Spectral representation of one-dimensional Liouville Brownian motion and Liouville Brownian excursion

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Abstract. In this paper we apply Krein's spectral theory of linear diffusions to study the one-dimensional Liouville Brownian motion and Liouville Brownian excursions from a given point. As an application we estimate the fractal dimensions of level sets of one-dimensional Liouville Brownian motion as well as various probabilistic asymptotic behaviours of Liouville Brownian motion and Liouville Brownian excursions.

1. Introduction

Liouville Brownian motion (LBM) was introduced by Garban, Rhodes and Vargas [15] and by Berestycki [6] as a way of understanding better the geometry of two-dimensional Liouville quantum gravity (LQG). Roughly speaking, planar Liouville Brownian motion is the time-change of a planar Brownian motion by the additive functional whose Revuz measure with respect to Lebesgue is the so-called Liouville measure

$$\mu_\gamma(\mathrm{d}z) = e^{\gamma\mathfrak{h}(x)}\,\mathrm{d}x, \quad x \in D,$$

where $\gamma \geq 0$ is a given parameter, D is a regular planar domain, and $\mathfrak{h}$ is a Gaussian free field (GFF) on D with certain boundary conditions. As GFFs are defined as random distributions or Gaussian processes on a certain space of measures which does not contain Dirac masses, $\mathfrak{h}(x)$ is not well-defined for individual points $x \in D$. Therefore, certain smooth approximations of $\mathfrak{h}$ are needed to define the measure μ_γ rigorously. This was done by Duplantier and Sheffield in [10] for $\gamma \in [0, 2)$ by using circle averages around given points. The resulting measure μ_γ is a random measure on D carried by a random fractal set whose fractal dimension is $2 - \gamma^2/2$.

Such random fractal measures with "log-Gaussian" densities obtained via a limiting procedure have a long history. The study was initiated by Mandelbrot [21] in the

1970s to analyse the energy dissipation phenomenon in fully developed turbulence. The mathematically rigorous foundation of these random measures was built later by Kahane [18] in 1985, which now is referred to as the Gaussian multiplicative chaos (GMC) theory. For a historical review of GMC and its relation to GFF and LQG, see, for example, the survey paper [23] of Rhodes and Vargas and the lecture notes [7] of Berestycki.

The study of planar Liouville Brownian motion was carried out in [1,14,23] with a focus on the regularity of the transition density function of LBM (so-called Liouville heat kernels). In this paper we shall continue the study but mainly focus on the case of one-dimensional Liouville Brownian motion, defined as a generalized linear diffusion process with natural scale function and speed measure ν, where ν is a boundary Liouville measure on $\mathbb{R}$ obtained from a GFF on the upper half-plane with Neumann boundary conditions.

The advantage of studying the one-dimensional case is that there exists in the literature a fully developed theory on the probabilistic interpretation of linear diffusions in terms of their scale functions and speed measures, namely the spectral theory of linear diffusions (see [11], for example). With the help of the spectral theory of linear diffusions, we are able to estimate various probabilistic asymptotic behaviours of one-dimensional LBM as well as that of Liouville Brownian excursions (LBE) from a given point. In fact, we shall study a more general family of time-changed Brownian motion with speed measure ν satisfying the following properties.

(A1) Regular support: ν is a Borel measure on $\mathbb{R}$ with no atoms, $0 < \nu([a,b]) < \infty$ for all $-\infty < a < b < \infty$, $\nu([0,x]) \to \infty$, and $\nu([-x,0]) \to \infty$ as $x \to \infty$.

(A2) Ergodicity: there exists a positive constant Z such that for every $a \in \mathbb{R}$,

$$\lim_{\eta \to \infty} \frac{\nu[a, a+\eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[a-\eta, a]}{\eta} = Z. \tag{1.1}$$

(A3) Multifractality: there exists an open interval I_ν, a family $\{\nu_q : q \in I_\nu\}$ of Borel measures on $\mathbb{R}$ and a family $\{\alpha(q) : q \in I_\nu\}$ of positive reals such that for $q \in I_\nu$, for ν_q-almost every $a \in \mathbb{R}$,

$$\lim_{r \to 0} \frac{1}{\log r} \log \nu(a-r, a+r) = \alpha(q). \tag{1.2}$$

Many stationary multifractal random measures satisfy these properties, for example, the log-infinitely divisible cascade measures constructed in [2, 4]. In particular, an instance of the boundary Liouville measure constructed in Section 3.1 satisfies (A1), (A2), and (A3).

For $w \in W = C(\mathbb{R}_+, \mathbb{R})$ let w_ν be the time change of w according to ν, and let $\mathbf{P}^x_\nu(\cdot) = \mathbf{P}^x_{\mathrm{BM}}(w_\nu \in \cdot)$ denote the law of time-changed Brownian motion starting from

$x \in \mathbb{R}$ (see Section 2.2 for precise definition). As a particular application, we proved the following theorem about the Hausdorff and packing dimension of the level sets of the time-changed Brownian motion.

Theorem 1.1. *Assume that v satisfies* (A1) *and* (A3). *For v_q-almost every $a \in \mathbb{R}$, for* $\mathbf{P}_v^a$-*almost every $w \in W$,*

$$\dim_H\{t \geq 0 \colon w(t) = a\} = \dim_P\{t \geq 0 \colon w(t) = a\} = \frac{1}{1 + \alpha(q)}.$$

The rest of the paper is organized as follows. In Section 2 we give a brief review of time-changed Brownian motion and Brownian excursions; in Section 3 we introduce the boundary Liouville quantum gravity as a one-dimensional Gaussian multiplicative chaos and use it to define one-dimensional LBM and LBE; in Section 4 we give a brief review of Krein's spectral theory of strings and list the spectral representation of time-changed BM and BE using the spectral theory of excursions of linear diffusions developed in [25, 28]; in Section 5 we study various probabilistic asymptotic behaviours of time-changed BM and BE.

2. One dimensional time-changed Brownian motion and Brownian excursion

2.1. Brownian motion and Brownian excursion

Let $W = C(\mathbb{R}_+, \mathbb{R})$ denote the Wiener space consisting of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. We regard W as a complete separable metric space. Let $\mathcal{W}$ denote its Borel σ-field. Let $\mathbf{P}_{BM}^0$ be the Wiener measure on $(W, \mathcal{W})$, under which the canonical process $w = \{w(t)\}_{t \geq 0}$ is a one-dimensional Brownian motion starting from 0. For $x \in \mathbb{R}$, let $\mathbf{P}_{BM}^x(\cdot)$ denote the measure $\mathbf{P}_{BM}^0(\cdot + x)$, that is, the law of the one-dimensional Brownian motion starting from x.

Let $\{L(t, x)\}_{t \geq 0, x \in \mathbb{R}}$ denote the joint-continuous version of the local time of the Brownian motion under $\mathbf{P}_{BM}^0$. For any bounded continuous function f on $\mathbb{R}$, one has

$$\int_0^t f(w(s)) \, ds = 2 \int_{\mathbb{R}} f(y) L(t, x) \, dx$$

for $\mathbf{P}_{BM}^0$-almost every $w \in W$. For $\ell \geq 0$ let

$$\tau(\ell) = \inf\{t \geq 0 \colon L(t, 0) > \ell\}$$

be the right-continuous inverse of the local time at 0. For $\ell \geq 0$ such that $\tau(\ell-) < \tau(\ell)$, we may define the path of the excursion at ℓ as

$$
e^{(\ell)}(t) = \begin{cases} |w(\tau(\ell-) + t)| & \text{if } 0 \leq t \leq \tau(\ell) - \tau(\ell-), \\ 0 & \text{if } t > \tau(\ell) - \tau(\ell-). \end{cases}
$$

The excursion $e^{(\ell)}$ takes values in the subspace $E \subset W$ consisting of continuous paths $e : [0, \infty) \mapsto [0, \infty)$ such that if $e(t_0) = 0$ for some $t_0 > 0$, then $e(t) = 0$ for all $t > t_0$. Let $\mathcal{E}$ denote its Borel σ-field. By Ito's excursion theory, there exists a σ-finite measure $\mathbf{n}_{\mathrm{BE}}$ on $(E, \mathcal{E})$ such that under $\mathbf{P}^0_{\mathrm{BM}}$, the point measure

$$
\sum_{\ell \geq 0 : \tau(\ell-) < \tau(\ell)} \delta_{\ell, e^{(\ell)}}(\mathrm{d}s \mathrm{d}e)
$$

is a Poisson measure on $\mathbb{R}_+ \times E$ with intensity $\mathrm{d}s \otimes \mathbf{n}_{\mathrm{BE}}(\mathrm{d}e)$. The measure $\mathbf{n}_{\mathrm{BE}}$ is called Ito's excursion measure of the Brownian motion. Here, we present the following four descriptions of $\mathbf{n}_{\mathrm{BE}}$ listed in [13]. For more details on Brownian excursions, see [22, Chapter XII], for example.

For $x > 0$, let $\mathbf{Q}^x_{\mathrm{BM}}$ denote the law of the one-dimensional Brownian motion starting from x and absorbed at 0. For $x \geq 0$, let $\mathbf{P}^x_{\mathrm{3B}}$ denote the law of the 3-dimensional Bessel process starting from x. Let $\mathbf{W}^x_{\mathrm{3B}}$ denote the law of the path obtained by piecing together two independent $\mathbf{P}^0_{\mathrm{3B}}$-processes up to their first hitting time to x (the second one runs backwards in time). These measures may be all considered to be defined on $(E, \mathcal{E})$. For $e \in E$, let $M(e) = \max_{t \geq 0} e(t)$ denote the maximum of e, and let $\zeta(e) = \inf\{t > 0 : e(t) = 0\}$ denote the lifetime of e, with the convention that $\inf \emptyset = \infty$.

