

# An extension of the Liouville theorem for Fourier multipliers to sub-exponentially growing solutions

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**Abstract.** We study the equation  $m(D)f = 0$  in a large class of sub-exponentially growing functions. Under appropriate restrictions on  $m \in C(\mathbb{R}^n)$ , we show that every such solution can be analytically continued to a sub-exponentially growing entire function on  $\mathbb{C}^n$  if, and only if,  $m(\xi) \neq 0$  for  $\xi \neq 0$ .

## 1. Introduction

The classical Liouville theorem for the Laplace operator  $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$  on  $\mathbb{R}^n$  says that every bounded (polynomially bounded) solution of the equation  $\Delta f = 0$  is in fact constant (is a polynomial). Recently, similar results have been obtained for solutions of more general equations of the form  $m(D)f = 0$ , where  $m(D) := \mathcal{F}^{-1}m(\xi)\mathcal{F}$ , and

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx \quad \text{and} \quad \mathcal{F}^{-1}u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} u(\xi) d\xi$$

are the Fourier and the inverse Fourier transforms, see [1–3, 12], and the references therein. Namely, it was shown that, under appropriate restrictions on  $m \in C(\mathbb{R}^n)$ , the implication

$$\begin{aligned} f \text{ is bounded (polynomially bounded) and } m(D)f = 0 \\ \implies f \text{ is constant (is a polynomial)} \end{aligned}$$

holds if, and only if,  $m(\xi) \neq 0$  for  $\xi \neq 0$ . Much of this research has been motivated by applications to infinitesimal generators of Lévy processes.

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In this paper, we study solutions of  $m(D)f = 0$  that can grow faster than any polynomial. Of course, one cannot expect such solutions to have a simple structure, not even in the case of  $\Delta f = 0$  in  $\mathbb{R}^2$ , see, e.g., [22, Chapter I, Section 2]. We consider sub-exponentially growing solutions whose growth is controlled by a submultiplicative function, cf. (2.1), satisfying the Beurling–Domar condition (2.3), and we show that, under appropriate restrictions on  $m \in C(\mathbb{R}^n)$ , every such solution admits analytic continuation to a sub-exponentially growing entire function on  $\mathbb{C}^n$  if, and only if,  $m(\xi) \neq 0$  for  $\xi \neq 0$ , see Corollary 4.4. Results of this type have been obtained for solutions of partial differential equations with constant coefficients by A. Kaneko and G. E. Šilov, see [17, 18, 26], [7, Chapter 10, Section 2, Theorem 2], and Section 5 below.

Keeping in mind applications to infinitesimal generators of Lévy processes, we do not assume that  $m$  is the Fourier transform of a distribution with compact support, so our setting is different from that in, e.g., [6] and [16, Chapter XVI].

The paper is organised as follows. In Chapter 2, we consider submultiplicative functions satisfying the Beurling–Domar condition. For every such function  $g$ , we introduce an auxiliary function  $S_g$ , see (2.11), (2.12), which appears in our main estimates. Chapter 3 contains weighted  $L^p$  estimates for entire functions on  $\mathbb{C}^n$ , which are a key ingredient in the proof of our main results in Chapter 4. Another key ingredient is the Tauberian Theorem 4.1, which is similar to [3, Theorem 7] and [24, Theorem 9.3]. The main difference is that the function  $f$  in Theorem 4.1 is not assumed to be polynomially bounded, and hence it might not be a tempered distribution. So, we avoid using the Fourier transform  $\hat{f} = \mathcal{F}f$  and its support (and non-quasianalytic-type ultradistributions). Although we are mainly interested in the case  $m(\xi) \neq 0$  for  $\xi \neq 0$ , we also prove a Liouville type result for  $m$  with compact zero set  $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\}$ , see Theorem 4.3. Finally, we discuss in Section 5 A. Kaneko’s Liouville-type results for partial differential equations with constant coefficients, cf. [17, 18], which show that the Beurling–Domar condition is in a sense optimal in our setting.

## 2. Submultiplicative functions and the Beurling–Domar condition

Let  $g: \mathbb{R}^n \rightarrow (0, \infty)$  be a locally bounded, measurable *submultiplicative* function, i.e., a locally bounded measurable function satisfying

$$g(x + y) \leq C g(x)g(y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

where the constant  $C \in [1, \infty)$  does not depend on  $x$  and  $y$ . Without loss of generality, we will always assume that  $g \geq 1$ , as otherwise we can replace  $g$  with  $g + 1$ . Also,

replacing  $g$  with  $Cg$ , we can assume that

$$g(x + y) \leq g(x)g(y) \quad \text{for all } x, y \in \mathbb{R}^n. \quad (2.1)$$

A locally bounded submultiplicative function is exponentially bounded, i.e.,

$$|g(x)| \leq Ce^{a|x|} \quad (2.2)$$

for suitable constants  $C, a > 0$ , see [25, Section 25] or [14, Chapter VII].

We will say that  $g$  satisfies the *Beurling–Domar* condition if

$$\sum_{l=1}^{\infty} \frac{\log g(lx)}{l^2} < \infty \quad \text{for all } x \in \mathbb{R}^n. \quad (2.3)$$

If  $g$  satisfies the Beurling–Domar condition, then it also satisfies the Gelfand–Raikov–Shilov condition

$$\lim_{l \rightarrow \infty} g(lx)^{1/l} = 1 \quad \text{for all } x \in \mathbb{R}^n,$$

while  $g(x) = e^{|x|/\log(e+|x|)}$  satisfies the latter but not the former condition, see [10]. It is also easy to see that  $g(x) = e^{|x|/\log^\gamma(e+|x|)}$  satisfies the Beurling–Domar condition if, and only if,  $\gamma > 1$ . The function

$$g(x) = e^{a|x|^b} (1 + |x|)^s (\log(e + |x|))^t$$

satisfies the Beurling–Domar condition for any  $a, s, t \geq 0$  and  $b \in [0, 1)$ , see [10].

**Lemma 2.1.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Then, for every  $\varepsilon > 0$ , there exists  $R_\varepsilon \in (0, \infty)$  such that*

$$\int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau < \varepsilon \quad \text{for all } x \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : |y| = 1\}.$$

*Proof.* Since  $g \geq 1$  is locally bounded,

$$0 \leq M := \sup_{|y| \leq 1} \log g(y) < \infty. \quad (2.4)$$

Take any  $x \in \mathbb{S}^{n-1}$ . It follows from (2.1) that

$$\log g((l+1)x) - M \leq \log g(\tau x) \leq \log g(lx) + M \quad \text{for all } \tau \in [l, l+1].$$

Hence,

$$\sum_{l=L}^{\infty} \frac{\log g((l+1)x) - M}{(l+1)^2} \leq \sum_{l=L}^{\infty} \int_l^{l+1} \frac{\log g(\tau x)}{\tau^2} d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx) + M}{l^2},$$

and this implies for all  $L \in \mathbb{N}$  that

$$\sum_{l=L+1}^{\infty} \frac{\log g(lx)}{l^2} - \frac{M}{L} \leq \int_L^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx)}{l^2} + \frac{M}{L-1}. \quad (2.5)$$

Let

$$\begin{aligned} \mathbf{e}_j &:= (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0), \quad j = 1, \dots, n, \quad \mathbf{e}_0 := \frac{1}{\sqrt{n}}(1, \dots, 1), \\ Q &:= \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}^n : \frac{1}{2\sqrt{n}} < y_j < \frac{2}{\sqrt{n}}, j = 1, \dots, n \right\}. \end{aligned} \quad (2.6)$$

For every  $x \in \mathbb{S}^{n-1}$ , there exists an orthogonal matrix  $A_x \in O(n)$  such that  $x = A_x \mathbf{e}_0$ . Hence,  $\{A_x Q\}_{A_x \in O(n)}$  is an open cover of  $\mathbb{S}^{n-1}$ . Let  $\{A_k Q\}_{k=1, \dots, K}$  be a finite subcover. Take an arbitrary  $\varepsilon > 0$ . It follows from (2.3) and (2.5) that there exists some  $R_\varepsilon > 0$  for which

$$\int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(\tau A_k \mathbf{e}_j)}{\tau^2} d\tau < \frac{\varepsilon}{2\sqrt{n}}, \quad k = 1, \dots, K, \quad j = 1, \dots, n.$$

For any  $x \in \mathbb{S}^{n-1}$ , there exist  $k = 1, \dots, K$  and  $a_j \in (\frac{1}{2\sqrt{n}}, \frac{2}{\sqrt{n}})$ ,  $j = 1, \dots, n$  such that

$$x = \sum_{j=1}^n a_j A_k \mathbf{e}_j.$$

Using (2.1), one gets

$$\begin{aligned} \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau &\leq \sum_{j=1}^n \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau a_j A_k \mathbf{e}_j)}{\tau^2} d\tau = \sum_{j=1}^n a_j \int_{a_j R_\varepsilon}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2} dr \\ &\leq \sum_{j=1}^n \frac{2}{\sqrt{n}} \int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2} dr < \sum_{j=1}^n \frac{2}{\sqrt{n}} \cdot \frac{\varepsilon}{2\sqrt{n}} = n \frac{\varepsilon}{n} = \varepsilon. \quad \blacksquare \end{aligned}$$

Let

$$\begin{aligned}
 I_{g,x}(r) &:= \int_{\max\{r,1\}}^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau < \infty, \\
 J_{g,x}(r) &:= \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x) d\tau < \infty, \\
 S_{g,x}(r) &:= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2} d\tau \quad r \geq 0, x \in \mathbb{S}^{n-1}.
 \end{aligned}$$

