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Bound states in bent soft waveguides

Pavel Exner and Semjon Vugalter

Abstract. The aim of this paper is to show that a two-dimensional Schrödinger operator with the potential in the form of a 'ditch' of a fixed profile can have a geometrically induced discrete spectrum. This occurs if such a potential channel has a single or multiple bends, being straight outside a compact. Moreover, under stronger geometric restrictions, the claim remains true in the presence of a potential bias at one of the channel 'banks.'

To Brian Davies on the occasion of his 80th birthday

1. Introduction

The behavior of quantum particles confined to tubular regions has attracted a lot of attention in recent decades, with motivation coming from two sources. On the physics side, it was the possibility to use such models to describe a guided dynamics in various condensed matter systems. At the same time, this appeared to be a source of interesting mathematical problems, in particular, those concerning spectral effects coming from the geometry of the confinement; for an introduction to the topic and a bibliography, we refer to the book [7].

There are different ways to localize a particle in the configuration space. One possibility is a hard confinement where the Hamiltonian is typically the Dirichlet Laplacian associated with a tube in $\mathbb{R}^d$ (or more complicated regions such as layers, networks, etc.). From the point of view of application to objects like semiconductor wires, such a model has a drawback: it does not take into account the tunneling between different parts of the waveguide. This fact motivated investigation of the 'leaky' confinement in which the Hamiltonian is instead a Schrödinger operator with an attractive singular interaction supported by a curve (or a surface, metric graph, etc.). To have it well defined, the codimension of the interaction support must not exceed three.

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If we stay for simplicity in the two-dimensional situation, both models exhibit *curvature-induced bound states*: whenever the strip, or the curve supporting the δ interaction, is non-straight but asymptotically straight, the corresponding Hamiltonian has a non-void discrete spectrum. This claim is valid universally modulo technical requirements on the regularity and asymptotic behavior.

The leaky guide model has another drawback in assuming that the interaction support has zero width. This recently motivated the investigation of a more realistic situation when the potential in the Schrödinger operator is regular in the form of a channel of a fixed profile [3, 4]. The term coined was *soft waveguides*; the analogous problem was studied in three dimensions [5] as well as for soft layers [2, 12]. One has to add that such operators were considered before [6, 16], however, the focus was then on the limit in which the potential shrinks transversally to a manifold; in the physics literature, the idea of determining the right 'quantization' on a manifold through such a limit was examined a long time ago [11, 15].

Not very surprisingly, soft waveguides were already shown to share properties with their hard and leaky counterparts, an example is the ground state optimization in a loop-shaped geometry [8]. Some results have also been obtained concerning the problem we are interested in here, the existence of curvature-induced bound states, however, so far they lack the universal character indicated above. In [3, 4] Birman–Schwinger principle was used to derive a sufficient condition under which the discrete spectrum is nonempty, expressed in terms of positivity of a certain integral which, in general, is not easy to evaluate. An alternative is to apply the variational method; in this way, the existence was established in the example of a particular geometry, often referred to as a 'bookcover' [10]. We note in passing that it is paradoxically easier to establish the existence in conic-shaped soft layers, where the discrete spectrum is infinite [2, 12].

The trouble with the variational approach is that it is not easy, beyond the simple example mentioned, to find a suitable trial function. The aim of this paper is to extend the existence result using a variational method to a much wider, even if still not optimal class of soft waveguides. The main restrictions in our analysis are the limitation of the curved part into a bounded region, a compact support of the potential defining the channel profile, and the requirement of the profile symmetry. The latter restriction can be relaxed in some situations, in particular, if the profile potential is sign-changing and the transverse part of the operator, the operator (2.2) below, has a zero-energy resonance.

We will also consider the situation when the system has a constant positive potential bias in one of the regions separated by the profile potential support. In this case, we have a stronger geometric restriction: we have to assume that one of the two regions is *convex*. If the bias potential is supported in it, we can again prove the existence of a discrete spectrum, even without the symmetry assumption. If the bias is supported

in the opposite region, we have the existence again, however, except in the situation when the operator (2.2) has a zero-energy resonance; this is in agreement with the result of [9] where we treated a system which can be regarded as a singular version of the present system. Let us stress that the convexity makes it also possible to prove the existence in the absence of the bias and the symmetry restriction, provided that operator (2.2) has a negative eigenvalue.

In the following section, we will state the problem in proper terms and present the main results. The rest of the paper is devoted to the proofs. The next two sections deal with the case without the bias; in Section 3 we prove Theorem 2.2 (a) which concerns the situation when the operator (2.2) has a zero-energy resonance, Section 4 provides the proof of Theorem 2.4 (a) which addresses the case when the operator has a negative eigenvalue and the channel profile is symmetric. Finally, in Section 5 we prove Theorem 2.2 (b) and Theorem 2.4 (b), which establish the existence results in the situation when one of the two regions to which the potential channel, not necessarily symmetric, divides the plane is convex. This holds even in the absence of the bias, except in the zero-energy resonance case.

2. Statement of the problem and main results

Let us now state the problem described in the introduction. We begin with the assumptions which are split into two groups: the first one concerns the support of the potential; the other the channel profile. The former is a strip built around a curve Γ, understood as the graph of a function $\Gamma\colon \mathbb{R} \to \mathbb{R}^2$ such that $|\Gamma'(s)| = 1$. Without repeating it at every occasion, we always exclude the trivial situation when Γ is a straight line. In addition, we suppose the following.

(s1) Γ is C^1, piecewise C^3-smooth, non-straight but straight outside a compact. Its curved part consists of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved.

(s2) $|\Gamma(s_+) - \Gamma(s_-)| \to \infty$ as $s_\pm \to \pm\infty$. In other words, the two straight parts of Γ are either not parallel, or if they are, they point in the opposite directions.

(s3) The strip neighborhood $\Omega^a := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < a\}$ of Γ with a halfwidth $a > 0$ does not intersect itself.

Assumption (s3) has various equivalent expressions: one can say, for instance, that the function $\operatorname{dist}(x, \Gamma(\cdot))$ has for any fixed $x \in \Omega^a$ a unique minimum, or that the map

$$x(s,t) \mapsto (\Gamma_1(s) - t\Gamma_2'(s), \Gamma_2(s) + t\Gamma_1'(s)) \tag{2.1}$$

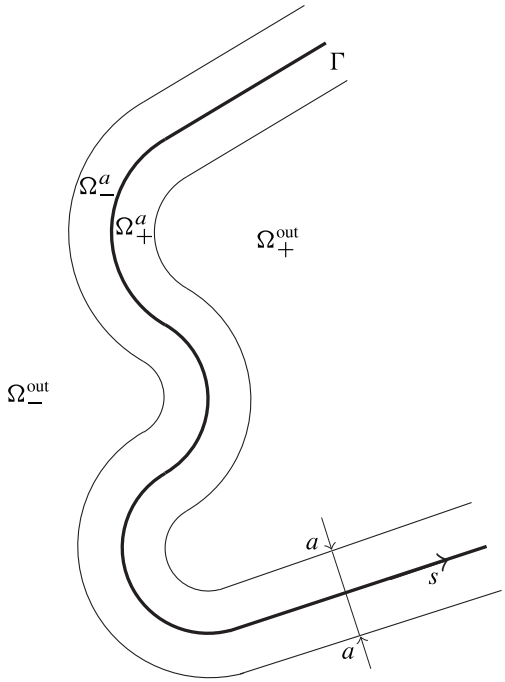

Figure 1. Scheme of the waveguide

from the straight strip $\Omega_0^a := \mathbb{R} \times (-a, a)$ to $\mathbb{R}^2$ is a bijection, in fact, a diffeomorphism; $\vec{n}(s) = (-\Gamma_2'(s), \Gamma_1'(s))$ is, of course, the (inward) normal to the curve at the point $\Gamma(s)$. Under assumption (s1), the signed curvature $\kappa(s) = (\Gamma_2'\Gamma_1'' - \Gamma_1'\Gamma_2'')(s)$ is piecewise smooth and compactly supported function; a necessary, but in general not sufficient, condition for (s3) to hold is $a\|\kappa\|_\infty < 1$, which ensures the local injectivity of the map. The curve divides the plane into open regions which we denote as $\Omega_\pm$; for the sake of definiteness, we assume that Ω_+ is *on the left side* when one looks in the direction of the increasing arc length variable s.

We also introduce $\Omega_\pm^a := \Omega_\pm \cap \Omega^a$ so that we have $\Omega^a = \Omega_+^a \cup \Gamma \cup \Omega_-^a$; given our choice of the normal orientation, the labels correspond to the sign of the transversal variable t. Finally, we will use a natural symbol for the complement of the strip, namely $\Omega_{\mathrm{out}} := \mathbb{R}^2 \setminus \Omega^a$, and its one-sided components will be denoted as $\Omega_\pm^{\mathrm{out}} := \Omega_\pm \setminus \Omega^a$ – cf. Figure 1.

The second group of assumptions concerns the potential. Its profile is determined by a function $v\colon \mathbb{R} \to \mathbb{R}$ of which we assume

(p1) $v \in L^2(\mathbb{R})$ and $\operatorname{supp} v \subset [-a, a]$.

In some situations, specifically in Theorem 2.4 (a) below, we will require it additionally to be mirror-symmetric,

(p2) $v(t) = v(-t)$ for $t \in [-a, a]$.

In addition to the potential defining the channel, we are going to consider, in general, also a one-sided potential bias of the system. To this aim, we introduce the one-dimensional Schrödinger operator

$$h := -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0 \chi_{[a,\infty)}(t), \quad V_0 \geq 0. \tag{2.2}$$

The crucial role will be played by the spectral bottom of this operator, specifically we will be concerned with the following two possibilities.

(p3) $\inf \sigma(h)$ is a negative (ground state) eigenvalue μ associated with a real-valued eigenfunction ϕ_0 which we may without loss of generality normalize by the requirement $\phi_0(-a) = 1$.

(p4) The operator h has a zero-energy resonance (virtual level), meaning that $h \geq 0$ and $-(1-\varepsilon)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0 \chi_{[a,\infty)}(t)$ has a negative eigenvalue for any $\varepsilon > 0$. In this case, the equation $h\phi = 0$ has a solution $\phi_0 \in \dot{H}^1(\mathbb{R})$ where $\dot{H}^1(\mathbb{R})$ is the Hilbert space with the norm $\|\phi\|^2 := \int_{|x|\leq 1} |\phi(x)|^2 \,\mathrm{d}\,x + \int_{\mathbb{R}} |\phi'(x)|^2 \,\mathrm{d}\,x$, cf. [1]. Since $\operatorname{supp} v \subset [-a, a]$, this solution is nonzero and constant for $|t| > a$ in the absence of the bias, for $V_0 > 0$ this is true for $t < -a$ only. The solution will be again supposed to satisfy the normalization condition $\phi_0(-a) = 1$.

