-q\|_{L^\infty(\mathbb{R},\mathbb{R}^k)}\geq\hat{\delta}.$$

By uniform continuity, there exists $\nu>0$ such that for all $n\in\mathbb{N}$ we have

$$\max_{x_1\in[x_{1,n}-\nu,x_{1,n}+\nu]}\|U(x_1,\cdot)-q\|_{L^\infty(\mathbb{R},\mathbb{R}^k)}\geq\frac{\hat{\delta}}{2}. \tag{5.31}$$

Let $A:=\bigcup_{n\in\mathbb{N}}[x_{1,n}-\nu,x_{1,n}+\nu]$. By (5.17) we have

$$\int_A\big(E(U(x_1,\cdot))-E(q)\big)\,dx_1<+\infty,$$

and since A has positive measure and it is unbounded above, we find a sequence $(y_{1,n})_{n\in\mathbb{N}}$ in A such that $y_{1,n}\to+\infty$ as $n\to\infty$ and $\lim_{n\to\infty}E(U(y_{1,n},\cdot))=E(q)$. Combining this fact with (5.18), we have that assumptions (5.24) and (5.25) in Lemma 5.4 hold, which means that

$$\lim_{n\to\infty}\|U(y_{1,n},\cdot)-q\|_{H^1(\mathbb{R},\mathbb{R}^k)}=0,$$

which contradicts (5.31). Therefore, we have shown that (5.19) holds. In order to prove (5.20), we first show that there exists $\underline{L}^+\leq L^+$ such that

$$\lim_{x_2\to\pm\infty}\|U(\cdot,x_2)-\hat{\sigma}_\pm\|_{L^\infty([\underline{L}^+,+\infty),\mathbb{R}^k)}=0. \tag{5.32}$$

We prove (5.32) by contradiction. The other case being handled in an analogous fashion, assume that there exists a sequence $(x_{2,n})_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $x_{2,n}\to+\infty$ as $n\to\infty$, a sequence $(x_{1,n})_{n\in\mathbb{N}}$ in $[L^+,+\infty)$ tending to $+\infty$ and $\hat{\delta}>0$ such that for all $n\in\mathbb{N}$,

$$|U(x_{1,n},x_{2,n})-\hat{\sigma}_+|\geq\hat{\delta}. \tag{5.33}$$

Since we have already proven that (5.19) holds, there exists $N_1\in\mathbb{N}$ such that for all $n\geq\mathbb{N}$ we have

$$\|U(x_{1,n},\cdot)-q\|_{L^\infty(\mathbb{R},\mathbb{R}^k)}\leq\frac{\hat{\delta}}{4} \tag{5.34}$$

and, since $q\in X(\hat{\sigma}_-,\hat{\sigma}_+)$, there exists $\hat{t}\in\mathbb{R}$ such that for all $t\geq\hat{t}$ we have

$$|q(t)-\hat{\sigma}_+|\leq\frac{\hat{\delta}}{4}. \tag{5.35}$$

Let $N_2 \in \mathbb{N}$ be such that for all $n \in \mathbb{N}$, $x_{2,n} \geq \hat{t}$. Taking any $n \geq \max\{N_1, N_2\}$, we obtain by (5.34) and (5.35) that

$$|U(x_{1,n}, x_{2,n}) - \hat{\sigma}_+| \leq \frac{\delta}{2},$$

which contradicts (5.33) and establishes (5.32). In order to establish (5.20), we handle the limit $x_2 \to +\infty$, as the other one is treated identically. Let $\rho_\Sigma^+ := \mathrm{dist}(\sigma^+, \Sigma \setminus \{\sigma^+\}) > 0$. We claim that for every $\tilde{L} \geq L^+$ we have that if

$$\lim_{x_2 \to \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([\tilde{L}, +\infty), \mathbb{R}^k)} = 0, \tag{5.36}$$

then

$$\lim_{x_2 \to \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([\tilde{L} - \eta_\Sigma^+, +\infty), \mathbb{R}^k)} = 0, \tag{5.37}$$

where

$$\eta_\Sigma^+ := \min\left\{\tilde{L} - L^+, \frac{\rho_\Sigma}{4\|D\mathfrak{U}\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^k)}}\right\}. \tag{5.38}$$

Such a claim allows us to easily complete the proof of (5.20) by a finite induction process, due to the fact that (5.32) holds. ■

It remains to establish one claim in the proof of Lemma 5.2.