(i)   We have $\mathbf{n}_{\mathrm{BE}}(M = 0) = 0$, and for every bounded continuous functional F on E supported by $\{M > x\}$ for some $x > 0$,

$$
\mathbf{n}_{\mathrm{BE}}(F) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mathbf{Q}^\varepsilon_{\mathrm{BM}}(F).
$$

(ii)   Under $\mathbf{n}_{\mathrm{BE}}$, the excursion process $e = \{e(t)\}_{t \geq 0}$ is a strong Markov process with transition kernel $\mathbf{Q}^x_{\mathrm{BM}}(e(t) \in \mathrm{d}y)$ and entrance law $(1/x)\mathbf{P}^0_{\mathrm{3B}}(e(t) \in \mathrm{d}x)$. In particular, for each positive stopping time τ and every measurable set Γ,

$$
\mathbf{n}_{\mathrm{BE}}(e(\tau + \cdot) \in \Gamma) = \int_{(0,\infty)} \frac{1}{x} \mathbf{P}^0_{\mathrm{3B}}(e(\tau) \in \mathrm{d}x)\mathbf{Q}^x_{\mathrm{BM}}(\Gamma).
$$

(iii)   For every measurable set Γ,

$$
\mathbf{n}_{\mathrm{BE}}(\Gamma) = \int_0^\infty \mathbf{W}^x_{\mathrm{3B}}(\Gamma) \frac{\mathrm{d}x}{x^2}.
$$

This means that $\mathbf{n}_{BE}(M \in dx) = dx/x^2$ and the law of $\mathbf{n}_{BE}$ conditioned on $M = x$ is $\mathbf{W}_{3B}^x$.

(iv) For every measurable set Γ,

$$\mathbf{n}_{BE}(\Gamma) = \int_0^\infty \mathbf{P}_{3B}^0(e_t \in \Gamma \mid e(t) = 0) \, p_{3B}(t, 0, 0) \, dt.$$

Here $e_t(\cdot) = e(t \wedge \cdot)$ is the path of e stopped at t, and $p_{3B}(t, 0, 0) = (2\pi t)^{-3/2}$ is obtained from the transition probability density $p_{3B}(t, x, y)$ of $\mathbf{P}_{3B}^x$ with respect to its speed measure $y^2 dy$ evaluated at $x = y = 0$. Since $\mathbf{P}_{3B}^0(\zeta(e) = t \mid e(t) = 0) = 1$, this description means that $\mathbf{n}_{BE}(\xi \in dt) = p_{3B}(t, 0, 0)dt$, and the law of $\mathbf{n}_{BE}$ conditioned on $\zeta = t$ is $\mathbf{P}_{3B}^0(e_t \in \cdot \mid e(t) = 0)$.

2.2. Time-changed Brownian motion

Let ν be a measure on $\mathbb{R}$ satisfying (A1). Define

$$A_\nu(t) = \int_{\mathbb{R}} L(t, x) \, \nu(dx), \quad t \geq 0.$$

Then $A_\nu = \{A_\nu(t)\}_{t \geq 0}$ forms an additive functional of the Brownian motion. Let

$$\tau_\nu(t) = \inf\{s \geq 0 : A_\nu(s) > t\}, \quad t \geq 0$$

be its right-continuous inverse. For $w \in W$ let

$$w_\nu(t) = w(\tau_\nu(t)), \quad t \geq 0$$

denote the time-change of w by τ_ν. Define a probability measure on W by

$$\mathbf{P}_\nu^x(\cdot) = \mathbf{P}_{BM}^x(w_\nu \in \cdot).$$

Then $\mathbf{P}_\nu^x$ is the law of the time-changed Brownian motion w_ν. Its joint-continuous transition density $p_\nu(t; x, y)$ satisfies

$$\mathbf{P}_\nu^x(w(t) \in B) = \int_B p_\nu(t; x, y) \, \nu(dy)$$

for $t > 0$, $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$. The process $\{L_\nu(t, x) = L(\tau_\nu(t), x)\}_{t \geq 0, x \in \mathbb{R}}$ is the joint-continuous local time of the time-changed BM under $\mathbf{P}_\nu^0$, that is, for any bounded continuous function f on $\mathbb{R}$ one has

$$\int_0^t f(w(s)) \, ds = 2 \int_{\mathbb{R}} f(y) L_\nu(t, x) \, \nu(dx)$$

for $\mathbf{P}_\nu^0$-almost every $w \in W$.

2.3. Time-changed Brownian excursion

Fix $a \in \mathbb{R}$. For $x \geq 0$, let us introduce the notations

$$m_{\nu,a,+}(x) = \nu([a, a+x]) \quad \text{and} \quad m_{\nu,a,-}(x) = \nu([a-x, a]).$$

Define

$$A_{\nu,a,\pm}(t) = \int_{(0,\infty)} L(t, a \pm x) \, dm_{\nu,a,\pm}(x), \quad t \geq 0.$$

Then $A_{\nu,a,\pm}$ forms an additive functional of the BM. Let

$$\tau_{\nu,a,\pm}(t) = \inf\{s \geq 0 : A_{\nu,a,\pm}(s) > t\}, \quad t \geq 0$$

be the right-continuous inverse of $A_{\nu,a,\pm}$. For $w \in W$, let

$$w_{\nu,a,\pm}(t) = w(\tau_{\nu,a,\pm}(t)), \quad t \geq 0$$

be the time change of w by $\tau_{\nu,a,\pm}$. For $x > 0$, define a probability measure on W by

$$\mathbf{P}^x_{\nu,a,\pm}(\cdot) = \mathbf{P}^{a \pm x}_{\mathrm{BM}}(\pm(w_{\nu,a,\pm} - a) \in \cdot).$$

We shall also use the same notation $L(t, x)$ to denote the joint-continuous version of the local time of the Brownian motion/excursion under $\mathbf{Q}^x_{\mathrm{BM}}$, $\mathbf{P}^x_{\mathrm{3B}}$, $\mathbf{W}^x_{\mathrm{3B}}$ and $\mathbf{n}_{\mathrm{BE}}$ on E. For $e \in E$, let

$$A_{\nu,a,\pm}(t) = \int_{(0,\infty)} L(t, x) \, dm_{\nu,a,\pm}(x), \quad t \geq 0$$

and

$$\tau_{\nu,a,\pm}(t) = \begin{cases} \inf\{s \geq 0 : A_{\nu,a,\pm}(s) > t\} & \text{if } 0 \leq t < A_{\nu,a,\pm}(\zeta), \\ \zeta & \text{if } t \geq A_{\nu,a,\pm}(\zeta), \end{cases}$$

as well as

$$e_{\nu,a,\pm}(t) = e(\tau_{\nu,a,\pm}(t)), \quad t \geq 0.$$

Define the measures on E by

$$\mathbf{Q}^x_{\nu,a,\pm}(\cdot) = \mathbf{Q}^x_{\mathrm{BM}}(e_{\nu,a,\pm} \in \cdot),$$
$$\mathbf{Q}^x_{h\text{-}\nu,a,\pm}(\cdot) = \mathbf{P}^x_{\mathrm{3B}}(e_{\nu,a,\pm} \in \cdot),$$
$$\mathbf{W}^x_{\nu,a,\pm}(\cdot) = \mathbf{W}^x_{\mathrm{BM}}(e_{\nu,a,\pm} \in \cdot),$$
$$\mathbf{n}_{\nu,a,\pm}(\cdot) = \mathbf{n}_{\mathrm{BE}}(e_{\nu,a,\pm} \in \cdot).$$

We have

(1) $\mathbf{P}^x_{v,a,\pm}$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from x and with 0 as an instantaneously reflecting boundary.

(2) $\mathbf{Q}^x_{v,a,\pm}$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from x and absorbed at 0.

(3) $\mathbf{Q}^x_{h\text{-}v,a,\pm}$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from x and conditioned never hit 0. Indeed, $\mathbf{Q}^x_{h\text{-}v,a,\pm}$ is Doob's h-transform of $\mathbf{Q}^x_{v,a,\pm}$ with $h(x) = x$. It is therefore the law of the generalized linear diffusion with speed measure $x^2 dm_{v,a,\pm}(x)$ and scale function $-1/x$.

(4) $\mathbf{W}^x_{v,a,\pm}$ is the law of the following process: consider two independent $\mathbf{Q}^0_{h\text{-}v,a,\pm}$-processes until they first hit x, and splice the two paths together (the second one runs backwards in time).

Finally, by applying [13, Theorem 2.5], we have the following descriptions of the Ito's excursion measure $\mathbf{n}_{v,a,\pm}$.

