Interfaces Free Bound. (Online first)
DOI 10.4171/IFB/521

Published by EMS Press

Time-global existence of generalized BV flow via the Allen–Cahn equation

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Abstract. We show that a mean curvature flow obtained as the limit of the Allen–Cahn equation is not only a Brakke flow but also a generalized BV flow proposed by Stuvard and Tonegawa (2024).

1. Introduction

One of the most important geometric flows, the mean curvature flow (henceforth referred to as MCF), has been studied in the mathematical literature since the 1970s. The unknown of MCF is a one-parameter family $\{M_t\}_{t\geq 0}$ of surfaces in the Euclidean space (or more generally, some Riemannian manifold) such that the normal velocity vector v of M_t equals its mean curvature vector h at each point for every time, that is, $v = h$ on M_t. Given a compact smooth surface, a unique smooth solution exists until singularities such as shrinkage and neck pinching occur. To consider the solutions that allow singularities, various frameworks of weak solutions of the MCF have been proposed; we mention the Brakke flow [1], the level set solution [2,3], the BV flow [17], the L^2 flow [20], the De Giorgi-type flow [6], and the generalized BV flow [24].

The Allen–Cahn equation is the following simple reaction-diffusion semilinear PDE that can produce a MCF in the singular perturbation limit:

$$\begin{cases} \partial_t \varphi^\varepsilon = \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} & \text{on } \mathbb{R}^n \times (0,\infty), \\ \varphi^\varepsilon(\cdot,0) = \varphi_0^\varepsilon(\cdot) & \text{on } \mathbb{R}^n. \end{cases} \tag{AC}$$

Here $\varepsilon > 0$ is a small parameter and W is a double-well potential with local minima at ± 1, for example, $W(s) := (1-s^2)^2/2$. We are interested in the characterization of the limit problem as $\varepsilon \to 0$, where one expects that $\varphi^\varepsilon \approx \pm 1$ for a bulk region, and the transition layer $\{\varphi^\varepsilon \approx 0\}$ moves by the mean curvature. It is known in the most general setting of geometric measure theory that the limit energy concentration measure μ_t is a Brakke flow [10, 30]. The purpose of this paper is to show that the MCF arising from the Allen–Cahn equation is a generalized BV flow in addition to being a Brakke flow. It is

Mathematics Subject Classification 2020: 53E10 (primary); 28A75 (secondary).
Keywords: Allen–Cahn equation, geometric measure theory, mean curvature flow.

natural to consider the relationship between the phase function φ and the Brakke flow μ_t, and we prove that φ satisfies the BV-type formula. More specifically, the main claim is the following, roughly speaking:

Theorem. *Let $\{\mu_t\}_{t\geq 0}$ be the Brakke flow and $\varphi(x,t) = \chi_{E_t}(x)$ be the phase function obtained as a limit of* (AC)*. Then, for all test functions $\phi \in C_c^1(\mathbb{R}^n)$ and $0 \leq t_1 < t_2 < \infty$, we have*

$$\int_{\mathbb{R}^n} \phi(x)\varphi(x,t)\,dx\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \phi(x)(h(x,t)\cdot\nu(x,t))\,d|\nabla\varphi(\cdot,t)|(x)dt, \tag{1.1}$$

where $h(\cdot,t)$ is the generalized mean curvature vector of μ_t, $|\nabla\varphi(\cdot,t)|$ is the perimeter measure of the phase function $\varphi(\cdot,t)$, and $\nu(\cdot,t)$ is the unit outer normal vector of $|\nabla\varphi(\cdot,t)|$.

Note that (1.1) gives an explicit formula for the volume change of the phase; assuming that the initial datum E_0 is bounded, we have

$$\mathcal{L}^n(E_{t_2}) - \mathcal{L}^n(E_{t_1}) = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} (h(x,t)\cdot\nu(x,t))\,d|\nabla\varphi(\cdot,t)|(x)dt.$$

For smooth MCFs, the above formula holds naturally, while it is not obvious for generalized MCFs. Luckhaus and Sturzenhecker [17] introduced the notion of BV flow by characterizing the motion law of the boundary using (1.1) and BV functions. More precisely, a family of sets of finite perimeter $\{E_t\}_{t\geq 0}$ is a BV flow if the perimeter measures $\{|\nabla\chi_{E_t}|\}_{t\geq 0}$ have the generalized mean curvature vector $h(\cdot,t)$ satisfying (1.1). The generalized BV flow proposed by Stuvard and Tonegawa [24] is a pair consisting of a phase function and Brakke flow, which allows possible higher-integer multiplicities (≥ 2). They proved the existence under a very general setting (even with multi-phase cases). If (1.1) holds, it is known by the work of Fischer et al. [4] that the BV flow is unique until some topological changes occur, thus partially resolving the issue of superfluous non-uniqueness of Brakke flows. There are some existence results of BV flows, such as [15–17], but these studies impose a reasonable (but non-trivial) assumption that the approximate solutions converge to the limit without loss of surface energy. In contrast, we prove the existence of (generalized) BV flows without any extra assumptions, with the caveat that the accompanying Brakke flow may possess possible higher-integer multiplicities. We mention that the similar conclusion can be derived [29] for flows obtained by the elliptic regularization [11].

Next we give a detailed account of the related works on the MCFs using the Allen–Cahn equation. As a pioneering work, Ilmanen [10] proved that the limit measure of (AC) is a rectifiable Brakke flow by using Huisken's monotonicity formula [8]. Additionally, Tonegawa [30] proved that the limit measure is an integer multiple. By modifying equation (AC), varifold solutions with various additional terms are studied and we mention [12,21,26–28]. As a conditional result, Laux and Simon [16] gave the time-local existence of BV flow with forcing term in the multi-phase case. Hensel and Laux [6] proposed the new concept of weak solution, called the De Giorgi-type flow, and studied the existence and the weak-strong uniqueness property. By using the relative entropy, the simplified proof of the singular limit of (AC) is given in [5] and the case of coupling with the Navier–Stokes equation [7] and volume preservation [14] have been studied.

The key observation of the present paper is that the two properties—being an L^2 flow (which follows from being Brakke flow) and the absolute continuity of phase boundary measure with respect to the Brakke flow (both as space-time measures)—lead to equality (1.1). To this end, we find the convergence of the velocity vector representing the motion of phase boundaries using the concept of measure function pairs by Hutchinson [9]. More precisely, if $\{\mu_t\}_{t\geq 0}$ is a Brakke flow arising from the Allen–Cahn equation, by the existence of the velocity, we may obtain $d|(\nabla, \partial_t)\varphi| \ll d\mu_t dt$, where $|(\nabla, \partial_t)\varphi|$ is the space-time perimeter measure of φ. Once this is done, we may recover formula (1.1) using a suitable version of the co-area formula (see [18, Theorem 13.4], for example) from geometric measure theory. The idea to prove the main results of this paper is similar to [20,24,29], though there are some fine differences. We also point out that similar conclusions can be derived for more general flows as in [12,21,25–28] by the same strategy.

The paper is organized as follows: in Section 2, we set our notation and explain the Allen–Cahn equation and the main result. In Section 3, we show that the absolute continuity between the perimeter measure of the phase function and the Brakke flow and the boundary motion by the mean curvature is expressed in equality (1.1), and we then prove that the Allen–Cahn equation gives a generalized BV flow in the limit.

2. Preliminaries and main results

2.1. Basic notation

We shall use the same notation for the most part adopted in [24, 29].

