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Normalized solutions of L^2-supercritical NLS equations on compact metric graphs

Xiaojun Chang, Louis Jeanjean, and Nicola Soave

Abstract. This paper is devoted to the existence of non-trivial bound states of prescribed mass for the mass-supercritical nonlinear Schrödinger equation on compact metric graphs. The investigation is based upon a min-max principle for some constrained functionals which combines the monotonicity trick and second-order information on the Palais–Smale sequences, and upon the blow-up analysis of bound states with prescribed mass and bounded Morse index.

1. Introduction and main results

In this paper we investigate the existence of non-constant critical points for the *mass supercritical* NLS energy functional $E(\cdot, \mathcal{G})\colon H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 \, dx - \frac{1}{p} \int_{\mathcal{G}} |u|^p \, dx, \quad p > 6 \tag{1.1}$$

under the mass constraint

$$\int_{\mathcal{G}} |u|^2 \, dx = \mu > 0, \tag{1.2}$$

where $\mathcal{G}$ is a *compact metric graph*. Critical points, also called *bound states*, solve the stationary nonlinear Schrödinger equation (NLS) on $\mathcal{G}$,

$$-u'' + \lambda u = |u|^{p-2} u,$$

for some Lagrange multiplier λ, coupled with the Kirchhoff condition at the vertexes (see (1.3) below). In turn, solutions to (1.3) give standing waves of the time-dependent focusing NLS on $\mathcal{G}$,

$$i \partial_t \psi(t, x) = -\partial_{xx} \psi(t, x) - |\psi(t, x)|^{p-2} \psi(t, x),$$

via the ansatz $\psi(t, x) = e^{i\lambda t} u(x)$. The constraint (1.2) is dynamically meaningful as the mass (or charge), as well as the energy, is conserved by the NLS flow.

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One of the main physical motivations to consider the Schrödinger equation on metric graphs is the study of propagation of optical pulses in nonlinear optics, or of matter waves (in the theory of Bose–Einstein condensates), in ramified structures, such as T-junctions or X-junctions. We refer the interested reader to the recent paper [26], to [5, 10, 29, 30], and to the references therein for more details. In addition, the problem on metric graphs presents interesting new mathematical features with respect to the Euclidean case. For these reasons, the problem of existence of bound states on metric graphs attracted a lot of attention in the past decade, mainly in the *subcritical* or *critical regimes*, which correspond to $p \in (2, 6)$ or $p = 6$, respectively. In such frameworks, a particularly relevant issue concerns the existence of *ground states*, that is, global minimizers of the energy under the mass constraint; see [1–4] for non-compact $\mathcal{G}$, and [13, 14] for the compact case. We also refer to [11, 16, 31–33, 35] and references therein for strictly related issues (problems with localized nonlinearities, combined nonlinearities, existence of critical points in absence of ground states), always in subcritical and critical regimes.

In striking contrast, the supercritical regime on general graphs is essentially untouched. We are only aware of [6] in which the assumptions allow the analysis to be reduced to the study of minimizing sequences living in a bounded subset of the constraint; see Remark 1.4. Actually, in the supercritical regime the energy is always unbounded from below (see the proof of Lemma 3.4 below), and ground states never exist. However, it is natural to discuss the existence of bound states, and in this paper we address this problem on any *compact* graph $\mathcal{G}$. An interesting feature of this setting is that there always exists a constrained constant (trivial) critical point of $E(\cdot, \mathcal{G})$, obtained by taking the constant function $\kappa_\mu := (\mu/\ell)^{1/2}$, where ℓ denotes the total length of $\mathcal{G}$. Thus, in order to obtain a non-trivial result, one has to focus on existence of non-constant bound states.

Basic notation and main result

We recall that a metric graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ is a connected metric space obtained by glueing together a number of closed line intervals, the edges in $\mathcal{E}$, by identifying some of their endpoints, the vertexes in $\mathcal{V}$. The peculiar way in which these identifications are performed defines the topology of $\mathcal{G}$. Any bounded edge e is identified with a closed bounded interval I_{e}, typically $[0, \ell_{\mathrm{e}}]$ (where ℓ_{e} is the length of e), while unbounded edges are identified with (a copy of) the closed half-line $[0, +\infty)$. A metric graph is compact if and only if it has a finite number of edges, and none of them is unbounded.

A function u on $\mathcal{G}$ is a map $u\colon \mathcal{G} \to \mathbb{R}$, which is identified with a vector of functions $\{u_{\mathrm{e}}\}$, where each u_{e} is defined on the corresponding interval I_{e}. Endowing each edge with Lebesgue measure, one can define L^p spaces over $\mathcal{G}$, denoted by $L^p(\mathcal{G})$, in a natural way, with norm

$$\|u\|_{L^p(\mathcal{G})}^p = \sum_{\mathrm{e}} \|u_{\mathrm{e}}\|_{L^p(\mathrm{e})}^p.$$

The Sobolev space $H^1(\mathcal{G})$ is defined as the set of functions $u\colon \mathcal{G} \to \mathbb{R}$ such that $u_{\mathrm{e}} \in H^1([0, \ell_{\mathrm{e}}])$ for every bounded edge e, $u_{\mathrm{e}} \in H^1([0, +\infty))$ for every unbounded edge e, and u is continuous on $\mathcal{G}$ (in particular, if a vertex v belongs to two or more edges e_i, the corresponding functions u_{e_i} take the same value on v); the norm in $H^1(\mathcal{G})$ is naturally defined as

$$\|u\|^2_{H^1(\mathcal{G})} = \sum_{\mathrm{e}} \big[\|u'_{\mathrm{e}}\|^2_{L^2(\mathrm{e})} + \|u_{\mathrm{e}}\|^2_{L^2(\mathrm{e})}\big].$$

We aim to prove the existence of non-constant critical points of the energy $E(\cdot,\mathcal{G})$, defined in (1.1), constrained on the L^2-sphere

$$H^1_\mu(\mathcal{G}) := \big\{u \in H^1(\mathcal{G}) \colon \textstyle\int_{\mathcal{G}} |u|^2\,dx = \mu\big\}.$$

If $u \in H^1_\mu(\mathcal{G})$ is such a critical point, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem:

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{for every edge } \mathrm{e} \in \mathcal{E}, \\ \sum_{\mathrm{e} \succ \mathrm{v}} u'_{\mathrm{e}}(\mathrm{v}) = 0 & \text{at every vertex } \mathrm{v} \in \mathcal{V}, \end{cases} \tag{1.3}$$

where $\mathrm{e} \succ \mathrm{v}$ means that the edge e is incident at v, and the derivative $u'_{\mathrm{e}}(v)$ is always an outer derivative. The second equation is the so-called *Kirchhoff condition*. Note that at external vertexes, namely vertexes which are reached by a unique edge, the Kirchhoff conditions reduce to purely Neumann conditions. Finally, notice that the positive constant function $\kappa_\mu = (\mu/\ell)^{1/2}$ trivially satisfies (1.3), for $\lambda = (\mu/\ell)^{(p-2)/2}$.

Our main existence result is as follows.

Theorem 1.1. *Let $\mathcal{G}$ be any compact metric graph, and $p > 6$. There exists $\mu_1 > 0$ depending on $\mathcal{G}$ and on p such that, for any $0 < \mu < \mu_1$, problem (1.3) with the mass constraint (1.2) has a positive non-constant solution which corresponds to a mountain pass critical point of $E(\cdot,\mathcal{G})$ on $H^1_\mu(\mathcal{G})$, at a strictly larger energy level than κ_μ.*

Remark 1.2. Note that the Lagrange multiplier associated with any positive solution u to (1.3) is positive. Indeed, by standard arguments, we know that $u \in C^2(\mathrm{e})$ on every edge. Then, integrating the first equation in (1.3) on every edge, summing over the edges, and making use of the Kirchhoff condition, we obtain

$$\lambda \int_{\mathcal{G}} |u|\,dx = \int_{\mathcal{G}} |u|^{p-1}\,dx,$$

whence we deduce that $\lambda > 0$.

Remark 1.3. The theorem is not a perturbation result, in the sense that the value μ_1 will not be obtained by any limit process, and can be explicitly estimated. We refer to Proposition 2.1 and Remark 2.2 for more details.

On the other hand, one may wonder whether or not the restriction $\mu < \mu_1$ can be removed. This is an open problem; our min-max approach fails for large masses. Observing that our solutions will have Morse index at most 2 as critical points of the associated action functional (see Section 3), another related issue could be to investigate whether it is possible to find solutions of (1.3), possibly non-positive, with any mass $\mu > 0$ and Morse index bounded by 2. For the NLS equations with Dirichlet conditions in bounded Euclidean domains, this question has a negative answer; see [34, Theorem 1.2]. Even if the two problems are not equivalent, this result suggests that a bound of type $\mu < \mu_1$ may be necessary.

The proof of Theorem 1.1 is divided into some intermediate steps. First, in Section 2, we observe the local minimality of the constant solution for $\mu < \mu_1$, following [13].

Since in addition $E(\cdot, \mathcal{G})$ is unbounded from below, as $p > 6$, this naturally suggests the possible existence of a second critical point, of mountain pass type. However, if $E(\cdot, \mathcal{G})$ indeed has a mountain pass geometry, it is unclear whether there exists a *bounded* Palais–Smale sequence at the mountain pass level. Since $\mathcal{G}$ is compact the existence of such a sequence would guarantee a corresponding critical point. We point out that, to establish its existence, the techniques based on scaling, usually employed in the Euclidean setting and related to the validity of a Pohozaev identity (see [23] or [7, 22]), do not work, since $\mathcal{G}$ is not scale invariant. To overcome this obstruction, we shall construct a special Palais–Smale sequence whose elements are exact critical points of some approximating problems. To this aim we introduce the family of functionals $E_\rho(\cdot, \mathcal{G})\colon H^1_\mu(\mathcal{G}) \to \mathbb{R}$ defined by

$$E_\rho(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{\rho}{p}\int_{\mathcal{G}} |u|^p\,dx, \quad \rho \in \left[\tfrac{1}{2}, 1\right].$$

Exploiting the monotonicity of $E_\rho(u, \mathcal{G})$ with respect to ρ as in [24, Theorem 1.1], it is relatively straightforward to show that $E_\rho(\cdot, \mathcal{G})|_{H^1_\mu(\mathcal{G})}$ has a bounded Palais–Smale sequence of mountain pass type, for almost every $\rho \in [1/2, 1]$. This ensures the existence of a critical point u_ρ of $E_\rho(u, \mathcal{G})$, for almost every $\rho \in [1/2, 1]$; see Lemma 3.1. At this point it is natural to take the limit of $\{u_{\rho_n}\}$ along a sequence $\rho_n \to 1^-$, in order to obtain a critical point of the original functional. Indeed, if $\{u_{\rho_n}\}$ is bounded, it proves to be a Palais–Smale sequence of the functional $E(\cdot, \mathcal{G})$. The idea behind the introduction of a family of approximating problems is that one expects to show more easily the boundedness of a sequence of exact critical points (of the approximating problems), than the boundedness of an arbitrary Palais–Smale sequence. We refer to [24, 25] for an exposition of this way to attack the boundedness of Palais–Smale sequences, a way which is inspired by the work of Struwe on the monotonicity trick [36].