Theorem 2.1. *Let $a \in \mathbb{R}$ be fixed.*

(i) *We have $\mathbf{n}_{v,a,\pm}(M = 0) = 0$ and for every bounded continuous functional F on E supported by $\{M > x\}$ for some $x > 0$,*

$$\mathbf{n}_{v,a,\pm}(F) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mathbf{Q}^{\varepsilon}_{v,a,\pm}(F).$$

(ii) *Under $\mathbf{n}_{v,a,\pm}$ the excursion process $\{e(t)\}_{t \geq 0}$ is a strong Markov process with the transition kernel $\mathbf{Q}^x_{v,a,\pm}(e(t) \in dy)$ and the entrance law $(1/x)\mathbf{Q}^0_{h\text{-}v,a,\pm}(e(t) \in dx)$. In particular, for each positive stopping time τ and every measurable set Γ,*

$$\mathbf{n}_{v,a,\pm}(e(\tau + \cdot) \in \Gamma) = \int_{(0,\infty)} \frac{1}{x} \mathbf{Q}^0_{h\text{-}v,a,\pm}(e(\tau) \in dx)\, \mathbf{Q}^x_{v,a,\pm}(\Gamma).$$

(iii) *For every measurable set Γ,*

$$\mathbf{n}_{v,a,\pm}(\Gamma) = \int_0^\infty \mathbf{W}^x_{v,a,\pm}(\Gamma)\, \frac{dx}{x^2}.$$

This means that $\mathbf{n}_{v,a,\pm}(M \in dx) = dx/x^2$, and the law of $\mathbf{n}_{v,a,\pm}$ conditioned on $M = x$ is $\mathbf{W}^x_{v,a,\pm}$.

3. One-dimensional Liouville Brownian motion

3.1. Gaussian multiplicative chaos and Liouville quantum gravity

Let $D \subset \mathbb{R}^k$ be a domain. Let $K(x, y)$ be a nonnegative definite kernel of the form

$$- \log |x - y| + g(x, y),$$

where g is continuous over $\overline{D} \times \overline{D}$. Let

$$\mathfrak{M}_+ = \left\{ \sigma\text{-finite measure } \rho \text{ on } D \text{ with } \int_D \int_D K(x, y) \, \rho(\mathrm{d}x)\rho(\mathrm{d}y) < \infty \right\},$$

and let $\mathfrak{M}$ be the set of the signed measures of the form $\rho = \rho_+ - \rho_-$, where $\rho_+, \rho_- \in \mathfrak{M}_+$. Let $\mathfrak{h} = \{\mathfrak{h}(\rho)\}_{\rho \in \mathfrak{M}}$ be a centered Gaussian process indexed by $\mathfrak{M}$ with covariance function

$$\operatorname{Cov}(\mathfrak{h}(\rho), \mathfrak{h}(\rho')) = \int_D \int_D K(x, y) \, \rho(\mathrm{d}x)\rho'(\mathrm{d}y).$$

The process $\mathfrak{h}$ is called a Gaussian field on D with covariance kernel K. Let θ be a smooth mollifier, and for $\varepsilon > 0$ let $\theta_\varepsilon(x) = \varepsilon^{-k} \theta(x/\varepsilon)$. Let $\mathfrak{h}_\varepsilon(x) = \mathfrak{h} * \theta_\varepsilon(x)$ be the smooth approximation of $\mathfrak{h}$. Then, for $\varepsilon > 0$, we may define a random measure

$$\mu_{\gamma,\varepsilon}(\mathrm{d}x) = e^{\gamma h_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}(\mathfrak{h}_\varepsilon(x))} \, \mathrm{d}x, \quad x \in D,$$

where $\gamma \geq 0$ is a given parameter. When $\gamma < \sqrt{2k}$, the sequence of measures $\mu_{\gamma,\varepsilon}$ converges weakly in probability to a limiting measure μ_γ called a Gaussian multiplicative chaos measure on D introduced by Kahane [18] (see [8], for example, for an elementary proof).

Here we shall focus on the case when $D = \mathbb{H}$ is the upper half-plane, and consider the boundary Liouville measures on $\mathbb{R}$ defined as follows. Let $g(x, y) = -\log |x - \bar{y}|$, then $\mathfrak{h} = \mathfrak{h}^{\mathrm{f}}$ is the Gaussian free field on $\mathbb{H}$ with Neumann boundary conditions. For $x \in \mathbb{R}$ and $\varepsilon > 0$ let $\rho_{x,\varepsilon}$ denote the Lebesgue measure on the semicircle $\{y \in \mathbb{H} : |y - x| = \varepsilon\}$ in $\mathbb{H}$ normalized to have mass 1. Let $\gamma \in [0, \sqrt{2})$ be fixed. For $n \geq 1$, define

$$\nu_n(\mathrm{d}x) = 2^{-n\frac{\gamma^2}{2}} e^{\frac{\gamma}{\sqrt{2}} \mathfrak{h}^{\mathrm{f}}(\rho_{x,2^{-n}})} \, \mathrm{d}x, \quad x \in \mathbb{R}.$$

Then almost surely ν_n converge weakly to a non-trivial measure ν as $n \to \infty$. The measure ν is called the boundary Liouville measure on $\mathbb{R}$ with parameter γ.

3.2. Boundary Liouville measures as speed measures

Let ν be an instance of the boundary Liouville measure on $\mathbb{R}$ with parameter γ. We may define one-dimensional Liouville Brownian motion as the time-changed BM w_ν according to ν. In other words, one-dimensional Liouville Brownian motion is a generalized linear diffusion process on $\mathbb{R}$ with natural scale function and speed measure ν. We have that ν satisfies (A1), (A2), and (A3).

(1)   The fact that ν has no atoms is due to Kahane [18] since the measure ν can not give positive mass to a set of Hausdorff dimension less than or equal to $1 - \gamma^2/2$. The measure ν has full support by non-degeneracy that for any open interval I, $\nu(I) > 0$ almost surely.

(2)   Since the boundary Liouville measure ν is a stationary positive measure on $\mathbb{R}$, that is, $nu(x + \cdot)$ has the same law as $\nu(\cdot)$ for any $x \in \mathbb{R}$ and $\nu(I) > 0$ for any open interval I, by Birkhoff ergodic theory there exists a positive random variable Z with finite mean such that almost surely for every $a \in \mathbb{R}$,

$$\lim_{\eta \to \infty} \frac{\nu[a, a + \eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[a - \eta, a]}{\eta} = Z.$$

(3)   For $q \in (-\sqrt{2}/\gamma, \sqrt{2}/\gamma)$ let ν_q be the boundary Liouville measure with parameter $q\gamma$ defined via the same GFF $\mathfrak{h}$ as ν. In particular, ν_0 is the Lebesgue measure on $\mathbb{R}$, and $\nu_1 = \nu$. By the multifractal analysis of ν (see [5], see also [23, Theorem 4.1] for a direct proof for positive q), we have that almost surely for ν_q-almost every $a \in \mathbb{R}$,

$$\lim_{r \to 0} \frac{1}{\log r} \log \nu(a - r, a + r) = 1 + \left(\frac{1}{2} - q\right)\frac{\gamma^2}{2}.$$

4. Spectral representation of time-changed Brownian motion and time-changed Brownian excursion

We shall use Krein's spectral theory to study the time-changed BM. This section is a review of Krein's theory based on [19].

4.1. Krein's spectral theory of strings

Let $\mathcal{M}$ be the set of non-decreasing right-continuous functions $m: [0, \infty] \mapsto [0, \infty]$ with $m(0-) = 0$ and $m(\infty) = \infty$. Each $m \in \mathcal{M}$ represents the mass distribution of a string. For $m \in \mathcal{M}$, let $l = \sup\{x \geq 0: m(x) < \infty\}$ denote the length of m. For $\lambda \in \mathbb{C}$, let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the unique solution of the following integral equations on

$[0, l)$ respectively:

$$\varphi(x, \lambda) = 1 + \lambda \int_{(0,x]} (x - y)\varphi(y, \lambda)\, \mathrm{d}m(y),$$

$$\psi(x, \lambda) = x + \lambda \int_{(0,x]} (x - y)\psi(y, \lambda)\, \mathrm{d}m(y).$$

The functions φ and ψ have the following explicit expressions. Let

$$\varphi_0(x) = 1, \qquad \varphi_{n+1}(x) = \int_{(0,x]} (x - y)\varphi_n(y)\, \mathrm{d}m(y) \quad \text{for } n \geq 0,$$

$$\psi_0(x) = x, \qquad \psi_{n+1}(x) = \int_{(0,x]} (x - y)\psi_n(y)\, \mathrm{d}m(y) \quad \text{for } n \geq 0,$$

then

$$\varphi(x, \lambda) = \sum_{n=0}^{\infty} \varphi_n(x)\lambda^n, \qquad \psi(x, \lambda) = \sum_{n=0}^{\infty} \psi_n(x)\lambda^n.$$

For each fixed $x \in [0, l)$, $\phi(x, \cdot)$, and $\psi(x, \cdot)$ are real entire functions, i.e., they are entire functions of λ, and they take real values if $\lambda \in \mathbb{R}$. Set

$$h(\lambda) = \int_0^l \frac{\mathrm{d}x}{\varphi(x, \lambda)^2} = \lim_{x \uparrow l} \frac{\psi(x, \lambda)}{\varphi(x, \lambda)}.$$

The function h is called Krein's correspondence of the string m.

Let $\mathcal{H}$ be the set of functions $h: (0, \infty) \mapsto \mathbb{C}$ such that $h(\lambda)$ can be extended to a homomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ such that $\operatorname{Im} h(\lambda) \leq 0$ for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$ and $h(\lambda) > 0$ for $\lambda > 0$. Introduce the topology on $\mathcal{M}$ such that $m_n \mapsto m$ if and only if $m_n(x) \mapsto m(x)$ on every continuous point of m, and the topology on $\mathcal{H}$ such that $h_n \mapsto h$ if and only if $h_n(\lambda) \mapsto h(\lambda)$ for every $\lambda > 0$.