Proof that (5.36) *implies* (5.37). Assume that (5.36) holds. We show that for every $\varepsilon \in (0, \eta_\Sigma^+)$ we have

$$\lim_{x_2 \to \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([\tilde{L} - \eta_\Sigma^+ + \varepsilon, +\infty), \mathbb{R}^k)} = 0, \tag{5.39}$$

which clearly implies (5.37) by uniform continuity. Then fix $\varepsilon \in (0, \eta_\Sigma^+)$. By assumption, there exists $\bar{x}_2^+ \in \mathbb{R}$ such that for all $x_2 \geq \bar{x}_2^+$ we have

$$|U(\tilde{L}, x_2) - \sigma^+| \leq \frac{\rho_\Sigma^+}{4},$$

which, by (5.38), implies that for all $(x_1, x_2) \in [\tilde{L} - \eta_\Sigma^+ + \varepsilon, \tilde{L}] \times [\bar{x}_2, +\infty)$, it holds that

$$|U(x_1, x_2) - \sigma^+| \leq \frac{\rho_\Sigma^+}{2} \tag{5.40}$$

and the definition of ρ_Σ^+ gives, in turn, that for all such (x_1, x_2) and $\sigma \in \Sigma \setminus \{\sigma^+\}$ we have

$$|U(x_1, x_2) - \sigma| \geq \frac{\rho_\Sigma^+}{2}. \tag{5.41}$$

Assume now that (5.39) does not hold. Then inequalities (5.40) and (5.41) imply that we can find a sequence $(x_{1,n}, x_{2,n})_{n \in \mathbb{N}}$ contained in $[\tilde{L} - \eta_\Sigma^+ + \varepsilon, \tilde{L}] \times [\bar{x}_2, +\infty)$, such that $x_{2,n} \to +\infty$ as $n \to \infty$ and $\hat{\delta} > 0$ such that for all $n \in \mathbb{N}$ and $\sigma \in \Sigma$,

$$|U(x_{1,n}, x_{2,n}) - \sigma| \geq \hat{\delta}.$$

By uniform continuity, we can find $\nu \in (0, \varepsilon)$ such that for all $n \in \mathbb{N}$ and

$$(x_1, x_2) \in B((x_{1,n}, x_{2,n}), \nu) \subset [\tilde{L} - \eta_\Sigma^+, \tilde{L}] \times [\bar{x}_2, +\infty),$$

we have for all $\sigma \in \Sigma$,

$$|U(x_1, x_2) - \sigma| \geq \frac{\hat{\delta}}{2}$$

or, equivalently,

$$V(U(x_1, x_2)) \geq V_{\hat{\delta}/2} := \min\{V(u) : u \in \mathbb{R}^k, \operatorname{dist}(u, \Sigma) \geq \tfrac{\hat{\delta}}{2}\}, \tag{5.42}$$

which is positive by (H1) and (H3). Up to an extraction and since $x_{2,n} \to +\infty$ as $n \to \infty$, we can assume that whenever $n \neq m$ we have