However, in the present situation once again the boundedness of $\{u_{\rho_n}\}$ is an issue. To overcome it, we shall look for solutions of the approximating problems having additional properties. We shall make use of [12, Theorem 1], recalled here as Theorem 3.10. Applying this abstract result to our problem we obtain the existence of a sequence $\{u_{\rho_n}\}$ of critical points for $E_{\rho_n}(\cdot, \mathcal{G})|_{H^1_\mu(\mathcal{G})}$ *with uniformly bounded Morse index.*

Roughly speaking, [12, Theorem 1] guarantees, for a parametrized family of functionals having a uniform mountain pass geometry on a constraint, the existence of a bounded Palais–Smale sequence with second-order, or Morse-type, properties, for almost every value of the parameter. It has been known since the pioneering work of Lions [27], see also [19, 20], that second-order information can turn out to be extremely useful for proving the compactness of Palais–Smale sequences. The proof of [12, Theorem 1] relies on a combination of the monotonicity trick, as presented in [24], on ideas from [19, 20], and on geometric considerations. Related results, but in unconstrained settings, were recently established in [9, 28].

In Section 4, we perform a detailed blow-up analysis for this type of sequence, in the spirit of [18] (see also [34]). We think that this analysis is of independent interest and, for the sake of generality, we perform it on graphs which are not necessarily compact. In Theorem 4.2, we characterize the blow-up behavior of solutions close to local maximum points, both when they accumulate in the interior of one edge and when they accumulate on a vertex; in the latter case, the limit problem is an NLS equation posed on a *star graph*, which is a new phenomenon with respect to the Euclidean case. In Theorem 4.6, we establish a relation between the upper bound on the Morse index and the number of maximum points of the solutions, and describe the behavior far away from them.

Afterwards, Theorems 4.2 and 4.6 are used in Section 5 to finally deduce, via a contradiction argument, that also the sequence $\{u_{\rho_n}\}$ is bounded, and converges to the non-constant mountain pass solution of Theorem 1.1.

Remark 1.4. In [6] the more general case of a graph with a compact core and a finite number of half-lines attached is considered. However, due to the presence of an external potential well or of attractive delta boundary conditions at the vertexes, the search for a solution can be reduced, following the strategy introduced in [8], to the search for a local minimum on a bounded subset of the mass constraint. So one avoids the issue of the boundedness of Palais–Smale sequences, which is the central difficulty in our problem.

2. Local minimality of the constant solution

Let $\kappa_\mu := (\mu/\ell)^{1/2}$ with $\ell := |\mathcal{G}|$ being the total length of the graph $\mathcal{G}$. Clearly, the constant function κ_μ is always a solution to (1.3) in $H^1_\mu(\mathcal{G})$ for some $\lambda \in \mathbb{R}$, and hence a constrained critical point of $E(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$. Furthermore, following [13], we can give a variational characterization of κ_μ.

Proposition 2.1. *Assume that $\mathcal{G}$ is a compact metric graph and $p > 2$. Then there exists $\mu_1 > 0$ depending on $\mathcal{G}$ and on p such that*

(i) *if $0 < \mu < \mu_1$, then κ_μ is a strict local minimizer of $E(u, \mathcal{G})$ in $H^1_\mu(\mathcal{G})$;*

(ii) *if $\mu > \mu_1$, then κ_μ is not a local minimizer of $E(u, \mathcal{G})$ in $H^1_\mu(\mathcal{G})$.*

Proof. To characterize the variational properties of κ_μ, we shall evaluate the sign of the quadratic form $\varphi \in T_{\kappa_\mu} H^1_\mu(\mathcal{G}) \mapsto d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu, \mathcal{G})[\varphi, \varphi]$, where $d^2|_{H^1_\mu(\mathcal{G})} E(u, \mathcal{G})$ denotes the constrained Hessian of $E(\cdot \mathcal{G})$ on $H^1_\mu(\mathcal{G})$ and $T_{\kappa_\mu} H^1_\mu(\mathcal{G})$ is the tangent space of $H^1_\mu(\mathcal{G})$ at κ_μ, defined as

$$T_{\kappa_\mu} H^1_\mu(\mathcal{G}) := \{\phi \in H^1(\mathcal{G}) \colon \textstyle\int_{\mathcal{G}} \phi \, dx = 0\}.$$

From [13, Proposition 4.1], which remains valid with the same proof for $p > 6$, we obtain

$$\begin{aligned} & d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu, \mathcal{G})[\phi, \phi] \\ & \quad = \int_{\mathcal{G}} |\phi'|^2 \, dx - (p-2)\kappa_\mu^{p-2} \int_{\mathcal{G}} |\phi|^2 \, dx \quad \forall \phi \in T_{\kappa_\mu} H^1_\mu(\mathcal{G}). \end{aligned} \tag{2.1}$$

Now denote by $\lambda_2(\mathcal{G})$ the smallest positive eigenvalue of the Kirchhoff Laplacian on $\mathcal{G}$ (that is, $-(\cdot)''$ on $\mathcal{G}$, coupled with the Kirchhoff condition at the vertexes), namely

$$\lambda_2(\mathcal{G}) = \inf_{\substack{\phi \in H^1(\mathcal{G}) \\ \int_{\mathcal{G}} \phi \, dx = 0}} \frac{\int_{\mathcal{G}} |\phi'|^2 dx}{\int_{\mathcal{G}} |\phi|^2 dx}.$$

Let us suppose that $0 < \mu < \mu_1$, where

$$\mu_1 := \ell \Big(\frac{\lambda_2(\mathcal{G})}{p-2} \Big)^{\frac{2}{p-2}}, \tag{2.2}$$

and let $\beta \in (0,1)$ be such that

$$\beta \lambda_2(\mathcal{G}) - (p-2) \Big(\frac{\mu}{\ell} \Big)^{\frac{p-2}{2}} > 0.$$

In view of (2.1), it follows that

$$d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu, \mathcal{G})[\phi, \phi] \geq (1-\beta) \int_{\mathcal{G}} |\phi'|^2 + \big[\beta \lambda_2(\mathcal{G}) - (p-2)\kappa_\mu^{p-2}\big] \int_{\mathcal{G}} |\phi|^2 \, dx$$

for every $\phi \in T_{\kappa_\mu} H^1_\mu(\mathcal{G})$, which implies that $d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu, \mathcal{G})$ is positive definite whenever $0 < \mu < \mu_1$. Hence, for any such μ, the constant κ_μ is a strict local minimizer of $E(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$.

If instead $\mu > \mu_1$, taking an eigenfunction ϕ_2 corresponding to $\lambda_2(\mathcal{G})$, we obtain

$$d^2|_{H^1_\mu(\mathcal{G})} E(\kappa_\mu, \mathcal{G})\big[\phi_2, \phi_2\big] = \big[\lambda_2(\mathcal{G}) - (p-2)\kappa_\mu^{p-2}\big] \int_{\mathcal{G}} |\phi_2|^2 \, dx < 0,$$

which implies that κ_μ is not a local minimizer of $E(u, \mathcal{G})$ in $H^1_\mu(\mathcal{G})$. ■

Remark 2.2. By [21, Theorem 1], we have $\lambda_2(\mathcal{G}) \geq \pi^2/\ell^2$. Then by (2.2) it follows that

$$\mu_1 \geq \ell^{\frac{p-6}{p-2}} \Big(\frac{\pi^2}{p-2} \Big)^{\frac{2}{p-2}}.$$

In particular, $\mu_1 \to +\infty$ as $\ell \to +\infty$.

3. Mountain pass solutions for approximating problems

When κ_μ is a local minimizer of the energy, and since the energy is unbounded from below on $H^1_\mu(\mathcal{G})$ in the supercritical regime, one may consider the question of finding a non-constant solution of mountain pass (MP) type. The existence of an MP solution will be the content of this and the next two sections. Before proceeding, it is convenient to recall a preliminary result and a definition.

Lemma 3.1 ([14, Proposition 3.1]). *Assume that $\mathcal{G}$ is a compact metric graph, $p > 2$, and $\{u_n\} \subset H^1_\mu(\mathcal{G})$ is a bounded Palais–Smale sequence of $E(\cdot, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$. Then there exists $u \in H^1(\mathcal{G})$ such that, up to a subsequence, $u_n \to u$ strongly in $H^1_\mu(\mathcal{G})$.*

Definition 3.2. For any graph $\mathcal{F}$ (not necessarily compact) and any solution $U \in C(\mathcal{F}) \cap H^1_{\mathrm{loc}}(\mathcal{F})$, not necessarily in $H^1(\mathcal{F})$, of

$$\begin{cases} -U'' + \lambda U = \rho |U|^{p-2} U & \text{in } \mathcal{F}, \\ \sum_{\mathrm{e} \succ \mathrm{v}} U'(\mathrm{v}) = 0 & \text{for any vertex v of } \mathcal{F}, \end{cases} \tag{3.1}$$

with $\lambda, \rho \in \mathbb{R}$, we consider

$$Q(\varphi; U, \mathcal{F}) := \int_{\mathcal{F}} \big(|\varphi'|^2 + (\lambda - (p-1)\rho |U|^{p-2}) \varphi^2 \big)\, dx \quad \forall \varphi \in H^1(\mathcal{F}) \cap C_c(\mathcal{F}). \tag{3.2}$$

The *Morse index of U*, denoted by $m(U)$, is the maximal dimension of a subspace $W \subset H^1(\mathcal{F}) \cap C_c(\mathcal{F})$ such that $Q(\varphi; U, \mathcal{F}) < 0$ for all $\varphi \in W \setminus \{0\}$.

Note that this is the definition of a Morse index as a solution to (3.1), and not as a critical point of the energy functional under the L^2 constraint (see Definition 3.9 below).

Lemma 3.1 is a useful result which exploits the compactness of the reference graph $\mathcal{G}$. However, as already anticipated in the introduction, in the present setting even the existence of a *bounded* Palais–Smale sequence at the mountain pass level is not straightforward. To overcome this issue, we introduce the family of functionals

$$E_\rho(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2\, dx - \frac{\rho}{p} \int_{\mathcal{G}} |u|^p\, dx,$$

depending on the parameter $\rho \in [1/2, 1]$. The idea is to adapt the monotonicity trick [24] on this family.