One has, for  $r > 1$  and any  $\beta \in (0, 1)$ ,

$$\begin{aligned}
 J_{g,x}(r) &= \frac{1}{r^2} \int_0^r \log g(\tau x) d\tau \\
 &= \frac{1}{r^2} \int_0^1 \log g(\tau x) d\tau + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{r^{2\beta}} d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{r^2} d\tau \\
 &\leq \frac{M}{r^2} + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{\tau^2} d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{\tau^2} d\tau \\
 &\leq \frac{M}{r^2} + \frac{I_{g,x}(1)}{r^{2(1-\beta)}} + I_{g,x}(r^\beta), \tag{2.7}
 \end{aligned}$$

see (2.4). Further, if  $r > 1$ , then

$$\begin{aligned}
 \pi S_{g,x}(r) &= \int_0^{\infty} \frac{\log g(\tau x)}{\tau^2 + r^2} d\tau + \int_0^{\infty} \frac{\log g(-\tau x)}{\tau^2 + r^2} d\tau \\
 &\leq \int_0^r \frac{\log g(\tau x)}{r^2} d\tau + \int_r^{\infty} \frac{\log g(\tau x)}{\tau^2} d\tau \\
 &\quad + \int_0^r \frac{\log g(-\tau x)}{r^2} d\tau + \int_r^{\infty} \frac{\log g(-\tau x)}{\tau^2} d\tau \\
 &= I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r), \tag{2.8}
 \end{aligned}$$

and, with a similar calculation,

$$\begin{aligned}
 \pi S_{g,x}(r) &\geq \int_0^r \frac{\log g(\tau x)}{2r^2} d\tau + \int_r^\infty \frac{\log g(\tau x)}{2\tau^2} d\tau \\
 &\quad + \int_0^r \frac{\log g(-\tau x)}{2r^2} d\tau + \int_r^\infty \frac{\log g(-\tau x)}{2\tau^2} d\tau \\
 &= \frac{1}{2}(I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r)). \tag{2.9}
 \end{aligned}$$

Since  $g$  is locally bounded, it follows from Lemma 2.1 that  $I_g$  defined by

$$I_g(r) := \sup_{x \in \mathbb{S}^{n-1}} I_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \int_{\max\{r,1\}}^\infty \frac{\log g(\tau x)}{\tau^2} d\tau < \infty,$$

is a decreasing function such that

$$I_g(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{2.10}$$

Let

$$\begin{aligned}
 J_g(r) &:= \sup_{x \in \mathbb{S}^{n-1}} J_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x) d\tau, \\
 S_g(r) &:= \sup_{x \in \mathbb{S}^{n-1}} S_{g,x}(r) = \sup_{x \in \mathbb{S}^{n-1}} \frac{1}{\pi} \int_{-\infty}^\infty \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2} d\tau. \tag{2.11}
 \end{aligned}$$

Then, in view of (2.7), (2.8), and (2.9),

$$\begin{aligned}
 J_g(r) &\leq \frac{M}{r^2} + \frac{I_g(1)}{r^{2(1-\beta)}} + I_g(r^\beta), \\
 \frac{1}{2\pi} \max\{I_g(r), J_g(r)\} &\leq S_g(r) \leq \frac{2}{\pi}(I_g(r) + J_g(r)).
 \end{aligned}$$

Thus,  $J_g(r) \rightarrow 0$ , and

$$S_g(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty, \tag{2.12}$$

see (2.10). It is clear that

$$S_g(r) = S_g(1) \text{ for } r \in [0, 1] \quad \text{and} \quad S_g \text{ is a decreasing function.} \tag{2.13}$$

**Examples.** (1) If  $g(x) = (1 + |x|)^s$ ,  $s \geq 0$ , then we have for all  $r \geq 1$

$$\begin{aligned}
 S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s \log(1 + |\tau|)}{\tau^2 + r^2} d\tau \\
 &= \frac{s}{\pi r} \int_{-\infty}^{\infty} \frac{\log(1 + r|\lambda|)}{\lambda^2 + 1} d\lambda \\
 &\leq \frac{s}{\pi r} \int_{-\infty}^{\infty} \frac{\log(1 + |\lambda|)}{\lambda^2 + 1} d\lambda + \frac{s \log(1 + r)}{\pi r} \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} d\lambda \\
 &= \frac{c_1 s}{r} + \frac{s \log(1 + r)}{r},
 \end{aligned} \tag{2.14}$$

where

$$c_1 := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1 + |\lambda|)}{\lambda^2 + 1} d\lambda < \infty.$$

(2) If  $g(x) = (\log(e + |x|))^t$ ,  $t \geq 0$ , then using the obvious inequality

$$u + v \leq 2uv, \quad u, v \geq 1,$$

yields for  $r \geq 1$

$$\begin{aligned}
 S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t \log \log(e + |\tau|)}{\tau^2 + r^2} d\tau = \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log \log(e + r|\lambda|)}{\lambda^2 + 1} d\lambda \\
 &\leq \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log(\log(e + |\lambda|) + \log(e + r))}{\lambda^2 + 1} d\lambda \\
 &\leq \frac{t}{\pi r} \int_{-\infty}^{\infty} \frac{\log(2 \log(e + |\lambda|))}{\lambda^2 + 1} d\lambda + \frac{t \log \log(e + r)}{\pi r} \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} d\lambda \\
 &= \frac{c_2 t}{r} + \frac{t \log \log(e + r)}{r},
 \end{aligned} \tag{2.15}$$

where

$$c_2 := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(2 \log(e + |\lambda|))}{\lambda^2 + 1} d\lambda < \infty.$$

(3) If  $g(x) = e^{a|x|^b}$ ,  $a \geq 0$ ,  $b \in [0, 1)$ , then we have for all  $r \geq 1$

$$\begin{aligned} S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a|\tau|^b}{\tau^2 + r^2} d\tau = \frac{ar^{b-1}}{\pi} \int_{-\infty}^{\infty} \frac{|\lambda|^b}{\lambda^2 + 1} d\lambda = \frac{2ar^{b-1}}{\pi} \int_0^{\infty} \frac{t^b}{t^2 + 1} dt \\ &= \frac{ar^{b-1}}{\pi} \int_0^{\infty} \frac{s^{\frac{b-1}{2}}}{s + 1} ds = \frac{ar^{b-1}}{\sin(\frac{1-b}{2}\pi)}, \end{aligned} \quad (2.16)$$

see, e.g., [4, Chapter V, Example 2.12].

(4) Finally, let  $g(x) = e^{|x|/\log^\gamma(e+|x|)}$ ,  $\gamma > 1$ . Since

$$\frac{\tau(e + \tau)}{\tau^2 + r^2} = \frac{1 + \frac{e}{\tau}}{1 + \frac{r^2}{\tau^2}} \leq 1 + \frac{e}{\tau} \leq 1 + \frac{e}{r} \quad \text{for } \tau \geq r,$$

then for any  $\beta \in (0, 1)$  and  $r \geq 1$

$$\begin{aligned} S_g(r) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\tau|}{(\tau^2 + r^2) \log^\gamma(e + |\tau|)} d\tau = \frac{2}{\pi} \int_0^{\infty} \frac{\tau}{(\tau^2 + r^2) \log^\gamma(e + \tau)} d\tau \\ &= \frac{2}{\pi} \left( \int_0^{r^\beta} + \int_{r^\beta}^r + \int_r^{\infty} \right) \frac{\tau}{(\tau^2 + r^2) \log^\gamma(e + \tau)} d\tau \\ &\leq \frac{2}{\pi} \int_0^{r^\beta} \frac{\tau}{\tau^2 + r^2} d\tau + \frac{2}{\pi \log^\gamma(e + r^\beta)} \int_{r^\beta}^r \frac{\tau}{\tau^2 + r^2} d\tau \\ &\quad + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \int_r^{\infty} \frac{1}{(e + \tau) \log^\gamma(e + \tau)} d\tau \\ &= \frac{1}{\pi} \log(\tau^2 + r^2)|_0^{r^\beta} + \frac{1}{\pi \log^\gamma(e + r^\beta)} \log(\tau^2 + r^2)|_{r^\beta}^r \\ &\quad + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{1 - \gamma} \log^{1-\gamma}(e + \tau)|_r^{\infty} \\ &\leq \frac{1}{\pi} \log(1 + r^{2(\beta-1)}) + \frac{\log 2}{\pi \log^\gamma(e + r^\beta)} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma - 1} \log^{1-\gamma}(e + r) \\ &\leq \frac{r^{2(\beta-1)}}{\pi} + \frac{\log 2}{\pi \log^\gamma(e + r^\beta)} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma - 1} \log^{1-\gamma}(e + r). \end{aligned}$$

Since

$$\lim_{r \rightarrow \infty} \frac{r^{2(\beta-1)} + (\log 2) \log^{-\gamma}(e + r^\beta)}{\log^{-\gamma}(e + r)} = \frac{\log 2}{\beta^\gamma} \quad \text{for all } \beta \in (0, 1),$$



one gets, if we take  $\beta \in ((\log 2)^{1/\gamma}, 1)$ , the following estimate

$$S_g(r) \leq \frac{\log^{-\gamma}(e+r)}{\pi} + \frac{2}{\pi} \left(1 + \frac{e}{r}\right) \frac{1}{\gamma-1} \log^{1-\gamma}(e+r) \quad (2.17)$$

for sufficiently large  $r$ .

### 3. Estimates for entire functions

Let  $1 \leq p \leq \infty$  and let  $\omega: \mathbb{R}^n \rightarrow [0, \infty)$  be a measurable function such that  $\omega > 0$  Lebesgue almost everywhere. We set

$$\|f\|_{L_\omega^p} := \|\omega f\|_{L^p} \quad \text{and} \quad L_\omega^p(\mathbb{R}^n) := \{f: \mathbb{R}^n \rightarrow \mathbb{C} \mid f \text{ measurable, } \|f\|_{L_\omega^p} < \infty\}. \quad (3.1)$$

**Lemma 3.1.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let  $\varphi$  be a measurable function such that for almost every  $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ ,  $\varphi(z_1, x')$  is analytic in  $z_1$  for  $\text{Im } z_1 > 0$  and continuous up to  $\mathbb{R}$ . Suppose also that  $\log |\varphi(z_1, x')| = O(|z_1|)$  for  $|z_1|$  large,  $\text{Im } z_1 \geq 0$ , and that the restriction of  $\varphi$  to  $\mathbb{R}^n$  belongs to  $L_{g^{\pm 1}}^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ . Finally, suppose that*

$$k_\varphi := \text{ess sup}_{x' \in \mathbb{R}^{n-1}} \left( \limsup_{0 < y_1 \rightarrow \infty} \frac{\log |\varphi(iy_1, x')|}{y_1} \right) < \infty.$$

Then

$$\|\varphi(\cdot + iy_1, \cdot)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_g e^{(k_\varphi + S_g(y_1))y_1} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y_1 > 0, \quad (3.2)$$

see (2.11), (2.12), where the constant  $C_g < \infty$  depends only on  $g$ .

