Rev. Mat. Iberoamericana **23** (2007), no. 2, 421–436

Integration Operators on Bergman Spaces with exponential weight

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Dedicated to the memory of my mother Nadežda Dostanić

Abstract

We study operators of the form $T_g f(z) = \int_0^z f(\xi)\, g'(\xi)\, d(\xi)$ (g is an analytic function unity disc) on weighted Bergman spaces $L_a^p(w)$ of the unit disc where symbol g is analytic function on the disc. For the case of

$$w(r) = \exp\left(\frac{-a}{(1-r)^\beta}\right) \qquad (a > 0, 0 < \beta \leq 1)$$

it is shown that operator T_g is bounded (compact) on $L_a^2(w)$ if and only if $(1-|z|)^{\beta+1}|g'(z)| = O(1)\,(= o(1))$ as $|z| \to 1-$, thus solving a problem formulated in [2].

1. Introduction and notation

Let D be the unit disc in the complex plane and $dA(z)$ the Lebesgue area measure on D. Let $w(r)$ $(0 \leq r < 1)$ be a strictly positive weight function which is integrable on $(0,1)$. Let $d\mu(z)$ be a measure on D defined by

$$d\mu(z) = |w(z)|\, dA(z).$$

For $1 \leq p < \infty$ the weighted Bergman space $L_a^p(w)$ is the space of all analytic functions $f : D \to \mathbb{C}$ such that

$$\|f\|_p^p = \int_D |f(z)|^p\, d\mu(z) < \infty.$$

2000 Mathematics Subject Classification: 47B38.
Keywords: Weighted Bergman's space, radial weight function, Tauberian theorem of Ingham.

Standard estimates show that point evaluations are bounded linear functionals on $L_a^p(w)$, and $L_a^p(w)$ is a Banach space. It is a Hilbert space for $p = 2$. We are interested in certain operators T_g acting on $L_a^p(w)$. They are defined by

$$T_g f(z) = \int_o^z f(\xi)\, g\prime(\xi)\, d\xi, \quad f \in L_a^p(w)$$

where g is an analytic function on D.

See [1] regarding reasons for studying the properties of such an operator. There, for a large class of weights it was shown that T_g is bounded on $L_a^p(w)$ if and only if g is in the Bloch space. Also, for the standard weight

$$w(r) = (1-r)^\alpha, \qquad \alpha > -1,$$

it was shown that operator T_g belongs to the Schatten p-class on $L_a^p(w)$ $(p > 1)$ if and only if g belongs to the analytic Besov p-class.

However, for the weight

$$w(r) = \exp\left(\frac{-a}{(1-r)^\beta}\right) \quad a > 0, \beta > 0$$

it was shown that the conditions

$$(1-|z|)^{\beta+1}|g'(z)| = O(1)\,(= o(1)) \text{ as } |z| \to 1-0$$

are sufficient for operator T_g to be bounded (compact) on $L_a^p(w)$, and a conjecture was formulated that these conditions are necessary as well. (this type of weight does not satisfy condition (P2) in [2], used to obtain the main result, i.e. Theorem 1, hence, the method used in [2] cannot be applied to establish that the above conditions are necessary.)

The study of similar problems can be traced back to the works of Hardy and Littlewood about fractional integration operators acting from one Hardy (or similar) space to another. Even though the problem is apparently very specific, this is deceiving because by varying the function g one gets many different operators, some of them important, notably the integration operator and the Cesáro operator. Among the relevant references we also mention [1], [2], [3] and [8]

In this paper we show that the above mentioned conjecture is correct for $p = 2$ and $0 < \beta \leq 1$.

The key of our proof is finding a suitable test function and a precise estimate of its norm as well as an estimate of the norm of the partial derivative (with respect to z) of the Bergman kernel.

From now on in this paper, even when not explicitly stated it is assumed that

$$w(r) = \exp\left(\frac{-a}{(1-r)^{\beta}}\right) \quad ,0 \le r < 1, a > 0, 0 < \beta \le 1.$$

for functions $f, g : [D, +\infty) \longrightarrow (0, +\infty)$ (sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$) we use the symbol $f(t) \asymp g(t)$, $t \longrightarrow +\infty$ ($a_n \asymp b_n$, $n \longrightarrow \infty$) if there exist positive constants C_1, C_2 such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for $t \ge t_0$ ($C_1 a_n \le b_n \le C_2 a_n$ for $n \ge n_0$).

2. Main result

Theorem 1 *Let*

$$w(r) = \exp\left(\frac{-a}{(1-r)^{\beta}}\right) \quad ,a > 0,\ 0 < \beta \le 1.$$

Then

a) *T_g is bounded on $L_a^2(w)$ if and only if*

$$(1-|z|)^{\beta+1} |g'(z)| = O(1),\ |z| \to 1-$$

b) *T_g is compact on $L_a^2(w)$ if and only if*

$$(1-|z|)^{\beta+1} |g'(z)| = o(1),\ |z| \to 1-.$$

Remark 1 *From Theorem 1, if T_g is bounded (compact) operator on $L_a^2(w)$ then*

$$(1-|z|)^{\beta+1} |g'(z)| = O(1)\,(= o(1)) \qquad as \quad |z| \to 1-$$

and hence (according to the sufficient condition proven in [1, p. 353]*) it follows that T_g is bounded (compact) on $L_a^p(w)$ for any $p \ge 1$.*

It would be interesting to establish the converse : If T_g bounded (compact) on $L_a^p(w)$ for some $p \ge 1$ does it follow that it is bounded (compact) on $L_a^2(w)$? If that were to be true, then , according to Theorem 1, it would follow that

$$(1-|z|)^{\beta+1} |g'(z)| = O(1)\,(= o(1)) \qquad \text{as } |z| \to 1-$$

and so this condition would be necessary and sufficient for operator T_g to be bounded (compact) on any space $L_a^p(w)$ for any $p \ge 1$.

