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Estimation of non-uniqueness and short-time asymptotic expansions for Navier–Stokes flows

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Abstract. There is considerable evidence that solutions to the three-dimensional Navier–Stokes equations in the natural energy space are not unique. Assuming this is the case, it becomes important to quantify how non-uniqueness evolves. In this paper we provide an algebraic estimate for how rapidly two possibly non-unique solutions can separate over a compact spatial region in which the initial data has sub-critical regularity. Outside of this compact region, the data is only assumed to be in the scaling critical weak Lebesgue space and can be large. To establish this upper bound on the separation rate, we develop a new spatially local, short-time asymptotic expansion which is of independent interest.

1. Introduction

The Navier–Stokes equations,

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0, \tag{1.1}$$

govern the evolution of a viscous incompressible flow's velocity field u and its associated scalar pressure p. The system is supplemented with a divergence-free initial datum u_0. We consider the problem on $\mathbb{R}^3 \times (0, T)$, where $0 < T \leq \infty$. A foundational mathematical treatment of the problem was given by Leray [38], where global weak solutions were constructed for finite energy data. Solutions with the properties of those constructed by Leray are referred to as Leray weak solutions. Recent work suggests that uniqueness does not hold in the class of Leray weak solutions. Indeed, non-uniqueness has been affirmed in weaker classes than the Leray class [14] and within the Leray class for the forced Navier–Stokes equations [2]. Within the Leray class and with no forcing, a research program of Jia and Šverák [27, 28] and the numerical work of Guillod and Šverák [25] support non-uniqueness. This program proposes non-uniqueness in a class of solutions with large $L^{3,\infty}$ data and then truncates the conjectured solutions to give non-unique Leray–Hopf solutions. We presently consider solutions in this critical space and give a precise definition below.

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While the evidence suggests non-uniqueness, there is no clear picture of how non-uniqueness would evolve. In this note, we take the view that solutions are not unique and seek to quantify how rapidly distinct solutions can separate as they evolve from a shared initial state. In particular, we are interested in the following question:

How can non-uniqueness be quantified in terms of local properties of the initial data?

To answer this question, we seek conditions so that, given some divergence-free u_0, ball B, positive exponent σ, time $T > 0$, and weak solutions u_1 and u_2 to (1.1) both evolving from u_0, we have

$$\|u_1 - u_2\|_{L^\infty(B)}(t) \lesssim t^\sigma$$

for all $0 < t < T$. We refer to bounds like the above as an "estimation of non-uniqueness" and the right-hand side as a "separation rate."

A preliminary perspective on this question follows from the local smoothing of Jia and Šverák [27]. Local smoothing says that, if u_0 is sufficiently regular in a ball B, then a solution u remains regular on $B' \times [0, T]$ for some $T > 0$, where $B' \Subset B$. This can be viewed as saying that the non-local effects of the pressure are not strong enough to overcome the local regularity of the data. Local regularity is proven in [27] by showing that, for solutions in the local Leray class,[1] if $u_0|_B \in L^p(B)$ for some $3 < p \leq \infty$ and U is the strong solution to the Navier–Stokes equations with initial data a divergence-free localization of u_0 to B, then the difference $u - U$ is in the parabolic Hölder space $C^\gamma_{\mathrm{par}}(B' \times [0, T])$, where $\gamma = \gamma(p) \in (0, 1)$. This space is endowed with the seminorm

$$[f]_{C^\gamma_{\mathrm{par}}(B' \times [0,T])} := [f]_{C_t^{\gamma/2}([0,T]; L^\infty(B'))} + [f]_{L^\infty([0,T]; C^\gamma_x(B'))}.$$

Let us point out that, given the parabolic scaling of (1.1), the exponent for the time variable is $\gamma/2$. Since U is uniquely determined by u_0, this implies that, for possibly distinct solutions u_1 and u_2 with the same data u_0, we have

$$\|u_1 - u_2\|_{L^\infty(B')}(t) \leq \|u_1 - U\|_{L^\infty(B')}(t) + \|U - u_2\|_{L^\infty(B')}(t) \lesssim t^{\frac{\gamma}{2}}. \tag{1.2}$$

Since $\gamma/2 < 1$, the derivative of the right-hand side blows up as $t \to 0^+$, allowing rapid separation.

A stronger separation rate is identified for discretely self-similar (DSS) solutions, i.e. solutions satisfying $u^\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t)$, for some $\lambda > 1$, with data in $L^p_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ for $3 < p \leq \infty$ in [7]. There, due to the global scaling properties of the solution,

$$|u_1 - u_2|(x, t) \lesssim \frac{t^{\frac{3}{2}}}{(|x| + \sqrt{t})^4},$$

[1] This is a more general class than the Leray class and was introduced by Lemarié-Rieusset. See [37] and the later papers [9, 27, 30, 32, 36] for useful properties. Local Leray solutions are sometimes referred to as local energy solutions.

outside of a space-time paraboloid, i.e. in the region $|x| \geq R_0\sqrt{t}$, for some $R_0 \geq 0$. Away from $x = 0$, this gives the separation rate $t^{3/2}$, which is stronger than (1.2) for $t \leq 1$. Although we do not have a proof, we expect the rate $t^{3/2}$ is optimal because it arises in [7] from pointwise bounds for gradients of the Oseen tensor [41] which seem unavoidable. The solutions in [7] have a great deal of structure due to their assumed scaling invariance and it is natural to seek separation rates under relaxed conditions.

In this paper, with no scaling assumptions we almost recover the separation rate $t^{3/2}$, which was obtained for DSS solutions in [7]. We take our initial data to be in $L^{3,\infty}(\mathbb{R}^3)$, which coincides with the weak Lebesgue space L^3_w and is a Lorentz space.[2] If, additionally, $u_0|_B \in L^p(B)$ for a ball B and some $3 < p \leq \infty$, then we show there exists a time $T > 0$ so that, for any $\sigma < 3/2$, any two weak solutions u_1 and u_2 in a certain class satisfy

$$\|u_1 - u_2\|_{L^\infty(B')}(t) \lesssim t^\sigma,$$

where $B' \Subset B$ and $0 < t < T$. This class of initial data is motivated by a natural type-I blow-up scenario wherein a strong solution u defined on $\mathbb{R}^3 \times (-1, 0)$ satisfying

$$|u(x,t)| \lesssim \frac{1}{|x| + \sqrt{-t}}$$

develops a singularity at the space-time origin. The singular profile would satisfy

$$|u(x,0)| \lesssim |x|^{-1} \in L^{3,\infty}.$$

Because uniqueness is not expected for large $L^{3,\infty}$ data, upon singularity formation the solution might branch into distinct flows. In this scenario, our theorem provides an upper bound on how fast the branching solutions can separate away from the singularity. Additionally, the initial data in [7] is only locally critical at the origin; it is locally sub-critical[3] everywhere else. In our theorem, the only sub-critical assumption is within the ball B; the data can have $L^{3,\infty}$ singularities anywhere else.

