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Cubic decomposition of a Laguerre–Hahn linear functional I

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Abstract. The aim of this contribution is to study orthogonal polynomials via cubic decomposition in the framework of the Laguerre–Hahn class. We consider two monic orthogonal polynomial sequences $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, and we let w and u be, respectively, the corresponding regular linear functionals such that $W_{3n}(x) = P_n(x^3)$, $n \geq 0$. We prove that if either w or u is a Laguerre–Hahn linear functional, then so is the other one. Based on this result, we deduce a complete analysis of the class s of the Laguerre–Hahn linear functional w. More precisely, we show that $3s' \leq s \leq 3s' + 6$, where s' is the class of u. An illustrative example of class 1 is analyzed.

1. Introduction

Laguerre–Hahn orthogonal polynomials are related to Stieltjes functions, S, that satisfy a Riccati differential equation with polynomial coefficients [17, 30, 34]

$$\Phi S' = BS^2 + CS + D, \quad \Phi \neq 0. \tag{1.1}$$

To define the so-called Laguerre–Hahn linear functionals, the authors in [17, 20, 30] gave a detailed formalism of the necessary operations, along with the suitable topological framework. More precisely, in [17, Theorem 3.1], the equivalence between the Riccati differential equation (1.1) for the Stieltjes function $S(z) = -\sum_{n\geq 0}(w)_n/z^{n+1}$, where $(w)_n$ denotes the nth moment of the corresponding linear functional w, and the distributional equation

$$(\Phi w)' + \Psi w + B(x^{-1}w^2) = 0, \quad \Psi = -\Phi' - C. \tag{1.2}$$

is given. Moreover, if w is a Laguerre–Hahn linear functional, then the class of w, denoted by s, is defined as

$$s := \min(\max(\deg \Psi - 1, \max(\deg \Phi, \deg B) - 2)),$$

where the minimum is taken over all triplets (Φ, Ψ, B) such that (1.2) holds.

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Looking back at the rich archive of orthogonal polynomials (OP), from a structural and constructive point of view, the Laguerre–Hahn polynomials are indeed one of the very remarkable families of OP, since most of the monic orthogonal polynomial sequences (MOPS) considered in the literature belong to this family. Namely, either one of the equations (1.1) or (1.2) can be simplified to become semiclassical. Indeed, if $B = 0$, then semiclassical case appears. However, the fact that B is not identically zero can also yield the semiclassical case, which is the case with the Stieltjes functions related to second-degree linear functionals [31]. In a more general context, the same thing can occur with the third-degree class. Indeed, a third degree linear functional belongs to the Laguerre–Hahn class [9], but the converse is not true in general.

As far as it concerns the techniques used to study the Laguerre–Hahn orthogonal polynomials, there is actually a variety of them: the modifications of linear functionals and the analysis of the corresponding perturbations on the sequences of orthogonal polynomials, to name a few. Another technique deals with the problem of classification of families of orthogonal polynomials in terms of classes of differential equations (1.1) whose goal was to describe the systems of difference equations for the recurrence relation coefficients of the corresponding sequence of orthogonal polynomials, the so-called Laguerre–Freud equations [4, 5, 18, 23, 28]. Some of them have been studied in the framework of discrete Painlevé equations (see [21, 22, 39]).

In general, the problem of determining in an explicit way the Laguerre–Hahn linear functional becomes a very difficult task when the class is greater than or equal to one. We should point out that several classes of Laguerre–Hahn linear functionals have been described: the class $s = 0$ [10, 11]; the symmetric class $s = 1$ [1, 3]; the symmetric class $s = 2$ when Φ and B vanish at zero [36]. Some of the families of Laguerre–Hahn MOPSs were also unfolded by using some processes or by solving some algebraic equations in the dual space of polynomials [2, 6, 9, 12, 13, 16, 19, 35]. One of these processes is the quadratic decomposition [29, 33].

On the other hand, a very important topic of research, often encountered in the literature of orthogonal polynomials, deals with the so-called cubic decomposition [32]. Nevertheless, questions related to cubic decompositions of orthogonal polynomial sequences satisfying some extra conditions as their Laguerre–Hahn character have not been considered in the literature up to the recent contributions [7, 14, 37, 38] for particular cases of semiclassical, second degree and third degree linear functionals of class one and two, respectively, and [27] for Laguerre–Hahn linear functionals of class $s = 1$.

Our work is focused on presenting a generator system of the set of Laguerre–Hahn linear functionals in a way that allows us to answer the following questions.

Consider two sequences of monic orthogonal polynomials

$$\{W_n\}_{n\geq 0} \quad \textit{and} \quad \{P_n\}_{n\geq 0},$$

let w and u be, respectively, the corresponding regular linear functionals such that

$$W_{3n}(x) = P_n(x^3), \quad n \geq 0. \tag{1.3}$$

Assuming that either w or u is a Laguerre–Hahn linear functional, the following questions can be posed.

(i) Can the same be said about the remaining one?

(ii) If so, Can a connection between their classes be stated?

In this direction, we mention the work in [14], where the authors proved that w is a semiclassical linear functional if and only if the linear functional u is a semiclassical linear functional. In addition, in [7] the authors prove that w is a second degree (resp., third degree) linear functional if and only if the first component $u = \sigma_{\varpi}(w)$ is a second degree (resp., third degree) linear functional. On the other hand, based either on spectral perturbations of the linear form [8] or on a cubic decomposition of the corresponding sequences of orthogonal polynomials (see [24–26], among others), a constructive approach to some families of TDRF is presented therein.

The main purpose of this paper is fully answering the previously raised questions which in their turn, constitute a generalization of all the results mentioned above. The paper is organized as follows. In Section 2, we review some basic tools concerning the general theory of OP's, focusing our attention on the cubic decomposition (CD) and on the theory of Laguerre–Hahn MOPS's. In Section 3, we deal with the stability, i.e., the preservation of the Laguerre–Hahn character. Indeed, if one of the two linear functionals w and u, such that $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ are, respectively, the corresponding sequences of orthogonal polynomials which are related by the cubic decomposition (1.3), is a Laguerre–Hahn linear functional then it is the same for the other one. In Section 4, a complete analysis of the class s of the Laguerre–Hahn linear functional w in terms of the class s' of the linear functional $u = \sigma_{\varpi}(w)$ is done. In particular, we show that $3s' \leq s \leq 3s' + 6$ and, more precisely, an accurate description of all the possible situations is given. Finally, in Section 5 we provide new examples of Laguerre–Hahn linear functionals of class 1. This is done by analyzing the case when $u = \sigma_{\varpi}(w)$, the first component of linear functional w in the cubic decomposition, is a singular Laguerre–Hahn linear functional of class zero.

2. Background

In this section, we summarize some basic ideas concerning Laguerre–Hahn OPS's, but first let us recall some basic tools about algebraic (topological) aspects in the theory of OP's and the cubic decomposition of sequences of orthogonal polynomials (CD, in short).

2.1. Basic tools

First of all, let us recall some basic notations from [30] that we will use throughout this paper. Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients, and let $\mathcal{P}'$ be its algebraic dual. The elements of $\mathcal{P}'$ will be called linear functionals (linear forms). By $\langle \cdot, \cdot \rangle$ we denote the duality brackets between $\mathcal{P}$ and $\mathcal{P}'$.

Let us introduce some useful operations in $\mathcal{P}'$. For any linear functional w, any polynomial g and any $(a,b,c) \in (\mathbb{C} - \{0\}) \times \mathbb{C}^2$, let w', gw, $h_a w$, $\tau_b w$, $(x-c)^{-1}w$ and δ_c be the forms defined by duality as

$$\begin{aligned}
\langle w', f\rangle &:= -\langle w, f'\rangle, \quad \langle gw, f\rangle := \langle w, gf\rangle, \quad f, g \in \mathcal{P},\\
\langle h_a w, f\rangle &:= \langle w, h_a f\rangle = \langle w, f(ax)\rangle,\\
\langle \tau_b w, f\rangle &:= \langle w, \tau_{-b} f\rangle = \langle w, f(x+b)\rangle, \quad f \in \mathcal{P},\\
\langle (x-c)^{-1} w, f\rangle &:= \langle w, \theta_c f\rangle = \left\langle w, \frac{f(x)-f(c)}{x-c}\right\rangle,\\
\langle \delta_c, f\rangle &:= f(c)\ (\delta_0 = \delta), \quad f \in \mathcal{P}.
\end{aligned}$$

For $f \in \mathcal{P}$ and $w \in \mathcal{P}'$, the product wf is the polynomial [30]

$$(wf)(x) := \left\langle w, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\right\rangle = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} (w)_{j-i} a_j \right) x^i, \tag{2.1}$$

where

$$f(x) = \sum_{i=0}^{n} a_i x^i.$$

This allows us to define the Cauchy product of two linear functionals v and w as follows:

$$\langle vw, f\rangle := \langle v, wf\rangle, \quad w \in \mathcal{P}',\ f \in \mathcal{P}.$$

In particular, the moments of the above Cauchy product are

$$(vw)_n = \sum_{k=0}^{n} (v)_k (w)_{n-k}, \quad n \geq 0. \tag{2.2}$$

The above product is commutative, associative and distributive with respect to the sum of linear functionals.

Thus, we have the well-known formulas [30]

$$\begin{aligned}
x(x^{-1}w) &= w, \quad x^{-1}(xw) = w - \delta,\\
x^{-(n+2)}w &= x^{-1}(x^{-(n+1)}w), \quad w \in \mathcal{P}',\ n \geq 0, &(2.3)\\
x^{-1}(vw) &= (x^{-1}v)w = v(x^{-1}w), \quad v, w \in \mathcal{P}', &(2.4)
\end{aligned}$$

where δ is the unit element for Cauchy product of two linear functionals, i.e., $\langle \delta, p(x)\rangle = p(0)$, $p \in \mathcal{P}$.

Now, we define the operator $\sigma_\varpi : \mathcal{P}' \to \mathcal{P}'$ by

$$\langle \sigma_\varpi(w), f\rangle := \langle w, \sigma_\varpi(f)\rangle, \quad w \in \mathcal{P}',\ f \in \mathcal{P}, \tag{2.5}$$

where the linear operator $\sigma_\varpi : \mathcal{P} \to \mathcal{P}$ is defined by

$$\sigma_\varpi(f)(x) := f(x^3)$$

for every $f \in \mathcal{P}$.

For any $f \in \mathcal{P}$ and $w \in \mathcal{P}'$, the following properties hold [37]:

$$f\sigma_{\varpi}(w) = \sigma_{\varpi}((f \circ \varpi)w), \tag{2.6}$$

$$\sigma_{\varpi}(w') = (\sigma_{\varpi}(\varpi' w))'. \tag{2.7}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$. It is defined by [30]

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}.$$

In what follows, we will call polynomial sequence PS for any sequence $\{W_n\}_{n\geq 0}$ such that $\deg W_n = n$, $n \geq 0$. We will also call monic polynomial sequence MPS for any PS such that all polynomials have a leading coefficient equal to one.

The linear functional w is called regular (or quasi-definite) if there exists a MPS $\{W_n\}_{n\geq 0}$ such that [15]

$$\langle w, W_n W_m \rangle = r_n \delta_{n,m}, \quad n \geq 0,$$

where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is Kronecker symbol.

The sequence $\{W_n\}_{n\geq 0}$ is then said to be orthogonal with respect to w. Henceforth, a monic orthogonal polynomial sequence $\{W_n\}_{n\geq 0}$ will be indicated as MOPS. It is well known (see [15]) that an MOPS is characterized by a three-term recurrence relation of the form

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1} W_n(x), \quad n \geq 0, \tag{2.8}$$

with initial conditions $W_0(x) = 1$ and $W_1(x) = x - \beta_0$, being $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ sequences of complex numbers such that $\gamma_{n+1} \neq 0$ for all $n \geq 0$.

2.2. Laguerre–Hahn linear functionals

Now, let us recall some features about the Laguerre–Hahn linear functionals.

Definition 2.1 ([3, 11, 34]). A linear functional w is said to be of Laguerre–Hahn class if its Stieltjes function satisfies a Riccati equation

$$\Phi(z)S'(w)(z) = B(z)S^2(w)(z) + C(z)S(w)(z) + D(z), \tag{2.9}$$

where $\Phi(z) \neq 0$, $B(z)$, $C(z)$ are polynomials with

$$D(z) = -(w\theta_0\Phi)'(z) - (w\theta_0\Psi)(z) - (w^2\theta_0^2 B)(z). \tag{2.10}$$

Remark 2.1. In particular, if $B = 0$, then the linear functional is said to be Laguerre–Hahn affine or semiclassical.

Proposition 2.1 ([3, 11, 34]). *Let w be a quasi-definite and normalized linear functional, i.e., $(w)_0 = 1$, and let $\{W_n\}_{n\geq 0}$ be its corresponding MOPS. The following statements are equivalent.*

(i) *w is a Laguerre–Hahn functional.*

(ii) *w satisfies the functional equation*

$$(\Phi w)' + \Psi w + B(x^{-1}w^2) = 0, \tag{2.11}$$

where $\Phi(x)$, $B(x)$, $C(x)$ are the polynomials in (2.9) and

$$\Psi(x) = -[\Phi'(x) + C(x)].$$

Notice that the above equation is not unique. Indeed, if w is Laguerre–Hahn and χ is an arbitrary polynomial, then w also satisfies the functional equation

$$(\chi\Phi w)' + (\chi\Psi - \chi'\phi)w + (\chi B)(x^{-1}w^2) = 0.$$

With this in mind, we give the following definition.

Definition 2.2 ([3,11,34]). The class of a Laguerre–Hahn functional w is the non-negative integer number defined as

$$s := \min\max\left\{\deg\Psi(x) - 1, \max\{\deg\Phi(x) - 1, \deg B(x) - 1\} - 2\right\},$$

where the minimum is taken among all polynomials $\Phi(x)$, $\Psi(x)$, and $B(x)$ such that w satisfies (2.11).

Taking into account that the class of a Laguerre–Hahn linear functional is very useful in order to state a hierarchy of such families, we need to give a simple way to characterize it.

Proposition 2.2 ([3, 34]). *Let w be a Laguerre–Hahn linear functional and let $\Phi(x)$ and $\Psi(x)$ be non-zero polynomials with $\deg\Phi(x) =: r$, $\deg\Psi(x) =: t$ and $\deg B(x) =: m$, such that (2.11) holds. Let $s = \max(t-1, d-2)$ with $d = \max\{r, m\}$. Then, s is the class of w if and only if*

$$\prod_{c\in\mathcal{Z}_\Phi}\left(|\Phi'(c) + \Psi(c)| + |B(c)| + |\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle|\right) \neq 0, \tag{2.12}$$

where $\mathcal{Z}_\Phi$ denotes the set of zeros of Φ.

With regard to the latter proposition, to show that whether or not, can the functional equation (2.11) can be simplified by $x - c$, where c is a zero of Φ, one must find $\Phi'(c) + \Psi(c)$, $B(c)$ and $\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle$. The computational work will indeed become more delicate due to the recurrence of the simplification process. As a matter of fact, the following lemma elucidates the simplification procedures.

