

Limits and colimits in the category of Banach L^0 -modules

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ABSTRACT – We prove that the category of Banach L^0 -modules over a given σ -finite measure space is both complete and cocomplete, which means that it admits all small limits and colimits.

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1. Introduction

In the well-established (but, nonetheless, still fast-growing) research field of analysis on metric measure spaces, a significant role is played by the theory of Banach L^0 -modules, which (in this context) was introduced by Gigli in his seminal work [9]. Therein, Banach L^0 -modules are used to supply an abstract notion of a “space of measurable tensor fields”. In this regard, an enlightening example is the so-called *cotangent module*, which we are going to describe. An effective Sobolev theory on metric measure spaces is available [1, 4, 15], so one can consider the “formal differentials” of Sobolev functions also in this nonsmooth framework. However, it is clear that in order to obtain an arbitrary 1-form this is not sufficient, not even on differentiable manifolds: one should also have the freedom to multiply differentials by functions, to sum the outcomes, and to take their limits. This corresponds to the fact that on a differentiable manifold M , the smooth 1-forms can be obtained as limits of $C^\infty(M)$ -linear combinations of differentials of smooth functions. The line of thought described above led to the following axiomatisation in [8]: a Banach L^0 -module is an algebraic

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module over the ring of L^0 -functions (i.e. of measurable functions, quotiented up to a.e. equality) endowed with a *pointwise norm* that induces a complete distance; see Definitions 2.2 and 2.3 for the details. As is evident from the literature on the topic, in order to achieve a deeper understanding of the structure of metric measure spaces, it is of pivotal importance to put the functional-analytic aspects of the tensor calculus via Banach L^0 -modules on firm ground. As an example of this fact, we recall that the finite-dimensionality (or, more generally, the reflexivity) of the cotangent module entails the density of Lipschitz functions in the Sobolev space. It is also worth pointing out that the interest in Banach L^0 -modules goes far beyond the analysis on metric measure spaces. Indeed, the essentially equivalent concept of a *randomly normed space* was previously introduced in [14], as a tool for studying ultrapowers of Lebesgue–Bochner spaces over a rather general class of measure spaces. Later on, the slightly different notion of a *random normed module* was investigated (see [13] and the references therein): the motivation comes from the theory of probabilistic metric spaces, and it has applications in finance optimisation problems, with connections to the study of conditional and dynamic risk measures. Due to the above reasons, in this paper we shall consider Banach L^0 -modules over an arbitrary σ -finite measure space.

The aim of the present paper is to study the category $\mathbf{BanMod}_{\mathbb{X}}$ of Banach $L^0(\mathbb{X})$ -modules, where $\mathbb{X} = (X, \Sigma, \mathfrak{m})$ is a given σ -finite measure space. The morphisms in $\mathbf{BanMod}_{\mathbb{X}}$ are those $L^0(\mathbb{X})$ -linear operators $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ that satisfy $|\varphi(v)| \leq |v|$ for every $v \in \mathcal{M}$. Our main result (namely, Theorem 3.14) states that $\mathbf{BanMod}_{\mathbb{X}}$ is both a complete category (i.e. all limits exist) and a cocomplete category (i.e. all colimits exist). This means, in particular, that $\mathbf{BanMod}_{\mathbb{X}}$ admits all equalisers, products, inverse limits, and pullbacks, as well as all coequalisers, coproducts, direct limits, and pushouts. The existence of inverse and direct limits in $\mathbf{BanMod}_{\mathbb{X}}$ was already known:

- (1) Inverse limits in $\mathbf{BanMod}_{\mathbb{X}}$, whose existence was proved in [12], were necessary to build the differential of a locally Sobolev map from a metric measure space to a metric space.
- (2) Direct limits in $\mathbf{BanMod}_{\mathbb{X}}$, whose existence was proved in the unpublished note [23], were used in [21] to obtain a “representation theorem” for separable Banach L^0 -modules. Since each separable Banach L^0 -module is the direct limit of a sequence of finitely generated Banach L^0 -modules, a representation of an arbitrary separable Banach L^0 -module as the space of L^0 -sections of a separable *measurable Banach bundle* could be deduced from the corresponding result for finitely generated modules, which was previously obtained in [19].

It is worth mentioning that the theory of Banach L^0 -modules extends the one of Banach spaces, as the latter correspond to Banach L^0 -modules over a measure space whose measure is a Dirac delta. In fact, the strategy of our proof of the (co)completeness of

$\mathbf{BanMod}_{\mathbb{X}}$ is inspired by that of the category $\mathbf{Ban}$ of Banach spaces, for which we refer to [3, 25, 26]. However, some other aspects of the Banach L^0 -module theory, such as the *inverse image functor* (see Section 3.3), are characteristic of Banach L^0 -modules and do not have a (nontrivial) counterpart in the Banach space setting.

We conclude by pointing out that the contents of this paper slightly overlap with those of the unpublished note [23]. More specifically, Examples 3.4, 3.21, 3.22, 3.23, and 4.3 are essentially taken from [23], but besides them the two papers are in fact independent. Indeed, the (co)completeness of $\mathbf{BanMod}_{\mathbb{X}}$ is proved *without* using the existence of inverse/direct limits.

2. Preliminaries

Throughout the paper, we denote by $\mathcal{P}_F(I)$ the family of all finite subsets of a given set $I \neq \emptyset$. To avoid pathological situations, all the measure spaces we consider are assumed to be non-null. We denote by $\mathbb{P} = (P, \Sigma_P, \delta_p)$ the probability space made of a unique point p , where $\Sigma_P = \{\emptyset, P\}$ is the only σ -algebra on P and δ_p is the Dirac measure at p , i.e. $\delta_p(\emptyset) = 0$ and $\delta_p(P) = 1$. Given two σ -finite measure spaces $\mathbb{X} = (X, \Sigma_X, \mathfrak{m}_X)$ and $\mathbb{Y} = (Y, \Sigma_Y, \mathfrak{m}_Y)$, we define their product $\mathbb{X} \times \mathbb{Y}$ as the σ -finite measure space $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mathfrak{m}_X \otimes \mathfrak{m}_Y)$, where $\Sigma_X \otimes \Sigma_Y$ and $\mathfrak{m}_X \otimes \mathfrak{m}_Y$ stand for the product σ -algebra and the product measure, respectively. We tacitly identify $\mathbb{X} \times \mathbb{P}$ with $\mathbb{X}$.

2.1 – Normed and Banach $L^0(\mathbb{X})$ -modules

In this section, we present the theory of Banach $L^0(\mathbb{X})$ -modules, which was first introduced in [9] and then refined further in [8]. See also [11].

2.1.1. The space $L^0(\mathbb{X})$. Let $\mathbb{X} = (X, \Sigma, \mathfrak{m})$ be a σ -finite measure space. We denote by $L_{\text{ext}}^0(\mathbb{X})$ the space of measurable functions from (X, Σ) to the extended real line $[-\infty, +\infty]$, quotiented up to $\mathfrak{m}$ -a.e. equality. The space $L_{\text{ext}}^0(\mathbb{X})$ is a lattice if endowed with the following partial order relation: given any $f, g \in L_{\text{ext}}^0(\mathbb{X})$, we declare that $f \leq g$ if and only if $f(x) \leq g(x)$ holds for $\mathfrak{m}$ -a.e. $x \in X$.

REMARK 2.1. Recall that a lattice $(A, \leq)$ is said to be *Dedekind complete* (see e.g. [6, 7]) if every nonempty subset of A having an upper bound admits a least upper bound, or equivalently every nonempty subset of A having a lower bound admits a greatest lower bound. It is well known that the lattice $(L_{\text{ext}}^0(\mathbb{X}), \leq)$ is order bounded, is Dedekind complete, and satisfies the following property: given any (possibly uncountable) nonempty subset $\{f_i\}_{i \in I}$ of $L_{\text{ext}}^0(\mathbb{X})$, there exist two (at most) countable subsets C, C' of I such that $\bigvee_{i \in I} f_i = \bigvee_{i \in C} f_i$ and $\bigwedge_{i \in I} f_i = \bigwedge_{i \in C'} f_i$.

We define the Riesz space $L^0(\mathbb{X})$ as

$$L^0(\mathbb{X}) := \{f \in L^0_{\text{ext}}(\mathbb{X}) \mid -\infty < f(x) < +\infty \text{ for } \mathfrak{m}\text{-a.e. } x \in \mathbb{X}\}.$$

Then $L^0(\mathbb{X})$ is a commutative algebra with respect to the usual pointwise operations, as well as a σ -sublattice of $L^0_{\text{ext}}(\mathbb{X})$, i.e. a sublattice of $L^0_{\text{ext}}(\mathbb{X})$ that is closed under countable suprema and infima. In particular, Remark 2.1 ensures that $(L^0(\mathbb{X}), \leq)$ is Dedekind complete and that each supremum (resp. infimum) in $L^0(\mathbb{X})$ can be expressed as a countable supremum (resp. infimum). Moreover, the space $L^0(\mathbb{X})$ is a topological algebra if endowed with the following complete distance:

$$\mathbf{d}_{L^0(\mathbb{X})}(f, g) := \int |f - g| \wedge 1 \, d\tilde{\mathfrak{m}} \quad \text{for every } f, g \in L^0(\mathbb{X}),$$

where $\tilde{\mathfrak{m}}$ is any finite measure on $(\mathbb{X}, \Sigma)$ satisfying $\mathfrak{m} \ll \tilde{\mathfrak{m}} \ll \mathfrak{m}$. While the distance $\mathbf{d}_{L^0(\mathbb{X})}$ depends on the specific choice of $\tilde{\mathfrak{m}}$, its induced topology does not. Moreover, a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^0(\mathbb{X})$ satisfies $\mathbf{d}_{L^0(\mathbb{X})}(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$ for some limit function $f \in L^0(\mathbb{X})$ if and only if we can extract a subsequence $(n_i)_{i \in \mathbb{N}}$ such that $f_{n_i}(x) \rightarrow f(x)$ as $i \rightarrow \infty$ for $\mathfrak{m}$ -a.e. $x \in \mathbb{X}$.

2.1.2. Definition and main properties of Banach $L^0(\mathbb{X})$ -modules. We begin with the relevant definitions:

DEFINITION 2.2 (Normed $L^0(\mathbb{X})$ -module). Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}$ be a module over $L^0(\mathbb{X})$. Then we say that $\mathcal{M}$ is a *seminormed $L^0(\mathbb{X})$ -module* if it is endowed with a *pointwise seminorm*, i.e. with a mapping $|\cdot|: \mathcal{M} \rightarrow L^0(\mathbb{X})$ that satisfies the following properties:

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathcal{M}, \\ |v + w| &\leq |v| + |w| && \text{for every } v, w \in \mathcal{M}, \\ |f \cdot v| &= |f| |v| && \text{for every } f \in L^0(\mathbb{X}) \text{ and } v \in \mathcal{M}. \end{aligned}$$

Moreover, we say that $\mathcal{M}$ is a *normed $L^0(\mathbb{X})$ -module* provided it holds that $|v| = 0$ if and only if $v = 0$.

Any pointwise seminorm on $\mathcal{M}$ induces a pseudometric $\mathbf{d}_{\mathcal{M}}$ on $\mathcal{M}$:

$$\mathbf{d}_{\mathcal{M}}(v, w) := \mathbf{d}_{L^0(\mathbb{X})}(|v - w|, 0) \quad \text{for every } v, w \in \mathcal{M}.$$

It holds that $\mathcal{M}$ is a normed $L^0(\mathbb{X})$ -module if and only if $\mathbf{d}_{\mathcal{M}}$ is a distance.

DEFINITION 2.3 (Banach $L^0(\mathbb{X})$ -module). Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}$ be a normed $L^0(\mathbb{X})$ -module. Then we say that $\mathcal{M}$ is a *Banach $L^0(\mathbb{X})$ -module* if the pointwise norm $|\cdot|$ is complete, meaning that the induced distance $\mathbf{d}_{\mathcal{M}}$ is complete.

The space $L^0(\mathbb{X})$ itself is a Banach $L^0(\mathbb{X})$ -module. Moreover, if the measure underlying $\mathbb{X}$ is a Dirac measure, then $L^0(\mathbb{X}) \cong \mathbb{R}$, and so the Banach $L^0(\mathbb{X})$ -modules are exactly the Banach spaces.

REMARK 2.4. A warning about the terminology: in this paper we distinguish between normed $L^0(\mathbb{X})$ -modules and Banach $L^0(\mathbb{X})$ -modules, while in the original papers [8, 9] only complete normed $L^0(\mathbb{X})$ -modules were considered (but they were called just “normed L^0 -modules”).

REMARK 2.5. Let us spend a few words on why this axiomatisation of Banach $L^0(\mathbb{X})$ -modules is useful in analysis on metric measure spaces. Differently from e.g. the theory of (Banach) modules over a Banach algebra (see [5]), we consider a *pointwise* norm operator that takes values into the space of functions $L^0(\mathbb{X})^+$, and not its “integrated version” taking values in $\mathbb{R}^+$. Heuristically, a Banach $L^0(\mathbb{X})$ -module $\mathcal{M}$ in the sense of Definition 2.3 can be thought of as a *measurable Banach bundle*, i.e. a collection of Banach spaces $\{\mathcal{M}_x\}_{x \in \mathbb{X}}$ that “vary in a measurable way”. In fact, this is really the case under suitable separability assumptions on $\mathcal{M}$ (see [21, Section 4.3]) and, in a weaker form, this is still true for arbitrary Banach $L^0(\mathbb{X})$ -modules (see [21, Section 3.3]). The reason why it is convenient to keep track of the “fibrewise behaviour” of the elements of $\mathcal{M}$ is that Banach $L^0(\mathbb{X})$ -modules are used to provide a generalised notion of a measurable 1-form, which has to be defined pointwise on the given metric measure space. The pointwise/fibrewise description of a Banach $L^0(\mathbb{X})$ -module turned out to be useful in order to characterise the dual and/of the pullback of a Banach $L^0(\mathbb{X})$ -module, which are tools of pivotal importance in vector calculus on metric measure spaces [10].

Given a Banach $L^0(\mathbb{X})$ -module $\mathcal{M}$, a partition $(E_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $\mathbb{X}$, and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$, the series $\sum_{n \in \mathbb{N}} \chi_{E_n} \cdot v_n$ converges unconditionally in $\mathcal{M}$ (by the dominated convergence theorem).

REMARK 2.6. Let $\mathbb{X} = (\mathbb{X}, \Sigma, \mathfrak{m})$ be a σ -finite measure space and $\mathcal{M}$ a Banach $L^0(\mathbb{X})$ -module. Then it follows from the properties of $\mathbf{d}_{L^0(\mathbb{X})}$ that for any partition $(E_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $\mathbb{X}$ it holds that

$$\mathcal{M}^{\mathbb{N}} \ni (v_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \chi_{E_n} \cdot v_n \in \mathcal{M} \quad \text{is a continuous map,}$$

where the source space is endowed with the product topology.

2.1.3. Examples of Banach $L^0(\mathbb{X})$ -modules. There are many ways to obtain a Banach $L^0(\mathbb{X})$ -module:

- (1) *Submodule*. By a *normed $L^0(\mathbb{X})$ -submodule* of a normed $L^0(\mathbb{X})$ -module $\mathcal{M}$ we mean an $L^0(\mathbb{X})$ -submodule $\mathcal{N}$ of $\mathcal{M}$ equipped with the restriction of the pointwise

norm of $\mathcal{M}$, which is easily seen to be a normed $L^0(\mathbb{X})$ -module. If in addition $\mathcal{M}$ is a Banach $L^0(\mathbb{X})$ -module and $\mathcal{N}$ is closed in $\mathcal{M}$, then $\mathcal{N}$ is also a Banach $L^0(\mathbb{X})$ -module. In this case, we say that $\mathcal{N}$ is a *Banach $L^0(\mathbb{X})$ -submodule* of $\mathcal{M}$.

- (2) *Closure.* Let $\mathcal{M}$ be a Banach $L^0(\mathbb{X})$ -module and $\mathcal{N}$ a normed $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$. Then the closure $\text{cl}_{\mathcal{M}}(\mathcal{N})$ of $\mathcal{N}$ is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$.
- (3) *Null space.* Let $\mathcal{M}, \mathcal{N}$ be Banach $L^0(\mathbb{X})$ -modules and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ an $L^0(\mathbb{X})$ -linear and continuous map. Then the space $\varphi^{-1}(\{0\}) := \{v \in \mathcal{M} : \varphi(v) = 0\}$ is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$.
- (4) *Range.* Let $\mathcal{M}, \mathcal{N}$ be Banach $L^0(\mathbb{X})$ -modules and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ an $L^0(\mathbb{X})$ -linear and continuous map. Then the space $\varphi(\mathcal{M}) := \{\varphi(v) : v \in \mathcal{M}\}$ is a normed $L^0(\mathbb{X})$ -submodule of $\mathcal{N}$. We point out that, in general, the space $\varphi(\mathcal{M})$ is not complete (see Example 3.4).
- (5) *Metric identification.* Let $\mathcal{M}$ be a seminormed $L^0(\mathbb{X})$ -module. The equivalence relation $\sim_{|\cdot|}$ on $\mathcal{M}$ is defined as follows: given any $v, w \in \mathcal{M}$, we declare that $v \sim_{|\cdot|} w$ if and only if $|v - w| = 0$. Then the quotient $\mathcal{M}/\sim_{|\cdot|}$ inherits a structure of normed $L^0(\mathbb{X})$ -module.
- (6) *Completion.* Let $\mathcal{M}$ be a normed $L^0(\mathbb{X})$ -module. Then there exists a unique couple $(\bar{\mathcal{M}}, \iota)$, where $\bar{\mathcal{M}}$ is a Banach $L^0(\mathbb{X})$ -module, while $\iota: \mathcal{M} \rightarrow \bar{\mathcal{M}}$ is an $L^0(\mathbb{X})$ -linear map that preserves the pointwise norm and satisfies $\text{cl}_{\bar{\mathcal{M}}}(\iota(\mathcal{M})) = \bar{\mathcal{M}}$. Uniqueness is up to a unique isomorphism: given any $(\bar{\mathcal{N}}, \tilde{\iota})$ with the same properties, there exists a unique $L^0(\mathbb{X})$ -linear bijection $\Phi: \bar{\mathcal{M}} \rightarrow \bar{\mathcal{N}}$ that preserves the pointwise norm (i.e. an *isomorphism* of Banach $L^0(\mathbb{X})$ -modules) and satisfies $\Phi \circ \iota = \tilde{\iota}$.
- (7) *Quotient.* Let $\mathcal{M}$ be a Banach $L^0(\mathbb{X})$ -module and $\mathcal{N}$ a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$. Then the quotient $\mathcal{M}/\mathcal{N}$ is a Banach $L^0(\mathbb{X})$ -module if endowed with the pointwise norm

$$|v + \mathcal{N}| := \bigwedge_{w \in \mathcal{N}} |v + w| \quad \text{for every } v \in \mathcal{M},$$

which we call the *quotient pointwise norm*. Whenever we refer to $\mathcal{M}/\mathcal{N}$ as the *quotient Banach $L^0(\mathbb{X})$ -module*, we always mean $\mathcal{M}/\mathcal{N}$ endowed with the quotient pointwise norm.