$$B((x_{1,n}, x_{2,n}), \nu) \cap B((x_{1,m}, x_{2,m}), \nu) = \emptyset,$$

which, due to the definition of η_Σ^+ in (5.38) and (5.42), implies that

$$\begin{aligned} \int_{L^+}^{+\infty} |E(U(x_1)) - E(q)|\, dx_1 &\geq \int_{\tilde{L}-\eta_\Sigma^+}^{\tilde{L}} E(U(x_1))\, dx_1 - \eta_\Sigma^+ E(q) \\ &\geq \sum_{n\in\mathbb{N}} \left(\int_{B((x_{1,n}, x_{2,n}), \nu)} V(U(x_1, x_2))\, dx_1\, dx_2 \right) - \eta_\Sigma^+ E(q) \\ &\geq \sum_{n\in\mathbb{N}} (\pi \nu^2 V_{\hat{\delta}/2}) - \eta_\Sigma^+ E(q) = +\infty, \end{aligned}$$

which is in contradiction with (5.17). Therefore, the claim has been proven. ∎

We now have all the necessary ingredients for completing the proof of Theorem 2:

Proof of Theorem 2 completed. Let $(c^\star, \mathfrak{U})$ be the solution given by Theorem 1, interpreted via the choices made in Lemma 5.1. We will invoke Lemma 5.2. The L^2 exponential convergence (2.13) given by Theorem 1 implies in particular that assumption (5.18) in Lemma 5.2 holds with $U = \mathfrak{U}$, $q = \mathfrak{q}^+(\cdot + \tau^+)$. Moreover, since $E_{2,c^\star}(\mathfrak{U}) = 0 < +\infty$, assumption (5.17) in Lemma 5.2 holds for all $L \in \mathbb{R}$ in view of the definition of $E_{2,c^\star}$ (recall that $c^\star > 0$). Finally, we have by Theorem 1 that $\mathfrak{U} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$, $\alpha \in (0,1)$, so that $\mathfrak{U}$ is uniformly continuous. As a consequence, Lemma 5.2 applies and we have (5.19) and (5.20) for all $L \in \mathbb{R}$, and this is exactly

$$\lim_{x_1 \to +\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} \tag{5.43}$$

and

$$\lim_{x_2 \to \pm\infty} \|\mathfrak{U}(\cdot, x_2) - \sigma^\pm\|_{L^\infty} \tag{5.44}$$

for all $L \in \mathbb{R}$. We now show that we can invoke Lemma 5.3. We have that (2.14) in Theorem 1 implies that (5.21) in Lemma 5.3 holds with $U = \mathfrak{U}$ and $q = \mathfrak{q}^-(\cdot + \tau^-)$.

Moreover, the abstract result Proposition 4.4 in combination with Lemma 5.1 implies in particular that for all $L \in \mathbb{R}$, (5.21) in Lemma 5.3 holds. Since $\mathfrak{U}$ is uniformly continuous, Lemma 5.3 applies, which means that (5.22) holds, so that we have proven

$$\lim_{x_1 \to -\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^-)\|_{L^\infty(\mathbb{R},\mathbb{R}^k)} = 0,$$

which in combination with (5.43) gives (2.15). Moreover, for all $L \in \mathbb{R}$ we have that (5.23) holds, which combined with (5.44) (which also holds for all $L \in \mathbb{R}$) gives (2.16) and completes the proof. ∎

5.4. Proof of Theorem 3 completed

Assume that (H6) holds. Arguing as in the proof of Theorem 1, we have that the assumptions of Theorem 6 are fulfilled if we choose as in Lemma 5.1. Notice that Theorem 3 is exactly Theorem 6 if we choose the abstract objects as in Lemma 5.1. Therefore, Theorem 3 is established. ∎

6. Examples of potentials verifying the assumptions

The purpose of this section is to exhibit a rather general and elementary method in order to produce examples of potentials for which the assumptions we make in this paper are satisfied. The idea is to modify a given multi-well potential $V_0 \colon \mathbb{R}^k \to \mathbb{R}$ satisfying (H1), (H2), (H3) and (H4) such that the associated energy possesses two minimizing heteroclinics (up to translations) in $X(\sigma^-, \sigma^+)$. Recall that σ^- and σ^+ are two wells of V_0 such that the strict triangle inequality for the infimums holds. Furthermore, we assume that the generic Schatzman spectral assumption [46] is satisfied for these heteroclinics, meaning that the constants defined in Section 2.5 (with the obvious modifications) also make sense here. That is, one can think of any potential V_0 satisfying the assumptions (H1), (H2), (H3) and (H4), as well as a modification of (H5) in which $\mathfrak{m}^+ = \mathfrak{m}^-$. For the reader's convenience, we will recall here some explicit examples of such potentials which are available in the existing literature.

