Random finite noncommutative geometries and topological recursion

Shahab Azarfar and Masoud Khalkhali

Abstract. In this paper, we investigate a model for quantum gravity on finite noncommutative spaces using the theory of blobbed topological recursion. The model is based on a particular class of random finite real spectral triples $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, called random matrix geometries of type $(1,0)$, with a fixed fermion space $(\mathcal{A}, \mathcal{H}, \gamma, J)$ and a distribution of the form $e^{-\mathcal{S}(D)}\, \mathrm{d}D$ over the moduli space of Dirac operators. The action functional $\mathcal{S}(D)$ is considered to be a sum of terms of the form $\prod_{i=1}^{s} \mathrm{Tr}(D^{n_i})$ for arbitrary $s \geq 1$. The Schwinger–Dyson equations satisfied by the connected correlators W_n of the corresponding multi-trace formal 1-Hermitian matrix model are derived by a differential geometric approach. It is shown that the coefficients $W_{g,n}$ of the large N expansion of W_n's enumerate discrete surfaces, called stuffed maps, whose building blocks are of particular topologies. The spectral curve $(\Sigma, \omega_{0,1}, \omega_{0,2})$ of the model is investigated in detail. In particular, we derive an explicit expression for the fundamental symmetric bidifferential $\omega_{0,2}$ in terms of the formal parameters of the model.

1. Introduction and basics

In metric noncommutative geometry, the formalism of *spectral triples* [15] encodes, in the commutative case, the data of a Riemannian metric on a spin manifold in terms of the Dirac operator. More precisely, by *Connes's reconstruction theorem* [18], one knows that a spin Riemannian manifold can be fully constructed if we are given a commutative spectral triple satisfying some natural conditions like reality and regularity. A simple manifestation of this fact is a distance formula [14] according to which one can recover the Riemannian metric from the interaction between the Dirac operator and the algebra of smooth functions on a spin manifold through their actions on the Hilbert space of L^2-spinors. This naturally leads to the view that spectral triples in general, without commutativity assumption, can be regarded as noncommutative spin Riemannian manifolds.

The data of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ consists of a *-algebra $\mathcal{A}$ together with a *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, a self-adjoint Dirac

operator D, a $\mathbb{Z}/2$-grading γ, and an anti-linear isometry J acting on $\mathcal{H}$. The above-mentioned operators should satisfy certain (anti-)commutation relations and technical functional analytic conditions (see [12, 15, 19] for the detailed axiomatic definition of a real spectral triple).

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *finite* if the Hilbert space $\mathcal{H}$ is finite dimensional, i.e., $\mathcal{H} \cong \mathbb{C}^n$. The data $(\mathcal{A}, \mathcal{H}, \gamma, J)$ corresponding to a finite real spectral triple is referred to as a *fermion space*. Given a fixed fermion space $(\mathcal{A}, \mathcal{H}, \gamma, J)$, the *moduli space* of Dirac operators of the finite real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ consists of all possible self-adjoint operators D (up to unitary equivalence) which satisfy the axiomatic definition of a real spectral triple [27]. It is considered as the space of all possible geometries, that is, Riemannian metrics, over the noncommutative space $(\mathcal{A}, \mathcal{H}, \gamma, J)$.

The theory of spectral triples has been used in constructing geometric models of matter coupled to gravity, using an action functional, called the *spectral action*, which is given in terms of the spectrum of the Dirac operator (see [11, 12, 16, 17]; see also [24] for a recent work in the spirit of matrix models).

This paper is about a second application of the idea of spectral triples by creating a connection with the recently emerged theory of *topological recursion* [21]. Roughly speaking, if we understand quantization of gravity as a path integral over the space of metrics, it is natural to consider models of Euclidean quantum gravity over a finite noncommutative space in which one integrates over the moduli space of Dirac operators for a fixed fermion space. Given a fermion space $(\mathcal{A}, \mathcal{H}, \gamma, J)$, the distribution over the moduli space of Dirac operators is considered to be of the form

$$e^{-\mathcal{S}(D)} \, \mathrm{d}D,$$

where the action functional $\mathcal{S}(D)$ is defined in terms of the spectrum of the Dirac operator D.

The investigation of the relation between models of quantum gravity on a certain class of finite noncommutative spaces and (anti-)Hermitian matrix ensembles started in the work of Barrett and Glaser ([3], cf. also [2]), although largely through numerical simulations. In this paper, we consider a much larger class of models and show that an analytic approach to analyzing these models is possible, using techniques of *topological recursion* and *blobbed topological recursion* pioneered by Eynard, Orantin [21, 22], Chekhov [13], and Borot [4, 6, 9].

In the following, we recall the definition of a particular type of finite real spectral triples whose Dirac operators are classified in terms of (anti-)Hermitian matrices in [2]. Denote the real Clifford algebra associated to the vector space $\mathbb{R}^n$ and the pseudo-Euclidean metric η of signature (p, q), given by

$$\eta(v, v) = v_1{}^2 + \cdots + v_p{}^2 - v_{p+1}{}^2 - \cdots - v_{p+q}{}^2, \quad v \in \mathbb{R}^n,$$

by $C\ell_{p,q}$.[1] Consider the complexification $\mathbb{C}\ell_n := C\ell_{p,q} \otimes_\mathbb{R} \mathbb{C}$ of $C\ell_{p,q}$. Let $\{e_i\}_{i=1}^n$ be the standard oriented orthonormal basis, i.e., $\eta(e_i, e_j) = \pm\delta_{ij}$, for $\mathbb{R}^n$. The *chirality operator* Γ is given by

$$\Gamma = i^{\frac{1}{2}s(s+1)} e_1 e_2 \cdots e_n,$$

where $s \equiv q - p \pmod 8$, $0 \leqslant s < 8$. We denote by $V_{p,q}$ the unique (up to unitary equivalence) irreducible complex $C\ell_{p,q}$-module, where, for $n = p + q$ odd, the chirality operator Γ acts trivially on $V_{p,q}$.[23] We refer to $\gamma^i = \rho(e_i)$, $i = 1, \ldots, n$, i.e., the Clifford multiplication by e_i's, as the *gamma matrices*. There exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $V_{p,q}$ such that the gamma matrices act unitarily with respect to it, i.e., $\langle \gamma^i u, \gamma^i v \rangle = \langle u, v \rangle$, $i = 1, \ldots, n$ [26].[4]

Consider the Hilbert space $(V_{p,q}, \langle \cdot, \cdot \rangle)$. Let $C : V_{p,q} \to V_{p,q}$ be a *real structure* of KO-dimension $s \equiv q - p \pmod 8$ (see, e.g., [27]) on $V_{p,q}$ such that

$$(\mathbb{C}\ell_n, V_{p,q}, \Gamma, C)$$

satisfies all the axioms of a fermion space.[5]

Definition 1.1. A *matrix geometry of type* (p, q) is a finite dimensional real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, where the corresponding fermion space is given by

- $\mathcal{A} = M_N(\mathbb{C})$,
- $\mathcal{H} = V_{p,q} \otimes M_N(\mathbb{C})$,
- $\langle v \otimes A, u \otimes B \rangle = \langle v, u \rangle \operatorname{Tr}(AB^*)$, $v, u \in V_{p,q}$, $A, B \in M_N(\mathbb{C})$,
- $\pi(A)(v \otimes B) = v \otimes (AB)$,
- $\gamma(v \otimes A) = (\Gamma v) \otimes A$,
- $J(v \otimes A) = (Cv) \otimes A^*$.

---

[1]Given a quadratic form q on a vector space V, we follow the convention to define the Clifford algebra $C\ell(V, q)$ associated to V and q as $C\ell(V, q) := \mathcal{T}(V)/\mathcal{I}_q(V)$, where $\mathcal{T}(V) = \sum_{r=0}^\infty V^{\otimes r}$ and $\mathcal{I}_q(V)$ denotes the two-sided ideal generated by elements of the form $v \otimes v - q(v)\mathbb{1}$ for $v \in V$.

[2]Let $\rho : C\ell_{p,q} \to \operatorname{Hom}_\mathbb{C}(V, V)$ be an irreducible complex unitary representation of $C\ell_{p,q}$. The following can be shown [26] that.

- If $n = p + q$ is even, then the representation ρ is unique up to unitary equivalence.
- If $n = p + q$ is odd, then either $\rho(\Gamma) = \mathbb{1}_V$ or $\rho(\Gamma) = -\mathbb{1}_V$. Both possibilities can occur, and the corresponding representations are inequivalent.

[3]We have $V_{p,q} \cong \mathbb{C}^k$, where $k = 2^{n/2}$ (resp., $k = 2^{(n-1)/2}$) for n even (resp., odd).

[4]For $i = 1, \ldots, p$ (resp., $i = p + 1, \ldots, n$), the gamma matrix γ^i is Hermitian (resp., anti-Hermitian) with respect to $\langle \cdot, \cdot \rangle$.

[5]In the physics literature, the operator C is called the *charge conjugation* operator.

The Dirac operators of type (p, q) matrix geometries are expressed in terms of gamma matrices γ^i and commutators or anti-commutators with given Hermitian matrices H and anti-Hermitian matrices L (see [2,3]). For a recent survey of interactions between fuzzy spectral triples and random matrix theory initiated in this paper, we recommend [25].

2. Random matrix geometries of type $(1, 0)$

In this section, we describe a model for Euclidean quantum gravity on finite noncommutative spaces corresponding to the random matrix geometries of type $(1, 0)$. The Dirac operator of type $(1, 0)$ matrix geometries is given by [3]

$$D = \{H, \cdot\}, \quad H \in \mathcal{H}_N,$$

where $\mathcal{H}_N$ denotes the space of $N \times N$ Hermitian matrices. The Dirac operator D acts on the Hilbert space $\mathcal{H} = \mathrm{M}_N(\mathbb{C})$ in the following way:

$$D(B) = \{H, B\} = HB + BH \quad \forall B \in \mathrm{M}_N(\mathbb{C}).$$

The moduli space of Dirac operators is isomorphic to the space of Hermitian matrices $\mathcal{H}_N$. A distribution of the form

$$d\rho = e^{-\mathcal{S}(D)} \, dD$$

is considered over $\mathcal{H}_N$, where

$$dD := dH = \prod_{i=1}^{N} dH_{ii} \prod_{1 \leqslant i < j \leqslant N} d(\mathrm{Re}(H_{ij})) \, d(\mathrm{Im}(H_{ij}))$$

is the canonical Lebesgue measure on $\mathcal{H}_N$. Let us describe the action functional $\mathcal{S}(D)$. Let

$$\mathbb{N}_{\uparrow}^{n} = \{\vec{l} \in \mathbb{Z}^n \mid 1 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_n\}.$$

Suggested by Connes' spectral action, we define the action functional $\mathcal{S}(D)$ of the model by

$$\mathcal{S}(D) = \mathcal{S}_{\mathrm{unstable}}(D) + \mathcal{S}_{\mathrm{stable}}(D), \tag{2.1}$$

where

$$\mathcal{S}_{\mathrm{unstable}}(D) = \mathrm{Tr}(\mathcal{V}(D)), \quad \mathcal{V}(x) = \frac{1}{2t}\left(\frac{x^2}{2} - \sum_{n=3}^{d} \alpha_n \frac{x^n}{n}\right) \tag{2.2}$$

and

$$S_{\text{stable}}(D) = -\sum_{s=1}^{\mathfrak{g}} (N/t)^{-4s} \sum_{n_I \in \mathbb{N}_\uparrow^s} \hat{\alpha}_{n_I} \prod_{i=1}^{s} \text{Tr}(D^{n_i}).$$

In the definition of the action functional, t is a fixed parameter ("temperature"), the $(\alpha_n, \hat{\alpha}_{n_I})_{n,n_I}$ are formal parameters, and, for each s, the summation over $n_I \in \mathbb{N}_\uparrow^s$ is a finite sum.[6] Let

$$\hbar := \frac{t}{N}.$$

For large N, the term $S_{\text{stable}}(D)$ can be considered as higher-order terms in $\hbar$-expansion of the action functional $S(D)$.

Using

$$D^n = (H \otimes \mathbb{1}_{(\mathbb{C}^N)^*} + \mathbb{1}_{\mathbb{C}^N} \otimes H^t)^n = \sum_{k=0}^{n} \binom{n}{k} H^{n-k} \otimes (H^k)^t,$$

we get

$$S_{\text{unstable}}(D) = \frac{N}{t}\left(\frac{1}{2}\text{Tr}(H^2) - \sum_{n=3}^{d} \frac{\alpha_n}{n}\text{Tr}(H^n) \right)$$
$$- \frac{1}{2}\left(-\frac{1}{t}(\text{Tr}(H))^2 + \sum_{n=3}^{d} \frac{\alpha_n}{nt} \sum_{r=1}^{n-1} \binom{n}{r} \text{Tr}\,(H^{n-r})\,\text{Tr}\,(H^r) \right). \quad (2.3)$$

In addition, we have

$$\prod_{i=1}^{s} \text{Tr}(D^{n_i}) =$$
$$\sum_{m=0}^{s} (2N)^{s-m} \sum_{\substack{J \subseteq I \\ |J|=m}} \prod_{i \in I\setminus J} \text{Tr}\,(H^{n_i})\left(\sum_{\substack{(r_{j_1},\ldots,r_{j_m}) \\ 1 \leqslant r_j \leqslant n_j-1,\ j\in J}} \prod_{j \in J} \binom{n_j}{r_j} \text{Tr}(H^{n_j-r_j})(H^{r_j}) \right),$$
$$(2.4)$$

where $I = \{1,\ldots,s\}$. By substituting (2.3) and (2.4) into (2.1), we get the expression for the action functional $S(D)$ in terms of the spectrum of the Hermitian matrix

$$H \in \mathcal{H}_N.$$

---

[6]The formal parameters $(\alpha_n, \hat{\alpha}_{n_I})_{n,n_I}$ play the role of coupling constants in physics litera-ture.

3. Topological expansion of the action functional

In the following, we rewrite the action functional $S(D)$, in succinct form, as a summation over a finite set of (properly defined) equivalence classes of surfaces.

We start by recalling the notion of a surface with polygonal boundary. A compact, connected, oriented surface C is said to have n polygonal boundary components of perimeters $\{\ell_i\}_{i=1}^n$, $\ell_i \geqslant 1$, if, for each $1 \leqslant i \leqslant n$, the i-th connected component of ∂C is equipped with a cellular decomposition into ℓ_i 0-cells and ℓ_i 1-cells. We refer to the 1-cells in each connected component of ∂C as the *sides* of that polygon. We define an equivalence relation between surfaces with polygonal boundaries in the following way.

Definition 3.1. Two compact, connected, oriented surfaces C_1, C_2 with polygonal boundaries are considered equivalent if there exists an orientation-preserving diffeomorphism $F : C_1 \to C_2$ which restricts to a *cellular* homeomorphism $f : \partial C_1 \to \partial C_2$ whose inverse is also a cellular map.

The set $\widehat{\mathcal{C}}$ of equivalence classes of compact, connected, oriented surfaces with polygonal boundaries is in bijective correspondence with the set

$$\{(g; \vec{\ell}) \mid g \geqslant 0, \ \vec{\ell} \in \mathbb{N}_\uparrow^n, \ n \geqslant 1\},$$

where g denotes the genus of the corresponding closed surface. Inspired by [4], we refer to the combinatorial data $(g; \vec{\ell})$, and its corresponding equivalence class $[C] \in \widehat{\mathcal{C}}$ of surfaces with polygonal boundaries, as the *elementary 2-cell* of type $(g; \vec{\ell})$.

We isolate the *free* part of the action functional and denote it by

$$S_0(H) = \frac{N}{2t} \operatorname{Tr}(H^2).$$

Let

$$P_\ell(H) := \frac{\operatorname{Tr}(H^\ell)}{\ell} \quad \forall \ell \in \mathbb{N}, \ H \in \mathcal{H}_N. \tag{3.1}$$

Consider the following two sets:

$$\mathfrak{L}_{\text{disk}} = \{\ell \in \mathbb{N} \mid 3 \leqslant \ell \leqslant d\},$$
$$\mathfrak{L}_{\text{cylinder}} = \{(\ell_1, \ell_2) \in \mathbb{N}_\uparrow^2 \mid 2 \leqslant \ell_1 + \ell_2 \leqslant d\}.$$

We rewrite $S_{\text{unstable}}(D)$ in the following form:

$$S_{\text{unstable}}(D) = S_0(H) - \frac{N}{t} \sum_{\ell \in \mathfrak{L}_{\text{disk}}} t_\ell^{(0)} P_\ell(H)$$

$$- \frac{1}{2} \sum_{(\ell_1, \ell_2) \in \mathfrak{L}_{\text{cylinder}}} t_{\ell_1, \ell_2}^{(0)} P_{\ell_1}(H) P_{\ell_2}(H), \tag{3.2}$$

where $t_\ell^{(0)} = \alpha_\ell$, $3 \leqslant \ell \leqslant d$; $t_{1,1}^{(0)} = -1/t$, and for $(\ell_1, \ell_2) \in \mathfrak{L}_{\text{cylinder}} \backslash \{(1, 1)\}$, $t_{\ell_1, \ell_2}^{(0)}$ is an integral multiple of $\alpha_{(\ell_1 + \ell_2)}/t$.