*Proof.* Let  $a^+ := \max\{a, 0\}$  for  $a \in \mathbb{R}$ . It follows from (2.1) that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log^+(g^{\mp 1}(t, x'))}{1+t^2} dt &\leq \int_{-\infty}^{\infty} \frac{\log(g(t, x'))}{1+t^2} dt \\ &\leq \int_{-\infty}^{\infty} \frac{\log(g(t, 0)) + \log(g(0, x'))}{1+t^2} dt \\ &\leq \pi(S_g(1) + \log(g(0, x'))) < +\infty. \end{aligned}$$

Since  $g^{\pm 1}\varphi \in L^p(\mathbb{R}^n)$ , Fubini's theorem implies that

$$g^{\pm 1}(\cdot, x')\varphi(\cdot, x') \in L^p(\mathbb{R})$$

for Lebesgue almost all  $x' \in \mathbb{R}^{n-1}$ . For such  $x' \in \mathbb{R}^{n-1}$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log^+ |\varphi(t, x')|}{1+t^2} dt &\leq \int_{-\infty}^{\infty} \frac{\log^+ (g^{\pm 1}(t, x') |\varphi(t, x')|)}{1+t^2} dt \\ &+ \int_{-\infty}^{\infty} \frac{\log^+ (g^{\mp 1}(t, x'))}{1+t^2} dt < \infty. \end{aligned}$$

Then

$$\log |\varphi(x_1 + iy_1, x')| \leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')|}{(t-x_1)^2 + y_1^2} dt, \quad x_1 \in \mathbb{R}, y_1 > 0,$$

cf. [20, Chapter III, G, 2], see also [22, Chapter V, Theorems 5 and 7].

Applying (2.1) again, one gets

$$\begin{aligned} \log g(x) &\leq \log g(t, x') + \log g(x_1 - t, 0), \\ \log g(t, x') &\leq \log g(x) + \log g(t - x_1, 0) \quad \text{for all } x = (x_1, x') \in \mathbb{R}^n, t \in \mathbb{R}. \end{aligned}$$

The latter inequality can be rewritten as follows

$$\log g^{-1}(x) \leq \log g^{-1}(t, x') + \log g(t - x_1, 0).$$

Hence,

$$\log g^{\pm 1}(x) \leq \log g^{\pm 1}(t, x') + \log g(\pm(x_1 - t), 0)$$

for all  $x = (x_1, x') \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ , and

$$\begin{aligned} &\log (|\varphi(x_1 + iy_1, x')| g^{\pm 1}(x)) \\ &\leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')|}{(t-x_1)^2 + y_1^2} dt + \log g^{\pm 1}(x) \\ &= k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t, x')| + \log g^{\pm 1}(x)}{(t-x_1)^2 + y_1^2} dt \\ &\leq k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log (|\varphi(t, x')| g^{\pm 1}(t, x'))}{(t-x_1)^2 + y_1^2} dt + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\pm(x_1 - t), 0)}{(t-x_1)^2 + y_1^2} dt \\ &= k_\varphi y_1 + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log (|\varphi(t, x')| g^{\pm 1}(t, x'))}{(t-x_1)^2 + y_1^2} dt + \frac{y_1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau. \end{aligned}$$

If  $y_1 \in [0, 1]$ , then

$$\begin{aligned} \frac{y_1}{\pi} \int_0^\infty \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau &\leq M \frac{y_1}{\pi} \int_0^1 \frac{1}{\tau^2 + y_1^2} d\tau + \frac{y_1}{\pi} \int_1^\infty \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau \\ &\leq M \frac{y_1}{\pi} \int_{\mathbb{R}} \frac{1}{\tau^2 + y_1^2} d\tau + \frac{1}{\pi} \int_1^\infty \frac{\log g(\tau, 0)}{\tau^2} d\tau \\ &\leq M + \frac{I_g(1)}{\pi}. \end{aligned}$$

It follows from (2.11) that for  $y_1 > 1$ ,

$$\frac{y_1}{\pi} \int_{-\infty}^\infty \frac{\log g(\tau, 0)}{\tau^2 + y_1^2} d\tau \leq y_1 S_g(y_1).$$

So,

$$\begin{aligned} \log(|\varphi(x_1 + iy_1, x')|g^{\pm 1}(x)) &\leq c_g + (k_\varphi + S_g(y_1))y_1 \\ &\quad + \frac{y_1}{\pi} \int_{-\infty}^\infty \frac{\log(|\varphi(t, x')|g^{\pm 1}(t, x'))}{(t - x_1)^2 + y_1^2} dt, \end{aligned}$$

where  $c_g := M + \frac{I_g(1)}{\pi}$ . Using Jensen's inequality, one gets

$$|\varphi(x_1 + iy_1, x')|g^{\pm 1}(x) \leq C_g e^{(k_\varphi + S_g(y_1))y_1} \frac{y_1}{\pi} \int_{-\infty}^\infty \frac{|\varphi(t, x')|g^{\pm 1}(t, x')}{(t - x_1)^2 + y_1^2} dt,$$

where

$$C_g := e^{M + \frac{I_g(1)}{\pi}}. \quad (3.3)$$

Estimate (3.2) now follows from Young's convolution inequality and (3.1).  $\blacksquare$

**Remark 3.2.** Let  $n = 1$ ,  $g: \mathbb{R} \rightarrow [1, \infty)$  be a Hölder continuous submultiplicative function satisfying the Beurling–Domar condition,  $g(0) = 1$ , and let

$$\begin{aligned} w(x + iy) &:= \frac{y}{\pi} \int_{-\infty}^\infty \frac{\log g(t)}{(t - x)^2 + y^2} dt \\ &\quad + \frac{i}{\pi} \int_{-\infty}^\infty \left( \frac{x - t}{(t - x)^2 + y^2} + \frac{t}{t^2 + 1} \right) \log g(t) dt, \quad x \in \mathbb{R}, y > 0. \end{aligned}$$

Then  $\varphi(z) := e^{w(z)}$  is analytic in  $z$  for  $\text{Im } z > 0$  and continuous up to  $\mathbb{R}$ ,

$$|\varphi(x)| = e^{\text{Re}(w(x))} = e^{\log g(x)} = g(x), \quad x \in \mathbb{R},$$

see, e.g., [8, Chapter III, Section 1], and

$$|\varphi(iy)| = e^{\text{Re}(w(iy))} = \exp\left(\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log g(t)}{t^2 + y^2} dt\right) = e^{S_g(y)y}, \quad y \geq 1.$$

So,

$$k_\varphi = \limsup_{0 < y \rightarrow \infty} \frac{\log |\varphi(iy)|}{y} = \limsup_{y \rightarrow \infty} S_g(y) = 0$$

see (2.12), and

$$\begin{aligned} \|\varphi(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R})} &\geq \frac{|\varphi(iy)|}{g(0)} = |\varphi(iy)| = e^{S_g(y)y} = e^{S_g(y)y} \|1\|_{L^\infty(\mathbb{R})} \\ &= e^{S_g(y)y} \|g^{-1}\varphi\|_{L^\infty(\mathbb{R})} = e^{S_g(y)y} \|\varphi\|_{L_{g^{-1}}^\infty(\mathbb{R})}, \end{aligned}$$

which shows that the factor  $e^{S_g(y_1)y_1}$  in the right-hand side of (3.2) is optimal in this case.

Clearly,

$$S_{\check{g}} = S_g, \quad C_{\check{g}} = C_g, \quad (3.4)$$

where  $\check{g}(x) := g(Ax)$  and  $A \in O(n)$  is an arbitrary orthogonal matrix, see (2.11), (3.3) and (2.4).

**Theorem 3.3.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let  $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}$  be an entire function such that  $\log |\varphi(z)| = O(|z|)$  for  $|z|$  large,  $z \in \mathbb{C}^n$ , and suppose that the restriction of  $\varphi$  to  $\mathbb{R}^n$  belongs to  $L_{g^{\pm 1}}^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ . Then, for every multi-index  $\alpha \in \mathbb{Z}_+^n$ ,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_\alpha e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (3.5)$$

where

$$\kappa_\varphi(\omega) := \sup_{x \in \mathbb{R}^n} \left( \limsup_{0 < t \rightarrow \infty} \frac{\log |\varphi(x + it\omega)|}{t} \right) < \infty, \quad \omega \in \mathbb{S}^{n-1}, \quad (3.6)$$

and the constant  $C_\alpha \in (0, \infty)$  depends only on  $\alpha$  and  $g$ .

*Proof* (Cf. the proof of [21, Lemma 9.29]). Take any  $y \in \mathbb{R}^n \setminus \{0\}$ . There exists an orthogonal matrix  $A \in O(n)$  such that  $A\mathbf{e}_1 = \omega := y/|y|$ , see (2.6). Define the functions  $\check{\varphi}(z) := \varphi(Az)$ ,  $z \in \mathbb{C}^n$ , and  $\check{g}(x) := g(Ax)$ ,  $x \in \mathbb{R}^n$ . Then  $\check{\varphi}: \mathbb{C}^n \rightarrow \mathbb{C}$  is an entire function, and one can apply to it Lemma 3.1 with  $\check{g}$  in place of  $g$ , see (3.4).

For any  $x \in \mathbb{R}^n$ , one has  $\varphi(x + iy) = \check{\varphi}(\tilde{x} + i|y|\mathbf{e}_1) = \check{\varphi}(\tilde{x}_1 + i|y|, \tilde{x}_2, \dots, \tilde{x}_n)$ , where  $\tilde{x} := A^{-1}x$ . Hence,

$$\begin{aligned} & \|\varphi(\cdot + iy)\|_{L_{g \pm 1}^p(\mathbb{R}^n)} \\ &= \|\check{\varphi}(\cdot + i|y|\cdot)\|_{L_{\check{g} \pm 1}^p(\mathbb{R}^n)} \leq C_{\check{g}} e^{(k_{\check{\varphi}} + S_{\check{g}}(|y|))|y|} \|\check{\varphi}\|_{L_{\check{g} \pm 1}^p(\mathbb{R}^n)} \\ &\leq C_g e^{(k_{\varphi}(|y|) + S_g(|y|))|y|} \|\check{\varphi}\|_{L_{\check{g} \pm 1}^p(\mathbb{R}^n)} = C_g e^{(k_{\varphi}(|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g \pm 1}^p(\mathbb{R}^n)}, \end{aligned}$$

see (3.4), which proves (3.5) for  $\alpha = 0$  and  $y \neq 0$ . This estimate is trivial for  $\alpha = 0$  and  $y = 0$ .