- (8) *Space of homomorphisms.* Let $\mathcal{M}, \mathcal{N}$ be Banach $L^0(\mathbb{X})$ -modules. Then we denote by $\text{Hom}(\mathcal{M}, \mathcal{N})$ the space of $L^0(\mathbb{X})$ -linear maps $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ for which there exists $g \in L^0(\mathbb{X})^+$ such that $|\varphi(v)| \leq g|v|$ holds for every $v \in \mathcal{M}$. If endowed with the pointwise norm

$$|\varphi| := \bigwedge \{g \in L^0(\mathbb{X})^+ \mid |\varphi(v)| \leq g|v| \text{ for every } v \in \mathcal{M}\}$$

and the usual pointwise operations, the space $\text{Hom}(\mathcal{M}, \mathcal{N})$ is a Banach $L^0(\mathbb{X})$ -module. Its elements are called the *homomorphisms* of Banach $L^0(\mathbb{X})$ -modules from $\mathcal{M}$ to $\mathcal{N}$.

(9) *Dual.* The dual of a Banach $L^0(\mathbb{X})$ -module $\mathcal{M}$ is given by $\mathcal{M}^* := \text{Hom}(\mathcal{M}, L^0(\mathbb{X}))$.

(10) *Hilbert modules.* Let $S \neq \emptyset$ be an arbitrary set. We define the space $\mathcal{H}_{\mathbb{X}}(S)$ as

$$\mathcal{H}_{\mathbb{X}}(S) := \{v \in L^0(\mathbb{X})^S \mid |v| := (\sum_{s \in S} |v(s)|^2)^{1/2} \in L^0(\mathbb{X})\}.$$

In particular, given $v \in \mathcal{H}_{\mathbb{X}}(S)$ we have $v(s) = 0$ for all but countably many $s \in S$. Then $(\mathcal{H}_{\mathbb{X}}(S), |\cdot|)$ is a Banach $L^0(\mathbb{X})$ -module with respect to the componentwise operations. Also, $\mathcal{H}_{\mathbb{X}}(S)$ is a *Hilbert $L^0(\mathbb{X})$ -module*, i.e. it verifies the pointwise parallelogram rule:

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2 \quad \text{for every } v, w \in \mathcal{H}_{\mathbb{X}}(S).$$

The elements $\{e_s\}_{s \in S} \subseteq \mathcal{H}_{\mathbb{X}}(S)$, defined as $e_s(t) := 0$ for every $t \in S \setminus \{s\}$ and $e_s(s) := \chi_X$, form an orthonormal basis of $\mathcal{H}_{\mathbb{X}}(S)$. Hence, given any $S_1, S_2 \neq \emptyset$, it holds that $\mathcal{H}_{\mathbb{X}}(S_1)$ and $\mathcal{H}_{\mathbb{X}}(S_2)$ are isomorphic as Banach $L^0(\mathbb{X})$ -modules if and only if $\text{card}(S_1) = \text{card}(S_2)$.

(11) *L^0 -Lebesgue–Bochner space.* Let $\mathcal{M}$ be a Banach $L^0(\mathbb{Y})$ -module, for some σ -finite measure space $\mathbb{Y}$. Then we denote by $L^0(\mathbb{X}; \mathcal{M})$ the space of all measurable maps $v: \mathbb{X} \rightarrow \mathcal{M}$ taking values in a separable subspace of $\mathcal{M}$ (that depends on v), quotiented up to $\mathfrak{m}_X$ -a.e. equality. Then it holds that $L^0(\mathbb{X}; \mathcal{M})$ is a Banach $L^0(\mathbb{X} \times \mathbb{Y})$ -module if equipped with the following operations:

$$\begin{aligned} (v + w)(x) &:= v(x) + w(x) \in \mathcal{M} && \text{for } \mathfrak{m}_X\text{-a.e. } x \in \mathbb{X}, \\ (f \cdot v)(x) &:= f(x, \cdot) \cdot v(x) \in \mathcal{M} && \text{for } \mathfrak{m}_X\text{-a.e. } x \in \mathbb{X}, \\ |v|(x, y) &:= |v(x)|(y) && \text{for } (\mathfrak{m}_X \otimes \mathfrak{m}_Y)\text{-a.e. } (x, y) \in \mathbb{X} \times \mathbb{Y}, \end{aligned}$$

for every $v, w \in L^0(\mathbb{X}; \mathcal{M})$ and $f \in L^0(\mathbb{X} \times \mathbb{Y})$. In particular, for any Banach space B we can regard $L^0(\mathbb{X}; B)$ as a Banach $L^0(\mathbb{X})$ -module. The space of *simple maps* from $\mathbb{X}$ to $\mathcal{M}$, i.e. of those elements of $L^0(\mathbb{X}; \mathcal{M})$ that can be written as $\sum_{i=1}^n \chi_{E_i} v_i$ for some $(E_i)_{i=1}^n \subseteq \Sigma_X$ and $(v_i)_{i=1}^n \subseteq \mathcal{M}$, is dense in $L^0(\mathbb{X}; \mathcal{M})$.

The above claims can be proved by adapting the arguments in [9, Section 1.2].

REMARK 2.7. Let $\mathbb{X}$ be a σ -finite measure space and B a Banach space. We denote by B' the dual of B in the sense of Banach spaces. Then it holds that the map $\iota_{\mathbb{X}, B}: L^0(\mathbb{X}; B') \rightarrow L^0(\mathbb{X}; B)^*$, given by

$$\iota_{\mathbb{X}, B}(\omega)(v) := \omega(\cdot)(v(\cdot)) \in L^0(\mathbb{X}) \quad \text{for every } \omega \in L^0(\mathbb{X}; B') \text{ and } v \in L^0(\mathbb{X}; B),$$

is a morphism of Banach $L^0(\mathbb{X})$ -modules that preserves the pointwise norm. It also holds that

$$\iota_{\mathbb{X},B} \text{ is an isomorphism} \Leftrightarrow B' \text{ has the Radon–Nikodým property};$$

see [16, Theorems 1.3.10 and 1.3.26], [9, Proposition 1.2.13], and [19, Appendix B]. We point out that, even in the case where B' does not have the Radon–Nikodým property, the space $L^0(\mathbb{X}; B)^*$ can be characterised in several ways; see e.g. [10] for (generalisations of) these kinds of results.

2.2 – A reminder on category theory

In this section we recall some important notions and results in category theory, mostly concerning limits and colimits. We refer to [2, 17, 20, 22] for a thorough account of these topics. Let us begin by fixing some useful terminology and notation.

Given a category $\mathbf{C}$, we denote by $\text{Ob}_{\mathbf{C}}$ and $\text{Hom}_{\mathbf{C}}$ the classes of its objects and morphisms, respectively. The domain and the codomain of a morphism $\varphi: X \rightarrow Y$ are denoted by $\text{dom}(\varphi) := X$ and $\text{cod}(\varphi) := Y$, respectively. Given two objects X, Y of $\mathbf{C}$, we denote by $\text{Hom}_{\mathbf{C}}(X, Y)$ the class of those morphisms φ in $\mathbf{C}$ such that $\text{dom}(\varphi) = X$ and $\text{cod}(\varphi) = Y$. We say that $\mathbf{C}$ is *small* if the classes $\text{Ob}_{\mathbf{C}}$ and $\text{Hom}_{\mathbf{C}}$ are sets, while we say that $\mathbf{C}$ is *locally small* if the class $\text{Hom}_{\mathbf{C}}(X, Y)$ is a set for every pair of objects X, Y of $\mathbf{C}$. The *opposite* (or *dual*) *category* $\mathbf{C}^{\text{op}}$ is obtained from $\mathbf{C}$ by “reversing the morphisms” (see e.g. [17, p. 12] for the precise definition of $\mathbf{C}^{\text{op}}$). A category $\mathbf{D}$ is said to be a *subcategory* of $\mathbf{C}$ provided the following conditions hold:

- (1) $\text{Ob}_{\mathbf{D}}$ is a subcollection of $\text{Ob}_{\mathbf{C}}$.
- (2) Given objects X, Y of $\mathbf{D}$, the class $\text{Hom}_{\mathbf{D}}(X, Y)$ is a subcollection of $\text{Hom}_{\mathbf{C}}(X, Y)$.
- (3) The composition in $\mathbf{D}$ is induced by the composition in $\mathbf{C}$.
- (4) The identity morphisms in $\mathbf{D}$ are identity morphisms in $\mathbf{C}$.

We say that $\mathbf{D}$ is a *full subcategory* of $\mathbf{C}$ if $\text{Hom}_{\mathbf{D}}(X, Y)$ coincides with $\text{Hom}_{\mathbf{C}}(X, Y)$ for every pair of objects X, Y of $\mathbf{D}$. Given two locally small categories $\mathbf{C}, \mathbf{D}$, each functor $F: \mathbf{C} \rightarrow \mathbf{D}$ induces a collection of maps $F_{X,Y}: \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(F(X), F(Y))$ for every pair of objects X, Y of $\mathbf{C}$. We say that F is *full* (resp. *faithful*) if $F_{X,Y}$ is surjective (resp. injective) for every pair of objects X, Y of $\mathbf{C}$. We say that F is *injective on objects* if $F(X)$ and $F(Y)$ are different whenever X and Y are different objects of $\mathbf{C}$. We say that $\mathbf{C}$ *has zero morphisms* if there is a collection of morphisms $0_{XY}: X \rightarrow Y$, indexed by the pairs X, Y of objects of $\mathbf{C}$, that satisfies the following property: given any three objects X, Y, Z of $\mathbf{C}$ and any two morphisms $\varphi: Y \rightarrow Z$ and $\psi: X \rightarrow Y$

in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{0_{XY}} & Y \\ \psi \downarrow & \searrow 0_{XZ} & \downarrow \varphi \\ Y & \xrightarrow{0_{YZ}} & Z \end{array}$$

commutes. If $\mathbf{C}$ has zero morphisms, then the zero morphisms 0_{XY} are uniquely determined.

An object I of a category $\mathbf{C}$ is said to be an *initial object* if for any object X of $\mathbf{C}$ there exists exactly one morphism $i_X: I \rightarrow X$. Dually, an object T of $\mathbf{C}$ is said to be a *terminal object* if for any object X of $\mathbf{C}$ there exists exactly one morphism $t_X: X \rightarrow T$. An object that is both initial and terminal is called a *zero object* of $\mathbf{C}$. Initial and terminal objects (thus, a fortiori, zero objects) are uniquely determined up to a unique isomorphism: given any two initial objects I_1 and I_2 of $\mathbf{C}$, there exists a unique isomorphism $I_1 \rightarrow I_2$, and similarly for terminal objects. A *pointed category* is a category $\mathbf{C}$ having a zero object, which we denote by $0_{\mathbf{C}}$. Each pointed category $\mathbf{C}$ has zero morphisms: for any two objects X, Y of $\mathbf{C}$, the zero morphism 0_{XY} is $i_Y \circ t_X$.

A morphism $\varphi: X \rightarrow Y$ in a category $\mathbf{C}$ is said to be a *monomorphism* if it is left cancellative, meaning that $\psi_1 = \psi_2$ whenever Z is an object of $\mathbf{C}$ and $\psi_1, \psi_2: Z \rightarrow X$ are two morphisms in $\mathbf{C}$ satisfying $\varphi \circ \psi_1 = \varphi \circ \psi_2$. Dually, a morphism $\varphi: X \rightarrow Y$ is said to be an *epimorphism* if it is right cancellative, meaning that $\psi_1 = \psi_2$ holds whenever Z is an object of $\mathbf{C}$ and $\psi_1, \psi_2: Y \rightarrow Z$ are two morphisms in $\mathbf{C}$ satisfying $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We say that a category $\mathbf{C}$ is *balanced* if every morphism in $\mathbf{C}$ that is both a monomorphism and an epimorphism is an isomorphism.

Some important examples of categories, which play a key role in this paper, are the following:

- (1) The category **Set**, whose objects are the sets and whose morphisms are the functions.
- (2) The category **Meas** $_{\sigma}$ of σ -finite measure spaces. Given any two σ -finite measure spaces $\mathbb{X} = (X, \Sigma_X, \mathfrak{m}_X)$ and $\mathbb{Y} = (Y, \Sigma_Y, \mathfrak{m}_Y)$, a morphism $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ is a (Σ_X, Σ_Y) -measurable map $\tau: X \rightarrow Y$ that satisfies $\tau_{\#}\mathfrak{m}_X \ll \mathfrak{m}_Y$. This notion of morphism differs from those of other authors, who require e.g. τ to be measure preserving or to verify $\tau_{\#}\mathfrak{m}_X \leq C \mathfrak{m}_Y$ for some constant $C > 0$. Notice also that the measure $\tau_{\#}\mathfrak{m}_X$ on Y need not be σ -finite.
- (3) The category **Ban**, whose objects are the Banach spaces and whose morphisms are the linear 1-Lipschitz operators. We refer e.g. to [3, 25] for a study of the category **Ban**.
- (4) Let $(I, \leq)$ be a *directed set*, i.e. a nonempty partially ordered set where any pair of elements admits an upper bound. Then $(I, \leq)$ can be regarded as a small category,

where the objects are the elements of I , while the morphisms are as follows: given any $i, j \in I$, we declare that there is a (unique) morphism $i \rightarrow j$ if and only if $i \leq j$.

Given two categories $\mathbf{C}, \mathbf{D}$ and two functors $F, G: \mathbf{D} \rightarrow \mathbf{C}$, a *natural transformation* from F to G is a collection η_* of morphisms $\eta_X: F(X) \rightarrow G(X)$, indexed by the objects X of $\mathbf{D}$, such that for any morphism $\varphi: X \rightarrow Y$ in $\mathbf{D}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(\varphi) \downarrow & & \downarrow G(\varphi) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes. We denote by $\mathbf{C}^{\mathbf{D}}$ the *functor category* from $\mathbf{D}$ to $\mathbf{C}$, whose objects are the functors from $\mathbf{D}$ to $\mathbf{C}$ and whose morphisms are the natural transformations between them. An isomorphism in $\mathbf{C}^{\mathbf{D}}$ is called a *natural isomorphism*. Given an index category $\mathbf{J}$, the *diagonal functor* $\Delta_{\mathbf{J}, \mathbf{C}}: \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{J}}$ is defined as follows: given any object X of $\mathbf{C}$, we set $\Delta_{\mathbf{J}, \mathbf{C}}(X)(i) := X$ for every object i of $\mathbf{J}$ and $\Delta_{\mathbf{J}, \mathbf{C}}(X)(\phi) := \text{id}_X$ for every morphism ϕ in $\mathbf{J}$; given any morphism $\varphi: X \rightarrow Y$ in $\mathbf{C}$, we define the natural transformation $\Delta_{\mathbf{J}, \mathbf{C}}(\varphi)_*$ as $\Delta_{\mathbf{J}, \mathbf{C}}(\varphi)_i := \varphi$ for every object i of $\mathbf{J}$.

The *arrow category* $\mathbf{C}^{\rightarrow}$ of a given category $\mathbf{C}$ is defined as follows. The objects of $\mathbf{C}^{\rightarrow}$ are the morphisms in $\mathbf{C}$, while a morphism $\varphi \rightarrow \psi$ in $\mathbf{C}^{\rightarrow}$ is given by a couple (α, β) of morphisms $\alpha: \text{dom}(\varphi) \rightarrow \text{dom}(\psi)$ and $\beta: \text{cod}(\varphi) \rightarrow \text{cod}(\psi)$ in $\mathbf{C}$ for which the following diagram commutes:

$$\begin{array}{ccc} \text{dom}(\varphi) & \xrightarrow{\alpha} & \text{dom}(\psi) \\ \varphi \downarrow & & \downarrow \psi \\ \text{cod}(\varphi) & \xrightarrow{\beta} & \text{cod}(\psi). \end{array}$$

More generally, given three categories $\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2$ and two functors $F: \mathbf{C}_1 \rightarrow \mathbf{C}$ and $G: \mathbf{C}_2 \rightarrow \mathbf{C}$, we define the *comma category* $(F \downarrow G)$ in the following way:

- (1) The objects of $(F \downarrow G)$ are the triples (X, Y, φ) , where X is an object of $\mathbf{C}_1$, Y is an object of $\mathbf{C}_2$, and $\varphi: F(X) \rightarrow G(Y)$ is a morphism in $\mathbf{C}$.
- (2) A morphism $(X_1, Y_1, \varphi_1) \rightarrow (X_2, Y_2, \varphi_2)$ in $(F \downarrow G)$ is given by a couple (ψ_1, ψ_2) , where $\psi_1: X_1 \rightarrow X_2$ is a morphism in $\mathbf{C}_1$ and $\psi_2: Y_1 \rightarrow Y_2$ is a morphism in $\mathbf{C}_2$ such that

$$\begin{array}{ccc} F(X_1) & \xrightarrow{F(\psi_1)} & F(X_2) \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ G(Y_1) & \xrightarrow{G(\psi_2)} & G(Y_2) \end{array}$$

is a commutative diagram.

If $\mathbf{C}_2 = \mathbf{1}$ (i.e. $\mathbf{C}_2$ is the one-object one-morphism category) and X is a given object of $\mathbf{C}$, we just write $(F \downarrow X)$ instead of $(F \downarrow \Delta_{\mathbf{1}, \mathbf{C}}(X))$, and similarly for $(X \downarrow G)$ in the case where $\mathbf{C}_1 = \mathbf{1}$. Observe also that the arrow category $\mathbf{C}^{\rightarrow}$ coincides with the comma category $(\text{id}_{\mathbf{C}} \downarrow \text{id}_{\mathbf{C}})$.