Before stating our result we define the class of solutions we have in mind, which was originally developed by Barker, Seregin, and Šverák [6] and extends ideas in [40]. This notion of solution has since been extended to non-endpoint critical Besov spaces of negative smoothness [1].

[2] $L^{3,\infty}$ includes all the DSS data considered in [7]. It is important in the analysis of the Navier–Stokes equations as an endpoint space, where many desirable features such as regularity or uniqueness are not known to hold. For example, there is a time-local unique strong solution when u_0 is possibly large in L^3 [31], but this is unknown in the larger space $L^{3,\infty}$. It is a *critical* space in that it is scaling invariant with respect to the scaling of (1.1).

[3] We say the space X is *sub-critical* if $\|u_0\|_X = \lambda^\alpha \|u_0^\lambda\|_X$, where $\alpha > 0$. Examples of sub-critical spaces are L^p for $p \in (3, \infty]$. Typically, inclusion in sub-critical spaces controls small scales and leads to regularity. For *super-critical* spaces, $\alpha < 0$ and small scales are typically not controlled. For critical spaces, small scales are usually controlled to an extent when C_c^∞ is dense in the space. Critical spaces where this fails, like $L^{3,\infty}$, are sometimes referred to as *ultra-critical*.

Definition 1.1 (Weak $L^{3,\infty}$-solutions). Let $T > 0$ be finite. Assume $u_0 \in L^{3,\infty}$ is divergence-free. We say that u and an associated pressure p comprise a weak $L^{3,\infty}$-solution if

- (u, p) satisfies (1.1) distributionally,
- u satisfies the local energy inequality of Scheffer [39] and Caffarelli, Kohn, and Nirenberg [15], i.e. for all non-negative $\phi \in C_c^\infty(\mathbb{R}^3 \times (0,T])$ and $0 < t < T$, we have

$$\int \phi(x,t)|u(x,t)|^2\,dx + 2\int_0^t \int |\nabla u|^2 \phi\,dx\,dt \\ \leq \int_0^t \int |u|^2(\partial_t \phi + \Delta\phi)\,dx\,dt + \int_0^t \int (|u|^2 + 2p)(u \cdot \nabla\phi)\,dx\,dt,$$

- for every $w \in L^2$, the following function is continuous on $[0, T]$,

$$t \to \int w(x) \cdot u(x,t)\,dx,$$

- $\tilde{u} := u - e^{t\Delta}u_0$ satisfies, for all $t \in (0, T)$,

$$\sup_{0<s<t} \|\tilde{u}\|_{L^2}^2(s) + \int_0^t \|\nabla\tilde{u}\|_{L^2}^2(s)\,ds < \infty,$$

  and

$$\|\tilde{u}\|_{L^2}^2(t) + 2\int_0^t \int |\nabla\tilde{u}|^2\,dx\,ds \leq 2\int_0^t \int (e^{s\Delta}u_0 \otimes \tilde{u} + e^{s\Delta}u_0 \otimes e^{s\Delta}u_0) : \nabla\tilde{u}\,dx\,ds.$$

  In particular, $\|\tilde{u}\|_{L^2}^2(t) \to 0$ as $t \to 0^+$.

In [6], weak solutions are constructed which satisfy the above definition for all $T > 0$. Also, due to their spatial decay, weak $L^{3,\infty}$-solutions are mild,[4] which means they satisfy the formula

$$u(x,t) = e^{t\Delta}u_0 - B(u,u),$$

where $\mathbb{P}$ is the Leray projection operator and

$$B(f,g) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \Big(\frac{1}{2}(f \otimes g + g \otimes f)\Big)\,ds,$$

which is symmetric.

An important observation in [6] is that the non-linear part of a weak $L^{3,\infty}$-solution satisfies a dimensionless energy estimate, namely

$$\sup_{0<s<t} \|\tilde{u}\|_{L^2}(s) + \Big(\int_0^t \|\nabla\tilde{u}\|_{L^2}^2(s)\,ds\Big)^{\frac{1}{2}} \lesssim_{u_0} t^{\frac{1}{4}}. \tag{1.3}$$

[4] Although this can be proved directly, it also follows from [10] or [37, p. 109].

We emphasize that the energy associated with $\tilde{u}$ vanishes at $t = 0$. This decay property will be essential in our work. It appeared earlier in the a priori estimates of the weak discretely self-similar solutions constructed in [8] as well as [40], which is the precursor to [6]. It is used in the Calderón-type splitting (see [16]) construction in [6] to deplete a time singularity. In [1], it is established in Besov spaces with $e^{t\Delta}u_0$ replaced by higher Picard iterates, which are defined below.

As pointed out in [6], (1.3) can be viewed as an estimate on the separation rate of two weak $L^{3,\infty}$-solutions since, denoting two such solutions with the same data by u_1 and u_2, we have

$$\|u_1 - u_2\|_{L^2}(t) \lesssim \|\tilde{u}_1\|_{L^2}(t) + \|\tilde{u}_2\|_{L^2}(t) \lesssim t^{\frac{1}{4}}.$$

Notably, this is a global estimate. For data in $L^{3,\infty}$, global estimates are confined to super-critical norms since we do not expect $\tilde{u}$ to be in a stronger space than u—indeed, any singularity at a positive time is carried by $\tilde{u}$ not by $e^{t\Delta}u_0$. Such singularities can possibly occur at arbitrarily small times. Therefore, if we seek a finer estimate (i.e. using a sub-critical norm) on the separation of the flows using the reasoning above, it should be confined to a local region where local smoothing holds, e.g. where the initial data is sub-critical. The following theorem provides such an estimate.

Theorem 1.2 (Estimation of non-uniqueness). *Assume $u_0 \in L^{3,\infty}$ and is divergence-free. Fix $x_0 \in \mathbb{R}^3$. Assume that $u_0|_B \in L^p(B)$, where $B = B_2(x_0)$ and $p \in (3, \infty]$. Let u_1 and u_2 be weak $L^{3,\infty}$-solutions with data u_0. Then there exists $T = T(p, u_0) > 0$ so that, for every $\sigma \in (0, 3/2)$ and $t \in (0, T)$,*

$$\|u_1 - u_2\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,\sigma,u_0} t^\sigma,$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^p(B)}$ and $\|u_0\|_{L^{3,\infty}}$.

Insofar as non-uniqueness in the Leray class is concerned, if $u_0 \in L^2 \cap L^{3,\infty}$, then any weak $L^{3,\infty}$-solution is also a Leray weak solution as discussed in [6]. Hence our result applies to a subset of the Leray class.