3. Proof

In order to prove Theorem 1, we first need to prove several Lemmas.

Lemma 1 *If* $0<\beta\le 1$ *and*

$$F(\lambda)=\int_0^1 (1-r)^{-s-1}\cdot r^{\lambda}\cdot \exp\left(\frac{-a}{(1-r)^{\beta}}\right)dr \quad (s\in\mathbb{R}, a>0)$$

then

$$F(\lambda)\asymp \lambda^{\frac{2s-\beta}{2\beta+2}}\exp\left(-c_0\lambda^{\frac{\beta}{\beta+1}}\right),\quad \lambda\to\infty$$

where

$$c_0=a^{\frac{1}{\beta+1}}\cdot\left(\beta^{\frac{1}{\beta+1}}+\beta^{-\frac{1}{\beta+1}}\right).$$

Proof. The proof is given for $0<\beta<1$. (It is similar for $\beta=1$)
Consider the function F and introduce the substitution $r=e^{-x}$. We obtain

$$F(\lambda)=G(\lambda+1)$$

where

$$G(\lambda)=\int_0^{\infty}\left(1-e^{-x}\right)^{-s-1}\cdot e^{-\lambda x}\cdot e^{-a\cdot x^{-\beta}\cdot\left(\frac{x}{1-e^{-x}}\right)^{\beta}}dx.$$

Let $G_0(\lambda)=\int_0^{\infty}(1-e^{-x})^{-s-1}\cdot e^{-\lambda x-ax^{-\beta}}dx$. Let us show that

$$G_0(\lambda)\asymp\lambda^{\frac{2s-\beta}{2\beta+2}}\cdot\exp\left(-c_0\lambda^{\frac{\beta}{\beta+1}}\right),\lambda\to+\infty \quad \text{and that} \quad \lim_{\lambda\to+\infty}\frac{G(\lambda)}{G_0(\lambda)}=1$$

from which Lemma 1 will follow.

Let us first show that

$$G_0(\lambda)\asymp\lambda^{\frac{2s-\beta}{2\beta+2}}\cdot\exp\left(-c_0\lambda^{\frac{\beta}{\beta+1}}\right),\lambda\to+\infty. \tag{3.1}$$

Introducing substitution $x=\left(\frac{a\beta}{\lambda}\right)^{\frac{1}{\beta+1}}\cdot t$ into integral

$$G_0(\lambda)=\int_0^{\infty}x^{-s-1}\exp\left(-\lambda x-ax^{-\beta}\right)dx$$

we get

$$G_0(\lambda)=\left(\frac{\lambda}{a\beta}\right)^{\frac{s}{\beta+1}}\int_0^{\infty}t^{-s-1}e^{S(t)\cdot\lambda^{\frac{\beta}{\beta+1}}}dt \tag{3.2}$$

where

$$S(t)=-(a\beta)^{\frac{1}{\beta+1}}\cdot t-a(a\beta)^{\frac{\beta}{\beta+1}}\cdot t^{-\beta}.$$

Let $A(\mu) = \int_0^\infty t^{-s-1} e^{\mu S(t)} dt$, $\mu = \lambda^{\frac{\beta}{\beta+1}}$. By applying the Laplace method ([5, p. 66]) to integral $\int_0^\infty t^{-s-1} e^{\mu S(t)} dt$ we get

$$A(\mu) = \text{const}\, \frac{e^{-\mu c_0}}{\sqrt{\mu}} (1 + o(1)), \quad \mu \to \infty \tag{3.3}$$

where const denotes a constant that does not depend on μ. From (3.2) and (3.3), (3.1) follows directly.

Let us now show that

$$\lim_{\lambda \to +\infty} \frac{G(\lambda)}{G_0(\lambda)} = 1. \tag{3.4}$$

Let

$$g(x) = \left(\frac{1 - e^{-x}}{x} \right)^{-s-1} \exp\left(-ax^{-\beta} \left(\frac{x}{1 - e^{-x}} \right)^{\beta} + ax^{-\beta} \right) - 1.$$

If $0 < \beta < 1$ then $\lim_{x \to 0^+} g(x) = 0$, and so, for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|g(x)| < \frac{\varepsilon}{2} \text{ for } x \in (0, \delta). \tag{3.5}$$