The first of the examples is due to Antonopoulos and Smyrnelis; see [7, Remark 3.6]. Their idea is to find a symmetric potential and exploit such symmetries in order to obtain multiplicity of globally minimizing heteroclinics. Let us sketch some details. First consider V_{GL}, which is the Ginzburg–Landau potential

$$V_{\mathrm{GL}} \colon u = (u_1, u_2) \in \mathbb{R}^2 \to \frac{(1 - |u|^2)^2}{4} \in \mathbb{R}.$$

The idea is to perturb V_{GL} in order to obtain a double-well potential with zero set

$$\{(-1, 0), (1, 0)\}$$

and symmetric with respect to the axis $\{u_2 = 0\}$. The perturbed potential has a globally minimizing heteroclinic $\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2)$ between $(-1, 0)$ and $(1, 0)$. By symmetry, the curve

$\hat{\mathfrak{q}} = (\mathfrak{q}_1, -\mathfrak{q}_2)$ is also a globally minimizing heteroclinic, which is not a translate of $\mathfrak{q}$ if and only if $\mathfrak{q}_2$ does not vanish identically on $\mathbb{R}$. In order to ensure this, one shows that the Lagrangian functional with Ginzburg–Landau potential V_{GL} has curves connecting $(-1,0)$ and $(1,0)$ with arbitrarily small energy. These curves are then used in order to define the perturbed potential, $\widetilde{V}_0$ and they have smaller Lagrangian energy (with respect to $\widetilde{V}_0$) than the infimum of the energy among curves of the type $(q_1, 0)$. We refer to [7] for more details. In a subsequent step, one performs an arbitrarily small perturbation on $\widetilde{V}_0$ so that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are nondegenerate, which is possible due to the fact that nondegeneracy is generic; see, for example, Schatzman [46].

Another example, this time in dimension $k = 3$, is provided by Zuñiga and Sternberg [50]. They consider the potential

$$\widetilde{V}_0 : u = (u_1, u_2, u_3) \to u_1^2(1-u_1^2)^2 + \Big(u_2^2 - \frac{1}{2}(1-u_1^2)^2\Big)^2 + \Big(u_3^2 - \frac{1}{2}(1-u_1^2)^2\Big)^2 \in \mathbb{R},$$

which vanishes exactly at the points

$$(-1,0,0), \quad (1,0,0),$$
$$\Big(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \ \Big(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \ \Big(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big), \ \Big(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big).$$

By explicit computations, they show that the potential $\widetilde{V}_0$ satisfies (H1), (H2), (H3) and (H4) with $\sigma^{\pm} := (\pm 1, 0, 0)$ and, moreover, that the infimum of the corresponding energy $\widetilde{E}_0$ in $X(\sigma^-, \sigma^+)$ is not attained by a curve with trace contained in $\{u_2 = u_3 = 0\}$. Using the reflections $(0, u_2, 0) \to (0, -u_2, 0)$ and $(0, 0, u_3) \to (0, 0, -u_3)$, one deduces the multiplicity up to translations of the globally minimizing heteroclinics for $\widetilde{E}_0$ in $X(\sigma^-, \sigma^+)$. As above, one can obtain V_0 arbitrarily close to $\widetilde{V}_0$ such that the globally minimizing heteroclinics satisfy the spectral assumption.

The core of this section is the following result:

Proposition 6.1. *Let V_0 be a multi-well potential as above. For each $\varepsilon > 0$, there exists V_ε which satisfies the assumptions* (H1), (H2), (H3), (H4), (H5) *and* (H6) *and such that* $\|V_\varepsilon - V_0\|_{\mathscr{C}^2(\mathbb{R}^k)} \le \varepsilon$.