Theorem 1.2 is a corollary of the following theorem, the proof of which constitutes the bulk of this paper. Before stating the theorem, we recall the definition of Picard iterates. Let $P_0 = P_0(u_0) = e^{t\Delta}u_0$ and define the kth Picard iterate to be $P_k = P_0 - B(P_{k-1}, P_{k-1})$. Classically, the Picard iterates converge to a solution to (1.1) whenever (1.1) can be viewed as a perturbation of the heat equation. This is not the case for large $L^{3,\infty}$ data, so we do not expect convergence of P_k to u when u is a weak $L^{3,\infty}$-solution. Nonetheless, the Picard iterates do capture some asymptotics at $t = 0$ of weak $L^{3,\infty}$-solutions, which is the point of the following theorem.

Theorem 1.3 (Local asymptotic expansion). *Assume $u_0 \in L^{3,\infty}$ and is divergence-free. Fix $x_0 \in \mathbb{R}^3$ and $p \in (3, \infty]$. Assume further that $u_0|_B \in L^p(B)$, where $B = B_2(x_0)$. Then there exist $\gamma = \gamma(p) \in (0, 1)$ and $T = T(p, \|u_0\|_{L^{3,\infty}}, \|u_0\|_{L^p(B)}) > 0$ so that, for any $\sigma \in (0, 3/2)$, $t \in (0, T)$, and $k = 0, 1, \ldots, k_0$,*

$$\|u - P_k\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,u_0,\sigma,k} t^{a_k},$$

where $a_0 = \min\{\gamma/2, 1/2 - 3/(2p)\}$, $a_{k+1} = \min\{\sigma, k(1/2 - 3/(2p)) + a_0\}$, *and* k_0 *is the smallest natural number so that*

$$k_0\Big(\frac{1}{2} - \frac{3}{2p}\Big) + a_0 \geq \sigma.$$

In particular, $a_{k_0} = \sigma$ *and* $a_k > a_{k-1}$ *for* $k = 1, \dots, k_0$. *It follows that, for* $(x,t) \in B_{1/4}(x_0) \times (0,T)$, *and letting* $a_{-1} = -3/(2p)$, *we have*

$$u(x,t) = P_0 + \sum_{k=0}^{k_0-1} \mathcal{O}(t^{a_k}) + \mathcal{O}(t^{\sigma}) = \sum_{k=-1}^{k_0} \mathcal{O}(t^{a_k}),$$

where the $\mathcal{O}(t^{a_k})$ *terms are exactly solvable for* $-1 \leq k < k_0$.

Note that Theorems 1.2 and 1.3 can be extended by replacing $B_{1/4}(x_0)$ with $B_\rho(x_0)$ for any $\rho < 2$ via rescaling and a covering argument. In this extension T degrades as ρ approaches 2.

Short-time asymptotic expansions have been examined by Brandolese for small self-similar flows [11] and by Brandolese and Vigneron for both small (in which case the expansion holds for all times) and large (in which case the data is globally sub-critical and the expansion is up to a finite time) non-self-similar flows [13]. A follow-up paper by Bae and Brandolese considers the forced Navier–Stokes equations [3]. In [34], Kukavica and Ries give an expansion in arbitrarily many terms assuming the solution is smooth. In all of the preceding papers, either the initial data is strong enough to generate smooth solutions (e.g. it is in a sub-critical class or is small in a critical class) or the solution is assumed to be smooth. Additionally, the terms of the asymptotic expansions depend on u.

The novelty of Theorem 1.3 is that it establishes time asymptotics without any scaling assumption (cf. [7]) or requirements implying global regularity on the relevant time domain (cf. [3, 11, 13, 34]). The asymptotics depend only on u_0—they are independent of u, which is necessary for Theorem 1.2. Because ours is a spatially local expansion, spatial asymptotics are not relevant.

Remark 1.4. If we take $|u_0|(x) \lesssim |x|^{-1}$, then, by a rescaling argument, it is possible to show that, for any $0 < \sigma < 3/2$ and $x \neq 0$,

$$|u - P_{k_0}|(x,t) \lesssim \frac{t^{\sigma}}{|x|^{2\sigma+1}},$$

where $0 < t \lesssim |x|^2$. This almost reaches the $t^{3/2}/|x|^4$ asymptotic bounds established for discretely self-similar solutions in [7].

Long-time asymptotic expansions have also been studied extensively; see e.g. [20, 21, 24, 26], the review article [12], and the references therein. The spatial asymptotics for the stationary problem have also been studied; see e.g. [33] and the references therein.

With Theorem 1.3 in hand, we quickly prove Theorem 1.2.

Proof of Theorem 1.2. Suppose u_1 and u_2 are weak $L^{3,\infty}$-solutions with data u_0. By Theorem 1.3, we have for $i = 1, 2$ that

$$\|u_i - P_{k_0}\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,\sigma,u_0} t^\sigma$$

for all $0 < t < T$. By the uniqueness of Picard iterates, we infer

$$\begin{aligned}\|u_1 - u_2\|_{L^\infty(B_{1/4}(x_0))}(t) &\leq \|u_1 - P_{k_0}\|_{L^\infty(B_{1/4}(x_0))}(t) + \|u_2 - P_{k_0}\|_{L^\infty(B_{1/4}(x_0))}(t)\\ &\lesssim_{p,\sigma,u_0} t^\sigma\end{aligned}$$

for all $0 < t < T$. ∎

Discussion of the proof: By local smoothing [27], it is not difficult to show that

$$\|u - P_0\|_{L^\infty(B_{1/2}(x_0))}(t) \lesssim t^{\frac{\gamma}{2}},$$

for some $\gamma = \gamma(p) \in (0, 1)$ and across some time interval. Our main insight is that this bound improves when P_0 is replaced by higher Picard iterates, a consequence of the self-improvement property of Picard iterates which has been used elsewhere, e.g. [1, 11, 23]. To see how this works, we note that

$$u - P_{k+1} = -B(u - P_k, u - P_k) - 2B(P_k, u - P_k). \tag{1.4}$$

Each term on the right-hand side locally has an algebraic decay rate at $t = 0$. The product structure and the time integral in the bilinear operator $B(f, g)$ leads to an improved algebraic decay rate for the left-hand side compared to that for $u - P_k$. This improvement is only local. The far-field contributions to the flow are managed using a new a priori bound for weak $L^{3,\infty}$-solutions—see Corollary 2.3. The properties of weak $L^{3,\infty}$-solutions [1,6] are used critically throughout.

Organization: In Section 2, we establish several key lemmas, most importantly the extension of the decay property (1.3) to other space-time Lebesgue norms. We also establish some elementary properties of Picard iterates. Section 3 contains the proof of Theorem 1.3.

Remark 1.5. It is worth mentioning that, for the Euler equations, Vasseur and Yang have explored separation rates for the energy [43] and Drivas, Elgindi, and La have explored rates in Gevrey spaces [18].

2. Preliminaries

In this section we prove new a priori bounds for weak $L^{3,\infty}$-solutions. See Lemmas 2.1 and 2.2. We then establish a property of Picard iterates in Lemma 2.5.

