

Analytic Besov functional calculus for several commuting operators

Charles Batty, Alexander Gomilko, Dominik Kobos, and Yuri Tomilov

Abstract. This paper investigates analytic Besov functions of n variables which act on the generators of n commuting C_0 -semigroups on a Banach space. The theory for $n = 1$ has already been published, and the present paper uses a different approach to that case as well as extending to the cases when $n \geq 2$. It also clarifies some spectral mapping properties and provides some operator norm estimates.

*Dedicated with much gratitude to E. Brian Davies
on the occasion of his 80th birthday*

1. Introduction

Let A be an (unbounded) operator on a complex Banach space X , and assume that $-A$ is the generator of a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X . The so-called Hille–Phillips functional calculus for A assigns a bounded operator $f(A)$ to each function f in the Hille–Phillips (HP) algebra \mathcal{LM} . If A is sectorial of angle less than $\pi/2$, so the semigroup is bounded and holomorphic on a sector in \mathbb{C}_+ , then there is a theory of a holomorphic calculus for A which may be applied to bounded holomorphic functions on a sector, but there is no guarantee that this calculus assigns a bounded operator to all bounded holomorphic functions. However, in the sectorial case there is a bounded calculus for A for all functions in the Banach algebra \mathcal{B}^1 of analytic Besov functions on \mathbb{C}_+ , as shown originally in [31]. We have recently developed a theory of a \mathcal{B}^1 -calculus, which applies to many semigroup generators, but not to all of them. There is a \mathcal{B}^1 -calculus for A if A is sectorial of angle less than $\pi/2$ or if X is a Hilbert space and the semigroup $(e^{-tA})_{t \geq 0}$ is bounded. The theory of the \mathcal{B}^1 -calculus is set out in detail in [7, 8], but readers may prefer to read the summary in [9] in order to understand the setting of \mathcal{B}^1 .

Mathematics Subject Classification 2020: 47A20 (primary); 32A70, 47B44, 47D03 (secondary).

Keywords: analytic Besov functions, functional calculus, commuting semigroups.

Some applications of the \mathcal{B}^1 -calculus are given in [7, 8], and others have been found subsequently. For example, a new variant of the Katznelson–Tzafriri theorem for the \mathcal{B}^1 -calculus has been proved in [11], provided that A has a \mathcal{B}^1 -calculus. Estimates involving functions in \mathcal{LM} can be improved by using the \mathcal{B}^1 -calculus norm, instead of the HP-norm; the difference is considerable in some cases.

Functional calculus is often used to investigate the various notions of joint spectrum for several commuting operators. Such work has had limited success, and work on functional calculi for two or more generators of C_0 -semigroups has been sparse. The Hille–Phillips calculus for bounded C_0 -semigroups extends to several commuting semigroups in a straightforward fashion. The holomorphic calculus for sectorial operators can be applied in some cases. Some partial results for several commuting holomorphic semigroups have been obtained in [5, 6, 25] and other papers, and a multi-variable version of the Bochner–Phillips functional calculus was set up in [28]. Some functional calculi for non-holomorphic functions have been introduced for operators with real spectrum (for example, in [15]), and versions for several operators have been studied in [2, 3, 29]. The \mathcal{B}^1 -calculus does not apply to generators of bounded C_0 -groups, except in very special cases. Nevertheless, Arveson’s theory of representations of locally abelian groups can be applied to generators of bounded C_0 -groups (see [14, Chapter 8]) and to finitely many commuting C_0 -groups, in ways which help to develop the theory of the \mathcal{B}^1 -calculus. In addition, the book [1, Chapter 3] develops a functional calculus for tuples of commuting generators of C_0 -groups from a different perspective and uses it for the study of commutators and other applications to mathematical physics.

A two-variable version \mathcal{B}^2 of \mathcal{B}^1 and a theory of the \mathcal{B}^2 -calculus for two commuting operators have been covered in the thesis [23], using the same methods as in [7, 8]. In this paper we extend the results to an arbitrary finite number of variables and commuting operators. Moreover, we use techniques which are quite different from those in [7, 8]. We base them on a reproducing formula for analytic Besov functions (see Corollary 3.3), which was first used in [7], and later used in different ways in [8, 10]. This formula provides a route to the essential properties of the \mathcal{B}^n -calculus: mapping \mathcal{B}^n into $L(X)$, showing it is a homomorphism, and that it is unique (see Section 5). This route is more direct, and clearer, than the arguments used in [7, 8].

We introduce the Banach algebras \mathcal{B}^n , and their elementary decomposition into 2^n subalgebras, in Section 2. In the case $n = 1$, the two subalgebras were an ideal \mathcal{B}_0^1 and the one-dimensional space of constant functions. In the general case, we need more sophisticated notation in our presentation of the decomposition, but the outcomes in this paper are natural generalisations from the one-variable and two-variable cases. In Section 3, we obtain a reproducing formula for the elementary components of a given function $f \in \mathcal{B}^n$ (Corollary 3.3). In Section 4, we define the \mathcal{B}^n -calculus initially as a function from \mathcal{B}^n to $L(X, X^{**})$, and then we prove in Section 5 that it

is a functional calculus. These proofs proceed in an unusual way, as did the corresponding proofs in [7]. In Sections 6–8, we establish various further properties of the family of \mathcal{B} -calculi, including compatibility across operators, spectral inclusions, and operator norm estimates.

1.1. Preliminaries

Notation. For $n \in \mathbb{N}$, we let $I_n := \{1, 2, \dots, n\}$ and \mathcal{P}_n be the power-set of I_n . We denote elements of \mathcal{P}_n by symbols such as Ω, Ψ . For $\Omega \in \mathcal{P}_n$, let $\Omega^c := I_n \setminus \Omega$, and $|\Omega|$ be the cardinality of Ω .

We use the notation \mathbb{R}_+ for the interval $[0, \infty)$, \mathbb{C}_+ for the open right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ and $\bar{\mathbb{C}}_+$ for the closed right half-plane. Real numbers may typically be represented by $s, t, \alpha, \beta, \gamma$, and complex numbers by z, λ, ν, ζ .

We use bold print to denote elements of \mathbb{R}^n and \mathbb{C}_+^n ; for example, \mathbf{t} denotes (t_1, \dots, t_n) in \mathbb{R}^n , \mathbf{z} denotes an n -tuple (z_1, z_2, \dots, z_n) in \mathbb{C}_+^n , and similarly $\boldsymbol{\lambda}$ denotes an n -tuple $(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Holomorphic functions. For a holomorphic function f on \mathbb{C}_+^n , we let $D_j f$ be the partial derivative of f with respect to the j th coordinate variable z_j where $j \in I_n$. For $\Omega \in \mathcal{P}_n$, we let \mathcal{D}_Ω be the composition of the derivatives D_j for $j \in \Omega$ (once each). If $\Omega = \emptyset$, then $\mathcal{D}_\emptyset f = f$. For $\Omega = I_n$, we may write \mathcal{D}_n instead of \mathcal{D}_{I_n} .

Let $f \in H^\infty(\mathbb{C}_+^n)$. We will make use of the following properties:

- for $\boldsymbol{\alpha} \in (0, \infty)^n$,

$$\sup_{\operatorname{Re} \mathbf{z} = \boldsymbol{\alpha}} |f(\mathbf{z})| = \sup_{\operatorname{Re} \mathbf{z} \geq \boldsymbol{\alpha}} |f(\mathbf{z})| \quad (\text{Maximum Principle});$$

- for $\mathbf{z} \in \mathbb{C}_+^n$,

$$|\mathcal{D}_\Omega f(\mathbf{z})| \leq \frac{\|f\|_\infty}{2^{|\Omega|} \prod_{j \in \Omega} \operatorname{Re} z_j} \quad (\text{Cauchy's inequality}).$$

Cauchy's inequality extends to higher-order derivatives, by repeated use of first-order cases.

Several variants of resolvent functions appear frequently, including the following:

- for $z, \lambda \in \mathbb{C}_+$, we put $r_\lambda(z) = (z + \lambda)^{-1}$;
- for $\lambda \in \mathbb{C}_+$, $j \in I_n$ and $\mathbf{z} \in \mathbb{C}_+^n$, we put $r_{\lambda, j}(\mathbf{z}) = (z_j + \lambda)^{-1}$;
- for $\mathbf{z}, \boldsymbol{\lambda} \in \mathbb{C}_+^n$, we put $\mathbf{r}_\lambda(\mathbf{z}) = \prod_{j=1}^n (z_j + \lambda_j)^{-1}$;
- for $z, \lambda \in \mathbb{C}_+$, we put $K(z, \lambda) = (-2/\pi)(z + \lambda)^{-2}$;
- for $\mathbf{z}, \boldsymbol{\lambda} \in \mathbb{C}_+^n$, we put $\mathcal{K}_n(\mathbf{z}, \boldsymbol{\lambda}) = \prod_{j=1}^n K(z_j, \lambda_j)$.

Measures. We may denote the Lebesgue measure on \mathbb{R}_+^n by $d\alpha$ or $d\mathbf{t}$, and the Lebesgue measure on \mathbb{C}_+^n by dS_n . We let

$$dV(\lambda) = \alpha d\beta d\alpha, \quad \lambda = \alpha + i\beta \in \mathbb{C}_+,$$

and dV_n be the n -fold product measure of dV , so

$$dV_n(\lambda) = \left(\prod_{j=1}^n \alpha_j \right) dS_n(\lambda), \quad \lambda = (\alpha_j + i\beta_j)_{j=1}^n \in \mathbb{C}_+^n.$$

Operators. In this paper an ‘‘operator’’ may be either a bounded linear operator on a Banach space X or an unbounded operator *with dense domain* in X . The notation $L(X)$ and $L(X, Y)$ denote the spaces of bounded linear operators from X to X and from X to Y , respectively. Bounded operators will typically be denoted by T and unbounded operators by A .

An operator A on X is sectorial of angle less than $\pi/2$ if and only if the spectrum $\sigma(A)$ of A is contained in $\mathbb{C}_+ \cup \{0\}$ and $M_A := \sup_{\lambda \in \mathbb{C}_+} \|\lambda(\lambda + A)^{-1}\| < \infty$. Equivalently, the operator $-A$ is the generator of a sectorially bounded holomorphic C_0 -semigroup on X .

2. The algebra \mathcal{B}^n

Let f be a holomorphic function on \mathbb{C}_+^n , $\Omega \in \mathcal{P}_n$ and $k = |\Omega|$. Let $H_\Omega f$ be the function of variables $\alpha_\Omega := (\alpha_j)_{j \in \Omega} \in (0, \infty)^k$ for $j \in \Omega$, defined by

$$(H_\Omega f)(\alpha_\Omega) := \sup_{\mathbf{z} \in W_{\alpha_\Omega}} |(\mathcal{D}_\Omega f)(\mathbf{z})|, \quad (2.1)$$

where

$$W_{\alpha_\Omega} := \{\mathbf{z} \in \mathbb{C}_+^n : \operatorname{Re} z_j = \alpha_j \text{ for all } j \in \Omega\}.$$

If f is bounded and $\alpha_j > 0$ for each $j \in \Omega$, then Cauchy’s inequality and the maximum principle imply that the supremum in (2.1) is finite, and $H_\Omega f$ is a decreasing function of each variable α_j for $j \in \Omega$.

We define

$$\|f\|_{\mathcal{B}_\Omega^n} := \int_{\mathbb{R}_+^k} (H_\Omega f)(\alpha_\Omega) d\alpha_\Omega,$$

where $d\alpha_\Omega$ denotes Lebesgue measure on \mathbb{R}_+^k . This integral may be infinite. When Ω is the empty set, $H_\emptyset f$ and $\|f\|_{\mathcal{B}_\emptyset^n}$ are both equal to the H^∞ -norm $\|f\|_\infty$ of f .

We define \mathcal{B}^n to be the vector space of all holomorphic functions f on \mathbb{C}_+^n such that $\|f\|_{\mathcal{B}_\Omega^n}$ is finite for all $\Omega \in \mathcal{P}_n$, so that $H_\Omega f$ is integrable over \mathbb{R}_+^k . Then $\|\cdot\|_{\mathcal{B}_\Omega^n}$ is a seminorm on \mathcal{B}^n , and there is a norm on \mathcal{B}^n defined by

$$\|f\|_{\mathcal{B}^n} := \sum_{\Omega \in \mathcal{P}_n} \|f\|_{\mathcal{B}_\Omega^n}. \quad (2.2)$$

For $n = 1$, this definition agrees with the norm on \mathcal{B}^1 defined in [7, p. 33], and, for $n = 2$, it agrees with the norm on \mathcal{B}^2 defined in [23, p. 37].

The definitions of \mathcal{B}^n and its norm are clearly invariant under permutation of the variables. For ease of presentation, in a proof of any statement involving one set $\Omega \in \mathcal{P}_n$, we may choose to assume that $\Omega = I_k$ or $\Omega^c = I_k$.

A useful property of functions $f \in \mathcal{B}^n$ is that they are bounded and uniformly continuous on \mathbb{C}_+^n , so they extend to bounded, uniformly continuous functions on $\overline{\mathbb{C}_+^n}$. This was noted for $n = 1$ in [7, Proposition 2.1] and shown for $n = 2$ in [23, Proposition 3.2.4]. In general, the uniform continuity can be proved by elementary methods (as in [23]) or by a special case of an argument in the proof of Remark 3.7 (1). As a preliminary taste, we will show here that both properties can be deduced using only the finiteness of the norms $\|f\|_{\mathcal{B}_\Omega^n}$ for every singleton subset Ω of I_n . However this is an exceptional case, and we need to include the H^∞ -norm in the definition of $\|f\|_{\mathcal{B}^n}$ in order that Proposition 2.2 is true and later proofs can be simplified.

Proposition 2.1. *Let f be a holomorphic function on \mathbb{C}_+^n , and assume that $\|f\|_{\mathcal{B}_\Omega^n}$ is finite for every singleton subset Ω of I_n . Then f is bounded and uniformly continuous on \mathbb{C}_+^n .*

Proof. Let \mathbf{z} and \mathbf{z}' be points in \mathbb{C}_+^n . To estimate $|f(\mathbf{z}) - f(\mathbf{z}')|$, we will consider the variables one at a time. The first step will be from \mathbf{z} to $\mathbf{z}^{[1]} = (z'_1, z_2, \dots, z_n)$, as follows.

Let $\theta \in (0, \pi/2)$ and $\kappa = \sec \theta > 1$. Consider the following path Γ_1 in \mathbb{C}_+ from z_1 to z'_1 :

- (a) the line-segment from z_1 to z'_1 if the gradient is in $[-\cot \theta, \cot \theta]$;
- (b) otherwise, two line-segments, one from z_1 and the other from z'_1 , one with gradient $\cot \theta$ and the other with gradient $-\cot \theta$, chosen so that the two lines cross at a point z''_1 to the right of z and z'_1 .

Then

$$\begin{aligned} |f(\mathbf{z}) - f(\mathbf{z}^{[1]})| &= \left| \int_{\Gamma_1} (D_1 f)(\lambda, z_2, \dots, z_n) d\lambda \right| \\ &\leq \kappa \int_{J_1} H_{\{1\}}(t) dt. \end{aligned}$$

Here J_1 represents the interval in \mathbb{R}_+ between $\operatorname{Re} z_1$ and $\operatorname{Re} z'_1$ in case (a). In case (b), it represents both the interval between $\operatorname{Re} z_1$ and $\operatorname{Re} z''_1$ and the interval between $\operatorname{Re} z'_1$ and $\operatorname{Re} z''_1$, with the two integrals added together. There is a similar estimate for $|f(\mathbf{z}^{[1]}) - f(\mathbf{z}^{[2]})|$, where $\mathbf{z}^{[2]} = (z'_1, z'_2, z_3, \dots, z_n)$, and so on. Eventually, we obtain

$$|f(\mathbf{z}) - f(\mathbf{z}')| \leq \kappa \sum_{j=1}^n \int_{J_j} H_{\{j\}}(t) dt \leq 2\kappa \sum_{j=1}^n \|f\|_{\mathcal{B}_{\{j\}}^n} < \infty.$$

By fixing \mathbf{z}' , this proves that f is bounded. By letting $\theta \rightarrow 0$, one may conclude that the estimate also holds for $\kappa = 1$.

To establish uniform continuity, we fix $\theta \in (0, \pi/2)$ and thereby we fix $\kappa > 1$. Take $\delta > 0$ and assume that $|\mathbf{z} - \mathbf{z}'| < \delta$. Then the length of each interval involved in the sets J_j is at most $\delta \cot \theta$. Since the functions $H_{\{j\}}$ are integrable over \mathbb{R}_+ , there exists $\delta > 0$ so small that all the integrals used above are less than ε , and then we can conclude that $|f(\mathbf{z}) - f(\mathbf{z}')| < 2\kappa n \varepsilon$ whenever $|\mathbf{z} - \mathbf{z}'| < \delta$. ■

We choose the norm $\|\cdot\|_{\mathcal{B}^n}$ because it makes \mathcal{B}^n into a Banach algebra. There are numerous norms on \mathcal{B}^n which are equivalent to $\|\cdot\|_{\mathcal{B}^n}$. Examples for $n = 1$ may be found in [7, p. 33] and for $n = 2$ in [23, Section 3.4].

Proposition 2.2. *The normed space $(\mathcal{B}^n, \|\cdot\|_{\mathcal{B}^n})$ is a Banach algebra.*

Proof. The proof of completeness of the space is a straightforward, but uninteresting, extension of the proofs for \mathcal{B}^1 in [7, Proposition 2.3] and \mathcal{B}^2 in [23, Proposition 3.2.1].

To show that \mathcal{B}^n is a Banach algebra, we need to show that, for any $f, g \in \mathcal{B}^n$, the product fg satisfies $\|fg\|_{\mathcal{B}^n} \leq \|f\|_{\mathcal{B}^n} \|g\|_{\mathcal{B}^n}$.

Let $\Omega \in \mathcal{P}_n$. By the product rule,

$$\mathcal{D}_\Omega(fg) = \sum_{\Psi \subseteq \Omega} (\mathcal{D}_\Psi f)(\mathcal{D}_{\Omega \setminus \Psi} g).$$

For $\Psi \subseteq \Omega$,

$$\begin{aligned} & \sup\{ |(\mathcal{D}_\Psi f)(\mathcal{D}_{\Omega \setminus \Psi} g)(\mathbf{z})| : \mathbf{z} \in \mathbb{C}_+^n, \operatorname{Re} z_j = \alpha_j (j \in \Omega) \} \\ & \leq \sup\{ |(\mathcal{D}_\Psi f)(\mathbf{z})| : \mathbf{z} \in \mathbb{C}_+^n, \operatorname{Re} z_j = \alpha_j (j \in \Psi) \} \\ & \quad \times \sup\{ |(\mathcal{D}_{\Omega \setminus \Psi} g)(\mathbf{z})| : \mathbf{z} \in \mathbb{C}_+^n, \operatorname{Re} z_j = \alpha_j (j \in \Omega \setminus \Psi) \}. \end{aligned}$$

The variables α_j for $j \in \Psi$ separate from those for $j \in \Omega \setminus \Psi$ on the right-hand side, so integration with respect to Lebesgue measure on \mathbb{R}_+^n shows that

$$\|fg\|_{\mathcal{B}_\Omega^n} \leq \sum_{\Psi \subseteq \Omega} \|f\|_{\mathcal{B}_\Psi^n} \|g\|_{\mathcal{B}_{\Omega \setminus \Psi}^n}.$$

Hence,

$$\begin{aligned} \|fg\|_{\mathcal{B}^n} &\leq \sum_{\Omega \in \mathcal{P}_n} \sum_{\Psi \subseteq \Omega} \|f\|_{\mathcal{B}_\Psi^n} \|g\|_{\mathcal{B}_{\Omega \setminus \Psi}^n} \leq \sum_{\Psi \in \mathcal{P}_n} \|f\|_{\mathcal{B}_\Psi^n} \sum_{\Psi \in \mathcal{P}_n} \|g\|_{\mathcal{B}_\Psi^n} \\ &= \|f\|_{\mathcal{B}^n} \|g\|_{\mathcal{B}^n}. \end{aligned} \quad \blacksquare$$

We note the following facts, which extend the corresponding statements for $n = 1$ or $n = 2$. The first result is almost immediate from the definitions. The other proofs are either similar to those in [7, 23], or, in some cases, they can be deduced from those results.

