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Quasi-stationary distribution for strongly Feller Markov processes by Lyapunov functions and applications to hypoelliptic Hamiltonian systems

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Abstract. We establish a general result on the existence and uniqueness of a quasi-stationary distribution $\mu_{\mathcal{D}}$ of a strongly Feller Markov process $(X_t, t \geq 0)$ killed when it exits a domain $\mathcal{D}$, under some Lyapunov function condition. Our result covers the case of hypoelliptic damped Hamiltonian systems. Our method is based on a characterization of the essential spectral radius by means of Lyapunov functions and measures of noncompactness.

Keywords: quasi-stationary distributions, Langevin process, Hamiltonian dynamics, metastability, molecular dynamics.

1. Introduction

1.1. Setting and literature

The notion of quasi-stationary distribution is a central object in the study of population processes or more generally of models derived from biological systems; see for instance [14, 17, 22, 24, 25, 60] and references therein. More recently, the notion of quasi-stationary distribution has attracted a lot of attention in the mathematical justification of very efficient accelerated dynamics algorithms [64, 71, 80, 81] (see also [65, 66]) which are widely used in practice and aim at simulating the atomistic evolution of statistical systems over long time scales (by accelerating the sampling of the exit event from a metastable macroscopic state $\mathcal{D}$). Let us be more precise on this. A typical process used in simulation

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in statistical physics to model the evolution of the positions of the particles of a system is a (stochastic) hypoelliptic damped Hamiltonian system $((x_t, v_t),\ t \geq 0)$ on $\mathbb{R}^d \times \mathbb{R}^d$ (see (6.1)), where $x_t \in \mathbb{R}^d$ gathers the positions of the particles of the system and $v_t \in \mathbb{R}^d$ their velocities at time $t \geq 0$ ($d = 3\mathsf{N}$, N being the number of particles). In most applications of the algorithms mentioned above, the macroscopic state is associated with a subdomain $\mathscr{D}$ of $\mathbb{R}^{2d}$ of the form $\mathscr{D} = \mathsf{O} \times \mathbb{R}^d$, where O is a subdomain of $\mathbb{R}^d$. The set O is defined in practice as a neighborhood (bounded or not) of some local minimum of the potential energy function $V : \mathbb{R}^d \to \mathbb{R}$ (the interatomic potential function), where the process $(x_t, t \geq 0)$ can spend a huge amount of time before leaving it[1] (in this case, $\mathscr{D}$ is called a metastable region). For such domains $\mathscr{D}$, it is thus expected that the conditional distribution of (x_t, v_t) before leaving $\mathscr{D}$ is close to a local equilibrium inside $\mathscr{D}$. This local equilibrium inside $\mathscr{D}$ is described by a quasi-stationary distribution (see (2.3)). The exit event from $\mathscr{D}$ can thus be studied starting from this quasi-stationary distribution [3, 29, 51, 56], which is at the heart of the mathematical analysis of the accelerated dynamics algorithms mentioned above (see [3, 29–31, 52, 53] when the process under study is the overdamped Langevin process, an elliptic diffusion).

The existence of a quasi-stationary distribution (as well as its uniqueness) for hypoelliptic processes is therefore of importance in molecular dynamics. This question is an open problem (as mentioned e.g. in [51, 56]) that we answer in our main result, Theorem 6.9. More precisely, Theorem 6.9 provides existence and uniqueness of a quasi-stationary distribution for hypoelliptic damped Hamiltonian systems (6.1) on domains $\mathscr{D}$ of the form $\mathsf{O} \times \mathbb{R}^d$. It also provides the exponential convergence of the law at time t of the conditioned process towards this quasi-stationary distribution.

The method we develop in this work allows us to deal with hypoelliptic damped Hamiltonian systems with coefficients which are only continuous. In addition, the damping coefficient can be unbounded and the position state space O is not necessarily bounded. The latter is of practical interest for the following reason. In many applications of the accelerated algorithms mentioned above, O is defined as the basin of attraction of a local minimum of the potential function V for the dynamics $\dot{x} = -\nabla V(x)$. In many situations, these basins of attraction are unbounded.

Quasi-stationary distributions for hypoelliptic diffusions have also been studied in [55, 68] for the Langevin dynamics with $\mathscr{C}^\infty$ coefficients, constant damping coefficient, and bounded position state space O. Hypoelliptic diffusions killed at the boundary of a subdomain $\mathscr{D}$ of $\mathbb{R}^m$ have also been studied in [7] when $\mathscr{D}$ is bounded and satisfies a noncharacteristic boundary condition, and the coefficients of the diffusions are smooth and satisfy some Hörmander conditions. None of these conditions are satisfied when $\mathscr{D} = \mathsf{O} \times \mathbb{R}^d$ and for the processes considered in Theorem 6.9. Let us also mention that nonconservative force fields are also considered in [55], and the approach developed in [7] allows dealling with some non-strongly-Feller processes.

Many different criteria have been given to ensure the existence and uniqueness of a quasi-stationary distribution for different Markov processes; see [18–21, 37, 42, 44, 49, 67,

[1] Because of the presence of energetic barriers.

72, 74, 86] and [4, 76] (based on the R-theory for Markov chains [77, 78]). We also mention [43] for the study of existence of quasi-stationary distributions through the notion of quasi-compact operators in the case of discrete time Markov chains; see also [39, 46]. In particular, for elliptic diffusions killed when exiting a bounded subdomain of $\mathbb{R}^d$, the existence and uniqueness of a quasi-stationary distribution are well known; see for instance [18, 21, 42, 45, 51, 67]. When considering unbounded domains, it is known that there might exist many quasi-stationary distributions [57]. We also refer to [32, 33] for the study of quasi-stationary distributions on a finite state space (see also [23] for a discrete state space) and to [11, 26, 38, 60, 79], and references therein, for the approximation of a quasi-stationary distribution using interacting particle systems in different settings (see also [8, 9]).

We finally refer to [62] for a spectral study of the kinetic Fokker–Planck operator on $L^2(\mathsf{O} \times \mathbb{R}^d)$ when O is bounded with several boundary conditions on $\partial\mathsf{O} \times \mathbb{R}^d$ (see also [1, 47] and references therein).

1.2. Purpose of this work

We recall that the main result of this work is Theorem 6.9. It provides existence and uniqueness of a quasi-stationary distribution for hypoelliptic damped Hamiltonian systems (6.1) on domains $\mathscr{D} = \mathsf{O} \times \mathbb{R}^d$, as well as the exponential convergence of the law at time t of the conditioned process towards this quasi-stationary distribution.

Theorem 6.9 is based on the general result of Theorem 2.2 which gives a general framework (see more precisely (**C1**)–(**C5**) in the next section) in which we can establish the following for general strongly Feller Markov processes $(X_t, t \geq 0)$ valued in a Polish space $\mathcal{S}$:

(1) the existence of a quasi-stationary distribution μ_D of the process $(X_t, t \geq 0)$ inside $\mathscr{D}$ (see (2.3) for definition);

(2) the uniqueness of a quasi-stationary distribution μ_D satisfying $\mu_D(\mathsf{W}^{1/p}) < +\infty$, where W is the Lyapunov functional appearing in (**C3**);

(3) the exponential convergence of the conditional distribution $\mathbb{P}_\nu(X_t \in \cdot \mid t < \sigma_D)$ towards μ_D, for any given initial distribution ν such that $\nu(\mathsf{W}^{1/p}) < +\infty$.

In particular, when the Lyapunov functional W is bounded (which is the case for instance when $\mathcal{S}$ is compact), the quasi-stationary distribution is unique (see more precisely Theorem 2.2 (b) and the discussion after Theorem 2.2). Our conditions on the semigroup of the killed Markov processes are quite general for strongly Feller Markov processes. We build our argument upon the literature on nonkilled Markov processes, where Lyapunov type conditions have already been widely investigated these past few years, with various aims:

- To study the stability of differential equations with random right hand side (since the pioneer works of Khasminskii, see the reference textbook [48]); we also refer to [2] where Lyapunov type conditions are used to characterize the stability of controlled diffusions or to obtain asymptotic flatness of controlled diffusions.

- To derive regularity estimates and upper bounds on the invariant measure, as in [12, Chapter 7].
- To obtain the existence of a spectral gap for the associated Markov semigroup (see for example [34] in weighted spaces and [5] in L^2).

Note that Lyapunov type conditions are alsoe used to study quasi-stationarity in the recent works [18,21,45].

Theorem 2.2 is then applied to a wide range of hypoelliptic damped Hamiltonian systems; this is the purpose of Theorem 6.9, for which the checking of assumptions (**C1**)–(**C5**) of Theorem 2.2 requires some extra fine analysis.

Finally, we point out that Theorem 2.2 can also be used to prove existence and uniqueness of a quasi-stationary distribution for elliptic diffusion processes, for which the assumptions of Theorem 2.2 are much easier to check.

1.3. Organization

In the next section we present the general framework and the main theoretical result, Theorem 2.2, for the quasi-stationary distribution. To prove the main result, a first key point is the existence of the spectral gap for the semigroup of the killed Markov process. To obtain it we will use the measures of non-weak-compactness of a positive kernel introduced in [84] to establish the formula for the essential spectral radius by means of the Lyapunov function for nonkilled Markov processes, which generalizes [84] from discrete time to continuous time case. The use of measures of noncompactness allows us to obtain the existence of a spectral gap for the semigroup of the killed Markov process. That is the content of Theorem 3.5.

The second key ingredient for the main result is a Perron–Frobenius type theorem (see Theorem 4.1) for a general Feller kernel. This is the purpose of Section 4.

With those preparations which should have independent interest, we prove the main result in Section 5. Finally, the applications to hypoelliptic damped Hamiltonian systems are developed in Section 6.

2. Main result

2.1. Framework: Notations and conditions

Let $(X_t,\, t\geq 0)$ be a time homogeneous Markov process valued in a metric complete separable (say Polish) space $\mathcal{S}$, with càdlàg paths and satisfying the strong Markov property, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_{\mathsf{x}})_{\mathsf{x}\in\mathcal{S}})$ where $\mathbb{P}_{\mathsf{x}}(X_0=\mathsf{x})=1$ for all $\mathsf{x}\in\mathcal{S}$ (and where the filtration satisfies the usual conditions). Its transition probability semigroup is denoted by $(P_t,\, t\geq 0)$. Given an initial distribution ν on $\mathcal{S}$, we write $\mathbb{P}_\nu(\cdot)=\int_{\mathcal{S}}\mathbb{P}_{\mathsf{x}}(\cdot)\,\nu(d\mathsf{x})$. Under $\mathbb{P}_\nu$, the distribution of X_0 is ν.

Let $\mathcal{B}(\mathcal{S})$ be the Borel σ-algebra of $\mathcal{S}$, and $b\mathcal{B}(\mathcal{S})$ the space of all bounded and Borel measurable functions f on $\mathcal{S}$ (its norm will be denoted by $b\mathcal{B}(\mathcal{S})\ni f\mapsto \|f\|_{b\mathcal{B}(\mathcal{S})}=$

$\sup_{\mathsf{x}\in\mathcal{S}}|f(\mathsf{x})|$). The space $\mathbb{D}([0,T],\mathcal{S})$ of $\mathcal{S}$-valued càdlàg paths defined on $[0,T]$ is equipped with the Skorokhod topology.

We suppose that the following conditions hold:

(**C1**) (*Strong Feller property*) There exists $t_0>0$ such that for each $t\geq t_0$, P_t is strong Feller, i.e. $P_t f$ is continuous on $\mathcal{S}$ for any $f\in b\mathcal{B}(\mathcal{S})$.

(**C2**) (*Trajectory Feller property*) For every $T>0$, $\mathsf{x}\mapsto\mathbb{P}_{\mathsf{x}}(X_{[0,T]}\in\cdot)$ (the law of $X_{[0,T]}:=(X_t)_{t\in[0,T]}$) is continuous from $\mathcal{S}$ to the space $\mathcal{M}_1(\mathbb{D}([0,T],\mathcal{S}))$ of probability measures on $\mathbb{D}([0,T],\mathcal{S})$, equipped with the weak convergence topology.