Due to scaling considerations, one predicts that if the energy-level quantities on the left-hand side of (1.3) are replaced by lower Lebesgue or Lorentz norms, then the exponent on the right-hand side will increase to preserve the scaling of the inequality. With our application in mind, it is natural to ask whether the following dimensionless estimate holds:

$$\sup_{0<s<t} \|u - P_k\|_{L^{\frac{3}{2},1}}(s) \lesssim t^{\frac{1}{2}}.$$

This estimate is motivated by the $L^{3/2,1}$–$L^{3,\infty}$ duality pairing and, looking forward, would improve the estimate (3.4) and allow us to achieve the separation rate $t^{3/2}$ in our main theorem. There are barriers to establishing the above decay rate, but nearby rates are within reach as the next two lemmas show. In the lemmas, we replace $u - P_k$ with terms from (1.4). The proofs of the lemmas also illustrate the barriers to getting the above estimate in $L^{3/2,1}$. Once we move away from the exponent $3/2$, it is sufficient to consider Lebesgue norms instead of Lorentz norms.

Lemma 2.1. *Fix $q \in (3/2, 3)$, $T > 0$, and $k \in \mathbb{N}_0$. Assume $u_0 \in L^{3,\infty}$ and is divergence-free. Let u be a weak $L^{3,\infty}$-solution with initial data u_0. Then, letting $r = \frac{2q}{2q-3}$,*

$$\|B(u - P_k, u - P_k)\|_{L^r(0,T;L^{q,1})} \lesssim_{k,q,u_0} T^{\frac{1}{2}}.$$

The above estimate is dimensionless. By Lorentz space embeddings we trivially infer

$$\|B(u - P_k, u - P_k)\|_{L^r(0,T;L^{q,\beta})} \lesssim_{k,q,u_0} T^{\frac{1}{2}}$$

for every $\beta \in (1, \infty]$. This includes $L^r(0, T; L^q)$ when $\beta = q$. In our application, we will only use the L^q version of this. However, the proof for the full scale of Lorentz spaces is no harder and we therefore include it in case it is useful elsewhere.

Proof of Lemma 2.1. By Yamazaki [44, Theorem 2.2],

$$\begin{aligned}\|B(u - P_k, u - P_k)\|_{L^{q,1}} &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|(u - P_k)^2\|_{L^{q,1}}(s)\, ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u - P_k\|^2_{L^{2q,2}}(s)\, ds. \qquad (2.1)\end{aligned}$$

Recall the extension of the Gagliardo–Nirenberg inequality to the Lorentz scale [17, Corollary 2.2] which states

$$\|f\|_{L^{\tilde{p},\beta}} \lesssim_{\tilde{p},\tilde{q},\beta} \|f\|^{\theta}_{L^{\tilde{q},\infty}} \|\nabla f\|^{1-\theta}_{L^2}, \qquad (2.2)$$

for $\beta > 0$ and

$$\frac{1}{\tilde{p}} = \frac{\theta}{\tilde{q}} + (1-\theta)\Big(\frac{1}{2} - \frac{1}{3}\Big),$$

where $1 \le \tilde{q} < \tilde{p} < \infty$ and $3/2 - 3/\tilde{p} < 1$. Let $\tilde{p} = 2q$ and $\tilde{q} = 2$. These satisfy the above conditions because $q < 3$. Then θ is given by

$$\frac{3}{2q} - \frac{1}{2} = \theta$$

and, provided $1 < q < 3$, the other conditions above are met. To summarize,

$$\|f\|_{L^{2q,2}} \lesssim_q \|f\|_{L^{2,\infty}}^{\theta} \|\nabla f\|_{L^2}^{1-\theta} \lesssim_q \|f\|_{L^2}^{\theta} \|\nabla f\|_{L^2}^{1-\theta},$$

where we used the continuous embedding $L^2 \subset L^{2,\infty}$. Returning to our main estimate, this gives

$$\begin{aligned}\|B(u-P_k, u-P_k)\|_{L^{q,1}}(t) &\lesssim_q \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u-P_k\|_{L^2}^{2\theta} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds \\ &\lesssim_{k,q,u_0} t^{\frac{\theta}{2}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds,\end{aligned}$$

where we used the fact that (1.3) applies also to $u - P_k$ as a consequence of [1, Lemma 2.2],[5] in which case the suppressed constant accrues a dependence on k. Note that for $t \in (0, T)$,

$$\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds = \int_{\mathbb{R}} \frac{1}{|t-s|^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}(s) \chi_{(0,T)}(s)\, ds,$$

and the right-hand side can be viewed as $I_{\frac{1}{2}}(\|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)})$, where $I_{\frac{1}{2}}$ is a Riesz potential in one dimension. The Hardy–Littlewood–Sobolev inequality states that

$$\left\| I_{\frac{1}{2}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \right\|_{L^r(\mathbb{R})} \lesssim_r \left\| \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \right\|_{L^{\tilde{r}}(\mathbb{R})},$$

where

$$\frac{1}{r} = \frac{1}{\tilde{r}} - \frac{1}{2}.$$

The selection

$$\tilde{r} = \frac{1}{1-\theta}, \quad r = \frac{2}{1-2\theta},$$

is valid for the Hardy–Littlewood–Sobolev inequality provided $3/2 < q$.[6] Letting $r = \frac{2q}{2q-3}$ and putting the above observations together leads to

$$\begin{aligned}\|B(u-P_k, u-P_k)\|_{L^r(0,T;L^{q,1})} &\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} \left\| \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \right\|_{L^{\tilde{r}}(\mathbb{R})} \\ &\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} \|\nabla(u-P_k)\|_{L^2(0,T;L^2)}^{\frac{2}{\tilde{r}}} \\ &\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} T^{\frac{1}{2\tilde{r}}} = T^{\frac{1}{2}},\end{aligned} \tag{2.3}$$

where we used the extension of (1.3) to $u - P_k$ again. ∎

[5] We will use the fact several times and presently elaborate on how it follows from [1, Lemma 2.2]. The bounds [1, (2.36)–(2.39)] allow us to extend (1.3) to $u - P_k$ for $k > 0$. Note that $\|\nabla(P_{k+1} - P_k)\|_{L^2(0,T;L^2)} \lesssim T^{1/4}$ is not mentioned in [1, (2.39)] but, upon inspecting the proof, it also holds as a consequence of the energy estimate for the Stokes equation and the above-listed bounds.

[6] If $q = 3/2$, then $\theta = 1/2$ and $r = \infty$, which is not permitted in the Hardy–Littlewood–Sobolev inequality.