Lemma 2.1 ([13]). *Let $c_1 \in \mathbb{C}$ be a zero of Φ such that*

$$|\Phi'(c_1) + \Psi(c_1)| + |B(c_1)| + |\langle w, \theta_{c_1}^2\Phi + \theta_{c_1}\Psi + w\theta_0\theta_{c_1}B\rangle| = 0.$$

Then, (2.11) can be simplified dividing by $x - c_1$ and it becomes

$$(\Phi_1 w)' + \Psi_1 w + B_1(x^{-1}w^2) = 0,$$

where

$$\Phi_1 = \theta_{c_1}\Phi, \quad \Psi_1 = \theta_{c_1}^2\Phi + \theta_{c_1}\Psi, \quad B_1 = \theta_{c_1}B.$$

Moreover, for all $c \in \mathbb{C}$, *we have*

$$(\Phi_1)'(c) + \Psi_1(c) = \frac{\Phi'(c) + \Psi(c)}{c - c_1}, \tag{2.13}$$

$$\langle w, \theta_c\Psi_1 + \theta_c^2\Phi_1 + w\theta_0\theta_c B_1\rangle = \frac{\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle}{c - c_1}. \tag{2.14}$$

From Proposition 2.2, there is an alternative way to find the class in terms of the polynomials involved in the Riccati equation (2.9). Indeed, we have the following corollary.

Corollary 2.1 ([3, 34]). *Let* w *be a Laguerre–Hahn functional satisfying* (2.9) *such that* $\deg \Phi(x) = r$, $\deg B(x) = m$ *and* $\deg \Psi(x) = t$ *with* $\Psi(x) = -[\Phi'(x) + C(x)]$. *Let* $s = \max(t-1, d-2)$ *with* $d = \max\{r, m\}$. *Then,* s *is the class of* w *if and only if the polynomials* $\Phi(x)$, $B(x)$, $C(x)$ *and* $D(x)$ *are coprime or, equivalently,*

$$\prod_{c \in \mathcal{Z}_\Phi} (|B(c)| + |C(c)| + |D(c)|) \neq 0.$$

Next, the concept of displacement is considered. Given $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, if a linear functional w of class s satisfies (2.11), then the linear functional $\hat{w} = (h_{a^{-1}} \circ \tau_{-b})w$ is of class s, and it satisfies

$$(\hat{\Phi}\hat{w})' + \hat{\Psi}\hat{w} + \hat{B}(x^{-1}\hat{w}^2) = 0, \tag{2.15}$$

where

$$\hat{\Phi}(x) = a^{-\deg\Phi}\Phi(ax+b), \quad \hat{\Psi}(x) = a^{1-\deg\Phi}\Psi(ax+b), \quad \hat{B}(x) = a^{-\deg\Phi}B(ax+b). \tag{2.16}$$

Hence, a displacement does not change neither the Laguerre–Hahn character nor the class of a Laguerre–Hahn linear functional [30]. Therefore, we can take canonical functional equations by re-situating the zeros of Φ in equation (2.11). This will be put in evidence in the sequel.

2.3. Cubic decomposition

In what follows, we are concerned with the following cubic decomposition (CD, in short) defined in [32]. Let us consider $\varpi(x) = x^3$.

For any MPS $\{W_n\}_{n\geq 0}$ there are three MPSs, $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, so that

$$W_{3n}(x) = P_n(x^3) + xa_{n-1}^1(x^3) + x^2a_{n-1}^2(x^3), \quad n \geq 0, \tag{2.17}$$

$$W_{3n+1}(x) = b_n^1(x^3) + xQ_n(x^3) + x^2b_{n-1}^2(x^3), \quad n \geq 0, \tag{2.18}$$

$$W_{3n+2}(x) = c_n^1(x^3) + xc_n^2(x^3) + x^2R_n(x^3), \quad n \geq 0, \tag{2.19}$$

with $\deg a^1_{n-1} \le n-1$, $\deg a^2_{n-1} \le n-1$, $\deg b^1_n \le n$, $\deg b^2_{n-1} \le n-1$, $\deg c^1_n \le n$ $\deg c^2_n \le n$ and $a^1_{-1}(x) = a^2_{-1}(x) = b^2_{-1}(x) = 0$. This is a particular case of the general cubic decomposition of any MPS presented in [32], where all the parameters involved are equal to zero. In this cubic decomposition (2.17)–(2.19) of $\{W_n\}_{n\ge 0}$, the sequences: $\{P_n\}_{n\ge 0}$, $\{Q_n\}_{n\ge 0}$, $\{R_n\}_{n\ge 0}$ are called the principal components; $\{a^1_{n-1}\}_{n\ge 0}$, $\{a^2_{n-1}\}_{n\ge 0}$, $\{b^1_n\}_{n\ge 0}$, $\{b^2_{n-1}\}_{n\ge 0}$, $\{c^1_n\}_{n\ge 0}$, $\{c^2_n\}_{n\ge 0}$ are called the secondary components, since they are sequences of polynomials, although not necessarily bases for the vector space of polynomials $\mathcal{P}$.

The next result is a particular scenery of [32, Theorem 5.4] characterizing the orthogonality case such that $W_{3n}(x) = P_n(\varpi(x))$, $n \ge 0$, with $\varpi(x) = x^3$ (i.e., the two secondary components $\{a^1_n\}_{n\ge 0}$ and $\{a^2_n\}_{n\ge 0}$ vanish).

Proposition 2.3 ([32]). *Let $\{W_n\}_{n\ge 0}$ be a MOPS with respect to the linear functional w defined by (2.17)–(2.19). The following statements are equivalent.*

(a) $a^1_n = a^2_n = 0$, $n \ge 0$.

(b) *The recurrence coefficients of $\{W_n\}_{n\ge 0}$ satisfy*

$$\beta_{3n} = \beta_0, \quad \beta_{3n+1} + \beta_{3n+2} = -\beta_0, \quad n \ge 0,$$
$$\gamma_{3n+2} = -\gamma_1 - \beta_0(\beta_{3n+1} + \beta_0) - \beta^2_{3n+1}, \quad n \ge 0,$$
$$\gamma_{3n} + \gamma_{3n+1} = \gamma_1, \quad n \ge 0, \quad \text{with } \gamma_0 = 0.$$

Moreover, $\{P_n\}_{n\ge 0}$ is orthogonal with respect to the linear functional $u = \sigma_\varpi(w)$.

Let us conclude this subsection with the following result that will be required in the sequel.

Proposition 2.4 ([37]). *Let $\{W_n\}_{n\ge 0}$ be a MOPS with respect to the linear functional w defined by (2.17)–(2.19) and such that $a^1_n = a^2_n = 0$, $n \ge 0$. Then,*

$$\sigma_\varpi((x - \beta_0)w) = 0, \tag{2.20}$$
$$\sigma_\varpi((x^2 - \gamma_1 - \beta_0^2)w) = 0. \tag{2.21}$$

Furthermore, the formal Stieltjes functions $S(w)$ and $S(\sigma_\varpi(w))$ associated with the linear functionals w and $u = \sigma_\varpi(w)$, respectively, are related by [7]

$$S(w)(z) = \rho(z) S(\sigma_\varpi(w))(z^3), \tag{2.22}$$

where $\rho(z) = z^2 + \beta_0 z + \gamma_1 + \beta_0^2$.

3. Stability of Laguerre–Hahn character via cubic decomposition

In this section, we deal with a stability problem, i.e., we show that w is a Laguerre–Hahn linear functional if and only if the first component $u = \sigma_\varpi(w)$ is also a Laguerre–Hahn linear functional. We start by stating some preliminary lemmas.

After that, we will give some properties related to the operator σ_ϖ which will be needed later.

Lemma 3.1. *Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w defined by (2.17)–(2.19) and such that $a_n^1 = a_n^2 = 0$, $n \geq 0$. For any $f \in \mathcal{P}$, we have*

$$\sigma_{\varpi}(xw) = \beta_0 \sigma_{\varpi}(w), \tag{3.1}$$

$$\sigma_{\varpi}(x^2 w) = (\gamma_1 + \beta_0^2)\sigma_{\varpi}(w), \tag{3.2}$$

$$\sigma_{\varpi}(f(x^3)w) = f(x)\sigma_{\varpi}(w), \tag{3.3}$$

$$\sigma_{\varpi}(xf(x^3)w) = \beta_0 f(x)\sigma_{\varpi}(w), \tag{3.4}$$

$$\sigma_{\varpi}(x^2 f(x^3)w) = (\gamma_1 + \beta_0^2) f(x)\sigma_{\varpi}(w). \tag{3.5}$$

Proof. Equations (3.1) and (3.2) follow immediately from (2.20) and (2.21). Equation (3.3) was stated in [37, Lemma 2.1]. Equations (3.4) and (3.5) can be computed similarly taking into account (3.1) and (3.2). ∎

Relying on Lemma 3.1, we prove the following results.

Lemma 3.2. *Under the hypotheses of Lemma 3.1, the following formulas hold:*

$$\sigma_{\varpi}(w^2) = (\sigma_{\varpi}(w))^2 + 2x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2 w), \tag{3.6}$$

$$\sigma_{\varpi}(xw^2) = 2\sigma_{\varpi}(w)\sigma_{\varpi}(xw) + x^{-1}(\sigma_{\varpi}(x^2 w))^2, \tag{3.7}$$

$$\sigma_{\varpi}(x^{-1}w^2) = x^{-1}\big(2\sigma_{\varpi}(w)\sigma_{\varpi}(x^2 w) + (\sigma_{\varpi}(xw))^2\big). \tag{3.8}$$

Proof. For each $n \geq 2$,

$$\begin{aligned}
&\langle \sigma_{\varpi}(w^2), x^n \rangle \\
&\overset{\text{by (2.2)}}{=} \sum_{k=0}^{3n} (w)_k (w)_{3n-k} = \sum_{k=0}^{n} (w)_{3k}(w)_{3n-3k} + \sum_{k=0}^{n-1} (w)_{3k+1}(w)_{3n-3k-1} \\
&\quad + \sum_{k=0}^{n-2} (w)_{3k+2}(w)_{3n-3k-2} \\
&= \sum_{k=0}^{n} (\sigma_{\varpi}(w))_k (\sigma_{\varpi}(w))_{n-k} + \sum_{k=0}^{n-1} (\sigma_{\varpi}(xw))_k (\sigma_{\varpi}(x^2 w))_{n-1-k} \\
&\quad + \sum_{k=0}^{n-2} (\sigma_{\varpi}(x^2 w))_k (\sigma_{\varpi}(x^4 w))_{n-2-k} \\
&\overset{\text{by (2.2)}}{=} \langle (\sigma_{\varpi}(w))^2, x^n \rangle + \langle \sigma_{\varpi}(xw)\sigma_{\varpi}(x^2 w), x^{n-1} \rangle + \langle \sigma_{\varpi}(x^2 w)\sigma_{\varpi}(x^4 w), x^{n-2} \rangle \\
&= \langle (\sigma_{\varpi}(w))^2 + x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2 w) + x^{-2}\sigma_{\varpi}(x^2 w)\sigma_{\varpi}(x^4 w), x^n \rangle \\
&= \langle (\sigma_{\varpi}(w))^2 + x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2 w) + x^{-2}\sigma_{\varpi}(x^2 w)(x\sigma_{\varpi}(xw)), x^n \rangle \\
&\overset{\text{by (2.3)–(2.4)}}{=} \langle (\sigma_{\varpi}(w))^2 + x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2 w) + x^{-1}\sigma_{\varpi}(x^2 w)[\sigma_{\varpi}(xw) - \delta], x^n \rangle \\
&= (\sigma_{\varpi}(w))^2 + 2x^{-1}(\sigma_{\varpi}(xw))(\sigma_{\varpi}(x^2 w)).
\end{aligned}$$

Notice that this equality is also true for $n=0$. Indeed, using (2.2) we obtain

$$\langle(\sigma_{\varpi}(w))^2+2x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2w),1\rangle=1=\langle\sigma_{\varpi}(w^2),1\rangle.$$

Finally, it is easy to check that

$$\langle\sigma_{\varpi}(w^2),x\rangle=2((w)_0(w)_3+(w)_1(w)_2).$$

On the other hand, from (2.2), we easily obtain

$$\langle(\sigma_{\varpi}(w))^2,x\rangle=2(w)_0(w)_3 \quad\text{and}\quad \langle x^{-1}\sigma_{\varpi}(xw)\sigma_{\varpi}(x^2w),x\rangle=2(w)_1(w)_2,$$

which correspond to (3.6) for $n=1$. Thus, we have proved (3.6). Equations (3.7) and (3.8) follow in a similar way. ■

We give now the following lemma for further use in the paper.

Lemma 3.3. *Let $\tilde{u}$ be a linear functional and let $\widetilde{\Phi}$, $\widetilde{\Psi}$ and $\widetilde{B}$ be three polynomials. If we deal with the cubic decompositions*

$$\widetilde{\Phi}(x)=\widetilde{\Phi}_1(x^3)+x\widetilde{\Phi}_2(x^3)+x^2\widetilde{\Phi}_3(x^3), \tag{3.9}$$

$$\widetilde{\Psi}(x)=\widetilde{\Psi}_1(x^3)+x\widetilde{\Psi}_2(x^3)+x^2\widetilde{\Psi}_3(x^3), \tag{3.10}$$

$$\widetilde{B}(x)=\widetilde{B}_1(x^3)+x\widetilde{B}_2(x^3)+x^2\widetilde{B}_3(x^3), \tag{3.11}$$

then one has

$$\begin{aligned}
&\sigma_{\varpi}\big((\widetilde{\Phi}\tilde{u})'+\widetilde{\Psi}\tilde{u}+\widetilde{B}(x^{-1}\tilde{u}^2)\big)\\
&\quad=\big(3x\widetilde{\Phi}_2(x)\sigma_{\varpi}(\tilde{u})\big)'+\widetilde{\Psi}_1(x)\sigma_{\varpi}(\tilde{u})+x\widetilde{B}_2(x)(x^{-1}(\sigma_{\varpi}(\tilde{u}))^2)\\
&\qquad+\big(3x\widetilde{\Phi}_3(x)\sigma_{\varpi}(x\tilde{u})\big)'+\widetilde{\Psi}_2(x)\sigma_{\varpi}(x\tilde{u})+\widetilde{B}_1(x)(x^{-1}(\sigma_{\varpi}(x\tilde{u}))^2)\\
&\qquad+\big(3\widetilde{\Phi}_1(x)\sigma_{\varpi}(x^2\tilde{u})\big)'+\widetilde{\Psi}_3(x)\sigma_{\varpi}(x^2\tilde{u})+\widetilde{B}_3(x)(x^{-1}(\sigma_{\varpi}(x^2\tilde{u}))^2)\\
&\qquad+2\big(\widetilde{B}_1(x)(x^{-1}\sigma_{\varpi}(\tilde{u})\sigma_{\varpi}(x^2\tilde{u}))+\widetilde{B}_2(x)(x^{-1}\sigma_{\varpi}(x\tilde{u})\sigma_{\varpi}(x^2\tilde{u}))\\
&\qquad+x\widetilde{B}_3(x)(x^{-1}\sigma_{\varpi}(\tilde{u})\sigma_{\varpi}(x\tilde{u}))\big).
\end{aligned}$$

Proof. From the linearity of the operator σ_{ϖ}, we have

$$\sigma_{\varpi}\big((\widetilde{\Phi}\tilde{u})'+\widetilde{\Psi}\tilde{u}+\widetilde{B}(x^{-1}\tilde{u}^2)\big)=\sigma_{\varpi}\big((\widetilde{\Phi}\tilde{u})'\big)+\sigma_{\varpi}(\widetilde{\Psi}\tilde{u})+\sigma_{\varpi}\big(\widetilde{B}(x^{-1}\tilde{u}^2)\big).$$

On the one hand, we use (2.7) to obtain

$$\begin{aligned}
\sigma_{\varpi}\big((\widetilde{\Phi}\tilde{u})'\big)&=3\big(\sigma_{\varpi}\big(x^2\widetilde{\Phi}(x)\tilde{u}\big)\big)'\\
&=3\big(\sigma_{\varpi}\big((x^2\widetilde{\Phi}_1(x^3)+x^3\widetilde{\Phi}_2(x^3)+x^4\widetilde{\Phi}_3(x^3))\tilde{u}\big)\big)'\\
&=3\big(\widetilde{\Phi}_1(x)\sigma_{\varpi}(x^2\tilde{u})\big)'+3\big(x\widetilde{\Phi}_2(x)\sigma_{\varpi}(\tilde{u})\big)'+3\big(x\widetilde{\Phi}_3(x)\sigma_{\varpi}(x\tilde{u})\big)'.
\end{aligned}$$