In order to obtain Proposition 6.1, one performs an arbitrarily small smooth perturbation of V_0 around the trace of one of the heteroclinics (see Figure 5), in such a way that its energy increases but a locally minimizing heteroclinic still exists (at least for small perturbations), which must necessarily have larger energy. One then chooses a perturbation which is not too large so that the upper bound on the difference of the energies is met.

Proof of Proposition 6.1. Let $\mathfrak{q}^-$ and $\mathfrak{q}^+$ in $X(\sigma^-, \sigma^+)$ be different up to translations and such that

$$E_0(\mathfrak{q}^-) = E_0(\mathfrak{q}^+) = \mathfrak{m}_0 := \inf_{q \in X(\sigma^-, \sigma^+)} E_0(q),$$

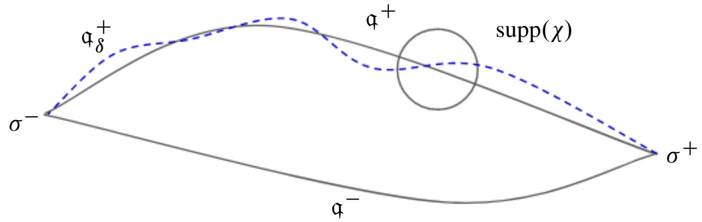

Figure 5. Representation of the cutoff function χ used in order to produce the family of perturbed functionals V_δ. We also draw the corresponding local minimizer $\mathfrak{q}_\delta^+$ (discontinuous curve).

where, for $q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$,

$$E_0(q) := \int_{\mathbb{R}} \Big[\frac{|q'(t)|^2}{2} + V_0(q(t))\Big]\, dt.$$

Recall that there exist $\rho_0^\pm$ such that

$$\forall q \in X(\sigma^-, \sigma^+), \quad \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm$$
$$\Rightarrow \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^\pm)^2 \leq \beta^\pm(E_0(q) - \mathfrak{m}_0),$$

where

$$\mathcal{F}^\pm := \big\{\mathfrak{q}^\pm(\cdot + \tau) : \tau \in \mathbb{R}\big\}.$$

Let $t_0 \in \mathbb{R}$ be such that $\mathrm{dist}(\mathfrak{q}^+(t_0), \Sigma) = \max_{t\in\mathbb{R}} \mathrm{dist}(\mathfrak{q}^+(t), \Sigma)$ for some $\mathfrak{q}^+ \in \mathcal{F}^+$, and set $u_0 := \mathfrak{q}^+(t_0)$. Let

$$r := \min\big\{\rho_0^+/2, \mathrm{dist}(\mathfrak{q}^+(t_0), \Sigma)/2\big\} > 0.$$

Define $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^k)$ as such that $0 \leq \chi \leq 1$, $\chi = 1$ on $B(u_0, r)$ and $\mathrm{supp}(\chi) \subset B(u_0, 2r)$. For each $\delta > 0$, consider the potential $V_\delta := V_0 + \delta\chi \geq 0$. Define

$$E_\delta(q) := \int_{\mathbb{R}} \Big[\frac{|q'(t)|^2}{2} + V_\delta(q(t))\Big]\, dt.$$

Notice that, by the choice of χ, V_δ vanishes exactly in $V_0^{-1}(\{0\})$. Now let $q \in X(\sigma^-, \sigma^+)$ be such that $\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^+) \leq \rho_0^+/2$. We have

$$\mathfrak{m}_0 + \frac{1}{\beta^+}\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q, \mathcal{F}^+)^2 \leq E_0(q) < E_0(q) + \delta \int_{\mathbb{R}} \chi(q) = E_\delta(q), \tag{6.1}$$

and notice that for $q \in \mathcal{F}^+$ we have $E_\delta(q) = \mathfrak{m}_0 + \delta A_\chi^+$ with