The main result of this section is the following:

Proposition 3.3. *Assume that $\mathcal{G}$ is a compact metric graph and $p > 6$. Let $\mu \in (0, \mu_1)$. For almost every $\rho \in [1/2, 1]$, there exists a critical point u_ρ of $E_\rho(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$, at* *level $c_\rho > E_\rho(\kappa_\mu, \mathcal{G})$, which solves*

$$\begin{cases} -u_\rho'' + \lambda_\rho u_\rho = \rho u_\rho^{p-1}, \ u_\rho > 0 & \text{in } \mathcal{G}, \\ \sum_{\mathrm{e}\succ \mathrm{v}} u_\rho'(\mathrm{v}) = 0 & \text{for any vertex } \mathrm{v}, \end{cases} \tag{3.3}$$

for some $\lambda_\rho > 0$. Moreover, its Morse index satisfies $m(u_\rho) \le 2$.

In the proof of the proposition, the value of $\mu \in (0, \mu_1)$ is fixed and will not change. As a first step, we show that the family of functionals $E_\rho(\cdot, \mathcal{G})$ has a mountain pass geometry on $H^1_\mu(\mathcal{G})$ around the constant local minimizer κ_μ, uniformly with respect to ρ.

Lemma 3.4. *Assume that $\mathcal{G}$ is a compact metric graph and $p > 6$. There exists $w \in H^1_\mu(\mathcal{G})$ such that, setting*

$$\Gamma := \{\gamma \in C([0,1], H^1_\mu(\mathcal{G})) \colon \gamma(0) = \kappa_\mu, \ \gamma(1) = w\},$$

we have

$$\begin{aligned} c_\rho &:= \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E_\rho(\gamma(t), \mathcal{G}) > E_\rho(\kappa_\mu, \mathcal{G}) \\ &= \max\{E_\rho(\gamma(0), \mathcal{G}), E_\rho(\gamma(1), \mathcal{G})\} \quad \forall \rho \in \left[\tfrac{1}{2}, 1\right]. \end{aligned}$$

Remark 3.5. Note that the functions κ_μ and w, and hence also Γ, are independent of ρ.

Proof of Lemma 3.4. Since $\rho \le 1$, and taking advantage of the monotonicity, we see from the proof of Proposition 2.1 that κ_μ remains a strict local minimizer of $E_\rho(\cdot, \mathcal{G})$ in $H^1_\mu(\mathcal{G})$ for all $\rho \in [1/2, 1]$.

More precisely, for any $\rho \in [1/2, 1]$ there exists a ball $B(\kappa_\mu, r_\rho)$ of center κ_μ in $H^1_\mu(\mathcal{G})$ and radius $r_\rho > 0$ such that κ_μ strictly minimizes $E_\rho(\cdot\, \mathcal{G})$ in $\overline{B(\kappa_\mu, r_\rho)}$, and

$$\inf_{u\in\partial B(\kappa_\mu, r_\rho)} E_\rho(u, \mathcal{G}) > E_\rho(\kappa_\mu, \mathcal{G}) > E_1(\kappa_\mu, \mathcal{G}). \tag{3.4}$$

Let e be any edge of $\mathcal{G}$; we identify e with the interval $[-\ell_\mathrm{e}/2, \ell_\mathrm{e}/2]$. Then any compactly supported H^1 function v on such an interval, with mass μ, can be seen as a function in $H^1_\mu(\mathcal{G})$. Denoting $v_t(x) := t^{1/2} v(tx)$, with $t > 1$, it is not difficult to check that $v_t \in H^1_\mu(\mathcal{G})$ (notice in particular that the support of v_t is shrinking as t becomes larger), and that

$$E_\rho(v_t, \mathcal{G}) = \frac{t^2}{2}\int_\mathrm{e} |v'|^2\,dx - \frac{\rho t^{\frac{p-2}{2}}}{p}\int_\mathrm{e} |v|^p\,dx \le \frac{t^2}{2}\left(\int_\mathrm{e} |v'|^2\,dx - \frac{t^{\frac{p-6}{2}}}{p}\int_\mathrm{e} |v|^p\,dx\right),$$

for every $\rho \in [1/2, 1]$. Since $p > 6$,

$$E_\rho(v_t, \mathcal{G}) < E_1(\kappa_\mu, \mathcal{G}) < E_\rho(\kappa_\mu, \mathcal{G})$$

for t sufficiently large (independent of ρ). Now taking $w = v_t$ with any such choice of t in the definition of Γ, the above estimate and the minimality of κ_μ in $\overline{B(\kappa_\mu, r_\rho)}$ imply that $w \notin B(\kappa_\mu, r_\rho)$. Therefore, by continuity, for any $\gamma \in \Gamma$ there exist $t_\gamma \in [0, 1]$ such that $\gamma(t_\gamma) \in \partial B(\kappa_\mu, r_\rho)$; and hence, by (3.4),

$$\max_{t\in[0,1]} E_\rho(\gamma(t), \mathcal{G}) \geq E_\rho(\gamma(t_\gamma), \mathcal{G}) > \inf_{u\in\partial B(\kappa_\mu, r_\rho)} E_\rho(u, \mathcal{G}) > E_\rho(\kappa_\mu, \mathcal{G})$$
$$= \max\{E_\rho(\kappa_\mu, \mathcal{G}), E_\rho(w, \mathcal{G})\},$$

which completes the proof. ∎

At this point we wish to use the monotonicity trick on the family of functionals $E_\rho(\cdot\,\mathcal{G})$, in order to obtain a bounded Palais–Smale sequence at level c_ρ for almost every $\rho \in [1/2, 1]$. In fact, we need a stronger result carrying also an "approximate Morse-index" information, Theorem 3.10 below, proved in [12].

We recall the general setting in which the theorem is stated. Let $(E, \langle\cdot,\cdot\rangle)$ and $(H, (\cdot,\cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that

$$E \hookrightarrow H \hookrightarrow E',$$

with continuous injections. For simplicity, we assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H. We also introduce

$$\begin{cases} \|u\|^2 = \langle u, u\rangle, \\ |u|^2 = (u, u), \end{cases} \qquad u \in E,$$

and, for $\mu \in (0, +\infty)$, we define

$$S_\mu = \{u \in E,\ |u|^2 = \mu\}.$$

For our application, it is plain that $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$.

In the following definition, we denote by $\|\cdot\|_*$ and $\|\cdot\|_{**}$, respectively, the operator norm of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$.

Definition 3.6. Let $\phi: E \to \mathbb{R}$ be a C^2-functional on E and $\alpha \in (0, 1]$. We say that ϕ' and ϕ'' are α-Hölder continuous on bounded sets if for any $R > 0$ one can find $M = M(R) > 0$ such that for any $u_1, u_2 \in B(0, R)$,

$$\|\phi'(u_1) - \phi'(u_2)\|_* \leq M\|u_2 - u_1\|^\alpha, \quad \|\phi''(u_1) - \phi''(u_2)\|_{**} \leq M\|u_1 - u_2\|^\alpha. \tag{3.5}$$

Definition 3.7. Let ϕ be a C^2-functional on E; for any $u \in E$ define the continuous bilinear map

$$D^2\phi(u) = \phi''(u) - \frac{\phi'(u)\cdot u}{|u|^2}(\cdot,\cdot).$$

Remark 3.8. If u is a critical point of the functional $\phi|_{S_\mu}$ then the restriction of $D^2\phi(u)$ to $T_u S_\mu$ coincides with the constrained *Hessian* of $\phi|_{S_\mu}$ at u (as introduced in Proposition 2.1.)

Definition 3.9. Let ϕ be a C^2-functional on E; for any $u \in S_\mu$ and $\theta > 0$, we define the *approximate Morse index* by

$$\tilde{m}_\theta(u) = \sup\big\{\dim L \mid L \text{ is a subspace of } T_u S_\mu \text{ such that } D^2|_{S_\mu}\phi(u)(\varphi,\varphi) < -\theta\|\varphi\|^2 \ \forall \varphi \in L \setminus \{0\}\big\}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the *Morse index of u as constrained critical point.*

Theorem 3.10 ([12, Theorem 1]). *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho \colon E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$, and

$$\text{either } A(u) \to +\infty \text{ or } B(u) \to +\infty \text{ as } u \in E \text{ and } \|u\| \to +\infty. \tag{3.6}$$

Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0, 1]$. Finally, suppose that there exist $w_1, w_2 \in S_\mu$ (independent of ρ) such that, setting

$$\Gamma = \big\{\gamma \in C([0,1], S_\mu) \colon \gamma(0) = w_1,\ \gamma(1) = w_2\big\},$$

we have

$$c_\rho := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\rho(\gamma(t)) > \max\big\{\Phi_\rho(w_1), \Phi_\rho(w_2)\big\}, \quad \rho \in I.$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;

(ii) $\|\Phi'_\rho|_{S_\mu}(u_n)\| \to 0$;

(iii) $\{u_n\}$ *is bounded in* E;

(iv) $\tilde{m}_{\zeta_n}(u_n) \leq 1$.

We are ready to give the proof of Proposition 3.3.

Proof of Proposition 3.3. We apply Theorem 3.10 to the family of functionals $E_\rho(\cdot, \mathcal{G})$, with $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H^1_\mu(\mathcal{G})$, and Γ defined in Lemma 3.4. Setting

$$A(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx \quad \text{and} \quad B(u) = \frac{\rho}{p}\int_{\mathcal{G}} |u|^p,$$

assumption (3.6) holds, since we have

$$u \in H^1_\mu(\mathcal{G}),\ \|u\| \to +\infty \quad \Longrightarrow \quad A(u) \to +\infty.$$

Moreover, assumption (3.5) holds since the unconstrained first and second derivatives of E_ρ are of class C^1, and hence locally Hölder continuous, on $H^1_\mu(\mathcal{G})$.

In this way, for almost every $\rho \in [1/2, 1]$ there exist a bounded Palais–Smale sequence $\{u_n\}$ for the constrained functional $E_\rho(\cdot, \mathcal{G})|_{H^1_\mu(\mathcal{G})}$ at level c_ρ, and $\zeta_n \to 0^+$, such that $\widetilde{m}_{\zeta_n}(u_n) \leq 1$. Moreover, as explained in [12, Remark 1.4], since $u \in S_\mu \mapsto |u| \in S_\mu$, $w_1, w_2 \geq 0$, the map $u \mapsto |u|$ is continuous, and $E_\rho(u, \mathcal{G}) = E_\rho(|u|, \mathcal{G})$, it is possible to choose $\{u_n\}$ with the property that $u_n \geq 0$ on $\mathcal{G}$. By Lemma 3.1, we have $u_n \to u_\rho$ strongly in $H^1(\mathcal{G})$, and $u_\rho \geq 0$ is a constrained critical point, thus a non-negative solution to (3.3), for $\lambda_\rho = \lambda(u_\rho)$ (Lemma 3.1 is stated for the particular value $\rho = 1$; however, it is immediate to check that this choice does not play any role in the proof). The case when u_ρ vanishes at one (or more) vertex can be easily ruled out by the Kirchhoff condition, the uniqueness theorem for ODEs, and the fact that $u_\rho \geq 0$. Thus, u_ρ is strictly positive on each vertex, whence $u_\rho > 0$ in $\mathcal{G}$ by the strong maximum principle.