On the other hand, from (3.3) and according to (3.10), we deduce

$$\begin{aligned}\sigma_{\varpi}(\widetilde{\Psi}\tilde{u}) &= \sigma_{\varpi}\big((\widetilde{\Psi}_1(x^3) + x\widetilde{\Psi}_2(x^3) + x^2\widetilde{\Psi}_3(x^3))\tilde{u}\big)\\ &= \widetilde{\Psi}_1(x)\sigma_{\varpi}(\tilde{u}) + \widetilde{\Psi}_2(x)\sigma_{\varpi}(x\tilde{u}) + \widetilde{\Psi}_3(x)\sigma_{\varpi}(x^2\tilde{u}).\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_{\varpi}\big(\widetilde{B}(x^{-1}\tilde{u}^2)\big) &= \sigma_{\varpi}\big(\widetilde{B}_1(x^3)(x^{-1}\tilde{u}^2) + \widetilde{B}_2(x^3)u^2 + x\widetilde{B}_3(x^3)\tilde{u}^2\big)\\ &= \widetilde{B}_1(x)\sigma_{\varpi}(x^{-1}\tilde{u}^2) + \widetilde{B}_2(x)\sigma_{\varpi}(\tilde{u}^2) + \widetilde{B}_3(x)\sigma_{\varpi}(x\tilde{u}^2)\\ &= \widetilde{B}_1(x)(x^{-1}(\sigma_{\varpi}(x\tilde{u}))^2) + x\widetilde{B}_2(x)(x^{-1}(\sigma_{\varpi}(\tilde{u}))^2)\\ &\quad + \widetilde{B}_3(x)(x^{-1}(\sigma_{\varpi}(x^2\tilde{u}))^2)\\ &\quad + 2\big(\widetilde{B}_1(x)(x^{-1}\sigma_{\varpi}(\tilde{u})\sigma_{\varpi}(x^2\tilde{u})) + \widetilde{B}_2(x)(x^{-1}\sigma_{\varpi}(x\tilde{u})\sigma_{\varpi}(x^2\tilde{u}))\\ &\quad + x\widetilde{B}_3(x)(x^{-1}\sigma_{\varpi}(\tilde{u})\sigma_{\varpi}(x\tilde{u}))\big).\end{aligned}$$

Hence, the desired statement follows. ∎

Proposition 3.1. *Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w fulfilling (2.17)–(2.19) with $a_n^1 = a_n^2 = 0$, $n \geq 0$. Let $u = \sigma_{\varpi}(w)$ be the regular functional associated with $\{P_n\}_{n\geq 0}$. If $u = \sigma_{\varpi}(w)$ is a Laguerre–Hahn linear functional of class s', then w is a Laguerre–Hahn linear functional of class $s \leq 3s' + 6$. Furthermore, if u satisfies*

$$(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2) = 0, \tag{3.12}$$

then w satisfies (2.11) with

$$\Phi(x) = \rho(x)\Phi^P(x^3), \tag{3.13}$$

$$\Psi(x) = 3x^2\rho(x)\Psi^P(x^3) - 2\rho'(x)\Phi^P(x^3), \tag{3.14}$$

$$B(x) = 3x^2 B^P(x^3), \tag{3.15}$$

where

$$\rho(x) = x^2 + \beta_0 x + \gamma_1 + \beta_0^2. \tag{3.16}$$

Proof. Set $\tilde{w} := (\Phi w)' + \Psi w + B(x^{-1}w^2)$. To prove that $\tilde{w} = 0$, it is enough to show that $\sigma_{\varpi}(\tilde{w}) = 0$, $\sigma_{\varpi}(x\tilde{w}) = 0$ and $\sigma_{\varpi}(x^2\tilde{w}) = 0$.

The components of the polynomials Φ, Ψ, and B in (3.13)–(3.15) are

$$\begin{aligned}&\Phi_1(x) = (\gamma_1 + \beta_0^2)\Phi^P(x), \quad \Phi_2(x) = \beta_0\Phi^P(x), \quad \Phi_3(x) = \Phi^P(x),\\ &\Psi_1(x) = 3\beta_0 x\Psi^P(x) - 2\beta_0\Phi^P(x), \quad \Psi_2(x) = 3x\Psi^P(x) - 4\Phi^P(x),\\ &\Psi_3(x) = 3(\gamma_1 + \beta_0^2)\Psi^P(x),\\ &B_1(x) = 0, \quad B_2(x) = 0, \quad B_3(x) = 3B^P(x).\end{aligned}$$

Then, from Lemma 3.3, we get

$$\begin{aligned}\sigma_{\varpi}(\tilde{w}) = {} & 3\beta_0(x\Phi^P(x)\sigma_{\varpi}(w))' + (3\beta_0 x\Psi^P(x) - 2\beta_0\Phi^P(x))\sigma_{\varpi}(w) \\ & + 3(x\Phi^P(x)\sigma_{\varpi}(xw))' + (3x\Psi^P(x) - 4\Phi^P(x))\sigma_{\varpi}(xw) \\ & + 6xB^P(x)(x^{-1}\sigma_{\varpi}(w)\sigma_{\varpi}(xw)) \\ & + 3(\gamma_1 + \beta_0^2)(\Phi^P(x)\sigma_{\varpi}(x^2w))' + 3(\gamma_1 + \beta_0^2)\Psi^P(x)\sigma_{\varpi}(x^2w) \\ & + 3B^P(x)(x^{-1}(\sigma_{\varpi}(x^2w))^2).\end{aligned}$$

Taking into account (3.1) and (3.2), we obtain

$$\sigma_{\varpi}(\tilde{w}) = 3(2\beta_0 x + (\gamma_1 + \beta_0^2)^2)[(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2)] = 0. \tag{3.17}$$

Following the same approach used to obtain (3.17), together with some straightforward computations, we get

$$\sigma_{\varpi}(x\tilde{w}) = 3(2\gamma_1 + 3\beta_0^2)x[(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2)] = 0, \tag{3.18}$$

$$\sigma_{\varpi}(x^2\tilde{w}) = 3(x + 2\beta_0(\gamma_1 + \beta_0^2))x[(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2)] = 0. \tag{3.19}$$

Summing up (3.17), (3.18) and (3.19), we get $\tilde{w} = 0$. Hence, w satisfies (2.11) with (3.13), (3.14) and (3.15). As a consequence, w is a Laguerre–Hahn linear functional.

To end the proof, it remains to prove that the class s of w is at most $3s' + 6$. Indeed, if $\deg \Phi = r$, $\deg \Psi = t$, $\deg B = m$, $\deg \Phi^P = r'$, $\deg \Psi^P = t'$, and $\deg B^P = m'$, then from (3.13), (3.14), and (3.15) it follows that $t = 3t' + 2$, $m = 3m' + 2$, and $r \leq \max(3r' + 4, 3t' + 1)$. As a consequence, we have the following cases.

(i) If either $t' = s' + 2$ or $m' = s' + 2$ and $p' \leq s' + 1$, then $t = 3s' + 8$ or $m = 3s' + 8$ and $r \leq 3s' + 7$.

(ii) If $t' \leq s' + 1$, $m' \leq s' + 1$ and $r' = s' + 1$, then $t \leq 3s' + 8$ or $m \leq 3s' + 8$ and $r = 3s' + 7$.

One than can deduce that, in any case, $s \leq 3s' + 6$, which completes the proof of the proposition. ■

Remark 3.1. Note that the above proposition provides only an upper bound to the class s of the linear functional w. In the sequel, a thorough investigation of the class of the linear functional w in terms of the class of the linear functional $u = \sigma_{\varpi}(w)$ will be carried out.

The following proposition states a result which is, in fact, the converse of the previous one. Indeed, we consider the Laguerre–Hahn linear functional w and we show that the first component $u = \sigma_{\varpi}(w)$ of the cubic decomposition is also a Laguerre–Hahn linear functional.

Proposition 3.2. *Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w fulfilling (2.17)–(2.19) with $a_n^1 = a_n^2 = 0$, $n \geq 0$. Let $u = \sigma_{\varpi}(w)$ be the regular functional* *associated with* $\{P_n\}_{n\geq 0}$. *If* w *is a Laguerre–Hahn linear functional satisfying* (2.11), *then* $u = \sigma_{\varpi}(w)$ *is a Laguerre–Hahn linear functional and satisfies*

$$(\Phi_k^P u)' + \Psi_k^P u + B_k^P(x^{-1}u^2) = 0, \quad k \in \{1,2,3\}, \tag{3.20}$$

where

$$\begin{aligned}
\Phi_1^P(x) &= 3((\gamma_1+\beta_0^2)\Phi_1(x) + x\Phi_2(x) + \beta_0 x\Phi_3(x)),\\
\Psi_1^P(x) &= \Psi_1(x) + \beta_0\Psi_2(x) + (\gamma_1+\beta_0^2)\Psi_3(x),\\
B_1^P(x) &= (2\gamma_1+3\beta_0^2)B_1(x) + (x+2\beta_0(\gamma_1+\beta_0^2))B_2(x) + (2\beta_0 x + (\gamma_1+\beta_0^2)^2)B_3(x)
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\Phi_2^P(x) &= 3((\gamma_1+\beta_0^2)x\Phi_3(x) + x\Phi_1(x) + \beta_0 x\Phi_2(x)),\\
\Psi_2^P(x) &= x\Psi_3(x) - \Phi_1(x) + \beta_0(\Psi_1(x) - \Phi_2(x)) + (\gamma_1+\beta_0^2)(\Psi_2(x) - \Phi_3(x)),\\
B_2^P(x) &= (2\gamma_1+3\beta_0^2)xB_3(x) + (x+2\beta_0(\gamma_1+\beta_0^2))B_1(x) + (2\beta_0 x + (\gamma_1+\beta_0^2)^2)B_2(x),
\end{aligned} \tag{3.22}$$

and finally,

$$\begin{aligned}
\Phi_3^P(x) &= 3((\gamma_1+\beta_0^2)x\Phi_2(x) + x^2\Phi_3(x) + \beta_0 x\Phi_1(x)),\\
\Psi_3^P(x) &= x\Psi_2(x) - 2x\Phi_3(x) + \beta_0(x\Psi_3(x) - 2\Phi_1(x)) + (\gamma_1+\beta_0^2)(\Psi_1(x) - 2\Phi_2(x)),\\
B_3^P(x) &= (2\gamma_1+3\beta_0^2)xB_2(x) + (x+2\beta_0(\gamma_1+\beta_0^2))xB_3(x) + (2\beta_0 x + (\gamma_1+\beta_0^2)^2)B_1(x).
\end{aligned} \tag{3.23}$$

Proof. Applying Lemma 3.3 to (2.11) and using (3.1) and (3.2) we get (3.20) and (3.21) with $k = 1$. Next, multiplying both sides of (2.11) by x (resp., by x^2) gives, respectively,

$$(x\Phi(x)w)' + (x\Psi(x) - \Phi(x))w + xB(x)(x^{-1}w^2) = 0, \tag{3.24}$$
$$(x^2\Phi(x)w)' + (x^2\Psi(x) - 2x\Phi(x))w + x^2B(x)(x^{-1}w^2) = 0. \tag{3.25}$$

In the same way, applying Lemma 3.3 to (3.24) (resp., to (3.25)) and using (3.1) and (3.2) one gets (3.20) and (3.22) with $k = 2$ (resp., (3.23) with $k = 3$).

Finally, notice that from (3.20)–(3.23) we cannot conclude that u is a Laguerre–Hahn linear functional since we have not proved that at least one of the polynomials Φ_k^P, Ψ_k^P, and B_k^P, $k \in \{1,2,3\}$ is not equal to zero, a fact that is not always true. As a matter of fact, let us suppose that $\Phi_1^P = \Psi_1^P = B_1^P = 0$, $\Phi_2^P = \Psi_2^P = B_2^P = 0$ and $\Phi_3^P = \Psi_3^P = B_3^P = 0$. Then, one has from (3.21), (3.22), and (3.23)

$$\begin{cases}
(\gamma_1+\beta_0^2)\Phi_1(x) + x\Phi_2(x) + \beta_0 x\Phi_3(x) = 0,\\
(\gamma_1+\beta_0^2)x\Phi_3(x) + x\Phi_1(x) + \beta_0 x\Phi_2(x) = 0,\\
(\gamma_1+\beta_0^2)x\Phi_2(x) + x^2\Phi_3(x) + \beta_0 x\Phi_1(x) = 0,
\end{cases} \tag{3.26}$$

which readily gives, by simple computations,

$$\Phi_1(x) = \Phi_2(x) = \Phi_3(x) = 0.$$

Then, it is clear that $\Phi(x) = 0$. Similarly, we use (3.21), (3.22), and (3.23), and proceed as above to deduce that

$$\Psi(x) = B(x) = 0,$$

which yields a contradiction. As a consequence, at least one of the polynomials

$$\Phi_1^P,\ \Phi_2^P,\ \Phi_3^P,\ \Psi_1^P,\ \Psi_2^P,\ \Psi_3^P,\ B_1^P,\ B_2^P,\ B_3^P$$

is not identically zero. Hence, u is a Laguerre–Hahn linear functional. ∎

To end this section, the polynomial coefficients A, B, C, and D of the Riccati equation (2.9) satisfied by the Stieltjes function corresponding to the linear functional w will be given in terms of those of the linear functional $u = \sigma_{\varpi}(w)$ which we denote by A^P, B^P, C^P, and D^P.

Proposition 3.3. *If $S(u)(z)$ satisfies*

$$A^P(z)S'(u)(z) = B^P(z)S^2(u)(z) + C^P(z)S(u)(z) + D^P(z), \tag{3.27}$$

then $S(w)(z)$ satisfies (2.9) with

$$A(z) = \rho(z)A^P(z^3), \tag{3.28}$$

$$B(z) = 3z^2 B^P(z^3), \tag{3.29}$$

$$C(z) = \rho'(z)A^P(z^3) + 3z^2\rho(z)C^P(z^3), \tag{3.30}$$

$$D(z) = 3z^2\rho^2(z)D^P(z^3). \tag{3.31}$$

Proof. Taking formal derivatives in (2.22), we get

$$S'(u)(z^3) = \frac{\rho(z)S'(w)(z) - \rho'(z)S(w)(z)}{3z^2\rho^2(z)}. \tag{3.32}$$

In (3.27), the change of variable $z \leftarrow z^3$ yields

$$A^P(z^3)S'(u)(z^3) = B^P(z^3)S^2(u)(z^3) + C^P(z^3)S(u)(z^3) + D^P(z^3). \tag{3.33}$$

Substituting (2.22) and (3.32) in (3.33) multiplying both sides of the resulting equation by $3z^2\rho^2(z)$, one obtains

$$\begin{aligned}\rho(z)A^P(z^3)S'(w)(z) = 3z^2B^P(z^3)S^2(w)(z) + (3z^2\rho(z)C^P(z^3)\\ + \rho'(z)A^P(z^3))S(w)(z) + 3z^2\rho^2(z)D^P(z^3),\end{aligned}$$

which is what we wanted to prove. ∎

4. The class of the linear functional w

The aim of this section is to optimize the results already established. As shown before, Proposition 3.1 does not specify the class s of the linear functional w. We focus our attention on the analysis of the class of the linear functional w in terms of the class of the linear functional $u = \sigma_{\varpi}(w)$. Thus, an accurate description of all the possible situations is given.