In particular, the ambient space we will be working in is the Euclidean space $\mathbb{R}^n$, and $\mathbb{R}^+$ will denote the interval $[0, \infty)$. The coordinates (x, t) are set in the product space $\mathbb{R}^n \times \mathbb{R}$, and t will be thought of and referred to as "time". The symbols $\mathbf{p}$ and $\mathbf{q}$ will denote the projections of $\mathbb{R}^n \times \mathbb{R}$ onto its factor, so that $\mathbf{p}(x, t) = x$ and $\mathbf{q}(x, t) = t$. If $A \subset \mathbb{R}^n$ is (Borel) measurable, $\mathcal{L}^n(A)$ will denote the Lebesgue measure of A, whereas $\mathcal{H}^k(A)$ denotes the k-dimensional Hausdorff measure of A. When $x \in \mathbb{R}^n$ and $r > 0$, the closed ball centered at x with radius r is denoted by $B_r(x)$. More generally, if k is an integer, then $B^k_r(x)$ will denote closed balls in $\mathbb{R}^k$. The symbols $\nabla, \nabla', \Delta, \nabla^2$ denote the spatial gradient and the full gradient in $\mathbb{R}^n \times \mathbb{R}$, the Laplacian, and the Hessian, respectively. The symbol ∂_t will denote the time derivative.

A positive Radon measure μ on $\mathbb{R}^n$ (or "space-time" $\mathbb{R}^n \times \mathbb{R}^+$) is always regarded as a positive linear functional on the space $C^0_c(\mathbb{R}^n)$ of continuous and compactly supported functions, with the pairing denoted by $\mu(\phi)$ for $\phi \in C^0_c(\mathbb{R}^n)$. The restriction of μ to a

Borel set A is denoted $\mu \llcorner_A$, so that $(\mu \llcorner_A)(E) := \mu(A \cap E)$ for any Borel sets $E \subset \mathbb{R}^n$. The support of μ is denoted suppμ, and it is the closed set defined by

$$\mathrm{supp}\mu := \{x \in \mathbb{R}^n \mid \mu(B_r(x)) > 0 \text{ for all } r > 0\}.$$

For $1 \leq p \leq \infty$, the space of p-integrable functions with respect to μ is denoted by $L^p(\mu)$. If $\mu = \mathcal{L}^n$, $L^p(\mathcal{L}^n)$ is simply written as $L^p(\mathbb{R}^n)$. For a signed or vector-valued measure μ, its total variation is denoted by $|\mu|$. For two Radon measures μ and $\overline{\mu}$, when the measure $\overline{\mu}$ is absolutely continuous with respect to μ, we write $\overline{\mu} \ll \mu$.

We say that a function $f \in L^1(\mathbb{R}^n)$ has a bounded variation, written $f \in BV(\mathbb{R}^n)$, if

$$\sup\left\{\int_{\mathbb{R}^n} f \,\mathrm{div} X \, dx \mid X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \|X\|_{C^0} \leq 1\right\} < \infty.$$

If $f \in BV(\mathbb{R}^n)$, then there exists an $\mathbb{R}^n$-valued Radon measure (which we will call the measure derivative of f, denoted by ∇f) satisfying

$$\int_{\mathbb{R}^n} f \,\mathrm{div} X \, dx = -\int_{\mathbb{R}^n} X \cdot d\nabla f \quad \text{for all } X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n).$$

For a set $E \subset \mathbb{R}^n$, χ_E is the characteristic function of E, defined by $\chi_E = 1$ if $x \in E$, and $\chi_E = 0$ otherwise. We say that E has a (locally) finite perimeter if $\chi_E \in BV(\mathbb{R}^n)$ ($\in BV_{\mathrm{loc}}(\mathbb{R}^n)$). When E is a set of (locally) finite perimeter, then the measure derivative $\nabla \chi_E$ is the associated Gauss–Green measure, and its total variation $|\nabla \chi_E|$ is the perimeter measure; by De Giorgi's structure theorem, $|\nabla \chi_E| = \mathcal{H}^{n-1} \llcorner_{\partial^* E}$, where $\partial^* E$ is the reduced boundary of E, and $\nabla \chi_E = -\nu_E |\nabla \chi_E| = -\nu_E \mathcal{H}^{n-1} \llcorner_{\partial^* E}$, where ν_E is the outer pointing unit normal vector field to $\partial^* E$.

A subset $M \subset \mathbb{R}^n$ is countably k-rectifiable if it admits a covering

$$M \subset Z \cup \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^k),$$

where $\mathcal{H}^k(Z) = 0$ and $f_k : \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz. If M is countably k-rectifiable, $\mathcal{H}^k$-measurable and $\mathcal{H}^k(M) < \infty$, M has a measure-theoretic tangent plane called the approximate tangent plane for $\mathcal{H}^k$-almost every $x \in M$ ([22, Theorem 11.6]), denoted by $T_x M$. We may simply refer to it as the tangent plane at $x \in M$ without fear of confusion. A Radon measure μ is said to be k-rectifiable if there is a countably k-rectifiable, $\mathcal{H}^k$-measurable set M and a positive function $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner_M)$ such that $\mu = \theta \mathcal{H}^k \llcorner_M$. The function θ is called multiplicity of μ. The approximate tangent plane of M in this case (which exists μ-almost everywhere) is denoted by $T_x \mu$. When θ is an integer for μ-almost everywhere, μ is said to be integral. The first variation $\delta\mu : C_c^1(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ of a rectifiable Radon measure μ is defined by

$$\delta\mu(X) = \int_{\mathbb{R}^n} \mathrm{div}_{T_x \mu} X \, d\mu,$$

where $P_{T_x\mu}$ is the orthogonal projection from $\mathbb{R}^n$ to $T_x\mu$, and $\mathrm{div}_{T_x\mu} X = \mathrm{tr}(P_{T_x\mu}\nabla X)$. For an open set $U \subset \mathbb{R}^n$, the total variation $|\delta\mu|(U)$ of μ is defined by

$$|\delta\mu|(U) = \sup\left\{\delta\mu(X) \mid X \in C_c^1(U;\mathbb{R}^n), \|X\|_{C^0} \leq 1\right\}.$$

If the total variation $|\delta\mu|(\widetilde{U})$ is finite for any bounded subset $\widetilde{U}$ of U, then $\delta\mu$ is called locally bounded, and we can regard $|\delta\mu|$ as a measure. If $|\delta\mu| \ll \mu$, then the Radon–Nikodým derivative (times -1) is called the generalized mean curvature vector h of μ, and we have

$$\delta\mu(X) = -\int_{\mathbb{R}^n} X \cdot h \, d\mu \quad \text{for all } X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

If μ is integral, then h and $T_x\mu$ are orthogonal for μ-almost everywhere by Brakke's perpendicularity theorem [1, Chapter 5].

2.2. Weak notions of mean curvature flow

In this subsection, we introduce some weak solutions to the MCF as well as [24, 29]. We briefly define and comment upon the three of interest in the present paper; we begin with the notion of Brakke flow introduced by Brakke [1].

Definition 2.1. A family of Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ in $\mathbb{R}^n$ is *an* $(n-1)$*-dimensional Brakke flow* if the following four conditions are satisfied:

(1) For almost every $t \in \mathbb{R}^+$, μ_t is integral and $\delta\mu_t$ is locally bounded and absolutely continuous with respect to μ_t (thus, the generalized mean curvature exists for almost every t, denoted by h).

(2) For all $s > 0$ and all compact sets $K \subset \mathbb{R}^n$, $\sup_{t\in[0,s]} \mu_t(K) < \infty$.

(3) The generalized mean curvature h satisfies $h \in L^2(d\mu_t dt)$.