For each $1 \leqslant s \leqslant \mathfrak{g}$, and $0 \leqslant m \leqslant s$, let

$$\Upsilon_{s,m} = \bigcup_{\substack{n_I, |I|=s \\ J \subseteq I, |J|=m \\ 1 \leqslant r_j \leqslant n_j - 1, j \in J}} \{(r_J, n_J - r_J, n_{I \backslash J})\}/\sim,$$

where two $(s + m)$-tuples are considered equivalent if there exists a permutation $\sigma \in \mathfrak{S}_{s+m}$ which maps one to the other. Consider the set

$$\mathfrak{L}_{s,m} = \{\vec{\ell} \in \mathfrak{N}_\uparrow^{s+m} \mid [\sigma \cdot \vec{\ell}] \in \Upsilon_{s,m} \text{ for some } \sigma \in \mathfrak{S}_{s+m}\}.$$

We rewrite $\mathcal{S}_{\text{stable}}(D)$ in the following form:

$$\mathcal{S}_{\text{stable}}(D) = -\sum_{\substack{1 \leqslant s \leqslant \mathfrak{g} \\ 0 \leqslant m \leqslant s}} \frac{(N/t)^{2-2(s+1)-(s+m)}}{(s+m)!} \sum_{\vec{\ell} \in \mathfrak{L}_{s,m}} t_{\vec{\ell}}^{(s+1)} \prod_{i=1}^{s+m} P_{\ell_i}(H), \qquad (3.3)$$

where, for each $\vec{\ell} \in \mathfrak{L}_{s,m}$, $t_{\vec{\ell}}^{(s+1)} = t^{s-m} \widetilde{\alpha}_{\vec{\ell}}$ and $\widetilde{\alpha}_{\vec{\ell}}$ is a finite linear combination of the formal parameters $\widehat{\alpha}_{n_I}$'s with integral coefficients.

Consider the following two sets of elementary 2-cells:

$$\mathcal{C}_{\text{unstable}} = \{(0; \vec{\ell}) \mid \vec{\ell} \in \mathfrak{L}_{\text{disk}} \cup \mathfrak{L}_{\text{cylinder}}\},$$

$$\mathcal{C}_{\text{stable}} = \bigcup_{\substack{1 \leqslant s \leqslant \mathfrak{g} \\ 0 \leqslant m \leqslant s}} \{(s+1; \vec{\ell}) \mid \vec{\ell} \in \mathfrak{L}_{s,m}\}.$$

We identify the set

$$\mathcal{C} = \mathcal{C}_{\text{unstable}} \cup \mathcal{C}_{\text{stable}} \qquad (3.4)$$

with the corresponding set of equivalence classes $[C]$ of surfaces with polygonal boundaries. We assign a *Boltzmann weight*, equal to $t_{\vec{\ell}}^{(g)}$, to each elementary 2-cell $[C] \in \mathcal{C}$ of type $(g; \vec{\ell})$. Note that the elementary 2-cells in $\mathcal{C}_{\text{unstable}}$ (resp., $\mathcal{C}_{\text{stable}}$) are represented by surfaces whose Euler characteristic satisfies $\chi \geqslant 0$ (resp., $\chi < 0$). For each elementary 2-cell $[C] \in \mathcal{C}$ of type $(g; \vec{\ell})$ with Boltzmann weight $t_{\vec{\ell}}^{(g)}$, $\vec{\ell} \in \mathbb{N}_\uparrow^n$, let

$$T_{[C]}(H) := t_{\vec{\ell}}^{(g)} \prod_{i=1}^n P_{\ell_i}(H), \qquad H \in \mathcal{H}_N, \qquad (3.5)$$

where $P_\ell(H)$ is defined by (3.1). We rewrite (3.2) and (3.3), respectively, in the following form:

$$\mathcal{S}_{\text{unstable}}(D) = \mathcal{S}_0(H) - \sum_{[C] \in \mathcal{C}_{\text{unstable}}} \frac{(N/t)^{\chi(C)}}{(\beta_0(\partial C))!} T_{[C]}(H),$$

$$S_{\text{stable}}(D) = -\sum_{[C]\in\mathcal{C}_{\text{stable}}} \frac{(N/t)^{\chi(C)}}{(\beta_0(\partial C))!}\, T_{[C]}(H),$$

where $\beta_0(\partial C)$ denotes the zeroth Betti number, i.e., the number of connected components, of the boundary ∂C of a surface C. Thus, we get the following proposition.

Proposition 3.2. *The action functional $S(D)$ for the random matrix geometries of type $(1,0)$, given by* (2.1)*, can be decomposed in the following form:*

$$S(D) = S_0(H) + S_{\text{int}}(H), \tag{3.6}$$

where

$$S_{\text{int}}(H) = -\sum_{[C]\in\mathcal{C}} \frac{(N/t)^{\chi(C)}}{(\beta_0(\partial C))!}\, T_{[C]}(H),$$

and $\mathcal{C}$ is given by (3.4).

4. The corresponding 1-Hermitian matrix model

From now on, we consider the multi-trace 1-Hermitian matrix model corresponding to the random matrix geometries of type $(1,0)$ with the distribution $d\rho = e^{-S(D)}\,dD$ in the sense of *formal* matrix integrals. In other words, we treat the term $S_{\text{int}}(H)$ in (3.6) as a perturbation of $S_0(H)$.

Consider the normalized Gaussian measure

$$d\rho_0 = ce^{-S_0(H)}\,dH = c\exp\left(-\frac{N}{2t}\operatorname{Tr}(H^2)\right)dH$$

over $\mathcal{H}_N$ with total mass one. Here,

$$c = \left[2^N\left(\frac{\pi t}{N}\right)^{N^2}\right]^{-1/2}.$$

Denote by $\mathbf{t}$ the sequence of Boltzmann weights $t_\ell^{(g)}$ in $S_{\text{int}}(H)$. We consider

$$\Phi(H) = \exp(-S_{\text{int}}(H))$$

as a formal power series in $\mathbf{t}$, i.e., an exponential generating function. The *partition function Z_N* of the model is defined by

$$Z_N = \rho_0[\Phi(H)] \overset{\text{formal}}{=} c\int_{\mathcal{H}_N} \Phi(H)\,e^{-S_0(H)}\,dH,$$

where the second integral is understood in the sense that we expand $\Phi(H)$ as a power series in $\mathbf{t}$ and *interchange* the integration with the summation.

The *disconnected n-point correlators* $\widehat{W}_n(x_1, \ldots, x_n)$, $n \geq 1$, of the model are defined as the joint moments of

$$\mathrm{X}_j = \mathrm{Tr}\big((x_j \mathbb{1}_N - H)^{-1}\big), \quad x_j \in \mathbb{C} \setminus \mathrm{Spec}(H),$$

i.e.,

$$\widehat{W}_n(x_1, \ldots, x_n) = \frac{1}{Z_N} \rho_0 \left[\Phi(H) \prod_{j=1}^n \mathrm{Tr}\big((x_j \mathbb{1}_N - H)^{-1}\big) \right]$$

$$\stackrel{\text{formal}}{=} \mathbb{E}\left[\prod_{j=1}^n \mathrm{Tr}\big((x_j \mathbb{1}_N - H)^{-1}\big) \right], \tag{4.1}$$

where $(x_j \mathbb{1}_N - H)^{-1}$ denotes the resolvent of H.[7] The *connected n-point correlators* $W_n(x_1, \ldots, x_n)$, $n \geq 1$, of the model are defined as the joint cumulants of X_j's, i.e.,

$$W_n(x_1, \ldots, x_n) = \sum_{K \vdash [\![1,n]\!]} (-1)^{[K]-1} ([K]-1)! \prod_{i=1}^{[K]} \widehat{W}_{|K_i|}(x_{K_i}). \tag{4.3}$$

In (4.3), the sum runs over partitions of $[\![1, n]\!] := \{1, 2, 3, \ldots, n\}$, the number of subsets in a partition K is denoted by $[K]$, and

$$\widehat{W}_{|K_i|}(x_{K_i}) \equiv \widehat{W}_{|K_i|}\big((x_j)_{j \in K_i}\big).$$

4.1. Topological expansion of the correlators

Proposition 4.1. *The connected n-point correlators $W_n(x_1, \ldots, x_n)$ of the random matrix geometries of type $(1, 0)$ with the distribution $\mathrm{d}\rho = e^{-\mathcal{S}(D)} \mathrm{d}D$ have a large N expansion of* topological type, *given by*

$$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} (N/t)^{2-2g-n} W_{g,n}(x_1, \ldots, x_n), \tag{4.4}$$

where $W_{g,n}(x_1, \ldots, x_n)$ is defined, in the following, by (4.8).[8]

---

[7]Strictly speaking, in the context of formal matrix integrals, one works with the formal series

$$\widetilde{\mathrm{X}}_j = \sum_{\ell=0}^{\infty} \frac{\mathrm{Tr}(H^\ell)}{x_j^{\ell+1}} \tag{4.2}$$

instead of $\mathrm{Tr}\big((x_j \mathbb{1}_N - H)^{-1}\big)$.

[8]One should *not* misinterpret (4.4) as the *asymptotic expansion* of the connected correlators as $N \to \infty$ (see [8, 20]).

Proof. We use *Wick's theorem* and the techniques of [10] to relate the formal matrix integrals in our model to the combinatorics of *stuffed maps* [4, 9]. Considering (4.1) and (4.2), the computation of $\hat{W}_n(x_1, \ldots, x_n)$ leads to Gaussian integrals of the following form:

$$\rho_0 \left[\prod_{j=1}^{n} \frac{\mathrm{Tr}(H^{\ell_j})}{x_j^{\ell_j+1}} \prod_{i=1}^{|\mathcal{C}|} \frac{1}{k_i!} \left(\frac{(N/t)^{\chi(C_i)}}{(\beta_0(\partial C_i))!} T_{[C_i]}(H) \right)^{k_i} \right]. \tag{4.5}$$

To compute (4.5) using Wick's theorem, we will represent each term of the form $\mathrm{Tr}(H^{\ell_j})/x_j^{\ell_j+1}$, in (4.5), by a marked face of perimeter ℓ_j with Boltzmann weight $x_j^{-(\ell_j+1)}$, $j = 1, \ldots, n$.[9] Also, we represent each term of the form

$$\frac{(N/t)^{\chi(C_i)}}{(\beta_0(\partial C_i))!} T_{[C_i]}(H),$$

in (4.5), by a surface C_i of Euler characteristic $\chi(C_i)$ and Boltzmann weight $t_{\vec{\ell}_i}^{(g_i)}$ representing the corresponding elementary 2-cell $[C_i] \in \mathcal{C}$ of type $(g_i; \vec{\ell}_i)$, $i = 1, \ldots, |\mathcal{C}|$. The orientation on each C_i induces an orientation on ∂C_i.

Let Ξ be the collection of all surfaces representing the terms in (4.5) in the above-mentioned way. Consider a pairing σ on the set of sides of the connected components of the boundary of surfaces in Ξ. We glue the surfaces in Ξ along the sides of their boundary, according to σ, such that the gluing map reverses the orientation. The resulting stuffed map

$$M = (S, G)$$

consists of an oriented, not necessarily connected surface S and a graph G embedded into S.[10] We denote by $\widetilde{S}$ the surface that one gets by deleting the marked faces from S.

It can be shown that each vertex (resp., edge) of G contributes a weight N (resp., t/N) [10]. In addition, each unmarked connected component $\mathfrak{U}$ of $S \backslash G$ contributes a weight $(N/t)^{\chi(\mathfrak{U})}$. Hence, the exponent of N, in the total contribution corresponding to $M = (S, G)$, equals $\chi(\widetilde{S})$.[11]

---

[9]A polygon of perimeter ℓ is an oriented 2-dimensional CW complex consisting of a 2-cell, homeomorphic to a disk, whose boundary is equipped with a cellular decomposition into ℓ 0-cells and ℓ 1-cells. A polygon is called rooted if we distinguish one of the 0-cells and its incident 1-cell on the boundary. We refer to a labeled rooted polygon of perimeter ℓ as a marked face of perimeter ℓ.

[10]If each connected component of $S \backslash G$ is homeomorphic to an open disk, then $M = (S, G)$ is called a *map*.

[11]This fact about the exponent of N, in the case of maps (or, equivalently, *ribbon graphs*), was first noticed by Gerard 't Hooft in [29] (see, e.g., [20]).

In addition to the pre-mentioned Boltzmann weights assigned to the 2-cells of M, we assign a Boltzmann weight equal to t to each vertex of G. The total Boltzmann weight of the isomorphism class $[M]$ of a stuffed map M, denoted by $\mathfrak{Bw}([M])$, is defined to be the product of all Boltzmann weights assigned to the cells in M divided by the order $|\mathrm{Aut}(M)|$ of the automorphism group of M. The contribution of the isomorphism class $[M]$ of a Boltzmann-weighted, not necessarily connected stuffed map $M = (S, G)$ to $\widehat{W}_n(x_1, \ldots, x_n)$ is given by

$$(N/t)^{\chi(\widetilde{S})} \mathfrak{Bw}([M]). \tag{4.6}$$

Let $\mathbb{M}_{g,n}(\mathcal{C})$ be the set of isomorphism classes of the Boltzmann-weighted connected closed stuffed maps $M = (S, G)$ of genus g with n marked faces such that the equivalence class of each unmarked connected component of $S \backslash G$ (in the sense of Definition 3.1) is in $\mathcal{C}$. Considering (4.6) and

$$\widehat{W}_n(x_1, \ldots, x_n) = \sum_{K \vdash [\![1,n]\!]} \prod_{i=1}^{[K]} W_{|K_i|}(x_{K_i}), \tag{4.7}$$

we have

$$W_n(x_1, \ldots, x_n) = \sum_{g \geqslant 0} \sum_{[M] \in \mathbb{M}_{g,n}(\mathcal{C})} (N/t)^{\chi(\widetilde{S})} \mathfrak{Bw}([M]).$$

Since, for a connected closed stuffed map $M = (S, G)$ of genus g with n marked faces,

$$\chi(\widetilde{S}) = 2 - 2g - n,$$

we get (4.4), where

$$W_{g,n}(x_1, \ldots, x_n) = \sum_{[M] \in \mathbb{M}_{g,n}(\mathcal{C})} \mathfrak{Bw}([M]) \in \mathbb{Q}[\mathbf{t}][\![t]\!][\![(x_j^{-1})_j]\!]. \tag{4.8}$$
∎

Let $\mathbb{M}_{g,n;\vec{\ell}}(\mathcal{C})$, $\vec{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$, be the set of isomorphism classes $[M] \in \mathbb{M}_{g,n}(\mathcal{C})$ of the stuffed maps with n marked faces of perimeters ℓ_j, $j = 1, \ldots, n$. By (4.8), the generating series $Q_{g;\vec{\ell}} \in \mathbb{Q}[\mathbf{t}][\![t]\!]$ of the stuffed maps, corresponding to our model, of genus g with n polygonal boundaries of perimeters ℓ_j, $j = 1, \ldots, n$, satisfies

$$W_{g,n}(x_1, \ldots, x_n) = \delta_{g,0}\,\delta_{n,1}\frac{t}{x_1} + \sum_{\vec{\ell} \in \mathbb{N}^n} \sum_{[M] \in \mathbb{M}_{g,n;\vec{\ell}}(\mathcal{C})} \mathfrak{Bw}([M])$$

$$= \delta_{g,0}\delta_{n,1}\frac{t}{x_1} + \sum_{\vec{\ell} \in \mathbb{N}^n} Q_{g;\vec{\ell}} \prod_{j=1}^{n} x_j^{-(\ell_j+1)}. \tag{4.9}$$

4.2. Large-N spectral distribution

Using (4.9), we see that the generating series $Q_{0;\ell}$, $\ell \in \mathbb{N}$, of the rooted planar stuffed maps with topology of a disk and perimeter ℓ, is given by

$$Q_{0;\ell} = \sum_{[M] \in \mathbb{M}_{0,1;\ell}(\mathcal{C}_{\text{unstable}})} \mathfrak{Bw}([M]) x^\ell \in \mathbb{Q}[t_{\vec{\ell}}^{(0)}][\![t]\!], \qquad (4.10)$$

where $t_{\vec{\ell}}^{(0)}$ denotes the sequence of Boltzmann weights $t_{\vec{\ell}}^{(0)}$, $\vec{\ell} \in \mathfrak{L}_{\text{disk}} \cup \mathfrak{L}_{\text{cylinder}}$.