Henceforth, we will assume that (3.12) satisfied by the linear functional w cannot be simplified. In other words, denoting by $s' = \max\{\deg \Phi^P - 2, \deg \Psi^P - 1, \deg B^P - 2\}$, in order to find s, condition (2.12) is in play.

We begin by stating the auxiliary lemmas that are crucial to our remaining results.

Lemma 4.1. *For each $c \in \mathbb{C}$, $f \in \mathcal{P}$ and all $v \in \mathcal{P}'$, we have*

$$(\theta_c \sigma_{\varpi}(f))(x) = (x^2 + cx + c^2)(\sigma_{\varpi}(\theta_{c^3} f))(x), \tag{4.1}$$

$$\langle v, \theta_c f - \theta_0 f\rangle = c\langle v, \theta_0\theta_c f\rangle. \tag{4.2}$$

Proof. The proof of this lemma is straightforward and will be omitted. ■

Lemma 4.2. *For all $f \in \mathcal{P}$, we have*

$$\langle w, f(x^3)\rangle = \langle u, f(x)\rangle, \tag{4.3}$$

$$\langle w, xf(x^3)\rangle = \beta_0\langle u, f(x)\rangle, \tag{4.4}$$

$$\langle w, x^2 f(x^3)\rangle = (\gamma_1 + \beta_0^2)\langle u, f(x)\rangle, \tag{4.5}$$

$$(w\sigma_{\varpi}(f))(x) = (uf)(x^3) + ((\gamma_1 + \beta_0^2)x + \beta_0 x^2)(u\theta_0 f)(x^3), \tag{4.6}$$

$$(wx\sigma_{\varpi}(f))(x) = (x + \beta_0)(uf)(x^3) + (\gamma_1 + \beta_0^2)x^2(u\theta_0 f)(x^3), \tag{4.7}$$

$$(wx^2\sigma_{\varpi}(f))(x) = \rho(x)(uf)(x^3). \tag{4.8}$$

Proof. Equations (4.3), (4.4), and (4.5) follow immediately from (2.5), (2.20), and (2.21). Equations (4.6), (4.7), and (4.8) are deduced in a straightforward way using (2.1). ■

The following lemma contains properties that will be used throughout the sequel.

Lemma 4.3. *For every $c \in \mathbb{C}$, we have*

$$\Psi(c) + \Phi'(c) = -\rho'(c)\Phi^P(c^3) + 3c^2\rho(c)(\Psi^P(c^3) + (\Phi^P)'(c^3)), \tag{4.9}$$

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2\rho^2(c)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3} B^P\big\rangle. \tag{4.10}$$

Proof. Equation (4.9) follows in a straightforward way from (3.13) and (3.14).

Observe that, from (3.13)–(3.15), we have

$$\begin{aligned}\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle &= \langle w, \theta_c(3x^2\rho(x)\Psi^P(x^3))\rangle \\ &\quad + \langle w, \theta_c^2(\rho(x)\Phi^P(x^3)) + \theta_c(-2\rho'(x)\Phi^P(x^3))\rangle \\ &\quad + \langle w, w\theta_0\theta_c(3x^2 B^P(x^3))\rangle. \end{aligned}\tag{4.11}$$

Using the definition of the operator θ_c, it is easy to see that, for two polynomials f and g, we have

$$\theta_c(fg)(x) = g(x)\theta_c(f)(x) + f(c)\theta_c(g)(x). \tag{4.12}$$

Taking $g(x) = \Psi^P(x^3)$ and $f(x) = 3x^2\rho(x)$ in (4.12), we get

$$\langle w, \theta_c(3x^2\rho(x)\Psi^P(x^3))\rangle = \langle w, \Psi^P(x^3)\theta_c(3x^2\rho(x)) + 3c^2\rho(c)\theta_c(\Psi^P(x^3))\rangle.$$

Replacing $g(x) = 3x^2$ and $f(x) = \rho(x)$ in (4.12), it is not difficult to check that

$$\theta_c(3x^2\rho(x)) = 3(x^3 + (c+\beta_0)x^2 + \rho(c)x + c\rho(c)).$$

Then, using (4.1), and from (4.3), (4.4), and (4.5), we obtain

$$\begin{aligned}\langle w, \theta_c(3x^2\rho(x)\Psi^P(x^3))\rangle = 3\langle u, x\Psi^P(x)\rangle + 3(c+\beta_0)(\rho(c)+\gamma_1+\beta_0^2)\langle u, \Psi^P(x)\rangle\\ + 3c^2\rho^2(c)\langle u, (\theta_{c^3}\Psi^P)(x)\rangle.\end{aligned} \tag{4.13}$$

Replacing $g(x) = \Phi^P(x^3)$ and $f(x) = \rho(x)$ in (4.12), and using (4.1), from (4.3), (4.4), and (4.5) we deduce that

$$\theta_c(\rho(x)\Phi^P(x^3)) = (x+c+\beta_0)\Phi^P(x^3) + \rho(c)(x^2+cx+c^2)(\theta_c\Phi^P)(x^3). \tag{4.14}$$

Again, as a consequence of (4.12), we get

$$\begin{aligned}\theta_c((x+c+\beta_0)\Phi^P(x^3)) &= \Phi^P(x^3) + (2c+\beta_0)(x^2+cx+c^2)(\theta_{c^3}\Phi^P)(x^3),\\ \theta_c((x^2+cx+c^2)(\theta_{c^3}\Phi^P)(x^3)) &= (x+2c)(\theta_{c^3}\Phi^P)(x^3)\\ &\quad + 3c^2(x^2+cx+c^2)\big(\theta_{c^3}^2\Phi^P\big)(x^3).\end{aligned}$$

Applying the operator θ_c to (4.14) and taking into account the last two equations,

$$\begin{aligned}\theta_c^2(\rho(x)\Phi^P(x^3)) = \Phi^P(x^3) + ((2c+\beta_0)(x^2+cx+c^2) + \rho(c)(x+2c))(\theta_{c^3}\Phi^P)(x^3)\\ + 3c^2\rho(c)(x^2+cx+c^2)\big(\theta_{c^3}^2\Phi^P\big)(x^3)\end{aligned}$$

holds. Then, using (4.3), (4.4), and (4.5), we have

$$\begin{aligned}\langle w, \theta_c^2(\rho(x)\Phi^P(x^3))\rangle = \langle u, \Phi^P(x)\rangle + 2\rho(c)(2c+\beta_0)\langle u, (\theta_{c^3}\Phi^P)(x)\rangle\\ + 3c^2\rho^2(c)\big\langle u, \big(\theta_{c^3}^2\Phi^P\big)(x)\big\rangle.\end{aligned} \tag{4.15}$$

On the other hand, if we consider in (4.12) $g(x) = \Phi^P(x^3)$ and $f(x) = -2\rho'(x)$, then,

$$\theta_c(-2\rho'(x)\Phi^P(x^3)) = -4\Phi^P(x^3) - 2(2c+\beta_0)(x^2+2c+c^2)(\theta_{c^3}\Phi^P)(x^3).$$

Using once more (4.3), (4.4), and (4.5), and after simple computations, one obtains

$$\langle w, \theta_c(-2\rho'(x)\Phi^P(x^3))\rangle = -4\langle u, \Phi^P(x)\rangle - 2(2c+\beta_0)\rho(c)\langle u, (\theta_{c^3}\Phi^P)(x)\rangle. \tag{4.16}$$

Hence, from (4.15) and (4.16), we have

$$\begin{aligned}&\langle w, \theta_c^2(\rho(x)\Phi^P(x^3)) + \theta_c(-2\rho'(x)\Phi^P(x^3))\rangle \\ &\quad = -3\langle u, \Phi^P(x)\rangle + 3c^2\rho^2(c)\big\langle u, \big(\theta_{c^3}^2\Phi^P\big)(x)\big\rangle.\end{aligned} \tag{4.17}$$

Now, replacing $g(x) = B^P(x^3)$ and $f(x) = 3x^2$ in (4.12), we get

$$\begin{aligned}\theta_c(3x^2B^P(x^3)) &= 3(x+c)B^P(x^3) + 3c^2(x^2+cx+c^2)(\theta_{c^3}B^P)(x^3),\\ \theta_0(\theta_c(3x^2B^P(x^3))) &= 3B^P(x^3) + 3cx^2(\theta_0B^P)(x^3) + 3c^2(x+c)(\theta_{c^3}B^P)(x^3)\\ &\quad + 3c^4x^2(\theta_0\theta_{c^3}B^P)(x^3).\end{aligned} \tag{4.18}$$

Using (4.6), (4.7), and (4.8), we get

$$\begin{aligned}w(B^P(x^3)) &= (uB^P)(x^3) + ((\gamma_1+\beta_0^2)x + \beta_0x^2)(u\theta_0B^P)(x^3),\\ w(x^2(\theta_0B^P)(x^3)) &= \rho(x)(u\theta_0B^P)(x^3),\\ w((x+c)(\theta_{c^3}B^P)(x^3)) &= (x+\beta_0+c)(u\theta_{c^3}B^P)(x^3)\\ &\quad + [c(\gamma_1+\beta_0^2)x + (c\beta_0+\gamma_1+\beta_0^2)x^2](u\theta_0\theta_{c^3}B^P)(x^3),\\ w(x^2(\theta_0\theta_{c^3}B^P)(x^3)) &= \rho(x)(u\theta_0\theta_{c^3}B^P)(x^3).\end{aligned}$$

In other words, from (4.18) and keeping in mind the last four equations, we have

$$\begin{aligned}w\theta_0(\theta_c(3x^2B^P(x^3))) &= 3(uB^P)(x^3) + [3(c(\gamma_1+\beta_0^2)x + c^2(c\beta_0+\gamma_1+\beta_0^2)x^2)\\ &\quad + 3c^4\rho(x)](u\theta_0\theta_{c^3}B^P)(x^3)\\ &\quad + [3((\gamma_1+\beta_0^2)x + \beta_0x^2) + 3c\rho(x)](u\theta_0B^P)(x^3)\\ &\quad + 3c^2(x+\beta_0+c)(u\theta_{c^3}B^P)(x^3).\end{aligned} \tag{4.19}$$

Using (4.3), (4.4), and (4.5) it is not hard to check that

$$\langle w, 3(uB^P)(x^3)\rangle = 3\langle u, uB^P(x)\rangle, \tag{4.20}$$

$$\begin{aligned}&\langle w, [3((\gamma_1+\beta_0^2)x + \beta_0x^2) + 3c\rho(x)](u\theta_0B^P)(x^3)\rangle\\ &\quad = 3[2(\beta_0+c)(\gamma_1+\beta_0^2) + \beta_0^2c]\langle u, u\theta_0B^P(x)\rangle,\end{aligned} \tag{4.21}$$

$$\begin{aligned}\langle w, 3c^2(x+\beta_0+c)(u\theta_{c^3}B^P)(x^3)\rangle &= 3c^2(2\beta_0+c)\langle u, u\theta_{c^3}B^P(x)\rangle\\ &\overset{\text{by (4.2)}}{=} 3c^2(2\beta_0+c)(c^3\langle u, u\theta_0\theta_{c^3}B^P(x)\rangle + \langle u, u\theta_0B^P(x)\rangle)\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\langle w, (3c^2[c(\gamma_1+\beta_0^2)x + (c\beta_0+\gamma_1+\beta_0^2)x^2] + 3c^4\rho(x))(u\theta_0\theta_{c^3}B^P)(x^3)\rangle\\ &\quad = 3c^2(2c\beta_0(\gamma_1+\beta_0^2) + 2c^2(\gamma_1+\beta_0^2) + (\gamma_1+\beta_0^2)^2 + c^2\beta_0^2)\langle u, (u\theta_0\theta_{c^3}B^P)(x)\rangle.\end{aligned} \tag{4.23}$$

Adding (4.20), (4.21), (4.22), (4.23) and taking into account (4.19) we deduce that

$$\langle w, w\theta_0(\theta_c(3x^2B^P(x^3)))\rangle = 3\langle u, uB^P(x)\rangle + 3c^2\rho^2(c)\langle u, (u\theta_0\theta_{c^3}B^P)(x)\rangle + 3(c+\beta_0)(\rho(c)+\gamma_1+\beta_0^2)\langle u, u\theta_0 B^P(x)\rangle. \quad (4.24)$$

Replacing (4.13), (4.17), and (4.24) in (4.11), we can conclude that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2\rho^2(c)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle + 3(c+\beta_0)(\rho(c)+\gamma_1+\beta_0^2)\langle u, \Psi^P(x) + u\theta_0 B^P(x)\rangle + 3\langle u, x\Psi^P(x) + uB^P(x)\rangle.$$

We conclude the proof by noting that (4.10) follows from $\langle u, \Psi^P(x) + u\theta_0 B^P(x)\rangle = \langle u, x\Psi^P(x) + uB^P(x)\rangle = 0$. ∎

Proposition 4.1. *The class of w depends only on the zeros $x = 0$ and $x = c$, of the polynomial $\Phi(x)$, where c is a root of $\rho(x)$ given by (3.16).*

Proof. Let c be a zero of Φ such that $c \neq 0$ and $\rho(c) \neq 0$. If $\Phi'(c) + \Psi(c)$ and $B(c) = 0$, then from (3.13), (3.15), and (4.9), c^3 is a common zero of Φ^P, B^P and $(\Phi^P)' + \Psi^P$. But, since (3.12) cannot be simplified, then $\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\rangle \neq 0$. Therefore, from (4.10) we conclude that $\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle \neq 0$. As a consequence, we cannot divide in (2.11) by $x - c$. ∎

In order to find the class of w, we will investigate the behavior of the polynomials Φ, Ψ and B at $x = c$, where c is either equal to zero or a root of $\rho(x)$ given by (3.16). So, we only need to analyze separately these three possible cases.

(A) $\rho(x)$ has two different zeros and one of them is zero, i.e., $\beta_0 \neq 0$ and $\rho(x) = x(x+\beta_0)$.
This means that $\beta_0^2 + \gamma_1 = 0$.

(B) $\rho(x)$ has a double zero, i.e., $\beta_0 \neq 0$ and $\rho(x) = (x-d)^2$ with $d = -\frac{\beta_0}{2}$.
This means that $\beta_0^2 + \gamma_1 = \frac{\beta_0^2}{4}$.

(C) $\rho(x) = (x-a)(x-b)$ with $ab \neq 0$, $a \neq b$.
This means that $\beta_0^2 + \gamma_1 \neq 0$ and $\beta_0^2 + \gamma_1 \neq \frac{\beta_0^2}{4}$.

Now, we are able to discuss in details the different situations. In fact, each of the three cases (A), (B), and (C) can be split into several sub-subcases.

4.1. Case A

Let us assume that $\beta_0^2 + \gamma_1 = 0$, i.e., $\beta_0 \neq 0$ and $\rho(x) = x(x+\beta_0)$.

First, observe that according to Proposition 3.1, the linear functional w satisfies (2.11) with $\Phi(x) = x(x+\beta_0)\Phi^P(x^3)$ and $B(x) = 3x^2B^P(x^3)$. The class of w is at most $3s' + 6$. From (4.9), we have

$$\Phi(0) = 0, \quad \Phi'(0) + \Psi(0) = -\beta_0\Phi^P(0), \quad B(0) = 0.$$

Further, from (4.10), we obtain

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^4(c+\beta_0)^2\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\rangle,$$

which readily gives

$$\langle w, \theta_0 \Psi + \theta_0^2 \Phi + w\theta_0^2 B\rangle = 0.$$

A.1. If $\Phi^P(0) \neq 0$, then we cannot divide by x in (2.11). For this reason, we can analyze the possible division by $x+\beta_0$. Due to Proposition 3.1 and (4.9)–(4.10) one has $\Phi(-\beta_0) = 0$, $B(-\beta_0) = 3\beta_0^2 B^P(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = \beta_0\Phi^P(-\beta_0^3)$, and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2\Phi + w\theta_0\theta_{-\beta_0}B\rangle = 0$, which brings up two subcases.