2.2.1. Limits and colimits. Let $\mathbf{J}$ be an index category. Then a *diagram* of type $\mathbf{J}$ in a category $\mathbf{C}$ is a functor $D: \mathbf{J} \rightarrow \mathbf{C}$. A *cone* to D is an object X of $\mathbf{C}$ together with a collection $\varphi_{\star}$ of morphisms $\varphi_i: X \rightarrow D(i)$, indexed by the objects i of $\mathbf{J}$, such that $D(\phi) \circ \varphi_i = \varphi_j$ for every morphism $\phi: i \rightarrow j$ in $\mathbf{J}$. A *limit* of D is a cone $(L, \lambda_{\star})$ to D that satisfies the following universal property: given any cone $(X, \varphi_{\star})$ to D , there exists a unique morphism $\Phi: X \rightarrow L$ such that the diagram

$$\begin{array}{ccc}
 & X & \\
 \varphi_i \swarrow & \downarrow \Phi & \searrow \varphi_j \\
 & L & \\
 \lambda_i \swarrow & & \searrow \lambda_j \\
 D(i) & \xrightarrow{D(\phi)} & D(j)
 \end{array}$$

commutes for every morphism $\phi: i \rightarrow j$ in $\mathbf{J}$. If a limit of D exists, then it is *essentially* unique (i.e. unique up to a unique isomorphism), so that we are entitled to refer to it as “the” limit of D . Alternatively, the cones of D can be identified with the objects of the category $(\Delta_{\mathbf{J}, \mathbf{C}} \downarrow D)$, which is thus called the *category of cones* to D , and a limit of D is a terminal object in $(\Delta_{\mathbf{J}, \mathbf{C}} \downarrow D)$.

Dually, by a *cocone* of D we mean an object X of $\mathbf{C}$, together with a collection $\varphi_{\star}$ of morphisms $\varphi_i: D(i) \rightarrow X$, indexed by the objects i of $\mathbf{J}$, such that $\varphi_j \circ D(\phi) = \varphi_i$ for every morphism $\phi: i \rightarrow j$ in $\mathbf{J}$. A *colimit* of D is a cocone $(C, c_{\star})$ of D that satisfies the following universal property: given any cocone $(X, \varphi_{\star})$ of D , there exists a unique morphism $\Phi: C \rightarrow X$ such that the diagram

$$\begin{array}{ccc}
 D(i) & \xrightarrow{D(\phi)} & D(j) \\
 \searrow c_i & & \swarrow c_j \\
 & C & \\
 \varphi_i \swarrow & \downarrow \Phi & \searrow \varphi_j \\
 & X &
 \end{array}$$

commutes for every morphism $\phi: i \rightarrow j$ in $\mathbf{J}$. Whenever it exists, a colimit of D is essentially unique, thus we can unambiguously call it “the” colimit of D . Alternatively, the cocones of D can be identified with the objects of the category $(D \downarrow \Delta_{\mathbf{J}, \mathbf{C}})$, which is thus called the *category of cocones* of D , and a colimit of D is an initial object in $(D \downarrow \Delta_{\mathbf{J}, \mathbf{C}})$.

The following are some of the most important examples of limits in a category $\mathbf{C}$:

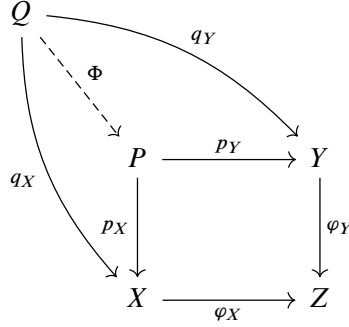
- (1) Let X, Y be two objects of $\mathbf{C}$ and $a, b: X \rightarrow Y$ two morphisms in $\mathbf{C}$. Then the *equaliser* of $a, b: X \rightarrow Y$ is an object $\text{Eq}(a, b)$ of $\mathbf{C}$ together with a morphism $\text{eq}(a, b): \text{Eq}(a, b) \rightarrow X$ with $a \circ \text{eq}(a, b) = b \circ \text{eq}(a, b)$ verifying the following universal property: if $u: E \rightarrow X$ is a morphism in $\mathbf{C}$ satisfying $a \circ u = b \circ u$, then there exists a unique morphism $\Phi: E \rightarrow \text{Eq}(a, b)$ such that $\text{eq}(a, b) \circ \Phi = u$. The equaliser $(\text{Eq}(a, b), \text{eq}(a, b))$ coincides with the limit of the diagram of type $\mathbf{J}_{\rightarrow}$ in $\mathbf{C}$, where $\mathbf{J}_{\rightarrow}$ is the category made of two objects (corresponding to X and Y) and (besides the identity morphisms) having only two parallel morphisms between them (corresponding to a and b). Observe also that $\text{eq}(a, b)$ is a monomorphism.
- (2) Suppose $\mathbf{C}$ has zero morphisms. Then the *kernel* of a morphism $\varphi: X \rightarrow Y$ in $\mathbf{C}$ is

$$(\text{Ker}(\varphi), \text{ker}(\varphi)) := (\text{Eq}(\varphi, 0_{XY}), \text{eq}(\varphi, 0_{XY})),$$

whenever the equaliser of $\varphi, 0_{XY}: X \rightarrow Y$ exists.

- (3) Let $X_{\star} = \{X_i\}_{i \in I}$ be a set of objects of $\mathbf{C}$. Then the *product* of $X_{\star}$ in $\mathbf{C}$ is an object $\prod^{\mathbf{C}} X_{\star} = \prod_{i \in I}^{\mathbf{C}} X_i$ together with a family of morphisms $\{\pi_i: \prod^{\mathbf{C}} X_{\star} \rightarrow X_i\}_{i \in I}$ verifying the following universal property: given an object Y of $\mathbf{C}$ and a family of morphisms $\{\varphi_i: Y \rightarrow X_i\}_{i \in I}$, there exists a unique morphism $\Phi: Y \rightarrow \prod^{\mathbf{C}} X_{\star}$ such that $\pi_i \circ \Phi = \varphi_i$ for every $i \in I$. The product $(\prod^{\mathbf{C}} X_{\star}, \{\pi_i\}_{i \in I})$ coincides with the limit of the diagram of type $\mathbf{J}_I$ in $\mathbf{C}$, where $\mathbf{J}_I$ is the discrete category whose objects are the elements of I .
- (4) Let $(I, \leq)$ be a directed set. By an *inverse* (or *projective*) *system* in $\mathbf{C}$ indexed by the directed set $(I, \leq)$ we mean a family of objects $\{X_i\}_{i \in I}$, together with a family $\{P_{ij}: i, j \in I, i \leq j\}$ of morphisms $P_{ij}: X_j \rightarrow X_i$ such that $P_{ii} = \text{id}_{X_i}$ for every $i \in I$ and $P_{ik} = P_{ij} \circ P_{jk}$ for every $i, j, k \in I$ with $i \leq j \leq k$. Then the *inverse* (or *projective*) *limit* of $(\{X_i\}_{i \in I}, \{P_{ij}\}_{i \leq j})$ is an object $\varprojlim X_{\star}$ of $\mathbf{C}$, together with a family $\{P_i\}_{i \in I}$ of morphisms $P_i: \varprojlim X_{\star} \rightarrow X_i$ such that $P_{ij} \circ P_j = P_i$ for every $i \leq j$ and verifying the following universal property: given an object Y of $\mathbf{C}$ and morphisms $Q_i: Y \rightarrow X_i$ with $P_{ij} \circ Q_j = Q_i$ for every $i \leq j$, there exists a unique morphism $\Phi: Y \rightarrow \varprojlim X_{\star}$ such that $P_i \circ \Phi = Q_i$ for every $i \in I$. Moreover, $(\varprojlim X_{\star}, \{P_i\}_{i \in I})$ coincides with the limit of the diagram of type $(I, \leq)^{\text{op}}$ in $\mathbf{C}$.

- (5) Let X, Y, Z be objects of $\mathbf{C}$. Let $\varphi_X: X \rightarrow Z$ and $\varphi_Y: Y \rightarrow Z$ be two given morphisms. Then the *pullback* of φ_X and φ_Y is an object $P = X \times_Z Y$ of $\mathbf{C}$, together with two morphisms $p_X: P \rightarrow X$ and $p_Y: P \rightarrow Y$ with $\varphi_X \circ p_X = \varphi_Y \circ p_Y$ verifying the following universal property: given an object Q of $\mathbf{C}$ and morphisms $q_X: Q \rightarrow X$ and $q_Y: Q \rightarrow Y$ with $\varphi_X \circ q_X = \varphi_Y \circ q_Y$, there exists a unique morphism $\Phi: Q \rightarrow P$ such that



is a commutative diagram. The pullback $(X \times_Z Y, p_X, p_Y)$ coincides with the limit of the diagram of type $\mathbf{J}_\perp$ in $\mathbf{C}$, where $\mathbf{J}_\perp$ is the category with three objects (corresponding to X, Y , and Z) and whose morphisms (besides the identity ones) are $X \rightarrow Z$ and $Y \rightarrow Z$.

Dually, the following are some of the most important examples of colimits in a category $\mathbf{C}$:

- (1) Let X, Y be two objects of $\mathbf{C}$ and $a, b: X \rightarrow Y$ two morphisms in $\mathbf{C}$. Then the *coequaliser* of $a, b: X \rightarrow Y$ is given by an object $\text{Coeq}(a, b)$ of $\mathbf{C}$ together with a morphism $\text{coeq}(a, b): Y \rightarrow \text{Coeq}(a, b)$ with $\text{coeq}(a, b) \circ a = \text{coeq}(a, b) \circ b$ verifying the following universal property: if $u: Y \rightarrow F$ is a morphism in $\mathbf{C}$ satisfying $u \circ a = u \circ b$, then there exists a unique morphism $\Phi: \text{Coeq}(a, b) \rightarrow F$ such that $\Phi \circ \text{coeq}(a, b) = u$. The coequaliser $(\text{Coeq}(a, b), \text{coeq}(a, b))$ coincides with the colimit of the diagram of type $\mathbf{J}_\rightarrow$ in $\mathbf{C}$. Observe also that $\text{coeq}(a, b)$ is an epimorphism.
- (2) Suppose $\mathbf{C}$ has zero morphisms. Then the *cokernel* of a morphism $\varphi: X \rightarrow Y$ in $\mathbf{C}$ is

$$(\text{Coker}(\varphi), \text{coker}(\varphi)) := (\text{Coeq}(\varphi, 0_{XY}), \text{coeq}(\varphi, 0_{XY})),$$

whenever the coequaliser of $\varphi, 0_{XY}: X \rightarrow Y$ exists.

- (3) Let $X_\star = \{X_i\}_{i \in I}$ be a set of objects of $\mathbf{C}$. Then the *coproduct* of $X_\star$ in $\mathbf{C}$ is an object $\coprod^{\mathbf{C}} X_\star = \coprod_{i \in I}^{\mathbf{C}} X_i$, together with a family of morphisms $\{\iota_i: X_i \rightarrow \coprod^{\mathbf{C}} X_\star\}_{i \in I}$ verifying the following universal property: given an object Y of $\mathbf{C}$ and a family

of morphisms $\{\varphi_i: X_i \rightarrow Y\}_{i \in I}$, there exists a unique morphism $\Phi: \coprod^{\mathbf{C}} X_{\star} \rightarrow Y$ such that $\Phi \circ \iota_i = \varphi_i$ for every $i \in I$. The coproduct $(\coprod^{\mathbf{C}} X_{\star}, \{\iota_i\}_{i \in I})$ coincides with the colimit of the diagram of type $\mathbf{J}_I$ in $\mathbf{C}$.

- (4) Let $(I, \leq)$ be a directed set. By a *direct* (or *inductive*) *system* in $\mathbf{C}$ indexed by the directed set $(I, \leq)$ we mean a family of objects $\{X_i\}_{i \in I}$, together with a family $\{\varphi_{ij} : i, j \in I, i \leq j\}$ of morphisms $\varphi_{ij}: X_i \rightarrow X_j$ such that $\varphi_{ii} = \text{id}_{X_i}$ for every $i \in I$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for every $i, j, k \in I$ with $i \leq j \leq k$. Then the *direct* (or *inductive*) *limit* of $(\{X_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ is an object $\varinjlim X_{\star}$ of $\mathbf{C}$, together with a family $\{\varphi_i\}_{i \in I}$ of morphisms $\varphi_i: X_i \rightarrow \varinjlim X_{\star}$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for every $i \leq j$ and verifying the following universal property: given an object Y of $\mathbf{C}$ and morphisms $\psi_i: X_i \rightarrow Y$ with $\psi_j \circ \varphi_{ij} = \psi_i$ for every $i \leq j$, there exists a unique morphism $\Phi: \varinjlim X_{\star} \rightarrow Y$ such that $\Phi \circ \varphi_i = \psi_i$ for every $i \in I$. Moreover, $(\varinjlim X_{\star}, \{\varphi_i\}_{i \in I})$ coincides with the colimit of the diagram of type $(I, \leq)$ in $\mathbf{C}$.
- (5) Let X, Y, Z be objects of $\mathbf{C}$. Let $\varphi_X: Z \rightarrow X$ and $\varphi_Y: Z \rightarrow Y$ be two given morphisms. Then the *pushout* of φ_X and φ_Y is an object $P = X \sqcup_Z Y$ of $\mathbf{C}$, together with two morphisms $i_X: X \rightarrow P$ and $i_Y: Y \rightarrow P$ with $i_X \circ \varphi_X = i_Y \circ \varphi_Y$ verifying the following universal property: given an object Q of $\mathbf{C}$ and morphisms $j_X: X \rightarrow Q$ and $j_Y: Y \rightarrow Q$ with $j_X \circ \varphi_X = j_Y \circ \varphi_Y$, there exists a unique morphism $\Phi: P \rightarrow Q$ such that the diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{\varphi_Y} & Y \\
 \varphi_X \downarrow & & \downarrow i_Y \\
 X & \xrightarrow{i_X} & P \\
 & \searrow j_X & \downarrow \Phi \\
 & & Q
 \end{array}$$

(Note: A curved arrow labeled j_Y also points from Y to Q , and a dashed arrow labeled Φ points from P to Q .)

commutes. The pushout $(X \sqcup_Z Y, i_X, i_Y)$ coincides with the colimit of the diagram of type $\mathbf{J}_{\Gamma}$ in $\mathbf{C}$, where by $\mathbf{J}_{\Gamma}$ we mean the category with three objects (corresponding to X, Y , and Z) whose morphisms (besides the identity ones) are $Z \rightarrow X$ and $Z \rightarrow Y$.

REMARK 2.8. A warning about the terminology: differently from other authors, we only consider (co)products that are indexed by sets (not by classes). These are sometimes called *small (co)products*. Moreover, by a projective (resp. an inductive) limit we mean the limit (resp. the colimit) of a diagram that is indexed by a directed

set, while for some authors “projective limit” (resp. “inductive limit”) is a synonym of (not necessarily directed) “limit” (resp. “colimit”).

Next we recall the notion of image and coimage of a morphism. Fix a category $\mathbf{C}$ that admits all finite limits and colimits. Let $\varphi: X \rightarrow Y$ be a morphism in $\mathbf{C}$. Then,

- (1) the *image* of φ is given by $(\text{Im}(\varphi), \text{im}(\varphi)) := (\text{Eq}(i_1, i_2), \text{eq}(i_1, i_2))$, where we denote by $(Y \sqcup_X Y, i_1, i_2)$ the pushout of φ and φ ;
- (2) the *coimage* of φ is given by $(\text{Coim}(\varphi), \text{coim}(\varphi)) := (\text{Coeq}(p_1, p_2), \text{coeq}(p_1, p_2))$, where we denote by $(Y \times_X Y, p_1, p_2)$ the pullback of φ and φ .

If in addition $\mathbf{C}$ has zero morphisms, we say that a monomorphism $\varphi: X \rightarrow Y$ in $\mathbf{C}$ is *normal* if there exist an object Z and a morphism $\eta: Y \rightarrow Z$ such that

$$(X, \varphi) \cong (\text{Ker}(\eta), \text{ker}(\eta)).$$

Dually, we say that an epimorphism $\psi: X \rightarrow Y$ in $\mathbf{C}$ is *conormal* if there exist an object W and a morphism $\theta: W \rightarrow X$ such that

$$(Y, \psi) \cong (\text{Coker}(\theta), \text{coker}(\theta)).$$

The category $\mathbf{C}$ is said to be *normal* if every monomorphism in $\mathbf{C}$ is normal, and *conormal* if every epimorphism in $\mathbf{C}$ is conormal. A category that is either normal or conormal is balanced; see e.g. [22, Proposition 14.3].

2.2.2. Completeness and cocompleteness. A category $\mathbf{C}$ is said to be *complete* if all small limits in $\mathbf{C}$ (i.e. limits of diagrams whose index category is small) exist. Dually, $\mathbf{C}$ is said to be *cocomplete* if all small colimits in $\mathbf{C}$ exist. A category that is both complete and cocomplete is called a *bicomplete category*. An example of a bicomplete category is **Set**. The product $\prod_{i \in I}^{\text{Set}} X_i$ of a given family of sets $X_\star = \{X_i\}_{i \in I}$ is the Cartesian product $\prod_{i \in I} X_i$, while the coproduct $\coprod_{i \in I}^{\text{Set}} X_i$ is the disjoint union $\bigsqcup_{i \in I} X_i$. Another example of a bicomplete category is **Ban**; see e.g. [26].

The following result provides a criterion to detect (co)complete categories. Albeit well known to the experts, we report its proof for the usefulness of the reader.

THEOREM 2.9. *A category in which all products and equalisers exist is complete. A category in which all coproducts and coequalisers exist is cocomplete.*

PROOF. We prove only the first statement, as the second one follows by a dual argument. Let $\mathbf{C}$ be a category in which all products and equalisers exist. Let $D: \mathbf{J} \rightarrow \mathbf{C}$ be a small diagram. Then the products $(\prod_{i \in \text{Ob}_{\mathbf{J}}}^{\mathbf{C}} D(i), \pi_\star)$ and $(\prod_{\phi \in \text{Hom}_{\mathbf{J}}}^{\mathbf{C}} D(\text{cod}(\phi)), \eta_\star)$ exist in $\mathbf{C}$. For brevity, let us set $Y := \prod_{i \in \text{Ob}_{\mathbf{J}}}^{\mathbf{C}} D(i)$ and $Z := \prod_{\phi \in \text{Hom}_{\mathbf{J}}}^{\mathbf{C}} D(\text{cod}(\phi))$.