We prove a similar result for $B(P_k, u - P_k)$. This requires the well-known fact that if $u_0 \in L^{3,\infty}$, then P_k is in the scaling-invariant Kato classes for $q \in (3, \infty]$, i.e.

$$\|P_k\|_{\mathcal{K}_q} := \sup_{0<t<\infty} t^{\frac{1}{2}-\frac{3}{2q}} \|P_k\|_q(t) \lesssim_{k,u_0} 1. \tag{2.4}$$

To check this, note that the above property is immediate for P_0 by the embedding $L^{3,\infty} \subset \dot{B}^{-1+(3/p)}_{p,\infty}$ for $3 < p \le \infty$, and the fact that $\|u_0\|_{\dot{B}^{-1+(3/p)}_{p,\infty}} \sim \|P_0\|_{\mathcal{K}_p}$ [4]. Then, by the standard bilinear estimate (see the original papers [19, 31] or [42, Chapter 5]),

$$\|B(f, g)\|_{L^p}(t) \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}} \|f \otimes g + g \otimes f\|_{L^q}(s)\, ds, \tag{2.5}$$

we have

$$\begin{aligned}
\|B(P_k, P_k)\|_{L^\infty}(t) &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2q}}} \|P_{k-1}\|_{L^q}(s) \|P_{k-1}\|_{L^\infty}(s)\, ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2q}} s^{1-\frac{3}{2q}}} \|P_{k-1}\|_{\mathcal{K}_q} \|P_{k-1}\|_{\mathcal{K}_\infty}\, ds \\
&\lesssim t^{-\frac{1}{2}} \|P_{k-1}\|_{\mathcal{K}_q} \|P_{k-1}\|_{\mathcal{K}_\infty}
\end{aligned}$$

and

$$\begin{aligned}
\|B(P_k, P_k)\|_{L^q}(t) &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|P_{k-1}\|_{L^q}(s) \|P_{k-1}\|_{L^\infty}(s)\, ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{1-\frac{3}{2q}}} \|P_{k-1}\|_{\mathcal{K}_q} \|P_{k-1}\|_{\mathcal{K}_\infty}\, ds \\
&\lesssim t^{-\frac{1}{2}+\frac{3}{2q}} \|P_{k-1}\|_{\mathcal{K}_q} \|P_{k-1}\|_{\mathcal{K}_\infty}.
\end{aligned}$$

Claim (2.4) follows from the above observations by induction.

Lemma 2.2. *Fix $q \in (3/2, 3)$, $T > 0$, and $k \in \mathbb{N}_0$. Assume $u_0 \in L^{3,\infty}$ and is divergence-free. Let u be a weak $L^{3,\infty}$-solution with initial data u_0. Then, letting $r = \frac{2q}{2q-3}$,*

$$\|B(P_k, u - P_k)\|_{L^r(0,T;L^{q,1})} \lesssim_{k,q,u_0} T^{\frac{1}{2}}.$$

Proof. By Yamazaki [44, Theorem 2.2] and the extension of the Hölder inequality to Lorentz spaces,

$$\begin{aligned}
\|B(P_k, u - P_k)\|_{L^{q,1}} &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|P_k(u - P_k)\|_{L^{q,1}}(s)\, ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} (\|P_k\|^2_{L^{2q,\infty}} + \|u - P_k\|^2_{L^{2q,1}})\, ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} (\|P_k\|^2_{L^{2q}} + \|u - P_k\|^2_{L^{2q,1}})\, ds.
\end{aligned}$$

Noting that we may choose $\beta = 1$ in the extension of the Gagliardo–Nirenberg inequality (2.2) to the Lorentz scale, we have the desired result for the $u - P_k$ term by the work done between (2.1) and (2.3) in the proof of Lemma 2.1.

We then consider P_k in L^{2q}. By the membership of P_k in the Kato class,

$$\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|P_k\|_{L^{2q}}^2 \, ds \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{1-\frac{3}{2q}}} \|P_k\|_{\mathcal{K}_{2q}}^2 \, ds \lesssim_{k,q,u_0} t^{\frac{3}{2q}-\frac{1}{2}}.$$

Then, using $r = \frac{2q}{2q-3}$,

$$\|B(P_k, u - P_k)\|_{L^r(0,T;L^{q,1})} \lesssim_{k,q,u_0} \left(\int_0^T (t^{\frac{3}{2q}-\frac{1}{2}})^r \, dt \right)^{\frac{1}{r}} \lesssim_{k,q,u_0} T^{\frac{3}{2q}-\frac{1}{2}+\frac{1}{r}} \lesssim_{k,q,u_0} T^{\frac{1}{2}}. \qquad \blacksquare$$

Together, the above two lemmas lead to the following corollary.

Corollary 2.3. *Fix $q \in (3/2, 3)$, $T > 0$, and $k \in \mathbb{N}$. Assume $u_0 \in L^{3,\infty}$ and is divergence-free. Let u be a weak $L^{3,\infty}$-solution with initial data u_0. Then, letting $r = \frac{2q}{2q-3}$, we have*

$$\|u - P_k\|_{L^r(0,T;L^q)} \lesssim_{k,q,u_0} T^{\frac{1}{2}}.$$

Proof. This is immediate given Lemmas 2.1 and 2.2 and the fact that

$$u - P_k = -B(u - P_{k-1}, u - P_{k-1}) - 2B(P_k, u - P_{k-1})$$

for $k \geq 1$. $\blacksquare$

Our next lemma is a technical statement about the decay at $t = 0$ of the heat semigroup.

Lemma 2.4. *Let $B = B_R(x_0)$ and $B' := B_r(x_0)$, where $0 < r < R < \infty$. Then, for $0 < t < \infty$,*

$$\Big\| \|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B)\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^\infty(B')} \lesssim_{R,r} e^{\frac{-(R-r)^2}{4t}}.$$

Proof. First, assume without loss of generality that $x_0 = 0$. Then, letting $x \in B'$,

$$\|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B)\|_{L_y^{\frac{3}{2},1}} = \frac{3}{2} \int_0^\infty \mu\big\{y : e^{-\frac{|x-y|^2}{4t}}(1-\chi_B(y)) \geq s\big\}^{\frac{2}{3}} \, ds,$$

where μ is Lebesgue measure. Note that the above set can be written as

$$A(x,s) = \big\{y : |x-y| \leq \sqrt{-4t\ln(s)},\ |y| > R\big\} = B(x, (-4t\ln(s))^{\frac{1}{2}}) \setminus B_R(0),$$

which is well defined because $t \geq 0$ and $s \leq 1$. Then

$$\begin{aligned}\Big\| \|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^\infty(B')} &\lesssim \left\| \int_0^\infty \mu(A(x,s))^{\frac{2}{3}}\, ds \right\|_{L_x^\infty(B)} \\ &\lesssim \int_0^{e^{\frac{-(R-r)^2}{4t}}} |-4t\ln(s)|\, ds \\ &\lesssim 4t\Big(e^{\frac{-(R-r)^2}{4t}} \frac{(R-r)^2}{4t} + e^{\frac{-(R-r)^2}{4t}} \Big) \\ &\lesssim_{R,r} e^{\frac{-(R-r)^2}{4t}}. \end{aligned}$$

■

The above lemma leads to a local a priori inclusion for Picard iterates.