The main object of our interest is the Schrödinger operator

$$H_{\Gamma,V} = -\Delta + V(x) \tag{2.3a}$$

on $L^2(\mathbb{R}^2)$ with the potential defined using the locally orthogonal coordinates (s, t) appearing in (2.1) as

$$V(x) = \begin{cases} v(t) & \text{if } x \in \Omega^a, \\ V_0 & \text{if } x \in \Omega_+ \setminus \Omega^a, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3b}$$

We will often drop the subscript of $H_{\Gamma,V}$ if it is clear from the context.

Proposition 2.1. *Under assumptions (s1)–(s3) and (p1), the operator (2.3) is self-adjoint, $D(H_{\Gamma,V}) = H^2(\mathbb{R}^2)$, and its essential spectrum is the same as for Γ being a straight line, $\sigma_{\mathrm{ess}}(H_{\Gamma,V}) = [\mu, \infty)$. The threshold μ is the lowest eigenvalue of h if (p3) is valid, while if the discrete spectrum of h is empty, we have $\mu = 0$.*

Proof. The self-adjointness is easy to check: it is sufficient to ascertain, using assumption (p1), that the potential (2.3b) is infinitely small with respect to $-\Delta$. We will do it in the form sense using the KLMN theorem [13, Section X.1] by checking that

to any $a > 0$ there is a $b > 0$ such that $(\psi, V\psi) \le a\|\nabla\psi\|^2 + b\|\psi\|^2$ holds for all $\psi \in H^1(\mathbb{R}^2)$. Suppose first that $V_0 = 0$. We decompose any such ψ into a sum $\psi = \psi_+ + \psi_0 + \psi_-$ of H^1 functions such that $\operatorname{supp} \psi_\pm\restriction_{\Omega^a}$ contains the straight parts of the strip and $\operatorname{supp} \psi_0\restriction_{\Omega^a}$ is bounded and contains the curved part. To the latter, the theorem applies directly since $V\restriction_{\operatorname{supp}\psi_0} \in L^2$. In the straight parts, we get first using (p1) the one-dimensional version of the inequality in the transverse variable; then we lift it to two dimensions using the inequality $\|\partial_t\psi\|^2 \le \|\nabla\psi\|^2$. The constants b_j, $j = 0, \pm$, in the obtained inequalities are in general different; we put $b := 3\max\{b_+, b_0, b_-\}$. Using then the triangle and Schwarz inequalities, which, in particular, give $\|\psi\|^2 \le 3\sum_j \|\psi_j\|^2$, we arrive at the desired conclusion. Finally, the self-adjointness is certainly not affected by adding the bounded potential $V_0\chi_{\Omega_+^{\mathrm{out}}}$.

The identification of the essential spectrum of $H_{\Gamma,V}$ with the interval $[\mu, \infty)$, where $\mu = \inf\sigma(h)$, was established in [3, Proposition 3.1] under slightly different assumptions. The argument can be easily modified for our present purpose; the requirement on the smoothness of Γ we made is stronger than there, and neither the substitution of a bounded negative v by a possibly sign indefinite square integrable one, nor the addition of a potential bias alters the conclusion. ∎

Note also that the above Hamiltonian can be investigated using the associate quadratic form $Q_{\Gamma,V}$, mostly written without the indices specifying the curve and the potential, and defined by

$$Q[\psi] := \|\nabla\psi\|^2 + \int_\Omega V(x)|\psi(x)|^2\,\mathrm{d}\,x, \quad D(Q) = H^1(\mathbb{R}^2). \tag{2.3c}$$

In the absence of the potential bias the potential of the profile v is integrated over Ω^a, while for $V_0 > 0$ we have in addition $V_0\int_{\Omega_+^{\mathrm{out}}} |\psi(x)|^2\,\mathrm{d}\,x$ on the right-hand side.

Now, we are in position to state our main results. The assumptions may appear in various combinations. We group them according to the spectral threshold μ starting with the situation when operator (2.2) has a zero-energy resonance.

Theorem 2.2 (Threshold resonance case). *Assume* (s1)–(s3), (p1), *and* (p4)*; then the following claims are valid.*

(a) *If the bias is absent,* $V_0 = 0$*, and*

$$[\phi_0(a)^2 - \phi_0(-a)^2]\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s \le 0 \tag{2.4}$$

holds, then $H_{\Gamma,V}$ *has at least one negative eigenvalue.*

(b) *The same is true if* $V_0 > 0$ *and* Ω_+ *is* convex.

Remark 2.3. Recall that by assumption κ does not vanish identically. Condition (2.4) is naturally satisfied if $\phi_0(a) = \phi_0(-a)$, in particular, under the mirror-symmetry assumption (p2). Consider further the asymmetric situation, $\phi_0(a) \neq \phi_0(-a)$, and recall that the integral in (2.4) equals $\pi - 2\theta$ where 2θ is the angle between the asymptotes. Consequently, at least one bound state exists then in the zero-energy resonance case if the asymptotes of Γ are parallel and pointing in the opposite directions, $\theta = \frac{1}{2}\pi$, or if they are not parallel and the resonance solution ϕ_0 is larger at the 'outer' side of the strip Ω^a.

If h has negative eigenvalues so that $\mu < 0$, the situation is more complicated and we have to make stronger restrictions on the profile or the shape of the waveguide.

Theorem 2.4 (Eigenvalue case). *Assume* (s1)–(s3) *together with* (p1) *and* (p3). *Then* $\sigma_{\mathrm{disc}}(H_{\Gamma,V})$ *is nonempty under any of the following conditions:*

(a) $V_0 = 0$ *and assumption* (p2) *is satisfied;*

(b) $V_0 \geq 0$ *and* one of the regions $\Omega_\pm$ *is* convex.

3. Proof of Theorems 2.2 (a) and 2.4 (a)

With the later purpose in mind, we will formulate the argument first in the general situation which involves both the bound-state and zero-energy-resonance cases as well as the possible potential bias. In view of Proposition 2.1, it is sufficient to construct a trial function $\psi \in H^1(\mathbb{R}^2)$ such that $Q[\psi] < \mu\|\psi\|^2$. Let us first fix the geometry. If the two straight parts of Γ are not parallel – cf. Figure 1 – their line extensions intersect at a point which we choose as the origin O, and use polar coordinates with this center, in which the two halflines correspond to the angles $\pm\theta_0$ for the appropriate $\theta_0 \in (0, \frac{1}{2}\pi)$. Furthermore, we fix the point $s = 0$ in such a way that for large $|s|$ the points with the coordinates $\pm s$ have the same Euclidean distance from O – cf. Figure 2.

If the asymptotes are parallel (and pointing in the opposite directions according to (s2)), we choose the origin as the point with equal distance from the endpoints of the two halflines. The point with $s = 0$ on the curve is likewise chosen so that those with the coordinates $\pm s$ have the same Euclidean distance from the origin; in both cases one can check easily that such a choice is unique.

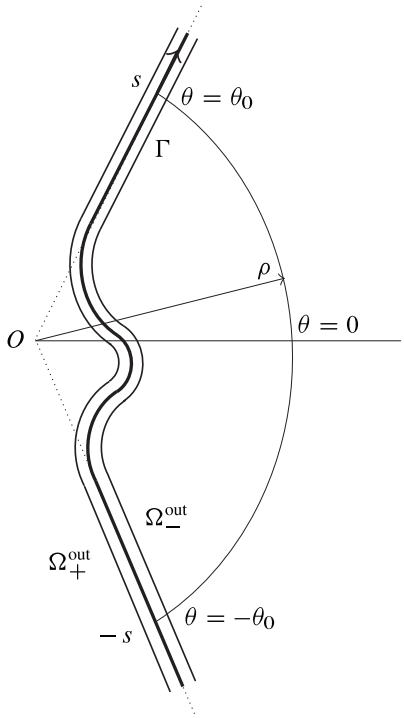

Figure 2. Parallel and polar coordinates used in the proof

3.1. Trial function inside the strip

For fixed values s_0, such that the points with coordinates $\pm s_0$ lay outside the curved part of Γ, and $s^* > s_0$, to be chosen later, we define

$$\chi_{\text{in}}(s) := \begin{cases} 1 & \text{if } |s| < s_0, \\ \ln \frac{s^*}{|s|} \left(\ln \frac{s^*}{s_0}\right)^{-1} & \text{if } s_0 \le |s| \le s^*, \\ 0 & \text{if } |s| > s^*. \end{cases} \tag{3.1}$$

Recalling that ϕ_0 is the ground-state eigenfunction or the zero-energy solution normalized by $\phi_0(-a) = 1$, we put

$$\psi(s,t) = \phi_0(t)\chi_{\text{in}}(s) + \nu g(s,t), \quad |t| \le a, \tag{3.2}$$

where the parameter ν and the function g, compactly supported within $(-s_0, s_0) \times (-a, a)$, will be chosen later. We denote by $Q_{\text{int}}[\psi]$ the contribution to the shifted quadratic form, $Q[\psi] - \mu\|\psi\|^2$, coming from the strip Ω^a, which can be using the

parallel coordinates expressed as

$$Q_{\mathrm{int}}[\psi] = \int\limits_{|t|\le a} \Big\{ \Big(\frac{\partial\psi}{\partial s}\Big)^2 (1-\kappa(s)t)^{-1} + \Big(\frac{\partial\psi}{\partial t}\Big)^2 (1-\kappa(s)t) + (v(t)-\mu)|\psi|^2(1-\kappa(s)t)\Big\}\,\mathrm{d}\,s\,\mathrm{d}\,t.$$

The first term on the right-hand side can be estimated as

$$\int\limits_{|t|\le a} \Big(\frac{\partial\psi}{\partial s}\Big)^2 (1-\kappa(s)t)^{-1}\,\mathrm{d}\,s\,\mathrm{d}\,t \le 2\tau_0^{-1}\|\phi_0\restriction_{[-a,a]}\|^2\|\chi_{\mathrm{in}}'\|^2 + C\nu^2,$$

where the norm refers to $L^2(\mathbb{R})$, $\tau_0 := 1 - a\|\kappa\|_\infty$ is positive by (s3), and C depends on the function g only; we will use the same letter for generic constants in the following. Note that choosing the parameter s^* in (3.1) large one can make for any fixed s_0 the norm $\|\chi_{\mathrm{in}}'\| = \big(\ln\frac{s^*}{s_0}\big)^{-1}\big(\frac{1}{s_0}-\frac{1}{s^*}\big)^{1/2}$ small. As for the other two terms, we have

$$\begin{aligned}
&\int\limits_{|t|\le a} \Big\{ \Big(\frac{\partial\psi}{\partial t}\Big)^2 (1-\kappa(s)t) + (v(t)-\mu)|\psi|^2(1-\kappa(s)t)\Big\}\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\quad = \int\limits_{|t|\le a} \big\{(\phi_0'(t))^2 + (v(t)-\mu)|\phi_0(t)|^2\big\}\chi_{\mathrm{in}}^2(s)(1-\kappa(s)t)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\qquad + 2\nu \int\limits_{|t|\le a} \Big\{\phi_0'\frac{\partial g}{\partial t} + (v(t)-\mu)\phi_0 g\Big\}\chi_{\mathrm{in}}(s)(1-\kappa(s)t)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\qquad + \nu^2 \int\limits_{|t|\le a} \Big\{\Big(\frac{\partial g}{\partial t}\Big)^2 + (v(t)-\mu)|g|^2\Big\}(1-\kappa(s)t)\,\mathrm{d}\,s\,\mathrm{d}\,t,
\end{aligned} \tag{3.3}$$

where the last term on right-hand side can be again estimated by $C\nu^2$ with a C depending on the function g only. Furthermore, integrating the middle term by parts with respect to t, we get

$$\begin{aligned}
&2\nu \int\limits_{|t|\le a} [-\phi_0'' + (v(t)-\mu)\phi_0]\chi_{\mathrm{in}}(s)g(s,t)(1-\kappa(s)t)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\quad + 2\nu \int\limits_{|t|\le a} \phi_0'(t)\chi_{\mathrm{in}}(s)g(s,t)\kappa(s)\,\mathrm{d}\,s\,\mathrm{d}\,t,
\end{aligned} \tag{3.4}$$

where the square bracket in the first integral is zero by assumption.