(4) For all $0 \leq t_1 < t_2 < \infty$ and all test functions $\phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R}^+)$,

$$\begin{aligned}\mu_{t_2}(\phi(\cdot,t_2)) - \mu_{t_1}(\phi(\cdot,t_1)) \\ \leq \int_{t_1}^{t_2}\int_{\mathbb{R}^n} (\nabla\phi(x,t) - \phi(x,t)h(x,t)) \cdot h(x,t) \\ + \partial_t\phi(x,t)\, d\mu_t(x) dt.\end{aligned} \tag{2.1}$$

Inequality (2.1) is motivated by the following identity:

$$\int_{M_t} \phi(x,t)\, d\mathcal{H}^{n-1}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{M_t} (\nabla\phi - \phi h)\cdot v + \partial_t\phi \, d\mathcal{H}^{n-1} dt, \tag{2.2}$$

where $\{M_t\}_{t\in[0,T)}$ is a family of $(n-1)$-dimensional smooth surfaces, $h(\cdot,t)$ is the mean curvature vector of M_t, and $v(\cdot,t)$ is the normal velocity vector of M_t. In particular, if $\{M_t\}_{t\in[0,T)}$ is a smooth MCF (hence, $v = h$), setting $\mu_t := \mathcal{H}^{n-1}\llcorner_{M_t}$ defines a Brakke flow for which (2.1) is satisfied with equality. Conversely, if $\mu_t = \mathcal{H}^{n-1}\llcorner_{M_t}$ with smooth M_t satisfies (2.1), then one can prove that $\{M_t\}_{t\in[0,T)}$ is a classical solution to the MCF. The notion of Brakke flow is equivalently (and originally, in [1]) formulated in the framework of varifolds, but we use the above slightly less general formulation using Radon measures, mainly for convenience.

The below definition of L^2 flow (modified slightly for our purpose) was given by Mugnai and Röger [20].

Definition 2.2 (L^2 flow). A family of Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ in $\mathbb{R}^n$ is *an* $(n-1)$-*dimensional* L^2 *flow* if it satisfies conditions (1)–(2) in Definition 2.1, as well as the following:

(a) The generalized mean curvature $h(\cdot,t)$ (which exists for almost every $t\in\mathbb{R}^+$, by condition (1)) satisfies $h(\cdot,t)\in L^2(\mu_t;\mathbb{R}^n)$, and $d\mu := d\mu_t dt$ is a Radon measure on $\mathbb{R}^n\times\mathbb{R}^+$.

(b) There exists a vector field $v\in L^2(\mu;\mathbb{R}^n)$ and a constant $C=C(\mu)>0$ such that

(b′1) $v(x,t)\perp T_x\mu_t$ for μ-almost everywhere $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$;

(b′2) For every test function $\phi\in C_c^1(\mathbb{R}^n\times\mathbb{R}^+)$, it holds that

$$\left|\int_0^\infty\int_{\mathbb{R}^n}\partial_t\phi(x,t)+\nabla\phi(x,t)\cdot v(x,t)\,d\mu_t(x)dt\right|\le C\|\phi\|_{C^0}. \tag{2.3}$$

The vector field v satisfying (2.3) is called the generalized velocity vector in the sense of L^2 flow. This definition interprets equality (2.2) as a functional expression of the area change.

Finally, we introduce the concept of generalized BV flow suggested by Stuvard and Tonegawa [24].

Definition 2.3 (Generalized BV flow). Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ and $\{E_t\}_{t\in\mathbb{R}^+}$ be families of Radon measures and sets of finite perimeter, respectively. The pair $(\{\mu_t\}_{t\in\mathbb{R}^+},\{E_t\}_{t\in\mathbb{R}^+})$ is a *generalized BV flow* if all of the following hold:

(i) $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a Brakke flow as in Definition 2.1.

(ii) For all $t\in\mathbb{R}^+$, $|\nabla\chi_{E_t}|\le\mu_t$.

(iii) For all $0\le t_1<t_2<\infty$ and all test functions $\phi\in C_c^1(\mathbb{R}^n\times\mathbb{R}^+)$,

$$\begin{aligned}\int_{E_t}\phi(x,t)\,dx\Big|_{t=t_1}^{t_2}\\ =\int_{t_1}^{t_2}\int_{E_t}\partial_t\phi(x,t)\,dxdt+\int_{t_1}^{t_2}\int_{\partial^*E_t}\phi(x,t)(h(x,t)\cdot\nu_{E_t}(x))\,d\mathcal{H}^{n-1}(x)dt.\end{aligned}$$

If μ_t and E_t satisfy the above definition, we say "$v=h$" in the sense of generalized BV flow. This definition expresses that the interface ∂^*E_t is driven by the mean curvature of μ_t. If $\mu_t=|\nabla\chi_{E_t}|$ for almost every t, the characterization (2.3) coincides with the notion of BV flow considered by Luckhaus–Struzenhecker in [17], since the mean curvature of $\partial^* E_t$ is naturally defined to be $h(\cdot,t)$ in this case. On the other hand, while the original BV flow is characterized only by (2.3), here μ_t is additionally a Brakke flow to which one can apply the local regularity theorems [13, 23, 31].

2.3. Assumptions and main result

First, we work under the following assumptions:

Assumption 2.4. Let $E_0 \subset \mathbb{R}^n$ be a set of finite perimeter with

$$\mathscr{L}^n(E_0) + |\nabla \chi_{E_0}|(\mathbb{R}^n) < \infty.$$

With this E_0 given, we can always have a sequence of good initial data for (AC), as shown in the next lemma.

Lemma 2.5. *There exist sequences of $\varepsilon_i \to 0$ and C^∞ functions $\varphi_0^{\varepsilon_i}$ such that*

$$\begin{aligned} &-1 \le \varphi_0^{\varepsilon_i} \le 1, \qquad \lim_{i\to\infty} \int_{\mathbb{R}^n} \Big| \frac{\varphi_0^{\varepsilon_i}+1}{2} - \chi_{E_0} \Big| \, d\mathscr{L}^n = 0, \\ &\frac{1}{\sigma}\Big(\frac{\varepsilon_i |\nabla \varphi_0^{\varepsilon_i}|^2}{2} + \frac{W(\varphi_0^{\varepsilon_i})}{\varepsilon_i} \Big) \, d\mathscr{L}^n \rightharpoonup |\nabla \chi_{E_0}| \quad \text{as } i \to \infty. \end{aligned} \tag{2.4}$$

Here $\sigma = \int_{-1}^{1} \sqrt{2W(s)}\, ds$ is the surface energy constant.

Proof. By [18, Theorem 13.8], there exists a sequence of open sets $E_0^i \subset \mathbb{R}^n$ with smooth boundary such that $\chi_{E_0^i} \to \chi_{E_0}$ in $L^1(\mathbb{R}^n)$ and $|\nabla \chi_{E_0^i}| \to |\nabla \chi_{E_0}|$ as measures. Let $\Psi : \mathbb{R} \to (-1,1)$ be the unique ODE solution for $\Psi' = \sqrt{2W(\Psi)}$ and $\Psi(0) = 0$. Define $\varphi_0^{\varepsilon_i}(x) := \Psi(\tilde{d}_i(x)/\varepsilon_i)$, where $\tilde{d}_i$ is the signed distance function from ∂E_0^i truncated so that it is smooth on $\mathbb{R}^n$. With suitably small choice of ε_i, one can show that all of the properties in (2.4) are satisfied for this $\varphi_0^{\varepsilon_i}$ (see [19]). ■

In fact, the particular form of $\varphi_0^{\varepsilon_i}$ described in the proof is not required, and only the properties in (2.4) matter in what follows. The following is extracted from [10, 30]:

Theorem 2.6. *Let $\varphi_0^{\varepsilon_i}$ (the index i being omitted in the following for simplicity) be a sequence satisfying (2.4), and let φ^ε be the solution of (AC). Define a time-parametrized measure*

$$\mu_t^\varepsilon := \frac{1}{\sigma}\Big(\frac{\varepsilon |\nabla \varphi^\varepsilon(\cdot,t)|^2}{2} + \frac{W(\varphi^\varepsilon(\cdot,t))}{\varepsilon} \Big) \, d\mathscr{L}^n.$$

Then, there exists a further subsequence (denoted φ^ε) such that

(a) *$\mu_t^\varepsilon \rightharpoonup \mu_t$ for all $t \in \mathbb{R}^+$ and $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a Brakke flow;*

(b) *$(1+\varphi^\varepsilon(\cdot,t))/2 \to \chi_{E_t}$ in locally $L^1(\mathbb{R}^n)$ for all $t \in \mathbb{R}^+$ and $|\nabla \chi_{E_t}| \le \mu_t$ for all $t \in \mathbb{R}^+$.*

We note that the key element of the proof is the vanishing of the discrepancy measure (see Lemma 3.3) which follows from the local point-wise estimate of [30, Lemma 3.3]. This gives a local Huisken's monotonicity formula and the verbatim proof of [10, 30] works to show the claim of Theorem 2.6. The following claim is the main result of the present paper:

Theorem 2.7. *The pair* $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$ *in Theorem 2.6 is a generalized BV flow as described in Definition 2.3.*

Remark 2.8. We comment here on the minimal properties for which formula (2.3) holds. To prove (2.3) for the limit flow $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$, we only need the properties of the L^2 flow, the upper density bound, and the absolute continuity of the phase boundary measure. More precisely, we prove the following in this paper: let $\{\mu_t\}_{t\in\mathbb{R}^+}$ be an L^2 flow with $v = h$ and let $\{E_t\}$ be a family of sets of finite perimeter with $|\nabla\chi_{E_t}| \le \mu_t$ for all $t \ge 0$. If the following hold for the pair $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$:

(a) $\Theta^{*n}(\mu, x) := \limsup_{r\to+0} \frac{\mu(B^n_r(x,t))}{r^n} < \infty$ for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$,

(b) $|\nabla'\chi_E| \ll \mu$,

where $E := \{(x,t) \mid x \in E_t\}$ and $d\mu := d\mu_t dt$, then the pair $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$ satisfies formula (2.3).

Remark 2.9. In [27] and [26], the existence theorem for volume-preserving MCFs and MCFs with transport and forcing term was proved in the L^2 flow sense, not in the Brakke sense. However, the argument of this paper to prove (2.3) can also be applied to [27] and [26], since the property of Brakke flow is only used for the weak solution of the MCF to become an L^2 flow.

3. Proof of Theorem 2.7

The pair $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$ in Theorem 2.6 satisfies properties (i) and (ii) of the Definition 2.3, due to Theorem 2.6. Therefore, the goal of this section is to prove formula (2.3). In what follows, let $E \subset \mathbb{R}^n \times \mathbb{R}^+$ denote the set

$$E := \Big\{(x,t) \mid x \in E_t, t \ge 0\Big\} \tag{3.1}$$

and note that $(1+\varphi^\varepsilon)/2 \to \chi_E$ locally in $L^1(\mathbb{R}^n \times \mathbb{R}^+)$ by the dominated convergence theorem and Theorem 2.6(b).

3.1. Absolute continuity of phase boundary measure

Even if a family of perimeter measures $\{|\nabla\chi_{E_t}|\}_{t\ge0}$ is a Brakke flow, the pair

$$(\{|\nabla\chi_{E_t}|\}_{t\ge0}, \{E_t\}_{t\ge0})$$

may not be a generalized BV flow. For example, define

$$E_t = \begin{cases} \{x \in \mathbb{R}^n \mid |x|^2 \le 1 - 2(n-1)t\} & 0 \le t < \frac{1}{4(n-1)}, \\ \emptyset & \frac{1}{4(n-1)} \le t; \end{cases}$$

this is a simple counterexample, that is, formula (2.3) fails at $t = 1/(4(n-1))$. We can expect such a phenomenon where formula (2.3) does not hold to occur due to a discontinuity to time direction in the measure-theoretic sense. In this subsection, in order to ensure that such examples do not occur, we prove $|\nabla' \chi_E| \ll \mu$.

First, we recall the below upper density bound ([10, Section 5.1]), which follows from Huisken's monotonicity formula adapted for the Allen–Cahn equation. Note that Ilmanen's result is for a "well-prepared" initial data, but the local discrepancy estimate of [30, Lemma 3.3] allows one to obtain Huisken's monotonicity formula with a small error term on $[T, \infty)$ for any $T > 0$ as follows:

Lemma 3.1. *For any $T > 0$, there exists $D = D(T, |\nabla \chi_{E_0}|(\mathbb{R}^n), n) > 0$ such that*

$$\Theta^{*(n-1)}(\mu_t, x) = \limsup_{r \to +0} \frac{\mu_t(B_r(x))}{r^{n-1}} \le D$$

for any $t \in [T, \infty)$. In particular, we have

$$\Theta^{*n}(\mu, (x,t)) = \limsup_{r \to +0} \frac{\mu(B_r^n(x,t))}{r^n} < \infty$$

for any $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $d\mu := d\mu_t dt$.

Next, we list the following energy bound:

Lemma 3.2. *Under the same assumptions of Theorem 2.6, we have*

$$\sup_{t \in [0,T]} \mu_t^\varepsilon(\mathbb{R}^n) + \frac{1}{\sigma} \int_0^T \int_{\mathbb{R}^n} \varepsilon(\partial_t \varphi^\varepsilon)^2 \, d\mathscr{L}^n dt \le \mu_0^\varepsilon(\mathbb{R}^n)$$

for any $0 < \varepsilon < 1$ and $0 < T < \infty$.

We also quote from [10, 30] the vanishing of discrepancy measure.

Lemma 3.3. *Under the same assumptions of Theorem 2.6, for any $T > 0$, we have*

$$\lim_{\varepsilon \to +0} \int_0^T \int_{\mathbb{R}^n} \left| \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon} \right| d\mathscr{L}^n dt = 0.$$

Finally, we prove the absolute continuity. Mugnai and Röger proved the following proposition for the Allen–Cahn functional, and the proof which follows it is almost identical to [20, Proposition 8.4]:

Proposition 3.4. *For the Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ and $\{E_t\}_{t\in\mathbb{R}^+}$ as in Theorem 2.6, let $d\mu = d\mu_t dt$ and let E be as in (3.1). Then, we have*

$$|\nabla' \chi_E| \ll \mu$$

and recall that ∇' is the gradient with respect to the time and space variables.