Since

$$\frac{G(\lambda)}{G_0(\lambda)} - 1 = \frac{1}{G_0(\lambda)} \int_0^\infty x^{-s-1} \cdot \exp\left(-\lambda x - ax^{-\beta}\right) \cdot g(x)\, dx$$

then, having in mind (3.5) we get

$$\begin{aligned} \left| \frac{G(\lambda)}{G_0(\lambda)} - 1 \right| &\le \frac{\varepsilon}{2} \cdot \frac{1}{G_0(\lambda)} \int_0^\infty x^{-s-1} \cdot \exp\left(-\lambda x - ax^{-\beta}\right) dx + \\ &\quad + \frac{1}{G_0(\lambda)} \int_0^\infty x^{-s-1} |g(x)| \cdot \exp\left(-\lambda x - ax^{-\beta}\right) dx. \end{aligned} \tag{3.6}$$

Since $|g(x)| \le c_1 x^{1+|s|}$ for $x \ge \delta$ (c_1 does not depend on x) from (3.6) we get

$$\begin{aligned} \left| \frac{G(\lambda)}{G_0(\lambda)} - 1 \right| &< \frac{\varepsilon}{2} + \frac{c_1}{G_0(\lambda)} \int_\delta^\infty x^{|s|-s} e^{-\lambda x - ax^{-\beta}}\, dx \\ &< \frac{\varepsilon}{2} + \frac{c_1}{G_0(\lambda)} \int_\delta^\infty x^{|s|-s} e^{\frac{-\lambda x}{2}} \cdot e^{\frac{-\lambda \delta}{2}} \cdot e^{-ax^{-\beta}} dx \\ &< \frac{\varepsilon}{2} + \frac{c_1}{G_0(\lambda)} e^{\frac{-\lambda\delta}{2}} \int_0^\infty x^{|s|-s} e^{\frac{-\lambda x}{2}} dx \\ &= \frac{\varepsilon}{2} + \frac{c_1 e^{\frac{-\lambda\delta}{2}}}{G_0(\lambda)} \cdot 2^{|s|-s+1} \frac{\Gamma(|s| - s + 1)}{\lambda^{|s|-s+1}} \end{aligned}$$

(Γ is the Euler Gamma-function).

Since

$$\lim_{\lambda\to+\infty}\frac{e^{-\frac{\lambda\delta}{2}}}{G_0(\lambda)}\cdot\frac{1}{\lambda^{|s|-s+1}}=0\,,$$

from the previous inequality it follows that

$$\left|\frac{G(\lambda)}{G_0(\lambda)}-1\right|<\varepsilon \text{ when } \lambda\geq\lambda_0.$$

This proves (3.4). Lemma 1 follows from(3.1) and (3.4). ∎

Lemma 2 *Let* $0<\beta<2,\ \ \alpha\in\mathbb{R}$ *and*

$$\varphi(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!\,\Gamma(k\beta+\alpha+1)}$$

then

$$\varphi(t)\asymp t^{-\frac{2\alpha+1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right),\qquad t\to+\infty$$

where

$$d_0=\beta^{\frac{1}{\beta+1}}+\beta^{-\frac{1}{\beta+1}}.$$

Proof. Let us first assume that $\alpha>0$. Let $A(t)=\int_0^t\varphi(s)\,ds$, $t>0$; then for any p, $Re\,p>0$ the following holds:

$$\int_0^{\infty}e^{-pt}\,dA(t)=\frac{1}{p}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{p}\right)^k}{\Gamma(k\beta+\alpha+1)}=\frac{1}{p}E_{\frac{1}{\beta}}\left(\frac{1}{p},\alpha+1\right)$$

where $E_\rho(z;\mu)=\sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu+\frac{k}{\rho}\right)}$ is the Mittag-Leffler function (see [4, p.117]). According to Lemma 3.4 from [4, p. 133], if $\frac{1}{\beta}>\frac{1}{2}$ i.e. $\beta<2$ the following holds:

$$\text{(3.7)}\quad \int_0^{\infty}e^{-pt}\,dA(t)=\frac{1}{\beta}\,p^{\frac{\alpha}{\beta}-1}\cdot\exp\left(\frac{1}{p^{\frac{1}{\beta}}}\right)-\sum_{k=1}^{r}\frac{p^{k-1}}{\Gamma(\alpha+1-k\beta)}+O\left(|p|^r\right)$$

as $p\to0$ remaining inside the angle $|\arg\,p|<\theta$ where $\frac{\pi\beta}{2}<\theta<\min\{\pi,\pi\beta\}$. Let $0<\theta_1<\theta$ such that

$$\frac{\theta_1}{\beta}<\frac{\pi}{2}.$$

Then, if $|\arg\,p|<\theta$, from (3.7) we get

$$\text{(3.8)}\qquad \int_0^{\infty}e^{-pt}\,dA(t)=\frac{1}{\beta}\,p^{\frac{\alpha}{\beta}-1}\cdot\exp\left(\frac{1}{p^{\frac{1}{\beta}}}\right)\cdot(1+o(1))$$

as $p\to0$ remaining inside the angle $|\arg\,p|<\theta_1$. Equality (3.8) holds uniformly over $\arg\,p$ inside the angle $|\arg\,p|<\theta_1$.