Notice next that ϕ_0' cannot vanish identically in the interval $[-a,a]$. Indeed, it is continuous in $\mathbb{R}$ and we have $v(t) = 0$ for $|t| > a$, hence should the derivative ϕ_0' be zero in $[-a,a]$, the function must have been a constant one; however, that is

impossible for an eigenfunction or a zero-energy resonance solution. This observation allows us to choose the function g in such a way that the last integral is negative. To this aim, it suffices to have it supported in a region where both ϕ_0' and κ do not change sign – recall that $\chi_{\rm in}(s) = 1$ holds on the support of κ – and to pick the sign of $g(s,t)$ accordingly. With such a choice, expression (3.4) will be equal to $-\delta\nu$, where $-\delta < 0$ is the value of the last integral. For small ν, this linear term will dominate over those estimated by $C\nu^2$, and, consequently, there is a $\nu_0 > 0$ such that, for all $\nu \in (0, \nu_0)$ and any s_0, s^* indicated in (3.1), the sum of the last two terms on the right-hand side of (3.3) will be smaller than $-\frac{1}{2}\delta\nu$. Note that, for a fixed v determining ϕ_0 and μ, and fixed $g(s,t)$, the value ν_0 depends continuously on κ. Hence, if we have two curves Γ and $\widetilde{\Gamma}$, the curvatures of which do not differ much, the corresponding values of ν_0 are close to each other too; we will use this fact later in Section 4.3.

It remains to deal with the first term on the right-hand side of (3.3). To simplify the notation, we introduce the following symbols:

$$\phi_+ = \phi_0(a), \quad \xi_+ = -\sqrt{|\mu| + V_0}, \quad \xi_- = \sqrt{|\mu|}, \tag{3.5}$$

which allows us to write $\phi_0'(a) = \xi_+\phi_+$ and $\phi'(-a) = \xi_-$. Recall that $\phi_0(-a) = 1$ holds by assumption. The expression in question then can be rewritten using integration by parts as follows:

$$\begin{aligned}
&\int\limits_{|t|\le a} \{(\phi_0'(t))^2 + (v(t)-\mu)|\phi_0(t)|^2\}\chi_{\rm in}^2(s)(1-\kappa(s)t)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\quad = \int\limits_{\mathbb{R}} \big[\xi_+\phi_+^2(1-\kappa(s)a) - \xi_-(1+\kappa(s)a)\big]\chi_{\rm in}^2(s)\,\mathrm{d}\,s \\
&\qquad + \int\limits_{|t|\le a} \big[-\phi_0''(t) + (v(t)-\mu)\phi_0(t)\big]\phi_0(t)(1-\kappa(s)t)\chi_{\rm in}^2(s)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\qquad + \int\limits_{|t|\le a} \phi_0'(t)\phi_0(t)\kappa(s)\chi_{\rm in}^2(s)\,\mathrm{d}\,s\,\mathrm{d}\,t \\
&\quad = [\xi_+\phi_+^2 - \xi_-]\|\chi_{\rm in}\|^2 - [\xi_+\phi_+^2 + \xi_-]a\int\limits_{\mathbb{R}} \kappa(s)\chi_{\rm in}^2(s)\,\mathrm{d}\,s \\
&\qquad + \frac{1}{2}(\phi_+^2 - 1)\int\limits_{\mathbb{R}} \kappa(s)\chi_{\rm in}^2(s)\,\mathrm{d}\,s,
\end{aligned} \tag{3.6}$$

where the norm in the last expression refers to $L^2(\mathbb{R})$ and we have used the identity $\phi_0'\phi_0 = \frac{1}{2}(\phi_0^2)'$. Since κ has a compact support and $\chi_{\rm in}^2(s) = 1$ holds on it by (3.1),

we can replace the integrals in the last part of (3.6) by $\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s$. Summarizing the estimate, we have obtained for all sufficiently small ν the inequality

$$Q_{\mathrm{int}}[\psi] \le -\frac{1}{2}\delta\nu + [\xi_+\phi_+^2 - \xi_-]\|\chi_{\mathrm{in}}\|^2 - [\xi_+\phi_+^2 + \xi_-]\,a\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s$$
$$+\frac{1}{2}(\phi_+^2 - 1)\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s + \tau_0^{-1}\|\phi_0\restriction_{[-a,a]}\|^2\|\chi'_{\mathrm{in}}\|^2. \tag{3.7}$$

Choosing for a fixed $\nu \in (0, \nu_0)$ and $\tau_0 = 1 - a\|\kappa\|_\infty$ the ratio at the right-hand side of (3.1) as $\frac{s^*}{s_0} \gg 1$, one can achieve that the last term in (3.7) will be smaller than $\frac{1}{4}\delta\nu$.

This general estimate simplifies in various situations indicated above. In particular, the assumptions (p2) and $V_0 = 0$ used in Theorem 2.4 (a) imply $\phi_+ = \phi_- = 1$ and $\xi_- = -\xi_+$, so that (3.7) becomes

$$Q_{\mathrm{int}}[\psi] \le -\frac{1}{4}\delta\nu - 2|\mu|^{1/2}\|\chi_{\mathrm{in}}\|^2 \tag{3.8}$$

for all $\nu \in (0, \nu_0)$, the ratio $\frac{s^*}{s_0}$ large enough, and s_0 such that the points referring to the curved part of Γ are inside $[-s_0, s_0]$. On the other hand, assumption (p4) used in Theorem 2.2 (a) together with the absence of the bias, $V_0 = 0$, means that $\xi_\pm = 0$, so the second and the third term on the right-hand side of (3.7) vanish, and since the fourth one is now supposed to be non-positive, the estimate reduces simply to $Q_{\mathrm{int}}[\psi] \le -\frac{1}{4}\delta\nu$.

3.2. Completing the proof of Theorem 2.2 (a)

In view of the last observation, to conclude the proof of Theorem 2.2 (a) we have thus to choose the outer part of trial function in such a way that its contribution to the quadratic form can be made smaller than any fixed positive number. Under assumption (p4), ϕ_0 is constant outside the potential support, $\phi_0(t) = \phi_\pm$ for $\pm t \ge a$; we recall that $\phi_- = 1$ by assumption. We choose the trial function outside Ω^a as the $\phi_\pm$ multiplier of the mollifier χ_{out} of which we require the following properties:

(i) in $\mathbb{R}^2 \setminus \Omega^a$ the function χ_{out} depends on $\rho = \mathrm{dist}(x, O)$ only;

(ii) we have continuity at the boundary: at the points $x(s, \pm a)$ the relation $\chi_{\mathrm{out}}(x) = \chi_{\mathrm{in}}(s)$ holds.

Let us consider the situation where the extensions of the asymptotes of Γ cross; the case of parallel asymptotes pointing in the opposite directions can be dealt with analogously. We again choose s_0 in such a way that the points $\Gamma(\pm s_0)$ belong to

the straight parts of the curve, then $\operatorname{dist}(\Gamma(s), O) = \rho_s := (|s| - s_0) + d_0$, where $d_0 = \operatorname{dist}(\Gamma(s_0), O)$ (recall that $\Gamma(-s_0) = \Gamma(s_0)$ holds by assumption).

Given that the distance of the points $x(s, \pm a)$ from the origin is $\sqrt{\rho_s^2 + a^2}$, in accordance with requirements (i) and (ii) we put

$$\chi_{\mathrm{out}}(\rho) := \begin{cases} \chi_{\mathrm{in}}(\sqrt{\rho^2 - a^2} - d_0 + s_0) & \text{if } \sqrt{\rho^2 - a^2} \geq d_0, \\ 1 & \text{if } \rho \leq \sqrt{d_0^2 + a^2} \end{cases}$$

This, in particular, means that χ_{out} vanishes if its argument exceeds s^*, in other words, for $\rho > \sqrt{(s^* - s_0 + d_0)^2 + a^2}$. Since $\mu = 0$ holds by assumption and the potential is zero away from Ω^a, the quantity to be estimated is the kinetic energy contribution to the form (2.3c) from the outer part of the trial function,

$$\int\limits_{\Omega\setminus\Omega^a} |\nabla\psi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x \leq 2\pi \max\{\phi_+^2, 1\} \int\limits_{\sqrt{d_0^2+a^2}}^{\sqrt{(s^*-s_0+d_0)^2+a^2}} \left|\frac{\mathrm{d}}{\mathrm{d}\,\rho}\chi_{\mathrm{in}}(\sqrt{\rho^2 - a^2} - d_0 + s_0)\right|^2 \rho\,\mathrm{d}\,\rho. \quad (3.9)$$

Relation (3.8) tells us that one can choose parameters δ and ν for which the inner contribution to the form is negative (using a sufficiently large s^*), hence to prove the claim it is enough to show that the integral on the right-hand side of (3.9) vanishes if $s_0,\ d_0 \to \infty$ with the difference $s_0 - d_0$ bounded and $\frac{s^*}{s_0} \to \infty$. The values of the integrated function on the support of $\nabla\psi_{\mathrm{out}}$ can be expressed using (3.1) to be

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}\,\rho}\chi_{\mathrm{in}}(\sqrt{\rho^2 - a^2} - d_0 + s)\right|^2 &= \left|\left(\ln\frac{s^*}{s_0}\right)^{-1} \frac{1}{\sqrt{\rho^2 - a^2} - d_0 + s_0}\frac{\partial s}{\partial\rho}\right|^2 \\ &= \left(\ln\frac{s^*}{s_0}\right)^{-2} (\sqrt{\rho^2 - a^2} - d_0 + s_0)^{-2}, \end{aligned}$$

because $\frac{\partial s}{\partial\rho} = 1$ in the considered region. Substituting from here to (3.9), we get

$$\int\limits_{\Omega\setminus\Omega^a} |\nabla\psi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x \leq 2\pi \max\{\phi_+^2, 1\}\left(\ln\frac{s^*}{s_0}\right)^{-2} \int\limits_{\sqrt{d_0^2+a^2}}^{\sqrt{(s^*-s_0+d_0)^2+a^2}} \frac{\rho\,\mathrm{d}\,\rho}{(\sqrt{\rho^2 - a^2} - d_0 + s_0)^2}$$

Since $s_0 - d_0$ is bounded and a is fixed, we can choose a $c \in (0, 1)$ in such a way that

$$\sqrt{\rho^2 - a^2} - d_0 + s_0 \geq c\rho$$

holds for all $\rho \geq \sqrt{d_0^2 + a^2}$, in which case we have

$$\int\limits_{\Omega\setminus\Omega^a} |\nabla\psi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x \leq 2\pi c^{-2} \max\{\phi_+^2, 1\}\Big(\ln\frac{s^*}{s_0}\Big)^{-2} \ln\rho\,\Big|_{\sqrt{d_0^2+a^2}}^{\sqrt{(s^*-s_0+d_0)^2+a^2}} . \tag{3.10}$$

Using again the fact that s^*, $d_0 \to \infty$ while a is fixed and $s_0 - d_0$ bounded, we see that the parameters can be chosen so that

$$\ln\frac{\sqrt{(s^*-s_0+d_0)^2+a^2}}{\sqrt{d_0^2+a^2}} \leq \ln\frac{c^*s^*}{c_0 s_0} = \ln\frac{s^*}{s_0} + \ln c^* - \ln c_0 \tag{3.11}$$

for $c^* > 1$ and $c_0 \in (0, 1)$. Substituting from (3.11) into (3.10), we get the needed result; this concludes the proof of Theorem 2.2 (a).