Iterating the standard Cauchy integral formula for one complex variable, one gets

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)} \left( \prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \dots d\theta_n, \\ \zeta &\in \Delta(z) := \{\eta \in \mathbb{C}^n : |\eta_k - z_k| < 1, k = 1, \dots, n\}, z \in \mathbb{C}^n \end{aligned}$$

(cf. [21, Chapter 1, Section 1]), which implies

$$\partial^\alpha \varphi(\zeta) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)^{\alpha_k + 1}} \left( \prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \dots d\theta_n.$$

Hence,

$$\partial^\alpha \varphi(z) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n e^{i\alpha_k \theta_k}} d\theta_1 \dots d\theta_n,$$

and

$$|\partial^\alpha \varphi(z)| \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} |\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})| d\theta_1 \dots d\theta_n. \quad (3.7)$$

Since  $g \geq 1$  is locally bounded,

$$1 \leq M_1 := \sup_{|s_k| \leq 1, k=1, \dots, n} g(s) < \infty.$$

Then it follows from (2.1) that

$$g^{\pm 1}(x_1 - \cos \theta_1, \dots, x_n - \cos \theta_n) \leq M_1 g^{\pm 1}(x). \quad (3.8)$$

According to the conditions of the theorem, there exists a constant  $c_\varphi \in (0, \infty)$  such that one has  $\log |\varphi(\zeta)| \leq c_\varphi |\zeta|$  for  $|\zeta|$  large. Then  $\kappa_\varphi(\omega) \leq c_\varphi$ , see (3.6). Let one set  $\varphi_y := \varphi(\cdot + iy)$ ,  $y = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$ . Then, similarly to the above inequality,  $\kappa_{\varphi_y}(\omega) \leq c_\varphi$ . Applying (3.5) with  $\alpha = 0$  to the function  $\varphi_y$  in place of  $\varphi$  and using (2.13), (3.8), one derives from (3.7)

$$\begin{aligned} & \|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \|\varphi(\cdot + iy_1 + e^{i\theta_1}, \dots, \cdot + iy_n + e^{i\theta_n})\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} d\theta_1 \dots d\theta_n \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 \|\varphi(\cdot + iy_1 + i \sin \theta_1, \dots, \cdot + iy_n + i \sin \theta_n)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \\ & \quad \otimes d\theta_1 \dots d\theta_n \\ & \leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} d\theta_1 \dots d\theta_n \\ & = \alpha! M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}. \end{aligned}$$

Applying (3.5) with  $\alpha = 0$  again, one gets

$$\begin{aligned} & \|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \\ & \leq \alpha! M_1 C_0^2 e^{(c_\varphi + S_g(1))\sqrt{n}} e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}. \quad \blacksquare \end{aligned}$$

**Corollary 3.4.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let  $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}$  be an entire function such that  $\log |\varphi(z)| = O(|z|)$  for  $|z|$  large,  $z \in \mathbb{C}^n$ , and that the restriction of  $\varphi$  to  $\mathbb{R}^n$  belongs to  $L_{g^{\pm 1}}^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ . Then, for every multi-index  $\alpha \in \mathbb{Z}_+^n$  and every  $\varepsilon > 0$ ,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)} \leq C_{\alpha, \varepsilon} e^{(\kappa_\varphi(y/|y|) + \varepsilon)|y|} \|\varphi\|_{L_{g^{\pm 1}}^p(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (3.9)$$

where  $\kappa_\varphi$  is defined by (3.6), and the constant  $C_{\alpha, \varepsilon} \in (0, \infty)$  depends only on  $\alpha$ ,  $\varepsilon$ , and  $g$ .

*Proof.* It follows from (2.12) that, for every  $\varepsilon > 0$ , there exists some  $c_\varepsilon$  such that

$$S_g(|y|)|y| \leq c_\varepsilon + \varepsilon|y| \quad \text{for all } y \in \mathbb{R}^n.$$

Hence, (3.5) implies (3.9). \blacksquare

## 4. Main results

We will use the notation  $\tilde{g}(x) := g(-x)$ ,  $x \in \mathbb{R}^n$ . It follows from submultiplicativity of  $\tilde{g}$  that  $L_{\tilde{g}}^1(\mathbb{R}^n)$  is a convolution algebra.

Taking  $y - x$  in place of  $y$  in (2.1) and rearranging, one gets

$$\frac{1}{g(x)} \leq \frac{g(y-x)}{g(y)}. \quad (4.1)$$

Using (4.1), one can easily show that  $f * u \in L_{g^{-1}}^\infty(\mathbb{R}^n)$  for every  $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$  and  $u \in L_{\tilde{g}}^1(\mathbb{R}^n)$ . The Fubini–Tonelli theorem implies that

$$f * (v * u) = (f * v) * u \quad \text{for all } f \in L_{g^{-1}}^\infty(\mathbb{R}^n) \text{ and } v, u \in L_{\tilde{g}}^1(\mathbb{R}^n). \quad (4.2)$$

Let  $A_{\tilde{g}} := \{c\delta + g \mid c \in \mathbb{C}, g \in L_{\tilde{g}}^1(\mathbb{R}^n)\}$ , where  $\delta$  is the Dirac measure on  $\mathbb{R}^n$ . This is the algebra  $L_{\tilde{g}}^1(\mathbb{R}^n)$  with a unit attached, cf. Rudin [24, 10.3 (d), 11.13 (e)]. Clearly, (4.2) holds for any  $v, u \in A_{\tilde{g}}$ .

**Theorem 4.1.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3),  $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$ , and  $Y$  be a linear subspace of  $L_{\tilde{g}}^1(\mathbb{R}^n)$  such that*

$$f * v = 0 \quad \text{for every } v \in Y. \quad (4.3)$$

Suppose the set

$$Z(Y) := \bigcap_{v \in Y} \{\xi \in \mathbb{R}^n \mid \hat{v}(\xi) = 0\}$$

is bounded, and  $u \in L_{\tilde{g}}^1(\mathbb{R}^n)$  is such that  $\hat{u} = 1$  in a neighbourhood of  $Z(Y)$ . Then  $f = f * u$ . If  $Z(Y) = \emptyset$ , then  $f = 0$ .

*Proof.* In order to prove the equality  $f = f * u$ , it is sufficient to show that

$$\langle f, h \rangle = \langle f * u, h \rangle \quad \text{for every } h \in L_g^1(\mathbb{R}^n). \quad (4.4)$$

Since the set of functions  $h$  with compactly supported Fourier transforms  $\hat{h}$  is dense in  $L_g^1(\mathbb{R}^n)$ , see [5, Theorem 1.52 and 2.11], it is enough to prove (4.4) for such  $h$ . Further,

$$\langle f, h \rangle = (f * \tilde{h})(0).$$

So, we have to show only that

$$f * w = f * u * w$$

for every  $w \in L^1_{\tilde{g}}(\mathbb{R}^n)$  with compactly supported Fourier transform  $\hat{w}$ . Take any such  $w$  and choose  $R > 0$  such that the support of  $\hat{w}$  lies in  $B_R := \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$ . It is clear that  $\tilde{g}$  satisfies the Beurling–Domar condition. Then there exists  $u_R \in L^1_{\tilde{g}}(\mathbb{R}^n)$  such that  $0 \leq \widehat{u}_R \leq 1$ ,  $\widehat{u}_R(\xi) = 1$  for  $|\xi| \leq R$ , and  $\widehat{u}_R(\xi) = 0$  for  $|\xi| \geq R + 1$ , see [5, Lemma 1.24].

If  $Z(Y) \neq \emptyset$ , let  $V$  be an open neighbourhood of  $Z(Y)$  such that  $\hat{u} = 1$  in  $V$ . Similarly to the above, there exists  $u_0 \in L^1_{\tilde{g}}(\mathbb{R}^n)$  such that  $0 \leq \widehat{u}_0 \leq 1$ ,  $\widehat{u}_0 = 1$  in a neighbourhood  $V_0 \subset V$  of  $Z(Y)$ , and  $\widehat{u}_0 = 0$  outside  $V$ , see [5, Lemma 1.24]. If  $Z(Y) = \emptyset$ , one can take  $u = u_0 = 0$  and  $V_0 = \emptyset$  below.

Since  $Y$  is a linear subspace, for every  $\eta \in B_{R+1} \setminus V_0 \subset \mathbb{R}^n \setminus Z(Y)$ , there exists  $v_\eta \in Y$  such that  $\widehat{v}_\eta(\eta) = 1$ . Since  $v_\eta \in L^1(\mathbb{R}^n)$ ,  $\widehat{v}_\eta$  is continuous, and there is a neighbourhood  $V_\eta$  of  $\eta$  such that  $|\widehat{v}_\eta(\xi) - 1| < 1/2$  for all  $\xi \in V_\eta$ . Similarly to the above, there exists  $u_\eta \in L^1_{\tilde{g}}(\mathbb{R}^n)$  such that  $\operatorname{Re}(\widehat{v}_\eta \widehat{u}_\eta) \geq 0$ , and  $\operatorname{Re}(\widehat{v}_\eta \widehat{u}_\eta) > \frac{1}{2}$  in a neighbourhood  $V_\eta^0 \subset V_\eta$  of  $\eta$ .

Since  $B_{R+1} \setminus V_0$  is compact, the open cover  $\{V_\eta^0\}_{\eta \in B_{R+1} \setminus V_0}$  has a finite subcover. So, there exist functions  $v_j \in Y$  and  $u_j \in L^1_{\tilde{g}}(\mathbb{R}^n)$ ,  $j = 1, \dots, N$ , such that

$$\operatorname{Re}(\sigma) > \frac{1}{2}, \quad \text{where } \sigma := \widehat{u}_0 + \sum_{j=1}^N \widehat{v}_j \widehat{u}_j + 1 - \widehat{u}_R.$$

Then there exists  $v \in A_{\tilde{g}}$  such that  $\hat{v} = 1/\sigma$ , see [5, Theorem 1.53].