A.1.1. If $\Phi^P(-\beta_0^3) \neq 0$ or $B^P(-\beta_0^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 6$.

A.1.2. If $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$, then the functional equation can be simplified to $x + \beta_0$.

Remark 4.1. Note that, for the sake of simplicity, regardless of how many times the simplification of the functional equation is repeated, we will always keep the same notations Φ, Ψ, and B for the resulting polynomials.

Then, in this case, the linear functional w satisfies (2.11), with $\Phi(x) = x\Phi^P(x^3)$ and $B(x) = 3(x-\beta_0)B^P(x^3) + 3\beta_0^2(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}B^P)(x^3)$. Using Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^4(c+\beta_0)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

It follows that $\Phi(-\beta_0) = 0$, $B(-\beta_0) = 9\beta_0^4(B^P)'(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = -3\beta_0^3\Psi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2\Phi + w\theta_0\theta_{-\beta_0}B\rangle = 0$. Here, two more subcases emerge.

A.1.2.1. If $(B^P)'(-\beta_0^3) \neq 0$ or $\Psi^P(-\beta_0^3) \neq 0$, then simplification of the functional equation cannot occur and $s = 3s' + 5$.

A.1.2.2. If $(B^P)'(-\beta_0^3) = \Psi^P(-\beta_0^3) = 0$, so it is possible to simplify the functional equation to $x+\beta_0$ and w satisfies (2.11), with $\Phi(x) = x(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\Phi^P)(x^3)$ and $B(x) = 3(x-\beta_0)(\theta_{-\beta_0^3}B^P)(x^3) + 3\beta_0^2(x^2-\beta_0 x+\beta_0^2)^2(\theta_{-\beta_0^3}^2 B^P)(x^3)$. Now, applying Lemma 2.1 yields

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^4\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

We conclude, in this case, that the functional equation can no longer be simplified. Indeed, otherwise, we have $\langle u, \theta_{-\beta_0^3}\Psi^P + \theta_{-\beta_0^3}^2\Phi^P + w\theta_0\theta_{-\beta_0^3}B^P\rangle = 0$, $(\theta_{-\beta_0^3}\theta_0\Phi^P)(-\beta_0^3) = 0$ and then, naturally, $(\theta_0\Phi^P)'(-\beta_0^3) = 0$. Therefore, $\Psi^P(-\beta_0^3) + (\Phi^P)'(-\beta_0^3) = 0$. On the other hand, we have $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$. Then, one can divide (3.12) by $x+\beta_0$ and this yields a contradiction. Hence, $s = 3s' + 4$.

A.2. If $\Phi^P(0) = 0$ then simplification of the functional equation of by x can occur, and w satisfies (2.11), with $\Phi(x) = (x + \beta_0)\Phi^P(x^3)$ and $B(x) = 3xB^P(x^3)$. Using Lemma 2.1 we infer that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^3(c + \beta_0)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(0) = 0$, $B(0) = 0$, $\Phi'(0) + \Psi(0) = 0$ and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$. Then, you can divide the functional equation by x and thus w satisfies (2.11), with

$$\Phi(x) = (x + \beta_0)x^2(\theta_0\Phi^P)(x^3), \quad B(x) = 3B^P(x^3). \tag{4.25}$$

From (4.10) and taking into account Lemma 2.1, we deduce

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c + \beta_0)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle. \tag{4.26}$$

Since $c = 0$, then $\Phi(0) = 0$, $B(0) = 3B^P(0)$, $\Phi'(0) + \Psi(0) = 0$ and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$. So, the current case can itself extend to two subcases.

A.2.1. If $B^P(0) \neq 0$ then no dividing by x in (2.11) is possible. Thus, we move to analyzing the possibility of dividing by $x + \beta_0$. Based on (4.25) and (4.26) we get $\Phi(-\beta_0) = 0$, $B(-\beta_0) = 3B^P(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = -\beta_0^2(\theta_0\Phi^P)(-\beta_0^3) = \beta_0^{-1}\Phi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2\Phi + w\theta_0\theta_{-\beta_0}B\rangle = 0$. So, two subcases present themselves.

A.2.1.1. If $\Phi^P(-\beta_0^3) \neq 0$ or $B^P(-\beta_0^3) \neq 0$, the functional equation remains non simplified and $s = 3s' + 4$.

A.2.1.2. If $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$, so dividing the functional equation by $x + \beta_0$ can indeed happen and w satisfies (2.11), with $\Phi(x) = x^2(\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}B^P)(x^3)$. It follows from Lemma 2.1 that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c + \beta_0)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

As a consequence, $\Phi(-\beta_0) = \beta_0^2(\theta_0\Phi^P)(-\beta_0^3) = -\beta_0^{-1}\Phi^P(-\beta_0^3) = 0$, $B(-\beta_0) = 9\beta_0^2 (B^P)'(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = -3\beta_0\Psi^P(-\beta_0^3)$, and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2\Phi + w\theta_0 \times \theta_{-\beta_0}B\rangle = 0$. At this point, two more subcases appear.

A.2.1.2.1. If $\Psi^P(-\beta_0^3) \neq 0$ or $(B^P)'(-\beta_0^3) \neq 0$, then again the functional equation cannot be simplified and $s = 3s' + 3$.

A.2.1.2.2. If $\Psi^P(-\beta_0^3) = (B^P)'(-\beta_0^3) = 0$, then it is possible for the functional equation to be simplified by $x + \beta_0$ and w satisfies (2.11), with $\Phi(x) = x^2(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 - \beta_0 x + \beta_0^2)^2(\theta_{-\beta_0^3}^2 B^P)(x^3)$.

In this case, the functional equation cannot be simplified. Indeed, suppose it does, then we have $\langle u, \theta_{-\beta_0^3}\Psi^P + \theta_{-\beta_0^3}^2\Phi^P + u\theta_0\theta_{-\beta_0^3}B^P\rangle = 0$ and $(\theta_{-\beta_0^3}\theta_0\Phi^P)(-\beta_0^3) = 0$. This shows that $(\theta_0\Phi^P)'(-\beta_0^3) = 0$, and so, $\Psi^P(-\beta_0^3) + (\Phi^P)'(-\beta_0^3) = 0$. On the other hand, we have $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$. Therefore, dividing in (3.12) by $x + \beta_0$ is possible and this makes a contradiction. Consequently, $s = 3s' + 2$.

A.2.2. If $B^P(0) = 0$, then simplifying by x the functional equation is possible and w satisfies (2.11), with

$$\Phi(x) = (x + \beta_0)x(\theta_0 \Phi^P)(x^3), \quad B(x) = 3x^2(\theta_0 B^P)(x^3). \tag{4.27}$$

Lemma 2.1 together with (4.10) yields

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B \rangle = 3c(c + \beta_0)^2 \big\langle u, \theta_{c^3} \Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P \big\rangle. \tag{4.28}$$

Considering $c = 0$ gives $\Phi(0) = 0$, $B(0) = 0$, $\Phi'(0) + \Psi(0) = \beta_0(3\Psi^P(0) + 2(\Phi^P)'(0))$ and $\langle w, \theta_0 \Psi + \theta_0^2 \Phi + w\theta_0^2 B \rangle = 0$. This provides us with two new subcases.

A.2.2.1. If $3\Psi^P(0) + 2(\Phi^P)'(0) \neq 0$ then (2.11) cannot be divided by x. Hence, as usual, an analysis of the possible division by $x + \beta_0$ is at play. Relying on (4.27) and (4.28) one gets $\Phi(-\beta_0) = 0$, $B(-\beta_0) = -3\beta_0^{-1} B^P(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = -\beta_0^{-2}\Phi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2 \Phi + w\theta_0\theta_{-\beta_0} B \rangle = 0$. Again, two subcases unfold.

A.2.2.1.1. If $\Phi^P(-\beta_0^3) \neq 0$ or $B^P(-\beta_0^3) \neq 0$, the functional equation cannot be simplified and $s = 3s' + 3$.

A.2.2.1.2. If $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$, then the functional equation can be simplified by $x + \beta_0$ and w satisfies (2.11), with $\Phi(x) = x(\theta_0\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\theta_0 B^P)(x^3)$. Again, using Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B \rangle = 3c(c + \beta_0)\big\langle u, \theta_{c^3} \Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P \big\rangle.$$

From the latter relation, it follows that $\Phi(-\beta_0) = -\beta_0(\theta_0\Phi^P)(-\beta_0^3) = 0$, $B(-\beta_0) = -9\beta_0(B^P)'(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = 3\Psi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2 \Phi + w\theta_0 \times \theta_{-\beta_0} B \rangle = 0$. This allows two new sub-subcases to be considered.

A.2.2.1.2.1. If $\Psi^P(-\beta_0^3) \neq 0$ or $(B^P)'(-\beta_0^3) \neq 0$, then simplification of the functional equation cannot take place, leading to $s = 3s' + 2$.

A.2.2.1.2.2. If $\Psi^P(-\beta_0^3) = (B^P)'(-\beta_0^3) = 0$, in this case, the functional equation can in fact be simplified by $x + \beta_0$ and w satisfies (2.11), with $\Phi(x) = x(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2 - \beta_0 x + \beta_0^2)^2(\theta_{-\beta_0^3}^2\theta_0 B^P)(x^3)$. By virtue of Lemma 2.1, we obtain

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B \rangle = 3c\big\langle u, \theta_{c^3} \Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P \big\rangle.$$

At this stage, the functional equation cannot be submitted to further simplification. For if one supposes the contrary, we have $\langle u, \theta_{-\beta_0^3}\Psi^P + \theta_{-\beta_0^3}^2\Phi^P + u\theta_0\theta_{-\beta_0^3} B^P \rangle = 0$ and $(\theta_{-\beta_0^3}\theta_0\Phi^P)(-\beta_0^3) = 0$. This yields $(\theta_0\Phi^P)'(-\beta_0^3) = 0$ and so $\Psi^P(-\beta_0^3) + (\Phi^P)'(-\beta_0^3) = 0$. But since one has $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$. Then, (3.12) can be divided by $x + \beta_0$ yielding a contradiction, and eventually $s = 3s' + 1$.

A.2.2.2. If $3\Psi^P(0) + 2(\Phi^P)'(0) = 0$, now, it is possible to simplify the functional equation by x and then w satisfies (2.11), with

$$\Phi(x) = (x + \beta_0)(\theta_0\Phi^P)(x^3), \quad B(x) = 3x(\theta_0 B^P)(x^3), \tag{4.29}$$

and it follows from (4.10) and Lemma 2.1 that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c + \beta_0)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle. \tag{4.30}$$

Hence, dividing in (2.11) by x is not possible, for the reason that if $\langle u, \theta_0\Psi^P + \theta_0^2\Phi^P + u\theta_0^2 B^P\rangle = 0$ and $(\theta_0\Phi^P)(0) = 0$. then one finds $(\Phi^P)'(0) = 0$. Taking into account that $3\Psi^P(0) + 2(\Phi^P)'(0) = 0$, we obtain $\Psi^P(0) = 0$. Therefore, $\Psi^P(0) + (\Phi^P)'(0) = 0$. But we also have $\Phi^P(0) = B^P(0) = 0$. Then, we can divide in (3.12) by x and this yields a contradiction.

As a result, we can now analyze the possible division by $x + \beta_0$. From (4.29) and (4.30) we have $\Phi(-\beta_0) = 0$, $B(-\beta_0) = -3\beta_0^{-2}B^P(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = \beta_0^{-3}\Psi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\Psi + \theta_{-\beta_0}^2\Phi + w\theta_0\theta_{-\beta_0}B\rangle = 0$. Here, two subcases arise.

A.2.2.2.1. If $\Phi^P(-\beta_0^3) \neq 0$ or $B^P(-\beta_0^3) \neq 0$, then the functional equation cannot be subject to further simplification and $s = 3s' + 2$.

A.2.2.2.2. If $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$, then the functional equation can be simplified by $x + \beta_0$ and w satisfies (2.11), with $\Phi(x) = (\theta_0\Phi^P)(x^3)$ and $B(x) = 3x(x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\theta_0 B^P)(x^3)$. By means of Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c + \beta_0)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Therefore, $\Phi(-\beta_0) = 0$, $B(-\beta_0) = -9\beta_0^* * (B^P)'(-\beta_0^3)$, $\Phi'(-\beta_0) + \Psi(-\beta_0) = -3\beta_0^{-1} \times \Psi^P(-\beta_0^3)$ and $\langle w, \theta_{-\beta_0}\widetilde{\Psi} + \theta_{-\beta_0}^2\Phi + w\theta_0\theta_{-\beta_0}B\rangle = 0$. By this fact, two new sub-subcases exist here.

A.2.2.2.2.1. If $\Psi^P(-\beta_0^3) \neq 0$ or $(B^P)'(-\beta_0^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 1$.

A.2.2.2.2.2. If $\Psi^P(-\beta_0^3) = (B^P)'(-\beta_0^3) = 0$, then simplifying the functional equation by $x + \beta_0$ can be done and w satisfies (2.11), with $\Phi(x) = (x^2 - \beta_0 x + \beta_0^2)(\theta_{-\beta_0^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3x(x^2 - \beta_0 x + \beta_0^2)^2(\theta_{-\beta_0^3}^2\theta_0 B^P)(x^3)$.

Furthermore, from Lemma 2.1, we infer that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

In this case, the functional equation cannot be simplified. Indeed, supposing the contrary, we have $\langle u, \theta_{-\beta_0^3}\Psi^P + \theta_{-\beta_0^3}^2\Phi^P + u\theta_0\theta_{-\beta_0^3}B^P\rangle = 0$ and $(\theta_{-\beta_0^3}\theta_0\Phi^P)(-\beta_0^3) = 0$. The last equality is equivalent to $(\theta_0\Phi^P)'(-\beta_0^3) = 0$, so $(\Phi^P)'(-\beta_0^3) = 0$, since $(\Phi^P)'(-\beta_0^3) = -\beta_0^3(\theta_0\Phi^P)'(-\beta_0^3)$. Hence, $\Psi^P(-\beta_0^3) + (\Phi^P)'(-\beta_0^3) = 0$. Further, we have $\Phi^P(-\beta_0^3) = B^P(-\beta_0^3) = 0$. Therefore, we can divide in (3.12) by $x + \beta_0$ which yields a contradiction, and so, $s = 3s'$.

4.2. Case B

Let us assume that $\beta_0^2 + \gamma_1 = \frac{\beta_0^2}{4}$, i.e., $\beta_0 \neq 0$ and $\rho(x) = (x-d)^2$ with $d = -\frac{\beta_0}{2}$.

Relying on Proposition 3.1, the linear functional w satisfies (2.11) with $\Phi(x) = (x-d)^2\Phi^P(x^3)$ and $B(x) = 3x^2B^P(x^3)$, where $d := -\frac{\beta_0}{2} \neq 0$. The class of w is at most $3s' + 6$. From (4.9) we have $\Phi(0) = d^2\Phi^P(0)$, $B(0) = 0$, $\Phi'(0) + \Psi(0) = 2d\Phi^P(0)$. But (4.10) states that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-d)^4\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Thus, this gives

$$\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0.$$

We then establish the following discussion.

B.1. If $\Phi^P(0) \neq 0$, then in (2.11) we cannot divide by x. Thus, we can analyze the possible division by $x - d$. According to Proposition 3.1 and (4.9)–(4.10) we have $\Phi(d) = 0$, $B(d) = 3d^2B^P(d^3)$, $\Phi'(d) + \Psi(d) = 0$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. This case is divided into two subcases.