Lemma 2.5. *Let $B = B_R(x_0)$ and $B' = B_r(x_0)$, where $0 < r < R < \infty$. Let $u_0 \in L^{3,\infty}$ with $u_0|_B \in L^q(B)$, for some $3 < q \leq \infty$. For each $k_0 \in \mathbb{N}_0$, it follows that $P_{k_0} \in L^\infty(0,\infty; L^q(B'))$ and*

$$\|P_{k_0}\|_{L^\infty(0,\infty;L^q(B'))} \leq C(\|u_0\|_{L^q(B)}, \|u_0\|_{L^{3,\infty}}, q, R, r, k_0).$$

Proof. Note that for any $\tau > 0$, $\sup_{\tau<t<\infty} \|P_k\|_q \lesssim_{\tau,k} \|u_0\|_{L^{3,\infty}}$ due to the fact that $P_k \in \mathcal{K}_q$ when $q > 3$. We therefore only need to prove the inclusion for a short period of time. Let $\{B_k\}$ be a collection of concentric balls about x_0 of radii $\alpha^{k+1}R$, for some $\alpha \in (0,1)$. Fix $k_0 \in \mathbb{N}_0$. Choose α so that $r = \alpha^{k_0+1}R$.

For $P_0 = e^{t\Delta}u_0$ we have

$$\begin{aligned}\|P_0\|_{L^q(B_0)}(t) &= \left\| \int_{\mathbb{R}^3} \frac{1}{t^{\frac{3}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \right\|_{L^q(B_0)} \\ &= \left\| \Big(\int_{B^c} + \int_B \Big) t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \right\|_{L^q(B_0)} \\ &\lesssim \|u_0\|_{L^{3,\infty}} t^{-\frac{3}{2}} \Big\| \|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^q(B_0)} \\ &\quad + \|e^{t\Delta}(\chi_B(y)u_0)\|_{L^q(\mathbb{R}^3)}. \end{aligned} \tag{2.6}$$

For the far-field term, by Lemma 2.4,

$$\begin{aligned}\Big\| \|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^q(B_0)} &\lesssim_{R,\alpha,q} \Big\| \|e^{-\frac{|x-y|^2}{4t}}(1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^\infty(B_0)} \\ &\lesssim_{R,\alpha} e^{\frac{-(R(1-\alpha))^2}{4t}}. \end{aligned} \tag{2.7}$$

For the near-field term,

$$\|e^{t\Delta}(u_0\chi_B)\|_{L^q(\mathbb{R}^3)} \lesssim \|u_0\chi_B\|_{L^q(\mathbb{R}^3)} \lesssim \|u_0\|_{L^q(B)}.$$

Therefore,

$$\|P_0\|_{L^\infty(0,t;L^q(B'))} \lesssim_{R,\alpha} \|u_0\|_{L^{3,\infty}} + \|u_0\|_{L^q(B)}.$$

If $k_0 = 0$, then we are done. If $k_0 > 0$, then we use induction. Observe that

$$B(P_{k-1}, P_{k-1}) = B(P_{k-1}\chi_{B_{k-1}}, P_{k-1}) + B(P_{k-1}(1-\chi_{B_{k-1}}), P_{k-1}).$$

For the first part,

$$\begin{aligned}\|B(P_{k-1}\chi_{B_{k-1}}, P_{k-1})\|_{L^q(B_k)}(t) &\lesssim_q \|P_k\|_{\mathcal{K}_\infty} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \|P_{k-1}\|_{L^q(B_{k-1})}(s)\, ds \\ &\lesssim_{k,q} \|u_0\|_{L^{3,\infty}} \|P_{k-1}\|_{L^\infty(0,t;L^q(B_{k-1}))}(t).\end{aligned}$$

For the other part, by the pointwise estimate for the kernel K of the Oseen tensor (see [41, 42]),

$$|D_x^m K(x,t)| \lesssim_m \frac{1}{(|x| + \sqrt{t})^{3+|m|}}, \tag{2.8}$$

where m is a multi-index, we have

$$\begin{aligned}&\|B(P_{k-1}(1-\chi_{B_{k-1}}), P_{k-1})\|_{L^q(B_k)}(t) \\ &\quad\lesssim \left\| \int_0^t \int_{B_{k-1}^c} \frac{P_{k-1} \otimes P_{k-1}(y,s)}{(|x-y| + \sqrt{t-s})^4}\, dy\, ds \right\|_{L^q(B_k)} \\ &\quad\lesssim_q \|\,|\cdot|^{-4}\|_{L^2(|\cdot|>R(\alpha^k - \alpha^{k+1}))} \int_0^t \|P_{k-1}\|_{L^4}^2\, ds \\ &\quad\lesssim_{R,\alpha,k,q} \|u_0\|_{L^{3,\infty}}^2 \int_0^t s^{(-1+\frac{3}{4})}\, ds \lesssim_{R,\alpha,k,q} \|u_0\|_{L^{3,\infty}}^2 t^{\frac{3}{4}},\end{aligned}$$

where we used the membership of P_{k-1} in the Kato class $\mathcal{K}_4$.

We know by our base case that P_0 is in $L^\infty(0,\infty; L^q(B_0))$. We have just shown $B(P_{k-1}, P_{k-1}) \in L^\infty(0,\infty; L^q(B_0))$ whenever P_{k-1} is in $L^\infty(0,\infty; L^q(B_{k-1}))$. Hence,

$$P_k = P_0 - 2B(P_{k-1}, P_{k-1}) \in L^\infty(0,\infty; L^q(B_k)).$$

This extends up to k_0 and so $P_{k_0} \in L^\infty(0,\infty; L^q(B'))$. Note that by tracing the proof, it is clear that $\|P_k\|_{L^\infty(0,\infty;L^q(B'))} \le C(\|u_0\|_{L^q(B)}, \|u_0\|_{L^{3,\infty}}, q, R, r, k_0)$. ∎

Remark 2.6. Under the assumptions of Lemma 2.5 and by classical estimates for the heat semigroup,

$$\|e^{t\Delta}(u_0\chi_B)\|_{L^\infty(B')}(t) \lesssim t^{-\frac{3}{2q}} \|u_0\|_{L^q(B)}.$$

Note that, combining (2.6) and (2.7),

$$\|e^{t\Delta}(u_0(1-\chi_B))\|_{L^\infty(B')}(t) \lesssim_T \|u_0\|_{L^{3,\infty}} t^{-\frac{3}{2q}},$$

provided $t < T$, for any given time T. Hence,

$$\|e^{t\Delta} u_0\|_{L^\infty(B')}(t) \lesssim_{u_0,T} t^{-\frac{3}{2q}}.$$

3. Proof of Theorem 1.3

Our foundation for the proof of Theorem 1.3 is the local smoothing result of Jia and Šverák [27], which we presently restate. Note that L^2_{uloc} is the space of uniformly locally square integrable functions and is defined by the norm

$$\|f\|^2_{L^2_{\mathrm{uloc}}} := \sup_{x_0\in\mathbb{R}^3} \int_{B_1(x_0)} |f|^2\,dx.$$

Let E^2 denote the closure of C_c^∞ in the L^2_{uloc} norm. Note that $L^{3,\infty}$ embeds in E^2 (see the appendix of [9]). Local smoothing as presented below refers to local energy solutions (a.k.a. local Leray solutions using the terminology of [27]; see also [9, 32, 37]). It is straightforward to show that weak $L^{3,\infty}$-solutions are local energy solutions.