Since  $\widehat{u}_0(1 - \hat{u}) = 0$  and  $(1 - \widehat{u}_R)\hat{w} = 0$ , one has

$$\begin{aligned} \left( \hat{u} + \sum_{j=1}^N \widehat{v}_j \widehat{u}_j \hat{v}(1 - \hat{u}) \right) \hat{w} &= (\hat{u} + (\sigma - (\widehat{u}_0 + 1 - \widehat{u}_R))\hat{v}(1 - \hat{u})) \hat{w} \\ &= (\hat{u} + (1 - \hat{u}) - (\widehat{u}_0 + 1 - \widehat{u}_R)\hat{v}(1 - \hat{u})) \hat{w} \\ &= (1 - (1 - \widehat{u}_R)\hat{v}(1 - \hat{u})) \hat{w} \\ &= \hat{w} - (1 - \widehat{u}_R)\hat{w}\hat{v}(1 - \hat{u}) = \hat{w}. \end{aligned}$$

It now follows from (4.2) and (4.3) that

$$\begin{aligned} f * w &= f * \left( u + \sum_{j=1}^N v_j * u_j * (v - v * u) \right) * w \\ &= f * u * w + f * \left( \sum_{j=1}^N v_j * u_j * (v - v * u) \right) * w \\ &= f * u * w + \sum_{j=1}^N (f * v_j) * u_j * (v - v * u) * w = f * u * w. \end{aligned}$$

If  $Z(Y) = \emptyset$ , one can take  $u = 0$ , and the equality  $f = f * u$  means that  $f = 0$ . ■



For a bounded set  $E \subset \mathbb{R}^n$ , let  $\text{conv}(E)$  denote its closed convex hull, and  $H_E$  denote its support function:

$$H_E(y) := \sup_{\xi \in E} y \cdot \xi = \sup_{\xi \in \text{conv}(E)} y \cdot \xi, \quad y \in \mathbb{R}^n.$$

Clearly,  $H_E$  is positively homogeneous and convex: for all  $x, y \in \mathbb{R}^n$  and  $\tau \geq 0$ , we have

$$H_E(\tau y) = \tau H_E(y), \quad H_E(y + x) \leq H_E(y) + H_E(x).$$

For every positively homogeneous convex function  $H$ ,

$$K := \{\xi \in \mathbb{R}^n \mid y \cdot \xi \leq H(y) \text{ for all } y \in \mathbb{R}^n\} \quad (4.5)$$

is the unique convex compact set such that  $H_K = H$ , see, e.g., [15, Theorem 4.3.2].

**Theorem 4.2.** *Let  $g$ ,  $f$ , and  $Y$  satisfy the conditions of Theorem 4.1, and let*

$$\mathcal{H}_Y(y) := H_{Z(Y)}(-y) = \sup_{\xi \in Z(Y)} (-y) \cdot \xi = - \inf_{\xi \in Z(Y)} y \cdot \xi, \quad y \in \mathbb{R}^n.$$

*Then  $f$  admits analytic continuation to an entire function  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  such that for every multi-index  $\alpha \in \mathbb{Z}_+^n$ ,*

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{\mathcal{H}_Y(y) + S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (4.6)$$

*see (2.11), (2.12), where the constant  $C_\alpha \in (0, \infty)$  depends only on  $\alpha$  and  $g$ .*

*Proof.* Take any  $\varepsilon > 0$ . There exists  $u \in L_{\frac{1}{g}}^1(\mathbb{R}^n)$  such that  $\hat{u} = 1$  in a neighbourhood of  $Z(Y)$ , and  $\hat{u} = 0$  outside the  $\frac{\varepsilon}{2}$ -neighbourhood of  $Z(Y)$ , see [5, Lemma 1.24]. It follows from the Paley–Wiener–Schwartz theorem, see, e.g., [15, Theorem 7.3.1] that  $u = \mathcal{F}^{-1} \hat{u}$  admits analytic continuation to an entire function  $u: \mathbb{C}^n \rightarrow \mathbb{C}$  satisfying the estimate

$$|u(x + iy)| \leq c_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|/2} \quad \text{for all } x, y \in \mathbb{R}^n$$

with some constant  $c_\varepsilon \in (0, \infty)$ . So,  $u$  satisfies the conditions of Corollary 3.4 with  $\tilde{g}$  in place of  $g$ , and

$$\|u(\cdot + iy)\|_{L_{\frac{1}{\tilde{g}}}^1(\mathbb{R}^n)} \leq C_{0, \varepsilon/2} e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|u\|_{L_{\frac{1}{\tilde{g}}}^1(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n.$$

Since

$$f(x) = \int_{\mathbb{R}^n} u(x - s) f(s) ds,$$

see Theorem 4.1,  $f$  admits analytic continuation

$$f(x + iy) := \int_{\mathbb{R}^n} u(x + iy - s) f(s) ds,$$

see Corollary 3.4, and

$$\begin{aligned} \|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} &\leq \|u(\cdot + iy)\|_{L_g^1(\mathbb{R}^n)} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \\ &\leq C_{0,\varepsilon/2} e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|u\|_{L_g^1(\mathbb{R}^n)} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \\ &=: M_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \end{aligned}$$

see (4.1). Since

$$\frac{|f(x + iy)|}{g(x)} \leq M_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)},$$

one has  $\log |f(x + iy)| = O(|x + iy|)$  for  $|x + iy|$  large, see (2.2), and

$$\begin{aligned} \limsup_{0 < t \rightarrow \infty} \frac{\log |f(x + it\omega)|}{t} &\leq \limsup_{0 < t \rightarrow \infty} \frac{\log(M_\varepsilon g(x) \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}) + t\mathcal{H}_Y(\omega) + \varepsilon t}{t} \\ &= \mathcal{H}_Y(\omega) + \varepsilon. \end{aligned}$$

Hence,

$$\kappa_f(\omega) := \sup_{x \in \mathbb{R}^n} \left( \limsup_{0 < t \rightarrow \infty} \frac{\log |f(x + it\omega)|}{t} \right) \leq \mathcal{H}_Y(\omega) + \varepsilon$$

for every  $\varepsilon > 0$ , i.e.,

$$\kappa_f(\omega) \leq \mathcal{H}_Y(\omega).$$

So, (4.6) follows from Theorem 3.3. ■

**Theorem 4.3.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3), and let  $m \in C(\mathbb{R}^n)$  be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \tilde{m}(D)\varphi := \mathcal{F}^{-1}(\tilde{m}\hat{\varphi})$$

*maps  $C_c^\infty(\mathbb{R}^n)$  into  $L_g^1(\mathbb{R}^n)$ . Suppose  $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$  is such that  $m(D)f = 0$  as a distribution, i.e.,*

$$\langle f, \tilde{m}(D)\varphi \rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \quad (4.7)$$

If  $K := \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$  is compact, then  $f$  admits analytic continuation to an entire function  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  such that for every multi-index  $\alpha \in \mathbb{Z}_+^n$ ,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{H(y) + S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (4.8)$$

see (2.11), (2.12), where  $H(y) := H_K(-y)$ , and the constant  $C_\alpha \in (0, \infty)$  depends only on  $\alpha$  and  $g$ .

Conversely, if every  $f \in L^\infty(\mathbb{R}^n)$  satisfying (4.7) admits analytic continuation to an entire function  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  such that

$$\|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq M_\varepsilon e^{H(y) + \varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (4.9)$$

holds for every  $\varepsilon > 0$  with a constant  $M_\varepsilon \in (0, \infty)$  that depends only on  $\varepsilon$ ,  $m$ , and  $g$ , then  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq K$ , where  $K$  is the unique convex compact set such that  $H_K(y) = H(-y)$ ; cf. (4.5).

*Proof.* Denote by  $(T_\nu\varphi)(x) := \varphi(x - \nu)$ ,  $x, \nu \in \mathbb{R}^n$  the shift by  $\nu$ . Since  $T_\nu\varphi \in C_c^\infty(\mathbb{R}^n)$  for every  $\varphi \in C_c^\infty(\mathbb{R}^n)$  and all  $\nu \in \mathbb{R}^n$ , it follows from (4.7) that

$$(f * \widetilde{m(D)\varphi})(\nu) = \langle f, T_\nu \widetilde{m(D)\varphi} \rangle = \langle f, \widetilde{m(D)}(T_\nu\varphi) \rangle = 0 \quad \text{for all } \nu \in \mathbb{R}^n.$$

Hence,

$$f * \widetilde{m(D)\varphi} = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

It is easy to see that

$$\begin{aligned} \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid \widetilde{m(D)\varphi}(\eta) = 0\} &= \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid \widetilde{m(D)\varphi}(-\eta) = 0\} \\ &= \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid m(\eta)\hat{\varphi}(-\eta) = 0\} \\ &= \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = K. \end{aligned}$$

Applying Theorem 4.2 with

$$Y := \{\widetilde{m(D)\varphi} \mid \varphi \in C_c^\infty(\mathbb{R}^n)\} \subset L_{\hat{g}}^1(\mathbb{R}^n)$$

and  $Z(Y) = K$ , one gets (4.8).