Proposition 2.3. *Let $f \in \mathcal{B}^n$, $\Omega \in \mathcal{P}_n$ with $|\Omega| = k$, and $\zeta_j \in \mathbb{C}_+$ be fixed for $j \in \Omega^c$. For $\mathbf{z} \in \mathbb{C}_+^n$, let $\mathbf{z}_\Omega = (z_j)_{j \in \Omega}$, and $\mathbf{z}_\Omega^+ = (z_j^*)_{j \in I_n}$, where*

$$z_j^* = \begin{cases} z_j, & j \in \Omega, \\ \zeta_j, & j \in \Omega^c. \end{cases}$$

Let Ψ be a subset of Ω^c , and

$$g(\mathbf{z}_\Omega) = (\mathcal{D}_\Psi f)(\mathbf{z}_\Omega^+).$$

Then $g \in \mathcal{B}^k$. In particular, the function $\mathbf{z}_\Omega \mapsto f(\mathbf{z}_\Omega^+)$ is in \mathcal{B}^k .

Proof. It suffices to consider the case when $\Omega = I_{n-1}$ and show that $g \in \mathcal{B}^{n-1}$, in the cases when Ψ is empty or $\Psi = \{n\}$. Then, we can conclude the result for general Ω by the symmetry of the variables, and considering the variables in Ω^c successively.

Let $\Upsilon \in \mathcal{P}_{n-1}$. If Ψ is empty, then it is clear that $(H_\Upsilon g)(\alpha_\Upsilon) \leq (H_\Upsilon f)(\alpha_\Upsilon)$, and so $g \in \mathcal{B}^{n-1}$. In the case when $\Psi = \{n\}$, we consider $\zeta_n \in \mathbb{C}_+$, $\Upsilon \in \mathcal{P}_{n-1}$ with $|\Upsilon| = m$ and $\Upsilon^+ = \Upsilon \cup \{n\}$. Since $f \in \mathcal{B}^n$, the function

$$(\alpha_\Upsilon, \alpha_n) \mapsto (H_{\Upsilon^+} f)(\alpha_\Upsilon, \alpha_n)$$

is integrable over \mathbb{R}_+^{m+1} . By Fubini's theorem, the function

$$\alpha_\Upsilon \mapsto (H_{\Upsilon^+} f)(\alpha_\Upsilon, \alpha_n)$$

is integrable over \mathbb{R}_+^m for almost all $\alpha_n \in (0, \infty)$. By the maximum principle, these functions of α_Υ form a family of non-negative functions which are non-increasing with respect to α_n , and the functions are integrable over \mathbb{R}_+^m for almost all $\alpha_n > 0$, so they are integrable for all $\alpha_n > 0$. Moreover,

$$(H_\Upsilon g)(\alpha_\Upsilon) \leq (H_{\Upsilon^+} f)(\alpha_\Upsilon, \operatorname{Re} \zeta_n),$$

so $H_\Upsilon g$ is integrable. \blacksquare

Let $f \in \mathcal{B}^n$, and $\Omega \in \mathcal{P}_n$. The following result was shown in [7, Proposition 2.2] for $n = 1$, and in [23, Proposition 3.2.2] for $n = 2$. The existence of iterated limits follows by applying the one-variable result repeatedly, but some additional arguments are needed to obtain a limit over several variables simultaneously.

Proposition 2.4. *Let $f \in \mathcal{B}^n$, $\Omega \in \mathcal{P}_n$, $|\Omega| = k$, and $\mathbf{z}_\Omega = (z_j)_{j \in \Omega} \in \mathbb{C}_+^k$. Then the following limit exists in \mathbb{C} :*

$$f_\Omega(\mathbf{z}_\Omega) := \lim_{\operatorname{Re} z_j \rightarrow \infty, j \in \Omega^c} f(\mathbf{z}).$$

Moreover, $f_\Omega \in \mathcal{B}^k$, and the map $f \mapsto f_\Omega$ is bounded and linear from \mathcal{B}^n to \mathcal{B}^k .

Proof. By symmetry of the variables, it suffices to consider the case when $\Omega = I_k$. First, we show the result when $k = n - 1$. For this case, we use the notation \mathbf{z} to denote $(z_1, \dots, z_{n-1}) \in \mathbb{C}_+^{n-1}$, and ζ for the variable z_n .

By Proposition 2.3, for each $\mathbf{z} \in \mathbb{C}_+^{n-1}$ the function $\zeta \mapsto f(\mathbf{z}, \zeta)$ is in \mathcal{B}^1 . By [7, Proposition 2.2], we know that

$$g_1(\mathbf{z}) := \lim_{\operatorname{Re} \zeta \rightarrow \infty} f(\mathbf{z}, \zeta)$$

exists. In fact,

$$|f(\mathbf{z}, \zeta) - g_1(\mathbf{z})| \leq \left| \int_{\operatorname{Re} \zeta}^{\infty} (D_n f)(\mathbf{z}, t + i \operatorname{Im} \zeta) dt \right| \leq \int_{\operatorname{Re} \zeta}^{\infty} (H_{\{n\}} f)(t) dt \rightarrow 0, \quad (2.3)$$

uniformly with respect to $\mathbf{z} \in \mathbb{C}_+^{n-1}$, as $\operatorname{Re} \zeta \rightarrow \infty$. It follows from Vitali's theorem for several variables that g_1 is holomorphic, and the partial derivatives of $f(\cdot, \zeta)$ converge (pointwise) to the corresponding derivatives of g_1 as $\operatorname{Re} \zeta \rightarrow \infty$. This implies that $H_\Upsilon g_1 \leq H_\Upsilon f$ on $(0, \infty)^{|\Upsilon|}$ for all $\Upsilon \in \mathcal{P}_{n-1}$, and therefore $g_1 \in \mathcal{B}^{n-1}$ and $\|g_1\|_{\mathcal{B}^{n-1}} \leq \|f\|_{\mathcal{B}^n}$.

If $\Omega = I_k$ where $k \in \{0, 1, \dots, n-2\}$, one may apply the argument above to g_1 to show that there is a function $g_2 \in \mathcal{B}^{n-2}$ such that

$$\lim_{\operatorname{Re} \zeta \rightarrow \infty} (g_1(\mathbf{z}', \zeta) - g_2(\mathbf{z}')) = 0,$$

uniformly for $\mathbf{z}' \in \mathbb{C}_+^{n-2}$. In combination with (2.3), it follows that the statement is proved for $\Omega = I_{n-2}$. Continuing in this way, it can be proved for $\Omega = I_k$. ■

Remarks 2.5. (1) In Proposition 2.4, f_\emptyset is the constant function $\lim_{\operatorname{Re} z_j \rightarrow \infty, j \in I_n} f(\mathbf{z})$ and $f_{I_n} = f$.

(2) Note that the limits involved in Proposition 2.4 are as $\operatorname{Re} z_j \rightarrow \infty$. Functions $f \in \mathcal{B}^n$ may not have limits as $|z_j| \rightarrow \infty$.

(3) The function f_Ω in Proposition 2.4 may be denoted in various ways. In particular, we may replace the variable z_j by the symbol ∞ for each $j \in \Omega^c$, and then regard f as being defined on an extended domain. Proposition 2.3 remains true if $\xi_j = \infty$ for some $j \in \Omega^c$. The function f_Ω is holomorphic, by a simple application of Vitali's theorem, and then the rest of the proof remains valid.

We denote by \mathcal{B}_0^n the ideal of \mathcal{B}^n consisting of those functions $f \in \mathcal{B}^n$ such that f_Ω is identically zero for all proper subsets Ω of I_n . It suffices that

$$\lim_{\operatorname{Re} z_j \rightarrow \infty} f(\mathbf{z}) = 0$$

whenever $j \in I_n$ and $z_k \in \mathbb{C}_+$ are fixed for each $j \neq k$. Adopting the practice described in Remark 2.5, it suffices that $f \in \mathcal{B}_0^n$ if and only if $f(\mathbf{z}) = 0$ whenever $z_j = \infty$ for at least one $j \in I_n$. We stress that the two sentences above mean precisely the same.

The seminorm $\|\cdot\|_{\mathcal{B}_0^n}$ is a norm on \mathcal{B}_0^n , and we denote this seminorm by $\|\cdot\|_{\mathcal{B}_0^n}$. Now, we show that the norms $\|\cdot\|_{\mathcal{B}_0^n}$ and $\|\cdot\|_{\mathcal{B}^n}$ are equivalent on \mathcal{B}_0^n . In addition, finiteness of $\|f\|_{\mathcal{B}_0^n}$, boundedness of f , and the condition above of vanishing (at infinity, to the right) imply that $f \in \mathcal{B}_0^n$.

Proposition 2.6. *Let $f \in H^\infty(\mathbb{C}_+^n)$, $\|f\|_{\mathcal{B}_0^n} < \infty$, and assume that*

$$\lim_{\operatorname{Re} z_j \rightarrow \infty} f(\mathbf{z}) = 0 \tag{2.4}$$

whenever $j \in I_n$ and z_k is fixed for all $k \neq j$. Then $f \in \mathcal{B}^n$ and $\|f\|_{\mathcal{B}^n} \leq 2^n \|f\|_{\mathcal{B}_0^n}$.

Proof. Let $\Omega \in \mathcal{P}_n$, $\Omega \neq I_n$, and $|\Omega| = k$. From the boundedness of f , (2.4), and Cauchy's inequality, we infer that, for all $\Psi \in \mathcal{P}_n$ such that $\Omega \subseteq \Psi$, any $j \in \Omega^c$ and z_k fixed for $k \neq j$,

$$\lim_{\operatorname{Re} z_j \rightarrow \infty} \mathcal{D}_\Psi f(\mathbf{z}) = 0,$$

and thus

$$\mathcal{D}_\Omega f(\mathbf{z}) = \int_{\mathbb{R}_+^{n-k}} \mathcal{D}_n f(\mathbf{z} + \boldsymbol{\alpha}_{\Omega^c}) d\boldsymbol{\alpha}_{\Omega^c}, \quad \mathbf{z} \in \mathbb{C}_+^n.$$

By the maximum principle for functions from $H^\infty(\mathbb{C}^n)$, we have

$$\begin{aligned} |\mathcal{D}_n f(\mathbf{z})| &\leq \int_{\mathbb{R}_+^{n-k}} \sup_{\substack{\lambda_j = z_j, j \in \Omega \\ \operatorname{Re} \lambda_j = \operatorname{Re} z_j, j \in \Omega^c}} |\mathcal{D}_n f(\boldsymbol{\lambda} + \boldsymbol{\alpha}_{\Omega^c})| d\boldsymbol{\alpha}_{\Omega^c} \\ &\leq \int_{\mathbb{R}_+^{n-k}} \sup_{\substack{\lambda_j = z_j, j \in \Omega \\ \operatorname{Re} \lambda_j = 0, j \in \Omega^c}} |\mathcal{D}_n f(\boldsymbol{\lambda} + \boldsymbol{\alpha}_{\Omega^c})| d\boldsymbol{\alpha}_{\Omega^c}, \end{aligned}$$

so that

$$\|f\|_{\mathcal{B}_\Omega^n} = \sup_{\mathbf{z} \in W_{\alpha_\Omega}} |\mathcal{D}_\Omega f(\mathbf{z})| \leq \int_{\mathbb{R}_+^{n-k}} \sup_{\mathbf{z} \in W_{\alpha_{I_n}}} |\mathcal{D}_n f(\mathbf{z})| d\alpha_{\Omega^c} = \|f\|_{\mathcal{B}_0^n}.$$

This implies that $f \in \mathcal{B}_0^n$ and

$$\|f\|_{\mathcal{B}^n} = \sum_{\Omega \in \mathcal{P}_n} \|f\|_{\mathcal{B}_\Omega^n} \leq 2^n \|f\|_{\mathcal{B}_0^n}. \quad \blacksquare$$

Note that $\|\cdot\|_{\mathcal{B}_{I_n}^n}$ is not a norm on \mathcal{B}^n , because some functions do not depend on all the n variables. Those functions satisfy $\mathcal{D}_n f = 0$ and hence $\|f\|_{\mathcal{B}_{I_n}^n} = 0$. However, linear combinations g of such functions may depend on all the variables while satisfying $\mathcal{D}_n g = 0$ (see Remark 2.9(2) below). Consequently, we will decompose \mathcal{B}^n into the direct sum of 2^n closed ideals, one for each subset of I_n . This was achieved for \mathcal{B}^1 in [7], where one summand contained only the constant functions, and also for \mathcal{B}^2 in [23], where two summands each contained functions of one of the two variables.

Let $f \in \mathcal{B}^n$, and let $\Omega_f \in \mathcal{P}_n$ be the subset of I_n representing the variables z_j on which f depends. It is easy to see that they are the variables for which $D_j f$ is not identically zero. We will use the following terminology:

- (a) Ω_f is the *support* of f ,
- (b) $|\Omega_f|$ is the *degree* of f ,
- (c) f is *elementary* if $f(\mathbf{z}) = 0$ whenever $z_j = \infty$ for at least one $j \in \Omega_f$.

For $\Omega \in \mathcal{P}_n$, we define subspaces of \mathcal{B}^n :

$$\begin{aligned} \mathcal{B}_\Omega^n &= \{f \in \mathcal{B}^n : \Omega_f \subseteq \Omega\} = \{f \in \mathcal{B}^n : \mathcal{D}_j f = 0 \text{ for all } j \in \Omega^c\}, \\ \mathcal{B}_{\Omega,0}^n &= \{f \in \mathcal{B}_\Omega^n : f(\mathbf{z}) = 0 \text{ whenever } z_j = \infty \text{ for at least one } j \in \Omega\}. \end{aligned}$$

It is easy to see that \mathcal{B}_Ω^n and $\mathcal{B}_{\Omega,0}^n$ are closed subalgebras of \mathcal{B}^n . The spaces $\mathcal{B}_{I_n}^n$ and $\mathcal{B}_{I_n,0}^n$ are the spaces \mathcal{B}^n and \mathcal{B}_0^n , respectively, and the space \mathcal{B}_\emptyset^n is the space of constant functions. Note that

- (i) $f \in \mathcal{B}^n$ is elementary if and only if $f \in \mathcal{B}_{\Omega_f,0}^n$, and
- (ii) if $f \in \mathcal{B}_0^n$ and $\mathcal{D}_n f = 0$, then $f = 0$.

For $f \in \mathcal{B}^n$, let

$$f_{\text{el}}(\mathbf{z}) = \sum_{\Omega \in \mathcal{P}_n} (-1)^{n-|\Omega|} f_\Omega(\mathbf{z}_\Omega),$$

where $f_\Omega(\mathbf{z}_\Omega)$ is defined in Proposition 2.4. For example, if $n = 2$ then

$$f_{\text{el}}(z_1, z_2) = f(z_1, z_2) - f(z_1, \infty) - f(\infty, z_2) + f(\infty, \infty).$$

Then

$$f = f_{\text{el}} + \sum_{\substack{\Omega \in \mathcal{P}_n \\ \Omega \neq I_n}} (-1)^{n-|\Omega|-1} f_{\Omega},$$

where f_{el} is elementary of degree n , and each function f_{Ω} for $\Omega \neq I_n$ is of degree less than n . The empty set is in \mathcal{P}_n , and $f_{\emptyset} = f_{\emptyset,0}$ is a constant equal to $f(\infty, \dots, \infty)$.

By simple relabelling of the variables, the space \mathcal{B}_{Ω}^n can be identified with \mathcal{B}^k and $\mathcal{B}_{\Omega,0}^n$ with \mathcal{B}_0^k , where $k = |\Omega|$. By considering those f_{Ω} for $\Omega \neq I_n$, and iterating the procedure until we have obtained only elementary functions, we can obtain an *elementary decomposition* of f as

$$f = \sum_{\Omega \in \mathcal{P}_n} f_{\Omega,0},$$

where $f_{\Omega,0}$ is either identically zero or an elementary function with support Ω .

The elementary decomposition of $f \in \mathcal{B}^n$ is unique. Suppose that $\sum_{\Omega \in \mathcal{P}_n} f_{\Omega} = 0$, where f_{Ω} is elementary with support Ω . By setting $z_j = \infty$ for all $j \in I_n$, one obtains that the constant function $f_{\emptyset} = 0$. By setting $z_j = \infty$ for all z_j except for $j = 1$, one obtains that $f_{\{1\}} = 0$. Continuing in this way, one obtains that $f_{\Omega} = 0$ for all Ω .

So, far the discussion has been somewhat heuristic. An alternative description is given in the following formulas, which can be readily verified.

Proposition 2.7. *The Banach algebra \mathcal{B}^n is the direct sum of the closed subalgebras $\mathcal{B}_{\Omega,0}^n$ for $\Omega \in \mathcal{P}_n$. In particular, the following properties hold.*

(i) For all $f \in \mathcal{B}^n$,

$$f(\mathbf{z}) = \sum_{\Omega \in \mathcal{P}_n} f_{\Omega,0}(\mathbf{z}_{\Omega}),$$

where

$$f_{\Omega,0}(\mathbf{z}_{\Omega}) = \sum_{\Psi \subseteq \Omega} (-1)^{|\Omega|-|\Psi|} f_{\Psi}(\mathbf{z}_{\Psi}) \in \mathcal{B}_{\Omega,0}^n.$$

(ii) For each $\Omega \in \mathcal{P}_n$, the map $f \mapsto f_{\Omega,0}$ for $\Omega \in \mathcal{P}_n$ is a contraction from \mathcal{B}^n to $\mathcal{B}^{|\Omega|}$.

(iii) For $f \in \mathcal{B}^n$, $\mathcal{D}_n f = 0$ if and only if f is a finite sum of holomorphic functions with degree less than n . In that case,

$$f(\mathbf{z}) = \sum_{\substack{\Omega \in \mathcal{P}_n \\ \Omega \neq I_n}} (-1)^{n-|\Omega|-1} f_{\Omega}(\mathbf{z}_{\Omega}).$$

(iv) If $g \in H^{\infty}(\mathbb{C}_+^n)$, $\mathcal{D}_n g = 0$, and $g(\mathbf{z}) = 0$ whenever $z_k = \infty$ for at least one $k \in I_n$, then $g = 0$.

Example 2.8. The space \mathcal{LM}^n of all functions of the form $\mathcal{L}\mu$, where μ is a bounded Borel measure on \mathbb{R}_+^n and $\mathcal{L}\mu$ is its Laplace transform, is a Banach algebra in the norm $\|\mathcal{L}\mu\|_{HP} = \|\mu\|$, and it is known as the *Hille–Phillips algebra*. It is continuously included in \mathcal{B}^n (see [7, Section 2.4] for the case $n = 1$).