Observe that there exist two (uniquely determined) morphisms $a, b: Y \rightarrow Z$ such that

$$\begin{array}{ccccc} Y & \xrightarrow{\quad a \quad} & Z & \xleftarrow{\quad b \quad} & Y \\ & \searrow \pi_{\text{cod}(\phi)} & \downarrow \eta_\phi & \swarrow D(\phi) \circ \pi_{\text{dom}(\phi)} & \\ & & D(\text{cod}(\phi)) & & \end{array}$$

is a commutative diagram for all $\phi \in \text{Hom}_{\mathbf{J}}$. Define $\lambda_i := \pi_i \circ \text{eq}(a, b): \text{Eq}(a, b) \rightarrow D(i)$ for every $i \in \text{Ob}_{\mathbf{J}}$. Given a morphism $\phi: i \rightarrow j$ in $\mathbf{J}$, one has that

$$\begin{aligned} D(\phi) \circ \lambda_i &= D(\phi) \circ \pi_{\text{dom}(\phi)} \circ \text{eq}(a, b) = \eta_\phi \circ b \circ \text{eq}(a, b) = \eta_\phi \circ a \circ \text{eq}(a, b) \\ &= \pi_{\text{cod}(\phi)} \circ \text{eq}(a, b) = \lambda_j. \end{aligned}$$

This shows that $(\text{Eq}(a, b), \lambda_\star)$ is a cone to D . Moreover, if $(X, \varphi_\star)$ is a cone to D , then there exists a unique morphism $u: X \rightarrow Y$ such that $\pi_i \circ u = \varphi_i$ for every $i \in \text{Ob}_{\mathbf{J}}$. In particular, the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{a} & Z & \xleftarrow{b} & Y & \xleftarrow{u} & X \\ & \searrow \varphi_{\text{cod}(\phi)} & \downarrow \pi_{\text{cod}(\phi)} & & \downarrow \eta_\phi & & \downarrow \pi_{\text{dom}(\phi)} & \swarrow \varphi_{\text{dom}(\phi)} & \\ & & D(\text{cod}(\phi)) & = & D(\text{cod}(\phi)) & \xleftarrow{D(\phi)} & D(\text{dom}(\phi)) & & \end{array}$$

commutes for every $\phi \in \text{Hom}_{\mathbf{J}}$. Given that $D(\phi) \circ \varphi_{\text{dom}(\phi)} = \varphi_{\text{cod}(\phi)}$, we deduce that $a \circ u = b \circ u$, thus there exists a unique morphism $\Phi: X \rightarrow \text{Eq}(a, b)$ such that $\text{eq}(a, b) \circ \Phi = u$. Observe that

$$\lambda_i \circ \Phi = \pi_i \circ \text{eq}(a, b) \circ \Phi = \pi_i \circ u = \varphi_i \quad \text{for every } i \in \text{Ob}_{\mathbf{J}}.$$

Finally, we claim that Φ is the unique morphism satisfying $\lambda_i \circ \Phi = \varphi_i$ for every $i \in \text{Ob}_{\mathbf{J}}$. To prove it, fix any morphism $\Psi: X \rightarrow \text{Eq}(a, b)$ such that $\lambda_i \circ \Psi = \varphi_i$ for every $i \in \text{Ob}_{\mathbf{J}}$. This means that $\pi_i \circ \text{eq}(a, b) \circ \Psi = \varphi_i$ holds for every $i \in \text{Ob}_{\mathbf{J}}$, which forces $\text{eq}(a, b) \circ \Psi = u$ by the uniqueness of u , and thus $\Psi = \Phi$ by the uniqueness of Φ . All in all, $(\text{Eq}(a, b), \lambda_\star)$ is the limit of D . $\blacksquare$

2.2.3. Limits and colimits as functors. If $\mathbf{J}$ is a small index category and all the diagrams of type $\mathbf{J}$ in a category $\mathbf{C}$ have limits, then there exists a unique functor $\text{Lim}_{\mathbf{J}, \mathbf{C}}: \mathbf{C}^{\mathbf{J}} \rightarrow \mathbf{C}$, called the *limit functor* from $\mathbf{J}$ to $\mathbf{C}$, such that $\text{Lim}_{\mathbf{J}, \mathbf{C}}(D)$ is (the object underlying) the limit of D for any diagram $D: \mathbf{J} \rightarrow \mathbf{C}$. Dually, if all the diagrams of type $\mathbf{J}$ in $\mathbf{C}$ have colimits, then there exists a unique functor $\text{Colim}_{\mathbf{J}, \mathbf{C}}: \mathbf{C}^{\mathbf{J}} \rightarrow \mathbf{C}$, called the *colimit functor* from $\mathbf{J}$ to $\mathbf{C}$, such that $\text{Colim}_{\mathbf{J}, \mathbf{C}}(D)$ is (the object underlying) the colimit of D for any diagram $D: \mathbf{J} \rightarrow \mathbf{C}$. In the special cases of inverse and direct limits, we write $\varprojlim_I$ and $\varinjlim_I$ instead of $\text{Lim}_{(I, \leq)^{\text{op}}, \mathbf{C}}$ and $\text{Colim}_{(I, \leq), \mathbf{C}}$, respectively.

REMARK 2.10. Let $\mathbf{C}$ be a category having zero morphisms and where all kernels exist, and $\mathbf{J}$ a small index category. Let $\theta_\star: D_1 \rightarrow D_2$ be a morphism in $\mathbf{C}^{\mathbf{J}}$. Then the kernel of $\theta_\star$ in $\mathbf{C}^{\mathbf{J}}$ exists:

- (1) The diagram $\text{Ker}(\theta_\star): \mathbf{J} \rightarrow \mathbf{C}$ is given by $\text{Ker}(\theta_\star)(i) = \text{Ker}(\theta_i)$ for every $i \in \text{Ob}_{\mathbf{J}}$, and for any morphism $\phi: i \rightarrow j$ in $\mathbf{J}$ we have that $\text{Ker}(\theta_\star)(\phi): \text{Ker}(\theta_i) \rightarrow \text{Ker}(\theta_j)$ is the unique morphism Φ satisfying $\text{ker}(\theta_j) \circ \Phi = D_1(\phi) \circ \text{ker}(\theta_i)$. Both the existence and the uniqueness of Φ are consequences of the commutativity of the following diagram:

$$\begin{array}{ccccccc} \text{Ker}(\theta_i) & \xrightarrow{\text{ker}(\theta_i)} & D_1(i) & \xrightarrow{\theta_i} & D_2(i) & \longrightarrow & \{0\} \\ \Phi \downarrow & & D_1(\phi) \downarrow & & D_2(\phi) \downarrow & & \parallel \\ \text{Ker}(\theta_j) & \xrightarrow{\text{ker}(\theta_j)} & D_1(j) & \xrightarrow{\theta_j} & D_2(j) & \longrightarrow & \{0\}. \end{array}$$

- (2) The morphism $\text{ker}(\theta_\star)$ coincides with the natural transformation

$$\text{ker}(\theta_\star)_\star: \text{Ker}(\theta_\star) \rightarrow D_1,$$

which is defined as $\text{ker}(\theta_\star)_i := \text{ker}(\theta_i): \text{Ker}(\theta_i) \rightarrow D_1(i)$ for every $i \in \text{Ob}_{\mathbf{J}}$.

The dual statement holds for the cokernel of $\theta_\star$ (when all cokernels exist in $\mathbf{C}$).

Given two categories $\mathbf{C}, \mathbf{D}$, and two functors $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$, we say that F is the *right adjoint* to G , or that G is the *left adjoint* to F (for short $G \dashv F$), if for any object X of $\mathbf{C}$ there exists a morphism $\varepsilon_X: G(F(X)) \rightarrow X$ with the following property: given an object Y of $\mathbf{D}$ and a morphism $\varphi: G(Y) \rightarrow X$, there exists a unique morphism $\psi: Y \rightarrow F(X)$ such that

$$\begin{array}{ccc} G(Y) & & \\ \downarrow F(\psi) & \searrow \varphi & \\ G(F(X)) & \xrightarrow{\varepsilon_X} & X \end{array}$$

commutes. Each right-adjoint functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is *continuous*, i.e. it preserves limits: if the limit $(L, \lambda_\star)$ of a diagram $D: \mathbf{J} \rightarrow \mathbf{C}$ exists, then the limit $(L^F, \lambda_\star^F)$ of the diagram $F \circ D: \mathbf{J} \rightarrow \mathbf{D}$ exists as well, and the unique morphism $\Phi: F(L) \rightarrow L^F$ such that

$$\begin{array}{ccc} F(L) & \xrightarrow{\Phi} & L^F \\ & \searrow F(\lambda_i) & \downarrow \lambda_i^F \\ & & F(D(i)) \end{array}$$

is a commutative diagram for every $i \in \text{Ob}_{\mathbf{J}}$ is an isomorphism. In particular, each right-adjoint functor is *left exact*, i.e. it commutes with finite limits. Dually, each left-adjoint functor is *cocontinuous*, i.e. it preserves colimits: if the colimit $(C, c_{\star})$ of a diagram $D: \mathbf{J} \rightarrow \mathbf{C}$ exists, then also the colimit $(C^F, c_{\star}^F)$ of $F \circ D: \mathbf{J} \rightarrow \mathbf{D}$ exists, and the unique morphism $\Psi: C^F \rightarrow F(C)$ such that

$$\begin{array}{ccc} F(D(i)) & & \\ c_i^F \downarrow & \searrow F(c_i) & \\ C^F & \xrightarrow[\Psi]{\quad} & F(C) \end{array}$$

is a commutative diagram for every $i \in \text{Ob}_{\mathbf{J}}$ is an isomorphism. In particular, each left-adjoint functor is *right exact*, i.e. it commutes with finite colimits. If $\mathbf{C}$ is a complete (resp. cocomplete) category, then for any small index category $\mathbf{J}$ it holds that $\Delta_{\mathbf{J}, \mathbf{C}} \dashv \text{Lim}_{\mathbf{J}, \mathbf{C}}$ (resp. $\text{Colim}_{\mathbf{J}, \mathbf{C}} \dashv \Delta_{\mathbf{J}, \mathbf{C}}$), thus in particular $\text{Lim}_{\mathbf{J}, \mathbf{C}}$ is continuous (resp. $\text{Colim}_{\mathbf{J}, \mathbf{C}}$ is cocontinuous).

3. The category of Banach $L^0(\mathbb{X})$ -modules

Let $\mathbb{X}$ be a given σ -finite measure space. Then we denote by $\mathbf{BanMod}_{\mathbb{X}}$ the category of Banach $L^0(\mathbb{X})$ -modules, where a morphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ between two Banach $L^0(\mathbb{X})$ -modules $\mathcal{M}$ and $\mathcal{N}$ is defined as an $L^0(\mathbb{X})$ -linear operator (i.e. a homomorphism of $L^0(\mathbb{X})$ -modules) that satisfies

$$|\varphi(v)| \leq |v| \quad \text{for every } v \in \mathcal{M}.$$

The morphisms in $\text{Hom}_{\mathbf{BanMod}_{\mathbb{X}}}(\mathcal{M}, \mathcal{N})$ are exactly those $\varphi \in \text{Hom}(\mathcal{M}, \mathcal{N})$ satisfying $|\varphi| \leq 1$. Also, the isomorphisms in $\mathbf{BanMod}_{\mathbb{X}}$ are exactly the isomorphisms of Banach $L^0(\mathbb{X})$ -modules.

REMARK 3.1. One could also consider the category $\mathbf{BanMod}_{\mathbb{X}}^0$, where the morphisms between two Banach $L^0(\mathbb{X})$ -modules $\mathcal{M}$ and $\mathcal{N}$ are given exactly by the elements of $\text{Hom}(\mathcal{M}, \mathcal{N})$. Observe that in this category a morphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism if and only if it is bijective and there exist $g_1, g_2 \in L^0(\mathbb{X})$ with $g_2 \geq g_1 > 0$ such that $g_1|v| \leq |\varphi(v)| \leq g_2|v|$ for every $v \in \mathcal{M}$. Our choice of working with $\mathbf{BanMod}_{\mathbb{X}}$ is due to the fact that $\mathbf{BanMod}_{\mathbb{X}}^0$ is neither complete nor cocomplete; see Example 3.13. However, by suitably adapting the results we are going to present, one can show that $\mathbf{BanMod}_{\mathbb{X}}^0$ is *finitely bicomplete* (i.e. it admits finite limits and colimits).

3.1 – Basic properties of $\mathbf{BanMod}_{\mathbb{X}}$

Let $\mathbb{X}$ be a σ -finite measure space. Then it can be readily checked that $\mathbf{BanMod}_{\mathbb{X}}$ is a pointed category, whose zero object $0_{\mathbf{BanMod}_{\mathbb{X}}}$ is given by the trivial Banach $L^0(\mathbb{X})$ -module consisting uniquely of the zero element. Moreover, $\mathbf{BanMod}_{\mathbb{X}}$ is locally small: given any two Banach $L^0(\mathbb{X})$ -modules $\mathcal{M}$ and $\mathcal{N}$, we observed that the morphisms $\mathcal{M} \rightarrow \mathcal{N}$ form a subset of $\text{Hom}(\mathcal{M}, \mathcal{N})$, thus in particular

$$\text{card}(\text{Hom}_{\mathbf{BanMod}_{\mathbb{X}}}(\mathcal{M}, \mathcal{N})) \leq \text{card}(\mathcal{N})^{\text{card}(\mathcal{M})}.$$

PROPOSITION 3.2. *Let $\mathbb{X}$ be a σ -finite measure space. Then the category $\mathbf{BanMod}_{\mathbb{X}}$ is not small.*

PROOF. Given any cardinal κ , we fix a set S_{κ} of cardinality κ . Recall that if κ_1, κ_2 are different cardinals, then $\mathcal{H}_{\mathbb{X}}(S_{\kappa_1})$ is not isomorphic to $\mathcal{H}_{\mathbb{X}}(S_{\kappa_2})$. Since the collection of all cardinals is a proper class (i.e. it is not a set), we conclude that the collection of all Hilbert $L^0(\mathbb{X})$ -modules is a proper class as well. This implies that the category $\mathbf{BanMod}_{\mathbb{X}}$ is not small, as we claimed. ■

PROPOSITION 3.3. *Let $\mathbb{X}$ be a σ -finite measure space and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ a morphism in $\mathbf{BanMod}_{\mathbb{X}}$. Then the following properties hold:*

- (1) *φ is a monomorphism if and only if it is injective.*
- (2) *φ is an epimorphism if and only if $\varphi(\mathcal{M})$ is dense in $\mathcal{N}$.*

PROOF. Suppose φ is a monomorphism. Denote by $\psi_1: \varphi^{-1}(\{0\}) \rightarrow \mathcal{M}$ and $\psi_2: \varphi^{-1}(\{0\}) \rightarrow \mathcal{M}$ the inclusion map and the null map, respectively. Since both $\varphi \circ \psi_1$ and $\varphi \circ \psi_2$ coincide with the null map from $\varphi^{-1}(\{0\})$ to $\mathcal{N}$, we deduce that $\psi_1 = \psi_2$ and thus $\varphi^{-1}(\{0\}) = \{0\}$, whence the injectivity of φ follows by linearity. Conversely, suppose φ is injective. Given any Banach $L^0(\mathbb{X})$ -module $\mathcal{P}$ and any two morphisms $\psi_1, \psi_2: \mathcal{P} \rightarrow \mathcal{M}$ satisfying $\varphi \circ \psi_1 = \varphi \circ \psi_2$, we have that $\psi_1 = \psi_2$, otherwise there would exist an element $z \in \mathcal{P}$ such that $\psi_1(z) \neq \psi_2(z)$ and thus accordingly $\varphi(\psi_1(z)) \neq \varphi(\psi_2(z))$ by the injectivity of φ . This shows that φ is a monomorphism.

Suppose φ is an epimorphism. Define $\mathcal{Q} := \mathcal{N}/\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$. Let us denote by $\psi_1: \mathcal{N} \rightarrow \mathcal{Q}$ the canonical projection map on the quotient and by $\psi_2: \mathcal{N} \rightarrow \mathcal{Q}$ the null map. Observe that

$$\psi_1(\varphi(v)) = \varphi(v) + \text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})) = \text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})) \quad \text{for every } v \in \mathcal{M},$$

thus $\psi_1 \circ \varphi$ and $\psi_2 \circ \varphi$ are the null map. Hence, we have $\psi_1 = \psi_2$, which implies $\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})) = \mathcal{N}$. Conversely, suppose $\varphi(\mathcal{M})$ is dense in $\mathcal{N}$. Choose any Banach $L^0(\mathbb{X})$ -module $\mathcal{P}$ and any two morphisms $\psi_1, \psi_2: \mathcal{N} \rightarrow \mathcal{P}$ with $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. This means

that $\psi_1 = \psi_2$ on the dense subspace $\varphi(\mathcal{M})$ of $\mathcal{N}$, thus accordingly $\psi_1 = \psi_2$ on the whole $\mathcal{N}$. This shows that φ is an epimorphism. ■

EXAMPLE 3.4. Let $\mathbb{X}$ be a given σ -finite measure space. Recall that c_0 stands for the space of all sequences $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$, and that c_0 is a separable Banach space if endowed with the componentwise operations and the supremum norm $\|(a_n)_{n \in \mathbb{N}}\|_{c_0} := \sup_{n \in \mathbb{N}} |a_n|$. Thanks to the density of simple maps in the Banach $L^0(\mathbb{X})$ -module $L^0(\mathbb{X}; c_0)$, we have that there exists a unique morphism $\varphi: L^0(\mathbb{X}; c_0) \rightarrow L^0(\mathbb{X}; c_0)$ in the category **BanMod** $_{\mathbb{X}}$ such that

$$\varphi(\chi_X(a_n)_{n \in \mathbb{N}}) = \chi_X(a_n/n)_{n \in \mathbb{N}} \quad \text{for every } (a_n)_{n \in \mathbb{N}} \in c_0.$$

Clearly, φ is injective and thus a monomorphism in **BanMod** $_{\mathbb{X}}$. Moreover, the range of φ contains all simple maps from $\mathbb{X}$ to c_{00} , where c_{00} stands for the space consisting of those sequences $(a_n)_{n \in \mathbb{N}}$ such that $a_n = 0$ holds for all but finitely many $n \in \mathbb{N}$. Since c_{00} is dense in c_0 , it follows that $\varphi(L^0(\mathbb{X}; c_0))$ is dense in $L^0(\mathbb{X}; c_0)$ and thus φ is an epimorphism in **BanMod** $_{\mathbb{X}}$. However, we have that $\chi_X(1/n)_{n \in \mathbb{N}} \in L^0(\mathbb{X}; c_0)$ does not belong to $\varphi(L^0(\mathbb{X}; c_0))$, which means that φ is not surjective. In particular, the morphism φ is not an isomorphism in **BanMod** $_{\mathbb{X}}$.