B.1.1. If $B^P(d^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 6$.

B.1.2. If $B^P(d^3) = 0$, then the functional equation can indeed be divided by $x - d$ and w satisfies (2.11), with $\Phi(x) = (x-d)\Phi^P(x^3)$, $B(x) = 3x^2(x^2 + dx + d^2)(\theta_{d^3}B^P)(x^3)$. Taking into account Lemma 2.1, we derive

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-d)^3\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

So, $\Phi(d) = 0$, $B(d) = 9d^4(B^P)'(d^3)$, $\Phi'(d) + \Psi(d) = -2\Phi^P(d^3)$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Here, two situations may arise.

B.1.2.1. If $\Phi^P(d^3) \neq 0$ or $B^P(d^3) \neq 0$, then no simplification of the functional equation can occur and $s = 3s' + 5$.

B.1.2.2. If $\Phi^P(d^3) = B^P(d^3) = 0$, then the functional equation can be simplified by $x - d$ and w satisfies (2.11), with $\Phi(x) = \Phi^P(x^3)$, $B(x) = 3x^2(x^2+dx+d^2)^2(\theta_{d^3}^2B^P)(x^3)$. Together with Lemma 2.1, we have

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-d)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

It is clear that $\Phi(d) = 0$, $B(d) = 27d^6(\theta_{d^3}B^P)'(d^3) = \frac{27d^6}{2}(B^P)''(d^3)$, $\Phi'(d) + \Psi(d) = 3d^2(\Psi(d^3) - (\Phi^P)'(d^3))$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Again, two situations may come up.

B.1.2.2.1. If $(B^P)''(d^3) \neq 0$ or $\Psi(d^3) - (\Phi^P)'(d^3) \neq 0$, then no simplification of the functional equation is possible and $s = 3s' + 4$.

B.1.2.2.2. If $(B^P)''(d^3) = \Psi(d^3) - (\Phi^P)'(d^3) = 0$, then the functional equation can be simplified by $x - d$ and w satisfies (2.11), with $\Phi(x) = (x^2 + dx + d^2)(\theta_{d^3}\Phi^P)(x^3)$, $B(x) = 3x^2(x^2 + dx + d^2)^3(\theta_{d^3}^3 B^P)(x^3)$. Using Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-d)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

It is easy to check that $\Phi(d) = 3d^2(\Phi^P)'(d^3)$, $B(d) = 81d^8(\theta_{d^3}^2 B^P)'(d^3) = \frac{27d^6}{3!}(B^P)'''(d^3)$, $\Phi'(d) + \Psi(d) = 6d\Psi(d^3) + 9d^4(\Psi^P)'(d^3)$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Now, we need to consider the the following sub-subcases.

B.1.2.2.2.1. If $|\Psi(d^3)| + |(\Psi^P)'(d^3)| + |(B^P)'''(d^3)| \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 3$.

B.1.2.2.2.2. If $|\Psi(d^3)| + |(\Psi^P)'(d^3)| + |(B^P)'''(d^3)| = 0$, then the functional equation can be simplified to $x - d$ and w satisfies (2.11), with

$$\Phi(x) = (x^2 + dx + d^2)^2\big(\theta_{d^3}^2\Phi^P\big)(x^3), \quad B(x) = 3x^2(x^2 + dx + d^2)^4\big(\theta_{d^3}^4 B^P\big)(x^3). \tag{4.31}$$

By Lemma 2.1, one can check that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle, \tag{4.32}$$

which confirms that the functional equation can no longer be subject to simplification. Indeed, by a contradiction argument, suppose that $\langle u, \theta_{d^3}\Psi^P + \theta_{d^3}^2\Phi^P + u\theta_0\theta_{d^3}B^P\rangle = 0$ and $\Psi^P(d^3) = (\Phi^P)'(d^3) = 0$. Therefore, $\Psi^P(d^3) + (\Phi^P)'(d^3) = 0$. But we also have $\Phi^P(d^3) = B^P(d^3) = 0$. Then, dividing in (3.12) by $x - d$ is possible which yields a contradiction. Then, $s = 3s' + 2$.

B.2. If $\Phi^P(0) = 0$, then the functional equation can be simplified to x and w satisfies (2.11), with $\Phi(x) = (x-d)^2x^2(\theta_0\Phi^P)(x^3)$, $B(x) = 3xB^P(x^3)$. Due to Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c(c-d)^4\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle,$$

which implies that $\Phi(0) = 0$, $B(0) = 0$, $\Phi'(0) + \Psi(0) = 0$ and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$. Then, the functional equation can indeed be simplified by x and w satisfies (2.11), with

$$\Phi(x) = (x-d)^2x(\theta_0\Phi^P)(x^3), \quad B(x) = 3B^P(x^3). \tag{4.33}$$

Now, by Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c-d)^4\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle. \tag{4.34}$$

Then, in (2.11), we cannot divide by x. Because, suppose we have $\langle u, \theta_0\Psi^P + \theta_0^2\Phi^P + u\theta_0^2B^P\rangle = 0$, $B^P(0) = 0$ and $(\Phi^P)'(0) + \Psi^P(0) = 0$. But, we also have $\Phi^P(0) = 0$. So, one can divide in (3.12) by x and this yields a contradiction. Therefore, an analysis of the possible division by $x - d$ is at play. From (4.33) and (4.33) we have $\Phi(d) = 0$, $B(d) = 3B^P(d^3)$, $\Phi'(d) + \Psi(d) = 0$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Here, two cases occur to discuss.

B.2.1. If $B^P(d^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 4$.

B.2.2. If $B^P(d^3) = 0$, then the functional equation can be simplified to $x - d$ and w satisfies (2.11), with $\Phi(x) = (x-d)x(\theta_0\Phi^P)(x^3)$, $B(x) = 3(x^2 + dx + d^2)(\theta_{d^3}B^P)(x^3)$. From Lemma 2.1 one can easily see that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c-d)^3\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

One then has $\Phi(d) = 0$, $B(d) = 9d^2(B^P)'(d^3)$, $\Phi'(d) + \Psi(d) = -2d^{-2}\Phi^P(d^3)$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Again, the following two subcases arise.

B.2.2.1. If $\Phi^P(d^3) \neq 0$ or $(B^P)'(d^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 3$.

B.2.2.2. If $\Phi^P(d^3) = (B^P)'(d^3) = 0$, then the functional equation can be simplified again to $x - d$ and w satisfies (2.11), with $\Phi(x) = x(\theta_0\Phi^P)(x^3)$, $B(x) = 3(x^2 + dx + d^2)^2(\theta_{d^3}^2 B^P)(x^3)$. Using Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c-d)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

So, $\Phi(d) = d^{-2}\Phi^P(d^3) = 0$, $B(d) = \frac{27d^2}{2}(B^P)''(d^3)$, $\Phi'(d) + \Psi(d) = -3(\Psi^P(d^3) - (\Phi^P)'(d^3))$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. In this case, we will consider the following two subcases.

B.2.2.2.1. If $(B^P)''(d^3) \neq 0$ or $\Psi(d^3) - (\Phi^P)'(d^3) \neq 0$, then no simplification of the functional equation is possible and then $s = 3s' + 2$.

B.2.2.2.2. If $(B^P)''(d^3) = \Psi(d^3) - (\Phi^P)'(d^3) = 0$, so it is possible to divide the functional equation by $x - d$ and w satisfies (2.11), with $\Phi(x) = x(x^2 + dx + d^2)(\theta_{d^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 + dx + d^2)^3(\theta_{d^3}^3 B^P)(x^3)$. Lemma 2.1 provides the fact that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c-d)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(d) = 3(\Phi^P)'(d^3) = 3\Psi(d^3)$, $B(d) = 81d^6(\theta_{d^3}^2 B^P)'(d^3) = \frac{27d^6}{6}(B^P)'''(d^3)$, $\Phi'(d) + \Psi(d) = 6d^{-1}\Psi(d^3) + 9d^2(\Psi^P)'(d^3)$ and $\langle w, \theta_d\Psi + \theta_d^2\Phi + w\theta_0\theta_d B\rangle = 0$. Again, two situations come up to be discussed.

B.2.2.2.2.1. If $|\Psi(d^3)| + |(\Psi^P)'(d^3)| + |(B^P)'''(d^3)| \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 1$.

B.2.2.2.2.2. If $|\Psi(d^3)| + |(\Psi^P)'(d^3)| + |(B^P)'''(d^3)| = 0$, then the functional equation can be simplified to $x - d$ and w satisfies (2.11) with $\Phi(x) = x(x^2 + dx + d^2)^2(\theta_{d^3}^2\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 + dx + d^2)^4(\theta_{d^3}^4 B^P)(x^3)$. Lemma 2.1 implies that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

In this case, the functional equation cannot be simplified. Suppose $\langle u, \theta_{d^3}\Psi^P + \theta_{d^3}^2\Phi^P + u\theta_0\theta_{d^3}B^P\rangle = 0$ and $\Psi^P(d^3) = (\Phi^P)'(d^3) = 0$. Therefore, $\Psi^P(d^3) + (\Phi^P)'(d^3) = 0$. But we also have

$$\Phi^P(d^3) = B^P(d^3) = 0.$$

Then, we can divide in (3.12) by $x - d$ and this yields a contradiction. Hence, $s = 3s'$.

4.3. Case C

Let us assume that $\beta_0^2 + \gamma_1 \neq 0$ and $\beta_0^2 + \gamma_1 \neq \frac{\beta_0^2}{4}$, i.e., $\rho(x) = (x-a)(x-b)$ with $ab \neq 0$, $a \neq b$.

According to Proposition 3.1, the linear functional w satisfies (2.11) with $\Phi(x) = (x - a)(x-b)\Phi^P(x^3)$ and $B(x) = 3x^2 B^P(x^3)$, with $a \neq 0, b \neq 0, a - b \neq 0$. The class of w is at most $3s' + 6$. From (4.9) we have $\Phi(0) = ab\Phi^P(0)$, $B(0) = 0$ and $\Phi'(0) + \Psi(0) = (a+b)\Phi^P(0)$. But due to (4.10),

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)^2(c-b)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

So,

$$\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0.$$

As done in the previous cases, an in-depth discussion of several subcases take place.

C.1. If $\Phi^P(0) \neq 0$, then dividing by x in (2.11) is not possible. Thus, naturally, we move on to analyzing the possibility of dividing by $x - a$ or by $x - b$. Indeed, Proposition 3.1 and relations (4.9)–(4.10) give that $\Phi(a) = 0$, $B(a) = 3a^2B^P(a^3)$, $\Phi'(a) + \Psi(a) = (b - a)\Phi^P(a^3)$ and $\langle w, \theta_a\Psi + \theta_a^2\Phi + w\theta_0\theta_a B\rangle = 0$. Therefore, two cases occur to discuss.

C.1.1. If $B^P(a^3) \neq 0$ or $\Phi^P(a^3) \neq 0$, this will indeed prevent us from dividing by $x - a$ in (2.11), which leads usually to an analysis of the possible division by $x - b$. Given that $\Phi(b) = 0$, $B(b) = 3b^2B^P(b^3)$, $\Phi'(b) + \Psi(b) = (a-b)\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$, The current case can itself provide two more subcases.

C.1.1.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 6$.

C.1.1.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, then the functional equation can be simplified to $x - b$ and w satisfies (2.11), with $\Phi(x) = (x-a)\Phi^P(x^3)$ and $B(x) = 3x^2(x^2 + bx + b^2)(\theta_{b^3}B^P)(x^3)$.

Now, using Lemma 2.1, one finds

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)^2(c-b)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(b) = (b-a)\Phi^P(b^3) = 0$, $B(b) = 9b^4(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = 3b^2(b - a)\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. This subcase splits into two sub-subcases.

C.1.1.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, then no simplification of the functional equation occurs and the class remains $s = 3s' + 5$.

C.1.1.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, this indeed allows the functional equation to be simplified by $x - b$ and then w satisfies (2.11), with $\Phi(x) = (x-a)(x^2 + bx + b^2)(\theta_{b^3}\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2 + bx + b^2)^2(\theta_{b^3}^2 B^P)(x^3)$. By Lemma 2.1 we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Note that at this stage, the functional equation cannot be subject to simplification anymore. As a matter of fact, suppose we have $\langle u, \theta_{b^3}\Psi^P + \theta_{b^3}^2\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$. Then, $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$, and since $\Phi^P(b^3) = B^P(b^3) = 0$. Hence, dividing in (3.12) by $x - b$ can occur which yields a contradiction and eventually, $s = 3s' + 4$.

C.1.2. If $B^P(a^3) = \Phi^P(a^3) = 0$, then the functional equation can be simplified by $x - a$ and w satisfies (2.11), with $\Phi(x) = (x-b)\Phi^P(x^3)$ and $B(x) = 3x^2(x^2 + ax + a^2)(\theta_{a^3}B^P)(x^3)$. Applying Lemma 2.1 yields

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)(c-b)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

So that $\Phi(a) = (a-b)\Phi^P(a^3) = 0$, $B(a) = 9a^4(B^P)'(a^3)$, $\Phi'(a) + \Psi(a) = 3a^2(a-b)\Psi^P(a^3)$ and $\langle w, \theta_a\Psi + \theta_a^2\Phi + w\theta_0\theta_a B\rangle = 0$. Hence, two cases present themselves.

C.1.2.1. If $(B^P)'(a^3) \neq 0$ or $\Psi^P(a^3) \neq 0$, then the functional equation cannot be simplified to $(x-a)$. So, an analysis of the possible division by $x - b$ is required. We have $\Phi(b) = 0$, $B(b) = 3b^2(b^2 + ab + a^2)(\theta_{a^3}B^P)(b^3) = \frac{3b^2}{b-a}B^P(b^3)$, $\Phi'(b) + \Psi(b) = -(b-a)(b^2 + ab + a^2)(\theta_{a^3}\Phi^P)(b^3) = -(b^3 - a^3)(\theta_{a^3}\Phi^P)(b^3) = -\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. Two situations arise to discuss.

C.1.2.1.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then the functional equation can not be simplified and $s = 3s' + 5$.

C.1.2.1.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, then the functional equation can be simplified to $x - b$ and w satisfies (2.11), with $\Phi(x) = \Phi^P(x^3)$ and $B(x) = 3x^2(x^2 + ax + a^2)(x^2 + bx + b^2)(\theta_{b^3}\theta_{a^3}B^P)(x^3)$. By Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)(c-b)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(b) = \Phi^P(b^3) = 0$, $B(b) = 9b^4(b^2 + ab + a^2)(\theta_{a^3}B^P)'(b^3) = \frac{9b^4}{b-a}(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = 3b^2\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. In this case, two more subcases arise.

C.1.2.1.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, then the simplification of the functional equation cannot take place which makes $s = 3s' + 4$.

C.1.2.1.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, in this case the functional equation can be simplified to $x - b$ and w satisfies (2.11), with $\Phi(x) = (x^2 + bx + b^2)(\theta_{b^3}\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2 + ax + a^2)(x^2 + bx + b^2)^2(\theta_{b^3}^2\theta_{a^3}B^P)(x^3)$.

Again, using Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-a)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

So, the functional equation cannot be simplified anymore. It suffice to see that if one has $\langle u, \theta_{b^3}\Psi^P + \theta_{b^3}^2\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$, then $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$. And since we have $\Phi^P(b^3) = B^P(b^3) = 0$. Then, one can divide by $x-b$ in (3.12) which brings us to a contradiction. Hence, $s = 3s' + 3$.

C.1.2.2. If $(B^P)'(a^3) = \Psi^P(a^3) = 0$, then the functional equation can be simplified by $x-a$ and w satisfies (2.11), with $\Phi(x) = (x-b)(x^2+ax+a^2)(\theta_{a^3}\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2+ax+a^2)^2(\theta_{a^3}^2 B^P)(x^3)$.