For the converse direction, we assume the contrary, that is, that the zero-set  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$  contains some  $\gamma \notin K$ , see (4.5). Then there exists a  $y_0 \in \mathbb{R}^n \setminus \{0\}$  such that  $y_0 \cdot \gamma > H_K(y_0) = H(-y_0)$ . It is easy to see that  $f(x) := e^{ix \cdot \gamma}$  satisfies  $m(D)e^{ix \cdot \gamma} = e^{ix \cdot \gamma} m(\gamma) = 0$  for all  $x \in \mathbb{R}^n$ . Take  $\varepsilon < (y_0 \cdot \gamma - H(-y_0))/|y_0|$ . Clearly,

$f \in L^\infty(\mathbb{R}^n)$ , and

$$\frac{\|f(\cdot - i\tau y_0)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}}{e^{H(-\tau y_0) + \varepsilon|\tau y_0|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}} = \frac{e^{\tau(y_0 \cdot \gamma)}}{e^{\tau(H(-y_0) + \varepsilon|y_0|)}} = e^{\tau(y_0 \cdot \gamma - H(-y_0) - \varepsilon|y_0|)} \xrightarrow{\tau \rightarrow \infty} \infty.$$

So,  $f$  does not satisfy (4.9).  $\blacksquare$

**Corollary 4.4.** *Let  $g: \mathbb{R}^n \rightarrow [1, \infty)$  be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3) and let  $m \in C(\mathbb{R}^n)$  be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \tilde{m}(D)\varphi := \mathcal{F}^{-1}(\tilde{m}\hat{\varphi})$$

maps  $C_c^\infty(\mathbb{R}^n)$  into  $L_g^1(\mathbb{R}^n)$ . Suppose  $f \in L_{g^{-1}}^\infty(\mathbb{R}^n)$  is such that  $m(D)f = 0$  as a distribution, i.e., (4.7) holds. If  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \{0\}$ , then  $f$  admits analytic continuation to an entire function  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  such that for every multi-index  $\alpha \in \mathbb{Z}_+^n$ ,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq C_\alpha e^{S_g(|y|)|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \quad (4.10)$$

where the constant  $C_\alpha \in (0, \infty)$  depends only on  $\alpha$  and  $g$ . If  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$ , then  $f = 0$ .

Conversely, if every  $f \in L^\infty(\mathbb{R}^n)$  satisfying (4.7) admits analytic continuation to an entire function  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  such that

$$\|f(\cdot + iy)\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)} \leq M_\varepsilon e^{\varepsilon|y|} \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n,$$

holds for every  $\varepsilon > 0$  with a constant  $M_\varepsilon \in (0, \infty)$  that depends only on  $\varepsilon$ ,  $m$ , and  $g$ , then  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq \{0\}$ .

*Proof.* The only part that does not follow immediately from Theorem 4.3 is that  $f = 0$  in the case  $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$ . In this case, one can take the same  $Y$  as in the proof of Theorem 4.3, note that  $Z(Y) = \emptyset$  and apply Theorem 4.1 to conclude that  $f = 0$ . (It is instructive to compare this result to [18, Proposition 2.2].)  $\blacksquare$

**Remark 4.5.** The condition that  $\tilde{m}(D)$  maps  $C_c^\infty(\mathbb{R}^n)$  to  $L_g^1(\mathbb{R}^n)$  is satisfied if  $m$  is a linear combination of terms of the form  $ab$ , where  $a = F\mu$ ,  $\mu$  is a finite complex Borel measure on  $\mathbb{R}^n$  such that

$$\int_{\mathbb{R}^n} \tilde{g}(y) |\mu|(dy) < \infty,$$

and  $b$  is the Fourier transform of a compactly supported distribution. Indeed, it is easy to see that  $\tilde{b}(D)$  maps  $C_c^\infty(\mathbb{R}^n)$  into itself, while the convolution operator  $\varphi \mapsto \tilde{\mu} * \varphi$  maps  $C_c^\infty(\mathbb{R}^n)$  to  $L_g^1(\mathbb{R}^n)$ .

A particular example is the characteristic exponent of a Lévy process (this is a stochastic process with stationary and independent increments, such that the trajectories are right-continuous with finite left limits, see, e.g., Sato [25])

$$m(\xi) = -ib \cdot \xi + \frac{1}{2}\xi \cdot Q\xi + \int_{0 < |y| < 1} (1 - e^{iy \cdot \xi} + iy \cdot \xi) \nu(dy) \\ + \int_{|y| \geq 1} (1 - e^{iy \cdot \xi}) \nu(dy),$$

where  $b \in \mathbb{R}^n$ ,  $Q \in \mathbb{R}^{n \times n}$  is a symmetric positive semidefinite matrix, and  $\nu$  is a measure on  $\mathbb{R}^n \setminus \{0\}$  such that  $\int_{0 < |y| < 1} |y|^2 \nu(dy) + \int_{|y| \geq 1} g(y) \nu(dy) < \infty$ . More generally, one can take

$$m(\xi) = \sum_{|\alpha|=0}^{2s} c_\alpha \frac{i^{|\alpha|}}{\alpha!} \xi^\alpha + \int_{0 < |y| < 1} \left[ 1 - e^{iy \cdot \xi} + \sum_{|\alpha|=0}^{2s-1} \frac{i^{|\alpha|}}{\alpha!} y^\alpha \xi^\alpha \right] \nu(dy) \\ + \int_{|y| \geq 1} (1 - e^{iy \cdot \xi}) \nu(dy)$$

with  $s \in \mathbb{N}$ ,  $c_\alpha \in \mathbb{R}$ , and a measure  $\nu$  on  $\mathbb{R}^n \setminus \{0\}$  such that  $\int_{0 < |y| < 1} |y|^{2s} \nu(dy) + \int_{|y| \geq 1} g(y) \nu(dy) < \infty$ . (As usual, for any  $\alpha \in \mathbb{N}_0^n$  and  $\xi \in \mathbb{R}^n$ , we define  $\alpha! := \prod_1^n \alpha_k!$  and  $\xi^\alpha := \prod_1^n \xi_k^{\alpha_k}$ .) Functions of this type appear naturally in positivity questions related to generalised functions (see, e.g., [9, Chapter II, Section 4] or [28, Chapter 8]). Some authors call the function  $-m$  for such an  $m$  (under suitable additional conditions on the  $c_\alpha$ 's) a *conditionally positive definite function*.

**Remark 4.6.** We are mostly interested in super-polynomially growing weights as polynomially growing ones have been dealt with in our previous paper [3]. Nevertheless, it is instructive to look at the behaviour of the factor  $e^{S_g(|y|)|y|}$  for typical super-polynomially, polynomially, and sub-polynomially growing weights.

It follows from (2.17) that if  $g(x) = e^{|x|/\log^\gamma(e+|x|)}$ ,  $\gamma > 1$ , then there exists a constant  $C_\gamma$  such that

$$e^{S_g(|y|)|y|} \leq C_\gamma e^{\frac{1}{\pi}|y|\log^{-\gamma}(e+|y|)(1+\frac{2}{\gamma-1}\log(e+|y|))} \\ = C_\gamma (e^{|y|/\log^\gamma(e+|y|)})^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))} \\ = C_\gamma (g(y))^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))}.$$

Similarly, if  $g(x) = e^{a|x|^b}$ ,  $a \geq 0$ ,  $b \in [0, 1)$ , then (2.16) implies

$$e^{S_g(|y|)|y|} = e^{a|y|^b(\sin(\frac{1-b}{2}\pi))^{-1}} = (g(y))^{(\sin(\frac{1-b}{2}\pi))^{-1}}. \quad (4.11)$$

If  $g(x) = (1 + |x|)^s$ ,  $s \geq 0$ , then (2.14) implies

$$e^{\mathcal{S}_g(|y|)|y|} \leq e^{c_1 s + s \log(1+|y|)} = C_s (1 + |y|)^s = C_s g(y). \quad (4.12)$$

Finally, if  $g(x) = (\log(e + |x|))^t$ ,  $t \geq 0$ , then (2.15) implies

$$e^{\mathcal{S}_g(|y|)|y|} \leq e^{c_2 t + t \log \log(e+|y|)} = C_t (\log(e + |y|))^t = C_t g(y).$$

**Remark 4.7.** If  $g$  is polynomially bounded in Corollary 4.4, then it follows from (4.10) and (4.12) that  $f$  is a polynomially bounded entire function on  $\mathbb{C}^n$ , hence a polynomial, see, e.g., [21, Corollary 1.7]. The fact that  $f$  is a polynomial in this case was established in [3, 12].

**Remark 4.8.** Let  $n = 2$ ,  $g(x) := (1 + |x|)^k$ ,  $k \in \mathbb{N}$ ,  $f(x_1, x_2) := (x_1 + ix_2)^k$  (or  $f(x_1, x_2) := (x_1 + ix_2)^k + (x_1 - ix_2)^k$  if one prefers to have a real-valued  $f$ ). Then  $f \in L_{g^{-1}}^\infty(\mathbb{R}^2)$ ,  $\Delta f = 0$ ,  $f(x + iy_1 \mathbf{e}_1) = (x_1 + iy_1 + ix_2)^k$  for any  $y_1 \in \mathbb{R}$ , see (2.6), and

$$\frac{\|f(\cdot + iy_1 \mathbf{e}_1)\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)}}{g(y_1 \mathbf{e}_1)} \geq \frac{|y_1|^k}{(1 + |y_1|)^k} \xrightarrow{|y_1| \rightarrow \infty} 1 = \|f\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)}.$$

So, the factor  $e^{\mathcal{S}_g(|y|)|y|} \leq C_k g(y)$ , see (4.12), is optimal in (4.10) in this case.