Now, from (3.8) by the Thauberian theorem of Ingham (see [9, p. 78-81] or [10]) we get

$$A\left(t\right)\asymp t^{\frac{2\beta=2\alpha-1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right)\quad t\rightarrow+\infty.$$

Therefore

$$\int_0^t\varphi\left(s\right)\,ds\asymp t^{\frac{2\beta=2\alpha-1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right)\quad t\rightarrow+\infty. \tag{3.9}$$

Since $\varphi'\left(t\right)=\sum\limits_{k=0}^{\infty}\frac{t^k}{k!\,\Gamma(k\beta+\alpha+\beta+1)}$, function φ' has the same form as function φ, except that α is replaced by $\alpha+\beta$ in the defining series. therefore from (3.9) it follows that

$$\int_0^t\varphi'\left(s\right)\,ds\asymp t^{\frac{2\beta=2(\alpha+\beta)-1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right)\quad t\rightarrow+\infty.$$

i.e.

$$\varphi\left(t\right)-\varphi\left(0\right)\asymp t^{-\frac{2\alpha+1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right)\quad t\rightarrow+\infty$$

and hence

$$\varphi\left(t\right)\asymp t^{-\frac{2\alpha+1}{2\beta+2}}\cdot\exp\left(d_0\,t^{\frac{1}{\beta+1}}\right)\quad t\rightarrow+\infty.$$

If $\alpha\leq 0$, the first finitely many (possible negative) terms of the series defining the function φ cannot change the asymptotic behavior of φ as $t\rightarrow+\infty$. ■

Let us now consider the function

$$z\longmapsto\left(1-z\right)^{-\alpha-1}\exp\left(\frac{-x}{\left(1-z\right)^{\beta}}\right)\quad\left(\alpha\in\mathbb{R},x\in\mathbb{R},0<\beta\leq1\right).$$

Here $\left(1-z\right)^{-\alpha-1}\big|_{z=0}=\left(1-z\right)^{\beta}\big|_{z=0}=1$. This function is analytic in D. Let its Taylor coefficients be $L_n^{(\alpha)}\left(x;\beta\right)$, *i.e.*

$$\left(1-z\right)^{-\alpha-1}\exp\left(\frac{-x}{\left(1-z\right)^{\beta}}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}\left(x;\beta\right)\;z^n \tag{3.10}$$

Lemma 3 *If $\alpha\in\mathbb{R},x>0$ and $0<\beta\leq1$ then*

$$L_n^{(\alpha)}\left(-x;\beta\right)\asymp n^{\frac{2\alpha-\beta}{2\beta+2}}\cdot\exp\left(d_0\,x^{\frac{1}{\beta+1}}\cdot n^{\frac{\beta}{\beta+1}}\right).$$

Proof. From

$$L_n^{(\alpha)}(x;\beta) = \frac{1}{2\pi i}\int_{|\xi|=r<1} \frac{(1-\xi)^{-\alpha-1}\exp\left(\frac{-x}{(1-\xi)^\beta}\right)}{\xi^{n+1}}d\xi,$$

by expanding the function $z \longmapsto (1-z)^{-\alpha-1}\exp\left(\frac{-x}{(1-z)^\beta}\right)$ into the series in $\frac{-x}{(1-z)^\beta}$ and integrating we get

$$L_n^{(\alpha)}(x;\beta) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}(-1)^n\binom{-k\beta-\alpha-1}{n}.$$

Keeping in mind that

$$(-1)^n\binom{-k\beta-\alpha-1}{n} = \frac{(k\beta+\alpha+1)(k\beta+\alpha+2)\cdots(k\beta+\alpha+n)}{n!}$$

we obtain

(3.11)
$$L_n^{(\alpha)}(-x;\beta) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{(k\beta+\alpha+1)(k\beta+\alpha+2)\cdots(k\beta+\alpha+n)}{n!};\ x>0.$$

Consider first $\alpha > 0$. Since for $s > 0$

$$\frac{(s+1)(s+2)\cdots(s+n)}{n!} \geq \frac{n^s}{\Gamma(s+1)},$$

by letting $s = k\beta+\alpha$ from (3.11) we obtain

$$L_n^{(\alpha)}(-x;\beta) \geq \sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{n^{k\beta+\alpha}}{\Gamma(k\beta+\alpha+1)} = n^\alpha\varphi\left(xn^\beta\right). \tag{3.12}$$

Since according to Lemma 2, $\varphi(t) \asymp t^{-\frac{2\alpha+1}{2\beta+2}}\cdot\exp\left(d_0\, t^{\frac{1}{\beta+1}}\right) \quad t\to+\infty$, from (3.12), for fixed $x > 0$, we get

$$L_n^{(\alpha)}(-x;\beta) \geq C_1\cdot n^{\frac{2\alpha-\beta}{2\beta+2}}\cdot\exp\left(d_0\, x^{\frac{1}{\beta+1}}\cdot n^{\frac{\beta}{\beta+1}}\right) \tag{3.13}$$

where $C_1 > 0$ and $n \geq n_1$ (C_1 does not depend on n).