It remains to show that the Morse index $m(u_\rho)$, defined in Definition 3.2 with $\lambda = \lambda(u_\rho)$ is at most 2. This result can be directly deduce from [12, Theorem 3] but we prove it here in our setting for completeness. To simplify the notation we omit the dependence of the functionals $E_\rho(\cdot, \mathcal{G})$ on $\mathcal{G}$. Defining

$$\bar{\lambda}_\rho := -\frac{1}{\mu} E'_\rho(u_\rho) \cdot u_\rho = -\lim_{n\to\infty} \frac{1}{\mu} E'_\rho(u_n) \cdot u_n,$$

we conclude from Theorem 3.10 (ii) that $\bar{\lambda}_\rho = \lambda_\rho$; we refer to [12, Remark 1.2] for more detail.

To show that $u_\rho \in S_\mu$ has Morse index at most 1 as a constrained critical point, see Definition 3.9, we assume by contradiction that there exists a $W_0 \subset T_u S_\mu$ with $\dim W_0 = 2$ such that

$$D^2 E_\rho(u_\rho)(w, w) < 0 \quad \forall w \in W_0 \setminus \{0\}.$$

Since W_0 is of finite dimension, by compactness and homogeneity, there exists a $\beta > 0$ such that

$$D^2 E_\rho(u_\rho)(w, w) < -\beta \|w\|^2 \quad \forall w \in W_0 \setminus \{0\}.$$

Now, from [12, Corollary 1] or using directly that E'_ρ and E''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, we deduce that there exists a $\delta_1 > 0$ such that, for any $v \in S_\mu$ such that $\|v - u\| \leq \delta_1$,

$$D^2 E_\rho(v)(w, w) < -\frac{\beta}{2} \|w\|^2 \quad \forall w \in W_0 \setminus \{0\}. \tag{3.7}$$

Since $\{u_n\} \subset S_\mu$ converges to u we have $\|u_n - u\| \leq \delta_1$ for $n \in \mathbb{N}$ large enough. Then since $\dim W_0 > 1$, (3.7) provides a contradiction with Theorem 3.10 (iv) where we recall that $\zeta_n \to 0^+$. Finally, recalling that S_μ is of codimension 1 in $H^1(\mathcal{G})$ and observing that, for any $w \in H^1(\mathcal{G})$,

$$\begin{aligned} D^2 E_\rho(u_\rho)(w, w) &:= E''_\rho(u)(w, w) + \lambda_\rho(w, w) \\ &= \int_{\mathcal{G}} \left[|w'|^2 + (\lambda_\rho - (p-1)|u_\rho|^{p-2}) w^2 \right] dx, \end{aligned}$$

we obtain that $m(u_\rho) \leq 2$. ■

4. Blow-up phenomena

Proposition 3.3 does not ensure the existence of a mountain pass solution for the original problem obtained when $\rho = 1$. However, it gives the existence of a sequence $\rho_n \to 1^-$, with a corresponding sequence of mountain pass critical points $u_{\rho_n} \in H^1_\mu(\mathcal{G})$ of $E_{\rho_n}(\cdot, \mathcal{G})$, constrained on $H^1_\mu(\mathcal{G})$. We aim to show that $\{u_{\rho_n}\}$ converges to a constrained critical point of $E_1(\cdot, \mathcal{G})$. For this purpose, it is sufficient to prove that $\{u_{\rho_n}\}$ is bounded in $H^1(\mathcal{G})$, thanks to Lemma 3.1. The advantage of working with $\{u_{\rho_n}\}$ is that this is a sequence of *solutions of approximating problems with uniformly bounded Morse index*. In this section we perform a blow-up analysis for this type of sequence, in the spirit of [18]. This analysis, of independent interest, will be used in the next section to gain the desired boundedness of $\{u_{\rho_n}\}$.

A somewhat related study, regarding least action solutions, was previously performed in [15].

General setting for the blow-up analysis

For the sake of generality, in what follows we consider a general metric graph satisfying the following assumption:

$$\mathcal{G} \text{ has a finite number of vertexes and edges (but is not necessarily compact).} \tag{4.1}$$

Let $\{u_n\} \subset H^1(\mathcal{G})$ be a sequence of positive solutions of the NLS equation, coupled with Kirchhoff condition at the vertexes:

$$\begin{cases} -u_n'' + \lambda_n u_n = \rho_n u_n^{p-1} & \text{on } \mathcal{G}, \\ u_n > 0 & \text{on } \mathcal{G}, \\ \sum_{\mathrm{e} \succ \mathrm{v}} u_{e,n}'(\mathrm{v}) = 0 & \forall \mathrm{v} \in \mathcal{V}, \end{cases} \tag{4.2}$$

where $\rho_n \to 1$ (in fact, it would be sufficient to ask that $\rho_n \to \rho > 0$, regardless of the value of ρ), and $\lambda_n \in \mathbb{R}$.

We denote $B_r(x_0) = \{x \in \mathcal{G} \colon \mathrm{dist}(x, x_0) < r\}$. Moreover, we denote by $\mathcal{G}_m$ the star graph with $m \geq 1$ half-lines glued together at their common origin 0 (note that $\mathcal{G}_1 = \mathbb{R}^+$, and $\mathcal{G}_2$ is isometric to $\mathbb{R}$).

It is also convenient to recall the definition of $Q(\phi; u, \mathcal{G})$; see (3.2).

First, we note that if $\lambda_n \to +\infty$, then u_n blows-up along any sequence of local maximum points.

Lemma 4.1. *Let $\mathcal{G}$ be a metric graph satisfying (4.1), $p > 2$, and $u_n \in H^1(\mathcal{G})$ be a solution to (4.2) for some $\lambda_n \in \mathbb{R}$ and $\rho_n \in (0, 1]$. Let $x_n \in \mathcal{G}$ be a local maximum point for u_n. Then*

$$u_n(x_n) \geq \lambda_n^{\frac{1}{p-2}}.$$

Proof. Let e be an edge of $\mathcal{G}$ such that $x_n \in \mathrm{e} \simeq [0, \ell_{\mathrm{e}}]$; it is plain that $u_n|_{\mathrm{e}} \in C^2([0, \ell_{\mathrm{e}}])$, by regularity. If x_n is in the interior of e, then $u_n''(x_n) \le 0$; if instead x_n is a vertex of e, then, by the Kirchhoff condition, $u_n'(x_n)$ must vanish, and hence $u_n''(x_n) \le 0$ again. In both cases, the equation for u_n (which holds on the whole closed interval $[0, \ell_{\mathrm{e}}]$) yields

$$\lambda_n u_n(x_n) - \rho_n u_n^{p-1}(x_n) = u_n''(x_n) \le 0,$$

whence the thesis follows. ■

The next theorem provides a precise behavior, close to a local maximum point, of the sequence $\{u_n\}$, as $\lambda_n \to +\infty$, while $m(u_n)$ remains bounded. In the statement and in the proof, we shall systematically identify an edge e with the interval $[0, \ell_{\mathrm{e}}]$, where ℓ_{e} denotes the length of e. Since in this section we allow $\mathcal{G}$ to be non-compact, it is admissible that $\ell_{\mathrm{e}} = +\infty$ (clearly, in such a case $\mathrm{e} \simeq [0, +\infty)$; unless it is necessary, we shall not distinguish these cases).

Theorem 4.2. *Let $\mathcal{G}$ be a metric graph satisfying (4.1), $p > 2$, and $\{u_n\} \subset H^1(\mathcal{G})$ be a sequence of solutions to (4.2) for some $\lambda_n \in \mathbb{R}$ and $\rho_n \in (0, 1]$. Suppose that*

$$\lambda_n \to +\infty \quad \text{and} \quad m(u_n) \le \bar{k} \;\text{ for some } \bar{k} \ge 1.$$

Let $x_n \in \mathcal{G}$ be such that, for some $R_n \to \infty$,

$$u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n, \quad \text{where } \tilde{\varepsilon}_n = (u_n(x_n))^{-\frac{p-2}{2}} \to 0.$$

Suppose moreover that

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} = +\infty. \tag{4.3}$$

Then, up to a subsequence, the following hold:

(i) *All the x_n lie in the interior of the same edge $\mathrm{e} \simeq [0, \ell_e]$.*

(ii) *Setting $\varepsilon_n = \lambda_n^{-\frac{1}{2}}$, we have*

$$\begin{aligned} &\frac{\tilde{\varepsilon}_n}{\varepsilon_n} \to (0, 1], \\ &\frac{\operatorname{dist}(x_n, \mathcal{V})}{\varepsilon_n} \to +\infty \quad \text{as } n \to \infty, \end{aligned} \tag{4.4}$$

and the scaled sequence

$$v_n(y) := \varepsilon_n^{\frac{2}{p-2}} u_n(x_n + \varepsilon_n y) \quad \text{for } y \in \frac{[0, \ell_e] - x_n}{\varepsilon_n} \tag{4.5}$$

converges to V in $C^2_{\mathrm{loc}}(\mathbb{R})$ as $n \to \infty$, where $V \in H^1(\mathbb{R})$ is the (unique) positive finite energy solution to

$$\begin{cases} -V'' + V = V^{p-1}, \ V > 0 & \text{in } \mathbb{R}, \\ V(0) = \max_{\mathbb{R}} V, \\ V(x) \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

(iii) *There exists* $\phi_n \in C_c^\infty(\mathcal{G})$*, with* $\operatorname{supp}\phi_n \subset B_{\bar{R}\varepsilon_n}(x_n)$ *for some* $\bar{R} > 0$*, such that*

$$Q(\phi_n; u_n, \mathcal{G}) < 0.$$

(iv) *For all* $R > 0$ *and* $q \geq 1$*, we have*

$$\lim_{n\to\infty} \lambda_n^{\frac{1}{2}-\frac{q}{p-2}} \int_{B_{R\varepsilon_n}(x_n)} u_n^q \, dx = \lim_{n\to\infty} \int_{B_R(0)} v_n^q \, dy = \int_{B_R(0)} V^q \, dy.$$

If, instead of (4.3)*, we suppose that*

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n} < +\infty, \tag{4.6}$$

then, up to a subsequence, the following hold:

(i′) $x_n \to \mathrm{v} \in \mathcal{V}$*, and all the* x_n *lie on the same edge* $\mathrm{e}_1 \simeq [0, \ell_1]$*, where the vertex* v *is identified by the coordinate* 0 *on* e_1.