If the Boltzmann weights $t_{\vec{\ell}}^{(0)}$ (or, equivalently, the formal parameters α_n, $3 \leqslant n \leqslant d$) have given values, then there exists a critical temperature $t_c > 0$ such that, for any $|t| < t_c$, we have $Q_{0;\ell} < \infty$, $\forall \ell \in \mathbb{N}$ [4]. From now on, we restrict ourselves to the case $0 < t < t_c$, where t_c is specified according to each set of given values to α_n's. Hence, we have the following *one-cut Lemma* [4, 5].

Lemma 4.2. *For given values to the Boltzmann weights* $t_{\vec{\ell}}^{(0)}$, *and* $0 < t < t_c$, *the series*

$$W_{0,1}(x) = \frac{t}{x} + \sum_{\ell=1}^{\infty} \frac{Q_{0;\ell}}{x^{\ell+1}} \qquad (4.11)$$

is the Laurent expansion at $x = \infty$ *of a holomorphic function, denoted by* $W_{0,1}(x)$, *on* $\mathbb{C} \backslash \Gamma$, *where* $\Gamma = [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R}$ *depends on* $t_{\vec{\ell}}^{(0)}$, t. *The limits* $\lim_{\varepsilon \to 0^+} W_{0,1}(s \pm i\varepsilon)$, $\forall s \in \Gamma^o$, *exist, and the jump discontinuity*

$$\varphi(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} (W_{0,1}(s - i\varepsilon) - W_{0,1}(s + i\varepsilon)) \qquad (4.12)$$

assumes positive values on the interior Γ^o *of the discontinuity locus* Γ *and vanishes at* $\partial \Gamma$.

Consider the measure $\mu = \varphi(s) \, ds$ on $\mathbb{R}$, where ds denotes the Lebesgue measure, and $\varphi(s)$ is given by (4.12). By the *Sokhotski–Plemelj theorem*, the function $W_{0,1}(x)$ is, indeed, the *Stieltjes transform* $\mathcal{S}[\mu](x)$ of μ, i.e.,

$$W_{0,1}(x) = \mathcal{S}[\mu](x) = \int \frac{\varphi(s)}{x - s} \, ds, \quad x \in \mathbb{C} \backslash \operatorname{supp}(\mu). \qquad (4.13)$$

In addition, consider the *empirical spectral distribution (empirical measure)*

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

on $\mathbb{R}$, where $\{\lambda_i\}_{i=1}^N$, $\lambda_i \in \mathbb{R}$, denotes the eigenvalues of the random Hermitian matrix $H \in \mathcal{H}_N$ corresponding to the random matrix geometries of type $(1, 0)$ with the

distribution $\mathrm{d}\rho = e^{-\mathcal{S}(D)}\,\mathrm{d}D$.[12] By (4.1), we have

$$\frac{1}{N}W_1(x) = \mathbb{E}[\mathcal{S}[\mu_N](x)]. \tag{4.14}$$

Motivated by (4.4), we assume that

$$\lim_{N\to\infty}\frac{1}{N}W_1(x) = W_{0,1}(x).$$

Therefore, considering (4.13) and (4.14), the expected distribution of the eigenvalues $\{\lambda_i\}_{i=1}^N$ is given by

$$\mu = \varphi(s)\,\mathrm{d}s,$$

up to terms with exponential decay, as $N \to \infty$.[13] We refer to the measure $\mu = \varphi(s)\,\mathrm{d}s$ as the *large-N spectral distribution*. In addition, from now on, we assume that the sequence of Boltzmann weights $\mathbf{t}$, and the parameter t, are *tame*, in the sense of [4, Definition 4.1]. Hence, each $W_{g,n}(x_1, \ldots, x_n)$, a priori defined as the generating series of stuffed maps, upgrades to a holomorphic function on $(\mathbb{C}\backslash\Gamma)^n$ which has a jump discontinuity when one of x_i's crosses Γ.

5. Schwinger–Dyson equations

Our main tool for analyzing the $W_{g,n}(x_1, \ldots, x_n)$'s is an infinite system of equations, called the *Schwinger–Dyson equations* (SDEs), satisfied by the n-point correlators of the model. In the matrix model framework, they were introduced by Migdal [28] and referred to as the *loop equations*. There are several versions of SDEs for matrix models (and some other closely related models in statistical physics, e.g., the *β-ensembles*) in the literature (see, e.g., [7,20]). However, the root of all of them is the invariance of the integral of a top degree differential form under a 1-parameter family of orientation-preserving diffeomorphisms on a manifold.

To put the above-mentioned differential geometric fact in a precise form, consider an oriented connected Riemannian n-manifold M with the Riemannian volume form ω. Let V be a smooth vector field on M with a local flow $\phi_t : M \to M$. Consider a smooth function $\Psi : M \to \mathbb{R}$. Let $\Omega \subset M$ be a compact n-dimensional submanifold of M. Since

$$\int_{\phi_t(\Omega)} \Psi\omega = \int_\Omega \phi_t{}^*(\Psi\omega) \quad \forall t \in (-\varepsilon, \varepsilon),$$

---

[12]The Dirac measure at $\lambda \in \mathbb{R}$ is denoted by δ_λ.

[13]The measure $\mu = \varphi(s)\,\mathrm{d}s$ plays the same role as what is called, in the context of *convergent* matrix integrals, the *equilibrium measure* (see, e.g., [1]).

using Cartan's magic formula and Stokes's theorem, we get

$$\int_\Omega (d\,\Psi(V) + \Psi \operatorname{div}(V))\omega - \int_{\partial\Omega} \Psi(\iota_V \omega) = 0. \tag{5.1}$$

In (5.1), the *exterior derivative*, the *interior product* by V, and the *divergence* of V are denoted by d, ι_V, and $\operatorname{div}(V)$, respectively.

The Schwinger–Dyson equations for the multi-trace 1-Hermitian matrix models are derived in [4], using *Tutte's decomposition* applied to the stuffed maps. In this section, we give a proof of them based on the above-mentioned differential geometric fact.[14]

Consider the action of the unitary group U_N on $\mathcal{H}_N$ by conjugation, i.e., $u \cdot H := uHu^{-1}$ for $u \in U_N$ and $H \in \mathcal{H}_N$. Since the action functional $\mathcal{S}(D) : \mathcal{H}_N \to \mathbb{R}$ is invariant under the pre-mentioned action of U_N on $\mathcal{H}_N$, we can rewrite $\mathcal{S}(D)$ as a function of the eigenvalues $\{\lambda_i\}_{i=1}^N$, $\lambda_i \in \mathbb{R}$, of the random Hermitian matrix $H \in \mathcal{H}_N$.

Let

$$(\widehat{\mu}_N)^k := \underbrace{\widehat{\mu}_N \times \cdots \times \widehat{\mu}_N}_{k\text{-times}}$$

be the product measure on $\mathbb{R}^k$ corresponding to the unnormalized empirical measure

$$\widehat{\mu}_N = \sum_{i=1}^N \delta_{\lambda_i}$$

on $\mathbb{R}$. For each elementary 2-cell $[C] \in \mathcal{C}$ of type $(g; \vec{\ell})$, $\vec{\ell} \in \mathbb{N}_\uparrow^k$, we rewrite $T_{[C]}(H)$, defined by (3.5), in the following form:

$$T_{[C]}(H) = \mathrm{t}_{\vec{\ell}}^{(g)} \prod_{j=1}^k \frac{\operatorname{Tr}(H^{\ell_j})}{\ell_j} = \frac{\mathrm{t}_{\vec{\ell}}^{(g)}}{\prod_{j=1}^k \ell_j} (\widehat{\mu}_N)^k [s_1^{\ell_1} s_2^{\ell_2} \cdots s_k^{\ell_k}],$$

where

$$(\widehat{\mu}_N)^k [s_1^{\ell_1} s_2^{\ell_2} \cdots s_k^{\ell_k}] = \sum_{i_1,\ldots,i_k=1}^N \lambda_{i_1}^{\ell_1} \lambda_{i_2}^{\ell_2} \cdots \lambda_{i_k}^{\ell_k}.$$

---

[14]In general, there are, at least, two approaches in the literature to deriving the SDEs for matrix models: one is usually referred to as "integration by parts" or "invariance under change of variable" which is basically the above-mentioned differential geometric approach; and the other one is combinatorial and based on Tutte's decomposition. The differential geometric approach has the advantage that it can be applied to a slightly more general class of models, e.g., the β-ensembles for arbitrary $\beta > 0$, which do not necessarily have a combinatorial interpretation.

For each $1 \leqslant k \leqslant 2\mathfrak{g}$, let

$$T_k(H) = -\delta_{k,1} \frac{N}{2t} \operatorname{Tr}(H^2) + \sum_{\substack{[C] \in \mathcal{C} \\ \beta_0(\partial C) = k}} (N/t)^{\chi(C)} T_{[C]}(H).$$

Let $\widetilde{T}_k(s_1, \ldots, s_k)$ be the polynomial in $\{s_j\}_{j=1}^k$ satisfying

$$(\widehat{\mu}_N)^k [\widetilde{T}_k(s_1, \ldots, s_k)] = T_k(H). \tag{5.2}$$

Because of the symmetry of $(\widehat{\mu}_N)^k$, we can replace $\widetilde{T}_k(s_1, \ldots, s_k)$ with its symmetrization $T_k(s_1, \ldots, s_k)$, i.e.,

$$T_k(s_1, \ldots, s_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \widetilde{T}_k(s_{\sigma(1)}, \ldots, s_{\sigma(k)}),$$

in (5.2). We refer to the symmetric polynomials $T_k(s_1, \ldots, s_k)$, $1 \leqslant k \leqslant 2\mathfrak{g}$, as the *k-point interactions*. We will need the following technical lemma on derivatives of T_k in the proof of SDEs.

Lemma 5.1. *For each fixed element $\widehat{\lambda}$ of $E = \{\lambda_i\}_{i=1}^N$, we have*

$$\partial_{\widehat{\lambda}}((\widehat{\mu}_N)^k [T_k(s_1, s_2, \ldots, s_k)]) = k(\widehat{\mu}_N)^{k-1}[T_k^{(1)}(\widehat{\lambda}, s_2, \ldots, s_k)],$$

where

$$T_k^{(1)}(s_1, s_2, \ldots, s_k) := \partial_{s_1} T_k(s_1, s_2, \ldots, s_k).$$

Proof. Let $\widehat{E} = E \backslash \{\widehat{\lambda}\}$. Let

$$\widehat{\Lambda}_1 = \{(\widehat{\lambda}, a_2, \ldots, a_k) \mid a_l \in \widehat{E}, \ \forall l\},$$

and, for each $2 \leqslant r \leqslant k$, let

$$\widehat{\Lambda}_r = \bigcup_{2 \leqslant m_2 < \cdots < m_r \leqslant k} \{(\widehat{\lambda}, a_2, \ldots, a_k) \mid a_{m_i} = \widehat{\lambda}, \ \forall i, \text{ and } a_l \in \widehat{E}, \ l \neq m_i\}.$$

We denote $(\widehat{\mu}_N)^k [T_k(s_1, \ldots, s_k)] = \sum_{i_1, \ldots, i_k = 1}^N T_k(\lambda_{i_1}, \ldots, \lambda_{i_k})$ by

$$\sum_{\lambda_I \in E^k} T_k(\lambda_I).$$

Since T_k is a symmetric polynomial, we have

$$\sum_{\lambda_I \in E^k} T_k(\lambda_I) = \sum_{\lambda_I \in \widehat{E}^k} T_k(\lambda_I) + \sum_{r=1}^k \frac{k}{r} \sum_{\lambda_I \in \widehat{\Lambda}_r} T_k(\lambda_I)$$

and

$$\partial_{\widehat{\lambda}}(T_k(\lambda_I)) = r T_k^{(1)}(\lambda_I) \quad \forall \lambda_I \in \widehat{\Lambda}_r.$$

Thus, we get

$$\partial_{\widehat{\lambda}} \sum_{\lambda_I \in E^k} T_k(\lambda_I) = k \sum_{r=1}^{k} \sum_{\lambda_I \in \widehat{\Lambda}_r} T_k^{(1)}(\lambda_I)$$

$$= k \sum_{\lambda_J \in E^{k-1}} T_k^{(1)}(\widehat{\lambda}, \lambda_J). \qquad \blacksquare$$

We rewrite the measure $d\rho = e^{-\mathcal{S}(D)} \, dD$ over $\mathcal{H}_N$ in the following form:

$$d\rho = e^{-\mathcal{S}(D)} \, dD = \exp\left(\sum_{k=1}^{2\mathfrak{g}} \frac{1}{k!} (\widehat{\mu}_N)^k [T_k(s_1, \ldots, s_k)] \right) dH.$$

By the *Weyl integration formula*, the measure $d\rho$ induces a measure $d\widetilde{\varrho}$, given by

$$d\widetilde{\varrho} = c_N \frac{\mathrm{Vol}(\mathrm{U}_N)}{N! \, (2\pi)^N} \Delta(\boldsymbol{\lambda})^2 \exp\left(\sum_{k=1}^{2\mathfrak{g}} \frac{1}{k!} (\widehat{\mu}_N)^k [T_k(s_1, \ldots, s_k)] \right) \prod_{i=1}^{N} d\lambda_i, \quad (5.3)$$

on the space $\mathbb{R}^N$ of eigenvalues of Hermitian matrices. In (5.3), $\Delta(\boldsymbol{\lambda})$ denotes the Vandermonde determinant, i.e.,

$$\Delta(\boldsymbol{\lambda}) = \prod_{1 \leqslant i < j \leqslant N} |\lambda_j - \lambda_i|,$$

and $c_N = 2^{\frac{N-N^2}{2}}$.[15]

Let

$$\Omega = \widehat{\Gamma}^N \subset \mathbb{R}^N, \qquad (5.4)$$

where $\widehat{\Gamma} \subset \mathbb{R}$ is a strict ε-enlargement of the support Γ of the large-N spectral distribution μ, i.e., $\Gamma \subset \widehat{\Gamma}^{\mathrm{o}}$, and $\widehat{\Gamma} \backslash \Gamma$ has small Lebesgue measure. We assume that if we replace $d\widetilde{\varrho}$ with

$$d\varrho = \mathbb{1}_\Omega \, d\widetilde{\varrho}$$

in the definition of the partition function and the correlators, then they get modified in terms of exponential decay as $N \to \infty$.

---

[15]In addition, the term $\mathrm{Vol}(\mathrm{U}_N)$ is the volume of the unitary group U_N with respect to the Riemannian volume form corresponding to the induced metric on U_N from the inner product $\langle A, B \rangle = \mathrm{Tr}(AB^*)$ on the ambient vector space $M_N(\mathbb{C})$.