Now let $\mathcal{D}$ be a nonempty open domain in $\mathcal{S}$, different from $\mathcal{S}$. Consider the first exit time from $\mathcal{D}$,

$$\sigma_{\mathcal{D}}:=\inf\{t\geq 0;\ X_t\in\mathcal{D}^c\},\tag{2.1}$$

where $\mathcal{D}^c=\mathcal{S}\setminus\mathcal{D}$ is the complement of $\mathcal{D}$. The transition semigroup of the killed process $(X_t, 0\leq t<\sigma_{\mathcal{D}})$ is defined for $t\geq 0$ and $\mathsf{x}\in\mathcal{D}$ by

$$P_t^{\mathcal{D}}f(\mathsf{x})=\mathbb{E}_{\mathsf{x}}[1_{t<\sigma_{\mathcal{D}}}f(X_t)]\tag{2.2}$$

for $f\in b\mathcal{B}(\mathcal{D})$. Let us now recall the definition of a quasi-stationary distribution.

Definition 2.1. A *quasi-stationary distribution* (QSD for short) of the Markov process $(X_t,\ t\geq 0)$ in the domain $\mathcal{D}$ is a probability measure on $\mathcal{D}$ such that

$$\mu_{\mathcal{D}}(A)=\mathbb{P}_{\mu_{\mathcal{D}}}(X_t\in A\,|\,t<\sigma_{\mathcal{D}})=\frac{\mathbb{P}_{\mu_{\mathcal{D}}}(X_t\in A,\ t<\sigma_{\mathcal{D}})}{\mathbb{P}_{\mu_{\mathcal{D}}}(t<\sigma_{\mathcal{D}})},\quad \forall t>0,\ A\in\mathcal{B}(\mathcal{D}),\tag{2.3}$$

where $\mathcal{B}(\mathcal{D}):=\{A\cap\mathcal{D};\ A\in\mathcal{B}(\mathcal{S})\}$.

2.2. *Main general result*

For a continuous time Markov process, what is given is often its generator $\mathcal{L}$, not its transition semigroup $(P_t,\ t\geq 0)$, which is unknown in general. We say that a continuous function f belongs to the *extended domain* $\mathbb{D}_e(\mathcal{L})$ of $\mathcal{L}$ if there is some measurable function g on $\mathcal{S}$ such that $\int_0^t|g|(X_s)\,ds<+\infty$, $\mathbb{P}_{\mathsf{x}}$-a.e. for all $\mathsf{x}\in\mathcal{S}$, and

$$M_t(f):=f(X_t)-f(X_0)-\int_0^t g(X_s)\,ds\tag{2.4}$$

is a $\mathbb{P}_{\mathsf{x}}$-local martingale for all x. Such a function g, denoted by $\mathcal{L}f$, is not unique in general. But it is unique up to equality *quasi-everywhere* (q.e.): two functions g_1,g_2 are said to be equal q.e. if $g_1=g_2$ almost everywhere with respect to the (resolvent) measure $R_1(\mathsf{x},\cdot)=\int_0^{+\infty}e^{-t}P_t(\mathsf{x},\cdot)\,dt$ for *every* $\mathsf{x}\in\mathcal{S}$.

Let us introduce the Lyapunov function condition:

(**C3**) (*Lyapunov condition*) There exist a continuous function $\mathsf{W}:\mathcal{S}\to[1,+\infty[$, belonging to the extended domain $\mathbb{D}_e(\mathcal{L})$, two sequences of positive constants, (r_n) and (b_n), where $r_n\to+\infty$, and an increasing sequence (K_n) of compact subsets of $\mathcal{S}$, such that

$$-\mathcal{L}\mathsf{W}(\mathsf{x}) \geq r_n \mathsf{W}(\mathsf{x}) - b_n 1_{K_n}(\mathsf{x}), \quad \text{q.e.},$$

where 1_{K_n} is the indicator function of K_n.

We say that a class $\mathcal{A}$ of bounded continuous functions on $\mathcal{D}$ is *measure-separable* if for any bounded (signed) measure ν on $\mathcal{D}$, $\nu(f) = 0$ for all $f \in \mathcal{A}$ implies $\nu = 0$.

Theorem 2.2. *Assume that* (**C1**)–(**C3**) *hold. Suppose that the killed process* $(X_t, 0 \leq t < \sigma_{\mathcal{D}})$ *satisfies:*

(**C4**) (Weak Feller property) *For* $t \geq 0$, $P_t^{\mathcal{D}}$ *is weakly Feller, i.e. for a measure-separable class* $\mathcal{A}$ *of continuous bounded functions* f *with support contained in* $\mathcal{D}$, $P_t^{\mathcal{D}} f$ *is continuous on* $\mathcal{D}$.

(**C5**) (Topological irreducibility and almost sure extinction) *There exists* $t_0 > 0$ *such that for all* $t \geq t_0$, *all* $\mathsf{x} \in \mathcal{D}$ *and all nonempty open subsets* O *of* $\mathcal{D}$,

$$P_t^{\mathcal{D}}(\mathsf{x}, O) > 0$$

(*we can assume* t_0 *is the same as in* (**C1**)), *and there is some* $\mathsf{x}_0 \in \mathcal{D}$ *such that* $\mathbb{P}_{\mathsf{x}_0}(\sigma_{\mathcal{D}} < +\infty) > 0$.

Then, for any $p \in]1, +\infty[$ *fixed:*

(a) *There is only one QSD* $\mu_{\mathcal{D}}^{(p)}$ *of the process* $(X_t, t \geq 0)$ *in* $\mathcal{D}$ *satisfying*

$$\mu_{\mathcal{D}}^{(p)}(\mathsf{W}^{1/p}) := \int_{\mathcal{D}} \mathsf{W}(\mathsf{x})^{1/p}\, \mu_{\mathcal{D}}^{(p)}(d\mathsf{x}) < +\infty. \tag{2.5}$$

(b) *In particular, if* W *is bounded over* $\mathcal{D}$, *the QSD inside* $\mathcal{D}$ *is unique.*

(c) *There exists* $\lambda_{\mathcal{D}}^{(p)} > 0$ (*often called the least Dirichlet eigenvalue of the killed Markov process*) *such that the spectral radius of* $P_t^{\mathcal{D}}$ *on* $b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D})$ *equals* $e^{-\lambda_{\mathcal{D}}^{(p)} t}$ *for all* $t \geq 0$. *Furthermore,* $\mu_{\mathcal{D}}^{(p)} P_t^{\mathcal{D}} = e^{-\lambda_{\mathcal{D}}^{(p)} t} \mu_{\mathcal{D}}^{(p)}$ *for all* $t \geq 0$, *and* $\mu_{\mathcal{D}}^{(p)}(O) > 0$ *for all nonempty open subsets* O *of* $\mathcal{D}$. *In addition, there is a unique continuous function* φ^p *bounded by* $c\mathsf{W}^{1/p}$ *such that* $\mu_{\mathcal{D}}^{(p)}(\varphi^{(p)}) = 1$ *and*

$$P_t^{\mathcal{D}} \varphi^{(p)} = e^{-\lambda_{\mathcal{D}}^{(p)} t} \varphi^{(p)} \quad \text{on } \mathcal{D}, \ \forall t \geq 0. \tag{2.6}$$

Moreover, $\varphi^{(p)} > 0$ *everywhere on* $\mathcal{D}$. *Here* $b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D})$ *is the Banach space of all* $\mathcal{B}(\mathcal{D})$*-measurable functions on* $\mathcal{D}$ *with norm*

$$\|f\|_{b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D})} := \sup_{\mathsf{x} \in \mathcal{D}} \frac{|f(\mathsf{x})|}{\mathsf{W}(\mathsf{x})^{1/p}} < +\infty.$$

(d) *There are some constants* $\delta > 0$ *and* $C \geq 1$ *such that for any initial distribution* ν *on* $\mathcal{D}$ *with* $\nu(\mathsf{W}^{1/p}) < +\infty$,

$$\left|\mathbb{P}_\nu(X_t \in A \mid t < \sigma_{\mathcal{D}}) - \mu_{\mathcal{D}}^{(p)}(A)\right| \leq C e^{-\delta t} \frac{\nu(\mathsf{W}^{1/p})}{\nu(\varphi^{(p)})}, \quad \forall A \in \mathcal{B}(\mathcal{D}), t > 0. \tag{2.7}$$

(e) $\mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}} < +\infty) = 1$ *for every* $\mathsf{x} \in \mathcal{D}$, $X_{\sigma_{\mathcal{D}}}$ *and* $\sigma_{\mathcal{D}}$ *are* $\mathbb{P}_{\mu_{\mathcal{D}}^p}$*-independent, and*

$$\mathbb{P}_{\mu_{\mathcal{D}}^p}(t < \sigma_{\mathcal{D}}) = e^{-\lambda_{\mathcal{D}}^{(p)} t}.$$

Notice that the set of initial distributions ν on $\mathcal{D}$ with $\nu(\mathsf{W}^{1/p}) < +\infty$ includes any initial distribution ν with compact support in $\mathcal{D}$, and thus includes in particular the Dirac measures δ_{x} ($\mathsf{x} \in \mathcal{D}$), which is for instance of interest to analyse the mathematical foundations of the accelerated algorithms we mentioned in the Introduction.[2] In addition, choosing $\nu = \delta_{\mathsf{x}}$ in (2.7) ($\mathsf{x} \in \mathcal{D}$) for each $p > 1$, one deduces that $\mu_{\mathcal{D}}^{(p)}$ is actually independent of $p > 1$ (and then so is $\lambda_{\mathcal{D}}^{(p)}$), i.e. for any $p, q > 1$, $\mu_{\mathcal{D}}^{(p)} = \mu_{\mathcal{D}}^{(q)}$ and $\lambda_{\mathcal{D}}^{(p)} = \lambda_{\mathcal{D}}^{(q)}$.

Theorem 2.2 is applied in Section 6 to hypoelliptic damped Hamiltonian systems; see (6.1) and Theorem 6.9. Let us mention that under the assumptions of Theorem 2.2, from Corollary 3.6 and (5.3), we find that for each $t > 2t_0$ (see (**C1**))

$$P_t^{\mathcal{D}} : b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D}) \to b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D}) \text{ is compact.}$$

Remark 2.3. From (2.3), $\mu_{\mathcal{D}}$ is a QSD if and only if

$$\mu_{\mathcal{D}} P_t^{\mathcal{D}} = \lambda(t)\mu_{\mathcal{D}}, \quad \lambda(t) = \mathbb{P}_{\mu_{\mathcal{D}}}(t < \sigma_{\mathcal{D}}),$$

in other words, $\mu_{\mathcal{D}}$ must be a common positive left-eigenvector of $P_t^{\mathcal{D}}$. Item (c) above says that $\lambda(t) = e^{-\lambda_{\mathcal{D}} t}$ is exactly the spectral radius of $P_t^{\mathcal{D}}$ on $b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{D})$.

We now discuss assumptions (**C1**) and (**C3**), and the uniqueness of the QSD in the whole space of measures on $\mathcal{D}$.

On assumption (**C1**). As already explained, a key point in the proof of Theorem 2.2 is the existence of the spectral gap for the semigroup of the killed Markov process. This spectral gap is obtained using formula (3.3) valid for a bounded nonnegative kernel P on $\mathcal{S}$ satisfying in particular the condition (see (3.1) and **(A1)**)

$$\exists N \geq 1, \quad \beta_\tau(1_K P^N) = 0 \text{ for all compact subsets } K \text{ of } \mathcal{S}. \tag{2.8}$$

This is where (**C1**) is used in this work (see Remark 3.3). Condition **(A1)** is not restricted to strongly Feller kernels but for them it is easier to check **(A1)**. We have decided to work with (**C1**) instead of (2.8) for ease of exposition and because most of the hypoelliptic processes we consider in Section 6 satisfy (**C1**). We also mention that one advantage of the method based on formula (3.3) is that it provides an explicit upper bound on the spectral radius (see Theorem 3.5). This can be used to derive a spectral gap when one has a sufficiently good lower bound on the spectral radius of the killed process, and we hope to develop this method for non-strongly-Feller processes in future work.