$$A_\chi^+ := \int_{\mathbb{R}} \chi(\mathfrak{q}^+(t))\, dt > 0.$$

A contradiction argument shows that

$$\mathfrak{m}_\delta^+ := \inf\{E_\delta(q) : q \in X(\sigma^-,\sigma^+),\ \mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q,\mathscr{F}^+) \leq \rho_0^+/2\} > \mathfrak{m}_0,$$

and we have $\mathfrak{m}_\delta^+ \leq E_\delta(\mathfrak{q}^+) = \mathfrak{m}_0 + \delta A_\chi^+$. Since the cutoff function is supported away from Σ, we can show by the usual concentration-compactness arguments that there exists $\mathfrak{q}_\delta^+ \in X(\sigma^-,\sigma^+)$ such that $\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(\mathfrak{q}_\delta^+,\mathscr{F}^+) \leq \rho_0^+/2$ and $E_\delta(\mathfrak{q}_\delta^+) = \mathfrak{m}_\delta^+$. If we show that $\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(\mathfrak{q}_\delta^+,\mathscr{F}^+) < \rho_0^+/2$, then the constraints of the minimization problem are not saturated and $\mathfrak{q}_\delta^+$ is an actual critical point. Notice that if $q \in X(\sigma^-,\sigma^+)$ is such that $\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(q,\mathscr{F}^+) = \rho_0^+/2$, then by (6.1) we obtain $E_0(q) \geq \mathfrak{m}_0 + (\rho_0^+)^2/(4\beta^+) > \mathfrak{m}_0$. Then, if we take $\delta < \delta_1$ with

$$\delta_1 := \frac{(\rho_0^+)^2}{4\beta^+ A_\chi^+} > 0,$$

it holds that $E_\delta(q) > E_0(q) \geq \mathfrak{m}_0 + \delta A_\chi^+ \geq \mathfrak{m}_\delta^+$, so that q cannot be a minimum. Therefore, for such a δ, items (1) and (2) in (H5) are satisfied for E_δ with minimizing heteroclinics $\mathfrak{q}^-$ and $\mathfrak{q}_\delta^+$, with the obvious modifications to the notation. Regarding item (3), which is the spectral assumption of Schatzman [46], it is a generic assumption, meaning that, arguing as in [46, Theorem 4.3], we find that V_δ can be modified with an arbitrarily small perturbation away from the traces of $\mathfrak{q}_\delta^+$ and $\mathfrak{q}^-$ so that (3) holds. As a consequence, we can assume that (H5) holds for all $\delta \in (0,\delta_1)$. Regarding (H6), compute the constant $E_{\max}$ as in (2.21), which by the choice of r and χ does not depend on δ, and set

$$\delta_2 := \frac{E_{\max}}{A_\chi^+} > 0,$$

so that for all $\delta \in (0,\delta_2)$ we have $\mathfrak{m}_\delta^+ - \mathfrak{m}_0 < E_{\max}$. Now define $\mathscr{F}^-_{\rho_0^-/2,\delta}$ as in (2.7) for the potential V_δ for $\delta \geq 0$. The choice of r and χ implies that $\mathscr{F}^-_{\rho_0^-/2,\delta}$ does not depend on δ, so that we rename it $\mathscr{F}_{\rho_0^-/2}$. As a consequence, we can find δ_3 such that for all $\delta \in (0,\delta_3)$ it holds that

$$\{q \in X(\sigma^-,\sigma^+) : E_\delta(q) < \mathfrak{m}_\delta^+\} \subset \mathscr{F}^-_{\rho_0^-/2},$$

meaning that (H6) holds for E_δ provided that $\delta \in (0,\delta_{\max})$ with

$$\delta_{\max} := \min\{\delta_1,\delta_2,\delta_3,\delta_4\} > 0$$

and δ_4 such that $m_\delta^+ - m_0 < (\mu^- \mathfrak{d}_0)/2$, which completes the proof. ∎

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