Theorem 4.1 (Krein's correspondence). *$\mathcal{M}$ and $\mathcal{H}$ are compact metric spaces and Krein's correspondence $m \in \mathcal{M} \leftrightarrow h \in \mathcal{H}$ defines a homeomorphism. Moreover, $h \in \mathcal{H}$ has a unique representation*

$$h(\lambda) = c + \int_{[0,\infty)} \frac{\sigma(\mathrm{d}\xi)}{\lambda + \xi},$$

where $c = \inf\{x > 0 : m(x) > 0\}$, and σ is a non-negative Borel measure on $[0, \infty)$ with $\int_{(0,\infty)} \sigma(\mathrm{d}\xi)/(1 + \xi) < \infty$.

The unique Borel measure σ is called the spectral measure of m. From the functional analysis point of view, σ is the unique measure on $[0, \infty)$ such that for $f \in L^2([0, l), \mathrm{d}m)$,

$$\|f\|_{L^2([0,l),\mathrm{d}m)} = \|\hat{f}\|_{L^2([0,\infty),\sigma)},$$

where

$$\hat{f}(\lambda) = \int_0^l f(x)\varphi(x,\lambda)\,dm(x)$$

is the generalized Fourier transform.

Note that the right-continuous inverse $m^*\colon t \geq 0 \mapsto \inf\{x > 0\colon m(x) > t\}$ also belongs to $\mathcal{M}$ with length $l^* = m(\infty-)$. It is called the dual string of m. Its Krein's correspondence is given by $h^*(\lambda) = 1/(\lambda h(\lambda))$, which also has a unique representation

$$h^*(\lambda) = c^* + \int_{[0,\infty)} \frac{\sigma^*(d\xi)}{\lambda + \xi},$$

where $c^* = m(0)$, and σ^* is a non-negative Borel measure supported on $[0, \infty)$ with $\int_{(0,\infty)} \sigma^*(d\xi)/(1+\xi) < \infty$. The measure σ^* is called the spectral measure of the dual string of m.

4.2. Spectral representation of time-changed Brownian motion

Recall that ν is a Borel measure on $\mathbb{R}$ satisfying (A1). Let $\mathcal{M}_c \subset \mathcal{M}$ be the set of functions $m\colon [0, \infty] \mapsto [0, \infty]$ that are continuous, strictly increasing functions with $m(0) = 0$ and $\sup\{x\colon m(x) < \infty\} = \infty$. For $x \geq 0$, define $m_{\nu,+}(x) = \nu([0, 0+x])$ and $m_{\nu,-}(x) = \nu([-x, 0])$. Then by (A1), both $m_{\nu,+}$ and $m_{\nu,-}$ belong to $\mathcal{M}_c$. Let $\varphi_{\nu,\pm}(x,\lambda)$ and $\psi_{\nu,\pm}(x,\lambda)$ be the unique solutions of the integral equations

$$\varphi_{\nu,\pm}(x,\lambda) = 1 + \lambda \int_{(0,x]} (x-y)\varphi_{\nu,\pm}(y,\lambda)\,dm_{\nu,\pm}(y),$$

$$\psi_{\nu,\pm}(x,\lambda) = x + \lambda \int_{(0,x]} (x-y)\psi_{\nu,\pm}(y,\lambda)\,dm_{\nu,\pm}(y),$$

and let

$$h_{\nu,\pm}(\lambda) = \int_0^\infty \frac{dx}{\varphi_{\nu,\pm}(x,\lambda)} = \lim_{x\to\infty} \frac{\psi_{\nu,\pm}(x,\lambda)}{\varphi_{\nu,\pm}(x,\lambda)}$$

be the Krein's correspondence of $m_{\nu,\pm}$. Let $\sigma_{\nu,\pm}$ be the spectral measure of $m_{\nu,\pm}$, that is, the unique non-negative Borel measure on $[0, \infty)$ with $\int_{(0,\infty)} \sigma_{\nu,\pm}(d\xi)/(1+\xi) < \infty$ such that

$$h_{\nu,\pm}(\lambda) = \int_0^\infty \frac{\sigma_{\nu,\pm}(d\xi)}{\lambda + \xi}.$$

Let h_ν be the Krein's correspondence of $m_\nu = m_{\nu,+} + m_{\nu,-}$, which satisfies

$$\frac{1}{h_\nu(\lambda)} = \frac{1}{h_{\nu,+}(\lambda)} + \frac{1}{h_{\nu,-}(\lambda)}.$$

Let σ_ν be the spectral measure of m_ν, that is, the unique non-negative Borel measure on $[0, \infty)$ with $\int_{(0,\infty)} \sigma_\nu(\mathrm{d}\xi)/(1 + \xi) < \infty$ such that

$$h_\nu(\lambda) = \int_0^\infty \frac{\sigma_\nu(\mathrm{d}\xi)}{\lambda + \xi}.$$

Define

$$\varphi_\nu(x, \lambda) = \begin{cases} \varphi_{\nu,+}(x, \lambda) & \text{if } x \geq 0, \\ \varphi_{\nu,-}(-x, \lambda) & \text{if } x < 0, \end{cases}$$

$$\psi_\nu(x, \lambda) = \begin{cases} \psi_{\nu,+}(x, \lambda) & \text{if } x \geq 0, \\ -\psi_{\nu,-}(-x, \lambda) & \text{if } x < 0. \end{cases}$$

For $\lambda > 0$, the λ-resolvent operator G_ν^λ of the time-changed BM is defined as

$$G_\nu^\lambda f(x) = \mathbf{E}_\nu^x \left(\int_0^\infty e^{-\lambda t} f(w(t)) \, \mathrm{d}t \right)$$

for any bounded continuous function f on $\mathbb{R}$. We have the following spectral representation:

$$G_\nu^\lambda f(x) = \int_{\mathbb{R}} g_\nu^\lambda(x, y) f(y) \, \nu(\mathrm{d}y),$$

where the λ-resolvent kernel $g_\nu^\lambda(x, y)$ is given by

$$g_\nu^\lambda(x, y) = h_\nu(\lambda)(\varphi_\nu(x, \lambda) + h_{\nu,+}(\lambda)^{-1} \psi_\nu(x, \lambda))(\varphi_\nu(y, \lambda) - h_{\nu,-}(\lambda)^{-1} \psi_\nu(y, \lambda)).$$

Many probabilistic quantities of the time-changed BM are related to the λ-resolvent kernel $g_\nu^\lambda(x, y)$. For example,

(i) it is the Laplace transform of the transition density $p_\nu(t; x, y)$,

$$g_\nu^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_\nu(t; x, y) \, \mathrm{d}t;$$

(ii) the right-continuous inverse of the local time $\ell_{\nu,0}(t) = \inf\{s \geq 0 \colon L_\nu(s, 0) > t\}$ at 0 is a Lévy subordinator, whose Lévy exponent is given by

$$\mathbf{E}_\nu^0\left(e^{-\lambda \ell_{\nu,0}(t)}\right) = e^{-t/g_\nu^\lambda(0,0)} = e^{-t/h_\nu(\lambda)};$$

(iii) for $a \in \mathbb{R}$, let $H_a = \inf\{t > 0 \colon w(t) = a\}$ denote the first hitting time at a. Then for $a, b \in \mathbb{R}$, we have

$$\mathbf{E}_\nu^a(e^{-\lambda H_b}) = \frac{g_\nu^\lambda(a, b)}{g_\nu^\lambda(b, b)}.$$

4.3. Spectral representation of time-changed Brownian excursion

Fix $a \in \mathbb{R}$. Recall the strings $m_{v,a,+}(x) = v([a, a + x])$ and $m_{v,a,-}(x) = v([a - x, a])$ for $x \geq 0$. Note that both $m_{v,a,+}$ and $m_{v,a,-}$ belong to $\mathcal{M}_c$. Let $\varphi_{v,a,\pm}(x, \lambda)$ and $\psi_{v,a,\pm}(x, \lambda)$ be the unique solutions of the integral equations

$$\varphi_{v,a,\pm}(x, \lambda) = 1 + \lambda \int_{(0,x]} (x - y)\varphi_{v,a,\pm}(y, \lambda) \, dm_{v,a,\pm}(y),$$

$$\psi_{v,a,\pm}(x, \lambda) = x + \lambda \int_{(0,x]} (x - y)\psi_{v,a,\pm}(y, \lambda) \, dm_{v,a,\pm}(y),$$

and let

$$h_{v,a,\pm}(\lambda) = \int_0^\infty \frac{dx}{\varphi_{v,a,\pm}(x, \lambda)} = \lim_{x \to \infty} \frac{\psi_{v,a,\pm}(x, \lambda)}{\varphi_{v,a,\pm}(x, \lambda)}$$

be the Krein's correspondence of $m_{v,a,\pm}$. Let $\sigma_{v,a,\pm}$ be the spectral measure of $m_{v,a,\pm}$. Let $m^*_{v,a,\pm} : t \geq 0 \mapsto \inf\{s \geq 0 : m_{v,a,\pm}(s) > t\}$ denote the dual string of $m_{v,a,\pm}$ and let $h^*_{v,a,\pm}$ denote its Krein's correspondence. We have

$$h^*_{v,a,\pm}(\lambda) = \frac{1}{\lambda h_{v,a,\pm}(\lambda)}.$$

Let $\sigma^*_{v,a,\pm}$ be the spectral measure of $m^*_{v,a,\pm}$. We have the following spectral representations of time-changed BM with different boundary conditions.