On assumption (**C3**). If we replace, in (**C3**), the assumption that $r_n \to \infty$ as $n \to \infty$ by $r_n \to r_\infty$ as $n \to \infty$ with $r_\infty > \lambda_{\mathcal{D}}$, then all the statements of Theorem 2.2 remain valid. The latter condition is used in [21]. Such a criterion is in practice quite hard to check since one does not know the Dirichlet eigenvalue $\lambda_{\mathcal{D}}$. Note that (**C3**) is known to imply various functional inequalities, depending on the growth of r_n to infinity such

[2]This is based on the comparison of the exit event from $\mathcal{D}$ when $X_0 = \mathsf{x} \in \mathcal{D}$ and when X_0 is distributed according to a quasi-stationary distribution of the process $(X_t, t \geq 0)$ in $\mathcal{D}$.

as logarithmic Sobolev inequalities or various F-Sobolev type inequalities [6, 15, 16]. If r_n goes to infinity slowly one may show that the F-Sobolev inequality implied by (**C3**) is weaker than ultracontractivity, a condition which was behind the "coming down from infinity" property used in [14] for example.

Let us also comment on the other assumptions imposed in [21]. The authors introduce condition (F) which includes in particular a variant of (**C3**), (F2) in their work, but also a local Harnack inequality (F3) which seems quite hard to verify in nonelliptic cases. We prefer to use our conditions (**C4**) and (**C5**), easily verified for muldimensional elliptic diffusion processes and thus recovering the results of [21, Section 4], but which will also be useful in hypoelliptic cases.

Let us also mention that Lyapunov type conditions to study quasi-stationary distributions have also been used before in [20]. The Lyapunov conditions in [20] imply uniqueness of the QSD, which does not hold in general assuming only (**C3**) (see the discussion below). Note also that (**C3**) only involves the nonkilled process.

We finally mention that (**C3**) implies that $\mathcal{L}\mathsf{W} \le b_n \mathsf{W}$, which ensures that the quasi-stationary measure $\mu_{\mathcal{D}}$ satisfies $\mu_{\mathcal{D}}(\mathsf{W}^{1/p}) < +\infty$, $p > 1$.

Uniqueness of the QSD in certain cases. We give two situations for which there will be a unique QSD of $(X_t, t \ge 0)$ in $\mathcal{D}$ (i.e. in the whole space of probability measures over $\mathcal{D}$).

- When in addition to (**C1**)–(**C5**) in Theorem 2.2, for some $t > 0$ and some $p > 1$ one has $P_t(b_{\mathsf{W}^{1/p}}\mathcal{B}(\mathcal{S})) \subset b\mathcal{B}(\mathcal{S})$, then the process $(X_t, t \ge 0)$ has a unique QSD in $\mathcal{D}$.
- When (**C1**), (**C2**), (**C4**), and (**C5**) hold, and $\mathcal{S}$ is compact, all the results of Theorem 2.2 are valid with $\mathsf{W} = 1$ (indeed (**C3**) is always satisfied with $\mathsf{W} = 1$, $K_n = \mathcal{S}$, and $r_n = b_n = n$ for instance). Thus, in this case, item (b) of Theorem 2.2 holds, and item (d) is satisfied for any initial distribution ν in $\mathcal{D}$.

Without extra assumptions in Theorem 2.2, $(X_t, t \ge 0)$ does not have a unique QSD in $\mathcal{D}$. Indeed, consider the Ornstein–Uhlenbeck process $dX_t = -X_t dt + dB_t$ on $\mathcal{S} = \mathbb{R}$ and with $\mathcal{D} = \mathbb{R}_-^*$ (for which it is known that there are infinitely many QSD [57]). One can easily check with much easier arguments than those used in Section 6 (since this process is elliptic) that Theorem 2.2 is valid with $\mathsf{W}(x) = e^{\varepsilon x^2/2}$ ($\varepsilon \in (0, 1)$). Theorem 2.2 allows us to catch its *minimal* QSD, namely $\nu_1 = 2xe^{-x^2}$ (see [57] and references therein).

Remark 2.4. The key, as in the current literature, consists in proving that $P_t^{\mathcal{D}}$ has a spectral gap at its spectral radius $\mathsf{r}_{\mathrm{sp}}(P_t^{\mathcal{D}})$ acting on some Banach lattice space $\mathbb{B}$ of functions. We will work on $\mathbb{B} = b_{\mathsf{W}}\mathcal{B}(\mathcal{S}), \mathcal{C}_{b\mathsf{W}}(\mathcal{S})$ (introduced in Section 3.3 below), which are well adapted to our Lyapunov function condition (**C3**). The problem is: $P_t^{\mathcal{D}}$ is not strongly continuous on such Banach spaces, and the domain $\mathbb{D}_{\mathbb{B}}(L)$ of the generator L is not dense in $\mathbb{B}$. So we cannot use the spectral theory of strongly continuous semigroups in functional analysis. Let us finally mention [58] for an analytical study (using the notions of minimal positive solutions and *weak generator*) of some classes of non-strongly-continuous or nonanalytical Markov semigroups induced by some degenerate second-order operators on $\mathbb{R}^N$ (the case of unbounded domains is also considered there for uniformly elliptic second-order operators).

3. Essential spectral radius of P_t

The purpose of this section is to prove Theorem 3.5 which aims at giving a lower bound on the essential spectral radius of P_t, $t > 0$. To this end, we first recall a characterization of the essential spectral radius of a semigroup of transition kernels (see Theorem 3.4) obtained in [84].

3.1. *Essential spectral radius*

Let $\mathbb{B}$ be a real Banach lattice and P a nonnegative, bounded linear operator on $\mathbb{B}$. A complex number $\lambda \in \mathbb{C}$ is said to be in the *resolvent set* $\rho(P)$ of P if the inverse $(\lambda I - P)^{-1}$ on the complexified Banach space $\mathbb{B}_{\mathbb{C}}$ of $\mathbb{B}$ exists and is bounded. The complementary set $\sigma(P) := \mathbb{C} \setminus \rho(P)$ is the *spectrum* of P on $\mathbb{B}$. The spectral radius of P is given by Gelfand's formula

$$\mathsf{r}_{\mathrm{sp}}(P|_{\mathbb{B}}) := \sup\{|\lambda|;\ \lambda \in \sigma(P)\} = \lim_{n\to\infty} (\|P^n\|_{\mathbb{B}\to\mathbb{B}})^{1/n}.$$

A complex number λ does not belong to the (Wolf) *essential spectrum* $\sigma_{\mathrm{ess}}(P|_{\mathbb{B}})$ of $P|_{\mathbb{B}}$ iff $\lambda I - P$ is a Fredholm operator on $\mathbb{B}_{\mathbb{C}}$. For a point λ_0 in the spectrum $\sigma(P|_{\mathbb{B}})$, $\lambda_0 \notin \sigma_{\mathrm{ess}}(P|_{\mathbb{B}})$ iff λ_0 is isolated in $\sigma(P|_{\mathbb{B}})$ and the associated eigenprojection

$$E_{\lambda_0} := \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - P)^{-1} d\lambda$$

(Dunford integral in the counter-clockwise way) is finite-dimensional, where Γ is a circumference of sufficiently small radius: $|\lambda - \lambda_0| = \delta$ such that the disk $|\lambda - \lambda_0| \le \delta$ contains no other spectral point than λ_0. Let us recall that the *algebraic multiplicity* of $\lambda_0 \notin \sigma_{\mathrm{ess}}(P|_{\mathbb{B}})$ is the dimension of the range of E_{λ_0}. Let us finally mention that since P is bounded, $\lambda_0 \notin \sigma_{\mathrm{ess}}(P|_{\mathbb{B}})$ is a pole of the resolvent [85, Theorem 4 in Section 8 of Chapter VIII].

Definition 3.1. The *essential spectral radius* of P on $\mathbb{B}$ is defined by

$$\mathsf{r}_{\mathrm{ess}}(P|_{\mathbb{B}}) = \sup\{|\lambda|;\ \lambda \in \sigma_{\mathrm{ess}}(P|_{\mathbb{B}})\}.$$

3.2. *Two parameters of non-weak-compactness and formulas for the essential spectral radius*

Our state space $\mathcal{S}$ is Polish with a compatible metric d (i.e., $(\mathcal{S}, \mathsf{d})$ is complete and separable), whose Borel σ-field is denoted by $\mathcal{B}(\mathcal{S})$. The notation $K \subset\subset \mathcal{S}$ means that K is compact in $\mathcal{S}$. Let $\mathcal{M}_b(\mathcal{S})$ (resp. $\mathcal{M}_b^+(\mathcal{S})$, $\mathcal{M}_1(\mathcal{S})$) be the space of all σ-additive (resp. σ-additive and nonnegative; probability) measures of bounded variation on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. The pairing between $\nu \in \mathcal{M}_b(\mathcal{S})$ and $f \in b\mathcal{B}(\mathcal{S})$ is

$$\langle \nu, f\rangle := \nu(f) := \int_{\mathcal{S}} f(\mathsf{x})\, d\nu(\mathsf{x}).$$

Using this pairing, $\mathcal{M}_b(\mathcal{S})$ is a subspace of the dual Banach space $(b\mathcal{B}(\mathcal{S}))^*$. For a non-negative kernel $P(\mathsf{x}, dy)$, bounded on $b\mathcal{B}(\mathcal{S})$, its adjoint operator P^* on $(b\mathcal{B}(\mathcal{S}))^*$ keeps $\mathcal{M}_b(\mathcal{S})$ stable, i.e., for each $\nu \in \mathcal{M}_b(\mathcal{S})$,

$$P^*\nu(\cdot) = (\nu P)(\cdot) := \int_{\mathcal{S}} \nu(d\mathsf{x})\, P(\mathsf{x}, \cdot) \in \mathcal{M}_b(\mathcal{S}).$$

Besides the variation norm $\|\nu\|_{\mathrm{TV}}$ topology, we shall also consider the following two weak topologies on $\mathcal{M}_b(\mathcal{S})$. The weak topology $\sigma(\mathcal{M}_b(\mathcal{S}), b\mathcal{B}(\mathcal{S}))$ (i.e., the weakest topology on $\mathcal{M}_b(\mathcal{S})$ for which $\nu \mapsto \nu(f)$ is continuous for all $f \in b\mathcal{B}(\mathcal{S})$), according to the usual language, will be called the *τ-topology*, denoted simply by τ. And the weak topology $\sigma(\mathcal{M}_b(\mathcal{S}), \mathcal{C}_b(\mathcal{S}))$ (the most often used *weak convergence* topology) will be denoted by w. The space $\mathcal{C}_b(\mathcal{S})$ is the space of all functions $f \in b\mathcal{B}(\mathcal{S})$ such that f is continuous on $\mathcal{S}$ (its norm is the one of $b\mathcal{B}(\mathcal{S})$ but we will sometimes write $\|\cdot\|_{\mathcal{C}_b(\mathcal{S})}$ when we want to emphasize that we work on $\mathcal{C}_b(\mathcal{S})$).

The following measures of non-weak-compactness of a positive (i.e. nonnegative and nonzero) kernel $P(\mathsf{x}, dy)$ were introduced by the third author [84].

Definition 3.2. (a) For a bounded subfamily $\mathcal{M}$ of $\mathcal{M}_b^+(\mathcal{S})$, define

$$\begin{aligned} \beta_w(\mathcal{M}) &:= \inf_{K \subset\subset \mathcal{S}} \sup_{\nu \in \mathcal{M}} \nu(K^c), \\ \beta_\tau(\mathcal{M}) &:= \sup_{(A_n)} \lim_{n \to \infty} \sup_{\nu \in \mathcal{M}} \nu(A_n), \end{aligned} \tag{3.1}$$

where $\sup_{(A_n)}$ is taken over all sequences $(A_n)_n \subset \mathcal{B}(\mathcal{S})$ decreasing to $\emptyset$.