Let $f = \mathcal{L}\mu \in \mathcal{LM}^n$. Then the elementary decomposition of f is given by $f_{\Omega,0} = \mathcal{L}\mu_\Omega$, where μ_Ω is the restriction of μ to the subset

$$\mathbb{R}_{+, \Omega} := \prod_{j=1}^n C_j, \quad C_j := \begin{cases} (0, \infty), & j \in \Omega, \\ \{0\}, & j \in \Omega^c. \end{cases}$$

Remarks 2.9. (1) In Propositions 2.3, 2.4, and 2.7, instead of letting $\operatorname{Re} z_j$ tend to ∞ , we could choose to take fixed points $\zeta_j \in \overline{\mathbb{C}}_+$. The results remain valid. Such a change would produce a different elementary decomposition of functions $f \in \mathcal{B}^n$. However, this is not a contradiction of the uniqueness, because the spaces $\mathcal{B}_{\Omega,0}^n$ would also change.

(2) In the paper [27], Marti constructed a functional calculus for two commuting closed operators under certain conditions. A subsequent short paper by Caradus [12] showed that the conditions could be put in a better form and pointed out that the construction could then be extended to n -tuples of commuting operators in a natural way. Marti’s construction involved a decomposition of a similar type to [23, Section 3.5], although the side conditions are different. Caradus did not specify a decomposition but it would undoubtedly have been of similar form to the elementary decomposition in Proposition 2.7 (i). Ichinose [22, p. 235] also used a similar decomposition in the context of two operators.

3. Reproducing formulas and shifts

3.1. Reproducing formulas

A reproducing formula for $f \in \mathcal{B}^1$ was given in [7, Proposition 2.20], and a version for $f \in \mathcal{B}^2$ was given in [23, Proposition 5.1.1]. In [10, Proposition 3.7], some formulas were obtained and the specific formula with $s = 1$ can be applied with $g = f'$ if $f \in \mathcal{B}^1$. We state that formula here.

Proposition 3.1. *Let g be a holomorphic function on \mathbb{C}_+ such that*

$$\int_{\mathbb{C}_+} \frac{|g(z)| \operatorname{Re} z}{|z|^2} dS_1(z) < \infty.$$

Define Qg on \mathbb{C}_+ by

$$(Qg)(z) = -\frac{2}{\pi} \int_{\mathbb{C}_+} g(\lambda) \frac{\operatorname{Re} \lambda}{(z + \bar{\lambda})^2} dS_1(\lambda).$$

Then Qg is holomorphic, and $g = (Qg)'$.

We will present the corresponding statement for functions of n variables in Proposition 3.2. We use the following kernel functions:

$$K(z, \lambda) = -\frac{2}{\pi(z + \lambda)^2}, \quad z, \lambda \in \mathbb{C}_+, \quad \mathcal{K}_n(\mathbf{z}, \boldsymbol{\lambda}) = \prod_{j=1}^n K(z_j, \lambda_j), \quad \mathbf{z}, \boldsymbol{\lambda} \in \mathbb{C}_+^n.$$

Let g be a holomorphic function on \mathbb{C}_+^n such that

$$\int_{\mathbb{C}_+^n} |g(\boldsymbol{\lambda})| \prod_{j=1}^n \frac{\operatorname{Re} \lambda_j}{|\lambda_j|^2} dS_n(\boldsymbol{\lambda}) < \infty. \quad (3.1)$$

Define the function $\mathcal{Q}_n g$ on \mathbb{C}_+^n by

$$(\mathcal{Q}_n g)(\mathbf{z}) := \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) g(\boldsymbol{\lambda}) \prod_{j=1}^n \operatorname{Re} \lambda_j dS_n(\boldsymbol{\lambda}), \quad \mathbf{z} \in \mathbb{C}_+^n.$$

This function is holomorphic on \mathbb{C}_+^n and the derivatives can be passed through the integral sign, the justification being that the derivatives of the integrand with respect to the \mathbf{z} variables can be estimated locally uniformly by constant multiples of the integrand in (3.1). Thus,

$$(\mathcal{D}_n \mathcal{Q}_n g)(\mathbf{z}) = \int_{\mathbb{C}_+^n} \mathcal{D}_n \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) g(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}), \quad \mathbf{z} \in \mathbb{C}_+^n,$$

where the derivatives of \mathcal{K}_n are with respect to the \mathbf{z} variables.

We now extend Proposition 3.1 to obtain the following reproducing formula for functions on \mathbb{C}_+^n .

Proposition 3.2. *Let g be a holomorphic function on \mathbb{C}_+^n such that (3.1) holds. Then $\mathcal{Q}_n g$ is holomorphic and*

$$g = \mathcal{D}_n \mathcal{Q}_n g. \quad (3.2)$$

Proof. The proof is by induction, and the case $n = 1$ is given in Proposition 3.1.

Let $n \geq 1$ and assume that (3.2) holds for n variables. We consider now the case of $n + 1$ variables (\mathbf{z}, z_{n+1}) . Assume that $g(\mathbf{z}, z_{n+1})$ satisfies (3.1) for $n + 1$ variables. For $\mathbf{z} \in \mathbb{C}_+^n$, let $h_{\mathbf{z}}(z_{n+1}) = g(\mathbf{z}, z_{n+1})$. By (3.1) and Fubini's theorem, we obtain that

$$\int_{\mathbb{C}_+} |h_{\mathbf{z}}(\lambda)| \frac{\operatorname{Re} \lambda}{|\lambda|^2} dS(\lambda) < \infty$$

for almost all $\mathbf{z} \in \mathbb{C}_+^n$. For those \mathbf{z} , we can apply Proposition 3.1 to the function $h_{\mathbf{z}}$. We obtain

$$g(\mathbf{z}, z_{n+1}) = h_{\mathbf{z}}(z_{n+1}) = \int_{\mathbb{C}_+} D_{n+1} K(z_{n+1}, \lambda_{n+1}) g(\mathbf{z}, \lambda_{n+1}) dV(\lambda_{n+1}).$$

Now, let $k_{z_{n+1}}(\mathbf{z}) = g(\mathbf{z}, z_{n+1})$. By applying Fubini's theorem and the inductive hypothesis to the functions $k_{z_{n+1}}$, we obtain that, for almost all $z_{n+1} \in \mathbb{C}_+$,

$$\begin{aligned} & g(\mathbf{z}, z_{n+1}) \\ &= \int_{\mathbb{C}_+^n} \mathcal{D}_n \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) k_{z_{n+1}}(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{C}_+^n} \mathcal{D}_n \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) \int_{\mathbb{C}_+} D_{n+1} K(z_{n+1}, \lambda_{n+1}) g(\boldsymbol{\lambda}, \lambda_{n+1}) dV(\lambda_{n+1}) dV_n(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{C}_+^{n+1}} \mathcal{D}_{n+1} \mathcal{K}_{n+1}(\mathbf{z}, z_{n+1}, \bar{\boldsymbol{\lambda}}_n, \bar{\lambda}_{n+1}) g(\boldsymbol{\lambda}, \lambda_{n+1}) dV_{n+1}(\boldsymbol{\lambda}, \lambda_{n+1}) \\ &= (\mathcal{D}_{n+1} \mathcal{Q}_{n+1} g)(\mathbf{z}, z_{n+1}). \end{aligned}$$

This establishes that $g = \mathcal{D}_{n+1} \mathcal{Q}_{n+1} g$ almost everywhere. Since both functions are holomorphic, it follows that they agree everywhere. ■

Corollary 3.3. *Let $f \in \mathcal{B}^n$. Then $\mathcal{D}_n f = \mathcal{D}_n \mathcal{Q}_n \mathcal{D}_n f$, and $f_{\text{el}} = \mathcal{Q}_n \mathcal{D}_n f$, so*

$$f_{\text{el}}(\mathbf{z}) = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) (\mathcal{D}_n f)(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}), \quad \mathbf{z} \in \mathbb{C}_+^n. \quad (3.3)$$

In particular, if $f \in \mathcal{B}_0^n$, then $f = \mathcal{Q}_n \mathcal{D}_n f$.

Proof. Let $g = \mathcal{D}_n f$. Then

$$\begin{aligned} \int_{\mathbb{C}_+^n} |g(\mathbf{z})| \prod_{j=1}^n \frac{\operatorname{Re} z_j}{|z_j|^2} dS_n(\mathbf{z}) &\leq \int_{\mathbb{R}_+^n} (H_{I_n} f)(\boldsymbol{\alpha}) \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\alpha_j}{\alpha_j^2 + t_j^2} d\boldsymbol{\beta} d\boldsymbol{\alpha} \\ &= \pi^n \|f\|_{\mathcal{B}_{I_n}^n} < \infty. \end{aligned}$$

Thus, g satisfies (3.1), so Proposition 3.2 shows that $g = \mathcal{D}_n \mathcal{Q}_n g$.

By the dominated convergence theorem, $(\mathcal{Q}_n \mathcal{D}_n f)(\mathbf{z}) \rightarrow 0$ if $\operatorname{Re} z_j \rightarrow \infty$ for any one j . Moreover, $f_{\text{el}} - \mathcal{Q}_n \mathcal{D}_n f$ is in $H^\infty(\mathbb{C}_+^n)$ and in the kernel of \mathcal{D}_n with zero limits as $\operatorname{Re} z_j \rightarrow \infty$, and then it follows that $f_{\text{el}} = \mathcal{Q}_n \mathcal{D}_n f$ (see Proposition 2.7 (iv)). \blacksquare

Example 3.4. In this example we establish the \mathcal{B}_k -norms and other properties of various resolvent functions.

For $z = \alpha + i\beta \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_+$, let $r_z(\lambda) = (\lambda + z)^{-1}$. Then $r_z \in \mathcal{B}_0^1$, and

$$\|r_z^k\|_{\mathcal{B}_0^1} = \alpha^{-k}, \quad \|r_z^k\|_{\mathcal{B}^1} = 2\alpha^{-k}, \quad k \in \mathbb{N}.$$

For $\mathbf{z} \in \mathbb{C}_+^n$ and $\boldsymbol{\lambda} \in \mathbb{C}_+^n$, let $r_{\mathbf{z}}(\boldsymbol{\lambda}) := \prod_{j=1}^n r_{z_j}(\lambda_j)$. Then $\mathbf{r}_{\mathbf{z}} \in \mathcal{B}_0^n$, and

$$\|\mathbf{r}_{\mathbf{z}}\|_{\mathcal{B}_0^n} = \prod_{j=1}^n \|r_{z_j}\|_{\mathcal{B}_0^1} = \prod_{j=1}^n \alpha_j^{-1}, \quad \|\mathbf{r}_{\mathbf{z}}^k\|_{\mathcal{B}_0^n} = \prod_{j=1}^n \|r_{z_j}^k\|_{\mathcal{B}_0^1} = \prod_{j=1}^n \alpha_j^{-k},$$

where $\alpha_j = \operatorname{Re} z_j$ and $k \in \mathbb{N}$.

For $z, \zeta \in \mathbb{C}_+$,

$$\|r_z - r_\zeta\|_{\mathcal{B}_0^1} = |z - \zeta| \|r_z r_\zeta\|_{\mathcal{B}_0^1} \leq |z - \zeta| \|r_z\|_{\mathcal{B}^1} \|r_\zeta\|_{\mathcal{B}^1} \leq \frac{4|z - \zeta|}{\operatorname{Re} z \operatorname{Re} \zeta} \rightarrow 0,$$

as $z \rightarrow \zeta$. So, the map $z \mapsto r_z$ is continuous from \mathbb{C}_+ to \mathcal{B}_0^1 .

For $j \in I_n$, the map $\mathbf{z} \mapsto r_{z_j}$ is continuous from \mathbb{C}_+^n to \mathcal{B}_0^1 . Let $r_{z,j}(\boldsymbol{\lambda}) = r_z(\lambda_j)$. Then $\mathbf{z} \mapsto r_{z,j}$ is continuous from \mathbb{C}_+^n to \mathcal{B}^n . Since $\mathbf{r}_{\mathbf{z}} = \prod_{j=1}^n r_{z,j}$, the map $\mathbf{z} \mapsto \mathbf{r}_{\mathbf{z}}$ is continuous from \mathbb{C}_+^n to \mathcal{B}_0^n .

In fact, the map $\mathbf{z} \mapsto r(\mathbf{z})$ is holomorphic from \mathbb{C}_+^n to \mathcal{B}_0^n , but we do not need this.

3.2. Shifts

We will use shifts of the variables z_j for the construction of the \mathcal{B} -calculus for operators. In [7,23], we used shifts not only in the horizontal direction but also in the vertical directions. In this paper, we define the functional calculus by a different route, and we do not need to use the vertical shifts. So, we present two lemmas here about the horizontal shifts on \mathcal{B}^n .

First, we will establish a variant of the reproducing formula (3.3) for shifted functions in \mathcal{B}_0^n . This result is a special case of [7, Lemma 2.18(2)] for $n = 1$, and [23, Proposition 5.4.1] for $n = 2$.

Lemma 3.5. *If $f \in \mathcal{B}_0^n$, then, for every $\mathbf{t} \in \mathbb{R}_+^n$,*

$$f(\mathbf{z} + \mathbf{t}) = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z} + \mathbf{t}, \bar{\boldsymbol{\lambda}})(\mathcal{D}_n f)(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}), \quad \mathbf{z} \in \mathbb{C}_+^n. \quad (3.4)$$

Proof. We begin with the case when $n = 1$. From Corollary 3.3,

$$f(z+t) = -\frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \frac{f'(\alpha+t+i\beta)}{(z+\alpha-i\beta)^2} d\beta d\alpha, \quad z \in \mathbb{C}_+.$$

Changing variables in the inner integral, noting that $f'(\alpha+\lambda)$ is bounded for $\lambda \in \mathbb{C}_+$ and a fixed $\alpha > 0$, and using Cauchy's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \frac{f'(\alpha+t+i\beta)}{(z+\alpha-i\beta)^2} d\beta &= -i \int_{t+i\mathbb{R}} \frac{f'(\alpha+\lambda)}{(z+\alpha+t-\lambda)^2} d\lambda \\ &= -i \int_{i\mathbb{R}} \frac{f'(\alpha+\lambda)}{(z+\alpha+t-\lambda)^2} d\lambda \\ &= \int_{\mathbb{R}} \frac{f'(\alpha+i\beta)}{(z+\alpha+t-i\beta)^2} d\beta. \end{aligned}$$

So, we obtain formula (3.4) for $n = 1$:

$$f(z+t) = -\frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \frac{f'(\alpha+i\beta)}{(z+t+\alpha-i\beta)^2} d\beta d\alpha, \quad z \in \mathbb{C}_+, t \geq 0.$$

For $n > 1$, we may consider the individual variables one at a time, applying the same argument as above to show that

$$\begin{aligned} &\int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z} + \mathbf{t}^{j-1}, \bar{\boldsymbol{\lambda}}) (\mathcal{D}_n f)(\boldsymbol{\lambda} + \mathbf{t}_j) dV_n(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z} + \mathbf{t}^j, \bar{\boldsymbol{\lambda}}) (\mathcal{D}_n f)(\boldsymbol{\lambda} + \mathbf{t}_{j+1}) dV_n(\boldsymbol{\lambda}), \end{aligned}$$

where

$$\mathbf{t}^j = (t_1, \dots, t_j, 0, 0, \dots, 0), \quad \mathbf{t}_j = (0, \dots, 0, t_j, t_{j+1}, \dots, t_n), \quad j \in I_n. \quad \blacksquare$$

Next, we show that horizontal shifts are strongly continuous on \mathcal{B}^n . For $f \in \mathcal{B}^n$, $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, and $\mathbf{z} \in \mathbb{C}_+^n$, let

$$(T(\mathbf{t})f)(\mathbf{z}) := f(\mathbf{z} + \mathbf{t}). \quad (3.5)$$

It is easy to see that $T(\mathbf{t})$ is a contraction on $(\mathcal{B}^n, \|\cdot\|_{\mathcal{B}^n})$. The following lemma was proved in [7, Lemma 2.6] for $n = 1$ and in [23, Proposition 4.2.2] for $n = 2$.

Lemma 3.6. *The map $T: \mathbb{R}_+^n \rightarrow L(\mathcal{B}^n)$ is strongly continuous.*

Proof. Let

$$(T_j(t)f)(\mathbf{z}) := f(z_1, \dots, z_j + t, \dots, z_n), \quad j \in I_n, t \geq 0.$$

Then

$$T(\mathbf{t}) = T_1(t_1) \dots T_n(t_n).$$

It suffices to show that the semigroups $(T_j(t))_{t \geq 0}$ are continuous in the strong operator topology on $L(\mathcal{B}^n)$. Without loss of generality, we will take $j = 1$, and we will write $\mathbf{z} = (z_1, \mathbf{z}') \in \mathbb{C}_+^n$.

Let $f \in \mathcal{B}^n$ and $\Omega \in \mathcal{P}_n$ with $k = |\Omega|$. We need to show that

$$\lim_{t \rightarrow 0} \|T_1(t)f - f\|_{\mathcal{B}_\Omega^n} = 0. \quad (3.6)$$

There are two cases: (1) $1 \in \Omega$; (2) $1 \notin \Omega$.

Case 1. $1 \in \Omega$. Let $\alpha_\Omega = (\alpha_j)_{j \in \Omega} \in (0, \infty)^k$, and let $\mathbf{z} \in W_{\alpha_\Omega}$. By an extended version of Cauchy's inequality,

$$|(D_1 \mathcal{D}_\Omega f)(\mathbf{z})| \leq \frac{2^{-(k-1)} \|f\|_\infty}{\alpha_1 \prod_{j \in \Omega} \alpha_j}.$$

This estimate is valid for $\operatorname{Re} z_1 > \alpha_1$, by the maximum principle. Hence,

$$\begin{aligned} |(\mathcal{D}_\Omega f)(z_1 + t, \mathbf{z}') - (\mathcal{D}_\Omega f)(z_1, \mathbf{z}')| &\leq \int_0^t |(D_1 \mathcal{D}_\Omega f)(z_1 + s, \mathbf{z}')| ds \\ &\leq \frac{2^{-(k-1)} t \|f\|_\infty}{\alpha_1 \prod_{j \in \Omega} \alpha_j}. \end{aligned}$$

Taking the supremum over all possible choices of $\mathbf{z} \in W_{\alpha_\Omega}$, this establishes that

$$H_\Omega(T_1(t)f - f)(\alpha_\Omega) \leq \frac{2^{-(k-1)} t \|f\|_\infty}{\alpha_1 \prod_{j \in \Omega} \alpha_j} \rightarrow 0, \quad t \rightarrow 0.$$

In addition,

$$H_\Omega(T_1(t)f - f)(\alpha_\Omega) \leq (H_\Omega T_1(t)f)(\alpha_\Omega) + H_\Omega f(\alpha_\Omega) \leq 2H_\Omega f(\alpha_\Omega),$$

by the maximum principle. Hence, the dominated convergence theorem implies that equation (3.6) holds.