(ii′) *Let* $\mathrm{e}_2 \simeq [0, \ell_2], \ldots, \mathrm{e}_m \simeq [0, \ell_m]$ *be the other edges of* $\mathcal{G}$ *having* v *as a vertex (if any), where* v *is identified by the coordinate* 0 *on each* e_i*. Setting* $\varepsilon_n = \lambda_n^{-\frac{1}{2}}$*, we have*

$$\begin{aligned} &\frac{\tilde{\varepsilon}_n}{\varepsilon_n} \to (0, 1], \\ &\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{V})}{\varepsilon_n} < +\infty, \end{aligned} \tag{4.7}$$

and the scaled sequence defined by

$$v_n(y) := \varepsilon_n^{\frac{2}{p-2}} u_n(\varepsilon_n y) \quad \text{for } y \in \frac{\mathrm{e}_i}{\varepsilon_n}, \text{for } i = 1, \ldots, m,$$

converges to a limit V *in* $C^0_{\mathrm{loc}}(\mathcal{G}_m)$ *as* $n \to \infty$*. Denoting by* V_i *the restriction of* V *to the* i*th half-line* ℓ_i *of* $\mathcal{G}_m$*, and by* $v_{i,n}$ *the restriction of* v_n *to* $\mathrm{e}_i/\varepsilon_n$*, we have moreover that* $v_{i,n} \to V_i$ *in* $C^2_{\mathrm{loc}}([0, +\infty))$*. Finally,* $V \in H^1(\mathcal{G}_m)$ *is a positive finite energy solution to the NLS equation on the star graph*

$$\begin{cases} -V'' + V = V^{p-1}, \; V > 0 & \text{in } \mathcal{G}_m, \\ \displaystyle\sum_{i=1}^m V_i'(0^+) = 0, & \\ V(x) \to 0 & \text{as } \operatorname{dist}(x, 0) \to \infty \end{cases}$$

with a global maximum point $\bar{x}$ *located on* ℓ_1*, whose coordinate is*

$$\bar{x} = \lim_{n\to\infty} \bar{x}_n \in [0, +\infty), \quad \text{where } \bar{x}_n := \frac{\operatorname{dist}(x_n, \mathcal{V})}{\varepsilon_n}.$$

(iii′) *There exists* $\phi_n \in C_c^\infty(\mathcal{G})$*, with* $\operatorname{supp}\phi_n \subset B_{\bar{R}\varepsilon_n}(x_n)$ *for some* $\bar{R} > 0$*, such that*

$$Q(\phi_n; u_n, \mathcal{G}) < 0.$$

(iv′) *For all* $R > 0$ *and* $q \geq 1$, *we have*

$$\begin{aligned}\lim_{n\to\infty} \lambda_n^{\frac{1}{2}-\frac{q}{p-2}} \int_{B_{R\varepsilon_n}(x_n)} u_n^q\,dx &= \lim_{n\to\infty} \int_{B_R(\bar{x}_n)} v_n^q\,dy \\ &= \int_{[0,\bar{x}+R]} V_1^q\,dy + \sum_{i=2}^{m} \int_{[0,R-\bar{x}]} V_i^q\,dy \\ &= \int_{B_R(\bar{x})} V^q\,dy\end{aligned}$$

(where $B_R(\bar{x}_n)$ *and* $B_R(\bar{x})$ *denote the balls in the scaled and in the limit graphs, respectively).*

The proof of the theorem is divided into several intermediate steps. We start with some preliminary results.

Lemma 4.3. *Let* $U \in H^1_{\mathrm{loc}}(\mathcal{G}_m)$ *be a solution to*

$$\begin{cases} -U'' + \lambda U = \rho U^{p-1} & \text{in } \mathcal{G}_m, \\ U > 0 & \text{in } \mathcal{G}_m, \\ \displaystyle\sum_{i=1}^{m} U_i'(0) = 0, \end{cases} \tag{4.8}$$

for some $p > 2$, $\rho, \lambda > 0$, *where* U_i *denotes the restriction of* U *on the* i*th half-line of* $\mathcal{G}_m$. *Suppose that* U *is stable outside a compact set* K, *in the sense that* $Q(\varphi; U, \mathcal{G}_m) \geq 0$ *for all* $\varphi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m \setminus K)$. *Then* $U(x) \to 0$ *as* $\operatorname{dist}(x, 0) \to +\infty$, *and* $U \in H^1(\mathcal{G})$.

The proof is analogous to that of [18, Theorem 2.3], and hence we omit it.

Remark 4.4. Clearly, by the density of $H^1([0,+\infty)) \cap C_c([0,+\infty))$ in $H^1([0,+\infty))$, any solution with finite Morse index is stable outside a compact set.

Lemma 4.5. *Let* $U \in H^1(\mathcal{G}_m)$ *be any non-trivial solution of* (4.8). *Then its Morse index* $m(U)$ *is strictly positive.*

Proof. Thanks to the Kirchhoff condition, it is not difficult to check that

$$\int_{\mathcal{G}_m} (|U'|^2 + \lambda U^2)\,dx = \int_{\mathcal{G}_m} \rho |U|^p\,dx.$$

Therefore

$$Q(U; U, \mathcal{G}_m) = (2-p) \int_{\mathcal{G}_m} |U|^p\,dx < 0,$$

and the thesis follows by density of $H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$ in $H^1(\mathcal{G}_m)$. ∎

Proof of Theorem 4.2 *under assumption* (4.3). This case is simpler than that when (4.6) holds, since, roughly speaking, after rescaling we do not see the vertexes of $\mathcal{G}$, and we obtain a limit problem on the line. We present in any case the proof for the sake of completeness. Since $\mathcal{G}$ has a finite number of edges, up to a subsequence all the points x_n belong to same edge e, and (i) holds. Let $\tilde{u}_n$ be defined by

$$\tilde{u}_n(y) := \tilde{\varepsilon}_n^{\frac{2}{p-2}} u_n(x_n + \tilde{\varepsilon}_n y) \quad \text{for } y \in \tilde{\mathrm{e}}_n := \frac{\mathrm{e} - x_n}{\tilde{\varepsilon}_n}.$$

Notice that any interval $[-a, a]$, with $a > 0$, is contained in $\tilde{\mathrm{e}}_n$ for sufficiently large n. Indeed, $(\mathrm{e} - x_n)/\tilde{\varepsilon}_n$ contains the set

$$\{y \in \mathbb{R} : |\tilde{\varepsilon}_n y| < \mathrm{dist}(x_n, \mathcal{V})\} = \{y \in \mathbb{R} : |y| < \mathrm{dist}(x_n, \mathcal{V})/\tilde{\varepsilon}_n\},$$

which exhausts the whole line $\mathbb{R}$ as $n \to \infty$, by (4.3).

Now, on every compact $[-a, a]$ we have $\tilde{u}_n(0) = 1 = \max_{[-a,a]} \tilde{u}_n$ for n large (since $u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n$ for some $R_n \to +\infty$), and

$$-\tilde{u}_n'' + \tilde{\varepsilon}_n^2 \lambda_n \tilde{u}_n = \rho_n \tilde{u}_n^{p-1}, \quad \tilde{u}_n > 0 \text{ in } \tilde{\mathrm{e}}_n.$$

Furthermore, by Lemma 4.1,

$$\tilde{\varepsilon}_n^2 \lambda_n \in (0, 1] \quad \forall n.$$

Thus, by elliptic estimates, we have $\tilde{u}_n \to \tilde{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R})$, and the limit $\tilde{u}$ solves

$$-\tilde{u}'' + \tilde{\lambda}\tilde{u} = \tilde{u}^{p-1}, \quad \tilde{u} \geq 0 \text{ in } \mathbb{R}, \tag{4.9}$$

for some $\tilde{\lambda} \in [0, 1]$. By local uniform convergence, $\tilde{u}(0) = 1$, and hence $\tilde{u} > 0$ in $\mathbb{R}$ by the strong maximum principle. We claim that

$$\text{the Morse index of } \tilde{u} \text{ is bounded by } \bar{k}. \tag{4.10}$$

If by contradiction this is false, then there exists $k > \bar{k}$ functions $\phi_1, \ldots, \phi_k \in H^1(\mathbb{R}) \cap C_c(\mathbb{R})$, linearly independent in $H^1(\mathbb{R})$, such that $Q(\phi_i; \tilde{u}, \mathbb{R}) < 0$ for every $i \in \{1, \ldots, k\}$. Then let

$$\phi_{i,n}(x) = \tilde{\varepsilon}_n^{\frac{1}{2}} \phi_i\Big(\frac{x - x_n}{\tilde{\varepsilon}_n}\Big).$$

Since ϕ_i has compact support, the functions $\phi_{i,n}$ can be regarded as functions in $H^1(\mathrm{e})$, and hence in $H^1(\mathcal{G})$, for every n large, thanks to (4.3). Indeed, if $\operatorname{supp} \phi_i \subset [-M, M]$, then

$$\begin{aligned}\{x \in \mathbb{R} : \tfrac{x - x_n}{\tilde{\varepsilon}_n} \subset [-M, M]\} &= [x_n - \tilde{\varepsilon} M, x_n + \tilde{\varepsilon}_n M] \\ &\subset \Big[x_n - \tilde{\varepsilon}_n \frac{\mathrm{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n}, x_n + \tilde{\varepsilon}_n \frac{\mathrm{dist}(x_n, \mathcal{V})}{\tilde{\varepsilon}_n}\Big] \subset \mathrm{e}.\end{aligned}$$

Moreover, $\phi_{1,n}, \ldots, \phi_{k,n}$ are linearly independent in $H^1(\mathcal{G})$, and, by scaling,

$$Q(\phi_{i,n}; u_n, \mathcal{G}) = Q(\phi_{i,n}; u_n, \mathrm{e}) = Q(\phi_i; \tilde{u}_n, \tilde{\mathrm{e}}_n) \to Q(\phi_i; \tilde{u}, \mathbb{R}) < 0.$$

This implies that $m(u_n) \geq k > \bar{k}$ for sufficiently large n, a contradiction. Therefore, claim (4.10) is proved. To sum up, $\tilde{u}$ is a finite Morse index non-trivial solution to (4.9), for some $\tilde{\lambda} \in [0, 1]$. Having $\tilde{\lambda} = 0$ is however not possible, since by phase plane analysis the equation $\tilde{u}'' + \tilde{u}^{p-1} = 0$ in $\mathbb{R}$ has only, as non-trivial solutions, periodic sign-changing solutions. Now, by Lemma 4.3, $\tilde{u} \to 0$ as $|x| \to +\infty$, and $\tilde{u} \in H^1(\mathbb{R})$. Therefore,

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \leq 1, \tag{4.11}$$

which proves the first estimate in (4.4). At this point it is equivalent, but more convenient, to work with v_n defined by (4.5) rather than with $\tilde{u}_n$. By (4.3) and (4.11),

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{V})}{\varepsilon_n} = +\infty.$$