REMARK 3.5. Example 3.4 shows that, given a σ -finite measure space $\mathbb{X}$, the category **BanMod** $_{\mathbb{X}}$ is not balanced, which implies that it is neither normal nor conormal.

3.2 – Limits and colimits in **BanMod** $_{\mathbb{X}}$

In this section we prove that **BanMod** $_{\mathbb{X}}$ is bicomplete.

3.2.1. Kernels and cokernels in **BanMod $_{\mathbb{X}}$.** We begin by proving that (co)kernels exist in **BanMod** $_{\mathbb{X}}$:

THEOREM 3.6 (Kernels in **BanMod $_{\mathbb{X}}$).** *Let $\mathbb{X}$ be a σ -finite measure space and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ a morphism between Banach $L^0(\mathbb{X})$ -modules $\mathcal{M}, \mathcal{N}$. Let us consider the null space $\varphi^{-1}(\{0\})$, which is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$. Then we have that the kernel of φ exists and is given by*

$$\text{Ker}(\varphi) \cong \varphi^{-1}(\{0\}),$$

together with the inclusion map $\ker(\varphi): \varphi^{-1}(\{0\}) \rightarrow \mathcal{M}$. In particular, the equaliser of any two morphisms $\varphi, \psi: \mathcal{M} \rightarrow \mathcal{N}$ exists and is given by

$$(3.1) \quad (\text{Eq}(\varphi, \psi), \text{eq}(\varphi, \psi)) \cong \left(\text{Ker}\left(\frac{\varphi - \psi}{2}\right), \ker\left(\frac{\varphi - \psi}{2}\right) \right).$$

PROOF. We know that $\varphi^{-1}(\{0\})$ is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M}$ and that the inclusion map $\iota: \varphi^{-1}(\{0\}) \rightarrow \mathcal{M}$ is a morphism of Banach $L^0(\mathbb{X})$ -modules with $\varphi \circ \iota = 0$. Now fix a Banach $L^0(\mathbb{X})$ -module $\mathcal{E}$ and a morphism $u: \mathcal{E} \rightarrow \mathcal{M}$ such that $\varphi \circ u = 0$. Define $\Phi: \mathcal{E} \rightarrow \varphi^{-1}(\{0\})$ as $\Phi(w) := u(w)$ for every $w \in \mathcal{E}$. Note that Φ is the unique map from $\mathcal{E}$ to $\varphi^{-1}(\{0\})$ with $\iota \circ \Phi = u$. Since Φ is a morphism, we conclude that the kernel of φ is $\varphi^{-1}(\{0\})$ together with the inclusion map. This proves the first part of the statement, whence (3.1) immediately follows. ■

THEOREM 3.7 (Cokernels in **BanMod $_{\mathbb{X}}$**). *Let $\mathbb{X}$ be a σ -finite measure space and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ a morphism between Banach $L^0(\mathbb{X})$ -modules $\mathcal{M}, \mathcal{N}$. Let us consider the Banach $L^0(\mathbb{X})$ -submodule $\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$ of $\mathcal{N}$ and the quotient Banach $L^0(\mathbb{X})$ -module $\mathcal{N}/\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$. Then we have that the cokernel of φ exists and is given by*

$$\text{Coker}(\varphi) \cong \mathcal{N}/\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})),$$

together with the canonical projection map $\text{coker}(\varphi): \mathcal{N} \rightarrow \mathcal{N}/\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$. In particular, the coequaliser of any two morphisms $\varphi, \psi: \mathcal{M} \rightarrow \mathcal{N}$ exists and is given by

$$(3.2) \quad (\text{Coeq}(\varphi, \psi), \text{coeq}(\varphi, \psi)) \cong \left(\text{Coker}\left(\frac{\varphi - \psi}{2}\right), \text{coker}\left(\frac{\varphi - \psi}{2}\right) \right).$$

PROOF. We know that the quotient $\mathcal{Q} := \mathcal{N}/\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$ is a Banach $L^0(\mathbb{X})$ -module and that the canonical projection $\pi: \mathcal{N} \rightarrow \mathcal{Q}$ is a morphism of Banach $L^0(\mathbb{X})$ -modules. Notice that $\pi \circ \varphi = 0$. Now fix a Banach $L^0(\mathbb{X})$ -module $\mathcal{F}$ and a morphism $u: \mathcal{N} \rightarrow \mathcal{F}$ such that $u \circ \varphi = 0$. Define the mapping $\Phi: \mathcal{Q} \rightarrow \mathcal{F}$ as $\Phi([w]) := u(w)$ for every $w \in \mathcal{N}$, where $[w] \in \mathcal{Q}$ stands for the equivalence class of w . Notice also that $\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})) \subseteq u^{-1}(\{0\})$. This implies that if two elements $v, \tilde{v} \in \mathcal{M}$ satisfy $v - \tilde{v} \in \text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$, then $u(v) - u(\tilde{v}) = u(v - \tilde{v}) = 0$. Hence, Φ is well defined. Moreover, if $w \in \mathcal{N}$ and $z \in u^{-1}(\{0\})$, then $|u(w)| = |u(w + z)| \leq |w + z|$, which gives

$$|\Phi([w])| = |u(w)| \leq \bigwedge_{z \in u^{-1}(\{0\})} |w + z| \leq \bigwedge_{z \in \text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))} |w + z| = |[w]|$$

for every $[w] \in \mathcal{Q}$. Since Φ is $L^0(\mathbb{X})$ -linear by construction, we deduce that it is a morphism of Banach $L^0(\mathbb{X})$ -modules. Observe that $\Phi: \mathcal{Q} \rightarrow \mathcal{F}$ is the unique map satisfying $\Phi \circ \pi = u$. Hence, the cokernel of φ is given by $\mathcal{Q}$ together with the canonical projection map on the quotient. This proves the first part of the statement, whence (3.2) immediately follows. ■

REMARK 3.8. In Theorems 3.6 and 3.7 we cannot write $\text{Eq}(\varphi, \psi) = \text{Ker}(\varphi - \psi)$ or $\text{Coeq}(\varphi, \psi) = \text{Coker}(\varphi - \psi)$, since $\varphi - \psi$ need not be a morphism in **BanMod $_{\mathbb{X}}$** . This is due to the fact that, in general, $|(\varphi - \psi)(v)| \leq 2|v|$ for every $v \in \mathcal{M}$ is the best

inequality one can have. In particular, we have that **BanMod** $_{\mathbb{X}}$ is not enriched over the category of Abelian groups.

3.2.2. Products and coproducts in $\mathbf{BanMod}_{\mathbb{X}}$. To construct (co)products in the category of Banach $L^0(\mathbb{X})$ -modules, we need to introduce the notion of ℓ_p -sum of a family of Banach $L^0(\mathbb{X})$ -modules.

DEFINITION 3.9 (ℓ_p -sum). Let $\mathbb{X}$ be a σ -finite measure space and $\mathcal{M}_{\star} = \{\mathcal{M}_i\}_{i \in I}$ a family of Banach $L^0(\mathbb{X})$ -modules. Fix $p \in [1, \infty]$. Given any $v_{\star} = (v_i)_{i \in I} \in \prod_{i \in I}^{\text{Set}} \mathcal{M}_i$, we define

$$|v_{\star}|_p := \begin{cases} \left\{ \bigvee \{ (\sum_{i \in J} |v_i|^p)^{1/p} \mid J \in \mathcal{P}_F(I) \} \in L^0_{\text{ext}}(\mathbb{X}) \right\} & \text{if } p < \infty, \\ \left\{ \bigvee \{ |v_i| \mid i \in I \} \in L^0_{\text{ext}}(\mathbb{X}) \right\} & \text{if } p = \infty. \end{cases}$$

Then we define the ℓ_p -sum of $\mathcal{M}_{\star}$ as

$$\ell_p(\mathcal{M}_{\star}) := \{v_{\star} \in \prod_{i \in I}^{\text{Set}} \mathcal{M}_i \mid |v_{\star}|_p \in L^0(\mathbb{X})\}.$$

In the case where I consists of finitely many elements, say $\mathcal{M}_{\star} = \{\mathcal{M}_1, \dots, \mathcal{M}_n\}$, we denote

$$\mathcal{M}_1 \oplus_p \dots \oplus_p \mathcal{M}_n := \ell_p(\mathcal{M}_{\star}).$$

Rather standard verifications show that $\ell_p(\mathcal{M}_{\star})$ has a Banach $L^0(\mathbb{X})$ -module structure:

PROPOSITION 3.10. *Let $\mathbb{X}$ be a σ -finite measure space and $\mathcal{M}_{\star} = \{\mathcal{M}_i\}_{i \in I}$ a family of Banach $L^0(\mathbb{X})$ -modules. Let $p \in [1, \infty]$ be fixed. Then $(\ell_p(\mathcal{M}_{\star}), |\cdot|_p)$ is a Banach $L^0(\mathbb{X})$ -module with respect to the following operations:*

$$\begin{aligned} v_{\star} + w_{\star} &:= (v_i + w_i)_{i \in I} \quad \text{for every } v_{\star}, w_{\star} \in \ell_p(\mathcal{M}_{\star}), \\ f \cdot v_{\star} &:= (f \cdot v_i)_{i \in I} \quad \text{for every } f \in L^0(\mathbb{X}) \text{ and } v_{\star} \in \ell_p(\mathcal{M}_{\star}). \end{aligned}$$

PROOF. First of all, we aim to prove that for any $v_{\star}, w_{\star} \in \ell_p(\mathcal{M}_{\star})$ and $f \in L^0(\mathbb{X})$ it holds that

$$(3.3) \quad |v_{\star} + w_{\star}|_p \leq |v_{\star}|_p + |w_{\star}|_p, \quad |f \cdot v_{\star}|_p = |f| |v_{\star}|_p.$$

We discuss only the case $p < \infty$, as the case $p = \infty$ is easier. Fix any $J \in \mathcal{P}_F(I)$ and notice that

$$\begin{aligned} \left(\sum_{i \in J} |v_i + w_i|^p \right)^{1/p} &\leq \left(\sum_{i \in J} |v_i|^p \right)^{1/p} + \left(\sum_{i \in J} |w_i|^p \right)^{1/p} \leq |v_{\star}|_p + |w_{\star}|_p, \\ \left(\sum_{i \in J} |f \cdot v_i|^p \right)^{1/p} &= |f| \left(\sum_{i \in J} |v_i|^p \right)^{1/p} \leq |f| |v_{\star}|_p. \end{aligned}$$

Thanks to the arbitrariness of $J \in \mathcal{P}_F(I)$, we deduce that $|v_\star + w_\star|_p \leq |v_\star|_p + |w_\star|_p$ and $|f \cdot v_\star|_p \leq |f| |v_\star|_p$, whence it follows that

$$|f| \left| |v_\star|_p - |\chi_{\{f \neq 0\}} \cdot v_\star|_p \right| \leq |f| |\chi_{\{f=0\}} \cdot v_\star|_p \leq \chi_{\{f=0\}} |f| |v_\star|_p = 0.$$

Therefore, letting $g := \chi_{\{f \neq 0\}} \frac{1}{f} \in L^0(\mathbb{X})$, we can estimate

$$|f| |v_\star|_p = |f| |\chi_{\{f \neq 0\}} \cdot v_\star|_p = |f| |(fg) \cdot v_\star|_p \leq |f| |g| |f \cdot v_\star|_p \leq |f \cdot v_\star|_p.$$

All in all, (3.3) is proved. In particular, $v_\star + w_\star$ and $f \cdot v_\star$ belong to $\ell_p(\mathcal{M}_\star)$ for every $v_\star, w_\star \in \ell_p(\mathcal{M}_\star)$ and $f \in L^0(\mathbb{X})$. It can be readily checked that $(\ell_p(\mathcal{M}_\star), |\cdot|_p)$ is a normed $L^0(\mathbb{X})$ -module. It remains to verify that $\mathbf{d}_{\ell_p(\mathcal{M}_\star)}$ is a complete distance. Let $(v_\star^n)_{n \in \mathbb{N}} \subseteq \ell_p(\mathcal{M}_\star)$ be any Cauchy sequence. Given any index $i \in I$, we have that $|v_i^n - v_i^m| \leq |v_\star^n - v_\star^m|_p$ for every $n, m \in \mathbb{N}$, whence it follows that $(v_i^n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_i$ is a Cauchy sequence. Let $v_i \in \mathcal{M}_i$ denote its limit as $n \rightarrow \infty$. Moreover, we have that $||v_\star^n|_p - |v_\star^m|_p| \leq |v_\star^n - v_\star^m|_p$ for every $n, m \in \mathbb{N}$, thus $(|v_\star^n|_p)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^0(\mathbb{X})$, which converges to some limit function $g \in L^0(\mathbb{X})$. For each $J \in \mathcal{P}_F(I)$ we have that $(\sum_{i \in J} |v_i^n|^p)^{1/p} \leq |v_\star^n|_p$ for every $n \in \mathbb{N}$, so that by letting $n \rightarrow \infty$ we deduce that $(\sum_{i \in J} |v_i|^p)^{1/p} \leq g$. Given that $J \in \mathcal{P}_F(I)$ was arbitrary, we obtain that $|v_\star|_p \leq g$, thus in particular $v_\star \in \ell_p(\mathcal{M}_\star)$. Finally, for any $\varepsilon > 0$ we can find $\bar{n} \in \mathbb{N}$ such that $\mathbf{d}_{L^0(\mathbb{X})}(|v_\star^n - v_\star^m|, 0) \leq \varepsilon$ for every $n, m \geq \bar{n}$. Since $(|v_\star^n - v_\star^m|)_{m \geq \bar{n}}$ is a Cauchy sequence in $L^0(\mathbb{X})$, it converges to some $g_n \in L^0(\mathbb{X})$ as $m \rightarrow \infty$. Notice that $\mathbf{d}_{L^0(\mathbb{X})}(g_n, 0) \leq \varepsilon$. For each $J \in \mathcal{P}_F(I)$ and $n, m \geq \bar{n}$, we have $(\sum_{i \in J} |v_i^n - v_i^m|^p)^{1/p} \leq |v_\star^n - v_\star^m|_p$, thus by letting $m \rightarrow \infty$ we deduce that $(\sum_{i \in J} |v_i^n - v_i|^p)^{1/p} \leq g_n$. This implies that $|v_\star^n - v_\star|_p \leq g_n$, and accordingly that $\mathbf{d}_{\ell_p(\mathcal{M}_\star)}(v_\star^n, v_\star) \leq \varepsilon$, for every $n \geq \bar{n}$. Therefore, $v_\star^n \rightarrow v_\star$ in $\ell_p(\mathcal{M}_\star)$ as $n \rightarrow \infty$, as desired. ■

With the concept of ℓ_p -sum at our disposal, we can describe products and coproducts in **BanMod** $_{\mathbb{X}}$:

THEOREM 3.11 (Products in **BanMod** $_{\mathbb{X}}$). *Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}_\star = \{\mathcal{M}_i\}_{i \in I}$ be a set of Banach $L^0(\mathbb{X})$ -modules. Then the product of $\mathcal{M}_\star$ in **BanMod** $_{\mathbb{X}}$ exists and is given by*

$$\prod_{i \in I}^{\mathbf{BanMod}_{\mathbb{X}}} \mathcal{M}_i \cong \ell_\infty(\mathcal{M}_\star),$$

together with the morphisms $\pi_i: \ell_\infty(\mathcal{M}_\star) \rightarrow \mathcal{M}_i$ defined as $\pi_i(v_\star) := v_i$ for every $v_\star \in \ell_\infty(\mathcal{M}_\star)$.

PROOF. First of all, observe that each mapping π_i is a morphism in $\mathbf{BanMod}_{\mathbb{X}}$. Now fix a Banach $L^0(\mathbb{X})$ -module $\mathcal{N}$ and a family $\{\varphi_i: \mathcal{N} \rightarrow \mathcal{M}_i\}_{i \in I}$ of morphisms. Let us define $\Phi: \mathcal{N} \rightarrow \ell_\infty(\mathcal{M}_\star)$ as $\Phi(w) := (\varphi_i(w))_{i \in I}$ for every $w \in \mathcal{N}$. Notice that Φ is the unique mapping from $\mathcal{N}$ to $\ell_\infty(\mathcal{M}_\star)$ satisfying $\pi_i \circ \Phi = \varphi_i$ for every $i \in I$. Clearly, Φ is a morphism of $L^0(\mathbb{X})$ -modules. Moreover, by passing to the supremum over $i \in I$, we deduce from $|\varphi_i(w)| \leq |w|$ that $|\Phi(w)|_\infty \leq |w|$ holds for every $w \in \mathcal{N}$. All in all, Φ is a morphism of Banach $L^0(\mathbb{X})$ -modules. The proof is complete. ■

THEOREM 3.12 (Coproducts in $\mathbf{BanMod}_{\mathbb{X}}$). *Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}_\star = \{\mathcal{M}_i\}_{i \in I}$ be a set of Banach $L^0(\mathbb{X})$ -modules. Then the coproduct of $\mathcal{M}_\star$ in $\mathbf{BanMod}_{\mathbb{X}}$ exists and is given by*

$$\coprod_{i \in I}^{\mathbf{BanMod}_{\mathbb{X}}} \mathcal{M}_i \cong \ell_1(\mathcal{M}_\star),$$

together with the morphisms $\iota_i: \mathcal{M}_i \rightarrow \ell_1(\mathcal{M}_\star)$ defined as $\iota_i(v) := (w_j)_{j \in I}$ for every $v \in \mathcal{M}_i$, where we set $w_i := v$ and $w_j := 0_{\mathcal{M}_j}$ for every $j \in I \setminus \{i\}$.