Theorem 5.2. *For any* $x, (x_i)_{i \in I} \in \mathbb{C} \backslash \widehat{\Gamma}$, *the rank n Schwinger–Dyson equation for the connected correlators of the model (up to the boundary term) is given by*

$$
W_{n+1}(x, x, x_I) + \sum_{J \subseteq I} W_{|J|+1}(x, x_J) W_{n-|J|}(x, x_{I \backslash J})
$$

$$
+ \sum_{i \in I} \oint_{C_{\widehat{\Gamma}}} \frac{d\xi}{2\pi i} \frac{W_{n-1}(\xi, x_{I \backslash \{i\}})}{(x - \xi)(x_i - \xi)^2}
$$

$$
+ \sum_{k=1}^{2\mathfrak{g}} \sum_{\substack{K \vdash \llbracket 1, k \rrbracket \\ J_1 \sqcup \cdots \sqcup J_{[K]} = I}} \oint_{C_{\widehat{\Gamma}}} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i} \right] \frac{\partial_{\xi_1} T_k(\xi_1, \ldots, \xi_k)}{(k-1)! (x - \xi_1)} \prod_{i=1}^{[K]} W_{|K_i|+|J_i|}(\xi_{K_i}, x_{J_i})
$$

$$
= 0, \tag{5.5}
$$

where $I = \{1, 2, \ldots, n-1\}$, $n \geq 1$, *and* $C_{\widehat{\Gamma}}$ *is a closed counter-clockwise-oriented contour in an ε-tubular neighborhood of $\widehat{\Gamma}$ which encloses $\widehat{\Gamma}$.*

Proof. Let $\{\tau_j\}_{j=1}^{n-1}$ be a sequence of parameters. Consider the random variables X_j, $j = 1, \ldots, n-1$, given by

$$
X_j = \sum_{i=1}^{N} \frac{1}{x_j - \lambda_i}, \quad x_j \in \mathbb{C} \backslash \widehat{\Gamma}.
$$

Let $P^{\vec{\tau}}$ be the probability measure on $\mathbb{R}^N$ defined by

$$
\mathbb{E}_{P^{\vec{\tau}}}[f] := \frac{1}{\mathbb{E}_P[\exp(\sum_{j=1}^{n-1} \tau_j X_j)]} \mathbb{E}_P \left[f \exp \left(\sum_{j=1}^{n-1} \tau_j X_j \right) \right],
$$

where $f \in C_0(\mathbb{R}^N)$, and

$$
dP = \frac{1}{\varrho[1]} d\varrho.
$$

We denote the joint cumulants (resp., moments) of $\{X_j\}_{j=1}^{r}$ with respect to the probability measure $P^{\vec{\tau}}$ by $W_r^{\vec{\tau}}(x_1, \ldots, x_r)$ (resp., $\widehat{W}_r^{\vec{\tau}}(x_1, \ldots, x_r)$).

Consider the Riemannian manifold $\mathbb{R}^N$ with the Euclidean metric and the Riemannian volume form

$$
d\lambda := \prod_{i=1}^{N} d\lambda_i.
$$

Let Ω be the compact subset of $\mathbb{R}^N$ given by (5.4). Consider the smooth vector field

$$
V = \sum_{i=1}^{N} \frac{1}{x - \lambda_i} \vec{e}_i, \quad x \in \mathbb{C} \backslash \widehat{\Gamma},
$$

where $\vec{e}_i$, $i = 1, \ldots, N$, denote the standard constant unit vector fields on $\mathbb{R}^N$. By considering the invariance of

$$Z_N^{\vec{\tau}} = \mathbb{E}_{\mathrm{P}}\left[\exp\left(\sum_{j=1}^{n-1} \tau_j X_j\right)\right] = \int \exp\left(\sum_{j=1}^{n-1} \tau_j \left(\sum_{i=1}^{N} \frac{1}{x_j - \lambda_i}\right)\right) d\mathrm{P},$$

under the flow of the vector field V, one gets the rank n Schwinger–Dyson equation for the connected correlators $W_n(x, x_1, \ldots, x_{n-1})$.

We rewrite $Z_N^{\vec{\tau}}$ in the following form:

$$Z_N^{\vec{\tau}} = \frac{1}{\varrho[1]} \int_{\Omega} \Psi(\lambda) \, d\lambda,$$

where $\Psi(\lambda) = \prod_{m=1}^{3} \psi_m(\lambda)$ is given by

$$\psi_1(\lambda) = \exp\left(\sum_{j=1}^{n-1} \tau_j \left(\sum_{i=1}^{N} \frac{1}{x_j - \lambda_i}\right)\right),$$

$$\psi_2(\lambda) = \exp\left(\sum_{k=1}^{2\mathfrak{g}} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{N} T_k(\lambda_{i_1}, \ldots, \lambda_{i_k})\right),$$

and

$$\psi_3(\lambda) = \prod_{1 \leq i \neq j \leq N} |\lambda_j - \lambda_i|.$$

By (5.1), the invariance of $Z_N^{\vec{\tau}}$ under the flow of V is equivalent to

$$\mathbb{E}_{\mathrm{P}^{\vec{\tau}}}\left[\sum_{m=1}^{3} \frac{d\psi_m(V)}{\psi_m} + \mathrm{div}(V)\right] = 0, \tag{5.6}$$

up to the boundary term.[16]

Using the Cauchy integral formula, the term $d\psi_1(V)/\psi_1$ can be expressed in the following way:[17]

$$\frac{d\psi_1(V)}{\psi_1} = \sum_{j=1}^{n-1} \tau_j \sum_{i=1}^{N} \frac{1}{(x - \lambda_i)(x_j - \lambda_i)^2}$$

$$= \sum_{j=1}^{n-1} \tau_j \oint_{C_{\widehat{\Gamma}}} \frac{d\xi}{2\pi i} \left(\frac{1}{(x - \xi)(x_j - \xi)^2}\right) \left(\sum_{i=1}^{N} \frac{1}{\xi - \lambda_i}\right).$$

---

[16] Since we are considering the case in which $\Gamma \subset \widehat{\Gamma}^{\circ}$, i.e., both edges of $\widehat{\Gamma}$ are *soft*, the boundary term is of exponential decay as $N \to \infty$ (see [7]).

[17] Strictly speaking, we should assume that x and x_j's are not in an ε-tubular neighborhood of $\widehat{\Gamma}$ so that the function $1/((x - \xi)(x_j - \xi)^2)$ is holomorphic on that neighborhood of $\widehat{\Gamma}$.

By interchanging the integration on $\mathbb{R}^N$ with the contour integral, we get

$$\mathbb{E}_{\mathrm{P}^{\vec{\tau}}}\left[\frac{d\psi_1(V)}{\psi_1}\right] = \sum_{j=1}^{n-1} \tau_j \oint_{C_{\widehat{\Gamma}}} \frac{d\xi}{2\pi i}\frac{W_1^{\vec{\tau}}(\xi)}{(x-\xi)(x_j-\xi)^2}. \tag{5.7}$$

Considering Lemma 5.1, we have

$$\left(d \sum_{i_1,\ldots,i_k=1}^{N} T_k(\lambda_{i_1},\ldots,\lambda_{i_k})\right)(V)$$

$$= k \sum_{i=1}^{N} \frac{1}{x-\lambda_i} \sum_{j_1,\ldots,j_{k-1}=1}^{N} T_k^{(1)}(\lambda_i,\lambda_{j_1},\ldots,\lambda_{j_{k-1}})$$

$$= k \oint_{C_{\widehat{\Gamma}}} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i}\right] \frac{\partial_{\xi_1} T_k(\xi_1,\ldots,\xi_k)}{x-\xi_1} \prod_{r=1}^{k}\left(\sum_{i=1}^{N} \frac{1}{\xi_r-\lambda_i}\right),$$

where the integration is a k-times iterated contour integral along $C_{\widehat{\Gamma}}$. Thus, using (4.7), we have

$$\mathbb{E}_{\mathrm{P}^{\vec{\tau}}}\left[\frac{d\psi_2(V)}{\psi_2}\right]$$

$$= \sum_{k=1}^{2\mathfrak{g}} \frac{1}{(k-1)!} \oint_{C_{\widehat{\Gamma}}} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i}\right] \frac{\partial_{\xi_1} T_k(\xi_1,\ldots,\xi_k)}{x-\xi_1} \, \widehat{W}_k^{\vec{\tau}}(\xi_1,\ldots,\xi_k)$$

$$= \sum_{k=1}^{2\mathfrak{g}} \sum_{K\vdash[\![1,k]\!]} \oint_{C_{\widehat{\Gamma}}} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i}\right] \frac{\partial_{\xi_1} T_k(\xi_1,\ldots,\xi_k)}{(k-1)!(x-\xi_1)} \prod_{i=1}^{[K]} W_{|K_i|}^{\vec{\tau}}(\xi_{K_i}). \tag{5.8}$$

By similar steps, we get

$$\mathbb{E}_{\mathrm{P}^{\vec{\tau}}}\left[\frac{d\psi_3(V)}{\psi_3}\right] = W_2^{\vec{\tau}}(x,x) + \left(W_1^{\vec{\tau}}(x)\right)^2 + \partial_x(W_1^{\vec{\tau}}(x)) \tag{5.9}$$

and

$$\mathbb{E}_{\mathrm{P}^{\vec{\tau}}}[\mathrm{div}(V)] = -\partial_x(W_1^{\vec{\tau}}(x)). \tag{5.10}$$

By substituting (5.7), (5.8), (5.9), and (5.10) into (5.6), we get

$$W_2^{\vec{\tau}}(x,x) + \left(W_1^{\vec{\tau}}(x)\right)^2 + \sum_{i\in I} \tau_i \oint_{C_{\widehat{\Gamma}}} \frac{d\xi}{2\pi i}\frac{W_1^{\vec{\tau}}(\xi)}{(x-\xi)(x_i-\xi)^2}$$

$$+ \sum_{k=1}^{2\mathfrak{g}} \sum_{K\vdash[\![1,k]\!]} \oint_{C_{\widehat{\Gamma}}} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i}\right] \frac{\partial_{\xi_1} T_k(\xi_1,\ldots,\xi_k)}{(k-1)!(x-\xi_1)} \prod_{i=1}^{[K]} W_{|K_i|}^{\vec{\tau}}(\xi_{K_i}) = 0, \tag{5.11}$$

where $I = \{1, 2, \ldots, n-1\}$. Considering the definition of joint cumulant, for given finite subsets $L_i \subset \mathbb{N}$, $i = 1, \ldots, \ell$, of $\mathbb{N}$, we have

$$\partial_{\vec{\tau}}\big|_{\vec{\tau}=0}\left(\prod_{i=1}^{\ell} W_{|L_i|}^{\vec{\tau}}(\xi_{L_i})\right) = \sum_{J_1 \sqcup \cdots \sqcup J_\ell = I} \prod_{i=1}^{\ell} W_{|L_i|+|J_i|}(\xi_{L_i}, x_{J_i}).$$

Therefore, by taking the derivative $\partial_{\vec{\tau}}\big|_{\vec{\tau}=0}$ of each term of (5.11), we get (5.5). ∎

The system of Schwinger–Dyson equations is not closed in the sense that, for each $n \geqslant 1$, the rank n SDE gives an expression for $W_n(x_1, \ldots, x_n)$ in terms of $W_r(x_1, \ldots, x_r)$'s with $1 \leqslant r \leqslant \max\{n+1, n+2\mathfrak{g}-1\}$. However, we will see that they are "asymptotically" closed as $N \to \infty$, and we can solve them to find the coefficients $W_{g,n}(x_1, \ldots, x_n)$ of the large N expansion of the correlators.

For each $1 \leqslant k \leqslant 2\mathfrak{g}$ and $h \geqslant 0$, let $T_{h,k}(s_1, \ldots, s_k)$ be the symmetric polynomial in $\{s_j\}_{j=1}^k$ satisfying

$$(\hat{\mu}_N)^k[T_{h,k}(s_1, \ldots, s_k)] = -\delta_{h,0}\delta_{k,1}\frac{\text{Tr}(H^2)}{2} + \sum_{\substack{[C]\in\mathcal{C} \\ \beta_0(\partial C)=k,\ g(C)=h}} T_{[C]}(H),$$

where $g(C)$ denotes the genus of a surface C. The k-point interactions $T_k(s_1, \ldots, s_k)$ of the model can be rewritten in the following form:

$$T_k(s_1, \ldots, s_k) = \sum_{h\geqslant 0}(N/t)^{2-2h-k}\, T_{h,k}(s_1, \ldots, s_k), \tag{5.12}$$

where the summation includes finitely many terms. Considering (5.12) and (4.4), for each $n \geqslant 1$, $g \geqslant 0$, the rank n Schwinger–Dyson equation to order N^{3-2g-n} gives

$$W_{g-1,n+1}(x, x, x_I) + \sum_{J\subseteq I,\ 0\leqslant f\leqslant g} W_{f,|J|+1}(x, x_J)W_{g-f,n-|J|}(x, x_{I\setminus J})$$

$$+ \sum_{i\in I}\oint_{C_{\hat{\Gamma}}}\frac{d\xi}{2\pi i}\frac{W_{g,n-1}(\xi, x_{I\setminus\{i\}})}{(x-\xi)(x_i-\xi)^2}$$

$$+ \sum_{\substack{1\leqslant k\leqslant 2\mathfrak{g} \\ 0\leqslant h}} \sum_{\substack{K\vdash[\![1,k]\!] \\ J_1\sqcup\cdots\sqcup J_{[K]}=I}} \sum_{\substack{0\leqslant f_1,\ldots,f_{[K]} \\ h+k-[K]+\sum_i f_i=g}} \oint_{C_{\hat{\Gamma}}}\left\{\left[\prod_{r=1}^{k}\frac{d\xi_r}{2\pi i}\right]\right.$$

$$\left.\times\left[\frac{\partial_{\xi_1}T_{h,k}(\xi_1, \ldots, \xi_k)}{(k-1)!\,(x-\xi_1)}\prod_{i=1}^{[K]}W_{f_i,|K_i|+|J_i|}(\xi_{K_i}, x_{J_i})\right]\right\} = 0. \tag{5.13}$$

Before continuing, we introduce the following notations which are used in this article. Let $V \subset X$ be an open subset of a Riemann surface X. We denote by $\mathcal{O}$, $\mathcal{O}^*$,

$\mathcal{M}$, Ω, and $\mathcal{Q}$ the sheaves on X defined by

- $\mathcal{O}(V) = $ holomorphic functions on V,
- $\mathcal{O}^*(V) = $ multiplicative group of nonzero holomorphic functions on V,
- $\mathcal{M}(V) = $ meromorphic functions on V,
- $\Omega(V) = $ holomorphic 1-forms on V,
- $\mathcal{Q}(V) = $ meromorphic 1-forms on V,

respectively.

6. Spectral curve

We start by analyzing the rank one Schwinger–Dyson equation to leading order in N, i.e., equation (5.13) for $n = 1$ and $g = 0$, given by

$$(W_{0,1}(x))^2 + \sum_{k=1}^{2} \oint_{C_\Gamma} \left[\prod_{r=1}^{k} \frac{d\xi_r}{2\pi i} \right] \frac{\partial_{\xi_1} T_{0,k}(\xi_1, \ldots, \xi_k)}{x - \xi_1} \prod_{r=1}^{k} W_{0,1}(\xi_r) = 0. \quad (6.1)$$

Recall that

$$T_{0,1}(\xi) = -2t \, \mathcal{V}(\xi) = -\frac{1}{2}\xi^2 + \sum_{\ell \in \mathcal{L}_{\text{disk}}} \frac{t_\ell^{(0)}}{\ell} \xi^\ell,$$

where $\mathcal{V}(\xi)$ is given by (2.2), and

$$T_{0,2}(\xi, \eta) = \frac{1}{2} \sum_{(\ell_1, \ell_2) \in \mathcal{L}_{\text{cylinder}}} \frac{t_{\ell_1, \ell_2}^{(0)}}{\ell_1 \ell_2} (\xi^{\ell_1} \eta^{\ell_2} + \xi^{\ell_2} \eta^{\ell_1}). \quad (6.2)$$

We fix a simply connected open neighborhood $U \subset \mathbb{C}$ such that $\Gamma \subset U$. Consider the following integral operator, called the *master operator* [4],

$$\mathcal{O} f(\xi) = \frac{1}{2\pi i} \oint_{C_\Gamma} R(\xi, \eta) f(\eta) \, d\eta, \quad \mathcal{O} : \mathcal{O}(U \setminus \Gamma) \to \mathcal{O}(U),$$

with the kernel

$$R(\xi, \eta) = \partial_\xi T_{0,2}(\xi, \eta).$$

We rewrite (6.1) in the following form:

$$(W_{0,1}(x))^2 + \oint_{C_\Gamma} \frac{d\xi}{2\pi i} \frac{Q(\xi)}{x - \xi} W_{0,1}(\xi) = 0, \quad (6.3)$$

where

$$Q(\xi) = (\partial_\xi T_{0,1}(\xi)) + \mathcal{O} W_{0,1}(\xi).$$

Since $W_{0,1}(x) \in \mathcal{O}(\mathbb{C} \setminus \Gamma)$, using (4.12) and (4.11), we have

$$\oint_{C_\Gamma} \frac{d\xi}{2\pi i} \xi^\ell W_{0,1}(\xi) = \mathfrak{m}_\ell = \oint_{\xi=\infty} \frac{d\xi}{2\pi i} \xi^\ell W_{0,1}(\xi) = Q_{0;\ell},$$

where

$$\mathfrak{m}_\ell = \int s^\ell \varphi(s) \, ds, \quad \ell \geqslant 1,$$

denotes the moments of the large-N spectral distribution $\mu = \varphi(s) \, ds$, and $Q_{0;\ell}$ is given by (4.10). Thus, the polynomial $Q(\xi)$ can be expressed in terms of $\mathfrak{m}_\ell$'s in the following way:

$$Q(\xi) = -\xi + \sum_{\ell \in \mathfrak{L}_{\text{disk}}} \mathfrak{t}_\ell^{(0)} \xi^{\ell-1}$$
$$+ \frac{1}{2} \sum_{(\ell_1,\ell_2) \in \mathfrak{L}_{\text{cylinder}}} \mathfrak{t}_{\ell_1,\ell_2}^{(0)} \left[\frac{\mathfrak{m}_{\ell_2}}{\ell_2} \xi^{\ell_1-1} + \frac{\mathfrak{m}_{\ell_1}}{\ell_1} \xi^{\ell_2-1} \right]. \tag{6.4}$$