4. Completing the proof of Theorem 2.4 (a)

Let us pass to the situation where there is again *no bias*, $V_0 = 0$, the channel profile is *symmetric*, and the transverse operator (2.2) is *subcritical*, $\mu < 0$.

4.1. Trial function outside Ω_a

This is the most difficult part of the argument. For the interior, we can use the result of Section 3.1 noting that in view of the assumption (p2) we have $\phi_+ = 1$ and $\xi_+ = -\xi_-$ which means that the inequality (3.8) is still valid. To begin with, we introduce in Ω_{out} function ϕ defined as

$$\phi(x) := \exp\{-\xi(\mathrm{dist}(x,\Gamma) - a)\}, \quad x \in \mathbb{R}^2 \setminus \Omega^a, \tag{4.1}$$

where $\xi := \xi_- = -\xi_+ = |\mu|^{1/2}$. The sought trial function will be then of the form $\psi_{\mathrm{out}} = \phi\chi_{\mathrm{out}}$ with the mollifier χ_{out} to be specified below. As before, we will focus on the situation where the asymptotes of Γ are not parallel, the case with $\theta_0 = \frac{\pi}{2}$ can be treated in a similar way.

Since $\theta_0 > 0$ by assumption, we can choose conical neighborhoods of the asymptotes which do not intersect, that is, to pick $\Delta\theta_0$ sufficiently small so that $[-\theta_0 - \Delta\theta_0, -\theta_0 + \Delta\theta_0] \cap [\theta_0 - \Delta\theta_0, \theta_0 + \Delta\theta_0] = \emptyset$. Furthermore, we pick an $r_0 > 0$ large enough to ensure that the curved part of Γ is contained in the disk of one half that radius, $B_{\frac{1}{2}r_0}(O)$, centered at the coordinate origin O. In addition, we assume that s_0 in (3.1) is sufficiently large so that the parts of Γ with $|s| > s_0$ are outside the disk $B_{r_0}(O)$ of the doubled radius. At the points of the corresponding conical sectors, $x = (\rho, \theta) \in \mathbb{R}^2 \setminus B_{r_0}(O)$ with $\theta \in [\theta_0 - \Delta\theta_0, \theta_0 + \Delta\theta_0]$ or $\theta \in [-\theta_0 - \Delta\theta_0, -\theta_0 + \Delta\theta_0]$,

we can use the (s,t) coordinates – cf. Figure 3 – and define the mollifier χ_{out} depending on the longitudinal variable only,

$$\chi_{\text{out}}(s,t) = \chi_{\text{in}}(s),$$

where the right-hand side is given by (3.1). Furthermore, at the points $x \in B_{r_0}(O) \setminus \Omega^a$ we put $\chi_{\text{out}}(x) = 1$, and finally, in the remaining part of the plane we choose χ_{out} independent of θ; in other words, as a function of the distance ρ from the origin O only, and such that χ_{out} is continuous in Ω_{out}. It is clear that the radial decay of such an external mollifier is determined by the behavior of the function (3.1).

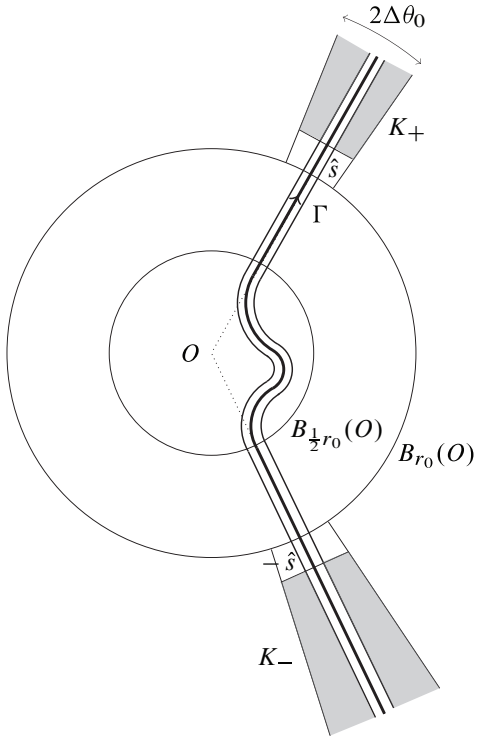

Figure 3. The regions used in the proof of Theorem 2.4

Since the potential is supported in Ω^a, the contribution to the quadratic form (2.3c) in the exterior region comes from the kinetic term only. The trial function factorizes into a product and our first goal is to show that the cross-term containing the integral of $2\nabla\phi \cdot \nabla\chi_{\text{out}}$ is small for large r_0; in particular, that one can make it smaller than $\frac{1}{16}\delta\nu$ with respect to the quantities appearing in (3.8).

Lemma 4.1. *We have*

$$\int_{\Omega_{\mathrm{out}}} |\nabla\psi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x \le \int_{\Omega_{\mathrm{out}}} |\nabla\phi(x)|^2 \chi_{\mathrm{out}}^2(x) \,\mathrm{d}\,x + \int_{\Omega_{\mathrm{out}}} |\phi(x)|^2 |\nabla\chi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x + \mathcal{O}(r_0^{-1}) \quad \textit{as} \;\; r_0 \to \infty. \tag{4.2}$$

Proof. Since $\psi_{\mathrm{out}} = \phi\chi_{\mathrm{out}}$ for $x \in \Omega_{\mathrm{out}}$, we have to estimate the integral $\int_{\Omega_{\mathrm{out}}} |\nabla\phi(x) \cdot \nabla\chi_{\mathrm{out}}(x)| \,\mathrm{d}\,x$ to deal with the cross-term. To this aim, we first note that $\chi_{\mathrm{out}} = 1$ holds inside $B_{r_0}(O)$ so we have to consider only the complement of the disk. In the conical sectors of $\mathbb{R}^2 \setminus B_{r_0}(O)$ with $\theta \in [\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0]$, the point nearest to (x, θ) lies on the straight part of Γ, as the distance to it is at most $\rho\Delta\theta < \frac{1}{2}\rho$ while that to the curved part is at least $\rho - \frac{1}{2}r_0 > \frac{1}{2}\rho$. This implies that $\nabla\phi$ is perpendicular to $\nabla\chi_{\mathrm{out}}$, and the corresponding contribution to the integral vanishes too. Finally, in view of our definition of χ_{out} in combination with (3.1), we see that $\nabla\chi_{\mathrm{out}}$ is bounded outside $B_{r_0}(O)$ and the two sectors, and, furthermore, we have $|\nabla\phi(x)| \le \xi\phi(x) \le C\,\mathrm{e}^{-\rho/2}$ which yields

$$\int_{\Omega_{\mathrm{out}}} |\nabla\phi(x) \cdot \nabla\chi_{\mathrm{out}}(x)| \,\mathrm{d}\,x \le C' \int_{\Omega_{\mathrm{out}}} |\nabla\phi(x)| \,\mathrm{d}\,x = \mathcal{O}(r_0^{-1})$$

as $r_0 \to \infty$ which we set out to prove. ∎

Let us turn to the second term on the right-hand side of (4.2).

Lemma 4.2. *We have*

$$\int_{\Omega_{\mathrm{out}}} |\phi(x)|^2 |\nabla\chi_{\mathrm{out}}(x)|^2 \,\mathrm{d}\,x = \mathcal{O}(r_0^{-1}) \quad \textit{as}\; r_0 \to \infty. \tag{4.3}$$

Proof. The integral over the disk $B_{r_0}(O)$ is again zero and using an argument analogous to that of the previous proof, one can check that the integral over the region outside the conical sectors is $\mathcal{O}(r_0^{-1})$ as $r_0 \to \infty$. Inside the sectors, we have

$$|\phi(x)|^2 |\nabla\chi_{\mathrm{out}}(x)|^2 = |\phi_0(t)|^2 |\chi'_{\mathrm{in}}(s)|^2 \le |\chi'_{\mathrm{in}}(s)|^2$$

with χ_{in} given by (3.1); recall that outside Ω^a the function ϕ_0 decays exponentially with the distance from Γ and $\phi_0(\pm a) = 1$ holds by assumption. Hence, the integral in (4.3) can be estimated by the squared norm of χ'_{in}, and since to a given r_0 the value of s_0 was chosen so that the parts of Γ with $|s| > s_0$ lay outside $B_{r_0}(O)$, one can pick an $s^* = s^*(r_0)$ in such a way that $\ln \frac{s^*}{s_0} > C r_0$ for some $C > 0$, and the claim follows. ∎

Combining these two lemmata, we see that for any fixed $\nu \in (0, \nu_0)$ one can choose r_0, s_0, and $\frac{s^*}{s_0}$ sufficiently large to satisfy the inequality

$$\int_{\Omega_{\text{out}}} |\nabla \psi_{\text{out}}(x)|^2 \, \mathrm{d}\, x \le \int_{\Omega_{\text{out}}} |\nabla \phi(x)|^2 \chi_{\text{out}}^2(x) \, \mathrm{d}\, x + \frac{1}{8}\delta\nu. \tag{4.4}$$