Case 2. $1 \notin \Omega$. Let $\Omega' = \Omega \cup \{1\}$, $\alpha_\Omega = (\alpha_j)_{j \in \Omega} \in (0, \infty)^k$, and $\mathbf{z} \in W_{\alpha_\Omega}$. Then $\mathbf{z}_{\Omega'} = (z_1, \mathbf{z}_\Omega)$. The definition of $H_{\Omega'}$ and its monotonicity show that

$$\begin{aligned} |(\mathcal{D}_\Omega f)(z_1 + t, \mathbf{z}') - (D_\Omega f)(z_1, \mathbf{z}')| &\leq \int_0^t |(\mathcal{D}_{\Omega'} f)(z_1 + s, \mathbf{z}')| ds \\ &\leq \int_0^t (H_{\Omega'} f)(\alpha_1 + s, \alpha_\Omega) ds \\ &\leq \int_0^t (H_{\Omega'} f)(s, \alpha_\Omega) ds. \end{aligned}$$

The final expression depends only on f , t and α_Ω , so we may take the supremum over all $\mathbf{z} \in W_{\alpha_\Omega}$ and infer that

$$H_\Omega(T_1(t)f - f)(\alpha_\Omega) \leq \int_0^t (H_{\Omega'} f)(s, \alpha_\Omega) ds. \quad (3.7)$$

Since $\|f\|_{\mathcal{B}_{\Omega'}^n}$ is finite, $H_{\Omega'} f$ is integrable over \mathbb{R}_+^{k+1} . Fubini's theorem implies that

$$\alpha_1 \mapsto \int_{\mathbb{R}_+^k} (H_{\Omega'} f)(\alpha_1, \alpha_\Omega) d\alpha_\Omega$$

is integrable over \mathbb{R}_+ , and then (3.7) and Fubini's theorem imply that

$$\begin{aligned} \|T_1(t)f - f\|_{\mathcal{B}_\Omega^n} &= \int_{\mathbb{R}_+^k} H_\Omega(T_1(t)f - f)(\alpha_\Omega) d\alpha_\Omega \\ &\leq \int_0^t \int_{\mathbb{R}_+^k} (H_{\Omega'} f)(\alpha_1, \alpha_\Omega) d\alpha_\Omega d\alpha_1 \rightarrow 0, \quad t \rightarrow 0. \quad \blacksquare \end{aligned}$$

Remarks 3.7. (1) Case 2 of the proof above can be applied when Ω is empty, to show that the left shifts are strongly continuous in the H^∞ -norm, and hence that f is uniformly continuous, as shown in Proposition 2.1.

(2) For $n = 2$, the proof above differs from [23, Proposition 4.2.2], where the proof relies on switching to an equivalent norm on \mathcal{B}^2 . That approach can also be applied for $n > 2$. It simplifies the proof of Case 2 above, provided one accepts the equivalence of norms.

(3) It was shown in [7, Proposition 4.6.2] that the shift semigroup on \mathcal{B}^1 extends to a bounded holomorphic semigroup of angle $\pi/2$.

3.3. Sums of variables

In this section, we consider how very simple changes of the variables z_j preserve \mathcal{B}^n . As noted in [7, Lemma 2.6(5)] a trivial scaling of variables shows that if $f \in \mathcal{B}^1$, $b > 0$ and $g(z) = f(bz)$, then $g \in \mathcal{B}^1$, and $\|g\|_{\mathcal{B}^1} = \|f\|_{\mathcal{B}^1}$, and the same applies for functions in \mathcal{B}^n with scaling of some or all variables.

In addition, the family of algebras \mathcal{B}^n is preserved by summing some or all of the variables z_j . In order to avoid heavy formulas, we present this only in the case where all the variables are summed. For $n = 2$, the result was proved in [23, Lemma 3.8.1].

Proposition 3.8. *Let $f \in \mathcal{B}^1$, $n \in \mathbb{N}$, and define g on \mathbb{C}_+^n by*

$$g(\mathbf{z}) = f(z_1 + \cdots + z_n), \quad \mathbf{z} \in \mathbb{C}_+^n. \quad (3.8)$$

Then $g \in \mathcal{B}^n$, and the map $f \mapsto g$ from \mathcal{B}^1 to \mathcal{B}^n is bounded.

Proof. We may assume that $f \in \mathcal{B}_0^1$. Clearly, g is holomorphic on \mathbb{C}_+^n and $g(\mathbf{z}) \rightarrow 0$ as $\operatorname{Re} z_j \rightarrow \infty$ for some $j \in I_n$. By Proposition 2.6, it suffices to show that $\|g\|_{\mathcal{B}_0^n} < \infty$. We have

$$\|g\|_{\mathcal{B}_0^n} = \int_{\mathbb{R}_+^n} \sup_{\beta \in \mathbb{R}} |f^{(n)}(\alpha_1 + \cdots + \alpha_n + i\beta)| d\alpha.$$

The change of variables

$$t_k = \sum_{j=1}^k \alpha_j, \quad k \in I_n,$$

establishes that

$$\|g\|_{\mathcal{B}_0^n} = \int_{S_n} \sup_{\beta \in \mathbb{R}} |f^{(n)}(t_n + i\beta)| dt,$$

where $S_n = \{(t_k)_{k \in I_n} \in \mathbb{R}_+^n : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n\}$. Integrating with respect to t_1, \dots, t_{n-1} and replacing t_n by t shows that

$$\|g\|_{\mathcal{B}_0^n} = \int_0^\infty \frac{t^{n-1}}{(n-1)!} \sup_{\beta \in \mathbb{R}} |f^{(n)}(t + i\beta)| dt.$$

A higher-order version of Cauchy's inequality gives

$$t^{n-1} \sup_{y \in \mathbb{R}} |f^{(n)}(t + i\beta)| \leq 2^{n-1} C_{n-1} \sup_{\beta \in \mathbb{R}} |f'(t/2 + i\beta)|,$$

where C_{n-1} is as in [7, Lemma 2.1 (4)]. Hence,

$$\|g\|_{\mathcal{B}_0^n} \leq \frac{2^{n-1} C_{n-1}}{(n-1)!} \int_0^\infty \sup_{\beta \in \mathbb{R}} |f'(t/2 + i\beta)| dt = \frac{2^n C_{n-1}}{(n-1)!} \|f\|_{\mathcal{B}_0^1}.$$

This establishes that $g \in \mathcal{B}_0^n$, and also that the map $f \mapsto g$ is bounded from \mathcal{B}^1 to \mathcal{B}^n . \blacksquare

If g as defined in (3.8) is in \mathcal{B}^n , then $f \in \mathcal{B}^1$, since $f(z) = g(z, 0, 0, \dots, 0)$.

4. The \mathcal{B}^n -calculus: set up

In the remainder of this paper, $\mathcal{A} := (A_1, \dots, A_n)$ will be an n -tuple of commuting operators on a Banach space X . The operators A_j commute in the sense that the resolvent set of each operator A_j is non-empty and their resolvents commute with each other. Moreover, we will always assume that the spectrum of each operator satisfies

$$\sigma(A_j) \subseteq \overline{\mathbb{C}}_+, \quad j \in I_n. \quad (4.1)$$

Assuming that (4.1) holds, let $\Omega \in \mathcal{P}_n$. We will say that \mathcal{A} satisfies the (GSF $_\Omega$) condition if the following holds for all $x \in X$ and $x^* \in X^*$:

$$\sup_{\alpha_j > 0, j \in \Omega} \int_{\mathbb{R}^{|\Omega|}} \left| \left\langle \prod_{j \in \Omega} \alpha_j (A_j + \alpha_j - i\beta_j)^{-2} x, x^* \right\rangle \right| d\boldsymbol{\beta} < \infty. \quad (\text{GSF}_\Omega)$$

By the closed graph theorem, if (GSF $_\Omega$) holds, there is a constant $\gamma_{\mathcal{A}, \Omega}$ such that

$$\left(\frac{2}{\pi}\right)^{|\Omega|} \int_{\mathbb{R}^{|\Omega|}} \left| \left\langle \prod_{j \in \Omega} \alpha_j (A_j + \alpha_j - i\beta_j)^{-2} x, x^* \right\rangle \right| d\boldsymbol{\beta} \leq \gamma_{\mathcal{A}, \Omega} \|x\| \|x^*\| \quad (4.2)$$

for all $x \in X$ and $x^* \in X^*$. If Ω is empty, then the conditions above are deemed to be satisfied with $\gamma_{\mathcal{A}, \emptyset} = 1$. For $\Omega = \{j\}$, the condition says that A_j satisfies the (GSF) condition for the single operator A_j as in [7, Section 4.1]. If $\Omega = I_n$, we will say that \mathcal{A} satisfies the (GSF $_n$) condition and we will write $\gamma_{\mathcal{A}}$ instead of γ_{I_n} .

We will say that \mathcal{A} satisfies the full (GSF) condition if \mathcal{A} satisfies (GSF $_\Omega$) for all $\Omega \in \mathcal{P}_n$. For $n = 2$, the full (GSF) condition agrees with [23, Section 6.1].

Definition 4.1. A \mathcal{B}^n -calculus for \mathcal{A} is a bounded algebra homomorphism $\Phi: \mathcal{B}^n \rightarrow L(X)$ such that $\Phi(r_{\lambda, j}) = (A_j + \lambda)^{-1}$ for all $j \in I_n$ and $\lambda \in \mathbb{C}_+$.

There is a similar definition of a \mathcal{B}_0^n -calculus. In this and the next section, we will show that the full (GSF) condition is a sufficient condition for \mathcal{A} to have a \mathcal{B}^n -calculus, but we first show that the full (GSF) condition is necessary. This was proved in [8, Theorem 6.1] for $n = 1$ and in [23, Theorem 6.3.1] for $n = 2$. The proof below is a variant of those proofs.

We will write $\mathbf{r}_z(\mathcal{A}) := \prod_{j=1}^n (A_j + z_j)^{-1}$.

Proposition 4.2. *Let \mathcal{A} be an n -tuple of commuting operators on a Banach space X satisfying (4.1), and assume that there is a \mathcal{B}_0^n -calculus Φ for \mathcal{A} . Then \mathcal{A} satisfies the (GSF $_n$) condition.*

Proof. Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{C}$ be a continuous function with compact support, and $\alpha \in (0, \infty)^n$. As shown in Example 3.4, the map $\beta \mapsto \mathbf{r}_{\alpha-i\beta}^2$ is continuous from \mathbb{R}^n to \mathcal{B}_0^n and $\|\mathbf{r}_{\alpha-i\beta}^2\|_{\mathcal{B}_0^n} \leq 4^n \prod_{j=1}^n \alpha_j^{-1}$. Hence,

$$G_{\alpha, \varphi} := \int_{\mathbb{R}^n} \varphi(\beta) \mathbf{r}_{\alpha-i\beta}^2 d\beta,$$

exists as a Bochner integral with values in \mathcal{B}_0^n . Since Φ is a bounded linear operator,

$$\Phi(G_{\alpha, \varphi}) = \int_{\mathbb{R}^n} \varphi(\beta) \mathbf{r}_{\alpha-i\beta}(\mathcal{A})^2 d\beta$$

and hence, for unit vectors $x \in X$ and $x^* \in X^*$,

$$\prod_{j=1}^n \alpha_j \left| \int_{\mathbb{R}^n} \varphi(\beta) \langle \mathbf{r}_{\alpha-i\beta}(\mathcal{A})^2 x, x^* \rangle d\beta \right| \leq 4^n \|\varphi\|_{\infty} \|\Phi\|.$$

Since this holds for all continuous functions φ with compact support, it follows that (GSF $_n$) holds. ■

Corollary 4.3. *Let \mathcal{A} be an n -tuple of commuting operators on a Banach space X satisfying (4.1), and assume that there is a bounded algebra homomorphism $\Phi: \mathcal{B}^n \rightarrow L(X)$ such that $\Phi(\mathbf{r}_z) = \mathbf{r}_z(\mathcal{A})$ for all $z \in \mathbb{C}_+^n$. Then \mathcal{A} satisfies the full (GSF) condition.*

Proof. For $\Omega \in \mathcal{P}_n$, we may deduce that (GSF $_{\Omega}$) holds, by applying Proposition 4.2 to \mathcal{B}_0^{Ω} and the restriction of Φ to that algebra. ■

Now, we give two classes of operators where \mathcal{A} satisfies the full (GSF) condition for all values of n . Proofs were given in [7] for $n = 1$ and in [23] for $n = 2$, and the general proofs are minor variants of those cases.

Examples 4.4. (1) Let $-A_j$ be the generator of a bounded C_0 -semigroup $(e^{-tA_j})_{t \geq 0}$ on a Hilbert space X , for each $j \in I_n$, and let $K_j = \sup_{t \geq 0} \|e^{-tA_j}\|$. Assume that the semigroups commute with each other (or equivalently, the resolvents of A_j commute with each other). Then \mathcal{A} satisfies the full (GSF) condition, with $\gamma_{\mathcal{A}} = 2^n \prod_{j=1}^n K_j^2$. This follows from applications of Plancherel's theorem and the Cauchy-Schwarz inequality. See the proofs for $n = 1$ in [7, Examples 4.1] and for $n = 2$ in [23, Example 6.1.1].

(2) Let A_j ($j \in I_n$) be sectorial operators of angle less than $\pi/2$ on a Banach space X , and assume that the resolvents commute with each other (equivalently, $-A_j$ is the generator of a bounded holomorphic C_0 -semigroup $(e^{-tA_j})_{t \geq 0}$ on a Banach space X , for $j \in I_n$, and the semigroups commute with each other). Let $M_j = \sup_{z \in \mathbb{C}_+} \|z(A_j + z)^{-1}\|$. Then \mathcal{A} satisfies the full (GSF) condition, with $\gamma_{\mathcal{A}} = 2^n \prod_{j=1}^n M_j^2$. This follows from standard resolvent estimates for sectorial operators. See the proofs for $n = 1$ in [7, Section 4.2] and for $n = 2$ in [23, Example 6.1.2].

Now, assume that \mathcal{A} satisfies the (GSF_n) condition, and let $f \in \mathcal{B}_0^n$. Recall from Corollary 3.3 that the following reproducing formula holds:

$$f(\mathbf{z}) = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}}) (\mathcal{D}_n f)(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}), \quad \mathbf{z} \in \mathbb{C}_+^n.$$

We define $f(\mathcal{A})$ by the corresponding formula:

$$\langle f(\mathcal{A})x, x^* \rangle := \int_{\mathbb{C}_+^n} \langle \mathcal{K}_n(\mathcal{A}, \bar{\boldsymbol{\lambda}})x, x^* \rangle (\mathcal{D}_n f)(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}), \quad (4.3)$$

where

$$\begin{aligned} \mathcal{K}_n(\mathcal{A}, \bar{\boldsymbol{\lambda}}) &:= \left(-\frac{2}{\pi}\right)^n \mathbf{r}_{\bar{\boldsymbol{\lambda}}}(\mathcal{A})^2 \\ &= \left(-\frac{2}{\pi}\right)^n \prod_{j=1}^n (A_j + \bar{\lambda}_j)^{-2}, \quad \boldsymbol{\lambda} \in \mathbb{C}_+^n. \end{aligned} \quad (4.4)$$

By the (GSF_n) condition, the integral is absolutely convergent, and the formula (4.3) defines $f(\mathcal{A})$ as a map in $L(X, X^{**})$, with norm at most $\gamma_{\mathcal{A}} \|f\|_{\mathcal{B}_0^n}$. In order to obtain a \mathcal{B}^n -calculus for \mathcal{A} , we need to extend the definition of $f(\mathcal{A})$ to functions $f \in \mathcal{B}^n$, show that $f(\mathcal{A})$ maps X into X for all $f \in \mathcal{B}^n$, and show that the map $f \mapsto f(\mathcal{A})$ is an algebra homomorphism from \mathcal{B}^n to $L(X)$.

If the operators A_j are the negative generators of n commuting C_0 -semigroups, and $f = \mathcal{L}\mu$ for some bounded measure μ on \mathbb{R}_+^n , then $f(\mathcal{A})$ as defined above agrees with the Hille–Phillips calculus for $\mathcal{L}\mathcal{M}^n$. This was shown in [7, Lemma 4.2] for $n = 1$, and in [23, Lemma 6.1.4] for $n = 2$. The proof for $n \geq 3$ is similar, using Example 2.8 and higher-order integrals. A consequence of this is that the definition of $\mathcal{K}_n(\mathcal{A}, \bar{\boldsymbol{\lambda}})$ in (4.4) agrees with the definition obtained by putting $f(\mathbf{z}) = \mathcal{K}_n(\mathbf{z}, \bar{\boldsymbol{\lambda}})$ in (4.3), for a fixed $\boldsymbol{\lambda}$.

Now, we assume that \mathcal{A} satisfies the full (GSF) condition. If f is an elementary function in \mathcal{B}^n , with order $k \geq 1$ and support Ω , we can define $f(\mathcal{A})$ by applying the

\mathcal{B}_0^k -calculus for $\mathcal{A}_\Omega := (A_j)_{j \in \Omega}$ to the function $f_\Omega \in \mathcal{B}_0^k$, so

$$\langle f(\mathcal{A})x, x^* \rangle := \int_{\mathbb{C}_+^k} \langle \mathcal{K}_k(\mathcal{A}_\Omega, \bar{\lambda})x, x^* \rangle (\mathcal{D}_k f_\Omega)(\lambda) dV_k(\lambda) \quad (4.5)$$

for all $x \in X$ and $x^* \in X^*$. In particular, if $\Omega = \{j\}$ for some $j \in I_n$, $g \in \mathcal{B}^1$, and we define $f(\mathbf{z}) = g(z_j)$, then we recover the \mathcal{B}^1 -calculus for the operator A_j . Thus, this definition establishes that $f(\mathcal{A}) = (A_j + \lambda)^{-1}$ if $f(\mathbf{z}) = r_\lambda(z_j)$.

Now, let f be an arbitrary function in \mathcal{B}^n . To define $f(\mathcal{A})$ we use the elementary decomposition of f as in Proposition 2.7 (i). We define $f_{\emptyset,0}(\mathcal{A}_\emptyset) = f_{\emptyset,0}I'$ and

$$f(\mathcal{A}) = \sum_{\Omega \in \mathcal{P}_n} f_{\Omega,0}(\mathcal{A}_\Omega). \quad (4.6)$$

Then we define a bounded linear map from $\Phi_{\mathcal{A}}: \mathcal{B}^n \rightarrow L(X, X^{**})$ by

$$\Phi_{\mathcal{A}}(f) = f(\mathcal{A}). \quad (4.7)$$

Then $\|\Phi_{\mathcal{A}}\| \leq \gamma_{\mathcal{A}} := \max_{\Omega \in \mathcal{P}_n} \gamma_{\mathcal{A}_\Omega}$, where $\gamma_{\mathcal{A}_\Omega}$ is as in (4.2). The norm of the restriction of $\Phi_{\mathcal{A}}$ to \mathcal{B}_0^n is the minimal value of $\gamma_{\mathcal{A}_n}$, and we will denote it by $\gamma_{\mathcal{A},0}$.

In the next section, we will prove that $\Phi_{\mathcal{A}}$ is an algebra homomorphism from \mathcal{B}^n to $L(X)$. It will suffice to show that $f(\mathcal{A}) \in L(X)$ and $(fg)(\mathcal{A}) = f(\mathcal{A})g(\mathcal{A})$ for all elementary functions f and g in \mathcal{B}^n . These properties were proved in [7] for $n = 1$ and in [23] for $n = 2$, by a rather complicated method, and the method used in Section 5 is simpler, and illuminating even in the case $n = 1$.