(1) Let $p_{v,a,\pm}(t; x, y)$ be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ and with 0 as an instantaneously reflecting boundary, that is, for $t > 0$, $x, y \in [0, \infty)$, and $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbf{P}^x_{v,a,\pm}(w(t) \in B) = \int_B p_{v,a,\pm}(t; x, y) \, dm_{v,a,\pm}(y).$$

Then

$$p_{v,a,\pm}(t; x, y) = \int_{(0,\infty)} e^{-t\lambda} \varphi_{v,a,\pm}(x, -\lambda)\varphi_{v,a,\pm}(y, -\lambda) \, \sigma_{v,a,\pm}(d\lambda).$$

The associated λ-resolvent kernel $g^\lambda_{v,a,\pm}(x, y)$ is given by

$$g^\lambda_{v,a,\pm}(x, y) = \int_{(0,\infty)} e^{-t\lambda} p_{v,a,\pm}(t; x, y) \, dt$$

$$= \int_{(0,\infty)} \frac{\varphi_{v,a,\pm}(x, -\xi)\varphi_{v,a,\pm}(y, -\xi)}{\lambda + \xi} \, \sigma_{v,a,\pm}(d\xi).$$

(2) Let $q_{v,a,\pm}(t;x,y)$ be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$, and with 0 as an absorbing boundary, that is, for $t > 0$, $x, y \in (0, \infty)$ and $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbf{Q}^x_{v,a,\pm}(e(t) \in B) = \int_B q_{v,a,\pm}(t;x,y)\, dm_{v,a,\pm}(y).$$

Then

$$q_{v,a,\pm}(t;x,y) = \int_{(0,\infty)} e^{-t\lambda} \psi_{v,a,\pm}(x,-\lambda) \psi_{v,a,\pm}(y,-\lambda)\, \lambda\sigma^*_{v,a,\pm}(d\lambda).$$

The associated λ-resolvent kernel $\hat{g}^\lambda_{v,a,\pm}(x,y)$ is given by

$$\hat{g}^\lambda_{v,a,\pm}(x,y) = \int_{(0,\infty)} e^{-t\lambda} q_{v,a,\pm}(t;x,y)\, dt$$
$$= \int_{(0,\infty)} \frac{\psi_{v,a,\pm}(x,-\xi)\psi_{v,a,\pm}(y,-\xi)}{\lambda+\xi}\, \xi\sigma^*_{v,a,\pm}(d\xi).$$

(3) Let $q_{h\text{-}v,a,\pm}(t;x,y)$ be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$, and conditioned to never hit 0, that is, for $t > 0$, $x, y \in (0, \infty)$ and $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbf{Q}^x_{h\text{-}v,a,\pm}(e(t) \in B) = \int_B q_{h\text{-}v,a,\pm}(t;x,y)\, dm_{v,a,\pm}(y).$$

Then

$$q_{h\text{-}v,a,\pm}(t;x,y) = \frac{q_{v,a,\pm}(t;x,y)}{xy}$$
$$= \int_{(0,\infty)} e^{-t\lambda} \frac{\psi_{v,a,\pm}(x,-\lambda)}{x} \frac{\psi_{v,a,\pm}(y,-\lambda)}{y}\, \lambda\sigma^*_{v,a,\pm}(d\lambda).$$

The associated λ-resolvent kernel $\widetilde{g}^\lambda_{v,a,\pm}(x,y)$ is given by

$$\widetilde{g}^\lambda_{v,a,\pm}(x,y) = \int_{(0,\infty)} e^{-t\lambda} q_{h\text{-}v,a,\pm}(t;x,y)\, dt$$
$$= \int_{(0,\infty)} \frac{\psi_{v,a,\pm}(x,-\xi)\psi_{v,a,\pm}(y,-\xi)}{xy(\lambda+\xi)}\, \xi\sigma^*_{v,a,\pm}(d\xi).$$

(4) The partial derivative of $q_{v,a,\pm}(t;x,y)$ at $y = 0$,

$$\pi_{v,a,\pm}(t;x) = \lim_{y\to 0+} \frac{q_{v,a,\pm}(t;,x,y)}{y} = \int_{(0,\infty)} e^{-t\lambda} \psi_{v,a,\pm}(x,-\lambda)\, \lambda\sigma^*_{v,a,\pm}(d\lambda),$$

is the density of the first hitting time $H_0 = \inf\{t > 0: w(t) = 0\}$ under $\mathbf{Q}^x_{v,a,\pm}$, that is

$$\mathbf{Q}^x_{v,a,\pm}(H_0 \in dt) = \pi_{v,a,\pm}(t;x)\, dt.$$

In particular,

$$\mathbf{Q}_{v,a,\pm}^{x}(H_0 > t) = \int_0^\infty e^{-t\lambda} \pi_{v,a,\pm}(t;x)\,\sigma_{v,a,\pm}^*(d\lambda).$$

It also defines an entrance law: for $t, s > 0$ and $y \in (0, \infty)$,

$$\int_{(0,\infty)} \pi_{v,a,\pm}(t;x) q_{v,a,\pm}(s;x,y)\,dm_{v,a,\pm}(x) = \pi_{v,a,\pm}(t+s;y).$$

(5) For $t > 0$, let $G_t = \sup\{s \le t : w(s) = 0\}$ and $D_t = \inf\{s \ge t : w(s) = 0\}$. Then for $u < t < v$ and $x > 0$,

$$\mathbf{P}_{v,a,\pm}^0(G_t \in du, w(t) \in dx, D_t \in dv)$$
$$= p_{v,a,\pm}(u;0,0)\pi_{v,a,\pm}(t-u;x)\pi_{v,a,\pm}(v-t;x)\,du\,dv\,dm_{v,a,\pm}(x).$$

(6) The partial derivative of $\pi_{v,a,\pm}(t;x)$ at $x = 0$,

$$n_{v,a,\pm}(t) = \lim_{x\to 0+} \frac{\pi_{v,a,\pm}(t;x)}{x} = \int_{(0,\infty)} e^{-t\lambda}\,\lambda\sigma_{v,a,\pm}^*(d\lambda),$$

is the density of the Lévy measure of the Lévy subordinator,

$$\{\ell_{v,a,\pm}(t) := \inf\{s \ge 0 : L(\tau_{v,a,\pm}(s), 0) > t\}\}_{t\ge 0}$$

under $\mathbf{P}_{v,a,\pm}^0$, that is,

$$\mathbf{E}_{v,a,\pm}^0(e^{-\lambda\ell_{v,a,\pm}(t)}) = e^{-t/h_{v,a,\pm}(\lambda)},$$

where the Lévy exponent $1/h_{v,a,\pm}(\lambda)$ takes the form

$$\frac{1}{h_{v,a,\pm}(\lambda)} = \int_0^\infty (1 - e^{-t\lambda})n_{v,a,\pm}(t)\,dt.$$

Let $\mathbf{Q}_{v,a,\pm}^{0,t,0}$ denote the law of the $\mathbf{Q}_{h-v,a,\pm}^0$-process pinned at 0 with lifetime t. Alternatively, $\mathbf{Q}_{v,a,\pm}^{0,t,0}$ is the weak limit of the law of the Markovian bridge $\mathbf{Q}_{v,a,\pm}^{x,t,y}$ as $y \to 0+$, $x \to 0+$ (see [12], for example). By applying the results in [25, 28], we have the following spectral representation of the Ito's excursion measure $\mathbf{n}_{v,a,\pm}$.

Theorem 4.2. *The Ito's excursion measure $\mathbf{n}_{v,a,\pm}$ has the following representation:*

$$\mathbf{n}_{v,a,\pm}(de) = \int_0^\infty \mathbf{n}_{v,a,\pm}(\zeta \in dt)\,\mathbf{Q}_{v,a,\pm}^{0,t,0}(de),$$

where the law of the lifetime ζ under $\mathbf{n}_{v,a,\pm}$ is equal to the Lévy measure of $\ell_{v,a,\pm}$:

$$\mathbf{n}_{v,a,\pm}(\zeta \in dt) = n_{v,a,\pm}(t)\,dt.$$

Moreover, we have the following finite dimensional distribution: for $0 < t_1 < t_2 < \cdots < t_n$ and $x_i > 0$, $i = 1, \ldots, n$,

$$\mathbf{n}_{v,a,\pm}(e(t_1) \in \mathrm{d}x_1, e(t_2) \in \mathrm{d}x_2, \ldots, e(t_n) \in \mathrm{d}x_n)$$
$$= \pi_{v,a,\pm}(t_1; x_1)\mathrm{d}m_{v,a,\pm}(x_1)q_{v,a,\pm}(t_2 - t_1; x_1, x_2)\mathrm{d}m_{v,a,\pm}(x_2)$$
$$\times \cdots \times q_{v,a,\pm}(t_n - t_{n-1}; x_{n-1}, x_n)\mathrm{d}m_{v,a,\pm}(x_n).$$