Proof. We define the function by

$$w(r) := \int_{-1}^{r} \sqrt{2W(s)}\, ds$$

and recall that $\sigma = w(1)$. For a point (x,t) with $|\nabla\varphi^\varepsilon(x,t)| > 0$, we have

$$|\nabla w(\varphi^\varepsilon)| = |\nabla\varphi^\varepsilon|\sqrt{2W(\varphi^\varepsilon)}, \quad |\partial_t w(\varphi^\varepsilon)| = |\partial_t\varphi^\varepsilon|\sqrt{2W(\varphi^\varepsilon)},$$

so that

$$|\nabla' w(\varphi^\varepsilon)| = \sqrt{1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2}\,|\nabla w(\varphi^\varepsilon)|. \tag{3.2}$$

By Lemma 3.2, for any $T > 0$, we have

$$\begin{aligned}
&\int_0^T \int_{\{|\nabla\varphi^\varepsilon(\cdot,t)|>0\}} \Big(\sqrt{1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2}\Big)^2 \varepsilon|\nabla\varphi^\varepsilon|^2 \, d\mathcal{L}^n dt \\
&\quad = \int_0^T \int_{\{|\nabla\varphi^\varepsilon(\cdot,t)|>0\}} \varepsilon|\nabla\varphi^\varepsilon|^2 + \varepsilon(\partial_t\varphi^\varepsilon)^2 \, d\mathcal{L}^n dt \le \sigma(2T+1)\mu_0^\varepsilon(\mathbb{R}^n).
\end{aligned} \tag{3.3}$$

Due to Lemma 3.3, $\varepsilon|\nabla\varphi^\varepsilon|^2\, d\mathcal{L}^n dt$ converges to μ. Hence, according to the compactness theorem of the measure-function pairs (Theorem A.3), there exists a μ-measurable function $f \ge 0$ such that

$$\lim_{\varepsilon\to+0} \frac{1}{\sigma} \int_{\{|\nabla\varphi^\varepsilon|>0\}} \phi \sqrt{1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2}\,\varepsilon|\nabla\varphi^\varepsilon|^2 \, d\mathcal{L}^n dt = \int_{\mathbb{R}^n\times\mathbb{R}^+} \phi f \, d\mu \tag{3.4}$$

for all $\phi \in C_c^0(\mathbb{R}^n \times (0,\infty))$. On the other hand, we have

$$\Big(\sqrt{\frac{2W(\varphi^\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon}|\nabla\varphi^\varepsilon|\Big)^2 \le 2\Big|\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big|.$$

Therefore, we have

$$\begin{aligned}
&\Big|\int_{\{|\nabla\varphi^\varepsilon|>0\}} \phi|\nabla' w(\varphi^\varepsilon)| - \int_{\{|\nabla\varphi^\varepsilon|>0\}} \phi \sqrt{1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2}\,\varepsilon|\nabla\varphi^\varepsilon|^2\Big| \\
&\quad = \Big|\int_{\{|\nabla\varphi^\varepsilon|>0\}} \phi \sqrt{1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2}\Big(\sqrt{\frac{2W(\varphi^\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon}|\nabla\varphi^\varepsilon|\Big)\sqrt{\varepsilon}|\nabla\varphi^\varepsilon|\Big| \\
&\quad \le \sqrt{2}\Big(\int_{\{|\nabla\varphi^\varepsilon|>0\}} \phi^2\Big(1 + \Big(\frac{\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\Big)^2\Big)\varepsilon|\nabla\varphi^\varepsilon|^2\Big)^{\frac{1}{2}} \\
&\qquad \cdot \Big(\int_{\{|\nabla\varphi^\varepsilon|>0\}} \Big|\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big|\Big)^{\frac{1}{2}},
\end{aligned} \tag{3.5}$$

where we used the Hölder inequality. Thanks to (3.2)–(3.5) and Lemma 3.3, we conclude that

$$\lim_{\varepsilon \to +0} \frac{1}{\sigma} \int_{\{|\nabla \varphi^\varepsilon|>0\}} \phi |\nabla' w(\varphi^\varepsilon)| \, d\mathcal{L}^n dt = \int_{\mathbb{R}^n \times \mathbb{R}^+} \phi f \, d\mu. \tag{3.6}$$

On the other hand, by using the Hölder inequality, we have

$$\begin{aligned}
\int_{\{|\nabla \varphi^\varepsilon|=0\}} |\nabla' w(\varphi^\varepsilon)| &= \int_{\{|\nabla \varphi^\varepsilon|=0\}} |\partial_t \varphi^\varepsilon| \sqrt{2W(\varphi^\varepsilon)} \\
&\le \sqrt{2} \Big(\int_{\mathbb{R}^n \times \mathbb{R}^+} \varepsilon (\partial_t \varphi^\varepsilon)^2 \Big)^{\frac{1}{2}} \Big(\int_{\{|\nabla \varphi^\varepsilon|=0\}} \frac{W(\varphi^\varepsilon)}{\varepsilon} \Big)^{\frac{1}{2}} \\
&\le \sqrt{2} \Big(\int_{\mathbb{R}^n \times \mathbb{R}^+} \varepsilon (\partial_t \varphi^\varepsilon)^2 \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n \times \mathbb{R}^+} \Big| \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon} \Big| \Big)^{\frac{1}{2}}.
\end{aligned}$$

Letting $\varepsilon \to 0$, again by Lemmas 3.2 and 3.3, we have $\int_{B^\varepsilon} |\nabla' w(\varphi^\varepsilon)| \, d\mathcal{L}^n dt \to 0$. Therefore, in general, equality (3.6) is valid even if we replace $\{|\nabla \varphi^\varepsilon| > 0\}$ by $\mathbb{R}^n \times (0, \infty)$. Thus, by [19, Proposition 1], the lower semi-continuity of the total variation measure, and (3.6), we obtain

$$\begin{aligned}
\sigma \int_{\mathbb{R}^n \times \mathbb{R}^+} \phi \, d|\nabla' \chi_E| &= \int_{\mathbb{R}^n \times \mathbb{R}^+} \phi \, d|\nabla' w(\varphi)| \, d\mathcal{L}^n dt \\
&\le \liminf_{\varepsilon \to +0} \int_{\mathbb{R}^n \times \mathbb{R}^+} \phi |\nabla' w(\varphi^\varepsilon)| \, d\mathcal{L}^n dt = \sigma \int_{\mathbb{R}^n \times \mathbb{R}^+} \phi f \, d\mu
\end{aligned}$$

for $\phi \ge 0$. This completes the proof of $|\nabla' \chi_E| \ll \mu$. ∎

3.2. Basic properties of L^2 flow and sets of finite perimeter

In this subsection, we state some properties of L^2 flow and sets of finite perimeter. The proof of Theorem 2.7 will follow from those properties. The arguments in this subsection are mostly contained in [20,24,29] and we include them for the convenience of the reader.

Proposition 3.5. *Let $\{\mu_t\}_{t \in \mathbb{R}^+}$ and $\{E_t\}_{t \in \mathbb{R}^+}$ be as in Theorem 2.6. We set $d\mu = d\mu_t dt$ and E as in (3.1). Then, $\mu \llcorner_{\partial^* E}$ is an n-dimensional rectifiable Radon measure and we have the following for $\mathcal{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$:*

(1) *the tangent space $T_{(x,t)}\mu$ exists, and $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$,*

(2) $\begin{pmatrix} h(x,t) \\ 1 \end{pmatrix} \in T_{(x,t)}\mu$,

(3) *$x \in \partial^* E_t$, and $T_x \mu_t = T_x(\partial^* E_t)$,*

(4) *$\mathbf{p}(\nu_E(x,t)) \ne 0$, and $\nu_{E_t}(x) = |\mathbf{p}(\nu_E(x,t))|^{-1} \mathbf{p}(\nu_E(x,t))$,*

(5) *$T_x(\partial^* E_t) \times \{0\}$ is linear subspace of $T_{(x,t)}\mu$.*

The key step of the proof of Theorem 2.7 is to prove the above proposition, for which the property of L^2 flow plays a central role, and this proposition is proved in detail by [24, Lemma 4.7]. In this paper, we will give a brief outline of the proof of Proposition 3.5.