PROOF. First of all, observe that each mapping ι_i is a morphism in $\mathbf{BanMod}_{\mathbb{X}}$ (with $|\iota_i(v)|_1 = |v|$ for every $v \in \mathcal{M}_i$). Now fix a Banach $L^0(\mathbb{X})$ -module $\mathcal{N}$ and a family $\{\varphi_i: \mathcal{M}_i \rightarrow \mathcal{N}\}_{i \in I}$ of morphisms. We claim that, given any element $v_\star \in \ell_1(\mathcal{M}_\star)$, the series $\sum_{i \in I} \varphi_i(v_i)$ is unconditionally convergent in $\mathcal{N}$ to some $\Phi(v_\star) \in \mathcal{N}$. To prove it, pick some partition $(E_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $\mathbb{X}$ that satisfies $\mathfrak{m}(E_n) < +\infty$ and $\chi_{E_n} |v_\star|_1 \leq n$ for every $n \in \mathbb{N}$. For each $J \in \mathcal{P}_F(I)$ we have

$$\begin{aligned} \sum_{i \in J} \int_{E_n} |\varphi_i(v_i)| \, d\mathfrak{m} &\leq \sum_{i \in J} \int_{E_n} |v_i| \, d\mathfrak{m} \\ &\leq \int_{E_n} |v_\star|_1 \, d\mathfrak{m} \leq n \mathfrak{m}(E_n). \end{aligned}$$

It follows that $\sum_{i \in I} \|\chi_{E_n} |\varphi_i(v_i)|\|_{L^1(\mathbb{X})} \leq n \mathfrak{m}(E_n) < +\infty$ for every $n \in \mathbb{N}$, thus in particular the series $\sum_{i \in I} \chi_{E_n} \cdot \varphi_i(v_i)$ is unconditionally convergent in $\mathcal{N}$. Recalling Remark 2.6, we conclude that the series $\sum_{i \in I} \varphi_i(v_i)$ converges unconditionally to some element $\Phi(v_\star) \in \mathcal{N}$, as we claimed.

One can readily check that the resulting mapping $\Phi: \ell_1(\mathcal{M}_\star) \rightarrow \mathcal{N}$ is a morphism of Banach $L^0(\mathbb{X})$ -modules and that it satisfies $\Phi \circ \iota_i = \varphi_i$ for every $i \in I$. It only remains to show that Φ is the unique mapping having these two properties. To this aim, fix a morphism $\Psi: \ell_1(\mathcal{M}_\star) \rightarrow \mathcal{N}$ such that $\Psi \circ \iota_i = \varphi_i$ for every $i \in I$. Given any $v_\star \in \ell_1(\mathcal{M}_\star)$, we have that the series $\sum_{i \in I} \iota_i(v_i)$ converges unconditionally to $v_\star$ in

$\ell_1(\mathcal{M}_\star)$. Hence, the linearity and the continuity of Ψ, Φ yield

$$\begin{aligned}\Psi(v_\star) &= \Psi\left(\sum_{i \in I} \iota_i(v_i)\right) = \sum_{i \in I} \Psi(\iota_i(v_i)) = \sum_{i \in I} \varphi_i(v_i) = \sum_{i \in I} \Phi(\iota_i(v_i)) \\ &= \Phi\left(\sum_{i \in I} \iota_i(v_i)\right) = \Phi(v_\star),\end{aligned}$$

which proves the uniqueness of Φ . Consequently, the statement is achieved. $\blacksquare$

EXAMPLE 3.13. Let $\mathbb{X}$ be a σ -finite measure space. We define $\mathcal{M}_\star = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$ as $\mathcal{M}_n := L^0(\mathbb{X})$ for every $n \in \mathbb{N}$. Then we claim that $\mathcal{M}_\star$ does not have a product in $\mathbf{BanMod}_\mathbb{X}^0$, whence it follows that $\mathbf{BanMod}_\mathbb{X}^0$ is not complete. In order to prove the claim, we argue by contradiction: suppose the product $(\mathcal{Q}, \{\pi_n: \mathcal{Q} \rightarrow \mathcal{M}_n\}_{n \in \mathbb{N}})$ of $\mathcal{M}_\star$ in $\mathbf{BanMod}_\mathbb{X}^0$ exists. In particular, for any $n \in \mathbb{N}$ there exists $g_n \in L^0(\mathbb{X})^+$ such that $|\pi_n(v)| \leq g_n|v|$ for every $v \in \mathcal{Q}$. Define the homomorphism $p_n \in \text{Hom}(\ell_\infty(\mathcal{M}_\star), \mathcal{M}_n)$ as $p_n(f_\star) := n(g_n + \frac{1}{n})f_n$ for every $f_\star \in \ell_\infty(\mathcal{M}_\star)$. Therefore, there exists a unique $\Phi \in \text{Hom}(\ell_\infty(\mathcal{M}_\star), \mathcal{Q})$ such that $p_n = \pi_n \circ \Phi$ for every $n \in \mathbb{N}$. Take any $g \in L^0(\mathbb{X})^+$ such that $|\Phi(f_\star)| \leq g|f_\star|_\infty$ holds for every $f_\star \in \ell_\infty(\mathcal{M}_\star)$. It follows that

$$n|f_n| = \frac{|p_n(f_\star)|}{g_n + \frac{1}{n}} = \frac{|(\pi_n \circ \Phi)(f_\star)|}{g_n + \frac{1}{n}} \leq g|f_\star|_\infty \quad \text{for every } f_\star \in \ell_\infty(\mathcal{M}_\star) \text{ and } n \in \mathbb{N}.$$

Picking $f_n := \chi_X$ for every $n \in \mathbb{N}$, we obtain $n \leq g$ for every $n \in \mathbb{N}$, which leads to a contradiction. Hence, the claim is proved. Similarly, one can also show, by using $\ell_1(\mathcal{M}_\star)$ instead of $\ell_\infty(\mathcal{M}_\star)$, that $\mathcal{M}_\star$ does not have a coproduct in $\mathbf{BanMod}_\mathbb{X}^0$, and thus $\mathbf{BanMod}_\mathbb{X}^0$ is not cocomplete.

3.2.3. The bicompleteness of $\mathbf{BanMod}_\mathbb{X}$. It is now immediate to obtain the main result of this paper:

THEOREM 3.14. *Let $\mathbb{X}$ be a σ -finite measure space. Then the category $\mathbf{BanMod}_\mathbb{X}$ is bicomplete.*

PROOF. It follows from Theorems 3.6, 3.7, 3.11, 3.12, and 2.9. $\blacksquare$

3.2.4. Description of other limits and colimits in $\mathbf{BanMod}_\mathbb{X}$. Theorem 3.14 ensures that inverse/direct limits and pullbacks/pushouts always exist in $\mathbf{BanMod}_\mathbb{X}$. However, we believe it is also useful to describe them explicitly. We shall only write the relevant statements, omitting their proofs.

PROPOSITION 3.15 (Pullbacks in $\mathbf{BanMod}_\mathbb{X}$). *Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}, \mathcal{N}$, and $\mathcal{Q}$ be Banach $L^0(\mathbb{X})$ -modules. Let $\varphi: \mathcal{M} \rightarrow \mathcal{Q}$ and $\psi: \mathcal{N} \rightarrow \mathcal{Q}$ be morphisms in*

BanMod $_{\mathbb{X}}$. Then $\{(v, w) \in \mathcal{M} \oplus_{\infty} \mathcal{N} \mid \varphi(v) = \psi(w)\}$ is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{M} \otimes_{\infty} \mathcal{N}$. Moreover, the pullback of φ and ψ in **BanMod $_{\mathbb{X}}$** is given by

$$\mathcal{M} \times_{\mathcal{Q}} \mathcal{N} \cong \{(v, w) \in \mathcal{M} \oplus_{\infty} \mathcal{N} \mid \varphi(v) = \psi(w)\},$$

together with the morphisms

$$\begin{aligned} p_{\mathcal{M}} &:= \pi_{\mathcal{M}}|_{\mathcal{M} \times_{\mathcal{Q}} \mathcal{N}}: \mathcal{M} \times_{\mathcal{Q}} \mathcal{N} \rightarrow \mathcal{M}, \\ p_{\mathcal{N}} &:= \pi_{\mathcal{N}}|_{\mathcal{M} \times_{\mathcal{Q}} \mathcal{N}}: \mathcal{M} \times_{\mathcal{Q}} \mathcal{N} \rightarrow \mathcal{N}, \end{aligned}$$

where $(\mathcal{M} \oplus_{\infty} \mathcal{N}, \pi_{\mathcal{M}}, \pi_{\mathcal{N}})$ stands for the product of $\{\mathcal{M}, \mathcal{N}\}$ in **BanMod $_{\mathbb{X}}$** .

PROPOSITION 3.16 (Pushouts in **BanMod $_{\mathbb{X}}$).** *Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}, \mathcal{N}, \mathcal{Q}$ be Banach $L^0(\mathbb{X})$ -modules. Let $\varphi: \mathcal{Q} \rightarrow \mathcal{M}$ and $\psi: \mathcal{Q} \rightarrow \mathcal{N}$ be morphisms in **BanMod $_{\mathbb{X}}$** . Then*

$$\mathcal{J} := \{(v, w) \in \mathcal{M} \oplus_1 \mathcal{N} \mid (\varphi(z), \psi(z)) = (-v, w) \text{ for some } z \in \mathcal{Q}\}$$

is a normed $L^0(\mathbb{X})$ -submodule of $\mathcal{M} \oplus_1 \mathcal{N}$. Moreover, the pushout of φ and ψ in **BanMod $_{\mathbb{X}}$** is given by

$$\mathcal{M} \sqcup_{\mathcal{Q}} \mathcal{N} \cong (\mathcal{M} \oplus_1 \mathcal{N}) / \text{cl}_{\mathcal{M} \oplus_1 \mathcal{N}}(\mathcal{J}),$$

together with the morphisms

$$\begin{aligned} i_{\mathcal{M}} &:= \pi \circ \iota_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M} \sqcup_{\mathcal{Q}} \mathcal{N}, \\ i_{\mathcal{N}} &:= \pi \circ \iota_{\mathcal{N}}: \mathcal{N} \rightarrow \mathcal{M} \sqcup_{\mathcal{Q}} \mathcal{N}, \end{aligned}$$

where $(\mathcal{M} \oplus_1 \mathcal{N}, \iota_{\mathcal{M}}, \iota_{\mathcal{N}})$ stands for the coproduct of $\{\mathcal{M}, \mathcal{N}\}$ in the category **BanMod $_{\mathbb{X}}$** , while by $\pi: \mathcal{M} \oplus_1 \mathcal{N} \rightarrow \mathcal{M} \sqcup_{\mathcal{Q}} \mathcal{N}$ we mean the canonical projection map on the quotient.

Furthermore, by combining Proposition 3.16 with Theorem 3.6 and, respectively, Proposition 3.15 with Theorem 3.7, one obtains the following two results:

COROLLARY 3.17 (Images in **BanMod $_{\mathbb{X}}$).** *Let $\mathbb{X}$ be a σ -finite measure space and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ a morphism in **BanMod $_{\mathbb{X}}$** . Recall that $\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M}))$ is a Banach $L^0(\mathbb{X})$ -submodule of $\mathcal{N}$ and that the inclusion map $\iota: \text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})) \hookrightarrow \mathcal{N}$ is a morphism in **BanMod $_{\mathbb{X}}$** . Then it holds that*

$$(\text{Im}(\varphi), \text{im}(\varphi)) \cong (\text{cl}_{\mathcal{N}}(\varphi(\mathcal{M})), \iota).$$

COROLLARY 3.18 (Coimages in **BanMod $_{\mathbb{X}}$).** *Let $\mathbb{X}$ be a σ -finite measure space and $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ a morphism in **BanMod $_{\mathbb{X}}$** . Consider the quotient Banach $L^0(\mathbb{X})$ -module*

$\mathcal{M}/\text{Ker}(\varphi)$ and recall that the canonical projection $\pi: \mathcal{M} \rightarrow \mathcal{M}/\text{Ker}(\varphi)$ is a morphism in $\mathbf{BanMod}_{\mathbb{X}}$. Then it holds that

$$(\text{Coim}(\varphi), \text{coim}(\varphi)) \cong (\mathcal{M}/\text{Ker}(\varphi), \pi).$$

Finally, we provide an explicit description of inverse and direct limits in $\mathbf{BanMod}_{\mathbb{X}}$. Recall that inverse and direct limits always exist in the category $R\text{-Mod}$ of modules over a commutative ring R ; see e.g. [18]. We shall denote by $f_{\mathbb{X}}: \mathbf{BanMod}_{\mathbb{X}} \rightarrow L^0(\mathbb{X})\text{-Mod}$ the forgetful functor.

PROPOSITION 3.19 (Inverse limits in $\mathbf{BanMod}_{\mathbb{X}}$). *Let $(I, \leq)$ be a directed set. Let $\mathbb{X}$ be a σ -finite measure space. Let $(\{\mathcal{M}_i\}_{i \in I}, \{P_{ij}\}_{i \leq j})$ be an inverse system in $\mathbf{BanMod}_{\mathbb{X}}$. We denote by $(M, \{\tilde{P}_i\}_{i \in I})$ the inverse limit of $(\{f_{\mathbb{X}}(\mathcal{M}_i)\}_{i \in I}, \{f_{\mathbb{X}}(P_{ij})\}_{i \leq j})$ in $L^0(\mathbb{X})\text{-Mod}$. Let us define the mapping $|\cdot|_M: M \rightarrow L^0_{\text{ext}}(\mathbb{X})$ as*

$$|v|_M := \bigvee_{i \in I} |\tilde{P}_i(v)| \quad \text{for every } v \in M.$$

Then $\tilde{M} := \{v \in M : |v|_M \in L^0(\mathbb{X})\}$ is an $L^0(\mathbb{X})$ -submodule of M . Moreover, $\tilde{M}$ is a Banach $L^0(\mathbb{X})$ -module if endowed with the restriction of $|\cdot|_M$, and the inverse limit of $(\{\mathcal{M}_i\}_{i \in I}, \{P_{ij}\}_{i \leq j})$ in $\mathbf{BanMod}_{\mathbb{X}}$ is given by

$$\varprojlim \mathcal{M}_\star \cong \tilde{M},$$

together with the morphisms $P_i := \tilde{P}_i|_{\tilde{M}}: \tilde{M} \rightarrow \mathcal{M}_i$.

PROPOSITION 3.20 (Direct limits in $\mathbf{BanMod}_{\mathbb{X}}$). *Let $(I, \leq)$ be a directed set and $\mathbb{X}$ a σ -finite measure space. Let $(\{\mathcal{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ be a direct system in $\mathbf{BanMod}_{\mathbb{X}}$. Let us denote by $(M, \{\tilde{\varphi}_i\}_{i \in I})$ the direct limit of $(\{f_{\mathbb{X}}(\mathcal{M}_i)\}_{i \in I}, \{f_{\mathbb{X}}(\varphi_{ij})\}_{i \leq j})$ in $L^0(\mathbb{X})\text{-Mod}$. Then*

$$|w| := \bigwedge \{ |v| \mid i \in I, v \in \mathcal{M}_i, \tilde{\varphi}_i(v) = w \} \quad \text{for every } w \in M$$

defines a pointwise seminorm $|\cdot|: M \rightarrow L^0(\mathbb{X})$, so that M is a seminormed $L^0(\mathbb{X})$ -module, its quotient $M/\sim_{|\cdot|}$ is a normed $L^0(\mathbb{X})$ -module, and $\overline{M/\sim_{|\cdot|}}$ is a Banach $L^0(\mathbb{X})$ -module, where we denote by $(\overline{M/\sim_{|\cdot|}}, \iota)$ the completion of $M/\sim_{|\cdot|}$. Moreover, the direct limit of $(\{\mathcal{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ in $\mathbf{BanMod}_{\mathbb{X}}$ is given by

$$\varinjlim \mathcal{M}_\star \cong \overline{M/\sim_{|\cdot|}},$$

together with the morphisms $\varphi_i := \iota \circ \pi \circ \tilde{\varphi}_i: \mathcal{M}_i \rightarrow \overline{M/\sim_{|\cdot|}}$, where $\pi: M \rightarrow M/\sim_{|\cdot|}$ is the canonical projection on the quotient.

Next we provide an elementary example of a nontrivial inverse system of Banach $L^0(\mathbb{X})$ -modules having a trivial inverse limit. This construction will be useful in Example 3.22.

EXAMPLE 3.21. Let $\mathbb{X}$ be a σ -finite measure space and $\mathcal{M} \neq \{0\}$ a given Banach $L^0(\mathbb{X})$ -module. We define $\mathcal{M}_n := \mathcal{M}$ for every $n \in \mathbb{N}$. Given any $n, m \in \mathbb{N}$ with $n \leq m$, we define the morphism $P_{nm}: \mathcal{M}_m \rightarrow \mathcal{M}_n$ as $P_{nm} := \frac{n}{m} \text{id}_{\mathcal{M}}$. Then $(\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{P_{nm}\}_{n \leq m})$ is an inverse system and

$$(3.4) \quad \varprojlim \mathcal{M}_\star \cong \{0\}.$$

Indeed, the inverse limit $(M, \{\tilde{P}_n\}_{n \in \mathbb{N}})$ of $(\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{P_{nm}\}_{n \leq m})$ in the category $L^0(\mathbb{X})\text{-}\mathbf{Mod}$ is given by $M := \{(kv)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}}^{\text{Set}} \mathcal{M}_k : v \in \mathcal{M}\}$, together with the morphisms $\tilde{P}_n: M \rightarrow \mathcal{M}_n$ defined as

$$\tilde{P}_n((kv)_{k \in \mathbb{N}}) := nv \quad \text{for every } (kv)_{k \in \mathbb{N}} \in M.$$

Using Proposition 3.19, we get $|(kv)_{k \in \mathbb{N}}| = \bigvee_{n \in \mathbb{N}} |nv| = (+\infty) \cdot \chi_{\{|v|>0\}}$, which gives (3.4).

We know that the inverse limit functor preserves limits, whereas the direct limit functor preserves colimits. On the contrary, the following two examples show that in $\mathbf{BanMod}_{\mathbb{X}}$ the inverse limit functor is not right exact and that the direct limit functor is not left exact, respectively.