Let us now prove the reverse inequality. A direct inspection shows that (by changing the order of integration and summation) we have

$$L_n^{(\alpha)}(-x;\beta) = \frac{1}{n!}\int_0^\infty t^{\alpha+n}e^{-t}\left(\sum_{k=0}^{\infty}\frac{\left(xt^\beta\right)^k}{k!\,\Gamma(k\beta+\alpha+1)}\right)dt.$$

From Lemma 2 it follows that there exists a constant A_0, which does not depend on s, such that

$$\sum_{k=0}^{\infty} \frac{s^k}{k!\,\Gamma(k\beta+\alpha+1)} \le A_0 \cdot s^{-\frac{2\alpha+1}{2\beta+2}} \exp\left(d_0\, s^{\frac{1}{\beta+1}}\right), \quad \forall s \ge 1$$

and so

$$\sum_{k=0}^{\infty} \frac{\left(xt^{\beta}\right)^k}{k!\,\Gamma(k\beta+\alpha+1)} \le A_0 \cdot x^{-\frac{2\alpha+1}{2\beta+2}} \cdot t^{-\frac{(2\alpha+1)\beta}{2\beta+2}} \exp\left(d_0\, x^{\frac{1}{\beta+1}} \cdot t^{\frac{\beta}{\beta+1}}\right) \tag{3.14}$$

if $xt^{\beta} \ge 1$, i.e. if $t \ge x^{-\frac{1}{\beta}}$.

Since

$$\frac{1}{n!}\int_0^{x^{-\frac{1}{\beta}}} t^{\alpha+n} e^{-t} \left(\sum_{k=0}^{\infty} \frac{\left(xt^{\beta}\right)^k}{k!\,\Gamma(k\beta+\alpha+1)}\right) dt = o(1), \quad n \to \infty$$

from (3.14) and the equality

$$L_n^{(\alpha)}(-x;\beta) = \frac{1}{n!}\left(\int_0^{x^{-\frac{1}{\beta}}} (\cdot)\, dt + \int_{x^{-\frac{1}{\beta}}}^{\infty} (\cdot)\, dt\right)$$

we obtain

$$L_n^{(\alpha)}(-x;\beta) \le o(1) + A_0\, x^{-\frac{2\alpha+1}{2\beta+2}} \cdot \frac{1}{n!}\int_0^{\infty} t^{\alpha+n-\frac{(2\alpha+1)\beta}{2\beta+2}} e^{-t+d_0 x^{\frac{1}{\beta+1}} \cdot t^{\frac{\beta}{\beta+1}}}\, dt.$$

To complete the proof of Lemma 3, it is therefore sufficient to demonstrate that for the sequence

$$c_n = \frac{1}{n!}\int_0^{\infty} t^{\alpha+n-\frac{(2\alpha+1)\beta}{2\beta+2}} \exp\left(-t + d_0\, x^{\frac{1}{\beta+1}} \cdot t^{\frac{\beta}{\beta+1}}\right) dt$$

the following holds:

$$c_n = O\left(n^{\frac{2\alpha-\beta}{2\beta+2}} \cdot \exp\left(d_0\, x^{\frac{1}{\beta+1}} \cdot n^{\frac{\beta}{\beta+1}}\right)\right), \quad n \to \infty.$$

Let

$$\begin{aligned} B &= \alpha - \frac{(2\alpha+1)\beta}{2\beta+2} \left(= \frac{2\alpha-\beta}{2\beta+2}\right) \\ C &= d_0\, x^{\frac{1}{\beta+1}} \text{ and } \delta = \frac{\beta}{\beta+1}. \end{aligned}$$

Hence we need to show that

$$c_n = O\left(n^B \exp\left(Cn^\delta\right)\right), \quad n \to \infty. \tag{3.15}$$

Introducing the substitution $t = ns$ to the integral $c_n = \frac{1}{n!}\int_0^\infty t^{n+B} e^{-t+C\delta} dt$, we obtain

$$c_n = \frac{n^{n+B+1}}{n!} e^{Cn^\delta} \cdot \int_0^\infty s^{n+B} e^{-ns+C\cdot n^\delta \cdot \left(s^\delta - 1\right)} ds. \tag{3.16}$$

Applying the generalized Laplace method (which concerns asymptotic behaviour of the integral of the form $\int f(x,\lambda) \exp S(x,\lambda)\, dx$, when $\lambda \to \infty$, see [5, p. 99-101]) to the integral

$$\int_0^\infty s^B \exp\left(n \ln s - ns + C\, n^\delta \left(s^\delta - 1\right)\right) ds$$

we get (if $\delta \le \frac{1}{2}$, which is true here because $\frac{\beta}{\beta+1}$ and $0 < \beta \le 1$):

$$\int_0^\infty s^{n+B} e^{-ns} \cdot e^{C\, n^\delta \left(s^\delta - 1\right)} ds = O\left(\frac{e^{-n}}{\sqrt{n}}\right), \quad n \to \infty$$

and so, from (3.16) it follows that

$$c_n = O\left(\frac{n^{n+B+1}}{n!} \frac{e^{C\, n^\delta - n}}{\sqrt{n}}\right).$$

From the above equality and the Stirling formula we obtain (3.15). This proves Lemma 3 for $\alpha > 0$.

If $\alpha \le 0$, only finitely many terms of the series

$$\sum_{k=0}^\infty \frac{x^k}{k!} \frac{(k\beta + \alpha + 1)(k\beta + \alpha + 2)\cdots(k\beta + \alpha + n)}{n!}$$

are negative and they increase polinomialy in n and cannot change the asymptotic behavior of $L_n^{(\alpha)}(-x;\beta)$. This proves Lemma 3. ■

Lemma 4 *Let* $f_\lambda(z) = \exp\left(\frac{A}{\left(1-\bar{\lambda}z\right)^\beta}\right)$, $f_\lambda(0) = e^A$ *where* $A = 2^\beta \cdot a$, $0 < \beta \le 1$, $\lambda \in D$. *Then*

$$\|f_\lambda\|_2 \le K_1 (1 - |\lambda|)^{1+\frac{\beta}{2}} \exp\left(\frac{A/2}{\left(1 - |\lambda|^2\right)^\beta}\right)$$

where K_1 *is constant which does not depend on* $\lambda \in D$.