Next, we note that in Theorem 2.4 (a) the bias is absent, $V_0 = 0$, which means that function (4.1) satisfies

$$|\nabla\phi|^2 - \mu|\phi|^2 = 2|\nabla\phi|^2$$

almost everywhere in Ω_{out}. This means that we can estimate the whole exterior contribution to the form $Q[\psi] - \mu\|\psi\|^2$ by doubling the kinetic term and neglecting the one containing the eigenvalue μ. Combining then estimates (3.8) and (4.4), we further infer that in order to prove the theorem it is sufficient to check that

$$2\int_{\Omega_{\text{out}}} |\nabla\phi(x)|^2 \chi_{\text{out}}^2(x) \, \mathrm{d}\, x \le 2|\mu|^{1/2}\|\chi_{\text{in}}\|^2 + \frac{1}{8}\delta\nu$$

holds for some $\nu \in (0, \nu_0)$ and sufficiently large r_0, s_0, and $\frac{s^*}{s_0}$ depending on the values of δ and ν. Moreover, this is, in view of $|\nabla\phi|^2 = -\mu|\phi|^2$, further equivalent to

$$\int_{\Omega_{\text{out}}} |\phi(x)\chi_{\text{out}}(x)|^2 \, \mathrm{d}\, x \le |\mu|^{-1/2}\|\chi_{\text{in}}\|^2 + \frac{1}{16}|\mu|^{-1}\delta\nu. \tag{4.5}$$

The rest of the proof consists of verification of inequality (4.5). To begin with, we estimate the contribution to its left-hand side from the parts of the plane adjacent to the straight parts of the waveguide; we choose them as conical sectors similar to those used in the proof of Lemma 4.1. We recall that for $x = (\rho, \theta)$ with $\rho \ge r_0$ and $\theta \in [\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0]$ we can use the (s, t) coordinates simultaneously with the polar ones. We choose an $\hat{s} \ge r_0$ so that the parts of Γ with $|s| \ge \hat{s}$ lay outside $B_{r_0}(O)$, and at the same time we choose s_0 of (3.1) in such a way that $s_0 > \hat{s}$. Then we define

$$K_\pm := \{x\colon |s| \ge \hat{s},\ |t| \ge a,\ \theta \in [\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0]\}. \tag{4.6}$$

Within these sets, the closest points of Γ are those on the straight parts of the curve with the same coordinate s. Then it is easy to see that

$$\int_{\Omega_{\text{out}}\cap\{K_+\cup K_-\}} |\phi(x)\chi_{\text{out}}(x)|^2 \, \mathrm{d}\, x \le |\mu|^{-1/2}\|\chi_{\text{in}}\|^2_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))} \tag{4.7}$$

It remains to integrate the function $|\phi\chi_{\text{out}}|^2$ over $\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}$. Obviously, the integral will increase if we replace χ_{out} by one, hence to complete the proof, it

is in view of (3.8) and (4.7) enough to check the following inequality for the fixed δ given by our choice of the function g, some $\nu \in (0, \nu_0)$, $r_0 > 0$ and $\Delta\theta_0 > 0$:

$$\int_{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x \le 2\hat{s}|\mu|^{-1/2} + \frac{1}{16}|\mu|^{-1}\delta\nu; \tag{4.8}$$

we have used here the fact that $\|\chi_{\mathrm{in}}\|^2_{L^2((-\hat{s},\hat{s}))} = 2\hat{s}$. Let us note that it is easy to see that if (4.8) holds for some r_0, the same is true for any $r_0' > r_0$.

4.2. Curves with a piecewise constant curvature

We divide the verification of inequality (4.8) into two parts, considering first a particular class of the generating curves assuming additionally that

(s4) Γ is a C^1 curve consisting of two halflines and a *finite array of circular arcs;* we allow some of these arcs to be straight segments of a finite length.

Consequently, the signed curvature $\kappa(\cdot)$ of such a curve is a step function, being zero on the straight parts of Γ. To estimate the indicated integral under the additional assumption (s4), the two parts of Γ corresponding to $|s| > \hat{s}$ will be considered as arcs of zero curvature, cf. Remark 4.4 below. First of all, we note that the function $d_x\colon \mathbb{R} \to \mathbb{R}_+$, defined by $d_x(s) := \mathrm{dist}(x, \Gamma(s))$, is C^1-smooth for any $x \in \mathbb{R}^2$, and under the assumption (s4) it is piecewise monotonous because on each arc it can have at most one extremum. At the same time, $d_x(s) \to \infty$ holds as $s \to \pm\infty$, hence the function has a global minimum at a point s_x^0, positive as long as x does not lie on the curve, and in view of its continuity it may also have a finite number of local extrema. Among those, the extrema with largest and smallest value of s are clearly both minima. As a result, the numbers of local extrema with $s < s_x^0$ and $s > s_x^0$ are both even; they come in pairs of an adjacent minimum and maximum. Let s_x^1 and s_x^2 be the coordinates of such a pair, respectively; then we have obviously $d_x(s_x^1) < d_x(s_x^2)$, and consequently

$$\exp\{-2\xi(d_x(s_x^1) - a)\} - \exp\{-2\xi(d_x(s_x^2) - a)\} \ge 0. \tag{4.9}$$

Let us denote by $M_x^{\uparrow}$ the subset of coordinates of the *local* maxima and by $M_x^{\downarrow}$ the subset of coordinates of *all* the minima. Summing the inequalities (4.9) over the pairs of *local* extrema, we get

$$\sum_{\substack{s_x^i \in M_x^{\downarrow} \\ s_x^i \ne s_x^0}} \exp\{-2\xi(d_x(s_x^i) - a)\} - \sum_{s_x^i \in M_x^{\uparrow}} \exp\{-2\xi(d_x(s_x^i) - a)\} \ge 0.$$

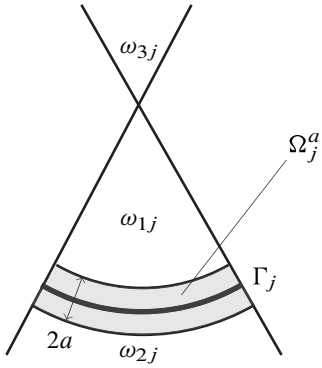

Figure 4. The regions appearing in Proposition 4.3

Adding finally $\exp\{-2\xi(d_x(s_x^0)-a)\}$ to both sides of this inequality, we arrive at

$$\exp\{-2\xi(d_x(s_x^0)-a)\} \le -\sum_{s_x^i\in M_x^{\uparrow}} \exp\{-2\xi(d_x(s_x^i)-a)\} + \sum_{s_x^i\in M_x^{\downarrow}} \exp\{-2\xi(d_x(s_x^i)-a)\}, \tag{4.10}$$

for all $x \in \Omega_{\text{out}}$, where the last sum now runs over all the minima. To estimate the integral in (4.8), we have to integrate the right-hand side of (4.10) over $\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}$. To this aim, let us first collect several simple geometric statements easy to check.

Proposition 4.3. *Let Γ_j be one of the arcs of Γ and denote by $\omega_{1j}, \omega_{2j}, \omega_{3j}$, and Ω_j^a the open regions shown in Figure 4. Then the following holds true.*

(i) *If $x \in \omega_{1j} \cup \omega_{2j}$, then $d_x(\cdot)$ has a minimum in the interior of Γ_j.*

(ii) *If $x \in \omega_{3j}$, then $d_x(\cdot)$ has a maximum in the interior of Γ_j.*

(iii) *$x \notin \bar{\omega}_{1j} \cup \bar{\omega}_{2j} \cup \bar{\omega}_{3j} \cup \bar{\Omega}_j^a$, then $d_x(\cdot)$ has no extremum on Γ_j.*

(iv) *$d_x(\cdot)$ cannot have more than one critical point in the interior of Γ_j.*

(v) *If $x \in \omega_{kj}$ for any of $k = 1, 2, 3$, then the one-sided derivative $d_x'(s) \neq 0$ at the endpoints of Γ_j.*

Remark 4.4. With an abuse of terminology, we include into (s4) also situations when a Γ_j is a straight segment, that is, $\kappa(s) = 0$ holds on Γ_j. In that case the wedge-shaped regions ω_{1j} and ω_{2j} become semi-infinite strips and ω_{3j} does not exist. This concerns, in particular, the two straight parts of Γ corresponding to $|s| > \hat{s}$.

Within the regions, we introduced the minimal and maximal distances are easily expressed; we have

$$\begin{aligned} d_x(s_x^i) &= \operatorname{dist}(x, \Gamma_j) && \text{if } s_x^i \in \Gamma_j \cap M_x^{\downarrow}; \\ d_x(s_x^i) &= |\kappa_j|^{-1} + \operatorname{dist}(x, O_j) && \text{if } s_x^i \in \Gamma_j \cap M_x^{\uparrow}, \end{aligned} \tag{4.11}$$

where O_j is the center of the corresponding circular arc.

Let $\iota_j^{1,2}$ and ι_j^3 be the characteristic functions of the sets $\omega_{1j} \cup \omega_{2j}$ and ω_{3j}, respectively. In view of Proposition 4.3 and relations (4.11), we can replace the first term at the right-hand side of (4.10), everywhere except the zero measure set referring to the boundaries of the regions ω_{kj}, $k = 1, 2, 3$, with

$$-\sum_j \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\iota_j^3(x) \tag{4.12}$$

and the second term similarly by

$$\sum_j \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\iota_j^{1,2}(x) \tag{4.13}$$

Integrating now (4.12) and (4.13) over $\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}$ and exchanging the order of integration over x with summation over j, using (4.10) we can estimate

$$\int_{\Omega_{\text{out}}\setminus\{K_+\cup K_-\}} \exp\{-2\xi(d_x(s_x^0) - a)\}\,\mathrm{d}\,x$$

from above by

$$\begin{aligned} &\sum_j \int_{(\omega_{1j}\cup\omega_{2j})\cap\{\Omega_{\text{out}}\setminus\{K_+\cup K_-\}\}} \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}\,x \\ &-\sum_j \int_{\omega_{3j}\cap\{\Omega_{\text{out}}\setminus\{K_+\cup K_-\}\}} \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\,\mathrm{d}\,x, \end{aligned} \tag{4.14}$$

where the sums run over all the indices of Γ_j including those of the straight segments of the curve with $|s| > \hat{s}$. Note that this estimate includes in general a double counting since the same x may belong to different ω_{kj}; this does not matter as long as we consider the contributions referring of a given Γ_j together.