(b) Let $P(\mathsf{x}, dy)$ be a nonnegative kernel on $\mathcal{S}$ such that $\sup_{\mathsf{x} \in E} P(\mathsf{x}, \mathcal{S}) = \|P1\|_{b\mathcal{B}(\mathcal{S})} < +\infty$ (i.e., the kernel is bounded). We call

$$\beta_w(P) := \beta_w(\mathcal{M}), \quad \beta_\tau(P) := \beta_\tau(\mathcal{M}), \tag{3.2}$$

where $\mathcal{M} = \{P(\mathsf{x}, \cdot);\ \mathsf{x} \in \mathcal{S}\}$, the *measure of non-$\tau$-compactness* and the *measure of non-w-compactness* of P, respectively.

Here and in the following, 1 will denote the constant function equal to 1 on $\mathcal{S}$. We introduce the following assumption

(**A1**) $\beta_w(1_K P) = 0$ and $\exists N \geq 1 :\ \beta_\tau(1_K P^N) = 0,\ \forall K \subset\subset \mathcal{S}$.

Remark 3.3. If P is Feller and P^k is strongly Feller on $\mathcal{S}$, i.e. $P^k(b\mathcal{B}(\mathcal{S})) \subset \mathcal{C}_b(\mathcal{S})$, then (**A1**) is satisfied with $N = k$.

In fact, for any sequence $A_n \downarrow \emptyset$ in $\mathcal{B}(\mathcal{S})$, $f_n(\mathsf{x}) = P^k(\mathsf{x}, A_n)$ is continuous and converges to zero for every $\mathsf{x} \in \mathcal{S}$. In addition $f_{n+1}(\mathsf{x}) \leq f_n(\mathsf{x})$ for all n and $\mathsf{x} \in \mathcal{S}$. Then, by Dini's monotone convergence theorem, for each $K \subset\subset \mathcal{S}$, $\lim_{n\to\infty} \sup_{\mathsf{x} \in \mathcal{S}} 1_K(\mathsf{x}) P^k(\mathsf{x}, A_n) = 0$. That yields $\beta_\tau(1_K P^k) = 0$.

When P is Feller, the fact that $\beta_w(1_K P) = 0$ for all $K \subset\subset \mathcal{S}$ is proved similarly using Prokhorov's theorem (see for instance [84, Lemma 3.1 (a.iii)]).

Theorem 3.4 ([84, Theorem 3.5]). *Let P be a bounded nonnegative kernel on $\mathcal{S}$ satisfying* (**A1**). *Then*

$$\mathsf{r}_{\mathrm{ess}}(P|_{b\mathcal{B}(\mathcal{S})}) = \lim_{n\to\infty} [\beta_w(P^n)]^{1/n}. \tag{3.3}$$

3.3. Lyapunov function criterion for the essential spectral radius of $(P_t,\ t \geq 0)$

The main objective of this section is to apply Theorem 3.4 to P_t, where we recall that $(P_t,\ t \geq 0)$ is the semigroup of the (nonkilled) process $(X_t, t \geq 0)$. Let us first introduce some notation. Given a continuous function $\mathsf{W} : \mathcal{S} \to [1, +\infty[$ (weight function), let

$$b_{\mathsf{W}}\mathcal{B}(\mathcal{S}) := \left\{ f : \mathcal{S} \to \mathbb{R} \text{ measurable; } \|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})} := \sup_{\mathsf{x}\in\mathcal{S}} \frac{|f(\mathsf{x})|}{\mathsf{W}(\mathsf{x})} < +\infty \right\},$$

and

$$\mathcal{C}_{b\mathsf{W}}(\mathcal{S}) = \{ f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{S});\ f : \mathcal{S} \to \mathbb{R} \text{ is continuous} \},$$

which are Banach spaces with norm $\|\cdot\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}$. Notice that if $\mathsf{W} = 1$, then $b_{\mathsf{W}}\mathcal{B}(\mathcal{S}) = b\mathcal{B}(\mathcal{S})$ and $\mathcal{C}_{b\mathsf{W}}(\mathcal{S}) = \mathcal{C}_b(\mathcal{S})$, where $b\mathcal{B}(\mathcal{S})$ and $\mathcal{C}_b(\mathcal{S})$ are introduced above. The space of measures

$$\mathcal{M}_{b\mathsf{W}}(\mathcal{S}) = \{ \nu \in \mathcal{M}_b(\mathcal{S});\ \mathsf{W}(\mathsf{x})\nu(d\mathsf{x}) \in \mathcal{M}_b(\mathcal{S}) \}$$

is a subspace of the dual Banach space $(b_{\mathsf{W}}\mathcal{B}(\mathcal{S}))^*$ by regarding each $\nu \in \mathcal{M}_{b\mathsf{W}}(\mathcal{S})$ as a bounded linear form $f \mapsto \nu(f)$ on $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$. We now turn to the main result of this section.

Theorem 3.5. *Assume* (**C1**) *and* (**C2**). *Assume that there is some continuous Lyapunov function* $\mathsf{W} : \mathcal{S} \to [1, +\infty[$ *such that for some $K \subset\subset \mathcal{S}$, $r > 0$, and $b > 0$,*

$$-\frac{\mathcal{L}\mathsf{W}}{\mathsf{W}} \geq r\mathbb{1}_{K^c} - b\mathbb{1}_K, \tag{3.4}$$

and for some $p > 1$,

$$\mathcal{L}\mathsf{W}^p \leq b\mathsf{W}^p. \tag{3.5}$$

Then for every $t > 0$,

$$\beta_w(P_{t,\mathsf{W}}) \leq e^{-rt} \quad \text{and in particular} \quad \mathsf{r}_{\mathrm{ess}}(P_t|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}) \leq e^{-rt}, \tag{3.6}$$

where for $\mathsf{x}, \mathsf{y} \in \mathcal{S}$, we set

$$P_{t,\mathsf{W}}(\mathsf{x}, d\mathsf{y}) = \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})} P_t(\mathsf{x}, d\mathsf{y}). \tag{3.7}$$

Proof. Consider the isomorphism $M_{\mathsf{W}} : f \mapsto \mathsf{W}f$ from $b\mathcal{B}(\mathcal{S})$ to $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$. For $t \geq 0$ we have $P_{t,\mathsf{W}} = M_{\mathsf{W}}^{-1} P_t M_{\mathsf{W}}$. Then

$$\mathsf{r}_{\mathrm{ess}}(P_t|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}) = \mathsf{r}_{\mathrm{ess}}(P_{t,\mathsf{W}}|_{b\mathcal{B}(\mathcal{S})}).$$

Then, $(P_{t,\mathsf{W}},\ t \geq 0)$ is again a semigroup of transition kernels, but it is not Markov in general.

Step 1: Proof that $Q = P_{t,\mathsf{W}}$ satisfies (**A1**) *for $t > 0$.* First, from (3.5), $(e^{-bt}\mathsf{W}^p(X_t), t \ge 0)$ is a supermartingale, and so for any $\mathsf{x} \in \mathcal{S}$ and $t \ge 0$,

$$(P_t \mathsf{W}^p)(\mathsf{x}) = e^{bt}\mathbb{E}_{\mathsf{x}}[e^{-bt}\mathsf{W}(X_t)^p] \le e^{bt}\mathsf{W}(\mathsf{x})^p. \tag{3.8}$$

Therefore, for any compact $K \subset\subset \mathcal{S}$, letting $q = p/(p-1)$ and using Hölder's inequality, we have

$$\begin{aligned}\beta_w(1_K Q) &= \inf_{K' \subset\subset \mathcal{S}} \sup_{\mathsf{x} \in K} Q(\mathsf{x}, \mathcal{S} \setminus K') \le \inf_{K' \subset\subset \mathcal{S}} \sup_{\mathsf{x} \in K} \frac{((P_t \mathsf{W}^p)(\mathsf{x}))^{1/p}}{\mathsf{W}(\mathsf{x})} P_t(\mathsf{x}, \mathcal{S} \setminus K')^{1/q} \\ &\le e^{bt/p}\Big(\inf_{K' \subset\subset \mathcal{S}} \sup_{\mathsf{x} \in K} P_t(\mathsf{x}, \mathcal{S} \setminus K')\Big)^{1/q} = 0\end{aligned}$$

by the Feller property of P_t (guaranteed by (**C2**), see Remark 3.3).

Let us check the second condition in (**A1**) with some N such that $Nt \ge t_0$ (see (**C1**)):

$$\begin{aligned}\beta_\tau(1_K Q^N) &= \sup_{(A_n)} \lim_{n \to \infty} \sup_{\mathsf{x} \in K} Q^N(\mathsf{x}, A_n) \\ &\le \sup_{(A_n)} \lim_{n \to \infty} \sup_{\mathsf{x} \in K} \frac{((P_{Nt} \mathsf{W}^p)(\mathsf{x}))^{1/p}}{\mathsf{W}(\mathsf{x})} P_{Nt}(\mathsf{x}, A_n)^{1/q} \\ &\le e^{Nbt/p}\Big(\sup_{(A_n)} \lim_{n \to \infty} \sup_{\mathsf{x} \in K} P_{Nt}(\mathsf{x}, A_n)\Big)^{1/q},\end{aligned}$$

where the sup above is taken over all sequences $(A_n)_n$ in $\mathcal{B}(\mathcal{S})$ decreasing to $\emptyset$. The last factor above, being $\beta_\tau(1_K P_{Nt})$, is equal to zero by the strong Feller property of P_{Nt} (see (**C1**) and Remark 3.3).

Step 2: Proof that $\beta_w(Q) \le e^{-rt}$. This yields $\beta_w(Q^n) \le \beta_w(Q)^n$ [84, Proposition 3.2.(e)] for all n and then the desired result by Theorem 3.4 (we can use these results since Q satisfies (**A1**)).

To prove that $\beta_w(Q) \le e^{-rt}$, we introduce the first hitting time $\tau_K := \inf\{s \ge 0; X_s \in K\}$ of the compact K for the process $(X_t,\ t \ge 0)$, where K is the compact set appearing in the Lyapunov condition (3.4). We then have

$$\beta_w(Q) = \inf_{K' \subset\subset \mathcal{S}} \sup_{\mathsf{x} \in \mathcal{S}} Q(\mathsf{x}, \mathcal{S} \setminus K') = \inf_{K' \subset\subset \mathcal{S}} \sup_{\mathsf{x} \in \mathcal{S}} \frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{X_t \notin K'}]. \tag{3.9}$$

Notice that for $\mathsf{x} \in \mathcal{S}$, we have

$$\begin{aligned}\frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{X_t \notin K'}] &\le \frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{X_t \notin K', \tau_K \le t}] + \frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{\tau_K > t}] \\ &\le e^{bt/p}\Big(\sup_{\mathsf{y} \in K} \mathbb{P}_{\mathsf{y}}(\exists s \in [0,t],\ X_s \notin K')\Big)^{1/q} + \frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{\tau_K > t}],\end{aligned} \tag{3.10}$$

where the second inequality follows by Hölder's inequality and the strong Markov property of the process $(X_t,\ t \ge 0)$.