First, we introduce the below property for Brakke flows [24, Theorem 4.3]. Note that the following claim holds for the generalized Brakke flows of [28] and [12] with a slight modification of the proof:

Proposition 3.6. *Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ be a Brakke flow as in Definition 2.1. Then, $\{\mu_t\}_{t\in\mathbb{R}^+}$ is an L^2 flow with the velocity $v = h$, that is, there exists $C = C(\mu) > 0$ such that*

$$\left|\int_{\mathbb{R}^+}\int_{\mathbb{R}^n} \partial_t\phi + \nabla\phi\cdot h\, d\mu_t dt\right| \le C\|\phi\|_{C^0},$$

for all $\phi \in C^1_c(\mathbb{R}^n \times (0,\infty))$.

The following is a simple property of L^2 flow ([20, Proposition 3.3]):

Proposition 3.7. *Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ be an L^2 flow with the velocity v as in Definition 2.2. Let μ be the space-time measure $d\mu = d\mu_t dt$. Then,*

$$\begin{pmatrix} v(x,t) \\ 1 \end{pmatrix} \in T_{(x,t)}\mu$$

at μ-almost every (x,t) wherever the tangent space $T_{(x,t)}\mu$ exists.

For the proof of Proposition 3.5, we need the following general facts about sets of finite perimeter ([18, Theorem 18.11]):

Lemma 3.8. *If $E \subset \mathbb{R}^n \times \mathbb{R}$ is a set of locally finite perimeter, then the horizontal section E_t is a set of locally finite perimeter in $\mathbb{R}^n$ for almost every $t \in \mathbb{R}$, and the following properties hold:*

(1) $\mathcal{H}^{n-1}(\partial^* E_t \Delta (\partial^* E)_t) = 0$,

(2) $\mathbf{p}(\nu_E(x,t)) \neq 0$ *for $\mathcal{H}^{n-1}$-almost every $x \in (\partial^* E)_t$,*

(3) $\nabla\chi_{E_t} = |\mathbf{p}(\nu_E(x,t))|^{-1}\mathbf{p}(\nu_E(x,t))\mathcal{H}^{n-1}\llcorner_{(\partial^* E)_t}$,

where $(\partial^* E)_t := \{x \in \mathbb{R}^n \mid (x,t) \in \partial^* E\}$.

Proof of Proposition 3.5. First of all, we will prove that $\mu\llcorner_{\partial^* E}$ is a rectifiable Radon measure. It is not difficult to see that $\mu \ll \mathcal{H}^n$. Indeed, let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a set with $\mathcal{H}^n(A) = 0$, and let the set $D_k := \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \mid \Theta^{*n}(\mu,(x,t)) \le k\}$ for each $k \in \mathbb{N}$. By [22, Theorem 3.2], we have

$$\mu(A \cap D_k) \le 2^n k \mathcal{H}^n(A \cap D_k) = 0$$

for all $k \in \mathbb{N}$. Furthermore, by the upper bound of mass density ratio (Lemma 3.1), we see that $\mu(A \setminus \bigcup_{k=1}^\infty D_k) = 0$. Thus, we obtain $\mu(A) = 0$, that is, $\mu \ll \mathcal{H}^n$ holds. Since $\mu \ll \mathcal{H}^n$, $|\nabla'\chi_E| = \mathcal{H}^n\llcorner_{\partial^* E}$, by Proposition 3.4, we see that

$$\mu\llcorner_{\partial^* E} \ll |\nabla'\chi_E|, \quad |\nabla'\chi_E| \ll \mu\llcorner_{\partial^* E}.$$

By Radon–Nikodým theorem, there exists an $L^1_{\mathrm{loc}}(|\nabla' \chi_E|)$-measurable function

$$f = \frac{d\mu\llcorner_{\partial^* E}}{d|\nabla' \chi_E|}$$

with $0 < f < \infty$ for $|\nabla' \chi_E|$-almost everywhere and $\mu\llcorner_{\partial^* E} = f|\nabla' \chi_E| = f\mathscr{H}^n\llcorner_{\partial^* E}$. This shows that $\mu\llcorner_{\partial^* E}$ is a rectifiable Radon measure and the tangent space $T_{(x,t)}(\mu\llcorner_{\partial^* E})$ with multiplicity f exists for $\mathscr{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$. For the next step, we prove that $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$ for $\mathscr{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$. Now, by [22, Theorem 3.5], we see that

$$\limsup_{r \to +0} \frac{\mu(B_r^{n+1}(x,t) \setminus \partial^* E)}{r^n} = 0 \quad \text{for } \mathscr{H}^n\text{-a.e. } (x,t) \in \partial^* E \cap \{t > 0\}.$$

Let then $\phi \in C_c^0(B_1^{n+1}(0))$ be arbitrary. We have

$$\begin{aligned}
&\lim_{r \to +0} \left| \int_{\mathbb{R}^n \times (0,\infty) \setminus \partial^* E} \frac{1}{r^n} \phi\Big(\frac{1}{r}(y-x, s-t)\Big)\, d\mu(y,s) \right| \\
&\quad \le \|\phi\|_{C^0} \limsup_{r \to +0} \frac{\mu(B_r^{n+1}(x,t) \setminus \partial^* E)}{r^n} = 0
\end{aligned}$$

for $\mathscr{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$. Thus, by $f \in L^1_{\mathrm{loc}}(|\nabla' \chi_E|)$, we obtain at each Lebesgue point of f

$$\begin{aligned}
&\lim_{r \to +0} \int_{\mathbb{R}^n \times (0,\infty)} \frac{1}{r^n} \phi\Big(\frac{1}{r}(y-x, s-t)\Big)\, d\mu(y,s) \\
&\quad = \lim_{r \to +0} \int_{\partial^* E} \frac{1}{r^n} \phi\Big(\frac{1}{r}(y-x, s-t)\Big) \frac{d\mu}{d|\nabla' \chi_E|}(y,s)\, d\mathscr{H}^n(y,s) \\
&\quad = f(x,t) \int_{T_{(x,t)}(\partial^* E)} \phi(y,s)\, d\mathscr{H}^n(y,s)
\end{aligned}$$

for all $\phi \in C_c^0(\mathbb{R}^n \times \mathbb{R})$ and $\mathscr{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$. This completes the proof of $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$.

By Propositions 3.6 and 3.7 and the above argument, parts (1) and (2) are proved. Next, we prove parts (3) and (4). By Lemma 3.8, we see that the following holds for almost every $t > 0$ and $\mathscr{H}^{n-1}$-almost every $x \in (\partial^* E)_t$:

$$\mathscr{H}^{n-1}(\partial^* E_t \Delta (\partial^* E)_t) = 0, \tag{3.7}$$

$$\mathbf{p}(\nu_E(x,t)) \neq 0, \tag{3.8}$$

$$\nu_{E_t}(x) = \frac{\mathbf{p}(\nu_E(x,t))}{|\mathbf{p}(\nu_E(x,t))|}. \tag{3.9}$$

Let $I := \{t > 0 \mid$ (3.7) fails$\}$ and set $A_t := \{x \in (\partial^* E)_t \mid x \notin \partial^* E_t$ or (3.8)–(3.9) fail$\}$ for every $t > 0$, so that $\mathcal{L}^1(I) = 0$ and $\mathcal{H}^{n-1}(A_t) = 0$ for every $t \in (0,\infty) \setminus I$. Consider then the characteristic function $\chi(x,t) := \chi_{A_t}(x)$ on $\mathbb{R}^n \times (0,\infty)$; since $\mathcal{L}^1(I) = 0$ and $\mathcal{H}^{n-1}(A_t) = 0$ for every $t \in (0,\infty) \setminus I$, we have

$$\begin{aligned}\int_{\partial^* E} \chi(x,t) |\nabla^{\partial^* E}(\mathbf{q}(x,t))| \, d\mathcal{H}^n(x,t) &= \int_0^\infty \int_{(\partial^* E)_t} \chi(x,t) \, d\mathcal{H}^{n-1} dt \\ &= \int_0^\infty \mathcal{H}^{n-1}(A_t) \, dt = \int_I \mathcal{H}^{n-1}(A_t) \, dt = 0,\end{aligned}$$

where we used the co-area formula in the first line, and where $\nabla^{\partial^* E}$ is the gradient on the tangent plane of $\partial^* E$, that is,

$$\nabla^{\partial^* E} \mathbf{q}(x,t) = P_{T_{(x,t)}(\partial^* E)}(\nabla \mathbf{q}(x,t)).$$