Thus, similarly to before, one can show that v_n converges to a limit function v in $C^2_{\mathrm{loc}}(\mathbb{R})$, such that

$$-v'' + v = v^{p-1}, \quad v \geq 0 \quad \text{in } \mathbb{R};$$

moreover, v has a positive global maximum $v(0) \geq 1$ (thus $v > 0$ in $\mathbb{R}$), has finite Morse index $m(v) \leq \bar{k}$, and hence, by Lemma 4.3, $v \to 0$ as $|x| \to \infty$, and $v \in H^1(\mathbb{R})$. It is well known that there exists only one such solution, denoted by V. Thus, (ii) is proved. Point (iv) follows directly by local uniform convergence. Finally, point (iii) is a consequence of the fact that the Morse index of V is positive (see Lemma 4.5; in fact, it is well known that in fact $m(V)$ is precisely equal to 1). This implies that there exists $\phi \in C^1_c(\mathbb{R})$ such that $Q(\phi; V, \mathbb{R}) < 0$; thus, defining

$$\phi_{i,n}(x) = \varepsilon_n^{\frac{1}{2}} \phi_i\left(\frac{x - x_n}{\varepsilon_n}\right),$$

we deduce that for sufficiently large n we have $Q(\phi_{i,n}; u_n, \mathcal{G}) < 0$, and $\operatorname{supp} \phi_{i,n} \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some $\bar{R} > 0$. ■

Proof of Theorem 4.2 under assumption (4.6). Since $\tilde{\varepsilon}_n \to 0$ and $\mathcal{G}$ has a finite number of vertexes and edges, up to a subsequence the maximum points x_n converge to a vertex v, and belong to same edge $\mathrm{e}_1 \simeq [0, \ell_1]$; thus, (i′) holds, and we can suppose that

$$\frac{d_n}{\tilde{\varepsilon}_n} \to \eta \in [0, +\infty), \quad d_n := \operatorname{dist}(x_n, \mathcal{V}) = x_n.$$

Let

$$\tilde{u}_n(y) := \tilde{\varepsilon}_n^{\frac{2}{p-2}} u_n(\tilde{\varepsilon}_n y) \quad \text{for } y \in \tilde{\mathrm{e}}_{i,n} := \frac{\mathrm{e}_i}{\tilde{\varepsilon}_n} \text{ for } i = 1, \dots, m.$$

Note that $\tilde{u}_n$ is defined on a graph $\mathcal{G}_{m,n}$ consisting of m expanding edges, glued together at their common origin, which is identified with the coordinate 0 on each edge $\tilde{e}_{i,n}$. In the limit $n \to \infty$, this graph converges to the star graph $\mathcal{G}_m$. Plainly, for every $a > \eta + 1$ and large n,

$$\tilde{u}_n\left(\frac{x_n}{\tilde{\varepsilon}_n}\right) = 1 = \max_{B_a(0)} \tilde{u}_n$$

(since $u_n(x_n) = \max_{B_{R_n \tilde{\varepsilon}_n}(x_n)} u_n$ for some $R_n \to +\infty$),

$$-\tilde{u}_n'' + \tilde{\varepsilon}_n^2 \lambda_n \tilde{u}_n = \rho_n \tilde{u}_n^{p-1}, \quad \tilde{u}_n > 0$$

on any edge of $\mathcal{G}_{m,n}$, and the Kirchhoff condition at the origin holds. Also, by Lemma 4.1,

$$\tilde{\varepsilon}_n^2 \lambda_n \in (0, 1] \quad \forall n.$$

Thus, by elliptic estimates, we have $\tilde{u}_n|_{\tilde{\mathrm{e}}_{i,n}} =: \tilde{u}_{i,n} \to \tilde{u}_i$ in $C^2_{\mathrm{loc}}([0, +\infty))$ for every i, and the limit $\tilde{u}_i$ solves

$$-\tilde{u}_i'' + \tilde{\lambda} \tilde{u}_i = \tilde{u}_i^{p-1}, \quad \tilde{u}_i \geq 0 \quad \text{in } (0, +\infty) \tag{4.12}$$

for some $\tilde{\lambda} \in [0, 1]$. Moreover, since $\tilde{u}_n$ is continuous on $\mathcal{G}_{m,n}$ and by uniform convergence, $\tilde{u}_i(0) = \tilde{u}_j(0)$ for every $i \neq j$, so that $\tilde{u} \simeq (\tilde{u}_1, \ldots, \tilde{u}_m)$ can be regarded as a function defined on $\mathcal{G}_m$. Since the convergence $\tilde{u}_{i,n} \to \tilde{u}_i$ takes place in C^2 up to the origin, the Kirchhoff condition also passes to the limit. Now we exclude the case that $\tilde{u} \equiv 0$ on some half-line of $\mathcal{G}_m$. By local uniform convergence, we have

$$\tilde{u}_1(\eta) = \lim_{n \to \infty} \tilde{u}_{1,n}\left(\frac{d_n}{\tilde{\varepsilon}_n}\right) = 1.$$

This implies that $\tilde{u}_1 > 0$ in $(0, +\infty)$, by the strong maximum principle. In turn, the Kirchhoff condition, the uniqueness theorem for ODEs, and the strong maximum principle again, ensure that $\tilde{u}_i > 0$ on $(0, +\infty)$ for every i. Finally, we claim that

$$\text{the Morse index of } \tilde{u} \text{ is bounded by } \bar{k}. \tag{4.13}$$

The proof of this claim is completely analogous to that of (4.10). If by contradiction this is false, then there exist $k > \bar{k}$ functions $\phi_1, \ldots, \phi_k \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$, linearly independent in $H^1(\mathcal{G}_m)$, such that $Q(\phi_i; \tilde{u}, \mathcal{G}_m) < 0$ for every $i \in \{1, \ldots, k\}$. Then let

$$\phi_{i,n}(x) = \tilde{\varepsilon}_n^{\frac{1}{2}} \phi_i\left(\frac{x}{\tilde{\varepsilon}_n}\right).$$

Since ϕ_i has compact support, the functions $\phi_{i,n}$ can be regarded as functions in $H^1(\mathcal{G}) \cap C_c(\mathcal{G})$ for every n large; precisely, $\mathrm{supp}(\phi_{i,n}) \subset B_{R\tilde{\varepsilon}_n}(x_n)$ for some $R > 2\rho$. Moreover, $\phi_{1,n}, \ldots, \phi_{k,n}$ are linearly independent in $H^1(\mathcal{G}_m)$ and, by scaling,

$$Q(\phi_{i,n}; u_n, \mathcal{G}) = Q(\phi_{i,n}; u_n, \mathrm{e}) = Q(\phi_i; \tilde{u}_n, \tilde{\mathrm{e}}_n) \to Q(\phi_i; \tilde{u}, \mathbb{R}) < 0.$$

This implies that $m(u_n) \geq k > \bar{k}$ for sufficiently large n, a contradiction. Therefore, claim (4.13) is proved.

To sum up, $\tilde{u}$ is a finite Morse index non-trivial solution to (4.12), for some $\tilde{\lambda} \in [0, 1]$. As before, the case $\tilde{\lambda} = 0$ can be ruled out by phase-plane analysis, and hence, by Lemma 4.3, $\tilde{u} \to 0$ as $|x| \to +\infty$, and $\tilde{u} \in H^1(\mathcal{G}_m)$. Therefore,

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \le \limsup_{n\to\infty} \frac{\lambda_n}{(u_n(x_n))^{p-2}} \le 1, \tag{4.14}$$

which proves the first estimate in (4.7). At this point it is equivalent, but more convenient, to work with v_n defined in point (ii′) of the theorem, rather than with $\tilde{u}_n$. By (4.6) and (4.14),

$$\limsup_{n\to\infty} \frac{\operatorname{dist}(x_n, \mathcal{V})}{\varepsilon_n} < +\infty.$$

Thus, similarly to before, one can show that v_n converges, in $C^0_{\mathrm{loc}}(\mathcal{G}_m)$ and in $C^2_{\mathrm{loc}}([0, +\infty))$ on every half-line, to a limit function $V \simeq (V_1, \dots, V_m)$, which solves

$$\begin{cases} -V'' + V = V^{p-1}, \ V \ge 0 \quad \text{in } \mathcal{G}_m, \\ \displaystyle\sum_{i=1}^m V_i'(0^+) = 0; \end{cases}$$

furthermore, V has a positive global maximum on the half-line ℓ_1, $V_1(\bar{x}) \ge 1$ (thus $V > 0$ in $\mathcal{G}_m$), and has finite Morse index $m(V) \le \bar{k}$. Moreover, by Lemma 4.3, $V \to 0$ as $|x| \to \infty$. Thus, (ii′) is proved. Point (iv′) follows directly by local uniform convergence. Finally, point (iii′) is a consequence of Lemma 4.5. This implies that there exists $\phi \in H^1(\mathcal{G}_m) \cap C_c(\mathcal{G}_m)$ such that $Q(\phi; V, \mathcal{G}_m) < 0$; thus, defining

$$\phi_{i,n}(x) = \varepsilon_n^{\frac{1}{2}} \phi_i\Big(\frac{x - x_n}{\varepsilon_n}\Big),$$

it is not difficult to deduce that for sufficiently large n we have $Q(\phi_{i,n}; u_n, \mathcal{G}) < 0$, and $\operatorname{supp} \phi_{i,n} \subset B_{\bar{R}\varepsilon_n}(x_n)$ for some positive $\bar{R}$. ■

Theorem 4.2 allows the pointwise blow-up behavior close to local maximum points to be described. In what follows, we focus on the global behavior, and, in particular, on what happens far away from local maxima.

Theorem 4.6. *Let $\mathcal{G}$ be a metric graph satisfying (4.1) and $p > 2$. Let $\{u_n\} \subset H^1(\mathcal{G})$ be a sequence of solutions to (4.2) such that $\lambda_n \to +\infty$ and $m(u_n) \le \bar{k}$ for some $\bar{k} \ge 1$. There exist $k \in \{1, \dots, \bar{k}\}$, and sequences of points $\{P_n^1\}, \dots, \{P_n^k\}$, such that*

$$\lambda_n \operatorname{dist}(P_n^i, P_n^j) \to +\infty \quad \forall i \ne j, \tag{4.15}$$

$$u_n(P_n^i) = \max_{B_{R_n\lambda_n^{-1/2}}(P_n^i)} u_n \quad \textit{for some } R_n \to +\infty, \textit{for every } i, \tag{4.16}$$

and constants $C_1, C_2 > 0$ and $R > 0$ such that

$$u_n(x) \le C_1 \lambda_n^{\frac{1}{p-2}} \sum_{i=1}^{k} e^{-C_2 \lambda_n^{\frac{1}{2}} \operatorname{dist}(x, P_n^i)} + C_1 \lambda_n^{\frac{1}{p-2}} \sum_{j=1}^{h} e^{-C_2 \lambda_n^{\frac{1}{2}} \operatorname{dist}(x, \mathrm{v}_j)} \quad \forall x \in \mathcal{G} \setminus \bigcup_{i=1}^{k} B_{R\lambda_n^{-1/2}}(P_n^i), \tag{4.17}$$

where $\mathrm{v}_1, \dots, \mathrm{v}_h$ are all the vertexes of $\mathcal{G}$.

Proof. The proof closely follows that of [18, Theorem 3.2], and is divided into two steps.