Before moving on to the main proof, we present two lemmas. The first one may help when considering (4.6).

Lemma 4.5. *Assume that $\mathcal{A} = (A_1, \dots, A_n)$ satisfies the full (GSF) condition, and let $\mathcal{A}_{n-1} = (A_1, \dots, A_{n-1})$, $x \in X$ and $x^* \in X^*$.*

(1) *For $f \in \mathcal{B}^n$ and $\Omega \in \mathcal{P}_n$ with $|\Omega| = k \geq 1$,*

$$\langle f_{\Omega,0}(\mathcal{A})x, x^* \rangle = \int_{\mathbb{C}_+^k} \langle \mathcal{K}_k(\mathcal{A}_\Omega, \bar{\lambda})x, x^* \rangle (\mathcal{D}_\Omega f_\Omega)(\lambda) dV_k(\lambda).$$

(2) *Let $f \in \mathcal{B}_0^n$. Then $(D_n f)(\cdot, \lambda_n) \in \mathcal{B}_0^{n-1}$ for all $\lambda_n \in \mathbb{C}_+$, and*

$$\langle f(\mathcal{A})x, x^* \rangle = \int_{\mathbb{C}_+} \langle K(A_n, \bar{\lambda}_n)(D_n f)(\mathcal{A}_{n-1}, \lambda_n)x, x^* \rangle dV(\lambda_n).$$

Proof. The first statement is proved in Corollary 3.3.

Proposition 2.3, with $\Omega = I_{n-1}$ and $\Psi = \{n\}$, establishes that $(D_n f)(\cdot, \lambda_n) \in \mathcal{B}_0^{n-1}$ for all $\lambda_n \in \mathbb{C}_+$. Fubini's theorem shows that

$$\begin{aligned} & \langle f(\mathcal{A})x, x^* \rangle \\ &= \int_{\mathbb{C}_+^n} \langle \mathcal{K}_{n-1}(\mathcal{A}_{n-1}, \bar{\lambda}_{n-1})K(A_n, \bar{\lambda}_n)x, x^* \rangle (\mathcal{D}_n f)(\lambda_{n-1}, \lambda_n) dV_n(\lambda_{n-1}, \lambda_n) \\ &= \int_{\mathbb{C}_+} \langle K(A_n, \bar{\lambda}_n)(\mathcal{D}_n f)(\mathcal{A}_{n-1}, \lambda_n)x, x^* \rangle dV(\lambda_n), \end{aligned}$$

for all $x \in X$ and $x^* \in X^*$. ■

In order to justify some applications of Fubini's theorem, and differentiation through integral signs, in the next section, we will apply estimates for n -tuples of operators of the form $\mathcal{A} + \mathbf{t} := (A_j + t_j)_{j \in I_n}$, where $t_j > 0$, and we will use the following fact. For $n = 1$, it is a special case of [7, Corollary 4.5]. Here $(T(\mathbf{t}))_{\mathbf{t} \in \mathbb{R}_+^n}$ are the shifts as defined in (3.5).

Lemma 4.6. *Assume that \mathcal{A} satisfies the (GSF $_n$) condition. Let $f \in \mathcal{B}_0^n$ and $\mathbf{t} \in \mathbb{R}_+^n$. Then*

$$f(\mathcal{A} + \mathbf{t}) = (T(\mathbf{t})f)(\mathcal{A}).$$

This can easily be proved by the same method as in Lemma 3.5, with $\mathcal{K}_n(\mathbf{z}, \bar{\lambda})$ being replaced by $\langle \mathcal{K}_n(\mathcal{A}, \bar{\lambda})x, x^* \rangle$, $\mathcal{K}_n(\mathbf{z} + \mathbf{t}, \bar{\lambda})$ by $\langle \mathcal{K}_n(\mathcal{A} + \mathbf{t}, \bar{\lambda})x, x^* \rangle$, and $(\mathcal{D}_n f)(\lambda)$ by $(\mathcal{D}_n f)(\lambda + \mathbf{t})$.

5. The \mathcal{B}^n -calculus: Developments

In this section, we assume that $\mathcal{A} := (A_1, \dots, A_n)$ is an n -tuple of commuting operators which satisfy the full (GSF) condition on a Banach space X , and we denote by $\Phi_{\mathcal{A}}$ the map $f \mapsto f(\mathcal{A})$ from \mathcal{B}^n to $L(X, X^{**})$, as defined in (4.5), (4.6), and (4.7). We will establish three properties:

- the operator $f(\mathcal{A})$ maps X into X ,
- the operator $\Phi_{\mathcal{A}}$ is an algebra homomorphism,
- $\Phi_{\mathcal{A}}$ is the unique \mathcal{B}^n -calculus for \mathcal{A} .

The first two properties establish that $\Phi_{\mathcal{A}}$ is a \mathcal{B}^n -calculus for \mathcal{A} . For $n = 1$, all three properties have been proved, the first two in [7, Theorem 4.4], and the third in [8, Theorem 6.6]. For $n = 2$, they have been proved in [23]. The proofs in those papers can probably be adapted to cover the same properties for arbitrary $n \in \mathbb{N}$, but we will present more direct proofs of all three properties even for $n = 1$.

We proceed with three lemmas which combine to prove the three properties, but the three proofs are inter-related. We start by considering two non-zero elementary functions f and g in \mathcal{B}^n . We denote their supports by Ω_f and Ω_g with cardinalities k and m , respectively. Thus, f and g belong to the spaces $\mathcal{B}_0^{\Omega_f}$ and $\mathcal{B}_0^{\Omega_g}$, respectively. Unlike the case when $n = 1$, the two supports may be different. As it turns out, this does not cause a serious complication.

Lemma 5.1. *Let $\mathcal{A} = (A_1, \dots, A_n)$ be an n -tuple of operators which satisfy the full (GSF) condition on a Banach space X . Let f and g be elementary functions in \mathcal{B}^n , and assume that $g(\mathcal{A} + \mathbf{t}) \in L(X)$ for $\mathbf{t} = (t, \dots, t)$, $t \geq 0$. Then fg is an elementary function in \mathcal{B}^n and*

$$(fg)(\mathcal{A}) = f(\mathcal{A})g(\mathcal{A}). \quad (5.1)$$

In particular,

$$(f\mathbf{r}_{\mathbf{z}})(\mathcal{A}) = f(\mathcal{A})\mathbf{r}_{\mathbf{z}}(\mathcal{A}), \quad \mathbf{z} \in \mathbb{C}_+^n.$$

Proof. If f or g are a constant function, the result holds. So, we assume that both f and g are not constant functions, and their supports have cardinalities k and m , respectively. It is clear that fg is an elementary function in \mathcal{B}^n , and its support is $\Omega := \Omega_f \cup \Omega_g$. Consequently, we can also assume that $\Omega = I_n$.

Let $\mathbf{t} = (t, \dots, t) \in (0, \infty)^n$, $f_{\mathbf{t}}(\mathbf{z}) = f(\mathbf{z} + \mathbf{t})$ and $g_{\mathbf{t}}(\mathbf{z}) = g(\mathbf{z} + \mathbf{t})$. Applying (4.3) to f and g , we obtain, for $\mathbf{z} \in \mathbb{C}_+^n$,

$$(f_{\mathbf{t}}g_{\mathbf{t}})(\mathbf{z}) = \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} [\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\mathbf{v}})] \\ \times (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) (\mathcal{D}_{\Omega_g} g)(\mathbf{v}) dV_k(\boldsymbol{\lambda}) dV_m(\mathbf{v}).$$

Differentiation through the integral (which is easily justified) gives

$$\mathcal{D}_n(f_{\mathbf{t}}g_{\mathbf{t}})(\mathbf{z}) = \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} \mathcal{D}_n[\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\mathbf{v}})] \\ \times (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) (\mathcal{D}_{\Omega_g} g)(\mathbf{v}) dV_k(\boldsymbol{\lambda}) dV_m(\mathbf{v}).$$

Combining (4.5), Lemma 3.5, and Fubini's theorem (see the Appendix for justification), we infer that

$$\langle (f_{\mathbf{t}}g_{\mathbf{t}})(\mathcal{A})x, x^* \rangle \\ = \int_{\mathbb{C}_+^n} \langle \mathcal{K}_n(\mathcal{A}, \bar{\mathbf{z}})x, x^* \rangle \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} \mathcal{D}_n(\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\mathbf{v}})) \\ \times (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) (\mathcal{D}_{\Omega_g} g)(\mathbf{v}) dV_k(\boldsymbol{\lambda}) dV_m(\mathbf{v}) dV_n(\mathbf{z}) \\ = \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) (\mathcal{D}_{\Omega_g} g)(\mathbf{v}) \mathbf{R}_{\mathbf{t}}(\boldsymbol{\lambda}, \mathbf{v}) dV_k(\boldsymbol{\lambda}) dV_m(\mathbf{v}), \quad (5.2)$$

where

$$R_{\mathbf{t}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \int_{\mathbb{C}_+^n} \langle \mathcal{K}_n(\mathcal{A}, \bar{\mathbf{z}})x, x^* \rangle \mathcal{D}_n(\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})) dV_n(\mathbf{z}).$$

For all $\boldsymbol{\lambda} \in \mathbb{C}_+^k$ and $\boldsymbol{\nu} \in \mathbb{C}_+^m$,

$$\mathcal{K}_k(\cdot + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \in \mathcal{LM}^k \cap \mathcal{B}_0^k, \quad \mathcal{K}_m(\cdot + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}}) \in \mathcal{LM}^m \cap \mathcal{B}_0^m,$$

as functions of \mathbf{z}_{Ω_f} and \mathbf{z}_{Ω_g} , respectively. Both these functions, and their product, are in \mathcal{LM}^n . Using (4.5) and the fact that the Hille–Phillips calculus is an algebra homomorphism, we see that

$$\begin{aligned} R_{\mathbf{t}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) &= \langle [\mathcal{K}_k(\cdot + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\cdot + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})](\mathcal{A})x, x^* \rangle \\ &= \langle \mathcal{K}_k(\mathcal{A}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathcal{A}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})x, x^* \rangle. \end{aligned}$$

Thus, by (5.2), (4.5), and Lemma 4.6,

$$\begin{aligned} &\langle (f_{\mathbf{t}} g_{\mathbf{t}})(\mathcal{A})x, x^* \rangle \\ &= \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) (\mathcal{D}_{\Omega_g} g)(\boldsymbol{\nu}) \\ &\quad \times \langle \mathcal{K}_k(\mathcal{A}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathcal{A}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})x, x^* \rangle dV_k(\boldsymbol{\lambda}) dV_m(\boldsymbol{\nu}) \\ &= \int_{\mathbb{C}_+^k} (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) \langle g(\mathcal{A} + \mathbf{t}) \mathcal{K}_k(\mathcal{A}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}})x, x^* \rangle dV_k(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{C}_+^k} (\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda}) \langle \mathcal{K}_k(\mathcal{A}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}})g(\mathcal{A} + \mathbf{t})x, x^* \rangle dV_k(\boldsymbol{\lambda}) \\ &= \langle f(\mathcal{A} + \mathbf{t})g(\mathcal{A} + \mathbf{t})x, x^* \rangle. \end{aligned}$$

Hence, for all $t > 0$,

$$(f_{\mathbf{t}} g_{\mathbf{t}})(\mathcal{A}) = f_{\mathbf{t}}(\mathcal{A})g_{\mathbf{t}}(\mathcal{A}). \quad (5.3)$$

Letting $t \rightarrow 0$ in (5.3) and using the strong continuity of the shifts on \mathcal{B}^n , we obtain the assertion (5.1). \blacksquare

Next we prove that for “smoothed” functions $f \in \mathcal{B}_0^n$ the reproducing formula (3.3) can be interpreted as a \mathcal{B}_0^n -valued Bochner integral. Here $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$.

Lemma 5.2. *Let $f \in \mathcal{B}_0^n$ and $\mathbf{t} = (t, \dots, t) \in (0, \infty)^n$. Define $F_{\mathbf{t}}: \mathbb{C}_+^n \rightarrow \mathcal{B}_0^n$ by*

$$F_{\mathbf{t}}(\boldsymbol{\lambda}) = (\mathcal{D}_n(f \mathbf{r}_{\mathbf{1}}^2))(\boldsymbol{\lambda}) \mathbf{r}_{\mathbf{t}+\bar{\boldsymbol{\lambda}}}^2, \quad \boldsymbol{\lambda} \in \mathbb{C}_+^n.$$

Then $F_{\mathbf{t}}$ is continuous and Bochner-integrable with respect to the measure $dV_n(\boldsymbol{\lambda})$. Moreover,

$$\left(-\frac{2}{\pi}\right)^n \int_{\mathbb{C}_+^n} F_{\mathbf{t}}(\boldsymbol{\lambda}) dV_n(\boldsymbol{\lambda}) = f_{\mathbf{t}} \mathbf{r}_{\mathbf{1}+\mathbf{t}}^2. \quad (5.4)$$

Proof. The function $\boldsymbol{\lambda} \mapsto \mathbf{r}_{\mathbf{t}+\bar{\boldsymbol{\lambda}}}^2$ is continuous from \mathbb{C}_+^n to \mathcal{B}_0^n , and hence $F_{\mathbf{t}}$ is continuous. If $\lambda_j = \alpha_j + i\beta_j$, then

$$\|\mathbf{r}_{\mathbf{t}+\bar{\boldsymbol{\lambda}}}^2\|_{\mathcal{B}_0^n} = \prod_{j=1}^n \frac{1}{(\alpha_j + t)^2}.$$

Moreover, $(\mathcal{D}_n(fr_1^2))(\boldsymbol{\lambda})$ is the sum of 2^n terms of the following form, indexed by $\Omega \in \mathcal{P}_n$, where $k = |\Omega|$,

$$J_{n,\Omega}(\boldsymbol{\lambda}) := (-2)^{n-k} (\mathcal{D}_{\Omega} f)(\boldsymbol{\lambda}) \prod_{j \in \Omega} r_1(\lambda_j)^2 \prod_{j \in \Omega^c} r_1(\lambda_j)^3,$$

and, by a standard estimate,

$$|(\mathcal{D}_{\Omega} f)(\boldsymbol{\lambda})| \leq \frac{\|f\|_{\infty}}{2^k \prod_{j \in \Omega} \alpha_j}.$$

It follows that

$$\begin{aligned} & \int_{\mathbb{C}_+^n} |J_{n,\Omega}(\boldsymbol{\lambda})| \|\mathbf{r}_{\mathbf{t}+\bar{\boldsymbol{\lambda}}}^2\|_{\mathcal{B}_0^n} dV_n(\boldsymbol{\lambda}) \\ & \leq 2^{n-2k} \|f\|_{\infty} \int_{\mathbb{C}_+^n} \prod_{j \in \Omega} \frac{\alpha_j}{\alpha_j (\alpha_j + t)^2 |1 + \lambda_j|^2} \prod_{j \in \Omega^c} \frac{\alpha_j}{(\alpha_j + t)^2 |1 + \lambda_j|^3} dS_n(\boldsymbol{\alpha}, \boldsymbol{\beta}) \\ & \leq 2^{n-2k} \pi^n \|f\|_{\infty} \left(\int_0^{\infty} \frac{d\alpha}{(\alpha + t)^2} \right)^n = \frac{(2\pi)^n}{4^k t^n} \|f\|_{\infty}. \end{aligned}$$

Hence,

$$\int_{\mathbb{C}_+^n} |\mathcal{D}_n(f \mathbf{r}_1)(\boldsymbol{\lambda})| \|\mathbf{r}_{\mathbf{t}+\bar{\boldsymbol{\lambda}}}^2\|_{\mathcal{B}_0^n} dV_n(\boldsymbol{\lambda}) \leq \frac{(2\pi)^n}{t^n} \sum_{k=0}^n \binom{n}{k} 4^{-k} \|f\|_{\infty} = \left(\frac{5\pi}{2t}\right)^n \|f\|_{\infty}.$$

Thus, $\int_{\mathbb{C}_+^n} \|F_{\mathbf{t}}(\boldsymbol{\lambda})\|_{\mathcal{B}_0^n} dV_n(\boldsymbol{\lambda}) < \infty$, so the Bochner integral of $F_{\mathbf{t}}$ exists. Since point evaluations are continuous linear functionals on \mathcal{B}_0^n , the function on the left-hand side of (5.4) maps $\mathbf{z} \in \mathbb{C}_+^n$ to

$$\left(-\frac{2}{\pi}\right)^n \int_{\mathbb{C}_+^n} (F_{\mathbf{t}}(\boldsymbol{\lambda}))(\mathbf{z}) dV_n(\boldsymbol{\lambda}) = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathbf{z} + \mathbf{t}, \bar{\boldsymbol{\lambda}}) \mathcal{D}_n(f \mathbf{r}_1^2) dV_n(\boldsymbol{\lambda}) = (f \mathbf{r}_1^2)(\mathbf{z} + \mathbf{t}),$$

using Lemma 3.5. Hence, (5.4) holds. \blacksquare

Next, we apply the mapping $\Phi_{\mathcal{A}}$ to both sides of (5.4), and then deduce the three properties of $\Phi_{\mathcal{A}}$ simultaneously. We say that

$$\Upsilon: \mathcal{B}_0^n \mapsto L(X, X^{**})$$

is a (bounded) semi-homomorphism if Υ is a bounded linear map satisfying

$$\Upsilon(f \cdot \mathbf{r}_z) = \Upsilon(f)(z + \mathcal{A})^{-1}, \quad f \in \mathcal{B}_0^n, \quad z \in \mathbb{C}_+^n. \quad (5.5)$$

Lemma 5.1 shows that Φ_A is a semi-homomorphism.

Lemma 5.3. *Let $\mathcal{A} = (A_1, \dots, A_n)$ be an n -tuple of commuting operators on X satisfying (4.1). Let*

$$\Upsilon: \mathcal{B}_0^n \mapsto \mathcal{L}(X, X^{**})$$

be a semi-homomorphism such that $\Upsilon(\mathbf{r}_z) = \mathbf{r}_z(\mathcal{A})$, $z \in \mathbb{C}_+^n$. Then, for all $f \in \mathcal{B}_0^n$,

$$\Upsilon(f)\mathbf{r}_1(\mathcal{A})^2 = \lim_{t \rightarrow 0} Q_t(f; \mathcal{A}), \quad (5.6)$$

in operator-norm, where $Q_t(f; \mathcal{A}) \in L(X)$ is defined for $t > 0$ by

$$Q_t(f; \mathcal{A}) := \left(-\frac{2}{\pi}\right)^n \int_{\mathbb{C}_+^n} (\mathcal{D}_n(f \mathbf{r}_1^2))(\boldsymbol{\lambda})(\mathcal{A} + \mathbf{t} + \bar{\boldsymbol{\lambda}})^{-2} dV_n(\boldsymbol{\lambda}). \quad (5.7)$$

Consequently, Υ is uniquely determined by its values on $\{\mathbf{r}_z : z \in \mathbb{C}_+^n\}$ and

$$\Upsilon(f) \in L(X), \quad f \in \mathcal{B}_0^n. \quad (5.8)$$

Proof. Using (5.5), Lemma 5.2, and Lemma 4.6, and applying Υ to both sides of (5.4), we have

$$\Upsilon(f_t)\mathbf{r}_{1+t}(\mathcal{A})^2 = \Upsilon(f_t \cdot \mathbf{r}_{1+t}^2) = Q_t(f; \mathcal{A}), \quad t > 0, \quad (5.9)$$

where $Q_t(f; \mathcal{A})$ is given by (5.7), and the integral converges in the norm of $L(X)$. By the continuity of shifts on \mathcal{B}_0^n , as in Lemma 3.6,

$$\lim_{t \rightarrow 0} \|\Upsilon(f_t) - \Upsilon(f)\|_{L(X, X^{**})} = 0,$$

and, by the resolvent identity,

$$\lim_{t \rightarrow 0} \|\mathbf{r}_{1+t}(\mathcal{A}) - \mathbf{r}_1(\mathcal{A})\|_{L(X)} = 0.$$

Thus, (5.9) implies (5.6) and (5.8).