Here, combining parts (1) and (2), we see that

$$\begin{pmatrix} h(x,t) \\ 1 \end{pmatrix} \in T_{(x,t)}(\partial^* E) \quad \text{at } \mathcal{H}^n\text{-a.e. } (x,t) \in \partial^* E \cap \{t > 0\},$$

which implies $|\nabla^{\partial^* E}(\mathbf{q}(x,t))| > 0$ for $\mathcal{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$. Hence, it must be that $\chi(x,t) = 0$ for $\mathcal{H}^n$-almost every $(x,t) \in \partial^* E \cap \{t > 0\}$; thus, the first part of (3) and (4) is proved. For the proof of the identity $T_x \mu_t = T_x(\partial^* E_t)$, it is obtained by Proposition 3.4, Definition 2.1(2), and repeating the argument of part (1) at fixed t.

Finally, we prove part (5). Taking the $(x,t) \in \partial^* E$ as satisfying (1)–(4) of this proposition, we can calculate

$${}^t(z,0) \cdot \nu_E(x,t) = z \cdot \mathbf{p}(\nu_E(x,t)) = |\mathbf{p}(\nu_E(x,t))| (z \cdot \nu_{E_t}(x)) = 0$$

for all $z \in T_x(\partial^* E_t)$. This completes the proof of part (5). ■

3.3. Boundaries move by mean curvature

In this subsection, we prove Theorem 2.7 by rephrasing the velocity v as the mean curvature, and by using geometric measure theory. The argument for this rephrasing corresponds to the proof of the area formula (see (2.3)).

Proof of Theorem 2.7. Let us fix a test function $\phi \in C_c^1(\mathbb{R}^n \times (0,\infty))$ arbitrarily. Then, by using Gauss–Green's theorem for sets of finite perimeter, we have

$$\int_{\mathbb{R}^n \times (0,\infty)} \partial_t \phi \chi_E \, dx dt = \int_{\partial^* E} \phi \mathbf{q}(\nu_E) \, d\mathcal{H}^n. \tag{3.10}$$

Let G be the set satisfying Proposition 3.5(1)–(5). Then, for all $(x,t) \in G$, we have

$$T_{(x,t)}\mu = (T_x(\partial^* E_t) \times \{0\}) \oplus \operatorname{span} \begin{pmatrix} h(x,t) \\ 1 \end{pmatrix} \quad \text{(by Proposition 3.5(2)).} \tag{3.11}$$

By $h(x,t) \perp T_x\mu_t$, (3.11), and Proposition 3.5(1) and (4), we have

$$\nu_E(x,t) = \frac{1}{\sqrt{1+|h(x,t)|^2}} \begin{pmatrix} \nu_{E_t}(x) \\ -h(x,t)\cdot \nu_{E_t}(x) \end{pmatrix}. \tag{3.12}$$

By (3.12) and $h(x,t) \perp T_x\mu_t$ again, for all $(x,t) \in G$, we can calculate the $i \times (n+1)$ component of the matrix $I_{n+1} - \nu_E \otimes \nu_E$ for $i = 1,\dots,n+1$ as

$$(I_{n+1} - \nu_E \otimes \nu_E)_{i,(n+1)}(x,t) = \begin{cases} \frac{-(\nu_{E_t}(x))_i (h(x,t)\cdot \nu_{E_t}(x))}{1+|h(x,t)|^2} & i = 1,\dots,n, \\ \frac{1}{1+|h(x,t)|^2} & i = n+1, \end{cases}$$

where I_{n+1} is the $(n+1)$-identity matrix and $(\nu_{E_t})_i$ is the i-th component of ν_{E_t}. According to this calculation and the facts that $\nabla \mathbf{q} = \mathbf{e}_{n+1}$ and $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$ on G, we obtain that the co-area factor of the projection $\mathbf{q}$ satisfies

$$|\nabla^{\partial^* E} \mathbf{q}(\nu_E(x,t))| = \frac{1}{\sqrt{1+|h(x,t)|^2}}. \tag{3.13}$$

Due to (3.10)–(3.13) and the co-area formula, we compute

$$\begin{aligned} \int_{\mathbb{R}^n \times (0,\infty)} \partial_t \phi \chi_E \, dxdt &= -\int_G \phi h \cdot \nu_{E_t} \frac{1}{\sqrt{1+|h|^2}} \, d\mathcal{H}^n \\ &= -\int_{\partial^* E} \phi h \cdot \nu_{E_t} |\nabla^{\partial^* E} \mathbf{q}(\nu_E)| \, d\mathcal{H}^n = -\int_0^\infty \int_{\partial^* E \cap \{\mathbf{q}=t\}} \phi h \cdot \nu_{E_t} \, d\mathcal{H}^{n-1} dt \\ &= -\int_0^\infty \int_{\partial^* E_t} \phi h \cdot \nu_{E_t} \, d\mathcal{H}^{n-1} dt, \end{aligned} \tag{3.14}$$

where we used $\mathcal{H}^n(\partial^* E \setminus G) = 0$. By a suitable approximation of ϕ in (3.14), we deduce

$$\begin{aligned} &\int_{E_{t_2}} \phi(x,t_2)\, dx - \int_{E_{t_1}} \phi(x,t_1)\, dx \\ &\qquad = \int_{t_1}^{t_2} \int_{E_t} \partial_t \phi \, dxdt + \int_{t_1}^{t_2} \int_{\partial^* E_t} \phi h \cdot \nu_{E_t} \, d\mathcal{H}^{n-1} dt \end{aligned} \tag{3.15}$$

for almost every $0 < t_1 < t_2 < \infty$. By (3.15) and the facts that $|\nabla \chi_{E_t}| \le \mu_t$ and $h \in L^2(d\mu_t dt)$, we have $1/2$-Hölder continuity of χ_{E_t} in L^1 for almost every $t \in \mathbb{R}^+$. Thus, if necessary, we may re-define E so that χ_{E_t} is $1/2$-Hölder continuous in L^1 for all $t \in \mathbb{R}^+$. Using the continuity of $\|\chi_{E_t}\|_{L^1(\mathbb{R}^n)}$, we obtain the above equality for all $0 \le t_1 < t_2 < \infty$ and all $\phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^+)$. This completes the proof. ∎

A. Measure-function pairs

Here, we recall the notion of measure-function pairs introduced by Hutchinson in [9].

Definition A.1. Let $E \subset \mathbb{R}^n$ be an open set and let μ be a Radon measure on E. Suppose $f \in L^1(\mu;\mathbb{R}^d)$. Then, we say that (μ, f) is an $\mathbb{R}^d$-valued measure-function pair over E.

Next, we define the notion of convergence for a sequence of $\mathbb{R}^d$-valued measure-function pairs over E.

Definition A.2. Let $\{(\mu_i, f_i)\}_{i=1}^{\infty}$ and let (μ, f) be $\mathbb{R}^d$-valued measure-function pairs over E. Suppose

$$\mu_i \rightharpoonup \mu$$

as Radon measures on E. Then, we say (μ_i, f_i) converges to (μ, f) in the weak sense if

$$\int_E f_i \cdot \phi \, d\mu_i \to \int_E f \cdot \phi \, d\mu$$

for all $\phi \in C_c^0(E;\mathbb{R}^d)$.