Considering $W_{0,1}(\xi) = O(1/\xi)$ as $\xi \to \infty$, we rewrite the contour integral in (6.3) as

$$\oint_{C_\Gamma} \frac{d\xi}{2\pi i} \frac{Q(\xi)}{x-\xi} W_{0,1}(\xi) = \oint_{C_\Gamma} \frac{d\xi}{2\pi i} \frac{Q(x) - (Q(x) - Q(\xi))}{x-\xi} W_{0,1}(\xi)$$
$$= Q(x) W_{0,1}(x) - \oint_{C_\Gamma} \frac{d\xi}{2\pi i} \Delta[Q](x,\xi) W_{0,1}(\xi),$$

where

$$\Delta[Q](x,\xi) = \frac{Q(x) - Q(\xi)}{x-\xi}$$

denotes the *noncommutative derivative*, aka "finite difference quotient", of $Q(\xi)$. The polynomial

$$P(x) = -\oint_{C_\Gamma} \frac{d\xi}{2\pi i} \Delta[Q](x,\xi) W_{0,1}(\xi)$$

has the following expression in terms of $\mathfrak{m}_\ell$'s:

$$P(x) = 1 - \sum_{\ell \in \mathfrak{L}_{\text{disk}}} \mathfrak{t}_\ell^{(0)} \sum_{n=0}^{\ell-2} \mathfrak{m}_{\ell-2-n} \, x^n$$
$$- \frac{1}{2} \sum_{(\ell_1,\ell_2) \in \mathfrak{L}_{\text{cylinder}}} \mathfrak{t}_{\ell_1,\ell_2}^{(0)} \left[\frac{\mathfrak{m}_{\ell_2}}{\ell_2} \left(\sum_{n=0}^{\ell_1-2} \mathfrak{m}_{\ell_1-2-n} \, x^n \right) \right.$$
$$\left. + \frac{\mathfrak{m}_{\ell_1}}{\ell_1} \left(\sum_{n=0}^{\ell_2-2} \mathfrak{m}_{\ell_2-2-n} \, x^n \right) \right]. \tag{6.5}$$

Therefore, we have proved the following proposition.

Proposition 6.1. *For the random matrix geometries of type* $(1,0)$ *with the distribution* $d\rho = e^{-\mathcal{S}(D)}\, dD$, *the Stieltjes transform* $W_{0,1}(x)$ *of the large-N spectral distribution* $\mu = \varphi(s)\, ds$ *of the model satisfies the following quadratic algebraic equation:*

$$y^2 + Q(x)y + P(x) = 0,$$

where the polynomials $Q(x)$ *and* $P(x)$ *are given by* (6.4) *and* (6.5), *respectively. The coefficients of* $Q(x)$ *and* $P(x)$ *depend on the Boltzmann weights* $t_{\ell}^{(0)}$ *and the moments* m_ℓ *of the large-N spectral distribution* μ.

Let the interval $\Gamma = [a, b] \subset \mathbb{R}$ and the open neighborhood $U \subset \mathbb{C}$, $\Gamma \subset U$, be the same as the above-mentioned ones. We recall the *Joukowski map* $x : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ given by

$$x(z) = \frac{a + b}{2} + \frac{b - a}{4}\left(z + \frac{1}{z}\right).$$

Denote by $\mathbb{T}$ and $\mathbb{D}$ the unit circle and the open unit disk in $\mathbb{C}$, respectively. The preimage $x^{-1}(U\backslash\Gamma)$ of $U\backslash\Gamma$ under the Joukowski map x has two connected components $V^e \subset \mathbb{C}\backslash\overline{\mathbb{D}}$ and $V^i \subset \mathbb{D}$, whose common boundary is the unit circle $\mathbb{T}$ (see Figure 1). The exterior neighborhood V^e is mapped to the interior neighborhood V^i under $\iota : z \mapsto 1/z$.

By Proposition 6.1 and Lemma 4.2, the function $W_{0,1}(x) \in \mathcal{O}(U\backslash\Gamma)$ can be expressed in the following way:

$$2W_{0,1}(x) = -Q(x) + M(x)\sqrt{(x - a)(x - b)}, \tag{6.6}$$

where $M(x) \in \mathcal{O}^*(U)$, and $\Gamma = [a, b] \subset \mathbb{R}$ is the support of the large-N spectral distribution μ. Let $W_{0,1}(x(z))$ be the pullback of $W_{0,1}(x) \in \mathcal{O}(U\backslash\Gamma)$ under the biholomorphism $x|_{V^e} : V^e \to U\backslash\Gamma$. Since

$$\sqrt{(x(z) - a)(x(z) - b)} = \frac{b - a}{4}\left(z - \frac{1}{z}\right),$$

considering (6.6), the function $W_{0,1}(x(z))$ has an analytic continuation to

$$V = x^{-1}(U) = V^e \cup V^i \cup \mathbb{T},$$

which is denoted by $W_{0,1}(x(z)) \in \mathcal{O}(V)$.

Consider the open neighborhood $\Upsilon = \mathbb{D}\backslash\overline{V^i}$ of the point $z = 0$. The Riemann surface

$$\Sigma = \mathbb{CP}^1\backslash\Upsilon,$$

which is homeomorphic to a disk, is called the *spectral curve* of the model. Let

$$x : \Sigma \to \mathbb{CP}^1, \quad x \in \mathcal{M}(\Sigma),$$

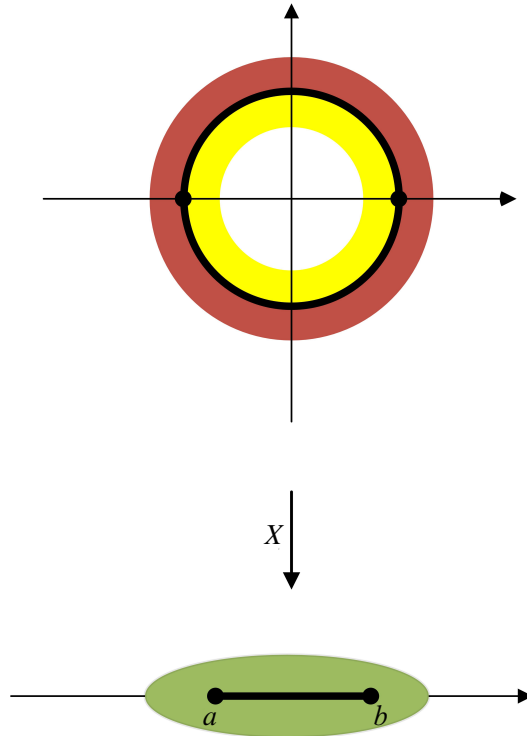**Figure 1.** Illustration of the Joukowski map in the case where the boundary of U is an ellipse with the foci $x = \mathfrak{a}$ and $x = \mathfrak{b}$. The open neighborhoods U, V^e, and V^i are colored green, brown, and yellow, respectively.

be the restriction of the Joukowski map to Σ. The map $x|_V : V \subset \Sigma \to U$ is a two-sheeted ramified covering map with the ramification points

$$\mathfrak{R} = \{z = 1, z = -1\},$$

and the branch points $x = \mathfrak{a}$, $x = \mathfrak{b}$. In addition, the spectral curve Σ is equipped with a *local* biholomorphic involution

$$\iota : V \to V$$
$$z \mapsto 1/z,$$

which satisfies $x \circ \iota = x$.

Using the Schwinger–Dyson equations (5.13) recursively, it can be shown [20] that each $W_{g,n}(x, x_I)$ for fixed $(x_i)_{i \in I} \in \mathbb{C} \backslash \Gamma$, initially defined as a holomorphic function on $\mathbb{C} \backslash \Gamma$, has a meromorphic continuation $W_{g,n}(x(z_1), x_I) \in \mathcal{M}(\Sigma)$ to Σ. By

doing the same process for the other arguments x_i, $i \in I$, of $W_{g,n}(x(z_1), x_1, \ldots, x_{n-1})$, one gets a meromorphic function $W_{g,n}(x(z_1), \ldots, x(z_n))$ on Σ^n.

Let $K_\Sigma \to \Sigma$ be the canonical line bundle, i.e., the holomorphic cotangent bundle, on the spectral curve Σ. Denote by $\pi_i : \Sigma^n \to \Sigma$ the projection map onto the i-th component. Let

$$K_\Sigma^{\boxtimes n} := (\pi_1^* K_\Sigma) \otimes \cdots \otimes (\pi_n^* K_\Sigma)$$

be the n-times external tensor product of K_Σ, where $\pi_i^* K_\Sigma$ denotes the pullback of K_Σ under π_i. The sections of the holomorphic line bundle $K_\Sigma^{\boxtimes n} \to \Sigma^n$ are referred to as the *differentials of degree n* over Σ^n. A differential of degree n is called *symmetric* if it is invariant under the natural action of the symmetric group $\mathfrak{S}_n$ on the line bundle $K_\Sigma^{\boxtimes n} \to \Sigma^n$.

In the theory of (blobbed) topological recursion [4, 6, 21], one constructs meromorphic symmetric differentials

$$\omega_{g,n}(z_1, \ldots, z_n) = W_{g,n}(x(z_1), \ldots, x(z_n)) dx(z_1)\, dx(z_2) \cdots dx(z_n)$$
$$+ \delta_{g,0}\delta_{n,2} \widehat{B}_0(z_1, z_2) \tag{6.7}$$

of degree n from the meromorphic functions $W_{g,n}(x(z_1), \ldots, x(z_n))$, where $dx(z_i)$ denotes the pullback $\pi_i^*(dx)$ of the 1-form dx under $\pi_i : \Sigma^n \to \Sigma$. In (6.7), the bidifferential, i.e., differential of degree two,

$$\widehat{B}_0(z_1, z_2) = \frac{dx(z_1)\, dx(z_2)}{[x(z_1) - x(z_2)]^2} \tag{6.8}$$

is the pullback of the *fundamental symmetric bidifferential of the second kind with biresidue 1* over $\mathbb{CP}^1 \times \mathbb{CP}^1$,[18] i.e.,

$$B_0(x_1, x_2) = \frac{dx_1\, dx_2}{(x_1 - x_2)^2},$$

under the map $(x, x) : \Sigma \times \Sigma \to \mathbb{CP}^1 \times \mathbb{CP}^1$. A couple (g, n) is called *stable* if $2 - 2g - n < 0$, i.e., $(g, n) \neq (0, 1), (0, 2)$.

---

[18]For a Torelli marked closed Riemann surface X of genus $g \geqslant 0$, the fundamental symmetric bidifferential $B(p, q)$ of the second kind with biresidue 1 over X^2 (also referred to as the *Bergman kernel*, in some references) is uniquely characterized by the following conditions (see, e.g., [23, 30, 31] for details):

- $B \in H^0(X^2, K_X^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2}$, and $\mathrm{Bires}|_\Delta B = 1$, where $\Delta \subset X^2$ denotes the diagonal divisor.

- For each $p \in X$, the 1-form $B(p, \cdot)$ has vanishing a_i-periods, where $\{a_i, b_i\}_{i=1}^g$ is the symplectic basis of $H_1(X, \mathbb{Z})$ specifying the Torelli making of X.

Before continuing, we introduce several operators in the following which are used in our investigation of the differentials $\omega_{g,n}(z_1, \ldots, z_n)$. Denote by $\mathcal{P}_+$ and $\mathcal{P}_-$ the orthogonal idempotents corresponding to the involution $\iota : V \to V$, given by

$$\mathcal{P}_+ = \frac{1}{2}(\mathbb{1} + \iota^*) \quad \text{and} \quad \mathcal{P}_- = \frac{1}{2}(\mathbb{1} - \iota^*), \tag{6.9}$$

respectively, where ι^* denotes the pullback under ι. Let

$$\widehat{\mathcal{P}}_\pm = 2\mathcal{P}_\pm. \tag{6.10}$$

The domain of the operators $\mathcal{P}_\pm$ can be $\mathcal{M}(V)$ or $\mathcal{Q}(V)$, depending on the context which they are used in. For a fixed $\varepsilon > 0$, consider the closed counter-clockwise-oriented contour

$$\gamma = \{z \in \mathbb{C} \mid |z| = 1 + \varepsilon\} \subset V^e.$$

Denote by $\widehat{\mathrm{F}} \subset \mathcal{Q}(V)$ the subspace of meromorphic 1-forms on $V \subset \Sigma$ which do not have poles on γ. Denote by $\Omega_{\mathrm{inv}}(V)$ (resp., $\mathcal{O}_{\mathrm{inv}}(V)$) the subspace of holomorphic 1-forms (resp., functions) on $V \subset \Sigma$ which are invariant under the involution ι^*. Let $T_{0,2}(x(z), x(\zeta))$, $z, \zeta \in V$, be the pullback of $T_{0,2}(\xi, \eta)$ under the map

$$(x, x) : V \times V \to U \times U.$$

Consider the following integral operator:

$$\widehat{\mathcal{O}}\phi(z) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \widehat{R}(z, \zeta)\, \phi(\zeta), \quad \widehat{\mathcal{O}} : \widehat{\mathrm{F}} \to \Omega_{\mathrm{inv}}(V), \tag{6.11}$$

with the kernel

$$\widehat{R}(z, \zeta) = d_z\, T_{0,2}(x(z), x(\zeta)) \in \Gamma(V \times V, \Omega \boxtimes \mathcal{O}),$$

where d_z denotes the exterior derivative operator acting on the first argument. The operator $\widehat{\mathcal{O}}$ is closely related to the master operator $\mathcal{O}$ in the sense that, for each fixed $z_I \in \Sigma^{n-1}$, we have

$$\mathcal{O}\, W_{g,n}(x(z), x_I)\, dx = \widehat{\mathcal{O}}\omega_{g,n}(z, z_I),$$

where $\mathcal{O}\, W_{g,n}(x(z), \cdot) \in \mathcal{O}_{\mathrm{inv}}(V)$ denotes the pullback of $\mathcal{O}\, W_{g,n}(x, \cdot)$ under $x : V \to U$.

7. Fundamental bidifferential $\omega_{0,2}(z, \zeta)$

As the next step, we investigate the *large-N spectral covariance* $W_{0,2}(x_1, x_2)$ of the model. Fix $\zeta \in \Sigma \backslash \overline{V^i}$. Denote by $W_{0,2}(x(z), x(\zeta))$, $z \in \Sigma$, the meromorphic continuation of the pullback of $W_{0,2}(x_1, x_2)$ under the biholomorphism $x|_{V^e} : V^e \to U \backslash \Gamma$

to Σ. From the definition of $W_{0,2}(x_1, x_2)$, we see that the possible singularities of the function $W_{0,2}(x(z), x(\zeta))$ can only occur in

$$\Re \cup V^i \subset \Sigma.$$

To investigate poles of $W_{0,2}(x(z), x(\zeta))$, we consider the rank two Schwinger–Dyson equation to leading order in N, i.e., equation (5.13) for $(g, n) = (0, 2)$, given by

$$2W_{0,1}(x_1)W_{0,2}(x_1, x_2) + \oint_{C_\Gamma} \frac{d\xi}{2\pi i} \frac{W_{0,1}(\xi)}{(x_1 - \xi)(x_2 - \xi)^2}$$

$$+ \oint_{C_\Gamma} \frac{d\xi}{2\pi i} \frac{1}{x_1 - \xi} \left([\partial_\xi T_{0,1}(\xi) + \mathcal{O} W_{0,1}(\xi)] W_{0,2}(\xi, x_2) + [\mathcal{O} W_{0,2}(\xi, x_2)] W_{0,1}(\xi) \right)$$

$$= 0, \tag{7.1}$$

for $x_1 \in U \backslash \Gamma$, and fixed $x_2 \in \mathbb{C} \backslash \Gamma$. We rewrite (7.1) in the following form:

$$- \left[2W_{0,1}(x_1) + \partial_{x_1} T_{0,1}(x_1) + \mathcal{O} W_{0,1}(x_1) \right] W_{0,2}(x_1, x_2)$$

$$= [\mathcal{O} W_{0,2}(x_1, x_2)] W_{0,1}(x_1) + \partial_{x_2} \left(\frac{W_{0,1}(x_1) - W_{0,1}(x_2)}{x_1 - x_2} \right) + P_{0,2}(x_1, x_2), \tag{7.2}$$

where $P_{0,2}(x_1, \cdot) \in \mathcal{O}(U)$. By (6.6), the function

$$2W_{0,1}(x(z)) + \partial_x T_{0,1}(x(z)) + \mathcal{O} W_{0,1}(x(z))$$

has a simple zero at the ramification points $\Re$. Therefore, considering equation (7.2), $W_{0,2}(x(z), x(\zeta))$ has a simple pole at $\Re$.