Our next goal is to show that expression (4.14) cannot decrease if we replace the integration domains by $(\omega_{1j} \cup \omega_{2j}) \setminus \{K_+ \cup K_-\}$ and $\omega_{3j} \setminus \{K_+ \cup K_-\}$, respectively. To this aim, consider a fixed arc Γ_{j_0} and the respective segment $\Omega_{j_0}^a$ of the strip Ω^a as indicated in Figure 4. For a point $x \in \Omega_{j_0}^a$, the function $d_x(\cdot)$ has the global

minimum on Γ_{j_0} with a coordinate s_x^0 and all the local extrema, if they exist, come in pairs situated outside Γ_{j_0}. This yields the estimate

$$0 \le -\sum_{s_x^i \in M_x^{\uparrow}} \exp\{-2\xi(d_x(s_x^i) - a)\} + \sum_{\substack{s_x^i \in M_x^{\downarrow} \\ s_x^i \neq s_x^0}} \exp\{-2\xi(d_x(s_x^i) - a)\}.$$

Using again Proposition 4.3, we get for any $x \in \Omega_{j_0}^a$ the inequality

$$\begin{aligned} 0 \le & -\sum_{j \neq j_0} \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\iota_j^3(x) \\ & + \sum_{j \neq j_0} \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\iota_j^{1,2}(x). \end{aligned} \tag{4.15}$$

Since $x \in \Omega_{j_0}^a$, we are able to replace the indicator functions in this expression by their restriction $\iota_j^3 \restriction \Omega_{j_0}^a$ and $\iota_j^{1,2} \restriction \Omega_{j_0}^a$, respectively. Noting further that $\omega_{kj_0} \cap \Omega_{j_0}^a = \emptyset$ holds for $k = 1, 2, 3$, we see that (4.15) remains valid if the summation is taken over all the j's. Integrating then the right-hand side over $x \in \Omega^a$ we arrive at the inequality

$$\begin{aligned} 0 \le & -\sum_j \int_{\omega_{3j} \cap \Omega_{j_0}^a} \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\,\mathrm{d}\,x \\ & + \sum_j \int_{(\omega_{1j} \cup \omega_{2j}) \cap \Omega_{j_0}^a} \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}\,x, \end{aligned}$$

and summing this result over j_0 we get

$$\begin{aligned} 0 \le & -\sum_j \int_{\omega_{3j} \cap \Omega^a} \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\,\mathrm{d}\,x \\ & + \sum_j \int_{(\omega_{1j} \cup \omega_{2j}) \cap \Omega^a} \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}\,x. \end{aligned} \tag{4.16}$$

Combining now (4.14) and (4.16), we obtain

$$\begin{aligned} & \int_{\Omega_{\mathrm{out}} \setminus \{K_+ \cup K_-\}} |\phi(x)|^2\,\mathrm{d}\,x \\ & \le \sum_j \int_{(\omega_{1j} \cup \omega_{2j}) \setminus \{K_+ \cup K_-\}} \exp\{-2\xi(\operatorname{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}\,x \\ & \quad - \sum_j \int_{\omega_{3j} \setminus \{K_+ \cup K_-\}} \exp\{-2\xi(|\kappa_j|^{-1} + \operatorname{dist}(x, O_j) - a)\}\,\mathrm{d}\,x. \end{aligned} \tag{4.17}$$

The summation in (4.17) runs over all the curve segments including the straight ones. Let us first estimate the contribution of these infinite 'arcs' to the positive part of (4.17) having in mind that in accordance with Remark 4.4 the segments with $\kappa = 0$ do not contribute to the negative one. We denote by Γ_+ the segment with $s > \hat{s}$ and by ω_{1+}, ω_{2+} the corresponding semi-infinite strips, then we have

$$\begin{aligned}
&\int\limits_{(\omega_{1+}\cup\omega_{2+})\setminus\{K_+\cup K_-\}} \exp\{-2\xi(\operatorname{dist}(x,\Gamma_+)-a)\}\,\mathrm{d}\,x \\
&\qquad\le 2\int\limits_{\omega_{1+}\setminus K_+} \exp\{-2\xi(\operatorname{dist}(x,\Gamma_+)-a)\}\,\mathrm{d}\,x \\
&\qquad= 2\int\limits_{\rho(\hat{s})\cos\Delta\theta}^{\infty}\ \int\limits_{s\sin\Delta\theta_0}^{\infty} \exp\{-2\xi(t-a)\}\,\mathrm{d}\,t\,\mathrm{d}\,s \\
&\qquad= \frac{\mathrm{e}^{2\xi a}}{4\xi^2\sin\Delta\theta_0}\mathrm{e}^{-\xi\sin 2\Delta\theta_0\cdot\rho(\hat{s})},
\end{aligned} \tag{4.18}$$

where $\rho(\hat{s})$ is the distance of the point $\Gamma(\hat{s})$ to the origin. In view of our choice of $\hat{s}$ we have $\rho(\hat{s}) \ge r_0$ and the integral at the right-hand side of (4.18) can be made arbitrarily small by choosing r_0 large enough. An analogous argument applies to the segment of Γ with $s < -\hat{s}$.

Denote now by $\sum_j^*$ the sum over all the Γ_j except of $\Gamma_\pm$. The conclusion just made allows us to replace the sum $\sum_j$ in (4.18) by $\sum_j^*$ with an error which can be made arbitrarily small by choosing an appropriately large r_0. Furthermore, we note that the positive part of (4.14) cannot decrease if we enlarge the integration domain in all the integrals there replacing $(\omega_{1j}\cup\omega_{2j})\setminus\{K_+\cup K_-\}$ by $\omega_{1j}\cup\omega_{2j}$.

Our next goal is to argue that we can do the same in the negative part of (4.14) replacing $\omega_{3j}\setminus\{K_+\cup K_-\}$ by ω_{3j}. In such a case, of course, the corresponding change of the integrals goes in the wrong way; our aim is to show that it again produces an error which can be made small if r_0 is large. Indeed, regions ω_{3j} exist only for the curved segments of Γ and those are by assumption inside $B_{\frac{1}{2}r_0}(O)$, while the regions $K_\pm$ are outside $B_{r_0}(O)$. Consequently, the contributions from the extended integration domains are

$$\begin{aligned}
&\int\limits_{\omega_{3j}\cap\{K_+\cup K_-\}} \exp\{-2\xi(|\kappa_j|^{-1}+\operatorname{dist}(x,O_j)-a)\}\,\mathrm{d}\,x \\
&\qquad\le \mathrm{e}^{2\xi a}\,|\Gamma_j|\int\limits_{\rho\ge\sqrt{3}r_0/2}^{\infty} \mathrm{e}^{-\sqrt{3}\,\xi\rho}\,\rho\,\mathrm{d}\,\rho = |\Gamma_j|\,\mathcal{O}(\mathrm{e}^{-3\xi r_0/2})
\end{aligned} \tag{4.19}$$

uniformly in j, and since the length of the curved part is finite, the error coming from the extension of the integration domain is $\mathcal{O}(\mathrm{e}^{-3\xi r_0/2})$. Combining (4.19) with (4.17) we get

$$\begin{aligned}\int\limits_{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}} &|\phi(x)|^2\,\mathrm{d}\,x \\ &\le \sum_j{}^{*} \int\limits_{\omega_{1j}\cup\omega_{2j}} \exp\{-2\xi(\mathrm{dist}(x,\Gamma_j)-a)\}\,\mathrm{d}\,x \\ &\quad - \sum_j{}^{*} \int\limits_{\omega_{3j}} \exp\{-2\xi(|\kappa_j|^{-1}+\mathrm{dist}(x,O_j)-a)\}\,\mathrm{d}\,x + \mathcal{O}(\mathrm{e}^{-3\xi r_0/2}). \qquad (4.20)\end{aligned}$$

It is not difficult to evaluate the integrals appearing at the right-hand side of (4.20): we have

$$\begin{aligned}\int\limits_{\omega_{2j}} \exp\{-2\xi(\mathrm{dist}(x,O_j)-a)\}\,\mathrm{d}\,x &= \Big(\frac{1}{2\xi}+\frac{a|\kappa_j|}{2\xi}+\frac{|\kappa_j|}{4\xi^2}\Big)|\Gamma_j| \\ &= \frac{|\Gamma_j|}{2\xi}+\frac{a}{2\xi}\int\limits_{\Gamma_j}|\kappa(s)|\,\mathrm{d}\,s+\frac{1}{4\xi^2}\int\limits_{\Gamma_j}|\kappa(s)|\,\mathrm{d}\,s \qquad (4.21)\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{\omega_{1j}} \exp\{-2\xi(\mathrm{dist}(x,O_j)-a)\}\,\mathrm{d}\,x \\ &\quad = \frac{|\Gamma_j|}{2\xi}-\frac{a}{2\xi}\int\limits_{\Gamma_j}|\kappa(s)|\,\mathrm{d}\,s-\frac{1}{4\xi^2}\int\limits_{\Gamma_j}|\kappa(s)|\,\mathrm{d}\,s+\frac{1}{4\xi^2}\int\limits_{\Gamma_j}\mathrm{e}^{-2\xi(|\kappa(s)|^{-1}-a)}|\kappa(s)|\,\mathrm{d}\,s\end{aligned}$$

for the positive part of the estimate, while in the negative one we use

$$\int\limits_{\omega_{3j}} \exp\{-2\xi(\kappa_j^{-1}+\mathrm{dist}(x,O_j))\}\,\mathrm{d}\,x = \frac{1}{4\xi^2}\int\limits_{\Gamma_j}\mathrm{e}^{-2\xi(|\kappa(s)|^{-1}-a)}|\kappa(s)|\,\mathrm{d}\,s.$$

Summing finally the contributions from given Γ_j, we get $|\Gamma_j|\xi^{-1}$; hence expression (4.14) is smaller that $2|\mu|^{-1/2}\hat{s}+o(r_0)$, which according to inequality (4.8) proves Theorem 2.4 (a) under the additional assumption (s4).

4.3. Extension to the case when condition (s4) does not hold

Before turning to this task, let us recall how the parameters of the trial function constructed above depend on each other. Assume that the potential v, and thus μ and ϕ_0,

and also $g(t,s)$ are fixed; then the parameters δ and ν_0 introduced in Section 3.1 depend on the curvature only. Note that this fact justifies *a posteriori* the possibility to use a fixed function $g(s,t)$, since having a family of curves the curvatures of which do not differ much, one can certainly find a common interval of the variable s on which these curvatures do not change sign. As for ν, it can be chosen as any number in $(0,\nu_0)$ noting, however, that if $\nu \to 0$, the parameters r_0, s_0, s^* and $\hat{s}$ which we pick after fixing ν must tend to infinity. The number $\Delta\theta_0$ in Section 4.1 depends on the geometry of Γ only, more specifically on the angle between the asymptotes of Γ.

Proceeding with the choice of the parameters, we pick $r_0 > 0$ depending on ν and $\Delta\theta_0$; it must be sufficiently large so that the curved part of Γ is contained in $B_{\frac{1}{2}r_0}(O)$ and, at the same time, the error in (4.8) coming from Lemmata 4.1 and 4.2 does not exceed $\frac{1}{16}|\mu|^{-1}\delta\nu$. Next, to the chosen r_0 we pick $\hat{s}$ and $s_0 > \hat{s}$ so that the parts of Γ with $|s| > \hat{s}$ are outside $B_{r_0}(O)$, and finally, we pick an $s^* \gg s_0$ depending on s_0 and ν.

After this preliminary, let us turn to the proof that (4.8) remains valid without assumption (s4). We will use the same trial function as before, in particular, its outer part will be again of the form $\psi_{\text{out}} = \phi\chi_{\text{out}}$ with ϕ given (4.1); the idea is to approximate the curve Γ satisfying (s1) by curves with a piecewise constant curvature, of the same length and with the same halfline asymptotes. Specifically, we are going to employ the following approximation result.