The case  $g(x) = e^{a|x|^b}$ ,  $a > 0$ ,  $b \in [0, 1)$ , is perhaps more interesting. Let us take  $b = \frac{1}{2}$ . Then it follows from (4.11) that  $e^{\mathcal{S}_g(|y|)|y|} = (g(y))^{\sqrt{2}}$ . Let us show that one cannot replace this factor in (4.10) with  $(g(y))^{\sqrt{2}(1-\varepsilon)}$ ,  $\varepsilon > 0$ . Take any  $\varepsilon > 0$ . Since

$$\sqrt[4]{1 + \tau^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau}\right) \xrightarrow{\tau \rightarrow 0, \tau > 0} \frac{1}{\sqrt{2}},$$

there exists some  $\tau_\varepsilon > 0$  such that

$$\sqrt[4]{1 + \tau_\varepsilon^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \leq \frac{1 + \varepsilon}{\sqrt{2}}.$$

Let us estimate  $\operatorname{Re} \sqrt{x_1 + i\kappa x_2}$ , where  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $\kappa > 0$  is a constant to be chosen later, and  $\sqrt{\cdot}$  is the branch of the square root that is analytic in  $\mathbb{C} \setminus (-\infty, 0]$  and positive on  $(0, +\infty)$ . If  $x_1 \geq \tau_\varepsilon \kappa |x_2|$ , then

$$\begin{aligned} \operatorname{Re} \sqrt{x_1 + i\kappa x_2} &\leq |\sqrt{x_1 + i\kappa x_2}| = \sqrt[4]{x_1^2 + \kappa^2 x_2^2} \\ &\leq \sqrt[4]{\left(1 + \frac{1}{\tau_\varepsilon^2}\right) x_1^2} \leq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{x_1} \leq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{|x|}. \end{aligned}$$

If  $0 < x_1 < \tau_\varepsilon \kappa |x_2|$ , then

$$\begin{aligned}
 \operatorname{Re} \sqrt{x_1 + i\kappa x_2} &= |\sqrt{x_1 + i\kappa x_2}| \cos\left(\frac{1}{2} \arctan \frac{\kappa |x_2|}{x_1}\right) \\
 &\leq |\sqrt{\tau_\varepsilon \kappa |x_2| + i\kappa x_2}| \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \\
 &= \kappa^{1/2} |x_2|^{1/2} \sqrt[4]{1 + \tau_\varepsilon^2} \cos\left(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\right) \\
 &\leq \frac{1 + \varepsilon}{\sqrt{2}} \kappa^{1/2} |x|^{1/2}.
 \end{aligned}$$

Now, take  $\kappa_\varepsilon \geq 1$  such that

$$\frac{1 + \varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} \geq \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4}.$$

Then

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leq \frac{1 + \varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2} \quad (4.13)$$

for  $x_1 > 0$ . If  $x_1 \leq 0$ , then the argument of  $\sqrt{x_1 + i\kappa_\varepsilon x_2}$  belongs to  $[\pi/4, \pi/2]$ , respectively  $[-\pi/2, -\pi/4]$ , depending on the sign of  $x_2$ . Hence,

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leq |\sqrt{x_1 + i\kappa_\varepsilon x_2}| \cos \frac{\pi}{4} \leq \frac{1}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2},$$

and (4.13) holds for all  $x = (x_1, x_2) \in \mathbb{R}^2$ .

Since the Taylor series of  $\cos w$  contains only even powers of  $w$ ,  $\cos(i\sqrt{z})$  is an analytic function of  $z \in \mathbb{C}$ . So,  $\cos(i\sqrt{x_1 + ix_2})$  is a harmonic function of the variable  $x = (x_1, x_2) \in \mathbb{R}^2$ . Hence,  $f(x_1, x_2) := \cos(i\sqrt{x_1 + i\kappa_\varepsilon x_2})$  is a solution of the elliptic partial differential equation

$$\left(\partial_{x_1}^2 + \frac{1}{\kappa_\varepsilon^2} \partial_{x_2}^2\right) f(x_1, x_2) = 0.$$

It follows from (4.13) that

$$|f(x_1, x_2)| \leq \frac{1}{2} (1 + e^{\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2}}) \leq e^{\frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2}}.$$

So,  $f \in L_{g-1}^\infty(\mathbb{R}^2)$ , where  $g(x) = e^{a|x|^{1/2}}$  with  $a = \frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2}$ . Clearly, the analytic continuation of  $f$  to  $\mathbb{C}^2$  is given by the formula

$$f(x_1 + iy_1, x_2 + iy_2) = \cos(i\sqrt{x_1 + iy_1 + i\kappa_\varepsilon(x_2 + iy_2)}).$$

Finally, see (2.6), letting  $(-\infty, 0) \ni y_2 \rightarrow -\infty$ , we arrive at

$$\begin{aligned} \frac{\|f(\cdot + iy_2 \mathbf{e}_2)\|_{L_{g^{-1}}^\infty(\mathbb{R}^2)}}{(g(y_2 \mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} &\geq \frac{|f(0 + iy_2 \mathbf{e}_2)|}{g(0)(g(y_2 \mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} = \frac{|\cos(i\sqrt{-\kappa_\varepsilon} y_2)|}{e^{\sqrt{2}(1-\varepsilon)\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} \\ &\geq \frac{e^{\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2e^{(1-\varepsilon^2)\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} = \frac{1}{2}e^{\varepsilon^2\kappa_\varepsilon^{1/2}|y_2|^{1/2}} \xrightarrow{y_2 \rightarrow -\infty} \infty. \end{aligned}$$

## 5. Concluding remarks

Corollary 4.4 shows that sub-exponentially growing solutions of  $m(D)f = 0$  admit analytic continuation to entire functions on  $\mathbb{C}^n$ . It is well known that no growth restrictions are necessary in the case when  $m(D)$  is an elliptic partial differential operator with constant coefficients, and every solution of  $m(D)f = 0$  in  $\mathbb{R}^n$  admits analytic continuation to an entire function on  $\mathbb{C}^n$ , see [6, 23].

**Remark 5.1.** The latter result has a local version similar to Hayman's theorem on harmonic functions, see [13, Theorem 1]: for every elliptic partial differential operator  $m(D)$  with constant coefficients there exists a constant  $c_m \in (0, 1)$  such that every solution of  $m(D)f = 0$  in the ball  $\{x \in \mathbb{R}^n : |x| < R\}$  of any radius  $R > 0$  admits continuation to an analytic function in the ball  $\{x \in \mathbb{C}^n : |x| < c_m R\}$ . Indeed, let  $m_0(D) = \sum_{|\alpha|=N} a_\alpha D^\alpha$  be the principal part of  $m(D) = \sum_{|\alpha| \leq N} a_\alpha D^\alpha$ . There exists  $C_m > 0$  such that

$$\sum_{|\alpha|=N} a_\alpha (a + ib)^\alpha = 0, \quad a, b \in \mathbb{R}^n \implies |a| \geq C_m |b|,$$

see, for example, [27, Section 7]. Then the same argument as in the proof of [19, Corollary 8.2] shows that  $f$  admits continuation to an analytic function in the ball  $\{x \in \mathbb{C}^n : |x| < (1 + C_m^{-2})^{-1/2} R\}$ . Note that in the case of the Laplacian, one can take  $C_m = 1$  and  $c_m = (1 + C_m^{-2})^{-1/2} = \frac{1}{\sqrt{2}}$ , which is the optimal constant for harmonic functions, see [13].

Let us return to equations in  $\mathbb{R}^n$ . Below,  $m(\xi)$  will always denote a polynomial with  $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} \subseteq \{0\}$ . For non-elliptic partial differential operators  $m(D)$ , one needs to place growth restrictions on solutions of  $m(D)f = 0$  to make sure that they admit analytic continuation to entire functions on  $\mathbb{C}^n$ .

We say that a function  $f$  defined on  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) is of *infra-exponential* growth, if for every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that

$$|f(z)| \leq C_\varepsilon e^{\varepsilon|z|} \quad \text{for all } z \in \mathbb{R}^n \text{ (} z \in \mathbb{C}^n \text{)}.$$



Let  $\mu: [0, \infty) \rightarrow [0, \infty)$  be an increasing function, which increases to infinity and satisfies

$$\mu(t) \leq At + B, \quad t \geq 0$$

for some  $A, B > 0$ , and

$$\int_1^{\infty} \frac{\mu(t)}{t^2} dt < \infty. \quad (5.1)$$

Suppose  $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} = \{0\}$ . Then, under additional restrictions on  $\mu$ , every solution  $f$  of  $m(D)f = 0$  that has growth  $O(e^{\varepsilon\mu(|x|)})$  for every  $\varepsilon > 0$  admits analytic continuation to an entire function of infra-exponential growth on  $\mathbb{C}^n$  see [18]. It is easy to see that (5.1) is equivalent to the Beurling–Domar condition (2.3) for the function  $g(x) := e^{\varepsilon\mu(|x|)}$ .

One cannot replace  $O(e^{\varepsilon\mu(|x|)})$  with  $O(e^{\varepsilon|x|})$  in the above result without placing a restriction on the complex zeros of  $m$ . If there exists  $\delta > 0$  such that  $m(\zeta)$  has no complex zeros in

$$|\operatorname{Im} \zeta| < \delta, \quad |\operatorname{Re} \zeta| > \delta^{-1}, \quad (5.2)$$

then every solution of  $m(D)f = 0$  that, together with its partial derivatives up to the order of  $m(D)$ , is of infra-exponential growth on  $\mathbb{R}^n$ , admits analytic continuation to an entire function of infra-exponential growth on  $\mathbb{C}^n$ , see [17, 18].

On the other hand, if for every  $\delta > 0$ , (5.2) contains complex zeros of  $m(\zeta)$ , then  $m(D)f = 0$  has a solution in  $C^\infty$  all of whose derivatives are of infra-exponential growth on  $\mathbb{R}^n$ , but which is not entire infra-exponential in  $\mathbb{C}^n$ . The proof of the latter result in [17, 18] is not constructive, and the author writes: “Unfortunately we cannot present concrete examples of such solutions.” However, it is not difficult to construct, for any  $\varepsilon > 0$ , a solution in  $C^\infty$  all of whose derivatives have growth  $O(e^{\varepsilon|x|})$ , but which is not real-analytic. Indeed, according to the assumption, there exist complex zeros

$$\zeta_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}^n, \quad k \in \mathbb{N}$$

of  $m(\zeta)$  such that

$$|\eta_k| < k^{-1}, \quad |\xi_k| > k. \quad (5.3)$$

Choosing a subsequence, we can assume that  $\omega_k := |\xi_k|^{-1}\xi_k$  converge to a point  $\omega_0 \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$  as  $k \rightarrow \infty$ , and that  $|\omega_k - \omega_0| < 1$  for all  $k \in \mathbb{N}$ . Then

$$\omega_k \cdot \omega_0 = \frac{|\omega_k|^2 + |\omega_0|^2 - |\omega_k - \omega_0|^2}{2} > \frac{1 + 1 - 1}{2} = \frac{1}{2}, \quad k \in \mathbb{N}. \quad (5.4)$$

Consider

$$f(x) := \sum_{k>\varepsilon^{-1}} \frac{e^{i\xi_k \cdot x}}{e^{|\xi_k|^{1/2}}} = \sum_{k>\varepsilon^{-1}} \frac{e^{i\xi_k \cdot x - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}}, \quad x \in \mathbb{R}^n. \quad (5.5)$$