Theorem 3.1 (Local smoothing [27, Theorem 3.1]). *Let $u_0 \in E^2$ be divergence-free. Suppose $u_0|_{B_2(0)} \in L^p(B_2(0))$ with $\|u_0\|_{L^p(B_2(0))} < \infty$ and $p > 3$. Decompose $u_0 = U_0 + U_0'$ with* $\operatorname{div} U_0 = 0$, *$U_0|_{B_{4/3}} = u_0$,* $\operatorname{supp} U_0 \Subset B_2(0)$, *and $\|U_0\|_{L^p(\mathbb{R}^3)} < C(p, \|u_0\|_{L^p(B_2(0))})$. Let U be the locally-in-time-defined mild solution to* (1.1) *with initial data U_0. Then there exists a positive $T = T(p, \|u_0\|_{L^2_{\mathrm{uloc}}}, \|u_0\|_{L^p(B_2(0))})$ such that any local energy solution u with data u_0 satisfies*

$$\|u - U\|_{C^\gamma_{\mathrm{par}}(\bar B_{\frac12}\times[0,T])} \le C(p, \|u_0\|_{L^2_{\mathrm{uloc}}}, \|u_0\|_{L^p(B_2(0))}),$$

for some $\gamma = \gamma(p) \in (0,1)$.

See also [5, 29, 30, 35] for more recent work on local smoothing which allows locally critical data which is also locally small; the above statement on the other hand is for locally sub-critical data. The dependence on $\|u_0\|_{L^2_{\mathrm{uloc}}}$ can be replaced with $\|u_0\|_{L^{3,\infty}}$, which is why L^2_{uloc} is not mentioned in Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality, assume $B := B_2(x_0)$ is centered at $x_0 = 0$. Assume $u_0|_B \in L^p(B)$. Let U_0 be a localization of the data to B such that $u_0 = U_0$ in $B_{4/3}(0) \subset B$, $\operatorname{supp} U_0 \Subset B$. This is done via a Bogovskii map [22] as per the decomposition in Theorem 3.1. Let U be the locally-in-time-defined mild solution to (1.1) with data U_0. Define $\{B_k\}_{k=0}^{k_0}$ to be a collection of nested balls centered at 0 with radii $\alpha^k/2$ so that $1/4 = \alpha^{k_0}/2$, where k_0 will be specified later (this is a slight abuse of notation in that B_{k_0} is the ball centered at the origin of radius $1/4$, which would usually be denoted $B_{1/4}$). Then, recalling $P_0 = e^{t\Delta}u_0$,

$$\begin{aligned}|u - P_0|(x,t) &\le |u - U|(x,t) + |U - e^{t\Delta}U_0|(x,t) + |e^{t\Delta}(U_0 - u_0)|(x,t)\\ &=: I_1(x,t) + I_2(x,t) + I_3(x,t).\end{aligned}$$

In the definition of $C^\gamma_{\mathrm{par}}(\bar B_{\frac12} \times [0,T])$, the exponent in the time-variable modulus of continuity is $\gamma/2$. By local smoothing (Theorem 3.1) and the fact that $\|u_0\|_{L^2_{\mathrm{uloc}}} \lesssim \|u_0\|_{L^{3,\infty}}$, there exists $T = T(p, u_0) > 0$ so that

$$I_1(x,t) \lesssim_{p,u_0} t^{\frac{\gamma}{2}},$$

for some $\gamma = \gamma(p) \in (0,1)$, $x \in B_0$, and $0 < t < T$.

For I_2, by (2.5), for any $p \in (3,\infty]$ and $0 < t < T$,

$$\begin{aligned} I_2(x,t) &\le \|B(U,U)\|_{L^\infty(\mathbb{R}^3)}(t) \\ &\lesssim t^{\frac{1}{2}-\frac{3}{2p}} \|U\|^2_{L^\infty(0,T;L^p)} \lesssim t^{\frac{1}{2}-\frac{3}{2p}} \|U_0\|^2_{L^p}, \end{aligned}$$

where we possibly redefine T to make it smaller than the timescale of existence for the strong solution to (1.1), i.e. $T \lesssim \|U_0\|_{L^p}^{-2p/(p-3)}$, and the timescale coming from Theorem 3.1.

Noting that $U_0 - u_0 = 0$ in $B_{4/3}$, the last part, I_3, is broken into integrals over a shell and a far-field region as

$$I_3(x,t) \lesssim \left(\int_{\frac{4}{3}\le|y|<2} + \int_{|y|\ge 2} \right) t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} |U_0 - u_0|(y)\, dy =: I_{31}(x,t) + I_{32}(x,t).$$

For I_{31}, using the fact that U_0 was solved for via a Bogovskii map, and therefore $\|U_0\|_{L^p(\mathbb{R}^3)} \lesssim \|u_0\|_{L^p(B)}$, we have for all $0 < t < T$ and $x \in B_0$ that

$$I_{31}(x,t) \lesssim t^{-\frac{3}{2}} e^{-\frac{(\frac{4}{3}-\frac{1}{2})^2}{4t}} \|U_0 - u_0\|_{L^p(\frac{4}{3}\le|y|<2)}(t) \lesssim_{u_0,p} t^{\frac{\gamma}{2}}.$$

For I_{32}, by Lemma 2.4, the fact that $U_0(y) \equiv 0$ for $|y| \ge 2$, and taking $x \in B_0$ and $0 < t < T$, we have

$$\begin{aligned} I_{32}(x,t) &\lesssim \int_{|y|\ge 2} t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} |u_0|(y)\, dy \\ &\lesssim t^{-\frac{3}{2}} \|u_0\|_{L^{3,\infty}} \Big\| \|e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \Big\|_{L_x^\infty(B_0)} \\ &\lesssim_{u_0} t^{-\frac{3}{2}} e^{\frac{-(2-\frac{1}{2})^2}{4t}} \lesssim_{p,u_0} t^{\frac{\gamma}{2}}. \end{aligned}$$

Therefore,

$$\|u - P_0\|_{L^\infty(B_0)}(t) \lesssim_{p,u_0} t^{\min\{\frac{\gamma}{2},\frac{1}{2}-\frac{3}{2p}\}},$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^p(B)}$ and $\|u_0\|_{L^{3,\infty}}$.