Step 1. *There exist $k \in \{1, \dots, \bar{k}\}$, and sequences of points $\{P_n^1\}$, ..., $\{P_n^k\}$, such that* (4.15) *and* (4.16) *hold, and moreover*

$$\lim_{R \to +\infty} \left(\limsup_{n \to \infty} \lambda_n^{-\frac{1}{p-2}} \max_{d_n(x) \ge R\lambda_n^{-1/2}} u_n(x) \right) = 0, \tag{4.18}$$

where $d_n(x) = \min\{\operatorname{dist}(x, P_n^i) : i = 1, \dots, k\}$ is the distance function from $\{P_1^n, \dots, P_k^n\}$.

Thanks to Theorem 4.2, we can adapt the proof of [18, Theorem 3.2] with minor changes (some details are actually simpler in the present setting, since here we deal with a constant potential, differently to [18]). In adapting [18, Theorem 3.2], it is important to point out that any limit of u_n, given by Theorem 4.2, tends to 0 at infinity. This fact is crucial in the proof of (4.18).

Moreover, if the reference graph is unbounded, it is important to observe that $u_n(x) \to 0$ as $|x| \to +\infty$ on each half-line, since $u_n \in H^1(\mathcal{G})$ by assumption. This implies that, if $\{P_n^1\}, \dots, \{P_n^h\}$ are local maximum points of u_n, then there exists a maximum point on $\mathcal{G} \setminus \bigcup_{i=1}^{h} B_{R\lambda_n^{-1/2}}(P_n^i)$.

Step 2. *Conclusion of the proof.* By (4.18), for every $\varepsilon \in (0, 1)$ small, to be chosen later, there exist $R > 0$ and $n_R \in \mathbb{N}$ large such that

$$\max_{d_n(x) > R\lambda_n^{-1/2}} u_n(x) \le \lambda_n^{\frac{1}{p-2}} \varepsilon \quad \forall n \ge n_R. \tag{4.19}$$

Thus, in the set $A_n := \{d_n(x) > R\lambda_n^{-1/2}\}$, in addition to (4.19) we also have

$$u_n'' = (\lambda_n - u_n^{p-2}) u_n \implies -u_n'' + \frac{\lambda_n}{2} u_n \le 0 \tag{4.20}$$

provided that $\varepsilon > 0$ is small enough.

We want to exploit (4.19) and (4.20) in a comparison argument, as in [18] (or [17, Theorem 3.1]). However, the presence of the vertexes makes the argument a little more involved in our setting.

Let us denote by $\{\mathrm{v}_j\}_{j=1}^{h_1}$ the set of vertexes which are not included in one of the balls $B_{R\lambda_n^{-1/2}}(P_n^i)$ for large n. On any such vertex, by (4.19),

$$u_n(\mathrm{v}_j) \le \lambda_n^{\frac{1}{p-2}} \varepsilon.$$

For any edge e, we consider the restriction of u_n on $\mathrm{e} \cap A_n$. Since k is independent of n, $\mathrm{e} \cap A_n$ consists of finitely many relatively open intervals (which may be unbounded, if $\mathcal{G}$ is non-compact).

Let I_n be any such *bounded* interval; then the following alternative holds: $\partial I_n \cap \{\mathrm{v}_j\}_{j=1}^{h_1}$ can either be empty (case 1), or be a single vertex , say v_1 (case 2), or be a pair of vertexes, say v_1 and v_2 (case 3).

Assume first that case 1 holds. Then there exist two indexes $i, j \in \{1, \dots, k\}$ such that ∂I_n consists of one point at distance $R\lambda_n^{-1/2}$ from P_n^i, and one point at distance $R\lambda_n^{-1/2}$ from P_n^j. Consider the function

$$\phi_n(x) = e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^i|} + e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^j|},$$

which solves $\phi_n'' = \gamma^2 \lambda_n \phi_n$ in I_n. By taking $\gamma < 1/4$, we have

$$-\phi_n'' + \frac{\lambda_n}{2} \phi_n \ge 0 \quad \text{in } I_n.$$

Moreover,

$$\left(e^{\gamma R} \lambda_n^{\frac{1}{p-2}} \phi_n - u_n\right)\Big|_{\partial I_n} \ge \lambda_n^{\frac{1}{p-2}} (1 - \varepsilon) > 0,$$

and hence, by the comparison principle, we have

$$u(x) \le e^{\gamma R} \lambda_n^{\frac{1}{p-2}} \phi_n(x) n \quad \forall x \in I_n,$$

which clearly implies the validity of the thesis on I_n in this case.

If case 2 holds, then there exists an index $i \in \{1, \dots, k\}$ such that ∂I_n consists of a point at distance $R\lambda_n^{-1/2}$ from P_n^i, plus the vertex v_1. Arguing as before, it is not difficult to check that

$$u(x) \le e^{\gamma R} \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^i|} + \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - \mathrm{v}_1|} \quad \forall x \in I_n,$$

which gives the thesis in case 2.

In case 3, an analogous argument ensures that

$$u(x) \le \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - \mathrm{v}_1|} + \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - \mathrm{v}_2|} \quad \forall x \in I_n,$$

whence the thesis follows once again.

Finally, let us consider the case when I_n is an *unbounded* interval of $\mathrm{e} \cap A_n$. Then we only have two possibilities: either ∂I_n consists of a point at distance $R\lambda_n^{-1/2}$ from P_n^i, or ∂I_n consists of a vertex, say v_1.

In the former case, we argue as before with the comparison function

$$\psi_n(x) = e^{-\gamma R \lambda_n^{\frac{1}{p-2}}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - P_n^i|},$$

where $\gamma < 1/4$. In the latter one, we can use

$$\psi_n(x) = \lambda_n^{\frac{1}{p-2}} e^{-\gamma \lambda_n^{\frac{1}{2}} |x - v_1|}.$$

To sum up, slightly modifying the choice of the comparison functions, according to the structure of ∂I_n, it is possible to prove the validity of (4.17) in all the possible cases. ■

5. Mountain pass solution for the original problem

In this section we complete the proof of the main existence result, Theorem 1.1. Let $\mu \in (0, \mu_1)$. As already anticipated in Section 4, Proposition 3.3 gives a sequence of mountain pass critical points $u_{\rho_n} \in H^1_\mu(\mathcal{G})$ of $E_{\rho_n}(\cdot, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$ with $\rho_n \to 1^-$ and $m(u_{\rho_n}) \le 2$. Moreover, the energy level c_{ρ_n} is bounded, since

$$E_1(\kappa_\mu, \mathcal{G}) \le E_\rho(\kappa_\mu, \mathcal{G}) \le c_\rho \le c_{1/2} \quad \forall \rho \in \left[\tfrac{1}{2}, 1\right]$$

(the first and the second inequalities are proved in Lemma 3.4; the third one follows directly from the monotonicity of c_ρ). Thus, Theorem 1.1 is a direct corollary of the next statement.

Proposition 5.1. *Let $\mathcal{G}$ be a metric graph satisfying (4.1) and $p > 6$. Let $\{u_n\} \subset H^1(\mathcal{G})$ be a sequence of solutions to (4.2) for some $\lambda_n \in \mathbb{R}$ and $\rho_n \to 1$. Suppose that*

$$\int_{\mathcal{G}} |u_n|^2\, dx = \mu, \quad m(u_n) \le \bar{k} \quad \forall n,$$

for some $\mu > 0$ and $\bar{k} \in \mathbb{N}$, and that

the sequence of the energy levels $\{c_n := E_{\rho_n}(u_n, \mathcal{G})\}$ is bounded.

Then the sequences $\{\lambda_n\} \subset \mathbb{R}$ and $\{u_n\} \subset H^1(\mathcal{G})$ must be bounded. In addition, $\{u_n\}$ is a (bounded) Palais–Smale sequence for $E_1(\cdot, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$.

Proof of Theorem 1.1. It is sufficient to apply Proposition 5.1 on the sequence $\{u_{\rho_n}\}$ which, as observed, fulfills the assumptions. Indeed, applying Lemma 3.1 we then deduce that $u_n \to \bar{u}$ strongly in $H^1(\mathcal{G})$. ■

Proof of Proposition 5.1. Since

$$\int_{\mathcal{G}} (|u_n'|^2 + \lambda_n u_n^2)\, dx = \rho_n \int_{\mathcal{G}} |u_n|^p\, dx,$$

it follows that

$$c_n = E_{\rho_n}(u_n, \mathcal{G}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |u_n'|^2 \, dx - \frac{\lambda_n \mu}{p};$$

therefore

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |u_n'|^2 \, dx = c_n + \frac{\lambda_n \mu}{p}. \tag{5.1}$$

This estimate gives the boundedness of $\{u_n\}$ in $H^1(\mathcal{G})$, provided that $\{\lambda_n\}$ is bounded (recall that $\{c_n\}$ is bounded as well). Once the boundedness of $\{u_n\}$ in $H^1(\mathcal{G})$ is proved, and since $\rho_n \to 1$, the fact that it is a Palais–Smale sequence for $E_1(\cdot, \mathcal{G})$ constrained on $H^1_\mu(\mathcal{G})$ is straightforward.

Therefore, we only have to show that $\{\lambda_n\}$ is bounded. By contradiction, we suppose that this is not the case. By (5.1), we have $\lambda_n \to +\infty$, up to a subsequence. Thus, Theorems 4.2 and 4.6 hold for $u_n := u_{\rho_n}$. For $\{P_n^1\}, \dots, \{P_n^k\}$ given by Theorem 4.6, Theorem 4.2 ensures the existence of blow-up limits, which can be either defined on $\mathbb{R}$, or on a star graph $\mathcal{G}_m$. In the rest of the proof,

- $\{v_n^i\}$ denotes the scaled sequence around P_n^i;
- V^i denotes the limit of $\{v_n^i\}$;
- $\bar{x}_n^i$ denotes the global maximum point of v_n^i;
- $\bar{x}^i$ denotes the global maximum point of V^i.

Then, for $R > 0$, on one hand we have

$$\left| \lambda_n^{\frac{1}{2} - \frac{2}{p-2}} \int_{\mathcal{G}} u_n^2 \, dx - \sum_{i=1}^k \int_{B_R(\bar{x}_n^i)} (v_n^i)^2 \, dx \right| \to +\infty \tag{5.2}$$

(in the second integral, the ball $B_R(\bar{x}_n^i)$ is the ball in the scaled graph). Indeed, the first term inside the absolute value satisfies

$$\lambda_n^{\frac{1}{2} - \frac{2}{p-2}} \int_{\mathcal{G}} u_n^2 \, dx = \lambda_n^{\frac{1}{2} - \frac{2}{p-2}} \mu \to +\infty,$$

since $p > 6$, while the second term is bounded, since by Theorem 4.2,

$$\sum_{i=1}^k \int_{B_R(\bar{x}_n^i)} (v_n^i)^2 \, dx \to \int_{B_R(\bar{x}^i)} (V^i)^2 \, dx,$$

and it is the sum of a finite number of bounded integrals, since $V^i \in H^1(\mathcal{G}_m)$.