Then, for every multi-index  $\alpha$  and every  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} |\partial^\alpha f(x)| &= \left| \sum_{k>\varepsilon^{-1}} \frac{(i\xi_k)^\alpha e^{i\xi_k \cdot x}}{e^{|\xi_k|^{1/2}}} \right| \leq \sum_{k>\varepsilon^{-1}} \frac{(|\xi_k| + 1)^{|\alpha|} e^{|\eta_k||x|}}{e^{|\xi_k|^{1/2}}} \\ &\leq e^{\varepsilon|x|} \sum_{k>\varepsilon^{-1}} \frac{(|\xi_k| + 1)^{|\alpha|}}{e^{|\xi_k|^{1/2}}} =: C_\alpha e^{\varepsilon|x|}, \end{aligned}$$

see (5.3). Further,

$$m(D)f(x) = \sum_{k>\varepsilon^{-1}} \frac{m(\zeta_k) e^{i\xi_k \cdot x}}{e^{|\xi_k|^{1/2}}} = 0.$$

On the other hand,  $f$  is not real-analytic. Before we prove this, note that formally putting  $x - it\omega_0$ ,  $t > 0$  in place of  $x$  in the right-hand side of (5.5), one gets a divergent series. Indeed, its terms can be estimated as follows:

$$\left| \frac{e^{i\xi_k \cdot x + t\xi_k \cdot \omega_0 - \eta_k \cdot x + it\eta_k \cdot \omega_0}}{e^{|\xi_k|^{1/2}}} \right| = \frac{e^{t|\xi_k||\omega_k \cdot \omega_0 - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}} \geq e^{-\varepsilon|x|} \frac{e^{t|\xi_k|/2}}{e^{|\xi_k|^{1/2}}} \rightarrow \infty$$

as  $k \rightarrow \infty$ , see (5.3), (5.4).

For any  $j > \varepsilon^{-1}$ , there exists  $\ell_j \in \mathbb{N}$  such that

$$\ell_j \leq |\xi_j|^{1/2} < \ell_j + 1. \quad (5.6)$$

It is clear that  $\ell_j \rightarrow \infty$  as  $j \rightarrow \infty$ , see (5.3). Note that

$$|\arg(\omega_0 \cdot \zeta_k)| \leq \frac{|\omega_0 \cdot \eta_k|}{|\omega_0 \cdot \xi_k|} \leq \frac{2}{k|\xi_k|}.$$

If  $|\xi_k| \geq 6\ell_j/\pi k$ , then

$$|\arg(\omega_0 \cdot \zeta_k)^{\ell_j}| \leq \frac{2\ell_j}{k|\xi_k|} \leq \frac{\pi}{3},$$

and

$$\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j} \geq \frac{1}{2} |\omega_0 \cdot \zeta_k|^{\ell_j} \geq \frac{1}{2^{\ell_j+1}} |\xi_k|^{\ell_j}.$$

Clearly,  $|\xi_j| \geq \frac{6\ell_j}{\pi j}$  for sufficiently large  $j$ , see (5.6). Hence, one has the following estimate for the directional derivative  $\partial_{\omega_0}$ :

$$\begin{aligned}
 |((-i\partial_{\omega_0})^{\ell_j} f)(0)| &\geq \sum_{k>\varepsilon^{-1}} \frac{\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j}}{e^{|\xi_k|^{1/2}}} \\
 &\geq - \sum_{k>\varepsilon^{-1}, |\xi_k| < \frac{6\ell_j}{\pi k}} \frac{|\zeta_k|^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \sum_{k>\varepsilon^{-1}, |\xi_k| \geq \frac{6\ell_j}{\pi k}} \frac{|\xi_k|^{\ell_j}}{2^{\ell_j+1} e^{|\xi_k|^{1/2}}} \\
 &\geq - \sum_{k>\varepsilon^{-1}, |\xi_k| < \frac{6\ell_j}{\pi k}} \frac{(|\xi_k| + \frac{1}{k})^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \frac{|\xi_j|^{\ell_j}}{2^{\ell_j+1} e^{|\xi_j|^{1/2}}} \\
 &\geq - \sum_{k>\varepsilon^{-1}, |\xi_k| < \frac{6\ell_j}{\pi k}} \frac{1}{e^{|\xi_k|^{1/2}}} \left(\frac{10\ell_j}{\pi k}\right)^{\ell_j} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1} e^{(\ell_j^2+1)^{1/2}}} \\
 &\geq -(10\ell_j)^{\ell_j} \sum_{k=1}^{\infty} \frac{1}{e^{|\xi_k|^{1/2}} k^2} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1} e^{\ell_j+1}} \\
 &= -C(10\ell_j)^{\ell_j} + (2e)^{-(\ell_j+1)} \ell_j^{2\ell_j}.
 \end{aligned}$$

Hence,

$$|((-i\partial_{\omega_0})^{\ell_j} f)(0)| \geq \ell_j^{\frac{3}{2\ell_j}}$$

for all sufficiently large  $j$ , which means that  $f$  is not real-analytic in a neighbourhood of 0.

The operator  $m(D)$  in the previous example is not hypoelliptic. If  $m(D)$  is hypoelliptic, then every solution of  $m(D)f = 0$ , such that  $|f(x)| \leq Ae^{a|x|}$ ,  $x \in \mathbb{R}^n$ , for some constants  $A, a > 0$ , admits analytic continuation to an entire function of order one on  $\mathbb{C}^n$ , see [11, Section 4, Corollary 2]. For elliptic operators, this result can be strengthened: every solution of  $m(D)f = 0$ , such that  $|f(x)| \leq Ae^{a|x|^\beta}$ ,  $x \in \mathbb{R}^n$ , for  $\beta \geq 1$  and some constants  $A, a > 0$ , admits analytic continuation to an entire function of order  $\beta$  on  $\mathbb{C}^n$ , see [11, Section 4, Corollary 3]. Let us show that for every  $\beta > 1$  there exists a semi-elliptic operator  $m(D)$ , see [16, Theorem 11.1.11], and a  $C^\infty$  solution of  $m(D)f = 0$ , all of whose derivatives have growth  $O(e^{a|x|^\beta})$ , but which does not admit analytic continuation to an entire function on  $\mathbb{C}^n$ .

A simple example of such a semi-elliptic operator is  $\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}$  with  $\ell \in \mathbb{N}$  satisfying  $1 + \frac{1}{2\ell} \leq \beta$ , i.e.,  $\ell \geq \frac{1}{2(\beta-1)}$ .

Let

$$f(x_1, x_2) := \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}x_1+kx_2}}{e^{k^{2\ell+1}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

If  $x_2 > 0$ , then the function  $t \mapsto tx_2 - t^{2\ell+1}$  achieves a maximum at  $t = (\frac{x_2}{2\ell+1})^{\frac{1}{2\ell}}$ , and this maximum is equal to

$$2\ell \left( \frac{1}{2\ell+1} \right)^{1+\frac{1}{2\ell}} x_2^{1+\frac{1}{2\ell}} =: c_\ell x_2^{1+\frac{1}{2\ell}}.$$

Hence, for every multi-index  $\alpha$ ,

$$\begin{aligned} |\partial^\alpha f(x_1, x_2)| &\leq \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{kx_2-k^{2\ell+1}} \\ &= \sum_{k=1}^{\lfloor x_2^{\frac{1}{2\ell}} \rfloor + 1} k^{(2\ell+1)|\alpha|} e^{kx_2-k^{2\ell+1}} + \sum_{k=\lfloor x_2^{\frac{1}{2\ell}} \rfloor + 2}^{\infty} k^{(2\ell+1)|\alpha|} e^{k(x_2-k^{2\ell})} \\ &\leq (\lfloor x_2^{\frac{1}{2\ell}} \rfloor + 1)^{(2\ell+1)|\alpha|+1} e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{-k} \\ &\leq 2^{(2\ell+1)|\alpha|+1} (x_2^{2|\alpha|+1} + 1) e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + c_{\ell, \alpha} \leq C_{\ell, \alpha} e^{(c_\ell+1)x_2^{1+\frac{1}{2\ell}}}. \end{aligned}$$

If  $x_2 \leq 0$ , then

$$|\partial^\alpha f(x_1, x_2)| \leq \sum_{k=1}^{\infty} \frac{k^{(2\ell+1)|\alpha|}}{e^{k^{2\ell+1}}} < \sum_{j=1}^{\infty} \frac{j^{|\alpha|}}{e^j} =: C_\alpha < \infty.$$

So,  $f \in C^\infty(\mathbb{R}^2)$ , and  $\partial^\alpha f(x_1, x_2) = O(e^{(c_\ell+1)|x_2|^{1+\frac{1}{2\ell}}}) = O(e^{(c_\ell+1)|x|^{1+\frac{1}{2\ell}}})$ . It is easy to see that  $(\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2})f(x_1, x_2) = 0$ .

The function  $f$  admits analytic continuation to the set

$$\Pi_1 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im } z_1 < 1\}.$$

Indeed, let

$$\begin{aligned} f(z_1, z_2) = f(x_1 + iy_1, x_2 + iy_2) &= \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}(x_1+iy_1)+k(x_2+iy_2)}}{e^{k^{2\ell+1}}} \\ &= \sum_{k=1}^{\infty} e^{i(ky_2-k^{2\ell+1}x_1)} e^{k^{2\ell+1}(y_1-1)+kx_2}. \end{aligned}$$

It is easy to see that the last series is uniformly convergent on compact subsets of  $\Pi_1$ . So,  $f$  admits analytic continuation to  $\Pi_1$ . On the other hand,  $f(iy_1, 0) \rightarrow \infty$  as  $y_1 \rightarrow 1 - 0$ . Indeed,

$$f(iy_1, 0) = \sum_{k=1}^{\infty} e^{k^{2\ell+1}(y_1-1)}.$$

Take any  $N \in \mathbb{N}$ . If  $y_1 > 1 - N^{-(2\ell+1)}$ , then

$$f(iy_1, 0) > \sum_{k=1}^{\infty} e^{-k^{2\ell+1}N^{-(2\ell+1)}} > \sum_{k=1}^N e^{-k^{2\ell+1}N^{-(2\ell+1)}} \geq \sum_{k=1}^N e^{-1} = \frac{N}{e}.$$

So,  $f(iy_1, 0) \rightarrow \infty$  as  $y_1 \rightarrow 1 - 0$ .

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