In particular,
$$\mathbf{n}_{v,a,\pm}(e(t) \in \mathrm{d}x) = \pi_{v,a,\pm}(t; x)\mathrm{d}m_{v,a,\pm}(x),$$

and it holds that

$$\mathbf{n}_{v,a,\pm}(\zeta > t) = \int_0^\infty \mathbf{n}_{v,a,\pm}(e(t) \in \mathrm{d}x) = \int_0^\infty \pi_{v,a,\pm}(t; x)\,\mathrm{d}m_{v,a,\pm}(x).$$

Remark 4.1. Note that $\mathbf{n}_{v,a,\pm}(\zeta > t)$ also has the form

$$\int_t^\infty n_{v,a,\pm}(s)\,\mathrm{d}s = \int_0^\infty e^{-\lambda t}\sigma_{v,a,\pm}^*(\mathrm{d}t).$$

This yields the identity (see [25, Proposition 3])

$$\int_0^t p_{v,a,\pm}(u; 0, 0)\,\mathrm{d}u \int_{t-u}^\infty n_{v,a,\pm}(v)\,\mathrm{d}v = 1.$$

Recall that $H_x = \inf\{t > 0 \colon w(t) = x\}$ is the first hitting time to x, and denote by $H^x = \sup\{t > 0 \colon w(t) = x\}$ the last exit time from x. Let $(\cdot)^\vee$ denote the time reverse operator on $E^0 = \{e \in E \colon e(0) = 0\}$, that is, for $e \in E^0$,

$$e^\vee(t) = e((\zeta - t)^+), \quad t \geq 0.$$

Let $\mathcal{E}^0$ denote the Borel σ-field of E^0, and let $\mathcal{E}^0_{(0, H_a)}$, $\mathcal{E}^0_{(H_a, H^a)}$ and $\mathcal{E}^0_{(H^a, \zeta)}$ denote the sub σ-fields with respect to the corresponding time intervals (see [28] for more precise definitions). Let $\theta_t(w)(\cdot) = w(t + \cdot)$ denote the left-shift operator. We have the following time reverse and first-entrance-last-exit decomposition of the excursion measure $\mathbf{n}_{v,a,\pm}$ from [28].

Theorem 4.3. *Let $a \in \mathbb{R}$ be fixed.*

(i)   *For $\Gamma \in \mathcal{E}^0$ one has*

$$\mathbf{n}_{v,a,\pm}(\Gamma^\vee) = \mathbf{n}_{v,a,\pm}(\Gamma).$$

(ii) *For $x > 0$ and $\Gamma_1 \in \mathcal{E}^0_{(0, H_x)}$, $\Gamma_2 \in \mathcal{E}^0_{(H_x, H^x)}$, $\Gamma_3 \in \mathcal{E}^0_{(H^x, \zeta)}$ one has*

$$\mathbf{n}_{v,a,\pm}(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3) = \frac{1}{x}\mathbf{P}^0_{h-v,a,\pm}(\Gamma_1)\mathbf{Q}^x_{v,a,\pm}(\theta_{H_x}(\Gamma_2))\mathbf{P}^0_{h-v,a,\pm}(\Gamma_3^\vee).$$

In particular,

$$\mathbf{n}_{v,a,\pm}\big(\{H_x \in dt_1\} \cap \{\{H^x - H_x\} \in dt_2\} \cap \{\zeta - H^x\} \in dt_3\big)$$

$$= \frac{1}{x}\mathbf{P}^0_{h\text{-}v,a,\pm}(H_x \in dt_1)\mathbf{Q}^x_{v,a,\pm}(H^a \in dt_2)\mathbf{P}^0_{h\text{-}v,a,\pm}(H_x \in dt_3).$$

Consequently, $\mathbf{n}_{v,a,\pm}(H_x \in dt) = (1/x)\mathbf{P}^0_{h\text{-}v,a,\pm}(H_x \in dt)$ *and*

$$\mathbf{n}_{v,a,\pm}(e^{-\lambda H_x}) = \frac{1}{x}\mathbf{P}^0_{h\text{-}v,a,\pm}(e^{-\lambda H_x}) = \frac{1}{\psi_{v,a,\pm}(x,\lambda)}.$$

5. Probabilistic asymptotic behaviours of time-changed Brownian motion and Brownian excursions

As an application of the spectral representation in Section 4.2 and 4.3, we shall study the probabilistic asymptotic behaviours of time-changed BM and BE.

The first application is Theorem 1.1 on the fractal dimensions of the level sets. Assume that v satisfies (A1) and (A3). We shall prove the following statement. For v_q-almost every $a \in \mathbb{R}$, for $\mathbf{P}^a_v$-almost every $w \in W$,

$$\dim_H\{t \geq 0 \colon w(t) = a\} = \dim_P\{t \geq 0 \colon w(t) = a\} = \frac{1}{1 + \alpha(q)}.$$

Proof of Theorem 1.1. For $a \in \mathbb{R}$, denote $m_{v,a} = m_{v,a,+} + m_{v,a,-}$, and let

$$V_{v,a}(r) = \int_0^r m_{v,a}(x)\,dx, \quad r \geq 0.$$

Let $h_{v,a}$ be the Krein's correspondence of $m_{v,a}$, which satisfies

$$\frac{1}{h_{v,a}(\lambda)} = \frac{1}{h_{v,a,+}(\lambda)} + \frac{1}{h_{v,a,-}(\lambda)}.$$

From [19] we have for $\eta > 0$

$$\frac{1}{4}h_{v,a}(1/\eta) \leq (V_{v,a})^{-1}(\eta) \leq 64h_{v,a}(1/\eta). \tag{5.1}$$

Recall the measure v_q in (A3). By (1.2), for v_q-almost every $a \in \mathbb{R}$ for every $\varepsilon > 0$ there exists $r_{a,\varepsilon} > 0$ such that

$$r^{\alpha(q)+\varepsilon} \leq v(a - r, a + r) \leq r^{\alpha(q)-\varepsilon}, \quad \forall r \leq r_{a,\varepsilon}.$$

This implies that

$$\frac{1}{1 + \alpha(q) + \varepsilon}r^{1+\alpha(q)+\varepsilon} \leq V_{v,a}(r) \leq \frac{1}{1 + \alpha(q) - \varepsilon}r^{1+\alpha(q)-\varepsilon}, \quad \forall r \leq r_{a,\varepsilon}. \tag{5.2}$$

Equivalently, for $\eta > 0$ small enough,

$$(1 + \alpha(q) - \varepsilon)^{\frac{1}{1+\alpha(q)-\varepsilon}} \eta^{\frac{1}{1+\alpha(q)-\varepsilon}} \leq (V_{\nu,a})^{-1}(\eta) \leq (1 + \alpha(q) + \varepsilon)^{\frac{1}{1+\alpha(q)+\varepsilon}} \eta^{\frac{1}{1+\alpha(q)+\varepsilon}}.$$

By (5.1), we deduce that there exist constants $0 < c_{a,\varepsilon}, C_{a,\varepsilon} < \infty$ such that for λ large enough,

$$c_{a,\varepsilon} \lambda^{-\frac{1}{1+\alpha(q)-\varepsilon}} \leq h_{\nu,a}(\lambda) \leq C_{a,\varepsilon} \lambda^{-\frac{1}{1+\alpha(q)+\varepsilon}}. \tag{5.3}$$

The local time $L_\nu(t, a) = L(\tau_\nu(t), a)$ at a is carried by the level set $\{t \geq 0 : w(t) = a\}$ for $\mathbf{P}_\nu^a$-almost every $w \in W$, and its right-continuous inverse $\ell_{\nu,a}(t) := \inf\{s \geq 0 : L_\nu(s, a) > t\}$ is a Lévy subordinator, whose Lévy exponent is given by

$$\mathbf{E}_\nu^a(e^{\lambda \ell_{\nu,a}(t)}) = e^{-t/h_{\nu,a}(\lambda)}.$$

By the general theory of fractal dimensions of images of Lévy subordinator (see [9, Chapter 5], for example), we have for $\mathbf{P}_\nu^a$-almost every $w \in W$,

$$\dim_H \{t \geq 0 : w(t) = a\} = \liminf_{\lambda \to \infty} \frac{-\log h_{\nu,a}(\lambda)}{\log \lambda},$$

$$\dim_P \{t \geq 0 : w(t) = a\} = \limsup_{\lambda \to \infty} \frac{-\log h_{\nu,a}(\lambda)}{\log \lambda}.$$

By (5.3), with $\varepsilon \to 0$ we get for $\mathbf{P}_\nu^a$-almost every $w \in W$,

$$\dim_H \{t \geq 0 : w(t) = a\} = \dim_P \{t \geq 0 : w(t) = a\} = \frac{1}{1 + \alpha(q)}. \qquad \blacksquare$$

Remark 5.1. Theorem 1.1 is linked to work [16] of Jackson on the Hausdorff dimension of the times that planar LBM spent on the thick points of the corresponding Gaussian free field. Theorem 1.1 estimates the size of the times that one-dimensional LBM spent at ν_q-almost every a, whereas [16] estimates the size of planar LBM spent in the support of μ_γ. So, roughly speaking, Theorem 1.1 can be considered as a fiber version of the result in [16] in dimension 1. Since in dimension 2 there does not exist the local time of BM/LBM at a given point, it seems difficult to derive an analogue of Theorem 1.1 in dimension two.