Let us first deal with the first term on the r.h.s. of (3.10). By condition (**C2**), for any $\varepsilon > 0$, there is some compact subset A_ε in $\mathbb{D}([0,t],\mathcal{S})$ such that

$$\sup_{\mathsf{y}\in K} \mathbb{P}_{\mathsf{y}}(X_{[0,t]} \notin A_\varepsilon^c) < \varepsilon.$$

By the well known property of the Skorokhod topology [36], the set

$$B_\varepsilon := \bigcup_{s\in[0,t]} \{\gamma(s);\ \gamma \in A_\varepsilon\}$$

is relatively compact in $\mathcal{S}$. Thus

$$\begin{aligned}\inf_{K'\subset\subset\mathcal{S}} \sup_{\mathsf{y}\in K} \mathbb{P}_{\mathsf{y}}(\exists s \in [0,t],\ X_s \notin K') &\le \sup_{\mathsf{y}\in K} \mathbb{P}_{\mathsf{y}}(\exists s \in [0,t],\ X_s \notin \overline{B_\varepsilon}) \\ &\le \sup_{\mathsf{y}\in K} \mathbb{P}_{\mathsf{y}}(X_{[0,t]} \notin A_\varepsilon^c) < \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $\inf_{K'\subset\subset\mathcal{S}} \sup_{\mathsf{y}\in K} \mathbb{P}_{\mathsf{y}}(\exists s \in [0,t], X_s \notin K') = 0$. Substituting it into (3.10), we see from (3.9) that it remains to show that

$$\frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{\tau_K>t}] \le e^{-rt}, \quad \forall \mathsf{x} \in \mathcal{S}. \tag{3.11}$$

This is the purpose of the next step.

Step 3: Proof of (3.11). To this end, we introduce, for $t \ge 0$,

$$M_t := \frac{\mathsf{W}(X_t)}{\mathsf{W}(X_0)} \exp\left(-\int_0^t \frac{\mathcal{L}\mathsf{W}}{\mathsf{W}}(X_s)\, ds\right).$$

The key ingredient is the fact that $(M_t,\ t \ge 0)$ is a local $\mathbb{P}_{\mathsf{x}}$-martingale (for every x), by Ito's formula. Thus, $(M_t,\ t \ge 0)$ is then a supermartingale by Fatou's lemma. Then, by the Lyapunov condition (3.4),

$$e^{rt}\frac{1}{\mathsf{W}(\mathsf{x})}\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_t)1_{\tau_K>t}] \le \mathbb{E}_{\mathsf{x}}[M_t] \le M_0 = 1.$$

This is (3.11). Therefore, the proof of Theorem 3.5 is complete. ■

Corollary 3.6. *Assume that* (**C1**)–(**C3**) *are satisfied. If* (3.5) *holds, then, for each* $t > 2t_0$ (*see* (**C1**)), $P_t^{\mathcal{D}} : b_{\mathsf{W}}\mathcal{B}(\mathcal{D}) \to b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$ *is compact.*

Proof. From the first step of the proof of Theorem 3.5, $P_{t,\mathsf{W}}$ satisfies (**A1**) with N such that $Nt \ge t_0$. For $t \ge 0$, one has

$$P_{t,\mathsf{W}}^{\mathcal{D}} = \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})}P_t^{\mathcal{D}}(\mathsf{x}, d\mathsf{y}) \le \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})}P_t(\mathsf{x}, d\mathsf{y}) = P_{t,\mathsf{W}}(\mathsf{x}, d\mathsf{y}).$$

Let us show that for $t \ge t_0$, $P_{t,\mathsf{W}}$ is strongly Feller like P_t, i.e. for any $f \in b\mathcal{B}(\mathcal{S})$, $P_t(\mathsf{W}f)$ is continuous. For any $n \ge 1$, let

$$f_n := \frac{\mathsf{W} \wedge n}{\mathsf{W}} f.$$

Since $\mathsf{W}f_n$ is bounded, $P_t(\mathsf{W}f_n)$ is continuous by the strong Feller property of P_t (by (**C1**)). Now for any compact $K \subset\subset \mathcal{S}$,

$$\sup_{\mathsf{x}\in K} |P_t(\mathsf{W}f)(\mathsf{x}) - P_t(\mathsf{W}f_n)(\mathsf{x})| \le \sup_{\mathsf{x}\in K} [(P_t\mathsf{W}^p)(\mathsf{x})]^{1/p} \sup_{\mathsf{x}\in K} [(P_t|f_n - f|^q)(\mathsf{x})]^{1/q}.$$

We have $|f_n - f| \le \|f\|_{b\mathcal{B}(\mathcal{S})}$ for all n, and $|f_n - f| \downarrow 0$ pointwise on $\mathcal{S}$. Since P_t is strongly Feller (by (**C1**)), the sequence of functions $h_n(\mathsf{x}) := (P_t|f_n - f|^q)(\mathsf{x})$ is continuous over $\mathcal{S}$. Moreover, $h_n \downarrow 0$ pointwise on $\mathcal{S}$. Consequently, by Dini's monotone convergence theorem, we have

$$\sup_{\mathsf{x}\in K} (P_t|f_n - f|^q)(\mathsf{x}) \to 0.$$

Thus, for $t \ge t_0$, $P_t(\mathsf{W}f)$ is continuous, which implies that $P_{t,\mathsf{W}}$ is strongly Feller.

From Theorem 3.5 (with $r = r_n \to \infty$ by (**C3**)), we obtain

$$\beta_w(P^{\mathcal{D}}_{t,\mathsf{W}}) \le \beta_w(P_{t,\mathsf{W}}) = 0 \quad \text{for each } t > 0. \tag{3.12}$$

Because for each $t \ge t_0$, $P_{t,\mathsf{W}}$ is strongly Feller on $\mathcal{S}$, for all $K \subset\subset \mathcal{D}$ we have

$$\beta_\tau(1_K P^{\mathcal{D}}_{t,\mathsf{W}}) \le \beta_\tau(1_K P_{t,\mathsf{W}}) = 0 \quad \text{for each } t \ge t_0. \tag{3.13}$$

Therefore, by [84, Proposition 3.2 (f)], for each $s > 0$,

$$\beta_\tau(P^{\mathcal{D}}_{s+t_0,\mathsf{W}}) \le \beta_w(P^{\mathcal{D}}_{s,\mathsf{W}})\beta_\tau(P^{\mathcal{D}}_{t_0,\mathsf{W}}) = 0.$$

Finally, applying [84, Proposition 3.2 (g)], $P^{\mathcal{D}}_{s+2t_0,\mathsf{W}} : b\mathcal{B}(\mathcal{D}) \to b\mathcal{B}(\mathcal{D})$ is compact. This concludes the proof of Corollary 3.6. ■

4. A Perron–Frobenius type theorem on $b_W\mathcal{B}$

In this section, we present a version of Perron–Frobenius' theorem we will need for Feller kernels Q on $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$ or $\mathcal{C}_{b\mathsf{W}}(\mathcal{S})$, which is of independent interest.

Theorem 4.1. *Let $Q = Q(\mathsf{x}, d\mathsf{y})$ be a positive bounded kernel on $\mathcal{S}$ and $\mathsf{W} \ge 1$ a continuous weight function on $\mathcal{S}$. Assume that:*

(1) *There exists $N_1 \ge 1$ such that Q^k is Feller for all $k \ge N_1$, i.e. $Q^k f \in \mathcal{C}_b(\mathcal{S})$ if $f \in \mathcal{C}_b(\mathcal{S})$.*

(2) *There exists $N_2 \ge 1$ such that for any $\mathsf{x} \in \mathcal{S}$ and any nonempty open subset O of $\mathcal{S}$,*

$$Q^{N_2}(\mathsf{x}, O) > 0.$$

(3) *For some $p > 1$ and constant $C > 0$,*

$$Q\mathsf{W}^p \le C\mathsf{W}^p.$$

Notice that this implies that Q is well defined and bounded on $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$.

(4) *Q has a spectral gap in $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$:*

$$\mathsf{r}_{\mathrm{ess}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}) < \mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}). \tag{4.1}$$

Then there exist a unique couple (μ, φ) where μ is a probability measure on $\mathcal{S}$ with $\mu(\mathsf{W}) < +\infty$ and $\varphi \in \mathcal{C}_{b\mathsf{W}}(\mathcal{S})$ is positive everywhere on $\mathcal{S}$ with $\mu(\varphi) = 1$, and constants $r \in]0, 1[$ and $C \geq 1$, such that

$$\mu Q = \mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}}(\mathcal{S}))\mu, \quad Q\varphi = \mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})})\varphi \tag{4.2}$$

and

$$\left\| \frac{1}{\mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})})^n} Q^n f - \varphi\mu(f) \right\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})} \leq C r^n \|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}, \quad \forall f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{S}). \tag{4.3}$$

In particular,

(a) *If $\nu \in \mathcal{M}_{b\mathsf{W}}(\mathcal{S})$ satisfies $\nu Q = \lambda\nu$ for some $\lambda \in \mathbb{R}$, and $\nu(\varphi) \neq 0$, then $\lambda = \mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})})$ and $\nu = c\mu$ for some constant c.*

(b) *If $f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$ satisfies $Qf = \lambda f$ for some $\lambda \in \mathbb{R}$, and $\mu(f) \neq 0$, then $\lambda = \mathsf{r}_{\mathrm{sp}}(Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})})$ and $f = c\varphi$ for some constant c.*

Remark 4.2. Let us mention that the standard Krein–Rutman theorem [35, Theorem 1.2] or its generalization [69, Theorem 7], with the natural choice of cone $K = \{\phi \in \mathbb{B}; \phi \geq 0\}$ (recall $\mathbb{B} = b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$ or $\mathcal{C}_{b\mathsf{W}}(\mathcal{S})$ with the norm $\sup_{\mathcal{S}} |f|/\mathsf{W}$), do not apply here in general, for the following reason. Let $\mathcal{S} \subset \mathbb{R}^d$ be a smooth bounded domain and $Q(\mathsf{x}, \cdot) = \mathbb{P}_{\mathsf{x}}(X_1 \in \cdot, 1 < \sigma_{\mathcal{S}})$ where $(X_t, t \geq 0)$ is a standard (d-dimensional) Brownian motion. It is well-known that Q has a smooth (positive) density $q(\mathsf{x}, \mathsf{y})$ in $\mathcal{S} \times \mathcal{S}$ with respect to the Lebesgue measure on $\mathcal{S}$, which moreover has a continuous extension to $\overline{\mathcal{S}} \times \overline{\mathcal{S}}$ which vanishes on $\partial(\mathcal{S} \times \mathcal{S})$. Thus, if $u \in \mathbb{B}$ is an eigenfunction for Q on $\mathbb{B}$ associated with an eigenvalue $r > 0$, then $u = r^{-1}Qu$ has a continuous extension to $\overline{\mathcal{S}}$ which vanishes on $\partial\mathcal{S}$. Thus, $u \notin \mathrm{int}(K) = \{\phi \in \mathbb{B}; \exists c > 0, \phi \geq c\}$ and therefore [69, Theorem 7 (2)] (see also [69, Theorem 1 (2)]) as well as [35, Theorem 1.2] cannot hold. Notice that one would naturally then want to work with $K_1 = \{\phi \in \mathcal{C}(\overline{\mathcal{S}}); \phi \geq 0, \phi = 0 \text{ on } \partial\mathcal{S}\}$, but K_1 has empty interior. Note also that Q satisfies (1)–(4).

We start the proof of Theorem 4.1 with the following lemma.

Lemma 4.3. *Let Q be a bounded (resp. bounded and Feller) kernel with $\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})}) = 1$. Then:*

(a) *For any λ in the resolvent set $\rho(Q|_{b\mathcal{B}(\mathcal{S})})$ with $|\lambda| > \mathsf{r}_{\mathrm{ess}}(Q|_{b\mathcal{B}(\mathcal{S})})$,*

$$R(\lambda) := (\lambda I - Q)^{-1}$$

is a bounded (resp. bounded and Feller) kernel.

(b) *If Q is Feller, then $\mathsf{r}_{\mathrm{sp}}(Q|_{\mathcal{C}_b(\mathcal{S})}) = \mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$, and $\mathsf{r}_{\mathrm{ess}}(Q|_{\mathcal{C}_b(\mathcal{S})}) = \mathsf{r}_{\mathrm{ess}}(Q|_{b\mathcal{B}(\mathcal{S})})$.*

Proof. (a) First, for $\lambda > \mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$,

$$R(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} Q^n$$

is a bounded (resp. Feller) kernel.