We present a less general version of [9, Theorem 4.4.2] to the extent that it can be used in this paper.

Theorem A.3. *Suppose that $\mathbb{R}^d$-valued measure-function pairs $\{(\mu_i, f_i)\}_{i=1}^{\infty}$ satisfy*

$$\sup_i \int_E |f_i|^2 \, d\mu_i < \infty.$$

Then, the following hold:

(1) *There exists a subsequence $\{(\mu_{i_j}, f_{i_j})\}_{j=1}^{\infty}$ and an $\mathbb{R}^d$-value measure-function pair (μ, f) such that (μ_{i_j}, f_{i_j}) converges to (μ, f) as a measure-function pair.*

(2) *If (μ_{i_j}, f_{i_j}) converges to (μ, f), then*

$$\int_E |f|^2 \, d\mu \le \liminf_{j\to\infty} \int_E |f_{i_j}|^2 \, d\mu_{i_j} < \infty.$$

Acknowledgments. The author would like to thank his supervisor Yoshihiro Tonegawa for his insightful feedback and careful reading and for improving the quality of this paper.

Funding. The author was supported by JST, the Establishment of University Fellowships towards the Creation of Science Technology Innovation, Grant Number JPMJFS2112.

References

[1] K. A. Brakke, *The motion of a surface by its mean curvature*. Math. Notes 20, Princeton University Press, Princeton, NJ, 1978 Zbl 0386.53047 MR 485012

[2] Y. G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* **33** (1991), no. 3, 749–786 Zbl 0696.35087 MR 1100211

[3] L. C. Evans and J. Spruck, Motion of level sets by mean curvature. I. *J. Differential Geom.* **33** (1991), no. 3, 635–681 Zbl 0726.53029 MR 1100206

[4] J. Fischer, S. Hensel, T. Laux, and T. Simon. The local structure of the energy landscape in multiphase mean curvature flow: Weak-strong uniqueness and stability of evolutions. [v1] 2020, [v2] 2021, arXiv:2003.05478v2, to appear in *J. Eur. Math. Soc.*

[5] J. Fischer, T. Laux, and T. M. Simon, Convergence rates of the Allen-Cahn equation to mean curvature flow: a short proof based on relative entropies. *SIAM J. Math. Anal.* **52** (2020), no. 6, 6222–6233 Zbl 1454.53079 MR 4187143

[6] S. Hensel and T. Laux. A new varifold solution concept for mean curvature flow: Convergence of the Allen-Cahn equation and weak-strong uniqueness. 2021, arXiv:2109.04233, to appear in *J. Differ. Geom.*

[7] S. Hensel and Y. Liu, The sharp interface limit of a Navier-Stokes/Allen-Cahn system with constant mobility: convergence rates by a relative energy approach. *SIAM J. Math. Anal.* **55** (2023), no. 5, 4751–4787 Zbl 07750775 MR 4645674

[8] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.* **31** (1990), no. 1, 285–299 Zbl 0694.53005 MR 1030675

[9] J. E. Hutchinson, Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana Univ. Math. J.* **35** (1986), no. 1, 45–71 Zbl 0561.53008 MR 825628

[10] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* **38** (1993), no. 2, 417–461 Zbl 0784.53035 MR 1237490

[11] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.* **108** (1994), no. 520 Zbl 0798.35066 MR 1196160

[12] G.-C. Jiang, C.-J. Wang, and G.-F. Zheng, Convergence of solutions of some Allen-Cahn equations to Brakke's mean curvature flow. *Acta Appl. Math.* **167** (2020), 149–169 Zbl 1448.35307 MR 4103921

[13] K. Kasai and Y. Tonegawa, A general regularity theory for weak mean curvature flow. *Calc. Var. Partial Differential Equations* **50** (2014), no. 1–2, 1–68 Zbl 1298.53063 MR 3194675

[14] M. Kroemer and T. Laux. Quantitative convergence of the nonlocal Allen–Cahn equation to volume-preserving mean curvature flow. 2023, arXiv:2309.12409

[15] T. Laux and F. Otto, Convergence of the thresholding scheme for multi-phase mean-curvature flow. *Calc. Var. Partial Differential Equations* **55** (2016), no. 5, article no. 129 Zbl 1388.35121 MR 3556529

[16] T. Laux and T. M. Simon, Convergence of the Allen-Cahn equation to multiphase mean curvature flow. *Comm. Pure Appl. Math.* **71** (2018), no. 8, 1597–1647 Zbl 1393.35122 MR 3847750

[17] S. Luckhaus and T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 253–271 Zbl 0821.35003 MR 1386964

[18] F. Maggi, *Sets of finite perimeter and geometric variational problems*. Cambridge Stud. Adv. Math. 135, Cambridge University Press, Cambridge, 2012 Zbl 1255.49074 MR 2976521

[19] L. Modica, The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* **98** (1987), no. 2, 123–142 Zbl 0616.76004 MR 866718

[20] L. Mugnai and M. Röger, The Allen-Cahn action functional in higher dimensions. *Interfaces Free Bound.* **10** (2008), no. 1, 45–78 MR 2383536

[21] Y. Qi and G.-F. Zheng, Convergence of solutions of the weighted Allen-Cahn equations to Brakke type flow. *Calc. Var. Partial Differential Equations* **57** (2018), no. 5, article no. 133 Zbl 1428.35180 MR 3844514

[22] L. Simon, *Lectures on geometric measure theory*. Proc. Centre Math. Anal. Austral. Nat. Univ. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983 Zbl 0546.49019 MR 756417

[23] S. Stuvard and Y. Tonegawa. End-time regularity theorem for Brakke flows. *Math. Ann.* (2024), online first. DOI 10.1007/s00208-024-02826-8

[24] S. Stuvard and Y. Tonegawa, On the existence of canonical multi-phase Brakke flows. *Adv. Calc. Var.* **17** (2024), no. 1, 33–78 Zbl 07786303 MR 4685059

[25] K. Takasao, Existence of weak solution for volume-preserving mean curvature flow via phase field method. *Indiana Univ. Math. J.* **66** (2017), no. 6, 2015–2035 Zbl 1383.53051 MR 3744817

[26] K. Takasao, Existence of weak solution for mean curvature flow with transport term and forcing term. *Commun. Pure Appl. Anal.* **19** (2020), no. 5, 2655–2677 Zbl 1439.35308 MR 4153526

[27] K. Takasao, The existence of a weak solution to volume preserving mean curvature flow in higher dimensions. *Arch. Ration. Mech. Anal.* **247** (2023), no. 3, article no. 52 Zbl 07687726 MR 4590203

[28] K. Takasao and Y. Tonegawa, Existence and regularity of mean curvature flow with transport term in higher dimensions. *Math. Ann.* **364** (2016), no. 3–4, 857–935 Zbl 1351.53083 MR 3466855

[29] K. Tashiro. Existence of BV flow via elliptic regularization. 2023, arXiv:2305.12374, to appear in *Hiroshima Math. J.*

[30] Y. Tonegawa, Integrality of varifolds in the singular limit of reaction-diffusion equations. *Hiroshima Math. J.* **33** (2003), no. 3, 323–341 Zbl 1059.35061 MR 2040901

[31] Y. Tonegawa, A second derivative Hölder estimate for weak mean curvature flow. *Adv. Calc. Var.* **7** (2014), no. 1, 91–138 Zbl 1283.53064 MR 3176585

Received 23 January 2024.

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