By considering (7.2) as $x_1 \to s \pm i\varepsilon$, $s \in \Gamma^\circ$, it can be shown [4] that

$$\lim_{\varepsilon \to 0^+} \left[W_{0,2}(s+i\varepsilon, x_2) + W_{0,2}(s-i\varepsilon, x_2) \right] + \mathcal{O} W_{0,2}(s, x_2) + \frac{1}{(s-x_2)^2} = 0 \quad \forall s \in \Gamma^\circ. \tag{7.3}$$

Using the *identity theorem for holomorphic functions* and the *Riemann's removable singularities theorem*, we deduce from (7.3) that

$$\widehat{\mathcal{P}}_+ W_{0,2}(x(z), x(\zeta)) + \mathcal{O} W_{0,2}(x(z), x(\zeta)) + \frac{1}{[x(z) - x(\zeta)]^2} = 0 \tag{7.4}$$

for all $z \in V \subset \Sigma$. Considering (7.4), we get the following equation:

$$\widehat{\mathcal{P}}_+ \widetilde{\omega}_{0,2}(z, \zeta) + \widehat{\mathcal{O}} \widetilde{\omega}_{0,2}(z, \zeta) + \widehat{B}_0(z, \zeta) = 0, \tag{7.5}$$

for the bidifferential

$$\widetilde{\omega}_{0,2}(z, \zeta) = W_{0,2}(x(z), x(\zeta)) \, dx(z) \, dx(\zeta),$$

where $\widehat{B}_0(z, \zeta)$ is given by (6.8).

Consider the function

$$F_\zeta(z) = W_{0,2}(x(z), x(\zeta))x'(z)x'(\zeta),$$

where $x'(z) \equiv \partial_z x(z)$. We have

$$\widetilde{\omega}_{0,2}(z, \zeta) = F_\zeta(z) \, dz \, d\zeta.$$

In the remaining part of this section, we consider the following set of assumptions and derive an explicit expression for $F_\zeta(z)$.

Hypothesis 7.1. (i) *For each fixed $\zeta \in \mathbb{C}\backslash\overline{\mathbb{D}}$, the function $F_\zeta(z)$ has a meromorphic continuation to the whole $\mathbb{CP}^1$.*

(ii) *The support Γ of the large-N spectral distribution μ of the model is of the form $\Gamma = [-\mathfrak{b}, \mathfrak{b}] \subset \mathbb{R}$.*[19]

To get an explicit expression for the meromorphic function $F_\zeta(z)$, $z \in \mathbb{CP}^1$, we find the principal part of the germ of $F_\zeta(z)$ at its poles on $\mathbb{CP}^1$ and then analyze the corresponding *Mittag–Leffler problem*. More precisely, we consider the following long exact sequence of Čech cohomology groups:

$$0 \to H^0(\mathbb{CP}^1, \mathcal{O}) \to H^0(\mathbb{CP}^1, \mathcal{M}) \to H^0(\mathbb{CP}^1, \mathcal{M}/\mathcal{O}) \to H^1(\mathbb{CP}^1, \mathcal{O}) \to \cdots.$$

It is well known that

$$H^0(\mathbb{CP}^1, \mathcal{O}) = \mathbb{C} \quad \text{and} \quad H^1(\mathbb{CP}^1, \mathcal{O}) = 0.$$

Therefore, given a section $\tilde{f} \in H^0(\mathbb{CP}^1, \mathcal{M}/\mathcal{O})$, there exists a meromorphic function

$$f \in H^0(\mathbb{CP}^1, \mathcal{M}),$$

unique up to a constant, whose local singular behavior is given by $\tilde{f}$.

The meromorphic function $F_\zeta(z)$ does not have any poles in $\mathbb{CP}^1\backslash\mathbb{D}$ because the simple zeros of $x'(z)$ at the ramification points $\mathfrak{R}$ cancel out the simple poles of $W_{0,2}(x(z), x(\zeta))$. We use (7.5) to analyze the possible poles of $F_\zeta(z)$ in $\mathbb{D}$. Rewrite the function $T_{0,2}(\xi, \eta)$, given by (6.2), in the following form:

$$T_{0,2}(\xi, \eta) = \sum_{k=1}^{d-1} \xi^k \left(\sum_{m=1}^{d-k} v_{k,m} \, \eta^m \right), \tag{7.6}$$

---

[19]Our approach for analyzing the function $F_\zeta(z)$ has a straightforward extension to the general case of $\Gamma = [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R}$. However, the computational part becomes more cumbersome.

where each $v_{k,m}$ is a linear combination of the Boltzmann weights $t^{(0)}_{\ell_1,\ell_2}$, $(\ell_1,\ell_2) \in \mathfrak{L}_{\text{cylinder}}$. We have

$$\hat{\mathcal{O}}\tilde{\omega}_{0,2}(z,\zeta) = \left(1 - \frac{1}{z^2}\right) \sum_{k=1}^{d-1} b_k(\zeta)\left(z + \frac{1}{z}\right)^{k-1} dz \, d\zeta, \qquad (7.7)$$

where, for each $1 \leqslant k \leqslant d-1$,

$$b_k(\zeta) = \frac{k}{2\pi i} \sum_{m=1}^{d-k} v_{k,m}(\mathfrak{b}/2)^{k+m} \oint_\gamma \left(\tau + \frac{1}{\tau}\right)^m F_\zeta(\tau) \, d\tau.$$

Considering (7.5) and (7.7), we get the following equation satisfied by $F_\zeta(z)$ on $\mathbb{C}\backslash\{0\}$:

$$F_\zeta(z) = \frac{1}{z^2} F_\zeta(\iota(z)) - \left[\left(1 - \frac{1}{z^2}\right)\sum_{k=1}^{d-1} b_k(\zeta)\left(z + \frac{1}{z}\right)^{k-1}\right] - \frac{x'(z)x'(\zeta)}{[x(z) - x(\zeta)]^2}.$$
$$(7.8)$$

We recall the following basic fact from the theory of *projective connections* on Riemann surfaces (see, e.g., [31]). Let $D \subset \mathbb{C}$ be an open neighborhood of a point $p \in \mathbb{C}$. Let f be a biholomorphism on D. Consider the holomorphic function $E_f(v,w)$ on $D \times D$ given by

$$E_f(v,w) = \frac{f'(v)f'(w)}{[f(v) - f(w)]^2}, \qquad v, w \in D.$$

Let $\hat{v} = v - p$, and $\hat{w} = w - p$. By expanding $E_f(v,w)$ as a series in $\hat{v}$ and $\hat{w}$, one gets

$$\frac{f'(v)f'(w)}{[f(v) - f(w)]^2} = \frac{1}{(v-w)^2} + \frac{1}{6}(Sf)(p) + H(\hat{v},\hat{w}), \qquad (7.9)$$

as $\hat{v}, \hat{w} \to 0$, where

$$(Sf)(p) = \frac{f'''(p)}{f'(p)} - \frac{3}{2}\left(\frac{f''(p)}{f'(p)}\right)^2$$

is called the *Schwarzian derivative* of f, and $H(\hat{v},\hat{w})$ is a sum of terms in $\hat{v}$ and $\hat{w}$ of strictly positive degree.

Considering (7.8), the function $F_\zeta(z)$ has a pole of order two at $z = \iota(\zeta)$. Using (7.9), we have

$$-\frac{x'(z)x'(\zeta)}{[x(z) - x(\zeta)]^2} = -\frac{x'(\zeta)}{x'(\iota(\zeta))} \cdot \frac{x'(z)x'(\iota(\zeta))}{[x(z) - x(\iota(\zeta))]^2}$$
$$= -\frac{x'(\zeta)}{x'(\iota(\zeta))}\left(\frac{1}{[z - \iota(\zeta)]^2} + O(1)\right),$$

as $z \to \iota(\zeta)$. Thus, the principal part of the germ of $F_\zeta(z)$ at $z = \iota(\zeta)$ is given by

$$Q_1(z, \zeta) = -\frac{x'(\zeta)}{x'(\iota(\zeta))} \cdot \frac{1}{[z - \iota(\zeta)]^2} = \frac{1}{(\zeta z - 1)^2}. \tag{7.10}$$

In addition, by (7.8), the function $F_\zeta(z)$ has a pole at $z=0$. Since $W_{0,2}(x(z), x(\zeta))$ has a zero of order 2 at $z = \infty$, the term $F_\zeta(\iota(z))/z^2$, in (7.8), is regular at $z = 0$. Therefore, the principal part of the germ of $F_\zeta(z)$ at $z = 0$ is the same as the principal part of the Laurent polynomial

$$-\left[\left(1 - \frac{1}{z^2}\right) \sum_{k=1}^{d-1} b_k(\zeta)\left(z + \frac{1}{z}\right)^{k-1}\right]$$

and is given by

$$Q_2(z, \zeta) = -\sum_{k=1}^{d-1} \frac{b_k(\zeta)}{z^{k+1}} \sum_{r=0}^{\left[\frac{k-1}{2}\right]} c_{k,r} z^{2r},$$

where

$$c_{k,r} = \frac{2r - k}{k}\binom{k}{r}.$$

By the Mittag–Leffler theorem, we have

$$F_\zeta(z) = Q_1(z, \zeta) + Q_2(z, \zeta) + c(\zeta), \quad c(\zeta) \in H^0(\mathbb{CP}^1, \mathcal{O}).$$

Since the function $F_\zeta(z)$ has a zero of order two at $z = \infty$, the constant function $c(\zeta)$ should be equal to zero. To get an explicit expression for $F_\zeta(z)$ in terms of z, ζ, it suffices to find $b_k(\zeta)$'s, as a function of ζ, explicitly. Note that each $b_k(\zeta)$ is a linear combination of the Fourier coefficients

$$a_n(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{F_\zeta(z)}{z^{n+1}}\, dz, \quad n \in \mathbb{Z},$$

of the restriction of $F_\zeta(z)$ to γ. Considering the degree of the Laurent polynomial $Q_2(z, \zeta)$, it suffices to find only $x_i = a_{-i-1}(\zeta)$, $1 \leqslant i \leqslant d - 1$. For each $1 \leqslant i \leqslant d - 1$, the identity

$$a_{-i-1}(\zeta) = \frac{1}{2\pi i}\oint_\gamma [Q_1(z, \zeta) + Q_2(z, \zeta)]z^i\, dz$$

leads to a linear equation of the following form satisfied by the x_i's:

$$\sum_{j=1}^{d-1} C_{ij}x_j = \frac{i}{\zeta^{i+1}},$$

where the coefficients C_{ij} depend only on $\mathfrak{b}$, and $t^{(0)}_{\ell_1,\ell_2}$, $(\ell_1, \ell_2) \in \mathfrak{L}_{\text{cylinder}}$.

We assume that the given values to the Boltzmann weights $t_\ell^{(0)}$ (or, equivalently, the formal parameters α_n, $3 \leqslant n \leqslant d$) of the model are such that the above-mentioned matrix (C_{ij}) is invertible. Hence, we get the explicit expression of

$$F(z, \zeta) = Q_1(z, \zeta) + Q_2(z, \zeta) \tag{7.11}$$

in terms of z, ζ, which depends on the support $\Gamma = [-\mathfrak{b}, \mathfrak{b}]$ of the large-N spectral distribution μ, and the Boltzmann weights $t_{\ell_1, \ell_2}^{(0)}$, $(\ell_1, \ell_2) \in \mathfrak{L}_{\text{cylinder}}$. Since the function

$$W_{0,2}(x(z), x(\zeta))x'(z)x'(\zeta) \tag{7.12}$$

is symmetric on its initial domain of definition $(\Sigma \backslash \overline{V^i}) \times (\Sigma \backslash \overline{V^i})$, the function $F(z, \zeta)$, given by (7.11), gives, indeed, the presumed meromorphic continuation of (7.12) to $\mathbb{CP}^1 \times \mathbb{CP}^1$.[20] From now on, we consider the restriction of $F(z, \zeta)$ to $\Sigma \times \Sigma$.

The bidifferential

$$\widetilde{\omega}_{0,2}(z, \zeta) = F(z, \zeta)\, dz\, d\zeta$$

over $\Sigma \times \Sigma$ has only a pole of order 2 at

$$\widetilde{\Delta} = \{(p, \iota(p)) \mid p \in V \subset \Sigma\} \subset \Sigma \times \Sigma.$$

Considering (7.10), the singular behavior of $\widetilde{\omega}_{0,2}(z, \zeta)$ at $\widetilde{\Delta}$ is as follows:

$$\widetilde{\omega}_{0,2}(z, \zeta) = -\frac{x'(\zeta)}{x'(\iota(\zeta))} \left(\frac{1}{[z - \iota(\zeta)]^2} + O(1) \right) dz\, d\zeta, \quad \text{as } z \to \iota(\zeta).$$

In addition, the polar divisor of the bidifferential

$$\widehat{B}_0(z, \zeta) = \frac{dx(z)\, dx(\zeta)}{[x(z) - x(\zeta)]^2}$$

over $\Sigma \times \Sigma$ is given by $2\Delta + 2\widetilde{\Delta}$, where

$$\Delta = \{(p, p) \mid p \in \Sigma\} \subset \Sigma \times \Sigma$$

denotes the diagonal divisor. The singular behavior of $\widehat{B}_0(z, \zeta)$ at Δ and $\widetilde{\Delta}$ is given by

$$\widehat{B}_0(z, \zeta) = \left(\frac{1}{[z - \zeta]^2} + O(1) \right) dz\, d\zeta, \quad \text{as } z \to \zeta,$$

---

[20]For the classical formal 1-Hermitian matrix models, in the one-cut regime, the meromorphic continuation of (7.12) to $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given by $Q_1(z, \zeta)$, and it is universal in the sense that it does not depend on the formal parameters of the model (see [20, Section 3.2.1]).

and

$$\hat{B}_0(z,\zeta) = \frac{\iota^*(d\zeta)}{d\zeta}\left(\frac{1}{[z-\iota(\zeta)]^2} + O(1)\right) dz\, d\zeta, \quad \text{as } z \to \iota(\zeta),$$

respectively. Therefore, the symmetric bidifferential

$$\omega_{0,2}(z,\zeta) = \tilde{\omega}_{0,2}(z,\zeta) + \hat{B}_0(z,\zeta) \tag{7.13}$$

has only a pole of order 2 at Δ, and its singular behavior is given by

$$\omega_{0,2}(z,\zeta) = \left(\frac{1}{[z-\zeta]^2} + O(1)\right) dz\, d\zeta, \quad \text{as } z \to \zeta.$$

Succinctly,

$$\omega_{0,2}(z,\zeta) \in H^0(\Sigma^2, K_\Sigma^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2} \tag{7.14}$$

and

$$\text{Bires}|_\Delta \omega_{0,2}(z,\zeta) = 1. \tag{7.15}$$

8. Local Cauchy kernel

Let $\tilde{V}$ be the simply connected domain which one gets by cutting the neighborhood $V \subset \Sigma$ along a radial line $\{z = re^{i\theta} \mid r \in \mathbb{R}_+, \text{ fixed } \theta\}$. Fix $p_0 \in \tilde{V}$. Considering (7.14) and (7.15), one gets a *local Cauchy kernel*

$$G(z,\zeta) = \int_{p_0}^\zeta \omega_{0,2}(z,\tau)$$

by integrating the 1-form $\omega_{0,2}(\cdot, \tau)$ on $\tilde{V}$ [6]. We have

$$G(z,\zeta) \in \Gamma(\Sigma \times V, \Omega \boxtimes \mathcal{O}(\Delta))$$

and

$$G(z,\zeta) = \left(\frac{1}{z-\zeta} + O(1)\right) dz, \quad \text{as } z \to \zeta. \tag{8.1}$$