Theorem 4.5 (Sabitov and Slovesnov [14]). *Let Γ be a C^3-smooth curve of a finite length consisting of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved. Then Γ can be approximated by a C^1-smooth function $\widehat{\Gamma}$ of the same length, the curvature of which is piecewise constant having jumps at the points $s_1 < s_2 < \cdots < s_N$, in the sense that the estimates*

$$\|\Gamma^{(m)} - \widehat{\Gamma}^{(m)}\|_\infty \le C \max_{1\le k\le N-1} (s_{k+1}-s_k)^{3-m}, \quad m = 0,1,2, \tag{4.22}$$

hold with some $C > 0$ for the function Γ and its two first derivatives. The endpoints of Γ and $\widehat{\Gamma}$ coincide, and the same is true for the tangent vectors at these points, and moreover, on each subinterval (s_k, s_{k+1}) the curvature $\hat{\kappa}$ of $\widehat{\Gamma}$ satisfies the inequality $\min_{s\in(s_k,s_{k+1})}\kappa(s) \le \hat{\kappa}(s) \le \max_{s\in(s_k,s_{k+1})}\kappa(s)$.

It is obvious that the hypotheses of Theorem 4.5 are satisfied under our assumption (s1) on the C^3 smooth parts of Γ.

Let $\{\Gamma_n\}$ by a sequence of curves with the following properties:

(i) Γ_n coincides with Γ at all parts of the curve where $\kappa(s) = 0$, in particular, all the Γ_n's have the same asymptotes as Γ;

(ii) on the curved parts of Γ the Γ_n's approximate Γ in the sense of Theorem 4.5 with $\max_k |s_{k+1} - s_k| < \frac{1}{n}$.

Recall that the curved part of Γ has finite length which allows us to use Theorem 4.5, and note also that we are abusing notation here using for the approximating curve the symbol which coincides with the one which in the previous section meant the arcs of a curve satisfying assumption (s4).

Let $\hat{\delta}_n$ and $\hat{\nu}_{0,n}$ be the quantities corresponding to $\widehat{\Gamma}_n$ in the same way as δ and ν_0, respectively, correspond to Γ. It is easy to see from (4.22) and the point (ii) above that $|\hat{\kappa}_n(s) - \kappa(s)| < Cn^{-1}$ holds for some $C > 0$ independent of s, which implies $\hat{\delta}_n \to \delta$ and $\hat{\nu}_{0,n} \to \nu_0$ as $n \to \infty$. By $\Omega_{\mathrm{out}}^{(n)}$, we denote the region exterior to the strip of halfwidth a built around $\widehat{\Gamma}_n$. The expression on the left-hand side of (4.8) can be in this case rewritten as

$$\int_{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x = \int_{\Omega_{\mathrm{out}}^{(n)}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x + \int_{\Omega_{\mathrm{out}}\setminus\Omega_{\mathrm{out}}^{(n)}} |\phi(x)|^2 \,\mathrm{d}\,x \tag{4.23}$$

Since Γ and $\widehat{\Gamma}_n$ differ on a bounded interval of s only, and since $|\Gamma(s) - \widehat{\Gamma}_n(s)|_\infty \to 0$ holds as $n \to \infty$ by Theorem 4.5, uniformly in s, in the second term on the right-hand side we integrate the function $|\phi(x)|^2 \le 1$ over a region the measure of which tends to zero as $n \to \infty$. For the first integral on the right-hand side of (4.23), we have

$$\begin{aligned}\int_{\Omega_{\mathrm{out}}^{(n)}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x &\le \mathrm{e}^{2\xi\|\Gamma-\widehat{\Gamma}_n\|_\infty} \int_{\Omega_{\mathrm{out}}^{(n)}\setminus\{K_+\cup K_-\}} \mathrm{e}^{-2\xi\{\mathrm{dist}(x,\widehat{\Gamma}_n)-a\}} \,\mathrm{d}\,x \\ &\le [1+\varepsilon_n^{(1)}]\Big[2\hat{s}|\mu|^{-1/2} + \frac{1}{16}|\mu|^{-1}\hat{\delta}_n\hat{\nu}_n\Big],\end{aligned} \tag{4.24}$$

where $\nu_n \in (0, \nu_{0,n})$ is fixed and $\varepsilon_n^{(1)} \to 0$ as $n \to \infty$; we have used the fact that (4.8) is valid for $\widehat{\Gamma}_n$ as the latter satisfies condition (s4) and the approximation preserves by Theorem 4.5 the length of the curve. Since the points $s = \pm\hat{s}$ are on the straight parts of Γ (coinciding with those of $\widehat{\Gamma}_n$), the distance between them along both Γ and $\widehat{\Gamma}_n$ is the same being equal to $2\hat{s}$.

Moreover, to derive (4.8) for curves satisfying condition (s4) we required r_0 and $\hat{s}$ which depend on ν to be large. For the approximating curves, r_0 depends on $\hat{\nu}_n$ but we avoid the situation when $\hat{\nu}_n \to 0$ which would imply $r_0 \to \infty$ as mentioned in the opening of this section. For the sake of definiteness, we fix $\hat{\nu}_n = \frac{1}{2}\hat{\nu}_{0,n}$ and consider n sufficiently large to have $\hat{\nu}_{0,n} \ge \frac{1}{2}\nu_0$. Then we can choose $\hat{s}$ in (4.24) independent of n, which yields

$$\int_{\Omega_{\mathrm{out}}^{(n)}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x \le 2\hat{s}|\mu|^{-1/2} + \frac{1}{16}|\mu|^{-1}\hat{\delta}_n\frac{1}{4}\nu_0 + \varepsilon_n^{(2)}, \tag{4.25}$$

where $\varepsilon_n^{(2)} \to 0$ as $n \to \infty$. Combining this with the fact that $\hat{\delta}_n \to \delta$ for $n \to \infty$ as mentioned above, we infer from (4.25) that relation (4.8) holds for $\nu = \frac{1}{8}\nu_0$ and all n large enough; this concludes the proof.

5. Concluding proofs of Theorems 2.2 and Theorem 2.4

It remains to establish parts (b) of both the main results. Let us begin with Theorem 2.4 and prove it in the situation when Ω_+ is *convex*; by assumption (p3), we have $\mu < 0$. Inside Ω^a we choose the trial function as in the previous proofs so that inequality (3.7) is valid for any $V_0 \geq 0$; recall that we derived it assuming the presence of a bias. Moreover, picking a suitable coordinate $s^* \gg s_0$ at the right-hand side of (3.1), the last term in (3.7) can be made as before smaller than $\frac{1}{4}\delta\nu$. Outside the strip Ω^a, we set

$$\phi(x) := \phi_\pm \exp\{-|\xi_\pm|(\mathrm{dist}(x,\Gamma) - a)\} \quad \text{if } x \in \Omega_\pm^{\mathrm{out}}, \tag{5.1}$$

recalling that $\phi_- = 1$, which is a natural generalization of (4.1), and we employ the same mollifier χ_{out} as before, cf. Section 4.2. Repeating the argument of this section, we arrive at inequality (4.4), however, now with the function ϕ given by (5.1). Let us split the outer contribution to the quadratic form into two parts referring, respectively, to $\Omega_\pm^{\mathrm{out}}$, for which we have

$$\begin{aligned} Q_{\mathrm{out}}^{(+)}[\psi_{\mathrm{out}}] &= \int_{\Omega_+^{\mathrm{out}}} |\nabla\psi_{\mathrm{out}}(x)|^2\,\mathrm{d}\,x + \int_{\Omega_+^{\mathrm{out}}} (V_0-\mu)|\psi_{\mathrm{out}}(x)|^2\,\mathrm{d}\,x \\ &\leq \int_{\Omega_+^{\mathrm{out}}} \big\{|\nabla\phi(x)|^2 + (V_0-\mu)|\phi(x)|^2\big\}\chi_{\mathrm{out}}(x)^2\,\mathrm{d}\,x + \frac{1}{16}\delta\nu, \qquad (5.2) \\ Q_{\mathrm{out}}^{(-)}[\psi_{\mathrm{out}}] &\leq \int_{\Omega_-^{\mathrm{out}}} \big\{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\big\}\chi_{\mathrm{out}}(x)^2\,\mathrm{d}\,x + \frac{1}{16}\delta\nu \end{aligned}$$

in view Lemmata 4.1 and 4.2 provided that r_0 is chosen large enough. As in the proof of the first part of Theorem 2.2, we choose an $\hat{s} \in [r_0, s_0)$ for which the parts of Γ with $|s| \geq \hat{s}$ are outside $B_{r_0}(O)$ and use the regions $K_\pm$ introduced by (4.6). By definition (5.1), we have

$$|\nabla\phi|^2 = \xi_\pm^2|\phi|^2 \quad \text{for } x \in \Omega_\pm^{\mathrm{out}}.$$

Within $\Omega_{\mathrm{out}} \cap \{K_+ \cup K_-\}$, we may use the (s,t) coordinates, and noting the ϕ is independent of s there, and, as a function of t it coincides with the eigenfunction ϕ_0 of h, cf. (2.2), we get

$$\int\limits_{\Omega_+^{\mathrm{out}}\cap\{K_+\cup K_-\}} \{|\nabla\phi(x)|^2 + (V_0-\mu)|\phi(x)|^2\}\chi_{\mathrm{out}}(x)^2\,\mathrm{d}\,x$$
$$\leq |\xi_+|\,\phi_+^2\,\|\chi_{\mathrm{in}}\|^2_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))} \tag{5.3a}$$

and

$$\int\limits_{\Omega_-^{\mathrm{out}}\cap\{K_+\cup K_-\}} \{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\}\chi_{\mathrm{out}}(x)^2\,\mathrm{d}\,x$$
$$\leq \xi_-\|\chi_{\mathrm{in}}\|^2_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))}. \tag{5.3b}$$

So far, we have not employed the convexity of Ω_+; we will need it from now on to estimate the integrals (5.3). As before, we will first prove the second claim of Theorem 2.2 under the additional assumption (s4) using again the notation introduced in Figure 4.