Proof. From (3.10) it follows that

$$\exp\left(\frac{A}{\left(1-\overline{\lambda} re^{i\theta}\right)^{\beta}}\right)=\sum_{n\geq 0} L_n^{(-1)}\left(-A;\beta\right)\overline{\lambda}^n r^n e^{in\theta}$$

and so keeping in mind that

$$\|f_\lambda\|_2^2=\int_D |f_\lambda(z)|^2\, d\mu(z)=\int_0^1 r\, w(r)\, dr\int_0^{2\pi}\left|\exp\left(\frac{A}{\left(1-\overline{\lambda} re^{i\theta}\right)^{\beta}}\right)\right|^2 d\theta$$

by applying the Parseval equality, we get

$$\begin{aligned}(3.17)\qquad \|f_\lambda\|_2^2 &= 2\pi\int_0^1 r\, w(r)\sum_{n=0}^{\infty}\left|L_n^{(-1)}\left(-A;\beta\right)\right|^2|\lambda|^{2n} r^{2n}\, dr\\ &= 2\pi\sum_{n=0}^{\infty}\left|L_n^{(-1)}\left(-A;\beta\right)\right|^2|\lambda|^{2n}\int_0^1 r^{2n+1} w(r)\, dr.\end{aligned}$$

From Lemmas 1 and 3 and (3.17), we obtain

$$\|f_\lambda\|_2^2\leq \text{const}\sum_n|\lambda|^{2n}\cdot\left(n^{-\frac{\beta+2}{2\beta+2}}\exp\left(n^{\frac{\beta}{\beta+1}}\cdot A^{\frac{1}{\beta+1}}\cdot d_0\right)\right)^2$$
$$\cdot\, n^{-\frac{\beta+2}{2\beta+2}}\cdot\exp\left(-d_0\cdot n^{\frac{\beta}{\beta+1}}\cdot 2^{\frac{\beta}{\beta+1}}\cdot a^{\frac{1}{\beta+1}}\right)$$

that is, after simplification

$$(3.18)\qquad \|f_\lambda\|_2^2\leq\text{const}\sum_n|\lambda|^{2n}\cdot n^{-\frac{3\beta+6}{2\beta+2}}\exp\left(d_0\cdot n^{\frac{\beta}{\beta+1}}\cdot 2^{\frac{\beta}{\beta+1}}\cdot a^{\frac{1}{\beta+1}}\right)$$

(const does not depend on $\lambda\in D$). Since according to Lemma 3

$$n^{-\frac{3\beta+6}{2\beta+2}}\exp\left(d_0\, n^{\frac{\beta}{\beta+1}}\left(2^\beta a\right)^{\frac{1}{\beta+1}}\right)\asymp L_n^{(-3-\beta)}\left(-2^\beta a;\beta\right)$$

then from (3.18) it follows that

$$\|f_\lambda\|_2^2\leq\text{const}\sum_n|\lambda|^{2n} L_n^{(-3-\beta)}\left(-A;\beta\right)$$

and keeping in mind the equality (3.10) we have

$$\|f_\lambda\|_2^2\leq\text{const}\left(1-|\lambda|^2\right)^{\beta+2}\exp\left(\frac{A}{\left(1-|\lambda|^2\right)^{\beta}}\right)$$

(where const constant which does not depend on $\lambda\in D$). From this, it follows that

$$\|f_\lambda\|_2\leq K_1\left(1-|\lambda|\right)^{1+\frac{\beta}{2}}\exp\left(\frac{A/2}{\left(1-|\lambda|^2\right)^{\beta}}\right)$$

and K_1 does not depend on $\lambda\in D$. This proves Lemma 4. ■

Let $\delta_n = \left(\int_D |z^n|^2 \, d\mu\right)^{1/2} = \left(2\pi \int_0^1 r^{2n+1} w(r) \, dr\right)^{1/2}$. The system of functions $\{z^n/\delta_n\}_{n=o}^{\infty}$ is an orthonormal basis of space $L_a^2(w)$ and function

$$K(z,\xi) = \sum_{n=0}^{\infty} \frac{z^n \, \overline{\xi}^n}{\delta_n^2}$$

is the corresponding Bergman reproductive kernel (see [6]).

Lemma 5 *The following inequality holds*

$$\left(\int_D |K_z(z,\xi)|^2 \, d\mu(\xi)\right)^{\frac{1}{2}} \le K_2 (1-|z|)^{-\frac{4+3\beta}{2}} \exp\left(\frac{A/2}{\left(1-|z|^2\right)^{\beta}}\right)$$

where the constant K_2 does not depend on $z \in D$. Here $K_z(z,\xi) = \frac{\partial K(z,\xi)}{\partial z}$.