Denote by $[\phi]_p \in \mathcal{Q}_p/\Omega_p$ the image of the germ of a meromorphic 1-form $\phi \in \mathcal{Q}(\Sigma)$ at a point $p \in \Sigma$ under the projection map $\pi_p : \mathcal{Q}_p \to \mathcal{Q}_p/\Omega_p$. Denote the set of poles of a meromorphic 1-form $\phi \in \mathcal{Q}(\Sigma)$ by $\mathfrak{P}(\phi)$. Let $\tilde{F} \subset \mathcal{Q}(V)$ be the subspace of meromorphic 1-forms which have finitely many poles on $V \subset \Sigma$. Consider the operator $\mathcal{K} : \tilde{F} \to \mathcal{Q}(\Sigma)$ given by

$$\mathcal{K}\phi(z) = \sum_{p \in V} \text{Res}_{\zeta=p} G(z,\zeta)\phi(\zeta) = \frac{1}{2\pi i} \sum_{p \in V} \oint_{|\zeta-p|=\varepsilon} G(z,\zeta)\phi(\zeta). \tag{8.2}$$

Note that, for each $\phi \in \widetilde{\mathrm{F}}$, a point $p \in V$ contributes to the summation in (8.2) iff $p \in \mathfrak{P}(\phi)$. In addition, by (8.1), we have

$$[\mathcal{K}\phi]_p = [\phi]_p \quad \forall p \in \mathfrak{P}(\mathcal{K}\phi) = \mathfrak{P}(\phi). \tag{8.3}$$

We use the same notation $\mathcal{K}$ to denote the operator $r_{\Sigma,V}\mathcal{K} : \widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}$, where

$$r_{\Sigma,V} : \mathcal{Q}(\Sigma) \to \mathcal{Q}(V)$$

is the restriction map. By (8.3), we have

$$(\mathbb{1} - \mathcal{K})(\widetilde{\mathrm{F}}) \subset \Omega(V).$$

Therefore, $\mathcal{K} : \widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}$ is an idempotent operator, and we have the decomposition

$$\widetilde{\mathrm{F}} = \mathcal{K}(\widetilde{\mathrm{F}}) \oplus (\mathbb{1} - \mathcal{K})(\widetilde{\mathrm{F}}).$$

Consider the operator $\widetilde{\mathcal{K}} : \widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}$ given by

$$\widetilde{\mathcal{K}}\phi(z) = \sum_{p \in V} \operatorname*{Res}_{\zeta=p} G(z, \iota(\zeta))\phi(\zeta).$$

The following simple lemma will be used several times later.

Lemma 8.1. *The operator $\mathcal{K} : \widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}$ can be decomposed as*

$$\mathcal{K} = \frac{1}{2}(\mathcal{K} + \widetilde{\mathcal{K}})\mathcal{P}_+ + \frac{1}{2}(\mathcal{K} - \widetilde{\mathcal{K}})\mathcal{P}_-,$$

where the orthogonal idempotents $\mathcal{P}_\pm : \widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}_\pm$ are given by (6.9).

Proof. We rewrite the operator $\mathcal{K}$ in the following form:

$$\mathcal{K} = \frac{1}{2}[(\mathcal{K} + \widetilde{\mathcal{K}}) + (\mathcal{K} - \widetilde{\mathcal{K}})][\mathcal{P}_+ + \mathcal{P}_-].$$

Since $\widetilde{\mathcal{K}} = \mathcal{K}\iota^*$, we have

$$(\mathcal{K} \pm \widetilde{\mathcal{K}})\mathcal{P}_\mp = 0. \qquad \blacksquare$$

In the remaining part of this section, we try to explain some elementary aspects of the blobbed topological recursion formula, which will be discussed in the next section, in a simpler setup. Consider the operator

$$\mathcal{T} = \widehat{\mathcal{P}}_+ + \widehat{\mathcal{O}}, \quad \mathcal{T} : \widehat{\mathrm{F}} \to \mathcal{Q}(V),$$

where the operators $\widehat{\mathcal{P}}_+$ and $\widehat{\mathcal{O}}$ are given by (6.10) and (6.11), respectively. Let $P \subset V$ be a fixed finite set of points in $V \subset \Sigma$ such that $P \cap \gamma = \emptyset$. Suppose we

are given a set of germs $s_i \in \mathfrak{Q}_{p_i}/\Omega_{p_i}$ at the points $p_i \in P$, $i = 1, \ldots, |P|$, and an ι-invariant holomorphic 1-form $\psi \in \Omega_{\mathrm{inv}}(V)$. Denote by $\mathrm{E} \subset H^0(\Sigma, K_\Sigma)$ the subspace of holomorphic 1-forms $\widetilde{\eta}$ on Σ whose restriction to $V \subset \Sigma$ satisfies the homogeneous equation

$$\mathcal{T}\widetilde{\eta} = 0, \quad \widetilde{\eta} \in \mathrm{E}.$$

Let $\mathrm{A} \subset \mathfrak{Q}(\Sigma)$ be the set of meromorphic 1-forms ϕ on Σ whose restriction to $V \subset \Sigma$ satisfies the inhomogeneous equation

$$\mathcal{T}\phi = \psi, \quad \phi \in \mathrm{A}, \tag{8.4}$$

and their singularities satisfy

$$\mathfrak{P}(\phi) = P \tag{8.5}$$

and

$$[\phi]_{p_i} = s_i \in \mathfrak{Q}_{p_i}/\Omega_{p_i} \quad \forall\, p_i \in P. \tag{8.6}$$

The set $\mathrm{A} \subset \mathfrak{Q}(\Sigma)$ is an affine space over the vector space E. In the following, we investigate the space A.

The operator $\mathcal{K}$ maps the space A to the meromorphic 1-form $\phi_0 \in \mathfrak{Q}(\Sigma)$, given by

$$\phi_0(z) = \sum_{p_i \in P} \operatorname*{Res}_{\zeta = p_i} G(z, \zeta)\widetilde{s}_i(\zeta), \tag{8.7}$$

where each $\widetilde{s}_i \in \mathfrak{Q}_{p_i}$ is a representative of $s_i \in \mathfrak{Q}_{p_i}/\Omega_{p_i}$, $\forall\, p_i \in P$. Using (8.4), we get

$$\mathcal{P}_+\phi \in \Omega_{\mathrm{inv}}(V) \quad \forall \phi \in \mathrm{A}. \tag{8.8}$$

Thus, by Lemma 8.1, we have

$$\phi_0 = \mathcal{K}\phi = \frac{1}{2}(\mathcal{K} - \widetilde{\mathcal{K}})\mathcal{P}_-\phi \quad \forall \phi \in \mathrm{A}. \tag{8.9}$$

Proposition 8.2. *The restriction of the 1-form $\phi_0 \in \mathfrak{Q}(\Sigma)$ to $V \subset \Sigma$ satisfies*

$$\mathcal{T}\phi_0 = 0.$$

Proof. Considering (7.5) and (7.13), since $\Omega_{\mathrm{inv}}(V) \subset \ker(\widehat{\mathcal{O}})$, we have

$$\mathcal{T}\omega_{0,2}(z, \zeta) = \widehat{B}_0(z, \zeta),$$

for each fixed $\zeta \in \widetilde{V}$. Thus,

$$\mathcal{T}G(z, \zeta) = \widehat{G}_0(z, \zeta), \tag{8.10}$$

where

$$\widehat{G}_0(z, \zeta) = \frac{dx(z)}{x(z) - x(\zeta)}$$

is the pullback of the canonical Cauchy kernel $\frac{dx}{x-y}$ over $\mathbb{CP}^1$ under the map

$$(x, x) : \Sigma \times \Sigma \to \mathbb{CP}^1 \times \mathbb{CP}^1.$$

Consider the operator $\mathcal{K}_0 : \widetilde{\mathbb{F}} \to \mathcal{Q}(\Sigma)$ given by

$$\mathcal{K}_0 \varphi(z) = \sum_{p \in V} \operatorname*{Res}_{\zeta = p} \widehat{G}_0(z, \zeta) \varphi(\zeta).$$

By (8.10), we have

$$\mathcal{T} \mathcal{K} = \mathcal{K}_0 \tag{8.11}$$

on the subspace $\mathbb{F} = \widetilde{\mathbb{F}} \cap \widehat{\mathbb{F}} \subset \mathcal{Q}(V)$ of meromorphic 1-forms which have finitely many poles on $V \subset \Sigma$, and their poles are not on the contour γ.

Let $\widetilde{A} = r_{\Sigma, V}(A)$. Considering (8.9) and (8.11), it suffices to show that

$$\widetilde{A} \subset \ker(\mathcal{K}_0). \tag{8.12}$$

By decomposing the operator $\mathcal{K}_0$ in the same way as in Lemma 8.1, using (8.8), we get

$$\mathcal{K}_0 \phi = \frac{1}{2}(\mathcal{K}_0 - \widetilde{\mathcal{K}}_0) \mathcal{P}_- \phi \quad \forall \phi \in \widetilde{A}.$$

On the other hand, we have $\mathcal{K}_0 = \widetilde{\mathcal{K}}_0$ because $\widehat{G}_0(z, \zeta) = \widehat{G}_0(z, \iota(\zeta))$. Thus, we get (8.12). ∎

Let

$$A_h = (\mathbb{1} - \mathcal{K})(A) \subset H^0(\Sigma, K_\Sigma)$$

be the image of A under the operator $\mathbb{1} - \mathcal{K}$. Considering Proposition 8.2, the space A_h consists of holomorphic 1-forms η on Σ whose restriction to $V \subset \Sigma$ satisfies

$$\mathcal{T} \eta = \psi, \quad \eta \in A_h.$$

Therefore, A_h is an affine subspace of $H^0(\Sigma, K_\Sigma)$ over the vector space E.

Lemma 8.3. *The holomorphic 1-form*

$$\eta_0(z) = -\frac{1}{2\pi i} \oint_{\partial \Sigma} G(z, \zeta) \psi(\zeta) \tag{8.13}$$

is an element of the affine subspace $A_h \subset H^0(\Sigma, K_\Sigma)$,[21] where the boundary $\partial \Sigma$ of the spectral curve Σ is assumed to have the induced orientation from Σ; i.e., it is clockwise oriented.

---

[21]We assume that $G(\cdot, \zeta)$ and $\psi(\zeta)$ have a continuous extension to $\overline{V} \subset \Sigma$.

Proof. By (8.10), the restriction of η_0 to $V \subset \Sigma$ satisfies

$$\mathcal{T}\eta_0(z) = -\frac{1}{2\pi i} \oint_{\partial\Sigma} \hat{G}_0(z,\zeta)\psi(\zeta).$$

In addition, we have

$$\frac{2}{2\pi i} \oint_{\partial\Sigma} \hat{G}_0(z,\zeta)\psi(\zeta) = \frac{1}{2\pi i} \oint_{\partial\Sigma} \hat{G}_0(z,\zeta)\psi(\zeta) + \frac{1}{2\pi i} \oint_{\iota(\partial\Sigma)} \hat{G}_0(z,\zeta)\psi(\zeta)$$
$$= \operatorname*{Res}_{\zeta=z} \hat{G}_0(z,\zeta)\psi(\zeta) + \operatorname*{Res}_{\zeta=\iota(z)} \hat{G}_0(z,\zeta)\psi(\zeta)$$
$$= -2\psi(z)$$

for each $z \in V \subset \Sigma$. Therefore, we get

$$\mathcal{T}\eta_0 = \psi. \qquad \blacksquare$$

Lemma 8.4. *In general, the subspace* $\mathrm{E} \subset H^0(\Sigma, K_\Sigma)$ *is non-trivial, and its dimension depends on the given values to the Boltzmann weights* $\mathrm{t}_{\frac{\ell}{\ell}}^{(0)}$ *(or, equivalently, the formal parameters* α_n, $3 \leqslant n \leqslant d$ *) of the model.*

Proof. For simplicity, we give a proof in the case where the support Γ of the large-N spectral distribution μ of the model is of the form $\Gamma = [-\mathfrak{b}, \mathfrak{b}] \subset \mathbb{R}$. Consider a holomorphic 1-form $\tilde{\eta} = f(z)\,dz \in \mathrm{E}$. We have

$$f(z) = \sum_{n=2}^{\infty} \frac{a_{-n}}{z^n}, \tag{8.14}$$

where

$$a_{-n} = \frac{1}{2\pi i} \oint_\gamma f(z) z^{n-1} \,\mathrm{d}z, \quad n \geqslant 2,$$

are the Fourier coefficients of the restriction of f to γ. Since

$$\mathcal{T}\tilde{\eta} = 0, \quad \text{on } V \subset \Sigma,$$

considering (7.6), we get

$$f(z) - \frac{1}{z^2} f\left(\frac{1}{z}\right) + \left(1 - \frac{1}{z^2}\right) \sum_{k=1}^{d-1} b_k \left(z + \frac{1}{z}\right)^{k-1} = 0, \tag{8.15}$$

on $V \subset \Sigma$, where, for each $1 \leqslant k \leqslant d-1$,

$$b_k = \frac{k}{2\pi i} \sum_{m=1}^{d-k} v_{k,m} (\mathfrak{b}/2)^{k+m} \oint_\gamma \left(\tau + \frac{1}{\tau}\right)^m f(\tau)\,\mathrm{d}\tau.$$

By substituting (8.14) into (8.15), we get $d - 1$ linear equations of the form

$$\sum_{j=1}^{d-1} C_{ij} x_j = 0,$$

satisfied by $x_i = a_{-i-1}$, $1 \leqslant i \leqslant d - 1$, where the coefficients C_{ij} depend only on $\mathfrak{b}$, and $t_{\ell_1,\ell_2}^{(0)}$, $(\ell_1, \ell_2) \in \mathfrak{L}_{\text{cylinder}}$. Thus, the dimension of the subspace $E \subset H^0(\Sigma, K_\Sigma)$ equals the dimension of the kernel of the matrix (C_{ij}). $\blacksquare$

Considering Proposition 8.2 and Lemmas 8.3 and 8.4, in general, a 1-form $\phi \in A$ can be written as

$$\phi = \phi_0 + \eta_0 + \tilde{\eta}$$

for some $\tilde{\eta} \in E$. From now on, we consider the following assumption.

Hypothesis 8.5. *The given values to the Boltzmann weights* $t_{\ell}^{(0)}$ *(or, equivalently, the formal parameters* α_n, $3 \leqslant n \leqslant d$ *) of the model are such that the subspace* $E \subset H^0(\Sigma, K_\Sigma)$ *of global holomorphic 1-forms* $\tilde{\eta}$, *whose restriction to* $V \subset \Sigma$ *satisfies the homogeneous equation*

$$\mathcal{T}\tilde{\eta} = 0,$$

is trivial.

Therefore, a 1-form $\phi \in \mathcal{Q}(\Sigma)$, satisfying (8.4)–(8.6), is uniquely given by

$$\phi = \phi_0 + \eta_0,$$

where the 1-forms ϕ_0 and η_0 are defined by (8.7) and (8.13), respectively.

9. Blobbed topological recursion formula

In this section, we show that all the stable coefficients $W_{g,n}(x_1, \ldots, x_n)$ of the large N expansion of the correlators of our model can be computed recursively using the blobbed topological recursion formula [4]. In the following, without further explicit mention, a couple (g, n) is assumed to be stable, i.e., $(g, n) \neq (0, 1), (0, 2)$.