On the other hand, by Theorem 4.6, for some positive constant C which changes from one line to another,

$$\left| \lambda_n^{\frac{1}{2} - \frac{2}{p-2}} \int_{\mathcal{G}} u_n^2 \, dx - \sum_{i=1}^k \int_{B_R(x_n^i)} (v_n^i)^2 \, dx \right|$$
$$= \lambda_n^{\frac{1}{2} - \frac{2}{p-2}} \left| \int_{\mathcal{G}} u_n^2 \, dx - \sum_{i=1}^k \int_{B_{R\lambda_n^{-1/2}}(P_n^i)} u_n^2 \, dx \right|$$

$$
\begin{aligned}
&= \lambda_n^{\frac{1}{2}-\frac{2}{p-2}} \int_{\mathcal{G}\setminus\bigcup_i B_{R\lambda_n^{-1/2}}(P_n^i)} u_n^2\,dx \\
&\le C_1\lambda_n^{\frac{1}{2}} \sum_{i=1}^{k} \int_{\mathcal{G}\setminus\bigcup_i B_{R\lambda_n^{-1/2}}(P_n^i)} e^{-C_2\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,P_n^i)}\,dx \\
&\quad + C_1\lambda_n^{\frac{1}{2}} \sum_{j=1}^{h} \int_{\mathcal{G}} e^{-C_2\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,\mathrm{v}_j)}\,dx \\
&\le C\lambda_n^{\frac{1}{2}} \sum_{i=1}^{k} \int_{\mathcal{G}\setminus B_{R\lambda_n^{-1/2}}(P_n^i)} e^{-C\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,P_n^i)}\,dx + C\lambda_n^{\frac{1}{2}} \sum_{j=1}^{h} \int_{\mathcal{G}} e^{-C\lambda_n^{\frac{1}{2}}\operatorname{dist}(x,\mathrm{v}_j)}\,dx \\
&\le C\lambda_n^{\frac{1}{2}} \int_{R\lambda_n^{-1/2}}^{+\infty} e^{-C\lambda_n^{\frac{1}{2}}y}\,dy + C\lambda_n^{\frac{1}{2}} \int_0^{+\infty} e^{-C\lambda_n^{\frac{1}{2}}y}\,dy \\
&\le C\int_R^{+\infty} e^{-Cz}\,dz + C\int_0^{\infty} e^{-Cz}\,dz \\
&\le Ce^{-CR} + C.
\end{aligned}
$$

By taking the limit as $n \to \infty$, we deduce that

$$
\limsup_{n\to\infty} \left| \lambda_n^{\frac{1}{2}-\frac{2}{p-2}} \int_{\mathcal{G}} u_n^2\,dx - \sum_{i=1}^{k} \int_{B_r(\bar{x}_i)} (V^i)^2\,dx \right| \le Ce^{-CR} + C,
$$

in contradiction with (5.2). ■

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References

[1] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, Constrained energy minimization and orbital stability for the NLS equation on a star graph. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **31** (2014), no. 6, 1289–1310 Zbl 1304.81087 MR 3280068

[2] R. Adami, E. Serra, and P. Tilli, NLS ground states on graphs. *Calc. Var. Partial Differential Equations* **54** (2015), no. 1, 743–761 Zbl 1330.35484 MR 3385179

[3] R. Adami, E. Serra, and P. Tilli, Threshold phenomena and existence results for NLS ground states on metric graphs. *J. Funct. Anal.* **271** (2016), no. 1, 201–223 Zbl 1338.35448 MR 3494248

[4] R. Adami, E. Serra, and P. Tilli, Negative energy ground states for the L^2-critical NLSE on metric graphs. *Comm. Math. Phys.* **352** (2017), no. 1, 387–406 Zbl 1372.35319 MR 3623262

[5] R. Adami, E. Serra, and P. Tilli, Nonlinear dynamics on branched structures and networks. *Riv. Math. Univ. Parma (N.S.)* **8** (2017), no. 1, 109–159 Zbl 1391.35381 MR 3706143

[6] A. H. Ardila, Orbital stability of standing waves for supercritical NLS with potential on graphs. *Appl. Anal.* **99** (2020), no. 8, 1359–1372 Zbl 1434.35174 MR 4097823

[7] T. Bartsch and N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.* **272** (2017), no. 12, 4998–5037 Zbl 1485.35173 MR 3639521

[8] J. Bellazzini, N. Boussaïd, L. Jeanjean, and N. Visciglia, Existence and stability of standing waves for supercritical NLS with a partial confinement. *Comm. Math. Phys.* **353** (2017), no. 1, 229–251 Zbl 1367.35150 MR 3638314

[9] J. Bellazzini and D. Ruiz, Finite energy traveling waves for the Gross-Pitaevskii equation in the subsonic regime. *Amer. J. Math.* **145** (2023), no. 1, 109–149 Zbl 07653692 MR 4545844

[10] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*. Mathematical Surveys and Monographs 186, American Mathematical Society, Providence, RI, 2013 Zbl 1318.81005 MR 3013208

[11] F. Boni and S. Dovetta, Prescribed mass ground states for a doubly nonlinear Schrödinger equation in dimension one. *J. Math. Anal. Appl.* **496** (2021), no. 1, article no. 124797 Zbl 1458.81017 MR 4186675

[12] J. Borthwick, X. Chang, L. Jeanjean and N. Soave, Bounded Palais-Smale sequences with Morse type information for some constrained functionals. 2022, arXiv:2210.12626

[13] C. Cacciapuoti, S. Dovetta, and E. Serra, Variational and stability properties of constant solutions to the NLS equation on compact metric graphs. *Milan J. Math.* **86** (2018), no. 2, 305–327 Zbl 1404.35470 MR 3877532

[14] S. Dovetta, Existence of infinitely many stationary solutions of the L^2-subcritical and critical NLSE on compact metric graphs. *J. Differential Equations* **264** (2018), no. 7, 4806–4821 Zbl 1391.35382 MR 3758538

[15] S. Dovetta, M. Ghimenti, A. M. Micheletti, and A. Pistoia, Peaked and low action solutions of NLS equations on graphs with terminal edges. *SIAM J. Math. Anal.* **52** (2020), no. 3, 2874–2894 Zbl 1454.35336 MR 4111673

[16] S. Dovetta and L. Tentarelli, L^2-critical NLS on noncompact metric graphs with localized nonlinearity: topological and metric features. *Calc. Var. Partial Differential Equations* **58** (2019), no. 3, article no. 108 Zbl 1432.35212 MR 3959930

[17] P. Esposito, G. Mancini, S. Santra, and P. N. Srikanth, Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. *J. Differential Equations* **239** (2007), no. 1, 1–15 Zbl 1143.35039 MR 2341546

[18] P. Esposito and M. Petralla, Pointwise blow-up phenomena for a Dirichlet problem. *Comm. Partial Differential Equations* **36** (2011), no. 9, 1654–1682 Zbl 1231.35084 MR 2825606

[19] G. Fang and N. Ghoussoub, Second-order information on Palais-Smale sequences in the mountain pass theorem. *Manuscripta Math.* **75** (1992), no. 1, 81–95 Zbl 0797.49036 MR 1156217

[20] G. Fang and N. Ghoussoub, Morse-type information on Palais-Smale sequences obtained by min-max principles. *Comm. Pure Appl. Math.* **47** (1994), no. 12, 1595–1653 Zbl 0829.58008 MR 1303222

[21] L. Friedlander, Extremal properties of eigenvalues for a metric graph. *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 1, 199–211 Zbl 1074.34078 MR 2141695

[22] N. Ikoma and K. Tanaka, A note on deformation argument for L^2 normalized solutions of nonlinear Schrödinger equations and systems. *Adv. Differential Equations* **24** (2019), no. 11-12, 609–646 Zbl 1437.35188 MR 4021261

[23] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.* **28** (1997), no. 10, 1633–1659 Zbl 0877.35091 MR 1430506
[24] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), no. 4, 787–809 Zbl 0935.35044 MR 1718530
[25] L. Jeanjean and K. Tanaka, A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^N$. *Indiana Univ. Math. J.* **54** (2005), no. 2, 443–464 Zbl 1143.35321 MR 2136816
[26] A. Kairzhan, D. Noja, and D. E. Pelinovsky, Standing waves on quantum graphs. *J. Phys. A* **55** (2022), no. 24, article no. 243001 Zbl 07618040 MR 4438617
[27] P.-L. Lions, Solutions of Hartree-Fock equations for Coulomb systems. *Comm. Math. Phys.* **109** (1987), no. 1, 33–97 Zbl 0618.35111 MR 879032
[28] R. López-Soriano, A. Malchiodi, and D. Ruiz, Conformal metrics with prescribed Gaussian and geodesic curvatures. *Ann. Sci. Éc. Norm. Supér. (4)* **55** (2022), no. 5, 1289–1328 Zbl 07629566 MR 4517687
[29] M. Lorenzo, M. Luccia, V. Merlo, I. Ottaviani, M. Salvato, M. Cirillo, M. Müller, T. Weimann, M. G. Castellano, F. Chiarello, and G. Torrioli, On Bose–Einstein condensation in Josephson junctions star graph arrays. *Phys. Lett. A* **378** (2014), no. 7-8, 655–658
[30] D. Noja, Nonlinear Schrödinger equation on graphs: recent results and open problems. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** (2014), no. 2007, article no. 20130002 Zbl 1322.35130 MR 3151078
[31] D. Noja and D. E. Pelinovsky, Standing waves of the quintic NLS equation on the tadpole graph. *Calc. Var. Partial Differential Equations* **59** (2020), no. 5, article no. 173 Zbl 1448.35474 MR 4150875
[32] D. Pierotti and N. Soave, Ground states for the NLS equation with combined nonlinearities on noncompact metric graphs. *SIAM J. Math. Anal.* **54** (2022), no. 1, 768–790 Zbl 1483.35308 MR 4371084
[33] D. Pierotti, N. Soave, and G. Verzini, Local minimizers in absence of ground states for the critical NLS energy on metric graphs. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), no. 2, 705–733 Zbl 1464.35329 MR 4241295
[34] D. Pierotti and G. Verzini, Normalized bound states for the nonlinear Schrödinger equation in bounded domains. *Calc. Var. Partial Differential Equations* **56** (2017), no. 5, article no. 133 Zbl 1420.35374 MR 3689156
[35] E. Serra and L. Tentarelli, Bound states of the NLS equation on metric graphs with localized nonlinearities. *J. Differential Equations* **260** (2016), no. 7, 5627–5644 Zbl 1408.35206 MR 3456809
[36] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.* **160** (1988), no. 1-2, 19–64 Zbl 0646.53005 MR 926524

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Xiaojun Chang
School of Mathematics and Statistics & Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, Jilin, P. R. China; changxj100@nenu.edu.cn

Louis Jeanjean
Université de Franche-Comté, Laboratoire de mathématiques de Besançon, UMR CNRS 6623,
16 route de Gray, 25000 Besançon, France; louis.jeanjean@univ-fcomte.fr

Nicola Soave
Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, 10123 Torino,
Italy; nicola.soave@unito.it