Proof. The proof is similar to proof of Lemma 4 (by applying the Parseval equality, Lemmas 1 and 3 and equality (3.10). ■

Lemma 6 *If T_g is a bounded operator on $L_a^2(w)$ then, for any $f \in L_a^2(w)$, the following inequalities hold*

a) $$(1-|z|)^{\beta+1} |g\prime(z)| \le D_1 \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|f(z)|} \cdot \exp\left(\frac{A/2}{\left(1-|z|^2\right)^{\beta}}\right) \cdot \|f\|_2 \cdot \|T_g\|$$

b) $$(1-|z|)^{\beta+1} |g\prime(z)| \le D_2 \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|f(z)|} \cdot \exp\left(\frac{A/2}{\left(1-|z|^2\right)^{\beta}}\right) \cdot \|T_g f\|_2$$

Here $A = 2^{\beta} a$. The constants D_1 and D_2 does not depend on $f \in L_a^2(w)$ nor on $z \in D$.

Proof. For a bounded operator T_g on $L_a^2(w)$ it is sufficient to prove the inequality **b)** since **a)** follows directly from **b)**.

As $K(z,\xi)$ the Bergman reproducing kernel we have

$$f(z) = \int_D K(z,\xi) f(\xi) \, d\mu(\xi), \quad f \in L_a^2(w)$$

and so, if $f \in L_a^2(w)$ and T_g is bounded operator on $L_a^2(w)$ we get

$$T_g f(z) = \int_D K(z,\xi) (T_g f)(\xi) \, d\mu(\xi).$$

Keeping in mind the way operator T_g is defined from the previous equality, by differentiating with respect to z, we get

$$f(z) g\prime(z) = \int_D K_z(z,\xi) (T_g f)(\xi) \, d\mu(\xi)$$

i.e.

$$(1-|z|)^{\beta+1} g\prime(z) = \frac{(1-|z|)^{1+\beta}}{f(z)} \int_D K_z(z,\xi)(T_g f)(\xi)\, d\mu(\xi).$$

Applying the Cauchy inequality to the integral on the right-hand side of the previous equality we get

$$(1-|z|)^{\beta+1} |g\prime(z)| \leq \frac{(1-|z|)^{1\beta+1}}{|f(z)|} \left(\int_D |K_z(z,\xi)|^2\, d\mu(\xi)\right) \cdot \|T_g f\|_2 .$$

The inequality, together with Lemma 5, proves Lemma 6 part **b)**. ∎

4. Proof of Theorem 1

It is enough to show that conditions listed in Theorem 1 are necessary. (It was demonstrated in [2] that they are sufficient.)

a) Let T_g be a bounded operator on $L_a^2(w)$.

Let us show that

$$(1-|z|)^{\beta+1} |g\prime(z)| = O(1) \text{ as } z \to 1-.$$

According to Lemma 6 part **a),** the following holds

$$(4.1) \qquad (1-|z|)^{\beta+1} |g\prime(z)| \leq D_1' \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|f(z)|} \cdot \exp\left(\frac{A/2}{\left(1-|z|^2\right)^{\beta}}\right) \cdot \|f\|_2$$

for any function $f \in L_a^2(w)$; constant D_1' does not depend on $f \in L_a^2(w)$ *nor on* $z \in D$. Replace f in (4.1) with $f_\lambda(z) = \exp\left(\frac{A}{(1-\bar{\lambda}z)^{\beta}}\right)$, $\lambda \in D$. Then

$$(1-|z|)^{\beta+1} |g\prime(z)| \leq D_1' \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|f_\lambda(z)|} \cdot \exp\left(\frac{A/2}{\left(1-|z|^2\right)^{\beta}}\right) \cdot \|f_\lambda\|_2$$

and so, letting $z = \lambda$, we get

$$(4.2) \quad (1-|\lambda|)^{\beta+1} |g\prime(\lambda)| \leq D_1' \frac{(1-|\lambda|)^{-1-\frac{\beta}{2}}}{|f_\lambda(\lambda)|} \cdot \exp\left(\frac{A/2}{\left(1-|\lambda|^2\right)^{\beta}}\right) \cdot \|f_\lambda\|_2$$

(D_1' does not depend on λ). According to Lemma 4, we have

$$(4.3) \qquad \|f_\lambda\|_2 \leq K_1 (1-|\lambda|)^{1+\frac{\beta}{2}} \exp\left(\frac{A/2}{\left(1-|\lambda|^2\right)^{\beta}}\right)$$

(K_1 does not depend on λ).

Since $f_\lambda(\lambda) = \exp\left(\frac{A}{(1-|\lambda|^2)^\beta}\right)$, from (4.2) and (4.3) it follows that

$$(1-|\lambda|)^{\beta+1} |g\prime(\lambda)| \le K_1 D_1' \text{ for } \lambda \in D$$

i.e.

$$(1-|\lambda|)^{\beta+1} |g\prime(\lambda)| = O(1), \quad |\lambda| \to 1- .$$

b) Let T_g be a compact operator on $L_a^2(w)$.