EXAMPLE 3.22. Fix any σ -finite measure space $\mathbb{X}$ and any Banach $L^0(\mathbb{X})$ -module $\mathcal{M} \neq \{0\}$. Given any $n \in \mathbb{N}$, we define $\mathcal{M}_n = \mathcal{N}_n := \mathcal{M}$. Given any $n, m \in \mathbb{N}$ with $n \leq m$, we define $P_{nm}: \mathcal{M}_m \rightarrow \mathcal{M}_n$ and $Q_{nm}: \mathcal{N}_m \rightarrow \mathcal{N}_n$ as $P_{nm} := \frac{n}{m} \text{id}_{\mathcal{M}}$ and $Q_{nm} := \text{id}_{\mathcal{M}}$, respectively. Then both $(\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{P_{nm}\}_{n \leq m})$ and $(\{\mathcal{N}_n\}_{n \in \mathbb{N}}, \{Q_{nm}\}_{n \leq m})$ are inverse systems in $\mathbf{BanMod}_{\mathbb{X}}$ and the collection $\theta_\star$ of morphisms $\theta_n: \mathcal{M}_n \rightarrow \mathcal{N}_n$ given by $\theta_n := \frac{1}{n} \text{id}_{\mathcal{M}}$ is a natural transformation, i.e. a morphism in $(\mathbf{BanMod}_{\mathbb{X}})^{(I, \leq)^{\text{op}}}$. Observe that trivially $\varprojlim_{\mathbb{N}} \mathcal{N}_\star \cong \mathcal{M}$, while $\varprojlim_{\mathbb{N}} \mathcal{M}_\star \cong \{0\}$ thanks to Example 3.21. Recalling Remark 2.10 and using the surjectivity of each θ_n , we obtain that $\text{Coker}(\theta_\star)(n) = \{0\}$ for all $n \in \mathbb{N}$. Hence, $\text{Coker}(\theta_\star)$ is the zero object of $(\mathbf{BanMod}_{\mathbb{X}})^{(I, \leq)^{\text{op}}}$ and thus $\varprojlim_{\mathbb{N}} \text{Coker}(\theta_\star) \cong \{0\}$. On the other hand, the cokernel of the morphism $\varprojlim_{\mathbb{N}} \theta_\star$ is

$$\begin{aligned} \text{Coker}(\varprojlim_{\mathbb{N}} \theta_\star) &\cong (\varprojlim_{\mathbb{N}} \mathcal{N}_\star) / \text{cl}_{(\varprojlim_{\mathbb{N}} \mathcal{N}_\star)}((\varprojlim_{\mathbb{N}} \theta_\star)(\varprojlim_{\mathbb{N}} \mathcal{M}_\star)) \cong \varprojlim_{\mathbb{N}} \mathcal{N}_\star \cong \mathcal{M} \\ &\neq \{0\}, \end{aligned}$$

thus the inverse limit functor $\varprojlim_{\mathbb{N}}$ on $(\mathbf{BanMod}_{\mathbb{X}})^{(I, \leq)^{\text{op}}}$ does not preserve cokernels.

EXAMPLE 3.23. Fix a σ -finite measure space $\mathbb{X}$. Let $\theta: \mathcal{H}_{\mathbb{X}}(\mathbb{N}) \rightarrow \mathcal{H}_{\mathbb{X}}(\mathbb{N})$ be the following morphism: given any $v \in \mathcal{H}_{\mathbb{X}}(\mathbb{N})$, we set $\theta(v)(1) := 0$ and $\theta(v)(i) := v(i)$ for every $i \in \mathbb{N}$ with $i \geq 2$. We define $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{\mathbb{X}}(\mathbb{N})$ as $v_1(i) := \frac{1}{i} \chi_X$ for every $i \in \mathbb{N}$ and $v_n := e_n$ for every $n \geq 2$. Moreover, we denote by $\mathcal{M}_n$ the closure in $\mathcal{H}_{\mathbb{X}}(\mathbb{N})$ of its $L^0(\mathbb{X})$ -submodule generated by $\{v_1, \dots, v_n\}$, while we define $\mathcal{N}_n := \mathcal{H}_{\mathbb{X}}(\mathbb{N})$. Finally, for any $n \leq m$ we denote by $\varphi_{nm}: \mathcal{M}_n \hookrightarrow \mathcal{M}_m$ the inclusion, by $\psi_{nm}: \mathcal{N}_n \rightarrow \mathcal{N}_m$ the identity map, and by $\theta_n: \mathcal{M}_n \rightarrow \mathcal{N}_n$ the map $\theta_n := \theta|_{\mathcal{M}_n}$. It holds that $(\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{\varphi_{nm}\}_{n \leq m})$ and $(\{\mathcal{N}_n\}_{n \in \mathbb{N}}, \{\psi_{nm}\}_{n \leq m})$ are direct systems in **BanMod** $_{\mathbb{X}}$, while θ_* is a morphism in **(BanMod)** $_{\mathbb{X}}^{(\mathbb{N}, \leq)}$ between them. Recalling Remark 2.10, we deduce that $\text{Ker}(\theta_*)$ is the zero object, so that $\lim_{\rightarrow \mathbb{N}} \text{Ker}(\theta_*) = \{0\}$. On the other hand, since the $L^0(\mathbb{X})$ -submodule of $\mathcal{H}_{\mathbb{X}}(\mathbb{N})$ generated by the sequence $(v_n)_{n \in \mathbb{N}}$ is dense in $\mathcal{H}_{\mathbb{X}}(\mathbb{N})$, one can easily check that $\lim_{\rightarrow \mathbb{N}} \mathcal{M}_* \cong \mathcal{H}_{\mathbb{X}}(\mathbb{N})$. Clearly, $\lim_{\rightarrow \mathbb{N}} \mathcal{N}_* \cong \mathcal{H}_{\mathbb{X}}(\mathbb{N})$ as well and $\lim_{\rightarrow \mathbb{N}} \theta_* \cong \theta$. Since θ is not injective, we conclude that $\text{Ker}(\lim_{\rightarrow \mathbb{N}} \theta_*) \neq \{0\}$, thus showing that the direct limit functor $\lim_{\rightarrow \mathbb{N}}$ on **(BanMod)** $_{\mathbb{X}}^{(\mathbb{N}, \leq)}$ does not preserve kernels.

3.3 – The inverse image functor

Here we introduce and study the “inverse image functor”.

3.3.1. The category BanMod. Let us now consider the category **BanMod**, which is defined as follows:

- (1) The objects of **BanMod** are given by the couples $(\mathbb{X}, \mathcal{M})$, where $\mathbb{X}$ is a σ -finite measure space and $\mathcal{M}$ is a Banach $L^0(\mathbb{X})$ -module.
- (2) A morphism in **BanMod** between two objects $(\mathbb{X}, \mathcal{M})$ and $(\mathbb{Y}, \mathcal{N})$ is a couple (τ, φ) , where $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ is a morphism in **Meas** $_{\sigma}$ and $\varphi: \mathcal{N} \rightarrow \mathcal{M}$ is a linear map such that

$$\begin{aligned} \varphi(f \cdot v) &= (f \circ \tau) \cdot \varphi(v) && \text{for every } f \in L^0(\mathbb{Y}) \text{ and } v \in \mathcal{N}, \\ |\varphi(v)| &\leq |v| \circ \tau && \text{for every } v \in \mathcal{N}. \end{aligned}$$

- (3) Given morphisms $(\tau, \varphi): (\mathbb{X}, \mathcal{M}) \rightarrow (\mathbb{Y}, \mathcal{N})$ and $(\eta, \psi): (\mathbb{Y}, \mathcal{N}) \rightarrow (\mathbb{W}, \mathcal{Q})$ in **BanMod**, their composition is $(\eta, \psi) \circ (\tau, \varphi) := (\eta \circ \tau, \varphi \circ \psi): (\mathbb{X}, \mathcal{M}) \rightarrow (\mathbb{W}, \mathcal{Q})$.

It holds that **Meas** $_{\sigma}$ and **Ban** $^{\text{op}}$ can be realised as full subcategories of **BanMod**, while each **BanMod** $_{\mathbb{X}}^{\text{op}}$ can be realised as a (not necessarily full) subcategory of **BanMod**. More precisely, we have the following:

- (1) Define the functor $I_M: \mathbf{Meas}_{\sigma} \rightarrow \mathbf{BanMod}$ as $I_M(\mathbb{X}) := (\mathbb{X}, \{0\})$ for every object $\mathbb{X}$ of **Meas** $_{\sigma}$ and $I_M(\tau) := (\tau, 0): (\mathbb{X}, \{0\}) \rightarrow (\mathbb{Y}, \{0\})$ for every morphism $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ in **Meas** $_{\sigma}$. Then I_M is a fully faithful functor that is injective on objects.

- (2) Let $\mathbb{X}$ be a σ -finite measure space. Define the functor $I_{\mathbb{X}}: \mathbf{BanMod}_{\mathbb{X}}^{\text{op}} \rightarrow \mathbf{BanMod}$ as $I_{\mathbb{X}}(\mathcal{M}) := (\mathbb{X}, \mathcal{M})$ for every object $\mathcal{M}$ of $\mathbf{BanMod}_{\mathbb{X}}$ and $I_{\mathbb{X}}(\varphi) := (\text{id}_{\mathbb{X}}, \varphi)$ for every morphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{BanMod}_{\mathbb{X}}^{\text{op}}$. Then $I_{\mathbb{X}}$ is faithful and injective on objects.
- (3) Recall that $\mathbf{BanMod}_{\mathbb{P}} = \mathbf{Ban}$ and notice that $\text{id}_{\mathbb{P}}: \mathbb{P} \rightarrow \mathbb{P}$ is the unique element of $\text{Hom}_{\mathbf{Meas}_{\sigma}}(\mathbb{P}, \mathbb{P})$, thus $I_{\mathbb{P}}$ is a full functor and $\mathbf{Ban}^{\text{op}}$ can be realised as a full subcategory of $\mathbf{BanMod}$.

We shall also consider the forgetful functor $\Pi_M: \mathbf{BanMod} \rightarrow \mathbf{Meas}_{\sigma}$, defined as

$$\begin{aligned} \Pi_M((\mathbb{X}, \mathcal{M})) &:= \mathbb{X} \quad \text{for every object } (\mathbb{X}, \mathcal{M}) \text{ of } \mathbf{BanMod}, \\ \Pi_M((\tau, \varphi)) &:= \tau \quad \text{for every morphism } (\tau, \varphi): (\mathbb{X}, \mathcal{M}) \rightarrow (\mathbb{Y}, \mathcal{N}) \text{ in } \mathbf{BanMod}. \end{aligned}$$

3.3.2. Inverse image versus pullback. We are now in a position to introduce the inverse image functor:

THEOREM 3.24 (Inverse image functor). *There exists a unique functor*

$$\text{InvIm}: (\text{id}_{\mathbf{Meas}_{\sigma}} \downarrow \Pi_M) \rightarrow \mathbf{BanMod}^{\rightarrow},$$

which we call the inverse image functor, such that the following properties are satisfied:

- (1) If $(\mathbb{X}, (\mathbb{Y}, \mathcal{M}), \tau)$ is a given object of the comma category $(\text{id}_{\mathbf{Meas}_{\sigma}} \downarrow \Pi_M)$, then there exist a Banach $L^0(\mathbb{X})$ -module $\tau^*\mathcal{M}$ and a linear operator $\tau^*: \mathcal{M} \rightarrow \tau^*\mathcal{M}$ such that

$$\text{InvIm}((\mathbb{X}, (\mathbb{Y}, \mathcal{M}), \tau)) = (\tau, \tau^*).$$

Moreover, the $L^0(\mathbb{X})$ -submodule of $\tau^*\mathcal{M}$ generated by the range $\tau^*(\mathcal{M})$ is dense in $\tau^*\mathcal{M}$ and it holds that $|\tau^*v| = |v| \circ \tau$ for every $v \in \mathcal{M}$.

- (2) If $(\eta_1, (\eta_2, \varphi)): (\mathbb{X}_1, (\mathbb{Y}_1, \mathcal{M}_1), \tau_1) \rightarrow (\mathbb{X}_2, (\mathbb{Y}_2, \mathcal{M}_2), \tau_2)$ is a morphism in the category $(\text{id}_{\mathbf{Meas}_{\sigma}} \downarrow \Pi_M)$, then there exists an operator $\psi: \tau_2^*\mathcal{M}_2 \rightarrow \tau_1^*\mathcal{M}_1$ such that

$$\text{InvIm}((\eta_1, (\eta_2, \varphi))) = ((\eta_1, \psi), (\eta_2, \varphi)).$$

The uniqueness of the functor InvIm is intended up to a unique natural isomorphism in the functor category from $(\text{id}_{\mathbf{Meas}_{\sigma}} \downarrow \Pi_M)$ to $\mathbf{BanMod}^{\rightarrow}$.

The statement of Theorem 3.24 is a reformulation of various results about “pullback modules” contained in [9, Section 1.6], or rather of their corresponding versions for Banach L^0 -modules, which can be obtained by suitably adapting the same proof arguments. Let us briefly comment on the terminology: in [9] the Banach module $\tau^*\mathcal{M}$ is called the *pullback module* of $\mathcal{M}$ with respect to τ , by analogy with the notion of pullback that is commonly used in differential geometry. Since in this paper we

are studying these topics from the perspective of category theory, we prefer to adopt the term “inverse image”, in order to avoid confusion with the categorical notion of pullback. Nevertheless, the two concepts are strongly related, as observed in [9, Remark 1.6.4]. Namely, we have the following theorem:

THEOREM 3.25. *Let $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ be a morphism in $\mathbf{Meas}_\sigma$. Let $\mathcal{M}$ be a Banach $L^0(\mathbb{Y})$ -module. Then it holds that the pullback of $(\text{id}_\mathbb{Y}, 0): (\mathbb{Y}, \mathcal{M}) \rightarrow (\mathbb{Y}, \{0\})$ and $(\tau, 0): (\mathbb{X}, \{0\}) \rightarrow (\mathbb{Y}, \{0\})$ exists in $\mathbf{BanMod}$:*

$$(\mathbb{Y}, \mathcal{M}) \times_{(\mathbb{Y}, \{0\})} (\mathbb{X}, \{0\}) \cong (\mathbb{X}, \tau^* \mathcal{M}),$$

together with $(\tau, \tau^*): (\mathbb{X}, \tau^* \mathcal{M}) \rightarrow (\mathbb{Y}, \mathcal{M})$ and $(\text{id}_\mathbb{X}, 0): (\mathbb{X}, \tau^* \mathcal{M}) \rightarrow (\mathbb{X}, \{0\})$.

Given a morphism $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ in $\mathbf{Meas}_\sigma$, the inverse image functor induces a functor

$$\text{InvIm}_\tau: \mathbf{BanMod}_\mathbb{Y} \rightarrow \mathbf{BanMod}_\mathbb{X}$$

as follows: for any Banach $L^0(\mathbb{Y})$ -module $\mathcal{M}$ we define $\text{InvIm}_\tau(\mathcal{M}) := \tau^* \mathcal{M}$ and for any morphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{BanMod}_\mathbb{Y}$ we define $\text{InvIm}_\tau(\varphi) := \tau^* \varphi$, where by $\tau^* \varphi: \tau^* \mathcal{M} \rightarrow \tau^* \mathcal{N}$ we mean the unique morphism in $\mathbf{BanMod}_\mathbb{X}$ with

$$\text{InvIm}((\text{id}_\mathbb{X}, (\text{id}_\mathbb{Y}, \varphi))) = ((\text{id}_\mathbb{X}, \tau^* \varphi), (\text{id}_\mathbb{Y}, \varphi)).$$

EXAMPLE 3.26. Let $\mathbb{X}$ be a finite measure space. Let $\mathcal{M}$ be a given Banach $L^0(\mathbb{Y})$ -module, for some σ -finite measure space $\mathbb{Y}$. We define $\pi_Y: X \times Y \rightarrow Y$ as $\pi_Y(x, y) := y$ for every $(x, y) \in X \times Y$. Clearly, π_Y is measurable and $(\pi_Y)_\#(\mathfrak{m}_X \otimes \mathfrak{m}_Y) = \lambda \mathfrak{m}_Y$, where we define $\lambda := \mathfrak{m}_X(X) \in (0, +\infty)$, so that $\pi_Y: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$ is a morphism in $\mathbf{Meas}_\sigma$. We define $c: \mathcal{M} \rightarrow L^0(\mathbb{X}; \mathcal{M})$ as $c(v) := \chi_X v$ for every $v \in \mathcal{M}$. Then we claim that

$$(L^0(\mathbb{X}; \mathcal{M}), c) \cong (\pi_Y^* \mathcal{M}, \pi_Y^*).$$

Its validity follows from the linearity of c , the fact that for any $v \in \mathcal{M}$ the identities

$$|c(v)|(x, y) = |c(v)(x)|(y) = |v|(y) = |v|(\pi_Y(x, y)) = (|v| \circ \pi_Y)(x, y)$$

are verified for $(\mathfrak{m}_X \otimes \mathfrak{m}_Y)$ -a.e. (x, y) , and the density of simple maps in $L^0(\mathbb{X}; \mathcal{M})$.

In the following result, we check (by a direct verification) that InvIm_τ preserves direct limits:

PROPOSITION 3.27. *Let $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ be a morphism in $\mathbf{Meas}_\sigma$. Let us consider a direct system $(\{\mathcal{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ in $\mathbf{BanMod}_\mathbb{Y}$, with direct limit $(\varinjlim \mathcal{M}_i, \{\varphi_i\}_{i \in I})$.*

Then $(\{\tau^*\mathcal{M}_i\}_{i \in I}, \{\tau^*\varphi_{ij}\}_{i \leq j})$ is a direct system in $\mathbf{BanMod}_{\mathbb{X}}$, whose direct limit is given by

$$(3.5) \quad \varinjlim \tau^*\mathcal{M}_\star \cong \tau^* \varinjlim \mathcal{M}_\star,$$

together with the family $\{\tau^*\varphi_i\}_{i \in I}$ of morphisms $\tau^*\varphi_i: \tau^*\mathcal{M}_i \rightarrow \tau^* \varinjlim \mathcal{M}_\star$.