Since $\text{dom}(A_j)$ is dense in X , $r_1(A_j)^2$ has dense range for $j \in I_n$, and then $\mathbf{r}_1(\mathcal{A})^2$ has dense range. The uniqueness of Υ follows from this and (5.6) and (5.7). Since X is closed in X^{**} , the assertion (5.8) holds. \blacksquare

Now, we can conclude that Φ_A , as defined in (4.5) and (4.6), is a \mathcal{B}^n -calculus for \mathcal{A} , as defined in Definition 4.1.

Theorem 5.4. *Let $\mathcal{A} = (A_1, \dots, A_n)$ satisfy the full (GSF) condition on a Banach space X . Then Φ_A is a \mathcal{B}^n -calculus for \mathcal{A} , and it is unique. Consequently, \mathcal{A} admits the \mathcal{B}^n -calculus if and only if \mathcal{A} satisfies the full (GSF) condition.*

Proof. Lemma 5.3 shows that $\Phi_{\mathcal{A}}$ maps \mathcal{B}_0^n into $L(X)$, and the same holds for the maps on $\mathcal{B}_{\Omega,0}^n$. By Lemma 5.1, $\Phi_{\mathcal{A}}$ is multiplicative on elementary functions and hence on \mathcal{B}^n . Then $\Phi_{\mathcal{A}}$ maps \mathcal{B}^n into $L(X)$, and is multiplicative on all functions in \mathcal{B}^n . In addition, Φ_A maps the function $\mathbf{z} \mapsto (z_j + \lambda)^{-1}$ to the operator $(A_j + \lambda)^{-1}$ for $\lambda \in \mathbb{C}_+$, $j \in I_n$, so $\Phi_{\mathcal{A}}$ is a \mathcal{B}^n -calculus for \mathcal{A} .

Let $f \in \mathcal{B}^n$. Then $f \mathbf{r}_1 \in \mathcal{B}_0^n$ and

$$\Phi_{\mathcal{A}}(f) \mathbf{r}_1(\mathcal{A}) = \Phi_A(f \mathbf{r}_1).$$

Since $\Phi_A(f \mathbf{r}_1)$ is uniquely determined (Lemma 5.3) and the range of $\mathbf{r}_1(\mathcal{A})$ is dense in X , it follows that $\Phi_{\mathcal{A}}(f)$ is uniquely determined.

The last statement follows from the earlier statement and Corollary 4.3. ■

6. Possible simplifications

In this section, we consider three possible types of simplification that might come into play when attempting to use the \mathcal{B} -calculus. The first one would be a dramatic simplification but it is not valid in general, while the other two are valid but restricted in scope.

6.1. Reduction to one variable

Let $\mathcal{A} = (A_1, \dots, A_n)$ be a commuting n -tuple where each A_j ($j \in I_n$) has a \mathcal{B}^1 -calculus. One might hope that this automatically implies that \mathcal{A} has a \mathcal{B}^n -calculus, but this is not valid in general.

Let $p \in [1, \infty)$, $p \neq 2$, and let \mathcal{C}_p be the p th Schatten-von Neumann ideal on a separable, infinite-dimensional, Hilbert space. Using [24, Proposition 3 and Example 1], together with results from [13, 16], and the equivalence in Theorem 5.4 for $n = 1$, it follows that there are commuting bounded operators A_1 and A_2 on \mathcal{C}_p , each of which admit the \mathcal{B}^1 -calculus, but $A_1 + A_2$ does not admit the \mathcal{B}^1 -calculus. Hence, (A_1, A_2) does not admit the \mathcal{B}^2 -calculus, since otherwise $A_1 + A_2$ would admit the \mathcal{B}^1 -calculus (see Proposition 6.2). Thus, the existence of \mathcal{B}^1 -calculi for operators A_1 and A_2 separately does not imply the existence of the \mathcal{B}^2 -calculus for (A_1, A_2) .

6.2. Mergers

In some cases, two or more of the operators in an n -tuple \mathcal{A} may coincide. We may merge some or all of the repeated operators to form an m -tuple $\tilde{\mathcal{A}} = (\tilde{A}_k)_{k \in I_m}$, where there is a surjective function π from I_n to I_m such that $A_j = \tilde{A}_{\pi(j)}$ for all $j \in I_n$. If \mathcal{A} satisfies (GSF), then it is trivial that $\tilde{\mathcal{A}}$ also satisfies the full (GSF), because $(\text{GSF}_{\tilde{\Omega}})$ for $\tilde{\mathcal{A}}$ coincides precisely with (GSF_{Ω}) for \mathcal{A} for some $\Omega \in \mathcal{P}_n$ where π is a bijection from $\tilde{\Omega}$ to Ω . We show in this section that the \mathcal{B} -calculi for $\tilde{\mathcal{A}}$ and \mathcal{A} are compatible, thus establishing that mergers of this type within the \mathcal{B} -calculus are valid. A similar result in a different context has been obtained in [3, Proposition 4.1].

Let $f \in \mathcal{B}^n$, and define

$$(\Upsilon f)(\mathbf{w}) := f(\mathbf{z}), \quad \mathbf{w} = (w_1, \dots, w_m) \in \mathbb{C}_+^m, \quad z_j = w_{\pi(j)}.$$

Proposition 6.1. *Let \mathcal{A} , $\tilde{\mathcal{A}}$ and Υ be as above, and assume that \mathcal{A} satisfies the full (GSF) condition. Let $f \in \mathcal{B}^n$. Then $\Upsilon(f) \in \mathcal{B}^m$, and $(\Upsilon f)(\tilde{\mathcal{A}}) = f(\mathcal{A})$ in the setting of their respective \mathcal{B} -calculi.*

Proof. To see that $\Upsilon f \in \mathcal{B}^m$, we first consider the case when $m = n - 1$, and $\pi(j) = \min(j, n - 1)$. Let $g = \Upsilon f$, and $\Omega \in \mathcal{P}_{n-1}$.

Firstly, assume that $z_{n-1} \notin \Omega$. Then $(\mathcal{D}_{\Omega} g)(\mathbf{z}) = (\mathcal{D}_{\Omega} f)(\mathbf{z}, z_{n-1})$, for $\mathbf{z} \in \mathbb{C}_+^{n-1}$, and $(H_{\Omega} g)(\alpha_{\Omega}) \leq (H_{\Omega} f)(\alpha_{\Omega})$, so $H_{\Omega} g$ is integrable. Secondly, assume that $n - 1 \in \Omega$. Let $\tilde{\Omega} = (\Omega \setminus \{n - 1\}) \cup \{n\}$. Then

$$(\mathcal{D}_{\Omega} g)(\mathbf{z}) = (\mathcal{D}_{\Omega} f)(\mathbf{z}, z_{n-1}) + (\mathcal{D}_{\tilde{\Omega}} f)(\mathbf{z}, z_{n-1}).$$

Hence,

$$(H_{\Omega} g)(\alpha_{\Omega}) \leq (H_{\Omega} f)(\alpha_{\Omega}) + (H_{\tilde{\Omega}} f)(\alpha_{\tilde{\Omega}}).$$

It follows that $H_{\Omega} g$ is integrable. Thus, $g \in \mathcal{B}^{n-1}$, and the map $\Upsilon: \mathcal{B}^n \rightarrow \mathcal{B}^{n-1}$ is bounded. By symmetry of the variables, this holds for any simple merger of just one pair of variables. The general case follows by carrying out $n - m$ simple mergers.

It is clear that Υ is a bounded algebra homomorphism from \mathcal{B}^n to \mathcal{B}^m . Let $\Upsilon(f) = \Phi_{\tilde{\mathcal{A}}}(\Upsilon f)$, so $\Upsilon: \mathcal{B}^n \rightarrow L(X)$ is a bounded algebra homomorphism. Let $r_{\lambda,j}(\mathbf{z}) = (z_j + \lambda)^{-1}$ where $\mathbf{z} \in \mathbb{C}_+^n$. Then, for $j \in I_n$ and $\lambda \in \mathbb{C}_+$,

$$\Upsilon r_{\lambda,j} = r_{\lambda,\pi(j)},$$

and

$$\Psi(r_{\lambda,j}) = (\tilde{A}_{\pi(j)} + \lambda)^{-1} = (A_j + \lambda)^{-1}.$$

It follows from the uniqueness statement in Theorem 5.4 that $\Upsilon = \Phi_{\mathcal{A}}$, as required. ■

6.3. Sums of operators

It was observed in [7, Lemma 2.6(5)] that if $f \in \mathcal{B}^1$, $b > 0$ and $g(z) = f(bz)$, then $g \in \mathcal{B}^1$ and $\|g\|_{\mathcal{B}^1} = \|f\|_{\mathcal{B}^1}$. Moreover, if A satisfies the (GSF) condition, then $g(A) = f(bA)$. This can be extended to the case of several variables, and the proofs are very simple changes of variables. Such scalings can be combined with mergers as above. For example, suppose that $f \in \mathcal{B}^2$ and $g(z) = (b_1z, b_2z)$, where $b_1, b_2 > 0$, and A satisfies the (GSF) condition, then $g(A) = f(b_1A, b_2A)$.

Here we discuss the case when the operators in \mathcal{A} may be added together, in the way that the variables z_j were added in Proposition 3.8. We present only the case where all the operators are added, but more general cases can be considered by the same techniques.

Let \mathcal{A} be an n -tuple of operators which satisfy the full (GSF) condition, $f \in \mathcal{B}^1$ and $f^{[n]}(\mathbf{z}) = f(z_1 + \cdots + z_n)$ for $\mathbf{z} \in \mathbb{C}_+^n$. By Proposition 3.8, $f^{[n]} \in \mathcal{B}^n$. Thus, $f^{[n]}(\mathcal{A})$ is defined in the \mathcal{B}^n -calculus, but $f(A_1 + \cdots + A_n)$ is not defined in the \mathcal{B}^1 -calculus (except in rare cases), because the operator $G_0 := A_1 + \cdots + A_n$, with domain $\text{dom}(G_0) = \text{dom}(A_1) \cap \cdots \cap \text{dom}(A_n)$, is not closed. However, $\text{dom}(G_0)$ is dense in X and the individual operators $-A_j$ generate commuting bounded C_0 -semigroups $(e^{-tA_j})_{t \geq 0}$, and the operators $T_{\mathcal{A}}(t) := \prod_{j=1}^n e^{-tA_j}$ form a bounded C_0 -semigroup whose negative generator G is the closure of G_0 (see [14, Theorem 1.9]).

The following result was proved for $n = 2$ in [23, Lemma 6.2.3].

Proposition 6.2. *Let \mathcal{A} be an n -tuple of operators which satisfy the full (GSF) condition, and let G be the closure of $A_1 + \cdots + A_n$. Then G satisfies (GSF₁) and $f(G) = f^{[n]}(\mathcal{A})$ for all $f \in \mathcal{B}^1$.*

Proof. The mapping $f \mapsto f^{[n]}(\mathcal{A})$ is a bounded algebra homomorphism from \mathcal{B}^1 to $L(X)$. It suffices to prove that $r_\lambda^{[n]}(\mathcal{A}) = r_\lambda(G)$ for $\lambda \in \mathbb{C}_+$. Then G has a \mathcal{B}^1 -calculus, so G satisfies (GSF₁), and uniqueness of the \mathcal{B}^1 -calculus implies that $f(G) = f^{[n]}(\mathcal{A})$.

Let μ be the measure on Borel subsets of \mathbb{C}_+^n given by $\mu(E) = \int_{\tilde{E}} e^{-\lambda t} dt$, where $\tilde{E} = \{t \in \mathbb{C}_+ : (t, t, \dots, t) \in E\}$. Then $r_\lambda^{[n]}$ is the Laplace transform of the measure μ . Since the \mathcal{B}^n -calculus agrees with the HP-calculus on \mathcal{LM}^n ,

$$r_\lambda^{[n]}(\mathcal{A})x = \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n e^{-t_j A_j} \right) x d\mu(\mathbf{t}) = \int_{\mathbb{R}_+} e^{-\lambda t} e^{-tG} x dt = r_\lambda(G)x$$

for all $x \in X$. ■

7. Spectral mapping properties

Let \mathcal{A} be an n -tuple of commuting operators on a Banach space X , satisfying the full (GSF) condition, so \mathcal{A} has a \mathcal{B}^n -calculus. We will use the Shilov joint spectrum of \mathcal{A} as in [29, Definition 9] based on [20, Theorem 16.3.1].

Let \mathfrak{A} be the (commutative) Banach subalgebra of $L(X)$ generated by all operators of the form $f(\mathcal{A})$ for $f \in \mathcal{B}^n$ and all their resolvents. The spectrum of $f(\mathcal{A})$ in \mathfrak{A} coincides with the spectrum in $L(X)$, and it will be denoted by $\sigma(f(\mathcal{A}))$. For each $j \in I_n$, the Gelfand spectrum of \mathfrak{A} splits into two subsets \mathfrak{M}_j and \mathfrak{N}_j and there is a continuous function η_j on \mathfrak{M}_j such that, for all $z \in \mathbb{C}_+$,

$$\chi((A_j + z)^{-1}) = \begin{cases} (\eta_j(\chi) + z)^{-1} & \text{if } \chi \in \mathfrak{M}_j, \\ 0 & \text{if } \chi \in \mathfrak{N}_j. \end{cases}$$

Then $\sigma(A_j) = \eta_j(\mathfrak{M}_j)$. The *Shilov joint spectrum* $\sigma(\mathcal{A})$ of \mathcal{A} is defined to be

$$\sigma(\mathcal{A}) = \left\{ (\eta_1(\chi), \dots, \eta_n(\chi)) : \chi \in \bigcap_{j \in I_n} \mathfrak{M}_j \right\}.$$

We have the following spectral inclusion theorem.

Theorem 7.1. *Let \mathcal{A} be an n -tuple of commuting operators which satisfy the full (GSF) condition, and let $f \in \mathcal{B}^n$ and $\lambda \in \sigma(\mathcal{A})$. Then $f(\lambda) \in \sigma(f(\mathcal{A}))$.*

Proof. First we will assume that the resolvents of A_j are bounded on the left half-plane. It follows from this assumption that there exists C such that, for all $z \in \mathbb{C}_+$,

$$\|(A_j + z)^{-2}(A_j + 1)^{-2}\| \leq C(1 + |z - 1|)^{-2}, \quad j \in I_n. \quad (7.1)$$

Let $f \in \mathcal{B}_0^n$. Then (4.5) and (7.1) can be used to obtain

$$f(\mathcal{A})r_1(\mathcal{A})^2 = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathcal{A}, \bar{\mathbf{v}})r_1(\mathcal{A})^2 (\mathcal{D}_n f)(\mathbf{v}) dV_n(\mathbf{v}), \quad (7.2)$$

where the integral converges in operator norm.

Let $\lambda \in \sigma(\mathcal{A})$. There is a character χ of \mathfrak{A} such that $\chi \in \mathfrak{M}_j$ and $\chi((A_j + 1)^{-1}) = (\lambda_j + 1)^{-1}$ for all $j \in I_n$. Then $\chi(r_1(\mathcal{A})) = r_1(\lambda)$ and $\chi(\mathcal{K}_n(\mathcal{A}, \mathbf{v})) = \mathcal{K}_n(\lambda, \mathbf{v})$. Applying χ to (7.2), we obtain

$$\chi(f(\mathcal{A}))r_1(\lambda)^2 = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\lambda, \bar{\mathbf{v}})r_1(\lambda)^2 (\mathcal{D}_n f)(\mathbf{v}) dV_n(\mathbf{v}) = f(\lambda)r_1(\lambda)^2.$$

Thus, $f(\lambda) = \chi(f(\mathcal{A}))$.

For a general $f \in \mathcal{B}^n$, one may consider the terms $f_{\Omega,0}$ in the elementary decomposition of f , using the same character χ , adapting the formulas above accordingly to show that $f_{\Omega,0}(\boldsymbol{\lambda}_\Omega) = \chi(f_{\Omega,0}(\mathcal{A}_\Omega))$, and then summing over $\Omega \in \mathcal{P}_n$ to deduce that $f(\boldsymbol{\lambda}) = \chi(f(\mathcal{A}))$.

Now, we cease to assume that the resolvents of A_j are bounded on the left half-plane. Instead, we can consider the operators $A_j + \varepsilon$, where $\varepsilon > 0$, and apply the result above. The same χ as above may be used for sufficiently small $\varepsilon > 0$ to show that $f(\boldsymbol{\lambda} + \boldsymbol{\varepsilon}) = \chi(f(\mathcal{A} + \boldsymbol{\varepsilon}))$, where $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon)$. Letting $\varepsilon \rightarrow 0$ and using Lemmas 3.6 and 4.6, it follows that $f(\boldsymbol{\lambda}) = \chi(f(\mathcal{A})) \in \sigma(f(\mathcal{A}))$. ■

There are other results on spectral mappings for the \mathcal{B}^n -calculus, as follows.

Proposition 7.2. *Let \mathcal{A} be an n -tuple of commuting operators on a Banach space X , satisfying the full (GSF) condition, and let $f \in \mathcal{B}^n$. The following results hold.*

- (1) *If $x \in X$ is an eigenvector of A_j with eigenvalue λ_j for each $j \in I_n$, then x is an eigenvector for $f(\mathcal{A})$ with eigenvalue $f(\boldsymbol{\lambda})$.*
- (2) *If $(x_k)_{k \in \mathbb{N}}$ are unit vectors in X forming an approximate eigenvector of A_j with approximate eigenvalue λ_j for each $j \in I_n$, then $f(\boldsymbol{\lambda})$ is an approximate eigenvalue of $f(\mathcal{A})$.*

Proof. The first statement can be seen very easily from the definition of $f(A)$ in (4.3).

The second statement follows from the first, using an F -product as in [4]. For this, let

$$\ell_T^\infty(X) = \{(x_k)_{k \in \mathbb{N}} : \sup_{k \in \mathbb{N}} \|x_k\| < \infty, \lim_{t \rightarrow 0} \sum_{j=1}^n \sup_{k \in \mathbb{N}} \|e^{-tA_j} x_k - x_k\| = 0\},$$

$$c_0(X) = \{(x_k)_{k \in \mathbb{N}} : \lim_{k \rightarrow \infty} \|x_k\| = 0\}.$$

Then $c_0(X)$ is a closed subspace of $\ell_T^\infty(X)$. Let $\mathcal{X}_T = \ell_T^\infty(X)/c_0(X)$, and $\pi: \ell_T^\infty(X) \rightarrow \mathcal{X}_T$ be the quotient map. For each $j \in I_n$, the C_0 -semigroup $(e^{-tA_j})_{t \geq 0}$ induces a C_0 -semigroup on \mathcal{X}_T . Let $-Z_j$ be the generator, and $\mathcal{Z} = (Z_1, \dots, Z_n)$. The operators $f(\mathcal{Z})$ on X induce operators on \mathcal{X}_T which form a \mathcal{B}^n -calculus for \mathcal{Z} .