As the second application, we shall estimate the asymptotic behaviours of the transition density $p_\nu(t; a, a)$ at a given point $a \in \mathbb{R}$. First note that $p_\nu(t; a, a)$ has the following spectral representation:

$$p_\nu(t; a, a) = \int_0^\infty e^{-t\lambda} \sigma_{\nu,a}(\mathrm{d}\lambda),$$

where $\sigma_{v,a}$ is the spectral measure of $m_{v,a}$, that is, the unique non-negative Borel measure on $[0, \infty)$ with $\int_{(0,\infty)} \sigma_{v,a}(\mathrm{d}\xi)/(1 + \xi) < \infty$ such that

$$h_{v,a}(\lambda) = \int_0^\infty \frac{\sigma_{v,a}(\mathrm{d}\xi)}{\lambda + \xi}.$$

This yields the following lemma of Tomisaki [26].

Lemma 5.1. *Let ϕ be a positive and non-increasing function on $(0, \delta)$ for some $\delta > 0$. Then*

$$\int_0^\delta \phi(t)\, p_v(t; a, a)\, \mathrm{d}t < \infty \Leftrightarrow \int_0^{(V_{v,a})^{-1}(\delta)} \phi(V_{v,a}(x))\, \mathrm{d}x < \infty.$$

We have the following result on the short term behaviour of $p_v(t; a, a)$.

Theorem 5.1. *For v_q-almost every $a \in \mathbb{R}$, for any $\beta > 1/(1 + \alpha(q))$,*

$$\int_{0+} t^{-\beta}\, p_v(t; a, a)\, \mathrm{d}t = \infty,$$

and for any $\beta < 1/(1 + \alpha(q))$,

$$\int_{0+} t^{-\beta}\, p_v(t; a, a)\, \mathrm{d}t < \infty.$$

In particular,

$$\liminf_{t \to 0} \frac{\log p_v(t; a, a)}{-\log t} \le 1 - \frac{1}{1 + \alpha(q)} \le \limsup_{t \to 0} \frac{\log p_v(t; a, a)}{-\log t}. \qquad (5.4)$$

Proof. As a direct consequence of Lemma 5.1 and (5.2), we have that for any $\beta > 1/(1 + \alpha(q))$,

$$\int_{0+} t^{-\beta}\, p_v(t; a, a)\, \mathrm{d}t = \infty,$$

and for any $\beta < 1/(1 + \alpha(q))$,

$$\int_{0+} t^{-\beta}\, p_v(t; a, a)\, \mathrm{d}t < \infty,$$

which implies (5.4). ∎

Remark 5.2. By using Tauberian theorem, it can be shown that if $v(a - r, a + r)$ is a regular varying function of r as $r \to 0$, then the inequalities in (5.4) become equalities, see [3], for example for the case when v is a Bernoulli measure on $[0, 1]$. However, due to the multifractal nature of GMC measures, it is the case that for v_q almost every a, $v(a - r, a + r)$ is not regular varying as $r \to 0$. So it is not clear whether the limit in (5.4) exists.

Remark 5.3. The short term behaviour (5.4) is quite different comparing to [24, Corollary 2.1]. The reason is that in [24] the one-dimensional Liouville Brownian motion is defined as a linear diffusion with scale function m_ν and speed measure dx, and the corresponding transition density $p_*(t; x, x)$ is defined with respect to $\nu(dx)$ rather than dx. Therefore, by change of variables, it is straightforward to verify that for every $x \in \mathbb{R}$,

$$\lim_{t \to 0} \frac{-\log p_*(t; x, x)}{\log t} = \frac{1}{2}.$$

From now on, we assume that ν satisfies (A1) and (A2). We have the following long term behaviour of $p_\nu(t; a, a)$.

Theorem 5.2. *For every $a \in \mathbb{R}$,*

$$\lim_{t \to \infty} \sqrt{2\pi t}\, p_\nu(t; a, a) = \frac{1}{\sqrt{Z}}.$$

Proof. By the ergodicity (1.1), for each $x \geq 0$ we have

$$m_{\nu,a,\pm}^{(\eta)}(x) := \frac{1}{\eta Z} m_{\nu,a,\pm}(\eta x) \to x \text{ as } \eta \to \infty.$$

By change of variables, it is easy to see that for constants $\eta, \xi > 0$ one has the following relation of the Krein's correspondence:

$$\frac{\xi}{\eta} m\left(\frac{x}{\eta}\right) \leftrightarrow \xi h(\eta \lambda).$$

Since Krein's correspondence is a homeomorphism when $m_{\nu,a,\pm}^{(\eta)}(x) \to x$ as $\eta \to \infty$ for each $x \geq 0$, the corresponding generalized linear diffusion process converges in law to one-dimensional Brownian motion. This implies that

$$\lim_{\eta \to \infty} \eta Z p_\nu(t\eta^2 Z; a, a) = p_{\mathrm{BM}}(t; a, a) = \frac{1}{\sqrt{2\pi t}}.$$

In other words,

$$\lim_{t \to \infty} \sqrt{2\pi t}\, p_\nu(t; a, a) = \frac{1}{\sqrt{Z}}.$$

∎

In the third application we shall study the first hitting/exit time of LBM. For $a \in \mathbb{R}$, recall that $H_a = \inf\{t > 0 : w(t) = a\}$ is the first hitting time at a. The following statement holds.

Theorem 5.3. *For $a \neq 0$ we have*

$$\lim_{t \to \infty} \sqrt{2\pi t}\, \mathbf{P}_\nu^0(H_a \geq t) = |a| Z.$$

Proof. Since

$$\lim_{\eta \to \infty} \frac{\nu[0, \eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[-\eta, 0]}{\eta} = Z,$$

the result is a direct application of [27, Theorem 4] with $\alpha = 1/2$ and $K(x) = Z$. ∎

Remark 5.4. Theorem 5.2 and 5.3 suggest that in long term one-dimensional LBM behaves exactly like one-dimensional Brownian motion.

For $a < 0 < b$, let

$$H_{a,b} = \inf\{t > 0 : w(t) \notin (a, b)\} = H_a \wedge H_b$$

denote the first exit time from (a, b). Define

$$C_{\nu,a,b} = \frac{1}{b-a} \int_a^b (b-x)(x-a)\,\nu(\mathrm{d}x).$$

Then we have the following statement.

Theorem 5.4. *If $\lambda < 1/C_{\nu,a,b}$, then $\mathbf{E}_\nu^0(e^{\lambda H_{a,b}}) < \infty$.*

Proof. This can be easily deduced by using the Kac's moment formula: for $n \geq 1$,

$$\mathbf{E}_\nu^0(H_{a,b}^n) = n \int_a^b \frac{(b - x \vee 0)(x \wedge 0 - a)}{b-a} \mathbf{E}_\nu^x(H_{a,b}^{n-1})\,\nu(\mathrm{d}x) = n!\,G_\nu^n 1(0),$$

where G_ν is the Green operator

$$G_\nu f(x) = \int_a^b \frac{(b - y \vee x)(y \wedge x - a)}{b-a} f(y)\,\nu(\mathrm{d}y).$$

See [20, Lemma 1.3], for example. ∎

Theorem 5.4 indicates that $A = -(\mathrm{d}/\mathrm{d}\nu)(\mathrm{d}/\mathrm{d}x)$, as a self-adjoint, non-negative definite operator on the Hilbert space $L^2((a, b), \nu)$, has a spectra gap. Indeed, let $\lambda_{\nu,a,b}$ denote the smallest eigenvalue of A on $L^2((a, b), \nu)$, and denote by

$$\widetilde{C}_{\nu,a,b} = \sup_{x \in (a,0]} (x-a)\nu((x, 0]) \vee \sup_{x \in [0,b)} (b-x)\nu([0, x)).$$

Then we have the following result of Katoni [19, Theorem 3, Appendix I] as an extension of the theorem of Kac and Krein [17].

Theorem 5.5. *For $a < 0 < b$,*

$$\widetilde{C}_{\nu,a,b} \leq \lambda_{\nu,a,b}^{-1} \leq 4\widetilde{C}_{\nu,a,b}.$$

In the last application we study the asymptotic behaviours of the lifetime of LBE. Recall that $n_{v,a,\pm}(t)$ is the density of the inverse local time $\ell_{v,a,\pm}$, which is also the density of the lifetime ζ under the excursion measure $\mathbf{n}_{v,a,\pm}$, that is,

$$\mathbf{n}_{v,a,\pm}(\zeta \in dt) = n_{v,a,\pm}(t)\,dt.$$

First we present the asymptotic behaviour of $\mathbf{n}_{v,a,\pm}(\zeta > t)$ as $t \to \infty$.