According to Lemma 6 part **b),** the following holds

$$(4.4) \quad (1-|z|)^{\beta+1} |g\prime(z)| \le D_2 \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|f(z)|} \cdot \exp\left(\frac{A/2}{\left(1-|z|^2\right)^\beta}\right) \cdot \|T_g f\|_2$$

for any $f \in L_a^2(w)$. Here constant D_2 does not depend on $f \in L_a^2(w)$ and $z \in D$. Let

$$\varphi_\lambda(z) = \exp\left(\frac{A}{\left(1-\overline{\lambda}z\right)^\beta}\right) \cdot (1-|\lambda|)^{-1-\frac{\beta}{2}} \cdot e^{-\frac{\frac{A}{2}}{(1-|\lambda|^2)^\beta}}.$$

From Lemma 4, it follows that

$$\|\varphi_\lambda\|_2 \le C_1$$

(C_1 does not depend on $\lambda \in D$). Replacing f in inequality (4.4) with φ_λ we get

$$(1-|z|)^{\beta+1} |g\prime(z)| \le D_2 \frac{(1-|z|)^{-1-\frac{\beta}{2}}}{|\varphi_\lambda(z)|} \cdot \exp\left(\frac{\frac{A}{2}}{\left(1-|z|^2\right)^\beta}\right) \cdot \|T_g \varphi_\lambda\|_2$$

(D_2 does not depend on λ and z). Letting $z = \lambda$ in the previous inequality and simplifying we get

$$(4.5) \qquad (1-|z|)^{\beta+1} |g\prime(z)| \le D_2 \|T_g \varphi_\lambda\|_2$$

(D_2 does not depend on $\lambda \in D$).

Let us show that φ_λ weakly converges to zero as $|\lambda| \to 1-$. Since $\|\varphi_\lambda\|_2 \le C_1$ it is sufficient to show that for every $n = 0, 1, 2, \dots$

$$(4.6) \qquad \langle \varphi_\lambda, z^n \rangle_{L_a^2(w)} \to 0 \text{ as } |\lambda| \to 1- .$$

(Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in the Hilbert space $L_a^2(w)$.)

Since

$$\langle \varphi_\lambda, \lambda^n \rangle_{L_a^2(w)} = \frac{(1-|\lambda|)^{-1-\frac{\beta}{2}}}{|\varphi_\lambda(\lambda)|} \cdot \exp\left(\frac{-A/2}{\left(1-|\lambda|^2\right)^\beta}\right) \cdot \overline{\lambda}^n \cdot L_n^{(-1)}(-A;\beta) \cdot \delta_n^2$$

it follows that (4.6) is true.

Since φ_λ weakly converges to zero as $|\lambda| \to 1-0$ and T_g is a compact operator, it follows that

$$\|T_g \varphi_\lambda\|_2 \to 0 \text{ as } |\lambda| \to 1-$$

and so from (4.5) we get

$$(1-|\lambda|)^{\beta+1} |g\prime(\lambda)| = o(1), \quad |\lambda| \to 1- .$$

This proves Theorem 1. ■

Remark 2 *Let* $w(r) = e^{-\frac{1}{(1-r)^\beta}}$. *If* T_g *is bounded on* $L_a^2(w)$ *then*

$$f(z) g'(z) = \int_D K_z(z,\xi) T_g f(\xi) d\mu(\xi)$$

so

$$|f(z) g'(z)|^2 \le \|T_g\|^2 \int_D |K_z(z,\xi)|^2 d\mu(\xi) \int_D |f|^2 d\mu(z)$$

whence by integration with respect to $dA(z)$, *we get*

$$\int_D |f(z)|^2 \frac{|g'(z)|^2}{\int_D |K_z(z,\xi)|^2 d\mu(\xi)} dA(z) \le \pi \|T_g\|^2 \int_D |f|^2 d\mu(z). \tag{4.7}$$

Let

$$d\nu(z) = \frac{|g'(z)|^2}{\int_D |K_z(z,\xi)|^2 d\mu(\xi)} dA(z).$$

Then from (4.7) it follows that

$$\int_D |f(z)|^2 d\nu(z) \le \pi \|T_g\|^2 \int_D |f|^2 d\mu(z)$$

for every $f \in L_a^2(w)$. *If* $\beta > 1$, *then by Oleinik's theorem [7] there holds*

$$\sup_{z\in D} (1-|z|)^{-2-\beta} \cdot \int_{|\xi-z|\le(1-|z|)^{1+\frac{\beta}{2}}} e^{\frac{1}{(1-|z|)^\beta}} d\nu(z) < +\infty.$$

In order that the preceding condition (via application of the sub-mean-value property) can be perhaps reduced to the form

$$\sup_{z\in D} (1-|z|)^{\beta+1} |g'(z)| < +\infty.$$

it is necessary to give a precise estimate from above of the function

$$\int_D |K_z(z,\xi)|^2 d\mu(\xi).$$

If $0 < \beta \le 1$ *we have done this by using Lemmas 3 and 4. However, if* $\beta > 1$, *this is connected with necessity of more precise (and mush more difficult) estimates of the function* F *and the sequence* $L_n^{(\alpha)}(-x,\beta)$.

Question. *Is the condition* $(1-|z|)^{\beta+1}|g'(z)| = o(1)$, $|z| \to 1-$, *necessary for the compactness of the operator* T_g *in the case* $\beta > 1$?

Acknowledgement. The author is grateful to the referee for many useful remarks.

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Recibido: 15 de abril de 2004
Revisado: 9 de diciembre de 2005

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Partially supported by MNZZS Grant, No. ON144010.