There are standard methods to show that (a) an approximate eigenvector $(x_k)_{k \in \mathbb{N}}$ for \mathcal{A} is mapped by π to an approximate eigenvector for \mathcal{Z} with eigenvalues $\boldsymbol{\lambda}$, and (b) $\boldsymbol{\lambda}$ is an eigenvalue for \mathcal{Z} . By the first statement (1), $f(\boldsymbol{\lambda})$ is an eigenvalue of $f(\mathcal{Z})$. It then follows directly that $\boldsymbol{\lambda}$ is an approximate eigenvalue of $f(\mathcal{A})$. Details for (a) and (b) in the cases $n = 1$ and $n = 2$ may be found in [4, p. 20 and p. 78] and [23, Propositions 2.3.17 and 6.4.4]. ■

If all the operators A_j are sectorial of angle less than $\pi/2$, then \mathcal{A} satisfies the full (GSF) condition and the formula (4.5) can be adapted to

$$f(\mathcal{A}) = \int_{\mathbb{C}_+^n} \mathcal{K}_n(\mathcal{A}, \bar{\nu}) (\mathcal{D}_n f)(\nu) dV_n(\nu), \quad (7.3)$$

where the integral is convergent in operator norm.

Theorem 7.3. *Let \mathcal{A} be an n -tuple of commuting sectorial operators of angle less than $\pi/2$ on a Banach space X , and let $f \in \mathcal{B}^n$. Then*

$$\sigma(f(\mathcal{A})) \subseteq \bigcup_{\Omega \in \mathcal{P}_n} f_{\Omega}(\sigma(\mathcal{A})).$$

Proof. Let χ be a character of \mathfrak{A} , and $\Omega_{\chi} = \{j \in I_n : \chi \in \mathfrak{M}_j\}$. For $j \in \Omega_{\chi}$, let λ_j satisfy $\chi((z + A_j)^{-1}) = (z + \lambda_j)^{-1}$ for all $z \in \mathbb{C}_+$.

Let $\Omega \in \mathcal{P}_n$, and $k = |\Omega|$. Assume first that $\Omega \subseteq \Omega_{\chi}$. Then $\lambda_{\Omega} \in \sigma(\mathcal{A}_{\Omega})$, and $\chi(\mathcal{K}_k(\mathcal{A}_{\Omega}, \bar{\nu})) = \mathcal{K}_k(\lambda_{\Omega}, \bar{\nu})$. It follows from (7.3) that

$$\chi(f_{\Omega,0}(\mathcal{A})) = \int_{\mathbb{C}_+^k} \mathcal{K}_k(\lambda_{\Omega}, \bar{\nu}) (\mathcal{D}_k f_{\Omega,0})(\nu) dV_n(\nu) = f_{\Omega,0}(\lambda_{\Omega}).$$

Note that this conclusion is valid even if Ω is the empty set. Then $f_{\Omega,0}$ is a scalar.

If Ω is not contained in Ω_{χ} , then, for some $j \in \Omega$, $\chi(r_z(A_j)) = 0$ for all $z \in \mathbb{C}_+$, so $\chi(\mathcal{K}_k(\mathcal{A}, \bar{\nu})) = 0$ and $\chi(f_{\Omega,0}(\mathcal{A})) = 0 = f_{\Omega,0}(\lambda)$.

Summing over all $\Omega \in \mathcal{P}_n$, we obtain that $\chi(f(\mathcal{A})) = f_{\Omega_{\chi}}(\lambda)$. ■

Corollary 7.4. *Let \mathcal{A} be an n -tuple of commuting sectorial operators of angle less than $\pi/2$ on a Banach space X , and let $f \in \mathcal{B}_0^n$. Then*

$$\sigma(f(\mathcal{A})) = f(\sigma(\mathcal{A})).$$

Proof. Theorem 7.1 shows that $f(\sigma(\mathcal{A})) \subseteq \sigma(f(\mathcal{A}))$. For the reverse inclusion, $f_{\Omega,0} = 0$ unless $\Omega = \mathcal{P}_n$. Then the proof of Theorem 7.3 shows that $\chi(f(\mathcal{A})) = f(\lambda)$ for some $\lambda \in \sigma(\mathcal{A})$, as required. ■

8. Estimates for functions and operators

Let $\mathcal{A} = (A_1, \dots, A_n)$ be an n -tuple on commuting operators satisfying the full (GSF) condition. In this section, we will use the \mathcal{B}^n -calculus constructed in Theorem 5.4 to provide several estimates for $f(\mathcal{A})$ in terms of $\|f\|_{\infty}$ for f belonging to substantial

subclasses of $H^\infty(\mathbb{C}_+^n)$. The estimates resemble the situation of a bounded H^∞ -calculus and thus could be potentially useful.

We will need a Fourier characterisation of functions in $H^\infty(\mathbb{C}_+^n)$ in terms of their boundary values. If $f \in H^\infty(\mathbb{C}_+^n)$, then $f^b(\boldsymbol{\beta}) := \lim_{\boldsymbol{\alpha} \rightarrow 0_+} f(\boldsymbol{\alpha} + i\boldsymbol{\beta})$ exists for almost all $\boldsymbol{\beta} \in \mathbb{R}^n$ (see [26, Theorem F]). Then $f^b \in L^\infty(\mathbb{R}_+^n)$. For $f \in L^\infty(\mathbb{R}_+^n)$, define the spectrum of f as $\text{sp}(f) := \text{supp}(\mathcal{F}^{-1}f)$, where \mathcal{F} stands for the (distributional) Fourier transform of f . Let $H^\infty(\mathbb{R}^n) := \{f \in L^\infty(\mathbb{R}^n) : \text{sp}(f) \subseteq \mathbb{R}_+^n\}$. Then $H^\infty(\mathbb{C}_+^n)$ is isometrically isomorphic to $H^\infty(\mathbb{R}^n)$ via the mapping $f \rightarrow f^b$, and the inverse mapping is given by

$$f(\boldsymbol{\alpha} + i\boldsymbol{\beta}) = (P_\alpha * f^b)(\boldsymbol{\beta}) = \int_{\mathbb{R}^n} P(\boldsymbol{\alpha}, \boldsymbol{\beta} - \mathbf{t}) f^b(\mathbf{t}) dt, \quad (8.1)$$

where $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, $\boldsymbol{\beta} \in \mathbb{R}^n$, and

$$P_\alpha(\boldsymbol{\beta}) = P(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{j=1}^n \frac{1}{\pi} \frac{\alpha_j}{\alpha_j^2 + \beta_j^2}$$

stands for the Poisson kernel. This fact is well known for $n = 1$ (see [19, Section II.1.5]). For $n > 1$, it follows from [26, Theorem 2.7].

For $\tau > 0$, let $H^\infty([\tau, \infty)^n) := \{f \in H^\infty(\mathbb{C}_+^n) : \text{sp}(f^b) \subseteq [\tau, \infty)^n\}$, and

$$e_\tau(\mathbf{z}) = e^{-\tau(z_1 + \dots + z_n)}, \quad \mathbf{z} \in \mathbb{C}_+^n.$$

Then

$$H^\infty([\tau, \infty)^n) = e_\tau H^\infty(\mathbb{C}_+^n), \quad (8.2)$$

so that from (8.1), for $f \in H^\infty([\tau, \infty)^n)$,

$$|f(\mathbf{z})| \leq e^{-\tau(\sum_{i=1}^n \text{Re } z_i)} \|f^b\|_\infty, \quad \mathbf{z} \in \mathbb{C}_+^n. \quad (8.3)$$

Indeed, $(e_{-\tau}f)^b \in L^\infty(\mathbb{R}^n)$ and $\text{sp}(e_{-\tau}f) \subseteq \mathbb{R}_+^n$. If $g(\boldsymbol{\alpha} + i\boldsymbol{\beta}) = (P_\alpha * (e_{-\tau}f)^b)(\boldsymbol{\beta})$, $\boldsymbol{\alpha} + i\boldsymbol{\beta} \in \mathbb{C}_+^n$, then by the above $g \in H^\infty(\mathbb{C}_+^n)$ and $(e_\tau g)^b = f^b$. Since both f and $e_\tau g$ are in $H^\infty(\mathbb{C}_+^n)$ we infer from (8.1) that $e_\tau(\mathbf{z})g(\mathbf{z}) = f(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}_+^n$. Taking into account (8.1), we obtain (8.3) which is well known for $n = 1$. It follows from a standard Fourier characterisation of $H^\infty(\mathbb{C}_+)$ and it was used essentially in [7], but its multivariate counterpart seems not to have been noted in the literature.

We will use an elementary variant of Bernstein's inequality, as follows (see [21, Theorem 7.3.1] or [30, Chapter 3, p. 116]). If $f \in H^\infty([0, \sigma]^n)$ then f is an entire function of exponential type (σ, \dots, σ) , as defined in [30, p. 98] and for all $\mathbf{z} = \boldsymbol{\alpha} + i\boldsymbol{\beta} \in \mathbb{C}_+^n$ and $\Omega \in \mathcal{P}_n$,

$$\sup_{\boldsymbol{\beta} \in \mathbb{R}^n} |\mathcal{D}_\Omega f(\boldsymbol{\alpha} + i\boldsymbol{\beta})| \leq \sigma^{|\Omega|} \sup_{\boldsymbol{\beta} \in \mathbb{R}^n} |f(\boldsymbol{\alpha} + i\boldsymbol{\beta})|. \quad (8.4)$$

In the following lemmas, we show that functions of various classes belong to \mathcal{B}_0^n , and we give estimates for their norms. Then Proposition 2.6 shows that $f \in \mathcal{B}^n$ and $\|f\|_{\mathcal{B}^n} \leq 2^n \|f\|_{\mathcal{B}_0^n}$. An alternative estimate of the \mathcal{B}^n -norms can be obtained by an extension of the method used for the \mathcal{B}_0^n -norms.

8.1. Holomorphic extensions to the left

We will show here that if $f \in \mathcal{B}^n$ extends to a bounded analytic function in a larger half-space, then some damped versions of f are in \mathcal{B}_0^n with norm dominated by an H^∞ -norm of f . For $n = 1$, functions of this type were considered in [7, Section 5.2].

Let $\omega > 0$ and $H_\omega^\infty = H^\infty(\{\mathbf{z} \in \mathbb{C}^n : \operatorname{Re} z_j > -\omega, j \in I_n\})$. If $f \in H_\omega^\infty$ and $\Omega \in \mathcal{P}_n$, then Cauchy's inequality gives

$$|\mathcal{D}_\Omega f(\mathbf{z})| \leq \frac{\|f\|_{H_\omega^\infty}}{2^k |\omega + \boldsymbol{\alpha}_\Omega|}, \quad \mathbf{z} = \boldsymbol{\alpha} + i\boldsymbol{\beta} \in \mathbb{C}_+^n, \quad (8.5)$$

where $k = |\Omega|$ and

$$|\omega + \boldsymbol{\alpha}_\Omega| = \prod_{j \in \Omega} (\omega + \alpha_j).$$

Lemma 8.1. *Let $f \in H_\omega^\infty$, $\omega > 0$ and let $\nu > 0$, $\lambda \in \mathbb{C}_+$, $m = \min\{\omega, \operatorname{Re} \lambda\}$. Let*

$$R_\lambda^\nu(\mathbf{z}) := \prod_{j=1}^n (\lambda + z_j)^{-\nu}, \quad S_\lambda^\nu(\mathbf{z}) := (\lambda + z_1 + \cdots + z_n)^{-\nu}, \quad \mathbf{z} \in \mathbb{C}_+^n. \quad (8.6)$$

Then $R_\lambda^\nu f \in \mathcal{B}_0^n$ and $S_\lambda^\nu f \in \mathcal{B}_0^n$. Moreover,

$$\begin{aligned} \|R_\lambda^\nu f\|_{\mathcal{B}_0^n} &\leq \|f\|_{H_\omega^\infty} \left(\frac{1}{2\nu} + \frac{1}{m^\nu} \right)^n, \\ \|S_\lambda^\nu f\|_{\mathcal{B}_0^n} &\leq \frac{\|f\|_{H_\omega^\infty}}{m^\nu} \left(\frac{n}{2\nu} + 1 \right)^n. \end{aligned}$$

Proof. From (2.2) and (8.5), we have

$$\begin{aligned} \|R_\lambda^\nu f\|_{\mathcal{B}_0^n} &\leq \sum_{\Omega \in \mathcal{P}_n} \int_{\mathbb{R}_+^n} \sup_{\mathbf{z} \in W_{\boldsymbol{\alpha}} I_n} |(D_\Omega R_\lambda^\nu)(\mathbf{z})| |(D_{\Omega^c} f)(\mathbf{z})| d\boldsymbol{\alpha} \\ &\leq \|f\|_{H_\omega^\infty} \sum_{\Omega \in \mathcal{P}_n} \frac{\nu^{|\Omega|}}{2^{n-|\Omega|}} \int_{\mathbb{R}_+^{|\Omega|}} \frac{d\boldsymbol{\alpha}}{|m + \boldsymbol{\alpha}_\Omega|^{1+\nu}} \\ &= \|f\|_{H_\omega^\infty} \sum_{j=0}^n \frac{\binom{n}{j}}{(2\nu)^{n-j}} \frac{1}{m^{\nu j}} = \|f\|_{H_\omega^\infty} \left(\frac{1}{2\nu} + \frac{1}{m^\nu} \right)^n. \end{aligned}$$

By a similar argument, we have (with $d\mathbf{t}_{n-j} = dt_1 \dots dt_{n-j}$),

$$\begin{aligned} & \|S_\lambda^\nu f\|_{\mathcal{B}_0^n} \\ & \leq \|f\|_{H_\omega^\infty} \sum_{j=0}^n \frac{\binom{n}{j}}{2^{n-j}} \prod_{k=0}^{j-1} (\nu + k) \int_{\mathbb{R}_+^n} \left(m + \sum_{k=1}^n t_k\right)^{-(\nu+j)} \prod_{k=1}^{n-j} (m + t_k)^{-1} d\mathbf{t} \\ & = \|f\|_{H_\omega^\infty} \sum_{j=0}^n \frac{\binom{n}{j}}{2^{n-j}} \int_{\mathbb{R}_+^{n-j}} \left(m + \sum_{k=1}^{n-j} t_k\right)^{-\nu} \prod_{k=1}^{n-j} (m + t_k)^{-1} d\mathbf{t}_{n-j}. \end{aligned}$$

By replacing t_k by mt_k , taking the inequality $\prod_{k=1}^{n-j} (1 + t_k) \leq (1 + \sum_{k=1}^{n-j} t_k)^{n-j}$ and raising it to the power $-\nu/(n-j)$, we arrive at

$$\begin{aligned} \|S_\lambda^\nu f\|_{\mathcal{B}_0^n} & \leq \frac{\|f\|_{H_\omega^\infty}}{m^\nu} \sum_{j=0}^n \frac{\binom{n}{j}}{2^{n-j}} \int_{\mathbb{R}_+^{n-j}} \prod_{k=1}^{n-j} (1 + t_k)^{-(1+\nu/(n-j))} d\mathbf{t}_{n-j} \\ & = \frac{\|f\|_{H_\omega^\infty}}{m^\nu} \sum_{j=0}^n \binom{n}{j} \left(\frac{n-j}{2\nu}\right)^{n-j} \\ & \leq \frac{\|f\|_{H_\omega^\infty}}{m^\nu} \left(\frac{n}{2\nu} + 1\right)^n. \quad \blacksquare \end{aligned}$$

Now, we show that the damping in Lemma 8.1 is not necessary if $\text{sp}(f)$ is separated from zero.

Lemma 8.2. *Let $f \in H^\infty([\tau, \infty)^n) \cap H_\omega^\infty$, $\tau > 0$, $\omega > 0$. Then $f \in \mathcal{B}_0^n$, and*

$$\|f\|_{\mathcal{B}_0^n} \leq \|f\|_{H_\omega^\infty} e^{-n\omega\tau} \left(1 + \frac{1}{2} \log\left(1 + \frac{1}{\tau\omega}\right)\right)^n.$$

Proof. By (8.3), there exists $g \in H_\omega^\infty$ such that $f(\mathbf{z}) = e^{-\tau(z_1 + \dots + z_n)} g(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}_+^n$, and thus $\|g\|_{H_\omega^\infty} = e^{-n\omega\tau} \|f\|_{H_\omega^\infty}$. From this and (8.5), we infer that

$$\begin{aligned} |\mathcal{D}_n f(\mathbf{z})| & \leq \sum_{\Omega \in \mathcal{P}_n} |(D_\Omega e^{-\tau(z_1 + \dots + z_n)})(D_{\Omega^c} g(\mathbf{z}))| \\ & \leq e^{-\tau(\alpha_1 + \dots + \alpha_n)} \sum_{\Omega \in \mathcal{P}_n} \left(\frac{\tau^{|\Omega|}}{2^{n-|\Omega|} |\omega + \alpha_{\Omega^c}|}\right) \|g\|_{H_\omega^\infty}, \end{aligned}$$

where $z_j = \alpha_j + i\beta_j$. Since the variables in the integrals below over \mathbb{R}_+^n split as the product of j integrals of one function and $n-j$ integrals of another function, we see

that

$$\begin{aligned} \|f\|_{\mathcal{B}_0^n} &\leq \|g\|_{H_\omega^\infty} \sum_{\Omega \in \mathcal{P}_n} \frac{\tau^{|\Omega|}}{2^{n-|\Omega|}} \int_{\mathbb{R}_+^n} e^{-\tau \sum_{j=1}^n \alpha_j} \left(\frac{1}{|\omega + \alpha_{\Omega^c}|} \right) d\alpha \\ &= \|g\|_{H_\omega^\infty} \sum_{j=0}^n \binom{n}{j} \frac{1}{2^{n-j}} \left(\int_0^\infty \tau e^{-\tau\alpha} d\alpha \right)^j \left(\int_0^\infty e^{-\tau\alpha} (\omega + \alpha)^{-1} d\alpha \right)^{n-j}. \end{aligned}$$

The first integral in the line above is equal to 1, and the second integral is shown in the proof of [7, Lemma 3.2 (2)] to be less than $\log(1 + (\tau\omega)^{-1})$, so

$$\begin{aligned} \|f\|_{\mathcal{B}_0^n} &\leq \|g\|_{H_\omega^\infty} \sum_{j=0}^n \binom{n}{j} \left(\frac{1}{2} \log\left(1 + \frac{1}{\tau\omega}\right) \right)^{n-j} \\ &= \|g\|_{H_\omega^\infty} \left(1 + \frac{1}{2} \log\left(1 + \frac{1}{\tau\omega}\right)\right)^n. \quad \blacksquare \end{aligned}$$

8.2. Functions with restricted spectrum

Instead of assuming that $f \in H_\omega^\infty$, we now assume that $\text{sp}(f)$ is separated from zero and infinity and we obtain an estimate of $\|f\|_{\mathcal{B}^n}$ in terms of $\|f\|_\infty$. We need the following technical lemma. For $\mathbf{t} = (t_1, \dots, t_k) \in (0, \infty)^k$, let $L_k(\mathbf{t}) = \sum_{j=1}^k t_j$ and $P_k(\mathbf{t}) = \prod_{j=1}^k t_j$.