Theorem 5.6. *For $a \in \mathbb{R}$, we have*

$$\lim_{t \to \infty} 2\sqrt{2\pi t^3} n_{v,a,\pm}(t) = \frac{1}{\sqrt{Z}}.$$

Consequently,

$$\lim_{t \to \infty} \sqrt{2\pi t}\, \mathbf{n}_{v,a,\pm}(\zeta > t) = \frac{1}{\sqrt{Z}}.$$

Proof. Similar to the proof of Theorem 5.2, for each $x \geq 0$,

$$m_{v,a,\pm}^{(\eta)}(x) := \frac{1}{\eta Z} m_{v,a,\pm}(\eta x) \to x \text{ as } \eta \to \infty.$$

This implies that the generalized diffusion process on $\mathbb{R}_+$ with natural scale function and speed measure $dm_{v,a,\pm}^{(\eta)}(x)$, and with 0 as an instantaneously reflecting boundary converges in law to the one-dimensional reflected Brownian motion as $\eta \to \infty$. Therefore, the corresponding local time

$$\ell_{v,a,\pm}^{(\eta)}(t) := \frac{1}{\eta^2 Z} \ell_{v,a,\pm}(t\eta)$$

converges in law to the $(1/2)$-stable Lévy subordinator as $\eta \to \infty$. Since

$$\mathbf{E}_{v,a,\pm}^0 \left(e^{-\lambda \frac{1}{\eta^2 Z} \ell_{v,a,\pm}(t\eta)} \right) = \exp\left(-t\eta \int_0^\infty (1 - e^{-s\lambda \frac{1}{\eta^2 Z}}) n_{v,a,\pm}(s)\,ds \right)$$

$$= \exp\left(-t \int_0^\infty (1 - e^{-u\lambda}) \eta^3 Z n_{v,a,\pm}(u\eta^2 Z)\,du \right),$$

we get that

$$\eta^3 Z \cdot n_{v,a,\pm}(u\eta^2 Z) \to \frac{1}{2\sqrt{2\pi u^3}} \text{ as } \eta \to \infty.$$

In other words,

$$\lim_{t \to \infty} 2\sqrt{2\pi t^3} \cdot n_{v,a,\pm}(t) = \frac{1}{\sqrt{Z}}.$$

Consequently,

$$\lim_{t \to \infty} \sqrt{2\pi t}\, \mathbf{n}_{v,a,\pm}(\zeta > t) = \frac{1}{\sqrt{Z}}. \qquad \blacksquare$$

Now we present the asymptotic behaviour of $\mathbf{n}_{v,a,\pm}(\zeta > t)$ as $t \to 0$.

Theorem 5.7. *For v_q-almost every $a \in \mathbb{R}$, for any $\beta > 1/(1 + \alpha(q))$,*

$$\int_{0+} t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt < \infty,$$

and for any $\beta < 1/(1 + \alpha(q))$,

$$\int_{0+} t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt = \infty.$$

In particular,

$$\liminf_{t \to 0+} \frac{\log \mathbf{n}_{v,a,\pm}(\zeta > t)}{-\log t} \le \frac{1}{1 + \alpha(q)} \le \limsup_{t \to 0+} \frac{\log \mathbf{n}_{v,a,\pm}(\zeta > t)}{-\log t}. \tag{5.5}$$

Proof. We have for $\lambda > 0$,

$$\frac{1}{\lambda h_{v,a,\pm}(\lambda)} = \int_0^\infty e^{-\lambda t} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt.$$

Therefore, for any $\delta > 0$ and $\beta > 0$

$$\int_\delta^\infty \frac{1}{\lambda^{\beta+1} h_{v,a,\pm}(\lambda)} \, d\lambda = \int_\delta^\infty \lambda^{-\beta} \int_0^\infty e^{-\lambda t} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt \, d\lambda$$

$$= \int_{\delta t}^\infty \lambda^{-\beta} e^{-\lambda} \, d\lambda \int_0^\infty t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt.$$

This implies that

$$\int^{\infty-} \frac{1}{\lambda^{\beta+1} h_{v,a,\pm}(\lambda)} \, d\lambda < \infty \iff \int_{0+} t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt < \infty.$$

By (5.3), this yields that for any $\beta > 1/(1 + \alpha(q))$,

$$\int_{0+} t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt < \infty,$$

and for any $\beta < 1/(1 + \alpha(q))$,

$$\int_{0+} t^{\beta-1} \mathbf{n}_{v,a,\pm}(\zeta > t) \, dt = \infty,$$

which implies (5.5). ∎

References

[1] S. Andres and N. Kajino, Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probab. Theory Related Fields* **166** (2016), no. 3-4, 713–752 Zbl 1353.60073 MR 3568038

[2] E. Bacry and J. F. Muzy, Log-infinitely divisible multifractal processes. *Comm. Math. Phys.* **236** (2003), no. 3, 449–475 Zbl 1032.60046 MR 2021198

[3] M. T. Barlow and T. Kumagai, Transition density asymptotics for some diffusion processes with multi-fractal structures. *Electron. J. Probab.* **6** (2001), article no. 9 Zbl 0974.60061 MR 1831804

[4] J. Barral and B. B. Mandelbrot, Multifractal products of cylindrical pulses. *Probab. Theory Related Fields* **124** (2002), no. 3, 409–430 Zbl 1014.60042 MR 1939653

[5] J. Barral and B. B. Mandelbrot, Non-degeneracy, moments, dimension, and multifractal analysis for random multiplicative measures (Random multiplicative multifractal measures. II). In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, pp. 17–52, Proc. Sympos. Pure Math. 72, Amer. Math. Soc., Providence, RI, 2004 Zbl 1117.28007 MR 2112120

[6] N. Berestycki, Diffusion in planar Liouville quantum gravity. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** (2015), no. 3, 947–964 Zbl 1325.60125 MR 3365969

[7] N. Berestycki, Introduction to the Gaussian free field and Liouville quantum gravity. Available on the author's website, 2016

[8] N. Berestycki, An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.* **22** (2017), article no. 27 Zbl 1365.60035 MR 3652040

[9] J. Bertoin, Subordinators: examples and applications. In *Lectures on probability theory and statistics (Saint-Flour, 1997)*, pp. 1–91, Lecture Notes in Math. 1717, Springer, Berlin, 1999 Zbl 0955.60046 MR 1746300

[10] B. Duplantier and S. Sheffield, Liouville quantum gravity and KPZ. *Invent. Math.* **185** (2011), no. 2, 333–393 Zbl 1226.81241 MR 2819163

[11] H. Dym and H. P. McKean, *Gaussian processes, function theory, and the inverse spectral problem*. Probability and Mathematical Statistics, Vol. 31, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976 Zbl 0327.60029 MR 448523

[12] P. Fitzsimmons, J. Pitman, and M. Yor, Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, pp. 101–134, Progr. Probab. 33, Birkhäuser Boston, Boston, MA, 1993 Zbl 0844.60054 MR 1278079

[13] P. J. Fitzsimmons and K. Yano, Time change approach to generalized excursion measures, and its application to limit theorems. *J. Theoret. Probab.* **21** (2008), no. 1, 246–265 Zbl 1148.60054 MR 2384481

[14] C. Garban, R. Rhodes, and V. Vargas, On the heat kernel and the Dirichlet form of Liouville Brownian motion. *Electron. J. Probab.* **19** (2014), article no. 96 Zbl 1334.60175 MR 3272329

[15] C. Garban, R. Rhodes, and V. Vargas, Liouville Brownian motion. *Ann. Probab.* **44** (2016), no. 4, 3076–3110 Zbl 1393.60015 MR 3531686

[16] H. Jackson, Liouville Brownian motion and thick points of the Gaussian free field. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (2018), no. 1, 249–279 Zbl 1396.60089 MR 3765889

[17] I. S. Kac and M. G. Kreĭn, Criteria for the discreteness of the spectrum of a singular string. *Izv. Vysš. Učebn. Zaved. Matematika* **1958** (1958), no. 2(3), 136–153 Zbl 1469.34111 MR 139804

[18] J.-P. Kahane, Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** (1985), no. 2, 105–150 Zbl 0596.60041 MR 829798

[19] S. Kotani and S. Watanabe, Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, pp. 235–259, Lecture Notes in Math. 923, Springer, Berlin-New York, 1982 Zbl 0496.60080 MR 661628

[20] O. Loukianov, D. Loukianova, and S. Song, Spectral gaps and exponential integrability of hitting times for linear diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011), no. 3, 679–698 Zbl 1233.60044 MR 2841071

[21] B. B. Mandelbrot, A possible refinement of the lognormal hypothesis concerning the distribution of energy in intermittent turbulence. In *Statistical Models and Turbulence*, pp. 333-351, Lecture Notes in Physisc 12, Springer, Berlin, Heidelberg, 1972 Zbl 0227.76079

[22] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*. Third edn., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 293, Springer, Berlin, 1999 Zbl 0917.60006 MR 1725357

[23] R. Rhodes and V. Vargas, Gaussian multiplicative chaos and applications: a review. *Probab. Surv.* **11** (2014), 315–392 Zbl 1316.60073 MR 3274356

[24] R. Rhodes and V. Vargas, Spectral dimension of Liouville quantum gravity. *Ann. Henri Poincaré* **15** (2014), no. 12, 2281–2298 Zbl 1305.83036 MR 3272822

[25] P. Salminen, P. Vallois, and M. Yor, On the excursion theory for linear diffusions. *Jpn. J. Math.* **2** (2007), no. 1, 97–127 Zbl 1160.60024 MR 2295612

[26] M. Tomisaki, On the asymptotic behaviors of transition probability densities of one-dimensional diffusion processes. *Publ. Res. Inst. Math. Sci.* **12** (1976/77), no. 3, 819–834 Zbl 0368.60089 MR 448589

[27] M. Yamazato, Hitting time distributions of single points for 1-dimensional generalized diffusion processes. *Nagoya Math. J.* **119** (1990), 143–172 Zbl 0745.60083 MR 1071905

[28] K. Yano, Excursion measure away from an exit boundary of one-dimensional diffusion processes. *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 3, 837–878 Zbl 1113.60084 MR 2266999

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