Now for any $\lambda \in \rho(Q|_{b\mathcal{B}(\mathcal{S})})$ with $|\lambda| > \mathsf{r}_{\mathrm{ess}}(Q|_{b\mathcal{B}(\mathcal{S})})$, there is a $\mathcal{C}^1$-curve $[0, 1] \ni t \mapsto \gamma(t) \in \mathbb{C}$ such that $\lambda = \gamma(1)$, $\gamma(0) > \mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$ and $\mathrm{Ran}(\gamma) \subset \rho(Q|_{b\mathcal{B}(\mathcal{S})})$. It is enough to show that there is some (common) $\delta > 0$ such that for any $t_0 \in [0, 1]$ such that $R(\gamma(t_0))$ is a bounded (resp. Feller) kernel, so is $R(\gamma(t))$ if $|t - t_0| < \delta$.

To this end, let $M = \sup_{t\in[0,1]} \|R(\gamma(t))\|_{b\mathcal{B}(\mathcal{S})}$ which is finite (where $\|\cdot\|_{b\mathcal{B}(\mathcal{S})}$ is the operator norm on $b\mathcal{B}(\mathcal{S})$). Let $t \in [0, 1]$ be such that $|t - t_0| \le \frac{1}{2M(|\gamma'|_{L^\infty}+1)}$, so that $|\gamma(t_0) - \gamma(t)| \le \frac{1}{2M}$. Then, for such t, we have

$$R(\gamma(t)) = \sum_{n=0}^{\infty} (\gamma(t_0) - \gamma(t))^n R(\gamma(t_0))^{n+1}.$$

Thus $R(\gamma(t))$ is a bounded (resp. bounded and Feller) kernel.

(b) The first equality follows by Gelfand's formula for the spectral radius and the fact that $\|Q^n\|_{b\mathcal{B}(\mathcal{S})} = \sup_{\mathsf{x}\in\mathcal{S}} Q^n(\mathsf{x}, \mathcal{S}) = \|Q^n\|_{\mathcal{C}_b(\mathcal{S})}$ (for all $n \ge 0$). The second equality is proved in [84, Proposition 4.7]. ■

Proof of Theorem 4.1. The proof is divided into several steps.

Step 1: Reduction to $\mathsf{W} = 1$. For $\mathsf{x} \in \mathcal{S}$, let

$$Q_{\mathsf{W}}(\mathsf{x}, d\mathsf{y}) := \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})} Q(\mathsf{x}, d\mathsf{y}).$$

By Hölder's inequality (see also Theorem 4.1 (3)) we have, for $\mathsf{x} \in \mathcal{S}$,

$$Q\mathsf{W}(\mathsf{x}) \le [Q1(\mathsf{x})]^{1/q}[Q\mathsf{W}^p(\mathsf{x})]^{1/p} \le \|Q1\|_{b\mathcal{B}(\mathcal{S})}^{1/q} C^{1/p}\mathsf{W}(\mathsf{x})$$

where $q = p/(p-1)$. Hence $Q_{\mathsf{W}}1 \le \|Q1\|_{b\mathcal{B}(\mathcal{S})}^{1/q} C^{1/p}$, i.e. Q_{W} is a bounded positive kernel on $\mathcal{S}$.

Let us prove that Q_{W}^k is again Feller for $k \ge N_1$, that is, for any $f \in \mathcal{C}_b(\mathcal{S})$, $Q^k(\mathsf{W}f)$ is continuous (notice that $\mathsf{W}f$ is continuous over $\mathcal{S}$ but not necessarily bounded on $\mathcal{S}$). To this end, set, for any $n \ge 1$ and $f \in \mathcal{C}_b(\mathcal{S})$,

$$f_n := \frac{\mathsf{W} \wedge n}{\mathsf{W}} f.$$

The function $Q^k(\mathsf{W}f_n)$ is continuous by the Feller property of Q^k. Now for any compact $K \subset\subset \mathcal{S}$, we have

$$\sup_{\mathsf{x}\in K} |Q^k(\mathsf{W}f)(\mathsf{x}) - Q^k(\mathsf{W}f_n)(\mathsf{x})| \le \sup_{\mathsf{x}\in K} [(Q^k\mathsf{W}^p)(\mathsf{x})]^{1/p} \sup_{\mathsf{x}\in K} [(Q^k|f_n - f|^q)(\mathsf{x})]^{1/q}.$$

By assumption (3) in Theorem 4.1, $\sup_{x\in K}[(Q^k \mathsf{W}^p)(x)]^{1/p} \le C^{k/p}\sup_{x\in K}\mathsf{W}(x)$. Since $|f_n - f| \le \|f\|_{b\mathcal{B}(\mathcal{S})}$ and $f_n \to f$ uniformly on compact sets in $\mathcal{S}$, by the tightness of $\{Q^k(x, dy); x \in K\}$ we have

$$\sup_{x\in K}(Q^k|f_n - f|^q)(x) \to 0.$$

Thus $Q^k(\mathsf{W}f)$ is continuous.

Finally, letting $M_{\mathsf{W}} f = \mathsf{W}f$, which is an isomorphism from $b\mathcal{B}(\mathcal{S})$ to $b_{\mathsf{W}}\mathcal{B}(\mathcal{S})$, we have $Q_{\mathsf{W}} = M_{\mathsf{W}}^{-1} Q M_{\mathsf{W}}$, i.e. $Q|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{S})}$ is similar to $Q_{\mathsf{W}}|_{b\mathcal{B}(\mathcal{S})}$. Hence it is enough to prove the theorem for Q_{W} on $b\mathcal{B}(\mathcal{S})$ (note that Q_{W} also satisfies Condition (2)).

From now on, we assume without loss of generality that $\mathsf{W} = 1$ and $\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})}) = 1$ (otherwise consider $Q/\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$).

Step 2: Existence of a positive eigenfunction and an eigen probability measure. The fact that $\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$ (= 1 by assumption) is in the spectrum of $Q|_{b\mathcal{B}(\mathcal{S})}$ is well known (see [70, Chapter V, Proposition 4.1]). In addition, by condition (4), $\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})}) \notin \sigma_{\mathrm{ess}}(Q|_{b\mathcal{B}(\mathcal{S})})$. We recall (see Section 3.1) that this implies that $\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})$ is isolated in the spectrum of $Q|_{b\mathcal{B}(\mathcal{S})}$, its associated eigenprojection $E_{\mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})})}$ has finite rank, and is a pole of the resolvent of $Q|_{b\mathcal{B}(\mathcal{S})}$. We can thus use [70, Chapter V, Theorem 5.5 and the subsequent note] (cyclic property of the peripheral spectrum) to deduce that there exists $m \ge \max\{N_1, N_2\}$ such that

$$\text{for any } \lambda \in \sigma(Q|_{b\mathcal{B}(\mathcal{S})}) \text{ with } |\lambda| = 1, \text{ we have } \lambda^m = 1. \tag{4.4}$$

For such an m, we have:

(1) $\mathsf{r}_{\mathrm{ess}}(Q^m|_{b\mathcal{B}(\mathcal{S})}) < \mathsf{r}_{\mathrm{sp}}(Q^m|_{b\mathcal{B}(\mathcal{S})}) = 1$ (which follows from the fact that for all $k \ge 1$, $\mathsf{r}_{\mathrm{sp}}(Q^k|_{b\mathcal{B}(\mathcal{S})})^{1/k} = \mathsf{r}_{\mathrm{sp}}(Q|_{b\mathcal{B}(\mathcal{S})}) = 1$ and $\mathsf{r}_{\mathrm{ess}}(Q^k|_{b\mathcal{B}(\mathcal{S})})^{1/k} = \mathsf{r}_{\mathrm{ess}}(Q|_{b\mathcal{B}(\mathcal{S})}) < 1$).

(2) $1 = \mathsf{r}_{\mathrm{sp}}(Q|^m_{b\mathcal{B}(\mathcal{S})}) \in \sigma(Q|^m_{b\mathcal{B}(\mathcal{S})})$. In particular, 1 is an isolated eigenvalue of Q^m and is a pole of the resolvent of $Q^m|_{b\mathcal{B}(\mathcal{S})}$.

(3) the peripheral spectrum of $Q^m|_{b\mathcal{B}(\mathcal{S})}$ is reduced to $\{1\}$ (by (4.4) and the fact that $\sigma(Q^m) = \sigma(Q)^m$), that is,

$$\{\lambda \in \mathbb{C};\ |\lambda| = 1\} \cap \sigma(Q^m|_{b\mathcal{B}(\mathcal{S})}) = \{1\}.$$

Let $\Gamma := \{\lambda \in \mathbb{C};\ |\lambda - 1| = \delta\}$ where $\delta > 0$ is such that

$$\{\lambda \in \mathbb{C}; 0 < |\lambda - 1| \le \delta\} \subset \rho(Q^m|_{b\mathcal{B}(\mathcal{S})}) \cap \{\lambda;\ |\lambda| > \mathsf{r}_{\mathrm{ess}}(Q^m|_{b\mathcal{B}(\mathcal{S})})\}. \tag{4.5}$$

Denote

$$\Pi = \frac{1}{2\pi i}\int_\Gamma (\lambda I - Q^m|_{b\mathcal{B}(\mathcal{S})})^{-1}\, d\lambda. \tag{4.6}$$

By the Riesz decomposition theorem, $Q^m|_{b\mathcal{B}(\mathcal{S})} = Q^m|_{b\mathcal{B}(\mathcal{S})}\Pi + Q^m|_{b\mathcal{B}(\mathcal{S})}(I - \Pi)$ where $I - \Pi$ is the Riesz projector associated with the spectrum of $Q^m|_{b\mathcal{B}(\mathcal{S})}$ in $\{\lambda \in \mathbb{C};\ |\lambda| < 1\}$, $\Pi(I - \Pi) = (I - \Pi)\Pi = 0$, $Q^m|_{b\mathcal{B}(\mathcal{S})}$ commutes with both Π and

$I - \Pi$, and $r_{sp}(Q^m|_{b\mathcal{B}(\mathcal{S})}(I - \Pi)) < 1$, i.e.

$$\mathrm{r}_{\mathrm{sp}}(Q^m|_{b\mathcal{B}(\mathcal{S})} - Q^m|_{b\mathcal{B}(\mathcal{S})}\Pi|_{b\mathcal{B}(\mathcal{S})}) < 1. \tag{4.7}$$

Notice that condition (2) still holds for all $k > N_2$. Indeed, (2) implies $Q(\mathsf{x}, \cdot)$ is a positive measure for every $\mathsf{x} \in \mathcal{S}$ (otherwise, if $Q(\mathsf{x}_1, \mathcal{S}) = 0$ for some x_1, then $Q^{N_2}(\mathsf{x}_1, \mathcal{S}) = 0$, which contradicts assumption (2) in Theorem 4.1). Consequently, for any nonempty open subset O of $\mathcal{S}$ and any $\mathsf{x} \in \mathcal{S}$,

$$Q^k(\mathsf{x}, O) = \int_{\mathcal{S}} Q^{N_2}(\mathsf{y}, O)\, Q^{k-N_2}(\mathsf{x}, d\mathsf{y}) > 0.$$

By applying [61, Theorem 4.1.4 and the subsequent note][3] to $Q^m|_{\mathcal{C}_b(\mathcal{S})}$ there are some nonnegative $\varphi \in \mathcal{C}_b(\mathcal{S})$ (with $\varphi \neq 0$) and some nonnegative $\psi \in (\mathcal{C}_b(\mathcal{S}))^*$ (with $\psi \neq 0$) such that

$$Q^m\varphi = \varphi \quad \text{and} \quad (Q^m)^*\psi = \psi.$$

By [84, Proposition 4.3], ψ is a positive bounded measure μ on $\mathcal{S}$. We may assume that μ is a probability measure. We claim that μ charges all nonempty open subsets O of $\mathcal{S}$. Indeed, as $\mu Q^m = (Q^m)^*\mu = \mu$, one has

$$\mu(O) = \int_{\mathcal{S}} Q^m(\mathsf{x}, O)\, \mu(d\mathsf{x}) > 0$$

since $Q^m(\mathsf{x}, O) > 0$ everywhere on $\mathcal{S}$, proved before.