Lemma 8.3. *For $k \in \mathbb{N}$ and $a \in (0, 1)$, let*

$$J_k(a) := \int_{\mathbb{R}_+^k} \frac{d\mathbf{t}}{P_k(\mathbf{t}) + ae^{L_k(\mathbf{t})}}.$$

Then

$$J_k(a) \leq 2^{2k} \log^k \left(1 + \frac{1}{a}\right). \quad (8.7)$$

Proof. We argue by induction. If $k = 1$, then (8.7) has been proved in [7, p. 37, (2.12)]. Next, assume that (8.7) holds for some $k \geq 1$. Note that J_k is a decreasing function, and by Fubini's theorem,

$$\begin{aligned} J_{k+1}(a) &= \int_0^\infty \int_{\mathbb{R}_+^k} \frac{d\mathbf{t} d\tau}{P_k(\mathbf{t})\tau + ae^{L_k(\mathbf{t})}e^\tau} \\ &= \int_0^\infty J_k(ae^\tau/\tau) \frac{d\tau}{\tau}. \end{aligned}$$

Using the monotonicity of J_k , a change of variable $u = e^{-\tau}$, and the inequalities $tJ_k(t) \leq 1$, and $(4 \log 2)^{k+1} \geq 2(k+1)$ if $k \geq 1$, we infer that

$$\begin{aligned}
 J_{k+1}(a) &\leq \int_0^1 J_k(a/\tau) \frac{d\tau}{\tau} + \int_1^\infty J_k(ae^\tau) d\tau \\
 &= \int_0^1 J_k(a/\tau) \frac{d\tau}{\tau} + \int_0^{1/e} J_k(a/u) \frac{du}{u} \\
 &\leq 2 \int_0^1 J_k(1/s) \frac{ds}{s} + 2 \int_1^{1/a} J_k(1/s) \frac{ds}{s} \\
 &\leq 2 + 2^{2k+1} \int_1^{1/a} \log^k(1+s) \frac{ds}{s} \\
 &\leq 2 + 2^{2k+2} \int_1^{1/a} \frac{\log^k(1+s)}{1+s} ds \\
 &= 2 + \frac{2^{2k+2}}{k+1} (\log^{k+1}(1+1/a) - \log^{k+1} 2) \\
 &\leq 2^{2(k+1)} \log^{k+1}(1+1/a),
 \end{aligned}$$

so (8.7) is true for $k+1$. The assertion follows. \blacksquare

The following lemma was proved in [7, Lemma 2.5] for $n=1$ and in [23, Lemma 4.1.1] for $n=2$. The functions considered in those cases played crucial roles in those papers.

Lemma 8.4. *Let $f \in H^\infty([\epsilon, \sigma]^n)$, where $0 < \epsilon < \sigma$. Then $f \in \mathcal{B}_0^n$ and*

$$\|f\|_{\mathcal{B}_0^n} \leq 2^{n+1} \|f\|_\infty \left(\log \left(1 + \left(\frac{2\sigma}{\epsilon} \right)^n \right) \right)^n. \quad (8.8)$$

Proof. From (8.4), we have

$$|\mathcal{D}_n f(\alpha + i\beta)| \leq \sigma^n \sup_{\beta \in \mathbb{R}^n} |f(\alpha + i\beta)| \leq \sigma^n e^{-\epsilon L_n(\alpha)} \|f\|_\infty. \quad (8.9)$$

From Cauchy's inequality,

$$|\mathcal{D}_n f(\alpha + i\beta)| \leq \frac{\|f\|_\infty}{2^n P_n(\alpha)}.$$

Thus,

$$\begin{aligned} |\mathcal{D}_n f(\boldsymbol{\alpha} + i\boldsymbol{\beta})| &\leq \|f\|_\infty \min\{\sigma^n e^{-\epsilon S_n(\boldsymbol{\alpha})}, 2^{-n} P_n(\boldsymbol{\alpha})^{-1}\} \\ &\leq \frac{2\|f\|_\infty}{2^n P_n(\boldsymbol{\alpha}) + \sigma^{-n} e^{\epsilon L_n(\boldsymbol{\alpha})}}. \end{aligned}$$

From Lemma, 8.3,

$$\int_{\mathbb{R}_+^n} \frac{d\boldsymbol{\alpha}}{P_n(\boldsymbol{\alpha}) + \sigma^{-n} e^{\epsilon L_n(\boldsymbol{\alpha})}} = J_n\left(\left(\frac{\epsilon}{2\sigma}\right)^n\right) \leq 2^n \left(\log\left(1 + \left(\frac{2\sigma}{\epsilon}\right)^n\right)\right)^n.$$

Now, (8.8) follows. ■

Lemmas 8.1, 8.2, and 8.4 imply the following operator-norm estimates for \mathcal{B}^n -functions of n -tuples of operators satisfying the full (GSF) condition. They were obtained in [7] for $n = 1$, providing direct alternative proofs for the results in [17, 18, 31] and their improvements. The estimates quantify a deviation of the \mathcal{B}^n -calculus from the bounded H^∞ -calculus, and they simplify operator norm-estimates in several situations of interest. In statements (i) and (ii), $f(\mathcal{A})$ is not necessarily a bounded operator, but it can be defined as a closed operator in the extended half-plane calculus for several variables. The proofs are consequences of Lemmas 8.1, 8.2, and 8.4. For (ii), one uses (8.2).

Theorem 8.5. *Let \mathcal{A} be an n -tuple of operators on a Banach space X , satisfying the full (GSF) condition, and let $\gamma_{\mathcal{A},0}$ be the norm of $\Phi_{\mathcal{A}}$ on \mathcal{B}_0^n . Then the following hold.*

- (i) *Let $f \in H_\omega^\infty$, $\omega > 0$, and let $\nu > 0$, $\lambda \in \mathbb{C}_+$, and $m = \min\{\omega, \operatorname{Re} \lambda\}$. Let R_λ^ν and S_λ^ν be as in (8.6). Then*

$$\|(R_\lambda^\nu f)(\mathcal{A})\| \leq \gamma_{\mathcal{A},0} \left(\frac{1}{2\nu} + \frac{1}{m}\right)^n \|f\|_{H_\omega^\infty}.$$

and

$$\|(S_\lambda^\nu f)(\mathcal{A})\| \leq \gamma_{\mathcal{A},0} m^{-\nu} \left(\frac{n}{2\nu} + 1\right)^n \|f\|_{H_\omega^\infty}.$$

- (ii) *Let $g \in H_\omega^\infty$, $\omega > 0$ and $\tau > 0$. Then*

$$\|(ge_{-\tau})(\mathcal{A})\| \leq \gamma_{\mathcal{A},0} e^{-n\omega\tau} \left(1 + \frac{1}{2} \log\left(1 + \frac{1}{\tau\omega}\right)\right)^n \|g\|_{H_\omega^\infty}.$$

- (iii) *Let $f \in H^\infty([\epsilon, \sigma]^n)$, $0 < \epsilon < \sigma$. Then*

$$\|f(\mathcal{A})\| \leq 2^{n+1} \gamma_{\mathcal{A},0} \left(\log\left(1 + \left(\frac{2\sigma}{\epsilon}\right)^n\right)\right)^n \|f\|_\infty.$$

9. Appendix

For the skeptical reader, we will justify the application of Fubini's theorem in the proof of Lemma 5.1 by showing that

$$J_{\mathbf{t}} := \int_{\mathbb{C}_+^m} \int_{\mathbb{C}_+^k} |(\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda})| |(\mathcal{D}_{\Omega_g} g)(\boldsymbol{\nu})| S_{\mathbf{t}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) dV_k(\boldsymbol{\lambda}) dV_m(\boldsymbol{\nu}) < \infty,$$

where f and g are elementary functions in \mathcal{B}^n with supports Ω_f and Ω_g with cardinalities k and m respectively, and $\Omega_f \cup \Omega_g = I_n$. Here $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are indexed by Ω_f and Ω_g , respectively, and

$$S_{\mathbf{t}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \int_{\mathbb{C}_+^n} | \langle \mathcal{K}_n(\mathcal{A}, \bar{\mathbf{z}}) x, x^* \rangle | | \mathcal{D}_n(\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})) | dV_n(\mathbf{z}),$$

where \mathcal{D}_n denotes differentiation with respect to all the z_j variables (once each).

We may assume that $\|x\| = \|x^*\| = 1$ and put $\beta_j = \operatorname{Re} \lambda_j$, $\eta_l = \operatorname{Re} \nu_l$. Using the (GSF) condition for \mathcal{A} , we can estimate

$$\begin{aligned} S_{\mathbf{t}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) &\leq \gamma_{\mathcal{A}} \int_{\mathbb{R}_+^n} \sup_{\operatorname{Re} z_j = \alpha_j} | \mathcal{D}_n(\mathcal{K}_k(\mathbf{z}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \bar{\boldsymbol{\lambda}}) \mathcal{K}_m(\mathbf{z}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \bar{\boldsymbol{\nu}})) | d\boldsymbol{\alpha} \\ &\leq \gamma_{\mathcal{A}} \left| \int_{\mathbb{R}_+^n} \mathcal{D}_n(\mathcal{K}_k(\boldsymbol{\alpha}_{\Omega_f} + \mathbf{t}_{\Omega_f}, \boldsymbol{\beta}_{\Omega_f}) \mathcal{K}_m(\boldsymbol{\alpha}_{\Omega_g} + \mathbf{t}_{\Omega_g}, \boldsymbol{\eta}_{\Omega_g})) d\boldsymbol{\alpha} \right| \\ &= \gamma_{\mathcal{A}} | \mathcal{K}_k(\mathbf{t}_{\Omega_f}, \boldsymbol{\beta}_{\Omega_f}) \mathcal{K}_m(\mathbf{t}_{\Omega_g}, \boldsymbol{\eta}_{\Omega_g}) | \leq \gamma_{\mathcal{A}} \left(\frac{2}{\pi} \right)^{k+m} \prod_{j \in \Omega_f} \frac{1}{2t\beta_j} \prod_{l \in \Omega_g} \frac{1}{2t\eta_l}. \end{aligned}$$

The inequality in the second line is obtained by straightforward estimates on the integrand on the function. Moving the absolute value signs outside the integral is justified, because the integrand is real-valued, and its sign depends only on k and m . The equality in the third line comes from the fundamental theorem of calculus for each of the variables. The final inequality is straightforward. Now,

$$\begin{aligned} J_{\mathbf{t}} &\leq \frac{\gamma_{\mathcal{A}}}{(\pi t)^{k+m}} \int_{\mathbb{R}_+^m} \int_{\mathbb{R}_+^k} \sup_{\substack{\operatorname{Re} \lambda_j = \beta_j \\ \operatorname{Re} \nu_l = \eta_l}} |(\mathcal{D}_{\Omega_f} f)(\boldsymbol{\lambda})| |(\mathcal{D}_{\Omega_g} g)(\boldsymbol{\nu})| d\boldsymbol{\beta} d\boldsymbol{\eta} \\ &= \frac{\gamma_{\mathcal{A}}}{(\pi t)^{k+m}} \|f\|_{\mathcal{B}_{\Omega_f, 0}} \|g\|_{\mathcal{B}_{\Omega_g, 0}}. \end{aligned}$$

Funding. This work was partially supported financially by an NCN grant UMO-2023/49/B/ST1/01961.

References

- [1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0 -groups, commutator methods and spectral theory of N -body Hamiltonians*. Progr. Math. 135, Birkhäuser, Basel, 1996 Zbl 0962.47500 MR 1388037
- [2] M. Andersson, H. Samuelsson, and S. Sandberg, *Operators with smooth functional calculi*. *J. Anal. Math.* **98** (2006), 221–247 Zbl 1152.47010 MR 2254486
- [3] M. Andersson and J. Sjöstrand, *Functional calculus for non-commuting operators with real spectra via an iterated Cauchy formula*. *J. Funct. Anal.* **210** (2004), no. 2, 341–375 Zbl 1070.47009 MR 2053491
- [4] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck, *One-parameter semigroups of positive operators*. Lecture Notes in Math. 1184, Springer, Berlin, 1986 Zbl 0585.47030 MR 0839450
- [5] O. Arrigoni and C. Le Merdy, *H^∞ -functional calculus for commuting families of Ritt operators and sectorial operators*. *Oper. Matrices* **13** (2019), no. 4, 1055–1090 Zbl 1447.47020 MR 4043874
- [6] O. Arrigoni and C. Le Merdy, *New properties of the multivariable H^∞ functional calculus of sectorial operators*. *Integral Equations Operator Theory* **93** (2021), no. 4, article no. 39 Zbl 1496.47027 MR 4277317
- [7] C. Batty, A. Gomilko, and Y. Tomilov, *A Besov algebra calculus for generators of operator semigroups and related norm-estimates*. *Math. Ann.* **379** (2021), no. 1-2, 23–93 Zbl 1492.47024 MR 4211082
- [8] C. Batty, A. Gomilko, and Y. Tomilov, *The theory of Besov functional calculus: developments and applications to semigroups*. *J. Funct. Anal.* **281** (2021), no. 6, article no. 109089 Zbl 1492.47025 MR 4257980
- [9] C. Batty, A. Gomilko, and Y. Tomilov, *Bounded functional calculi for unbounded operators*. In *Operators, semigroups, algebras and function theory*, pp. 27–60, Oper. Theory Adv. Appl. 292, Birkhäuser/Springer, Cham, 2023 MR 4696662
- [10] C. Batty, A. Gomilko, and Y. Tomilov, *Functional calculi for sectorial operators and related function theory*. *J. Inst. Math. Jussieu* **22** (2023), no. 3, 1383–1463 Zbl 1517.47025 MR 4574175
- [11] C. Batty and D. Seifert, *A continuous-parameter Katznelson–Tzafriri theorem for algebras of analytic functions*. *Studia Math.* **270** (2023), no. 2, 229–239 Zbl 1529.47025 MR 4570193
- [12] S. R. Caradus, *A note on a paper of J. T. Marti*. *Comment. Math. Helv.* **44** (1969), 282–283 Zbl 0177.41104 MR 0243372
- [13] P. A. Cojuhari and A. M. Gomilko, *On the characterization of scalar type spectral operators*. *Studia Math.* **184** (2008), no. 2, 121–132 Zbl 1163.47029 MR 2365805
- [14] E. B. Davies, *One-parameter semigroups*. London Mathematical Society Monographs 15, Academic Press, Inc. Harcourt Brace Jovanovich, Publishers, London-New York, 1980 Zbl 0457.47030 MR 0591851
- [15] E. B. Davies, *The functional calculus*. *J. London Math. Soc. (2)* **52** (1995), no. 1, 166–176 Zbl 0858.47012 MR 1345723

- [16] I. Doust and T. A. Gillespie, [Well-boundedness of sums and products of operators](#). *J. London Math. Soc. (2)* **68** (2003), no. 1, 183–192 Zbl [1077.47034](#) MR [1980251](#)
- [17] M. Haase, [Transference principles for semigroups and a theorem of Peller](#). *J. Funct. Anal.* **261** (2011), no. 10, 2959–2998 Zbl [1235.47038](#) MR [2832588](#)
- [18] M. Haase and J. Rozendaal, [Functional calculus for semigroup generators via transference](#). *J. Funct. Anal.* **265** (2013), no. 12, 3345–3368 Zbl [1440.47012](#) MR [3110505](#)
- [19] V. Havin and B. Jöricke, [The uncertainty principle in harmonic analysis](#). *Ergeb. Math. Grenzgeb. (3)* 28, Springer, Berlin, 1994 Zbl [0827.42001](#) MR [1303780](#)
- [20] E. Hille and R. S. Phillips, [Functional analysis and semi-groups](#). Amer. Math. Soc. Colloq. Publ. 31, American Mathematical Society, Providence, RI, 1957 Zbl [0078.10004](#) MR [0089373](#)
- [21] L. Hörmander, [The analysis of linear partial differential operators. I](#). Second edn., Grundlehren Math. Wiss. 256, Springer, Berlin, 1990 Zbl [0712.35001](#) MR [1065993](#)
- [22] T. Ichinose, [Spectral properties of tensor products of linear operators. II. The approximate point spectrum and Kato essential spectrum](#). *Trans. Amer. Math. Soc.* **237** (1978), 223–254 Zbl [0364.47018](#) MR [0479372](#)
- [23] D. Kobos, [Besov functional calculus for two commuting operators](#). Ph.D. thesis, University of Oxford, Oxford, 2020 <https://ora.ox.ac.uk/objects/uuid:3a1c21b5-74ed-4ada-96d2-6297cbd1b46e> visited on 14 April 2024
- [24] S. Kosowicz, [Remarks on a characterization of generators of bounded \$C_0\$ -semigroups](#). In *Semigroups of operators—theory and applications*, pp. 69–78, Springer Proc. Math. Stat. 325, Springer, Cham, 2020 Zbl [1501.47068](#) MR [4115838](#)
- [25] F. Lancien, G. Lancien, and C. Le Merdy, [A joint functional calculus for sectorial operators with commuting resolvents](#). *Proc. London Math. Soc. (3)* **77** (1998), no. 2, 387–414 Zbl [0904.47015](#) MR [1635157](#)
- [26] H.-C. Li, G.-T. Deng, and T. Qian, [Fourier spectrum characterizations of \$H^p\$ spaces on tubes over cones for \$1 \leq p \leq \infty\$](#) . *Complex Anal. Oper. Theory* **12** (2018), no. 5, 1193–1218 Zbl [06909439](#) MR [3800967](#)
- [27] J. T. Marti, [Operational calculus for two commuting closed operators](#). *Comment. Math. Helv.* **43** (1968), 87–97 Zbl [0155.46103](#) MR [0223920](#)
- [28] A. R. Mirotin, [Multivariable \$\mathcal{T}\$ -calculus in generators of \$C_0\$ -semigroups](#). *Algebra i Analiz* **11** (1999), no. 2, 142–169; English transl. in *St. Petersburg Math. J.* **11** (2000), no. 2, 315–335 Zbl [0954.47032](#) MR [1702567](#)
- [29] A. R. Mirotin, [On joint spectra of families of unbounded operators](#). *Ross. Akad. Nauk Ser. Mat.* **79** (2015), no. 6, 145–170; English transl. in *Izv. Math.* **79** (2015), no. 6, 1235–1259; Zbl [1380.47033](#) MR [3438468](#)
- [30] S. M. Nikol’skiĭ, [Approximation of functions of several variables and imbedding theorems](#). Grundlehren Math. Wiss. 205, Springer, Berlin etc., 1975 Zbl [0307.46024](#) MR [0374877](#)
- [31] P. Vitse, [A Besov class functional calculus for bounded holomorphic semigroups](#). *J. Funct. Anal.* **228** (2005), no. 2, 245–269 Zbl [1101.47026](#) MR [2175408](#)

Charles Batty

St John's College, University of Oxford, St Giles, Oxford OX1 3JP, UK;
charles.batty@sjc.ox.ac.uk

Alexander Gomilko

Faculty of Mathematics and Computer Science, Nicolas Copernicus University,
Chopin Street 12/18, 87-100 Toruń, Poland; alex@gomilko.com

Dominik Kobos

School of Philosophy, University of Southern California, Mudd Hall of Philosophy (MHP),
Los Angeles, CA 90089-0451, USA; dkobos@usc.edu

Yuri Tomilov

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw;
Faculty of Mathematics and Computer Science, Nicolas Copernicus University,
Chopin Street 12/18, 87-100 Toruń, Poland; ytomilov@impan.pl