Let $f(x) \in \mathcal{O}(\mathbb{C}\backslash\Gamma)$ be a holomorphic function on $\mathbb{C}\backslash\Gamma$ with a jump discontinuity on Γ. We denote by $\delta f(s)$ and $\sigma f(s)$ the functions given by

$$\delta f(s) = \lim_{\varepsilon \to 0^+} [f(s + i\varepsilon) - f(s - i\varepsilon)], \quad s \in \Gamma^{\circ} \subset \mathbb{R}$$

and

$$\sigma f(s) = \lim_{\varepsilon \to 0^+} [f(s + i\varepsilon) + f(s - i\varepsilon)], \quad s \in \Gamma^{\circ} \subset \mathbb{R},$$

respectively. Using the rank n Schwinger–Dyson equation to order N^{3-2g-n}, given by (5.13), as $x \to s \pm i\varepsilon$, $s \in \Gamma^\circ$, we get

$$
\delta W_{0,1}(s)\{\sigma W_{g,n}(s, x_I) + \mathcal{O}W_{g,n}(s, x_I) + \partial_s V_{g,n}(s; x_I)\}
$$
$$
+ \delta W_{g,n}(s, x_I)\{\sigma W_{0,1}(s) + \mathcal{O}W_{0,1}(s) + \partial_s T_{0,1}(s)\}
$$
$$
+ \sum_{i \in I} \delta W_{g,n-1}(s, x_{I\setminus\{i\}})\left\{\sigma W_{0,2}(s, x_i) + \mathcal{O}W_{0,2}(s, x_i) + \frac{1}{(s - x_i)^2}\right\}
$$
$$
+ \sum_{\substack{J \subseteq I, 0 \leqslant f \leqslant g \\ (J,f) \neq (\emptyset,0),(I,g), \\ (I\setminus\{i\},g)}} \delta W_{f,|J|+1}(s, x_J)\{\sigma W_{g-f,n-|J|}(s, x_{I\setminus J})
$$
$$
+ \mathcal{O}W_{g-f,n-|J|}(s, x_{I\setminus J}) + \partial_s V_{g-f,n-|J|}(s; x_{I\setminus J})\}
$$
$$
+ \delta_{s,2}\{\sigma_{s,1} W_{g-1,n+1}(s, s, x_I) + \mathcal{O}W_{g-1,n+1}(s, s, x_I) + \partial_s V_{g-1,n+1}(s; s, x_I)\}
$$
$$
= 0, \tag{9.1}
$$

where $\delta_{s,2}$ (resp., $\sigma_{s,1}$) means that the operator δ (resp., σ) is acting on the second (resp., first) argument. In (9.1), the functions $V_{g,n}(x; x_I)$, called the *potentials for higher topologies* [4],[22] are given by

$$
V_{g,n}(x; x_I) = \delta_{g,2}\delta_{n,1}T_{2,1}(x) + \sum_{\substack{2 \leqslant k \leqslant 2\mathfrak{g} \\ 0 < h}} \sum_{\substack{K \vdash [\![2,k]\!] \\ 0 \leqslant f_1,\ldots,f_{[K]} \\ h+(k-1)-[K]+\sum_i f_i = g \\ J_1 \sqcup \cdots \sqcup J_{[K]} = I}} \oint_{C_\Gamma}\left\{\left[\prod_{r=2}^{k} \frac{d\xi_r}{2\pi i}\right]\right.
$$
$$
\left. \times \left[\frac{T_{h,k}(x, \xi_2, \ldots, \xi_k)}{(k-1)!} \prod_{i=1}^{[K]} W_{f_i,|K_i|+|J_i|}(\xi_{K_i}, x_{J_i})\right]\right\}. \tag{9.2}
$$

Let $\mathcal{V}_{g,n}(z; z_I)$ be the differential of degree $n - 1$ over Σ^n given by

$$
\mathcal{V}_{g,n}(z; z_I) = V_{g,n}(x(z); x(z_I)) \prod_{i \in I} dx(z_i).
$$

Consider the differential

$$
d_z \mathcal{V}_{g,n}(z; z_I) = \left[\partial_{x(z)} V_{g,n}(x(z); x(z_I))\right] dx(z) \prod_{i \in I} dx(z_i)
$$

of degree n over Σ^n. From now on, we consider a *fixed* $z_I \in \Sigma^{n-1}$. The restriction of the exact 1-form

$$
d_z \mathcal{V}_{g,n}(z; z_I) = \left[\partial_{x(z)} V_{g,n}(x(z); x(z_I))\right] dx(z)
$$

---

[22]We consider a slightly modified version of Equation (4.13) in [4] to get (9.2).

to $V \subset \Sigma$ is in $\Omega_{\mathrm{inv}}(V)$. Using (9.1), by induction on $2g + n - 2$, it can be shown [4] that each 1-form $\omega_{g,n}(z, z_I)$ satisfies

$$\mathcal{T}\omega_{g,n}(z, z_I) = -d_z \mathcal{V}_{g,n}(z; z_I)$$

on $V \subset \Sigma$.

By recasting the Schwinger–Dyson equations, one can show [4] that

$$\omega_{g-1,n+1}(z, \iota(z), z_I) + \sum_{J \subseteq I,\, 0 \leqslant f \leqslant g} \omega_{f,|J|+1}(z, z_J)\, \omega_{g-f,n-|J|}(\iota(z), z_{I \setminus J})$$

$$= \mathcal{Q}_{g,n}(z; z_I), \tag{9.3}$$

where $\mathcal{Q}_{g,n}(z; z_I)$ is a *local* holomorphic quadratic differential, i.e., a local section of $K_\Sigma^{\otimes 2} \to \Sigma$ over $V \subset \Sigma$, with double zeros at the ramification points $\mathfrak{R}$. One can rewrite (9.3) in the following form:

$$\mathcal{P}_- \omega_{g,n}(z, z_I) = \frac{1}{\widehat{\mathcal{P}}_- \omega_{0,1}(z)} \big[\mathcal{E}_{g,n}(z, \iota(z); z_I) + \tilde{\mathcal{Q}}_{g,n}(z; z_I) \big], \tag{9.4}$$

where

$$\mathcal{E}_{g,n}(z, \iota(z); z_I) = \omega_{g-1,n+1}(z, \iota(z), z_I)$$

$$+ \sum_{\substack{J \subseteq I,\, 0 \leqslant f \leqslant g \\ (J,f) \neq (\emptyset, 0),\, (I,g)}} \omega_{f,|J|+1}(z, z_J) \omega_{g-f,n-|J|}(\iota(z), z_{I \setminus J})$$

and

$$\tilde{\mathcal{Q}}_{g,n}(z; z_I) = \mathcal{P}_+ \omega_{g,n}(z, z_I) \widehat{\mathcal{P}}_+ \omega_{0,1}(z) - \mathcal{Q}_{g,n}(z; z_I).$$

Considering (6.6), the zeros of the 1-from $\widehat{\mathcal{P}}_- \omega_{0,1}(z)$ in $V \subset \Sigma$ occur only at the ramification points $\mathfrak{R}$, and their order is exactly two. Since the quadratic differential $\tilde{\mathcal{Q}}_{g,n}(z; z_I)$ has double zeros at $\mathfrak{R}$, the 1-form

$$\frac{1}{\widehat{\mathcal{P}}_- \omega_{0,1}(z)} \tilde{\mathcal{Q}}_{g,n}(z; z_I)$$

is holomorphic on V. Therefore, the singularities of $\omega_{g,n}(z, z_I)$ in V are the same as the 1-form

$$\frac{1}{\widehat{\mathcal{P}}_- \omega_{0,1}(z)} \mathcal{E}_{g,n}(z, \iota(z); z_I).$$

Using (9.4), by induction on $2g + n - 2$, it can be shown that the poles of the 1-form $\omega_{g,n}(z, z_I)$ occur only at the ramification points $\mathfrak{R}$.

Considering (9.4) and (8.9), we can use the local Cauchy kernel $G(z, \zeta)$ to construct a meromorphic 1-form

$$\widetilde{\omega}_{g,n}(z, z_I) = \frac{1}{2}(\mathcal{K} - \widetilde{\mathcal{K}})\left[\frac{1}{\widehat{\mathcal{P}}_{-\omega_{0,1}(z)}}\mathcal{E}_{g,n}(z, \iota(z); z_I)\right]$$

such that

$$\Phi_{g,n}(z; z_I) = \omega_{g,n}(z, z_I) - \widetilde{\omega}_{g,n}(z, z_I)$$

is a holomorphic 1-form on Σ, satisfying

$$\mathcal{T}\Phi_{g,n}(z; z_I) = -d_z \mathcal{V}_{g,n}(z; z_I) \quad \text{on } V \subset \Sigma.$$

The 1-form $\widetilde{\omega}_{g,n}(z, z_I)$ can be expressed in terms of the *recursion kernel* (see [6, 21])

$$K(z, \zeta) = \frac{1}{2}\frac{\int_{\iota(\zeta)}^{\zeta} \omega_{0,2}(z, \tau)}{\omega_{0,1}(\zeta) - \omega_{0,1}(\iota(\zeta))} \in \Gamma(\Sigma \times V, \Omega \boxtimes \Omega^{-1})$$

in the following form:

$$\widetilde{\omega}_{g,n}(z, z_I) = \sum_{p \in \mathfrak{R}} \operatorname*{Res}_{\zeta = p} K(z, \zeta)\mathcal{E}_{g,n}(\zeta, \iota(\zeta); z_I). \tag{9.5}$$

Considering Lemma 8.3 and Hypothesis 8.5, the 1-form $\Phi_{g,n}(z; z_I)$ is uniquely given by

$$\Phi_{g,n}(z; z_I) = \frac{1}{2\pi i}\oint_{\partial\Sigma} G(z, \zeta)d_\zeta \mathcal{V}_{g,n}(\zeta; z_I)$$

$$= -\frac{1}{2\pi i}\oint_{\partial\Sigma} \omega_{0,2}(z, \zeta)\mathcal{V}_{g,n}(\zeta; z_I). \tag{9.6}$$

Note that, for each stable (g, n), the right-hand side of (9.5) and (9.6) involves only $\omega_{g',n'}$ with

$$2g' + n' - 2 < 2g + n - 2.$$

Therefore, we have the following theorem.

Theorem 9.1. *For the random matrix geometries of type $(1, 0)$ with the distribution $d\rho = e^{-\mathcal{S}(D)}\,dD$, all the stable $\omega_{g,n}$, $2g + n - 2 > 0$, can be computed recursively, using the blobbed topological recursion formula given by*

$$\omega_{g,n}(z, z_I) = -\frac{1}{2\pi i}\oint_{\partial\Sigma} \omega_{0,2}(z, \zeta)\mathcal{V}_{g,n}(\zeta; z_I)$$

$$+ \sum_{p \in \mathfrak{R}} \operatorname*{Res}_{\zeta = p} K(z, \zeta)\mathcal{E}_{g,n}(\zeta, \iota(\zeta); z_I). \tag{9.7}$$

The initial data for the recursion relation (9.7) *is the 1-form* $\omega_{0,1}(z)$ *and the fundamental bidifferential* $\omega_{0,2}(z, \zeta)$.

References

[1] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*. Cambridge Stud. Adv. Math. 118, Cambridge University Press, Cambridge, 2010 Zbl 1184.15023 MR 2760897

[2] J. W. Barrett, Matrix geometries and fuzzy spaces as finite spectral triples. *J. Math. Phys.* **56** (2015), no. 8, article no. 082301, 25 pp. Zbl 1364.81157 MR 3455331

[3] J. W. Barrett and L. Glaser, Monte Carlo simulations of random non-commutative geometries. *J. Phys. A* **49** (2016), no. 24, article no. 245001, 27 pp. Zbl 1344.81111 MR 3512080

[4] G. Borot, Formal multidimensional integrals, stuffed maps, and topological recursion. *Ann. Inst. Henri Poincaré D* **1** (2014), no. 2, 225–264 Zbl 1297.05013 MR 3229944

[5] G. Borot, J. Bouttier, and E. Guitter, More on the $O(n)$ model on random maps via nested loops: loops with bending energy. *J. Phys. A* **45** (2012), no. 27, article no. 275206, 32 pp. Zbl 1246.82041 MR 2947230

[6] G. Borot, B. Eynard, and N. Orantin, Abstract loop equations, topological recursion and new applications. *Commun. Number Theory Phys.* **9** (2015), no. 1, 51–187 Zbl 1329.14074 MR 3339853

[7] G. Borot and A. Guionnet, Asymptotic expansion of β matrix models in the one-cut regime. *Comm. Math. Phys.* **317** (2013), no. 2, 447–483 Zbl 1344.60012 MR 3010191

[8] G. Borot, A. Guionnet, and K. K. Kozlowski, Large-N asymptotic expansion for mean field models with Coulomb gas interaction. *Int. Math. Res. Not. IMRN* **2015** (2015), no. 20, 10451–10524 Zbl 1332.82069 MR 3455872

[9] G. Borot and S. Shadrin, Blobbed topological recursion: Properties and applications. *Math. Proc. Cambridge Philos. Soc.* **162** (2017), no. 1, 39–87 Zbl 1396.14031 MR 3581899

[10] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, Planar diagrams. *Comm. Math. Phys.* **59** (1978), no. 1, 35–51 Zbl 0997.81548 MR 0471676

[11] A. H. Chamseddine and A. Connes, The spectral action principle. *Comm. Math. Phys.* **186** (1997), no. 3, 731–750 Zbl 0894.58007 MR 1463819

[12] A. H. Chamseddine, A. Connes, and M. Marcolli, Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.* **11** (2007), no. 6, 991–1089 Zbl 1140.81022 MR 2368941

[13] L. Chekhov, B. Eynard, and N. Orantin, Free energy topological expansion for the 2-matrix model. *J. High Energy Phys.* (2006), no. 12, article no. 053, 31 pp. Zbl 1226.81250 MR 2276699

[14] A. Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems* **9** (1989), no. 2, 207–220 Zbl 0718.46051 MR 1007407

[15] A. Connes, Geometry from the spectral point of view. *Lett. Math. Phys.* **34** (1995), no. 3, 203–238 Zbl 1042.46515 MR 1345552

[16] A. Connes, Gravity coupled with matter and the foundation of non-commutative geometry. *Comm. Math. Phys.* **182** (1996), no. 1, 155–176 Zbl 0881.58009 MR 1441908

[17] A. Connes, Noncommutative geometry and the standard model with neutrino mixing. *J. High Energy Phys.* (2006), no. 11, article no. 081, 19 pp. MR 2270385

[18] A. Connes, On the spectral characterization of manifolds. *J. Noncommut. Geom.* **7** (2013), no. 1, 1–82 Zbl 1287.58004 MR 3032810

[19] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ. 55, American Mathematical Society, Providence, RI; Hindustan Book Agency, New Delhi, 2008 Zbl 1159.58004 MR 2371808

[20] B. Eynard, *Counting surfaces*. Prog. Math. Phys. 70, Birkhäuser/Springer, Cham, 2016 Zbl 1338.81005 MR 3468847

[21] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.* **1** (2007), no. 2, 347–452 Zbl 1161.14026 MR 2346575

[22] B. Eynard and N. Orantin, Topological recursion in enumerative geometry and random matrices. *J. Phys. A* **42** (2009), no. 29, article no. 293001, 117 pp. Zbl 1177.82049 MR 2519749

[23] J. D. Fay, *Theta functions on Riemann surfaces*. Lecture Notes in Mathematics 352, Springer, Berlin, 1973 Zbl 0281.30013 MR 0335789

[24] E. Gesteau, Renormalizing Yukawa interactions in the standard model with matrices and noncommutative geometry. *J. Phys. A* **54** (2021), no. 3, article no. 035203, 18 pp. Zbl 1519.81341 MR 4209134

[25] H. Hessam, M. Khalkhali, N. Pagliaroli, and L. S. Verhoeven, From noncommutative geometry to random matrix theory. *J. Phys. A* **55** (2022), no. 41, article no. 413002, 38 pp. Zbl 1519.58005 MR 4492421

[26] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*. Princeton Math. Ser. 38, Princeton University Press, Princeton, NJ, 1989 Zbl 0688.57001 MR 1031992

[27] M. Marcolli, *Noncommutative cosmology*. World Scientific, Hackensack, NJ, 2018 Zbl 1401.81005 MR 3752173

[28] A. A. Migdal, Loop equations and 1N expansion. *Phys. Rept.* **102** (1983), no. 4, 199–290

[29] G. 't Hooft, A planar diagram theory for strong interactions. *Nucl. Phys. B* **72** (1974), no. 3, 461–473

[30] L. A. Takhtajan, Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. Current correlation functions. *Lett. Math. Phys.* **56** (2001), no. 3, 181–228 Zbl 1072.14521 MR 1855262

[31] A. N. Tyurin, On periods of quadratic differentials. *Russ. Math. Surv.* **33** (1978), no. 6, 169–221 Zbl 0429.30037

Shahab Azarfar
Department of Mathematics, University of Western Ontario, 1151 Richmond Street, London,
N6A-5B7, Canada; sazarfar@uwo.ca

Masoud Khalkhali
Department of Mathematics, University of Western Ontario, 1151 Richmond Street, London,
N6A-5B7, Canada; masoud@uwo.ca