Using Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-b)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Here, the functional equation cannot be simplified by $x-a$. Indeed, in the contrary, we have $\langle u, \theta_{a^3}\Psi^P + \theta_{a^3}^2\Phi^P + u\theta_0\theta_{a^3}B^P\rangle = 0$ and $\Psi^P(a^3) = (\Phi^P)'(a^3) = 0$. Hence, we derive $\Psi^P(a^3) + (\Phi^P)'(a^3) = 0$. But we also have $\Phi^P(a^3) = B^P(a^3) = 0$. Therefore, we can divide in (3.12) by $x-b$ and this yields a contradiction.

Thus, we need to see if a division by $x-b$ is possible. Since $\Phi(b) = 0$, $B(b) = 3b^2(b^2+ab+a^2)^2(\theta_{a^3}^2 B^P)(b^3) = \frac{3b^2}{(b-a)^2}B^P(b^3)$, $\Phi'(b) + \Psi(b) = (b^2+ab+a^2)(\theta_{a^3}\Phi^P)(b^3) = \frac{-1}{b-a}\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. Then, two subcases come up to discussion.

C.1.2.2.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 4$.

C.1.2.2.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, then the functional equation can be simplified by $x-b$ and w satisfies (2.11), with $\Phi(x) = (x^2+ax+a^2)(\theta_{a^3}\Phi^P)(x^3)$ and $B(x) = 3x^2(x^2+ax+a^2)^2(x^2+bx+b^2)(\theta_{b^3}\theta_{a^3}^2 B^P)(x^3)$.

Using Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2(c-b)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Then, it is easy to check that $\Phi(b) = (b^2+ab+a^2)(\theta_{a^3}\Phi^P)(b^3) = 0$, $B(b) = 9b^4(b^2+ab+a^2)^2(\theta_{a^3}^2 B^P)'(b^3) = \frac{9b^4}{(b-a)^2}(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = \frac{3b^2}{b-a}\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$, and so, we need to consider the two ensuing sub-subcases.

C.1.2.2.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, the functional equation cannot be subject to simplification and then $s = 3s' + 3$.

C.1.2.2.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, in this case, the functional equation can indeed be divided by $x-b$ and w satisfies (2.11), with

$$\Phi(x) = (x^2+ax+a^2)(x^2+bx+b^2)\big(\theta_{b^3}\theta_{a^3}\Phi^P\big)(x^3),$$
$$B(x) = 3x^2(x^2+ax+a^2)^2(x^2+bx+b^2)^2\big(\theta_{b^3}^2\theta_{a^3}^2 B^P\big)(x^3).$$

Lemma 2.1 yields

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c^2 \big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P\big\rangle.$$

At this point, the functional equation is no longer subject to simplification. By a contradiction argument, suppose we have $\langle u, \theta_{b^3}\Psi^P + \theta_{b^3}^2\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$. Therefore, $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$. But since $\Phi^P(b^3) = B^P(b^3) = 0$. Then, we can divide in (3.12) by $x - b$ which yields a contradiction. Eventually, $s = 3s' + 2$.

C.2. If $\Phi^P(0) = 0$, then the functional equation can be simplified by x and w satisfies (2.11), with $\Phi(x) = (x-a)(x-b)x^2(\theta_0\Phi^P)(x^3)$ and $B(x) = 3xB^P(x^3)$. We derive from Lemma 2.1 that

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3c(c-a)^2(c-b)^2 \big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P\big\rangle,$$

which implies that $\Phi(0)=0$, $B(0)=0$, $\Phi'(0)+\Psi(0)=0$ and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$. The functional equation can then be simplified by x and w satisfies (2.11), with $\Phi(x) = (x-a)(x-b)x(\theta_0\Phi^P)(x^3)$ and $B(x) = 3B^P(x^3)$.

Again, by Lemma 2.1, we get

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3(c-a)^2(c-b)^2 \big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P\big\rangle.$$

So, $\Phi(0) = 0$, $B(0) = 3B^P(0)$, $\Phi'(0) + \Psi(0) = 3ab((\Phi^P)'(0) + \Psi^P(0))$ and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle \neq 0$. Then, dividing by x in (2.11) cannot occur. So, as customary, we analyze the potential division by $x - a$ and by $x - b$. We have $\Phi(a) = 0$, $B(a) = 3B^P(a^3)$, $\Phi'(a) + \Psi(a) = -a^{-2}(a-b)\Phi^P(a^3)$ and $\langle w, \theta_a\Psi + \theta_a^2\Phi + w\theta_0\theta_a B\rangle = 0$. Consequently, two cases are unfolded.

C.2.1. If $B^P(a^3) \neq 0$ or $\Phi^P(a^3) \neq 0$, then the functional equation cannot be simplified by $x - a$. Thus, we can analyze the possible division by $x - b$. We have $\Phi(b) = 0$, $B(b) = 3B^P(b^3)$, $\Phi'(b) + \Psi(b) = -(b-a)b^{-2}\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. Two different sub-use cases should be considered.

C.2.1.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then no further simplification of the functional equation is possible and $s = 3s' + 4$.

C.2.1.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, this infers that the functional equation can be simplified to $x - b$ and w satisfies (2.11), with $\Phi(x) = (x-a)x(\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 + bx + b^2)(\theta_{b^3}B^P)(x^3)$. Both (4.10) and Lemma 2.1 imply that

$$\langle w, \theta_c \Psi + \theta_c^2 \Phi + w\theta_0\theta_c B\rangle = 3(c-a)^2(c-b) \big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2 \Phi^P + u\theta_0\theta_{c^3} B^P\big\rangle.$$

Hence, $\Phi(b) = (b-a)b(\theta_0\Phi^P)(b^3) = 0$, $B(b) = 9b^2(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = 3(b-a)\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. This situation shades the light on two other subcases.

C.2.1.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 3$.

C.2.1.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, then the functional equation can be simplified to $x - b$ and w satisfies (2.11), with $\Phi(x) = x(x-a)(x^2+bx+b^2)(\theta_{b^3}\theta_0\Phi^P)(x^3)$, $B(x) = 3(x^2+bx+b^2)^2(\theta_{b^3}^2 B^P)(x^3)$.

Using Lemma 2.1, we obtain

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(c-a)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

In this case, the functional equation is no longer subject to simplification. Indeed, on the contrary, if $\langle u, \theta_{b^3}\Psi^P + \theta_{b^3}^2\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$. Therefore, $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$. But we also have $\Phi^P(b^3) = B^P(b^3) = 0$. Then, one can divide in (3.12) by $x - b$ yielding a contradiction. Hence, $s = 3s' + 2$.

C.2.2. If $B^P(a^3) = \Phi^P(a^3) = 0$, then the functional equation can be simplified by $x - a$ and w satisfies (2.11), with $\Phi(x) = (x-b)x(\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2+ax+a^2)(\theta_{a^3}B^P)(x^3)$.

Using Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(x-a)(x-b)^2\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(a) = (a-b)a(\theta_0\Phi^P)(a^3) = 0$, $B(a) = 9a^2(B^P)'(a^3)$, $\Phi'(a) + \Psi(a) = 3(a-b)\Psi^P(a^3)$ and $\langle w, \theta_a\Psi + \theta_a^2\Phi + w\theta_0\theta_a B\rangle = 0$. This leads to the following subcases.

C.2.2.1. If $(B^P)'(a^3) \neq 0$ or $\Psi^P(a^3) \neq 0$, then the functional equation cannot be simplified by $x - a$. Thus, we can analyze the possible division by $x - b$. We have $\Phi(b) = 0$, $B(b) = 3(b^2+ab+a^2)(\theta_{a^3}B^P)(b^3) = \frac{3}{b-a}B^P(b^3)$, $\Phi'(b) + \Psi(b) = -3b^3(b-a)(\theta_{a^3}\theta_0\Phi^P)(b^3) = -3(b^2+ab+a^2)^{-1}\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. There are two subcases that may arise in this case.

C.2.2.1.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 3$.

C.2.2.1.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, then the functional equation can be simplified by $x - b$ and w satisfies (2.11), with $\Phi(x) = x(\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2+ax+a^2)(x^2+bx+b^2)(\theta_{b^3}\theta_{a^3}B^P)(x^3)$.

Lemma 2.1 implies that

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(x-a)(x-b)\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(b) = b(\theta_0\Phi^P)(b^3) = b^{-2}\Phi^P(b^3) = 0$, $B(b) = 9b^2(b^2+ab+a^2)(\theta_{a^3}B^P)'(b^3) = \frac{9b^2}{b-a}(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = 3\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. This fact provides us with two more sub-cases.

C.2.2.1.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, then no simplification of the functional equation may occur and $s = 3s' + 2$.

C.2.2.1.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, then it is possible to simplify the functional equation by $x - b$ and w satisfies (2.11), with $\Phi(x) = x(x^2 + bx + b^2)(\theta_{b^3}\theta_0\Phi^P)(x^3)$, $B(x) = 3(x^2 + ax + a^2)(x^2 + bx + b^2)^2(\theta^2_{b^3}\theta_{a^3}B^P)(x^3)$.

According to Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(x-a)\big\langle u, \theta_{c^3}\Psi^P + \theta^2_{c^3}\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

In this case, the functional equation cannot be simplified. Indeed, by a contradiction argument, we have $\langle u, \theta_{b^3}\Psi^P + \theta^2_{b^3}\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$. Hence, $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$. In other words $\Phi^P(b^3) = B^P(b^3) = 0$. Then, we can divide in (3.12) by $x - b$ and this brings us to a contradiction. As a consequence, $s = 3s' + 1$.

C.2.2.2. If $(B^P)'(a^3) = \Psi^P(a^3) = 0$, then the functional equation can be simplified by $x - a$ and w satisfies (2.11), with $\Phi(x) = x(x-b)(x^2 + ax + a^2)(\theta_{a^3}\theta_0\Phi^P)(x^3)$, $B(x) = 3(x^2 + ax + a^2)^2(\theta^2_{a^3}B^P)(x^3)$.

We use Lemma 2.1 to find

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(x-b)^2\big\langle u, \theta_{c^3}\Psi^P + \theta^2_{c^3}\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Here, the functional equation cannot longer be divided by $x - a$. Indeed, on the contrary, we have $\langle u, \theta_{a^3}\Psi^P + \theta^2_{a^3}\Phi^P + u\theta_0\theta_{a^3}B^P\rangle = 0$ and $\Psi^P(a^3) = (\Phi^P)'(a^3) = 0$. This implies that $\Psi^P(a^3) + (\Phi^P)'(a^3) = 0$. Also, we have $\Phi^P(a^3) = B^P(a^3) = 0$. Then, dividing in (3.12) by $x - b$ is not possible and this yields a contradiction.

So, we move now to inspect the possible division by $x - b$. We have $\Phi(b) = 0$, $B(b) = 3(b^2 + ab + a^2)^2(\theta^2_{a^3}B^P)(b^3) = \frac{3}{(b-a)^2}B^P(b^3)$, $\Phi'(b) + \Psi(b) = -b(b^2 + ab + a^2)(\theta_{a^3}\theta_0\Phi^P)(b^3) = \frac{-1}{b^2(b-a)}\Phi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. Then, two situations may emerge.

C.2.2.2.1. If $B^P(b^3) \neq 0$ or $\Phi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 2$.

C.2.2.2.2. If $B^P(b^3) = \Phi^P(b^3) = 0$, then in this case, it is possible to divide the functional equation by $x - b$ and w satisfies (2.11), with $\Phi(x) = x(x^2 + ax + a^2)(\theta_{a^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2 + ax + a^2)^2(x^2 + bx + b^2)(\theta_{b^3}\theta^2_{a^3}B^P)(x^3)$. Relying on Lemma 2.1 we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3(x-b)\big\langle u, \theta_{c^3}\Psi^P + \theta^2_{c^3}\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

Hence, $\Phi(b) = b(b^2 + ab + a^2)(\theta_{a^3}\Phi^P)(b^3) = 0$, $B(b) = 9b^2(b^2 + ab + a^2)^2(\theta^2_{a^3}B^P)'(b^3) = \frac{9b^2}{(b-a)^2}(B^P)'(b^3)$, $\Phi'(b) + \Psi(b) = \frac{3b^2}{b-a}\Psi^P(b^3)$ and $\langle w, \theta_b\Psi + \theta_b^2\Phi + w\theta_0\theta_b B\rangle = 0$. As a consequence, we split the current subcase into two sub-subcases.

C.2.2.2.2.1. If $(B^P)'(b^3) \neq 0$ or $\Psi^P(b^3) \neq 0$, then the functional equation cannot be simplified and $s = 3s' + 1$.

C.2.2.2.2.2. If $(B^P)'(b^3) = \Psi^P(b^3) = 0$, then the functional equation can be simplified to $x-b$ and w satisfies (2.11), with $\Phi(x) = x(x^2+ax+a^2)(x^2+bx+b^2)(\theta_{b^3}\theta_0\Phi^P)(x^3)$ and $B(x) = 3(x^2+ax+a^2)^2(x^2+bx+b^2)^2(\theta_{b^3}^2\theta_{a^3}^2 B^P)(x^3)$. Using Lemma 2.1, we get

$$\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\rangle = 3\big\langle u, \theta_{c^3}\Psi^P + \theta_{c^3}^2\Phi^P + u\theta_0\theta_{c^3}B^P\big\rangle.$$

In this case, the functional equation can no longer be submitted to further simplification. Indeed, suppose we have $\langle u, \theta_{b^3}\Psi^P + \theta_{b^3}^2\Phi^P + u\theta_0\theta_{b^3}B^P\rangle = 0$ and $\Psi^P(b^3) = (\Phi^P)'(b^3) = 0$. Therefore, $\Psi^P(b^3) + (\Phi^P)'(b^3) = 0$. But we have $\Phi^P(b^3) = B^P(b^3) = 0$. Then, we can divide in (3.12) by $x-b$ and this yields a contradiction. Hence, $s = 3s'$.

Next, we will introduce some notations in order to enlight the presentation of our main result. Let

$$X^P(x) := |\Phi^P(x)| + |B^P(x)|, \quad Y^P(x) := |\Psi^P(x)| + |(B^P)'(x)|,$$
$$Z^P(x) := 3\Psi^P(x) + 2(\Phi^P)'(x), \tag{4.35}$$
$$M^P(x) := |\Phi^P(x)| + |(B^P)'(x)|,$$
$$N^P(x) := |\Psi^P(x) - (\Phi^P)'(x)| + |(B^P)''(x)|, \tag{4.36}$$
$$R^P(x) := |\Psi^P(x)| + |(\Psi^P)'(x)| + |(B^P)'''(x)|. \tag{4.37}$$

Then, we have the following theorem.