The part $\Omega_{\mathrm{out}}^{(-)}$ consists then of a finite number of sectors ω_{2j} which in view of the convexity assumption do not overlap mutually. Moreover, since Γ is a C^1 curve, the neighboring sectors have common boundaries which are the halflines normal to Γ at the points where the curvature changes value, and as a result, the closures of sectors ω_{2j} cover the region $\Omega_{\mathrm{out}}^{(-)}$. Let $\Gamma_\pm$ and $\omega_{k\pm}$, $k=1,2$, be the same as in Theorem 2.4 (a). By the same reasoning as in the proof of the latter, cf. (4.18), one can check that the contribution of the regions $\omega_{k\pm}\setminus\{K_+\cup K_-\}$ to the integrals (5.3) can be made arbitrarily small by choosing r_0 sufficiently large. Using further the fact that $|\chi_{\mathrm{out}}|\leq 1$ in combination with (4.21), we get

$$\int\limits_{\Omega_{\mathrm{out}}^{(-)}\setminus\{K_+\cup K_-\}} \{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\}\chi_{\mathrm{out}}(x)^2\,\mathrm{d}\,x$$
$$\leq 2|\mu|\left[\frac{2\hat{s}}{2\xi_-} + \frac{a}{2\xi_-}\int\limits_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}\,s + \frac{1}{4\xi_-^2}\int\limits_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}\,s\right] + \mathcal{O}(\mathrm{e}^{-cr_0})$$
$$= 2\xi_-\hat{s} + a\xi_-\int\limits_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}\,s + \frac{1}{2}\int\limits_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}\,s + \mathcal{O}(\mathrm{e}^{-cr_0}) \tag{5.4}$$

for some $c>0$, and since $\kappa(s)=0$ for $|s|>\hat{s}$, we can let the variable s in the above integrals run over the whole $\mathbb{R}$. Comparing now the right-hand side of (5.4) with that

of (3.7), we see that the terms containing ξ_- in the latter have their counterparts here with the opposite sign, hence they cancel mutually.

Next, we estimate the contribution to (5.3a) coming from $\Omega_{\mathrm{out}}^{(+)}$. We note that $|\nabla\phi|^2 = (-\mu + V_0)|\phi|^2 = |\xi_+|^2|\phi(x)|^2$ holds almost everywhere in Ω_+^{out} which means that the integral at the right-hand side of (5.2) can be rewritten as

$$2|\xi_+|^2 \int\limits_{\Omega_+^{\mathrm{out}}} |\phi(x)|^2 \chi_{\mathrm{out}}(x)^2 \,\mathrm{d}\,x.$$

In analogy with (4.10), we can estimate the function ϕ using local extrema of the distance function, namely

$$\begin{aligned} |\phi(x)|^2 &= \phi_+^2 \exp\{-2|\xi_+|(d_x(s_x^0) - a)\} \\ &\le \phi_+^2 \Big[-\sum_{s_x^i \in M_x^{\uparrow}} \exp\{-2|\xi_+|(d_x(s_x^i) - a)\} + \sum_{s_x^i \in M_x^{\downarrow}} \exp\{-2|\xi_+|(d_x(s_x^i) - a)\} \Big]. \end{aligned} \tag{5.5}$$

As in Theorem 2.4 (a), we want to replace the integral of the expression at the right-hand side of (5.5) over $\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}$ by the sum of the integrals over the regions ω_{1j} and ω_{3j} corresponding to the partition of the curve segment with $s \in [-\hat{s}, \hat{s}]$ into circular arcs. In analogy with relation (4.14), we get

$$\begin{aligned} &\int\limits_{\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x \\ &\quad \le \phi_+^2 \sum_j \Bigg\{ \int\limits_{\omega_{1j} \cap \{\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(\mathrm{dist}(x, \Gamma_j) - a)\} \,\mathrm{d}\,x \\ &\qquad\qquad - \int\limits_{\omega_{3j} \cap \{\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\} \,\mathrm{d}\,x \Bigg\}, \end{aligned} \tag{5.6}$$

where in contrast to (4.14) the right-hand side (5.6) does not involve integrals over ω_{2j} because in view of the convexity assumption we have $\Omega_{\mathrm{out}}^{(+)} \cap \omega_{2j} = \emptyset$ holds for any j.

Following the strategy used in the proof of Theorem 2.4 (a), we want to replace integrals over $\omega_{kj} \cap \{\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}\}$, $k = 1, 3$, with those over the extended regions $\omega_{kj} \setminus \{K_+ \cup K_-\}$, respectively. To this aim, we employ the following simple geometric result.

Lemma 5.1. *Suppose that $x \in \Omega_-$ does not belong to the boundaries of ω_{kj}, $k = 1, 2, 3$, for any j. Let further the distance function $d_x(s)$ reach a minimum which is not global at a point of the curve belonging to an arc Γ_{j^*}; then we have $x \in \omega_{1j^*}$.*

The lemma in fact says that if Ω_+ is convex, it cannot happen that $x \in \omega_{2j^*}$, which is obviously equivalent to the following claim.

Lemma 5.1′. *Let $x \in \Omega_-$. For any distance function extremum, except the global minimum, the segment L_x^i connecting the points x and $\Gamma(s_x^i)$ approaches the curve from the side of Ω_+.*

Proof. The point of global minimum is obviously approached from the region where x lies, that is, from Ω_-. The next two extrema on both sides of s_x^0, provided they exist, are necessarily maxima, and in view of the assumed convexity of Ω_+ the segments L_x^i cannot approach $\Gamma(s_x^i)$ from the side of Ω_-. We denote by $L(s)$ the segment connecting the point x with $\Gamma(s)$. The side from which $L(s)$ approaches the curve can change only at the points where the angle $\beta(s)$ between the segment $L(s)$ and L_x^0 corresponding to the global minimum of $d_x(\cdot)$ has, as a function of s, a local maximum or minimum. Since the curve Γ is by assumption C^1-smooth, and so is $\beta(\cdot)$, the lines connecting such points with x are tangent to Ω, however, a convex region cannot cross its own tangent, hence the extrema of the function $\beta(\cdot)$ are global, one maximum and one minimum. The corresponding points s_x^i, provided both of them exist, lie on both sides of s_x^0 because a convex region can have only two tangents passing through an exterior point x and the point $\Gamma(s_x^i)$ lies between the two tangent points on the boundary of Ω_+. The same tangent argument shows that once the $L(s)$ switches the side from which it approached Γ it can never come back. ∎

As before, all the local extrema of $d_x(\cdot)$ for $x \in \Omega_-$ except the global minimum come in pairs, so in analogy with (4.14) we are able to estimate the expression

$$\phi_+^2 \int\limits_{\Omega_- \setminus \{K_+ \cup K_-\}} \exp\{-2|\xi_+|(d_x(s_x^0) - a)\}\, \mathrm{d}\, x$$

from above by

$$\phi_+^2 \sum_j \Bigg\{ \int\limits_{\omega_{1j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(\mathrm{dist}(x, \Gamma_j) - a)\}\, \mathrm{d}\, x \\ - \int\limits_{\omega_{3j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\, \mathrm{d}\, x \Bigg\}, \tag{5.7}$$

where in view of Lemma 5.1 the first part does not include integration over $\omega_{2j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}$. Adding (5.7) to (5.6), we get

$$\begin{aligned}
&\int_{\Omega_{\mathrm{out}}^{(+)}\setminus\{K_+\cup K_-\}} |\phi(x)|^2 \,\mathrm{d}\,x \\
&\quad \le \phi_+^2 \sum_j \Bigg\{ \int_{\omega_{1j}\cap\widetilde{\Omega}} \exp\{-2|\xi_+|(\mathrm{dist}(x,\Gamma_j)-a)\}\,\mathrm{d}\,x \\
&\qquad\qquad - \int_{\omega_{3j}\cap\widetilde{\Omega}} \exp\{-2|\xi_+|(|\kappa_j|^{-1}+\mathrm{dist}(x,O_j)-a)\}\,\mathrm{d}\,x \Bigg\}, \qquad (5.8)
\end{aligned}$$

where $\widetilde{\Omega} := \Omega_- \cup \{\Omega_{\mathrm{out}}^{(+)} \setminus \{K_+ \cup K_-\}\}$. Moreover, applying again the argument that lead to (4.19) we infer that one can replace $\omega_{1j} \cap \widetilde{\Omega}$ and $\omega_{3j} \cap \widetilde{\Omega}$ in (5.8) by $\omega_{1j} \setminus \{K_+ \cup K_-\}$ and $\omega_{3j} \setminus \{K_+ \cup K_-\}$, respectively, with an error which can be made arbitrarily small by choosing r_0 large enough. The rest of the proof of Theorem 2.4 (b) for a convex Ω_+ repeats the corresponding part of the proof of the part (a); in the final step we take into account that a convex Γ can be approximated by convex curves of piecewise constant curvature so we can proceed as in Section 4.3; note that by (4.22) the curvature of Γ is approximated pointwise by the those of curves $\widehat{\Gamma}_n$.

To complete the proof of Theorem 2.4 (b), assume next that Ω_+ is *concave*. This case is already easy given the fact that in the first part of the proof we have not used the difference between $|\xi_+|$ and $|\xi_-|$, or between ϕ_+ and ϕ_-; the latter was set to one for convenience only. The role of the convexity was just to help us to distinguish the extrema of the distance function referring to the two outer parts of the trial function; if Ω_- is convex, we can repeat the argument step by step interchanging the roles of Ω_- and Ω_+ arriving thus at the sought claim.

It remains to prove Theorem 2.2 (b) where we have $\mu = 0$ by assumption and Ω_+ is again convex. Since $V_0 > 0$, the equation $h\phi = 0$ has a resonance solution ϕ_0 which is constant for $t \le -a$ and decays exponentially for $t > a$; as before, we normalize it putting $\phi_- = 1$. We have to construct a trial function $\psi \in H^2(\mathbb{R}^2)$ which makes the quadratic form (2.3c), now containing the potential bias, negative. We use elements of the previous proofs. In particular, inside Ω^a the function will be given by (3.2) and (3.1). Outside Ω^a, the trial function in Ω_- will be the same as in the proof of Theorem 2.2 (a), cf. Section 3.2, while in Ω_+ we choose it as in the of Theorem 2.4 (b) discussed above, putting there $\mu = 0$, in other words, as (5.1) in which in view of (3.5) we set $\xi_+ = -\sqrt{V_0}$. Repeating then the estimates used to prove Theorem 2.2 (a) in

Ω_- and Theorem 2.4 (a) in Ω_+, we obtain

$$Q[\psi] = -\frac{1}{8}\delta\nu - \int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s + o(\psi), \tag{5.9}$$

where the error term can be made arbitrarily small by choosing large r_0 and s^* in (3.1). In view of the assumed convexity of Ω_+ we have $\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}\,s > 0$, hence choosing the parameters properly we can make the form negative; this concludes the proof of Theorem 2.2 (b).

Remark 5.2. As we have noted in the introduction, the 'two-sided' validity of Theorem 2.4 (b) does not extend to the zero-energy resonance case. The above proof indicates the source of this difference. While for $\mu < 0$ we can use the trial function from the proof of Theorem 2.2 (b) and simply switch the roles of Ω_+ and Ω_-, a similar interchange does not work if $\mu = 0$ because it leads to the sign change of the second term on the right-hand side of (5.9) and we are obviously not free to choose $\delta\nu$ to compensate this positive number.

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Pavel Exner
Doppler Institute for Mathematical Physics and Applied Mathematics,
Czech Technical University, Břehová 7, 11519 Prague; Department of Theoretical Physics,
NPI, Academy of Sciences, 25068 Řež near Prague, Czechia; exner@ujf.cas.cz

Semjon Vugalter
Institute for Analysis, Karlsruhe Institute of Technology, Englerstrasse 2, 76131 Karlsruhe,
Germany; semjon.wugalter@kit.edu