We inductively extend this estimate to higher Picard iterates. Fix σ as in the statement of the theorem. Recursively define the sequence $\{a_k\}$ by

$$\begin{aligned} a_{k+1} &= \min\{\sigma, 1/2 - 3/(2p) + a_k\}, \\ a_0 &= \min\{\gamma/2, 1/2 - 3/(2p)\}. \end{aligned}$$

Assume for induction that

$$\|u - P_k\|_{L^\infty(B_k)} \lesssim_{k,\alpha,p,u_0} t^{a_k}$$

for $0 < t < T$, and the dependence on u_0 is via the same quantities as above. Note that

$$\begin{aligned} |u - P_{k+1}|(x,t) &\leq |B(u - P_k, u - P_k)| + 2|B(P_k, u - P_k)| \\ &=: J(x,t) + K(x,t). \end{aligned}$$

We split J further as

$$\begin{aligned} J(x,t) &\leq |B((u - P_k)\chi_{B_k}, u - P_k)| + |B((u - P_k)(1 - \chi_{B_k}), u - P_k)| \\ &=: J_1(x,t) + J_2(x,t). \end{aligned}$$

For the near field, J_1, we use the inductive hypothesis to obtain that, for $0 < t < T$,

$$\begin{aligned} \|J_1\|_{L^\infty(B_{k+1})}(t) &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u - P_k\|^2_{L^\infty(B_k)}\, ds \\ &\lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}+2a_k} \lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}-\frac{3}{2p}+a_k}. \end{aligned} \tag{3.1}$$

Considering J_2, for $0 < t < T$, we have by (2.8) that

$$\begin{aligned} \|J_2\|_{L^\infty(B_{k+1})}(t) &\lesssim \int_0^t \int_{|x-y|>\frac{1}{2}\alpha^k - \frac{1}{2}\alpha^{k+1}} \frac{1}{|x-y|^4} |u - P_k|^2(y,s)\, dy\, ds \\ &\lesssim \frac{t}{(\alpha^k - \alpha^{k+1})^4} \|u - P_k\|^2_{L^2}(t) \lesssim_{\alpha,k,u_0} t^{\frac{3}{2}}, \end{aligned} \tag{3.2}$$

where we used the version of (1.3) for higher Picard iterates from [1].

Attending now to K, we split and bound it as

$$\begin{aligned} K(x,t) &\leq 2|B(P_k, (u - P_k)\chi_{B_k})| + 2|B(P_k, (u - P_k)(1 - \chi_{B_k}))| \\ &=: K_1(x,t) + K_2(x,t). \end{aligned}$$

For the near-field K_1 and for $0 < t < T$ we have

$$\|K_1\|_{L^\infty(B_{k+1})}(t) \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2p}}} \|u - P_k\|_{L^\infty(B_k)}(s) \|P_k\|_{L^p(B_k)}(s)\, ds.$$

By Lemma 2.5, $\sup_{0<t<\infty} \|P_k\|_{L^p(B_k)} < \infty$. Note that $1/2 + 3/(2p) < 1$ precisely if $3 < p$. Hence,

$$\|K_1\|_{L^\infty(B_{k+1})}(t) \lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}-\frac{3}{2p}+a_k} \tag{3.3}$$

for $0 < t < T$, by the inductive hypothesis. For the far-field K_2, using Corollary 2.3 and taking $x \in B_{k+1}$, $0 < t < T$, and $q \in (3/2, 3)$, we have by (2.8) and O'Neil's inequality that

$$\begin{aligned} K_2(x,t) &\lesssim \int_0^t \int_{B_k^c} \frac{1}{(|x-y| + \sqrt{t-s})^4} |u - P_k|\,|P_k|\, dy\, ds \\ &\lesssim \left\| \frac{1 - \chi_{B_k}(\cdot)}{|x - \cdot|^4} \right\|_{L^{r'}(0,T;L^{q',q''})} \|P_k\|_{L^\infty(0,T;L^{3,\infty})} \|u - P_k\|_{L^r(0,T;L^q)} \\ &\lesssim_{k,q,u_0} t^{\frac{1}{r'}+\frac{1}{2}}, \end{aligned}$$

where

$$1=\frac{1}{q}+\frac{1}{q'}+\frac{1}{3},\quad 1=\frac{1}{q}+\frac{1}{q''},\quad \text{and}\quad 1=\frac{1}{r}+\frac{1}{r'}.$$

Because in Corollary 2.3 we take

$$r=\frac{2q}{2q-3},$$

we have

$$\frac{1}{r'}=\frac{3}{2q}.$$

Observe that $1/r'<1$ and

$$\lim_{q\to\frac{3}{2}^+}\frac{1}{r'}=1.$$

Therefore, for any $\sigma<3/2$, by taking $q>3/2$ sufficiently close to $3/2$,

$$K_2(x,t)\lesssim_{k,\sigma,u_0,q} t^{\sigma}. \tag{3.4}$$

Altogether, (3.1), (3.2), (3.3), and (3.4) imply that, for $0<t<T$,

$$\|u-P_{k+1}\|_{L^\infty(B_{k+1})}(t)\lesssim_{k,\alpha,p,u_0,q} t^{a_{k+1}},$$

for $k\geq 0$ and any $\sigma<3/2$, where

$$a_{k+1}=\min\Big\{\sigma,(k+1)\Big(\frac{1}{2}-\frac{3}{2p}\Big)+a_0\Big\}.$$

Choose k_0 to be the smallest natural number so that

$$k_0\Big(\frac{1}{2}-\frac{3}{2p}\Big)+a_0\geq\sigma.$$

Then $a_{k_0}=\sigma$ and $a_k<a_{k-1}$ for $k=1,\dots,k_0$. Because $B_{k_0}:=B_{1/4}(0)$, it follows that

$$\|u-P_{k_0}\|_{L^\infty(B_{1/4}(x_0))}(t)\lesssim_{p,\sigma,u_0} t^{\sigma}.$$

Regarding the asymptotic expansion, we observe that for $1\leq k\leq k_0$ and $(x,t)\in B_{1/4}(x_0)\times(0,T)$,

$$u=P_{k_0}+\mathcal{O}(t^{\sigma}),$$

and

$$|P_k-P_{k-1}|(x,t)\leq|u-P_k|(x,t)+|u-P_{k-1}|(x,t)=\mathcal{O}(t^{a_{k-1}}).$$

Hence,

$$\begin{aligned} u(x,t)&=\underbrace{P_0+\sum_{k=1}^{k_0}(P_k-P_{k-1})(x,t)}_{=P_{k_0}}+\mathcal{O}(t^{\sigma})\\ &=\mathcal{O}(t^{-\frac{3}{2p}})+\sum_{k=0}^{k_0-1}\mathcal{O}(t^{a_k})+\mathcal{O}(t^{\sigma})=\sum_{k=-1}^{k_0}\mathcal{O}(t^{a_k}),\end{aligned}$$

where we are letting $a_{-1} = -3/(2p)$ and are using Remark 2.6 to obtain the asymptotics for P_0. ■

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