In the same way, for any $\mathsf{x} \in \mathcal{S}$, since $\varphi \neq 0$ is continuous,

$$\varphi(\mathsf{x}) = Q^m\varphi(\mathsf{x}) = \int_{\mathcal{S}} \varphi(\mathsf{y}) Q^m(\mathsf{x}, d\mathsf{y}) > 0,$$

i.e. φ is everywhere positive on $\mathcal{S}$.

By considering $\varphi/\mu(\varphi)$ if necessary, we may assume without loss of generality that $\mu(\varphi) = 1$.

Step 3: Proof that $\mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$ *is one-dimensional, i.e., generated by* φ. Let $f \in \mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$, i.e. $f \in \mathcal{C}_b(\mathcal{S})$ and $Q^m f = f$. Then $Q^m|f| \geq |f|$. Since

$$\mu(Q^m|f|) = \mu(|f|),$$

and since the function $Q^m|f| - |f|$ is nonnegative and continuous over $\mathcal{S}$, and μ charges all nonempty open subsets of $\mathcal{S}$, one deduces that $Q^m|f| = |f|$ everywhere on $\mathcal{S}$. In other words, $|f| \in \mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$, that is, $\mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$ is a lattice.

[3] Because $1 = \mathrm{r}_{\mathrm{sp}}(Q^m|_{\mathcal{C}_b(\mathcal{S})})$ is a pole of the resolvent of $Q^m|_{\mathcal{C}_b(\mathcal{S})}$. Indeed, by Lemma 4.3 (b), $1 = \mathrm{r}_{\mathrm{sp}}(Q^m|_{b\mathcal{B}(\mathcal{S})}) = \mathrm{r}_{\mathrm{sp}}(Q^m|_{\mathcal{C}_b(\mathcal{S})})$. In addition, $\mathrm{r}_{\mathrm{sp}}(Q^m|_{\mathcal{C}_b(\mathcal{S})}) \in \sigma(Q^m|_{\mathcal{C}_b(\mathcal{S})})$ (see [70, Chapter V, Proposition 4.1]). Finally, 1 is a pole of the resolvent of $Q^m|_{\mathcal{C}_b(\mathcal{S})}$ because $\mathrm{r}_{\mathrm{ess}}(Q^m|_{\mathcal{C}_b(\mathcal{S})}) = \mathrm{r}_{\mathrm{ess}}(Q^m|_{b\mathcal{B}(\mathcal{S})}) < \mathrm{r}_{\mathrm{sp}}(Q^m|_{b\mathcal{B}(\mathcal{S})}) = \mathrm{r}_{\mathrm{sp}}(Q^m|_{\mathcal{C}_b(\mathcal{S})}) = 1$.

If $0 \neq f \in \mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$ were linearly independent of φ, as f/φ is not constant, we could find $c \in \mathbb{R}$ such that the open sets

$$O_+ = \{f > c\varphi\} \quad \text{and} \quad O_- := \{f < c\varphi\}$$

are both nonempty. Since $(f - c\varphi)^+ \in \mathrm{Ker}(I - Q^m)$, we obtain, for $\mathsf{x} \in O_-$,

$$0 = (f - c\varphi)^+(\mathsf{x}) = \int_{\mathcal{S}} (f - c\varphi)^+(\mathsf{y})\, Q^m(\mathsf{x}, d\mathsf{y}) > 0.$$

This contradiction shows that $\mathrm{Ker}(I - Q^m) \cap \mathcal{C}_b(\mathcal{S})$ is generated by φ.

Step 4: Proof that the algebraic multiplicity and the geometric multiplicity of 1 *of the eigenvalue* $Q^m|_{b\mathcal{B}(\mathcal{S})}$ *coincide.* Let us prove that $(Q^m|_{b\mathcal{B}(\mathcal{S})} - I)\Pi = 0$. To this end, consider the Laurent series of $(\lambda I - Q^m|_{b\mathcal{B}(\mathcal{S})})^{-1}$ in a neighborhood of 1 in $\mathbb{C}$,

$$(\lambda I - Q^m|_{b\mathcal{B}(\mathcal{S})})^{-1} = A_{-l}(\lambda - 1)^{-l} + \cdots + A_{-1}(\lambda - 1)^{-1} + \sum_{k=0}^{\infty} A_k(\lambda - 1)^k,$$

where (see (4.6))

$$A_{-1} = \Pi$$

and $A_{-k-1} = (Q^m|_{b\mathcal{B}(\mathcal{S})} - I)^k \Pi$ [85, Chapter VIII, Section 8]. Notice that Π is a bounded Feller kernel by Lemma 4.3 and its definition, and thus so are $A_{-2}, \ldots, A_{-l}$.

We must prove that $l = 1$. For any bounded measurable function f over $\mathcal{S}$ such that $|f| \leq c\varphi$ for some $c > 0$, we have, for any $\lambda > 1$,

$$\begin{aligned}
|(\lambda - 1)(\lambda I - Q^m|_{b\mathcal{B}(\mathcal{S})})^{-1} f| &= \left|(\lambda - 1)\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} Q^{mn} f\right| \\
&\leq (\lambda - 1)\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} Q^{mn}|f| \\
&\leq c(\lambda - 1)\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} Q^{mn}\varphi = c\varphi,
\end{aligned}$$

i.e. $\{(\lambda - 1)(\lambda I - Q^m|_{b\mathcal{B}(\mathcal{S})})^{-1} f;\ \lambda > 1\}$ is uniformly bounded. Letting $\lambda \to 1^+$, we obtain $A_{-k} f = 0$ for any $k \geq 2$. Because A_{-k} is a bounded kernel and $A_{-k} f(\mathsf{x}) = 0$ for all $\mathsf{x} \in \mathcal{S}$ and such f, it follows that $A_{-k} = 0$ for all $k \geq 2$.

Step 5: Proof of (4.3). By Lemma 4.3 (b) and (4.7),

$$\mathsf{r}_{\mathrm{sp}}((Q^m - Q^m\Pi)|_{\mathcal{C}_b(\mathcal{S})}) < 1.$$

Since $Q^m\Pi|_{\mathcal{C}_b(\mathcal{S})} = \Pi|_{\mathcal{C}_b(\mathcal{S})}$ by Step 4 (because $A_{-2} = 0$ implies that $Q^m\Pi|_{b\mathcal{B}(\mathcal{S})} = \Pi|_{b\mathcal{B}(\mathcal{S})}$), one has $\mathsf{r}_{\mathrm{sp}}((Q^m - \Pi)|_{\mathcal{C}_b(\mathcal{S})})) < 1$ and for all $n \geq 1$, $Q^{mn}|_{\mathcal{C}_b(\mathcal{S})} - \Pi|_{\mathcal{C}_b(\mathcal{S})} = (Q^m - \Pi)^n|_{\mathcal{C}_b(\mathcal{S})}$. Therefore, by Gelfand's formula, there exist $C \geq 1$ and $r \in]0, 1[$ such that

$$\|Q^{mn} - \Pi\|_{\mathcal{C}_b(\mathcal{S})} = \|(Q^m - \Pi)^n\|_{\mathcal{C}_b(\mathcal{S})} \leq C r^n, \quad \forall n \geq 1.$$

Thus Π is a nonnegative (Feller) kernel and $\mu\Pi = \mu$.

As $\Pi|_{\mathcal{C}_b(\mathcal{S})}$ is a one-dimensional projection to $\{c\varphi;\ c\in\mathbb{R}\}$ by Steps 3 and 4, there is some $\psi\in(\mathcal{C}_b(\mathcal{S}))^*$ such that for any $f\in\mathcal{C}_b(\mathcal{S})$,

$$\Pi f=\psi(f)\varphi.$$

Integrating it with respect to μ and since $\mu(\varphi)=1$, we obtain $\mu(f)=\psi(f)$ for all $f\in\mathcal{C}_b(\mathcal{S})$, i.e. $\Pi f=\mu(f)\varphi$ for all $f\in\mathcal{C}_b(\mathcal{S})$ (and thus also for all $f\in b\mathcal{B}(\mathcal{S})$). In other words,

$$\Pi(\mathsf{x},d\mathsf{y})=\varphi(\mathsf{x})\mu(d\mathsf{y}).$$

In addition, we have

$$\|Q^{mn}-\Pi\|_{b\mathcal{B}(\mathcal{S})}\le Cr^n,\quad \forall n\ge 1. \tag{4.8}$$

because for a Feller kernel, such as $Q^{mn}-\Pi$, its norm on $b\mathcal{B}(\mathcal{S})$ coincides with its norm on $\mathcal{C}_b(\mathcal{S})$. This implies in particular that $\mathrm{Ker}(I-Q^m|_{b\mathcal{B}(\mathcal{S})})=\{c\varphi;\ c\in\mathbb{R}\}$ and $\mathrm{Ker}(I-(Q^m)^*|_{M_b(\mathcal{S})})=\{c\mu;\ c\in\mathbb{R}\}$.

Now for any eigenfunction f of Q in $b\mathcal{B}(\mathcal{S})$ associated with 1, we have $Q^m f=f$, so $f=c\varphi$. Thus $Q\varphi=\varphi$. Thus $Q\Pi=\Pi$ on $b\mathcal{B}(\mathcal{S})$.

Finally, the desired geometric convergence (4.3) follows from (4.8), because for $0\le k\le m-1$,

$$\|Q^{mn+k}-\Pi\|_{b\mathcal{B}(\mathcal{S})}=\|Q^k(Q^{mn}-\Pi)\|_{b\mathcal{B}(\mathcal{S})}\le\max_{k\le m-1}\|Q^k\|_{b\mathcal{B}(\mathcal{S})}\cdot\|Q^{mn}-\Pi\|_{b\mathcal{B}(\mathcal{S})}.$$

Step 6: Proofs of (a) *and* (b). By (4.3), if $\nu\in\mathcal{M}_{b\mathsf{W}}(\mathcal{S})$ is such that $\nu Q=\lambda\nu$, and $\nu(\varphi)\neq 0$, we have

$$\|\nu Q^n-\nu(\varphi)\mu\|_{\mathrm{TV}}=\|\lambda^n\nu-\nu(\varphi)\mu\|_{\mathrm{TV}}\to 0$$

as $n\to\infty$. As $\nu(\varphi)\neq 0$, $\lambda=1$ and $\nu=\nu(\varphi)\cdot\mu$. That is part (a). In the same way we get (b). This concludes the proof of Theorem 4.1. ∎

5. Proof of Theorem 2.2

5.1. Preliminary results

Let us start with the following proposition.

Proposition 5.1. *If $a<f\in\mathbb{D}_e(\mathcal{L})$ where $a\in[-\infty,+\infty[$, then for any $\mathcal{C}^2$ concave function $\varphi:]a,+\infty[\to\mathbb{R}$, we have $\varphi\circ f\in\mathbb{D}_e(\mathcal{L})$ and*

$$\mathcal{L}\varphi\circ f\le\varphi'(f)\mathcal{L}f. \tag{5.1}$$

Proof. For $t\ge 0$, let

$$M_t=f(X_t)-f(X_0)-\int_0^t\mathcal{L}f(X_s)\,ds,$$

which is a local martingale. By Ito's formula (Dellacherie–Meyer [28, p. 350, Théorème 27]), $\varphi \circ f \in \mathbb{D}_e(\mathcal{L})$ and

$$d\varphi \circ f(X_t) = \varphi'(f)(X_{t-})[\mathcal{L} f(X_{t-})dt + dM_t] + \frac{1}{2}\varphi''(X_{t-})d[M_c, M_c]_t + dS_t,$$

where M_c is the continuous martingale part of M, and

$$S_t = \sum_{0<s\le t} \big(\varphi \circ f(X_s) - \varphi \circ f(X_{s-}) - \varphi'(f)(X_{s-})[f(X_s) - f(X_{s-})]\big).$$

As φ is concave, $d\varphi \circ f(X_t) \le \varphi'(f)(X_{t-})[\mathcal{L} f(X_t)dt + dM_t]$. Thus (5.1) holds. ■

The following lemma establishes the strong Feller property of $P_t^{\mathcal{D}}$ on $\mathcal{D}$ for $t \ge t_0$.