PROOF. Clearly, we have $(\tau^*\varphi_j) \circ (\tau^*\varphi_{ij}) = \tau^*\varphi_i$ for all $i, j \in I$ with $i \leq j$. Fix any $(\mathcal{N}, \{\psi_i\}_{i \in I})$, where $\mathcal{N}$ is a Banach $L^0(\mathbb{X})$ -module and $\psi_i: \tau^*\mathcal{M}_i \rightarrow \mathcal{N}$ are morphisms satisfying $\psi_j \circ (\tau^*\varphi_{ij}) = \psi_i$ for every $i \leq j$. We claim that there exists a unique morphism $\Phi: \tau^* \varinjlim \mathcal{M}_\star \rightarrow \mathcal{N}$ such that

$$(3.6) \quad \Phi(\tau^*(\varphi_i(v))) = \psi_i(\tau^*v) \quad \text{for every } i \in I \text{ and } v \in \mathcal{M}_i,$$

whence the statement follows. Given that $\bigcup_{i \in I} \varphi_i(\mathcal{M}_i)$ is a dense $L^0(\mathbb{Y})$ -submodule of $\varinjlim \mathcal{M}_\star$ by Proposition 3.20, the $L^0(\mathbb{X})$ -submodule of $\tau^* \varinjlim \mathcal{M}_\star$ generated by $\bigcup_{i \in I} \{\tau^*(\varphi_i(v)) : v \in \mathcal{M}_i\}$ is dense in $\tau^* \varinjlim \mathcal{M}_\star$, which forces the uniqueness of the morphism Φ . We now pass to the verification of (the well-posedness and of) the existence of Φ . We denote by $(M, \{\tilde{\varphi}_i\}_{i \in I})$ the direct limit of $(\{f_{\mathbb{Y}}(\mathcal{M}_i)\}_{i \in I}, \{f_{\mathbb{Y}}(\varphi_{ij})\}_{i \leq j})$ in $L^0(\mathbb{Y})\text{-}\mathbf{Mod}$. Note that the map $\tilde{\Psi}: M \rightarrow \mathcal{N}$, which we define as $\tilde{\Psi}(\tilde{\varphi}_i(v)) := \psi_i(\tau^*v)$ for every $i \in I$ and $v \in \mathcal{M}_i$, is well posed and linear. Indeed, if $z \in M$ can be written as $z = \tilde{\varphi}_i(v) = \tilde{\varphi}_j(w)$, then (recalling how direct limits in $L^0(\mathbb{Y})\text{-}\mathbf{Mod}$ are constructed) we have $\varphi_{ik}(v) = \varphi_{jk}(w)$ for some $k \in I$ with $i, j \leq k$, so that the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{M}_i & \xrightarrow{\varphi_{ik}} & \mathcal{M}_k & \xleftarrow{\varphi_{jk}} & \mathcal{M}_j \\ \tau^* \downarrow & & \tau^* \downarrow & & \tau^* \downarrow \\ \tau^*\mathcal{M}_i & \xrightarrow{\tau^*\varphi_{ik}} & \tau^*\mathcal{M}_k & \xleftarrow{\tau^*\varphi_{jk}} & \tau^*\mathcal{M}_j \\ & \searrow \psi_i & \downarrow \psi_k & \swarrow \psi_j & \\ & & \mathcal{N} & & \end{array}$$

implies that $\psi_i(\tau^*v) = \psi_j(\tau^*w)$, thus showing that $\tilde{\Psi}(z)$ is well posed. The linearity of $\tilde{\Psi}$ follows by construction. Moreover, we can estimate $|\tilde{\Psi}(z)| \leq |v| \circ \tau$ for every $z \in M, i \in I$, and $v \in \mathcal{M}_i$ with $\tilde{\varphi}_i(v) = z$, whence it follows that $|\tilde{\Psi}(z)| \leq |z| \circ \tau$ for every $z \in M$. It is then easy to check that there is a unique linear map $\Psi: \overline{M/\sim_{|\cdot|}} \rightarrow \mathcal{N}$ satisfying $\Psi((\iota \circ \pi)(z)) = \tilde{\Psi}(z)$ for every $z \in M$, where $\pi: M \rightarrow M/\sim_{|\cdot|}$ is the projection and $(\overline{M/\sim_{|\cdot|}}, \iota)$ is the completion of $M/\sim_{|\cdot|}$. Notice also that $|\Psi(z)| \leq |z| \circ \tau$

for every $z \in \overline{M/\sim_{|\cdot|}}$. Recalling from Proposition 3.20 that $\varinjlim \mathcal{M}_\star \cong \overline{M/\sim_{|\cdot|}}$, we conclude that there exists a unique morphism $\Phi: \tau^* \varinjlim \mathcal{M}_\star \rightarrow \mathcal{N}$ for which the diagram

$$\begin{array}{ccc} \varinjlim \mathcal{M}_\star & \xrightarrow{\Psi} & \mathcal{N} \\ \tau^* \downarrow & \nearrow \Phi & \\ \tau^* \varinjlim \mathcal{M}_\star & & \end{array}$$

commutes, which is equivalent to requiring the validity of (3.6). This proves the statement. $\blacksquare$

It also holds that if $\tau: \mathbb{X} \rightarrow \mathbb{Y}$ is a morphism in $\mathbf{Meas}_\sigma$ and $(\{\mathcal{M}_i\}_{i \in I}, \{\mathcal{P}_{ij}\}_{i \leq j})$ is an inverse system in $\mathbf{BanMod}_{\mathbb{Y}}$, then $(\{\tau^* \mathcal{M}_i\}_{i \in I}, \{\tau^* \mathcal{P}_{ij}\}_{i \leq j})$ is an inverse system in $\mathbf{BanMod}_{\mathbb{X}}$. Nevertheless, it can happen that $\varprojlim \tau^* \mathcal{M}_\star \not\cong \tau^* \varprojlim \mathcal{M}_\star$. In other words, the functor InvIm_τ does not necessarily preserve inverse limits, as is shown in Example 4.3 below.

4. The hom-functors in $\mathbf{BanMod}_{\mathbb{X}}$

Let $\mathbb{X}$ be a given σ -finite measure space. Let $\mathcal{M}, \mathcal{N}$ be Banach $L^0(\mathbb{X})$ -modules. The functors

$$\text{Hom}(\mathcal{M}, -): \mathbf{BanMod}_{\mathbb{X}} \rightarrow \mathbf{BanMod}_{\mathbb{X}}, \quad \text{Hom}(-, \mathcal{N}): \mathbf{BanMod}_{\mathbb{X}}^{\text{op}} \rightarrow \mathbf{BanMod}_{\mathbb{X}},$$

which we call the *hom-functors* in $\mathbf{BanMod}_{\mathbb{X}}$, are defined in the following way:

- (1) Given an object $\mathcal{Q}$ of $\mathbf{BanMod}_{\mathbb{X}}$, we define $\text{Hom}(\mathcal{M}, -)(\mathcal{Q}) := \text{Hom}(\mathcal{M}, \mathcal{Q})$. Given a morphism $\varphi: \mathcal{Q} \rightarrow \mathcal{R}$ in $\mathbf{BanMod}_{\mathbb{X}}$, we define

$$\text{Hom}(\mathcal{M}, -)(\varphi): \text{Hom}(\mathcal{M}, \mathcal{Q}) \rightarrow \text{Hom}(\mathcal{M}, \mathcal{R})$$

as $\text{Hom}(\mathcal{M}, -)(\varphi)(T) := \varphi \circ T$ for every $T \in \text{Hom}(\mathcal{M}, \mathcal{Q})$.

- (2) Given an object $\mathcal{Q}$ of $\mathbf{BanMod}_{\mathbb{X}}$, we define $\text{Hom}(-, \mathcal{N})(\mathcal{Q}) := \text{Hom}(\mathcal{Q}, \mathcal{N})$. Given a morphism $\varphi: \mathcal{R} \rightarrow \mathcal{Q}$ in $\mathbf{BanMod}_{\mathbb{X}}$, we define

$$\text{Hom}(-, \mathcal{N})(\varphi^{\text{op}}): \text{Hom}(\mathcal{Q}, \mathcal{N}) \rightarrow \text{Hom}(\mathcal{R}, \mathcal{N})$$

as $\text{Hom}(-, \mathcal{N})(\varphi^{\text{op}})(T) := T \circ \varphi$ for every $T \in \text{Hom}(\mathcal{Q}, \mathcal{N})$.

In the following result, we obtain the expected continuity properties of the two hom-functors. We prove the statement directly, rather than by applying a general principle, which would amount to showing that hom-functors are left/right adjoints to a suitable

notion of tensor product. For a study of tensor products of Banach L^0 -modules, we refer to [24].

PROPOSITION 4.1 (Continuity properties of the hom-functors). *Let $\mathbb{X}$ be a σ -finite measure space. Let $\mathcal{M}, \mathcal{N}$ be given Banach $L^0(\mathbb{X})$ -modules. Then the following properties are verified:*

- (1) *Let $D: \mathbf{J} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$ be a small diagram and denote by $(\mathcal{L}, \lambda_{\star})$ its limit. Then the limit of the diagram $\tilde{D} := \text{Hom}(\mathcal{M}, -) \circ D: \mathbf{J} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$ is given by*

$$(\text{Hom}(\mathcal{M}, \mathcal{L}), \text{Hom}(\mathcal{M}, -)(\lambda_{\star})).$$

- (2) *Let $D: \mathbf{J} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$ be a small diagram and denote by $(\mathcal{C}, c_{\star})$ its colimit. We define the diagram $\tilde{D}: \mathbf{J}^{\text{op}} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$ as $\tilde{D}(i) := \text{Hom}(D(i), \mathcal{N})$ for every $i \in \text{Ob}_{\mathbf{J}}$, while we set $\tilde{D}(\phi^{\text{op}}) := \text{Hom}(-, \mathcal{N})(D(\phi)^{\text{op}})$ for every $\phi \in \text{Hom}_{\mathbf{J}}$. Then the limit of $\tilde{D}$ is given by*

$$(\text{Hom}(\mathcal{C}, \mathcal{N}), \text{Hom}(-, \mathcal{N})(c_{\star}^{\text{op}})).$$

PROOF. We focus only on the first item, since the second one can be proved via similar arguments. For brevity, we denote $\Phi_i := \text{Hom}(\mathcal{M}, -)(\lambda_i)$ for every $i \in \text{Ob}_{\mathbf{J}}$. Given a morphism $\phi: i \rightarrow j$ in $\mathbf{J}$ and any $T \in \text{Hom}(\mathcal{M}, \mathcal{L})$, we have

$$(\tilde{D}(\phi) \circ \Phi_i)(T) = D(\phi) \circ \lambda_i \circ T = \lambda_j \circ T = \Phi_j(T),$$

which shows that $(\text{Hom}(\mathcal{M}, \mathcal{L}), \Phi_{\star})$ is a cone to $\tilde{D}$. Now fix an arbitrary cone $(\mathcal{Q}, \Psi_{\star})$ to $\tilde{D}$. Given any $z \in \mathcal{Q}$, we define $f_z := \chi_{\{|z|>0\}} \frac{1}{|z|} \in L^0(\mathbb{X})$. Observe that for any $i \in \text{Ob}_{\mathbf{J}}$ the homomorphism $T_i^z := f_z \cdot \Psi_i(z): \mathcal{M} \rightarrow D(i)$ satisfies $|T_i^z| \leq 1$, thus in particular it is a morphism in $\mathbf{BanMod}_{\mathbb{X}}$. Moreover, for any morphism $\phi: i \rightarrow j$ in $\mathbf{J}$ and for any $v \in \mathcal{M}$ we can compute

$$\begin{aligned} (D(\phi) \circ T_i^z)(v) &= (D(\phi) \circ \Psi_i(z))(f_z \cdot v) = (\tilde{D}(\phi) \circ \Psi_i)(z)(f_z \cdot v) \\ &= \Psi_j(z)(f_z \cdot v) = T_j^z(v), \end{aligned}$$

which shows that $(\mathcal{M}, T_{\star}^z)$ is a cone to D . Hence, there exists a unique morphism $\tilde{\Phi}(z): \mathcal{M} \rightarrow \mathcal{L}$ such that $\lambda_i \circ \tilde{\Phi}(z) = T_i^z$ for every $i \in \text{Ob}_{\mathbf{J}}$. Letting

$$\Phi(z) := |z| \cdot \tilde{\Phi}(z) \in \text{Hom}(\mathcal{M}, \mathcal{L}) \quad \text{for every } z \in \mathcal{Q},$$

we deduce that $\Phi: \mathcal{Q} \rightarrow \text{Hom}(\mathcal{M}, \mathcal{L})$ is the unique morphism in $\mathbf{BanMod}_{\mathbb{X}}$ verifying the identity $\Phi_i \circ \Phi = \Psi_i$ for every $i \in \text{Ob}_{\mathbf{J}}$. This proves that $(\text{Hom}(\mathcal{M}, \mathcal{L}), \Phi_{\star})$ is the limit of $\tilde{D}$. $\blacksquare$

The next result immediately follows from Proposition 4.1 (plugging in $\mathcal{N} = L^0(\mathbb{X})$):

COROLLARY 4.2. *Let $\mathbb{X}$ be a σ -finite measure space and let $\mathcal{M}$ be a Banach $L^0(\mathbb{X})$ -module. Fix a small diagram $D: \mathbf{J} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$, whose colimit we denote by $(\mathcal{C}, c_{\star})$. Define $\tilde{D}: \mathbf{J}^{\text{op}} \rightarrow \mathbf{BanMod}_{\mathbb{X}}$ as $\tilde{D}(i) := D(i)^*$ for every $i \in \text{Ob}_{\mathbf{J}}$ and*

$$\tilde{D}(\phi^{\text{op}}) := \text{Hom}(-, L^0(\mathbb{X}))(D(\phi)^{\text{op}}) \quad \text{for every } \phi \in \text{Hom}_{\mathbf{J}}.$$

Then the limit of the diagram $\tilde{D}$ is given by $(\mathcal{C}^, \text{Hom}(-, L^0(\mathbb{X}))(c_{\star}^{\text{op}}))$.*

In particular, if $(\{\mathcal{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ is a direct system in $\mathbf{BanMod}_{\mathbb{X}}$ and we define

$$\varphi_{ij}^{\text{Lim}} := \text{Hom}(-, L^0(\mathbb{X}))(\varphi_{ij}^{\text{op}}): \mathcal{M}_j^* \rightarrow \mathcal{M}_i^* \quad \text{for every } i, j \in I \text{ with } i \leq j,$$

then $(\{\mathcal{M}_i^*\}_{i \in I}, \{\varphi_{ij}^{\text{Lim}}\}_{i \leq j})$ is an inverse system in $\mathbf{BanMod}_{\mathbb{X}}$ whose inverse limit is given by

$$(4.1) \quad \varprojlim \mathcal{M}_{\star}^* \cong (\varinjlim \mathcal{M}_{\star})^*,$$

together with the morphisms $\varphi_i^{\text{Lim}} := \text{Hom}(-, L^0(\mathbb{X}))(\varphi_i^{\text{op}}): (\varinjlim \mathcal{M}_{\star})^* \rightarrow \mathcal{M}_i^*$, where $(\varinjlim \mathcal{M}_{\star}, \{\varphi_i\}_{i \in I})$ stands for the direct limit of $(\{\mathcal{M}_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ in $\mathbf{BanMod}_{\mathbb{X}}$.

We conclude the paper with an example. We denote by ℓ^1 the space of all those sequences $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ satisfying $\sum_{n \in \mathbb{N}} |a_n| < +\infty$, which is a Banach space if endowed with the componentwise operations and the norm $\|(a_n)_{n \in \mathbb{N}}\|_{\ell^1} := \sum_{n \in \mathbb{N}} |a_n|$. Its dual Banach space is the space ℓ^{∞} of all bounded sequences in $\mathbb{R}$, endowed with the supremum norm $\|(b_n)_{n \in \mathbb{N}}\|_{\ell^{\infty}} := \sup_{n \in \mathbb{N}} |b_n|$. It is well known that the space ℓ^{∞} does not have the Radon–Nikodým property, thus in particular

$$(4.2) \quad L^0(\mathbb{X}; \ell^1)^* \not\cong L^0(\mathbb{X}; \ell^{\infty}) \quad \text{for every finite measure space } \mathbb{X},$$

as follows from the discussion in Remark 2.7.

EXAMPLE 4.3. Let $\mathbb{X} = (X, \Sigma, \mathfrak{m})$ be any given finite measure space. We denote by $\pi: X \rightarrow \{p\}$ the constant map, so that $\pi: \mathbb{X} \rightarrow \mathbb{P}$ is a morphism in $\mathbf{Meas}_{\sigma}$. We know from Example 3.26 that

$$(4.3) \quad \pi^* B \cong L^0(\mathbb{X}; B) \quad \text{for every Banach space } B.$$

We define the elements $(e_n)_{n \in \mathbb{N}} \subseteq \ell^1$ as $e_n := (\delta_{nk})_{k \in \mathbb{N}}$ for every $n \in \mathbb{N}$. Then one can readily check that $(\{B_n\}_{n \in \mathbb{N}}, \{\iota_{nm}\}_{n \leq m})$ is a direct system in $\mathbf{Ban}$, where B_n stands for the subspace of ℓ^1 generated by $\{e_1, \dots, e_n\}$ and $\iota_{nm}: B_n \hookrightarrow B_m$ denotes the inclusion map, and that $\varinjlim_{\mathbb{N}} B_n \cong \ell^1$. Each space B_n is finite-dimensional, thus its dual B'_n

is finite-dimensional as well, and in particular it has the Radon–Nikodým property. Therefore, we deduce (recalling Remark 2.7) that

$$(4.4) \quad L^0(\mathbb{X}; B_n)^* \cong L^0(\mathbb{X}; B'_n) \quad \text{for every } n \in \mathbb{N}.$$

Observe that $(\{B'_n\}_{n \in \mathbb{N}}, \{\iota_{nm}^{\text{Lim}}\}_{n \leq m})$ and $(\{\pi^* B'_n\}_{n \in \mathbb{N}}, \{\pi^* \iota_{nm}^{\text{Lim}}\}_{n \leq m})$ are inverse systems in **Ban** and in **BanMod** $_{\mathbb{X}}$, respectively. Nonetheless, it holds that $\pi^* \varprojlim B'_\star \not\cong \varprojlim \pi^* B'_\star$, since

$$\begin{aligned} \pi^* \varprojlim B'_\star &\stackrel{(4.1)}{\cong} \pi^* (\varinjlim B_\star)' \cong \pi^* (\ell^1)' \cong \pi^* \ell^\infty \stackrel{(4.3)}{\cong} L^0(\mathbb{X}; \ell^\infty) \stackrel{(4.2)}{\not\cong} L^0(\mathbb{X}; \ell^1)^* \\ &\stackrel{(4.3)}{\cong} (\pi^* \ell^1)^* \cong (\pi^* \varinjlim B_\star)^* \stackrel{(3.5)}{\cong} (\varinjlim \pi^* B_\star)^* \stackrel{(4.1)}{\cong} \varprojlim (\pi^* B_\star)^* \\ &\stackrel{(4.3)}{\cong} \varprojlim L^0(\mathbb{X}; B_\star)^* \stackrel{(4.4)}{\cong} \varprojlim L^0(\mathbb{X}; B'_\star) \stackrel{(4.3)}{\cong} \varprojlim \pi^* B'_\star, \end{aligned}$$

which shows that the functor InvIm_π does not preserve inverse limits.

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