Theorem 4.1. *Let s, s' be the class of w and $u = \sigma_{\varpi}(w)$, respectively. Then, we distinguish the following cases.*

Case A. $\rho(x) = x(x+\beta_0)$, $\beta_0 \neq 0$. *Then,*

$$\Phi^P(0) \neq 0 \Rightarrow \begin{cases} X^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+6, \\ X^P(-\beta_0^3) = 0 \Rightarrow \begin{cases} Y^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+5, \\ Y^P(-\beta_0^3) = 0 \Rightarrow s = 3s'+4. \end{cases} \end{cases}$$

$\Phi^P(0) = 0 \Rightarrow$

$$\begin{cases} B^P(0) \neq 0 \Rightarrow \begin{cases} X^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+4, \\ X^P(-\beta_0^3) = 0 \Rightarrow \begin{cases} Y^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+3, \\ Y^P(-\beta_0^3) = 0 \Rightarrow s = 3s'+2, \end{cases} \end{cases} \\ B^P(0) = 0 \Rightarrow \begin{cases} Z^P(0) \neq 0 \Rightarrow \begin{cases} X^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+3, \\ X^P(-\beta_0^3) = 0 \Rightarrow \begin{cases} Y^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+2, \\ Y^P(-\beta_0^3) = 0 \Rightarrow s = 3s'+1, \end{cases} \end{cases} \\ Z^P(0) = 0 \Rightarrow \begin{cases} X^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+4, \\ X^P(-\beta_0^3) = 0 \Rightarrow \begin{cases} Y^P(-\beta_0^3) \neq 0 \Rightarrow s = 3s'+1, \\ Y^P(-\beta_0^3) = 0 \Rightarrow s = 3s'. \end{cases} \end{cases} \end{cases} \end{cases}$$

Case B. $\rho(x) = (x-d)^2$ *with* $d = -\frac{\beta_0}{2} \neq 0$. *Then,*

$\Phi^P(0) \neq 0 \Rightarrow$

$$\begin{cases} B^P(d^3) \neq 0 \Rightarrow s = 3s' + 6, \\ B^P(d^3) = 0 \Rightarrow \begin{cases} M^P(d^3) \neq 0 \Rightarrow s = 3s' + 5, \\ M^P(d^3) = 0 \Rightarrow \begin{cases} N^P(d^3) \neq 0 \Rightarrow s = 3s' + 4, \\ N^P(d^3) = 0 \Rightarrow \begin{cases} R^P(d^3) \neq 0 \Rightarrow s = 3s' + 3, \\ R^P(d^3) = 0 \Rightarrow s = 3s' + 2. \end{cases} \end{cases} \end{cases} \end{cases}$$

$\Phi^P(0) = 0 \Rightarrow$

$$\begin{cases} B^P(d^3) \neq 0 \Rightarrow s = 3s' + 4, \\ B^P(d^3) = 0 \Rightarrow \begin{cases} M^P(d^3) \neq 0 \Rightarrow s = 3s' + 3, \\ M^P(d^3) = 0 \Rightarrow \begin{cases} N^P(d^3) \neq 0 \Rightarrow s = 3s' + 2, \\ N^P(d^3) = 0 \Rightarrow \begin{cases} R^P(d^3) \neq 0 \Rightarrow s = 3s' + 1, \\ R^P(d^3) = 0 \Rightarrow s = 3s'. \end{cases} \end{cases} \end{cases} \end{cases}$$

Case C. $\rho(x) = (x-a)(x-b)$ *with* $ab \neq 0$, $a \neq b$. *Then,*

$\Phi^P(0) \neq 0 \Rightarrow$

$$\begin{cases} X^P(a^3) \neq 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 6, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 5, \\ Y^P(b^3) = 0 \Rightarrow s = 3s' + 4, \end{cases} \end{cases} \\ X^P(a^3) = 0 \Rightarrow \begin{cases} Y^P(a^3) \neq 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 5, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 4, \\ Y^P(b^3) = 0 \Rightarrow s = 3s' + 3, \end{cases} \end{cases} \\ Y^P(a^3) = 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 4, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 3, \\ Y^P(b^3) = 0 \Rightarrow s = 3s' + 2. \end{cases} \end{cases} \end{cases} \end{cases}$$

$\Phi^P(0) = 0 \Rightarrow$

$$\begin{cases} X^P(a^3) \neq 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 4, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 3, \\ Y^P(b^3) = 0 \Rightarrow s = 3s' + 2, \end{cases} \end{cases} \\ X^P(a^3) = 0 \Rightarrow \begin{cases} Y^P(a^3) \neq 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 3, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 2, \\ Y^P(b^3) = 0 \Rightarrow s = 3s' + 1, \end{cases} \end{cases} \\ Y^P(a^3) = 0 \Rightarrow \begin{cases} X^P(b^3) \neq 0 \Rightarrow s = 3s' + 2, \\ X^P(b^3) = 0 \Rightarrow \begin{cases} Y^P(b^3) \neq 0 \Rightarrow s = 3s' + 1, \\ Y^P(b^3) = 0 \Rightarrow s = 3s'. \end{cases} \end{cases} \end{cases} \end{cases}$$

5. An example of Laguerre–Hahn linear functional of class 1

In this section, as an application of the results of the previous sections, we determine all the Laguerre–Hahn orthogonal polynomial sequences $\{W_n\}_{n\geq 0}$ of class 1 obtained via cubic transformations $W_{3n}(x) = P_n(x^3)$, requiring $\{P_n\}_{n\geq 0}$ to be a singular Laguerre–Hahn sequence of class zero.

Proposition 3.1 and Theorem 4.1 state that w is a Laguerre–Hahn linear functional of class $s = 1$ if and only if the first component $u = \sigma_{\varpi}(w)$, is a Laguerre–Hahn linear functional of class $s' = 0$ and one of the following conditions hold:

A.2.2.2.2.1. $\rho(x) = x(x + \beta_0), \quad \beta_0 \neq 0; \quad \Phi^P(0) = 0, \quad B^P(0) = 0, \quad Z^P(0) = 0,$
$X^P(-\beta_0^3) = 0, \quad Y^P(-\beta_0^3) \neq 0.$

A.2.2.1.2.2. $\rho(x) = x(x + \beta_0), \quad \beta_0 \neq 0; \quad \Phi^P(0) = 0, \quad B^P(0) = 0, \quad Z^P(0) \neq 0,$
$X^P(-\beta_0^3) = 0, \quad Y^P(-\beta_0^3) = 0.$

B.2.2.2.2.1. $\rho(x) = (x - d)^2, \quad d = -\dfrac{\beta_0}{2} \neq 0; \quad \Phi^P(0) = 0, \quad B^P(d^3) = 0,$
$M^P(d^3) = 0, \quad N^P(d^3) = 0, \quad R^P(d^3) \neq 0.$

C.2.2.1.2.2. $\rho(x) = (x - a)(x - b), \quad ab \neq 0, \quad a \neq b; \quad \Phi^P(0) = 0, \quad X^P(a^3) = 0,$
$Y^P(a^3) \neq 0, \quad X^P(b^3) = 0, \quad Y^P(b^3) = 0.$

C.2.2.2.2.1. $\rho(x) = (x - a)(x - b), \quad ab \neq 0, \quad a \neq b; \quad \Phi^P(0) = 0, \quad X^P(a^3) = 0,$
$Y^P(a^3) = 0, \quad X^P(b^3) = 0, \quad Y^P(b^3) \neq 0.$

First, we will analyze the conditions in A.2.2.2.2.1. Let us consider a complex number $\beta_0 \neq 0$ and according to (4.35), it is clear that these conditions are equivalent to $\rho(x) = x(x + \beta_0)$. As a consequence,

$$\Phi^P(0) = 0, \quad B^P(0) = 0, \quad 3\Psi^P(0) + 2(\Phi^P)'(0) = 0, \quad \Phi^P(-\beta_0^3) = 0,$$
$$B^P(-\beta_0^3) = 0, \quad \Psi^P(-\beta_0^3) \neq 0 \quad \text{or} \quad (B^P)'(-\beta_0^3) \neq 0.$$

Since $s' = 0$ implies $\deg \Phi^P \leq 2$, then from the conditions $\Phi^P(0) = \Phi^P(-\beta_0^3) = 0$ we have

$$\Phi^P(x) = x(x + \beta_0^3). \tag{5.1}$$

Therefore, from (2.15) and (2.16), $u = \sigma_{\varpi}(w)$ can be obtained by shifting the linear functional satisfying (3.12) with $\Phi^P(x) = x^2 - 1$. In fact,

$$u = h_{\beta_0^3/2} \circ \tau_{-1}\mathfrak{J}(\alpha, \rho, \nu, \mu),$$

where $\mathfrak{J}(\alpha, \rho_1, \nu, \mu)$ is the singular Laguerre–Hahn linear functional of class zero analogous to the classical Jacobi. Indeed, it satisfies [10]

$$(\phi\mathfrak{J}(\alpha, \rho, \nu, \mu))' + \psi\mathfrak{J}(\alpha, \rho, \nu, \mu) + \varphi(x^{-1}\mathfrak{J}^2(\alpha, \rho, \nu, \mu)) = 0,$$

with

$$\phi(x) = x^2 - 1,$$
$$\psi(x) = (\alpha - 2)x + \mu,$$
$$\varphi(x) = (1-\alpha)x^2 + \left(\alpha\nu - 2\mu\frac{\alpha-1}{\alpha-2}\right)x - \frac{\mu^2}{\alpha-2} + \mu\nu + \rho(\alpha+1) - 1.$$

$\mathfrak{J}(\alpha, \rho, \nu, \mu)$ is regular if and only if $\rho \neq 0$, $\alpha \neq 2$, $\alpha \neq -n$, $n \geq 0$, and $\alpha \neq \pm\mu - 2n$, $n \geq 1$.

In this case,

$$\Psi^P(x) = (\alpha-2)x + \frac{\beta_0^3}{2}(\alpha - 2 + \mu), \tag{5.2}$$

$$B^P(x) = \left(\frac{2}{\beta_0^3}\right)^{-2}\left[(1-\alpha)\left(\frac{2}{\beta_0^3}x+1\right)^2 + \left(\alpha\nu - 2\mu\frac{\alpha-1}{\alpha-2}\right)\left(\frac{2}{\beta_0^3}x+1\right) - \frac{\mu^2}{\alpha-2} + \mu\nu + \rho(\alpha+1) - 1\right]. \tag{5.3}$$

From $3\Psi^P(0) + 2(\Phi^P)'(0) = 0$ and the fact that $(\Phi^P)'(0) = \beta_0^3$, $\Psi^P(0) = \frac{\beta_0^3}{2}(\alpha - 2 + \mu)$, we get

$$\mu = \frac{2}{3} - \alpha,$$

which readily yields

$$\Psi^P(x) = (\alpha-2)x - \frac{2}{3}\beta_0^3. \tag{5.4}$$

Moreover, conditions $B^P(0) = 0$ and $B^P(-\beta_0^3) = 0$ are equivalent to

$$\nu = \frac{2}{3}\frac{(2-3\alpha)(\alpha-1)}{\alpha(\alpha-2)}, \quad \rho = \frac{4}{9}\frac{(3\alpha-1)}{\alpha(\alpha+1)}.$$

As a consequence,

$$B^P(x) = (1-\alpha)x(x+\beta_0^3). \tag{5.5}$$

Therefore, it follows from (5.1), (5.4), and (5.5) and Proposition 3.1 that

$$\Phi(x) = x^4(x+\beta_0)(x^3+\beta_0^3),$$
$$\Psi(x) = 3x^3(x+\beta_0)\left[(\alpha-2)x^3 - \frac{2}{3}\beta_0^3\right] - 2(2x+\beta_0)x^3(x^3+\beta_0^3),$$
$$B(x) = 3(1-\alpha)x^5(x^3+\beta_0^3).$$

Thus, by simple computations we obtain, after division by $x^4(x+\beta_0)$

$$\Phi(x) = x^3 + \beta_0^3,$$
$$\Psi(x) = (3\alpha-5)x^2 + \beta_0 x - \beta_0^2,$$
$$B(x) = 3(1-\alpha)x(x^2 - \beta_0 x + \beta_0^2). \tag{5.6}$$

So, here, w is a Laguerre–Hahn linear functional of class $s = 1$ and satisfying (2.11) with Φ, Ψ, B given in (5.6).

The linear functionals described above are the unique Laguerre–Hahn linear functionals of class 1 obtained via cubic transformations of the form $W_{3n}(x) = P_n(x^3)$, requiring $\{P_n\}_{n\geq 0}$ to be a singular Laguerre–Hahn sequence of class zero (see [27]). In fact, we will show that the conditions in A.2.2.1.2.2., B.2.2.2.2.1., C.2.2.1.2.2., and C.2.2.2.2.1. are not compatible with the regularity of u. Indeed, let us assume that there exists a regular linear functional w satisfying (2.11) under the conditions in A.2.2.1.2.2.

By the notations (4.35), it is clear that these conditions are equivalent to, $\rho(x) = x(x+\beta_0)$, $\beta_0 \neq 0$;

$$\Phi^P(0) = 0, \quad B^P(0) = 0, \quad 3\Psi^P(0) + 2(\Phi^P)'(0) \neq 0, \quad \Phi^P(-\beta_0^3) = 0,$$
$$B^P(-\beta_0^3) = 0, \quad \Psi^P(-\beta_0^3) = 0, \quad (B^P)'(-\beta_0^3) = 0.$$

Since $s' = 0$ implies $\deg \Phi^P \leq 2$, then by the conditions $\Phi^P(0) = 0$ and $\Phi^P(-\beta_0^3) = 0$ we have

$$\Phi^P(x) = x(x+\beta_0^3).$$

Thus, in this case, Ψ^P and B^P are given by equations (5.2) and (5.3). Conditions $B^P(0) = B^P(-\beta_0^3) = (B^P)'(-\beta_0^3) = 0$ and on account of $\deg B^P \leq 2$, we assume that B^P is identically null, which gives $\alpha = 1$, $\nu = 0$ and

$$\mu^2 + 2\rho - 1 = 0. \tag{5.7}$$

On the other hand, from the condition $\Psi^P(-\beta_0^3) = 0$ and taking into account $\beta_0 \neq 0$, it is clear that $\mu = -1$. Hence, from (5.7) we get $\rho = 0$ which contradicts the regularity of the linear functional u.

Next, let us assume that there exists a regular linear functional w satisfying (2.11) under conditions in B.2.2.2.2.1.

Since $s' = 0$ implies $\deg \Phi^P \leq 2$, then by the conditions $\Phi^P(0) = \Phi^P(d^3) = 0$ we have

$$\Phi^P(x) = x(x-d^3).$$

Therefore,

$$u = h_{-d^3/2} \circ \tau_{-1}\mathfrak{J}(\alpha,\rho,\nu,\mu).$$

In this case,

$$\Psi^P(x) = (\alpha-2)x - \frac{d^3}{2}(\alpha - 2 + \mu),$$
$$B^P(x) = \left(\frac{2}{d^3}\right)^{-2}\left[(1-\alpha)\left(-\frac{2}{d^3}x+1\right)^2 + \left(\alpha\nu - 2\mu\frac{\alpha-1}{\alpha-2}\right)\left(-\frac{2}{d^3}x+1\right) - \frac{\mu^2}{\alpha-2} + \mu\nu + \rho(\alpha+1) - 1\right].$$

Taking into account that conditions

$$B^P(d^3) = (B^P)'(d^3) = (B^P)''(d^3) = 0$$

hold, and since $\deg B^P \leq 2$, we assume that the polynomial B^P is identically zero, which then gives $\alpha = 1$, $\nu = 0$ and

$$\mu^2 + 2\rho - 1 = 0. \tag{5.8}$$

On the other hand, based on $\Psi^P(d^3) - (\Phi^P)'(d^3) = 0$, the fact that $(\Phi^P)'(d^3) = d^3$ and $\Psi^P(d^3) = -\frac{d^3}{2}(\mu + 1)$, we get

$$\frac{d^3}{2}(\alpha - \mu - 4) = 0,$$

which gives $\mu = -1$. Hence, combined with (5.8) we then have $\rho = 0$ which is in contradiction with the regularity of the linear functional u.

Finally, let us assume that there exists a regular linear functional w satisfying (2.11) under the conditions in C.2.2.1.2.2. or those in C.2.2.2.2.1. Since $s' = 0$ implies $\deg \Phi^P \leq 2$, so from the conditions

$$\Phi^P(0) = \Phi^P(a^3) = \Phi^P(b^3) = 0,$$

we conclude that, in this case, $\deg \Phi^P \geq 3$ which yields a contradiction.

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