Lemma 5.2. *Under* (**C1**) *and* (**C4**), $P_t^{\mathcal{D}}$ *is strongly Feller on* $\mathcal{D}$ *for all* $t \ge t_0$.

Proof. Let $(\mathsf{x}_n)_{n\ge 0}$ be a sequence of points in $\mathcal{D}$ converging to $\mathsf{x} \in \mathcal{D}$. Let us prove that $P_t^{\mathcal{D}} f(\mathsf{x}_n) \to P_t^{\mathcal{D}} f(\mathsf{x})$ for any $f \in b\mathcal{B}(\mathcal{D})$ and $t \ge t_0$ fixed. Let $K = \{\mathsf{x}, \mathsf{x}_n;\, n \ge 0\}$. One has

$$\beta_\tau(1_K P_t^{\mathcal{D}}) \le \beta_\tau(1_K P_t) = 0$$

by the strong Feller property of $(P_t,\ t \ge 0)$ on $\mathcal{S}$ (see (**C1**)). In other words, the family $\{P_t^{\mathcal{D}}(\mathsf{x}_n, \cdot);\ n \ge 0\}$ is relatively compact in the τ-topology. That is equivalent to saying that

$$\left\{h_n := \frac{dP_t^{\mathcal{D}}(\mathsf{x}_n, \cdot)}{dm};\ n \ge 0\right\}$$

is relatively compact in the weak topology $\sigma(L^1(m), L^\infty(m))$, where m is the reference measure given by

$$m(\cdot) = P_t^{\mathcal{D}}(\mathsf{x}, \cdot) + \sum_{n=0}^{\infty} \frac{1}{2^n} P_t^{\mathcal{D}}(\mathsf{x}_n, \cdot).$$

By the well known equivalence of the relative compactness and the sequential compactness in $\sigma(L^1, L^\infty)$ (by the Dunford–Pettis theorem [27, Théorème 25, p. 43]), we have only to prove that the limit point in the τ-topology of $P_t^{\mathcal{D}}(\mathsf{x}_n, \cdot)$ is unique and coincides with $P_t^{\mathcal{D}}(\mathsf{x}, \cdot)$, i.e. if $P_t^{\mathcal{D}}(\mathsf{x}_{n_k}, \cdot) \to \nu$ in the τ-topology for a subsequence (n_k), then $\nu = P_t^{\mathcal{D}}(\mathsf{x}, \cdot)$. By (**C4**), for any $f \in \mathcal{A}$ we have

$$\nu(f) = \lim_{k\to\infty} P_t^{\mathcal{D}} f(\mathsf{x}_{n_k}) = P_t^{\mathcal{D}} f(\mathsf{x}).$$

By the measure-separability of $\mathcal{A}$, $\nu = P_t^{\mathcal{D}}(\mathsf{x}, \cdot)$. ■

5.2. *Proof of the main result*

We will formulate a weaker version of Theorem 2.2.

Theorem 5.3. *Assume that* (**C1**)–(**C5**) *hold. Suppose moreover that for some* $p > 1$ *and* $M > 0$,

$$\mathcal{L}\mathsf{W}^p \le M\mathsf{W}^p. \tag{5.2}$$

Then all claims in Theorem 2.2 *hold with* $\mathsf{W}^{1/p}$ *replaced by* W.

Admitting this result, we give the proof of Theorem 2.2.

Proof of Theorem 2.2. For any $p > 1$ fixed, by Proposition 5.1 and (**C3**) one has

$$\mathcal{L}\mathsf{W}^{1/p} \le \frac{1}{p}\mathsf{W}^{1/p-1}\mathcal{L}\mathsf{W} \le -\frac{r_n}{p}\mathsf{W}^{1/p} + \frac{b_n}{p}1_{K_n}.$$

In other words, $\tilde{\mathsf{W}} = \mathsf{W}^{1/p}$ satisfies (**C3**). Furthermore, for each n,

$$\mathcal{L}\tilde{\mathsf{W}}^p = \mathcal{L}\mathsf{W} \le b_n 1_{K_n} \le b_n \mathsf{W} = b_n \tilde{\mathsf{W}}^p. \tag{5.3}$$

Thus applying Theorem 5.3 to $\tilde{\mathsf{W}}$, we obtain Theorem 2.2. ∎

Let us now prove Theorem 5.3.

Proof of Theorem 5.3. The proof is divided into several steps.

Step 1: Proof that for any $t > 0$ fixed, $P_t^{\mathcal{D}}$ satisfies assumptions (1)–(4) *in Theorem* 4.1 *with $\mathcal{S} = \mathcal{D}$.* First, reasoning as in (3.8) one deduces, using (5.2) together with the fact that $P_t^{\mathcal{D}} \le P_t$, that $(P_t^{\mathcal{D}}\mathsf{W}^p)(\mathsf{x}) \le e^{Mt}\mathsf{W}(\mathsf{x})^p$ for all $\mathsf{x} \in \mathcal{D}$. Thus $P_t^{\mathcal{D}}$ satisfies assumption (3) in Theorem 4.1. In addition (**C5**) implies that $P_t^{\mathcal{D}}$ satisfies assumption (2). Recall that

$$P_{t,\mathsf{W}}^{\mathcal{D}} = \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})}P_t^{\mathcal{D}}(\mathsf{x}, d\mathsf{y}) \le \frac{\mathsf{W}(\mathsf{y})}{\mathsf{W}(\mathsf{x})}P_t(\mathsf{x}, d\mathsf{y}) = P_{t,\mathsf{W}}(\mathsf{x}, d\mathsf{y})$$

and $P_{t,\mathsf{W}}^{\mathcal{D}}$ also satisfies (**A1**) (by (3.12) and (3.13)). Then, by Theorem 3.4 and (3.12),

$$\mathsf{r}_{\mathrm{ess}}(P_t^{\mathcal{D}}|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}) = \lim_{n\to\infty}[\beta_w(P_{nt,\mathsf{W}}^{\mathcal{D}})]^{1/n} = 0.$$

Furthermore, by Lemma 5.2, $P_t^{\mathcal{D}}$ is strongly Feller on $\mathcal{D}$ for all $t \ge t_0$ (and thus in particular $P_t^{\mathcal{D}}$ satisfies assumption (1) in Theorem 4.1). This together with the topological transitivity in (**C5**) implies that for any $t > 0$, $P_t^{\mathcal{D}}$ is m_1-irreducible, where $m_1 = \int_{t_0}^{+\infty} e^{-s}P_s^{\mathcal{D}}(\mathsf{x}_1, \cdot)\,ds$ for some $\mathsf{x}_1 \in \mathcal{D}$. Indeed, let $A \in \mathcal{B}(\mathcal{D})$ be such that $m_1(A) > 0$. The function $g_1(\mathsf{x}) := \int_{t_0}^{+\infty} e^{-s}P_s^{\mathcal{D}}(\mathsf{x}, A)\,ds$ is continuous (since $P_t^{\mathcal{D}}$ is strongly Feller on $\mathcal{D}$ for all $t \ge t_0$) and positive at x_1 (by choice of A). Then, by (**C5**), if $Nt \ge t_0$ we have

$$P_{Nt}^{\mathcal{D}}g_1(\mathsf{x}) > 0, \quad \forall \mathsf{x} \in \mathcal{D}.$$

By Nummelin [63, Theorem 3.2],

$$\mathsf{r}_{\mathrm{sp}}(P_t^{\mathcal{D}}|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}) = \lim_{n\to\infty}\left(\sup_{\mathsf{x}\in\mathcal{D}}\frac{\mathbb{E}_{\mathsf{x}}[\mathsf{W}(X_{nt})1_{nt<\sigma_{\mathcal{D}}}]}{\mathsf{W}(\mathsf{x})}\right)^{1/n} > 0.$$

Thus, we have proved that $P_t^{\mathcal{D}}$ satisfies assumption (4) in Theorem 4.1.

Step 2. Let $\lambda_{\mathcal{D}} := -\log \mathsf{r}_{\rm sp}(P_1^{\mathcal{D}}|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})})$. Applying Theorem 4.1 to $Q = P_1^{\mathcal{D}}$ on $b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$, there is a unique couple $(\mu_{\mathcal{D}}, \varphi)$ where $\mu_{\mathcal{D}}$ is a probability measure on $\mathcal{D}$ with $\mu_{\mathcal{D}}(\mathsf{W}) < +\infty$, $\varphi \in \mathcal{C}_{b\mathsf{W}}(\mathcal{S})$ is positive everywhere on $\mathcal{D}$, $\mu_{\mathcal{D}}(\varphi) = 1$ and

$$\mu_{\mathcal{D}} P_1^{\mathcal{D}} = e^{-\lambda_{\mathcal{D}}} \mu_{\mathcal{D}}, \quad P_1^{\mathcal{D}} \varphi = e^{-\lambda_{\mathcal{D}}} \varphi,$$

and for all $f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$ and $n \ge 1$,

$$\|e^{n\lambda_{\mathcal{D}}} P_n^{\mathcal{D}} f - \mu_{\mathcal{D}}(f)\varphi\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})} \le C e^{-\delta n} \|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}, \tag{5.4}$$

where $C \ge 1$ and $\delta > 0$ are independent of f and n. In addition, for any $t > 0$, since $(\mu_{\mathcal{D}} P_t^{\mathcal{D}}) P_1^{\mathcal{D}} = (\mu_{\mathcal{D}} P_1^{\mathcal{D}}) P_t^{\mathcal{D}} = e^{-\lambda_{\mathcal{D}}} \mu_{\mathcal{D}} P_t^{\mathcal{D}}$ and $\mathrm{Ker}(e^{-\lambda_{\mathcal{D}}} I - (P_1^{\mathcal{D}})^*)$ in $\mathcal{M}_{b\mathsf{W}}(\mathcal{D})$ is one-dimensional and $\mu_{\mathcal{D}} P_t^{\mathcal{D}} \in \mathcal{M}_{b\mathsf{W}}(\mathcal{D})$, one deduces that $\mu_{\mathcal{D}} P_t^{\mathcal{D}} = \lambda(t)\mu_{\mathcal{D}}$. By the semigroup property, $\lambda(t+s) = \lambda(t)\cdot\lambda(s)$. As $\lambda(1) = e^{-\lambda_{\mathcal{D}}}$, one obtains

$$\lambda(t) = e^{-\lambda_{\mathcal{D}} t}, \quad t \ge 0.$$

By Theorem 4.1 (a),

$$\mathsf{r}_{\rm sp}(P_t^{\mathcal{D}}|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}) = \lambda(t) = e^{-\lambda_{\mathcal{D}} t}.$$

Since

$$\mathbb{P}_{\mu_{\mathcal{D}}}(t < \tau_D) = \mu_{\mathcal{D}} P_t^{\mathcal{D}} \mathsf{1} = e^{-\lambda_{\mathcal{D}} t} \mu_{\mathcal{D}}(\mathsf{1}) = e^{-\lambda_{\mathcal{D}} t},$$

we have $\lambda_{\mathcal{D}} \ge 0$ and

$$\mathbb{P}_{\mu_{\mathcal{D}}}(X_t \in \cdot \,|\, t < \tau_D) = e^{\lambda_{\mathcal{D}} t} \mu_{\mathcal{D}} P_t^{\mathcal{D}}(\cdot) = \mu_{\mathcal{D}},$$

i.e. $\mu_{\mathcal{D}}$ is a QSD.

Let us now prove the uniqueness of the QSD of $(X_t, t \ge 0)$ in $\mathcal{D}$ in the set of measures ν such that $\nu(\mathsf{W}) < +\infty$. To this end, let us consider a QSD $\nu_{\mathcal{D}}$ satisfying $\nu_{\mathcal{D}}(\mathsf{W}) < +\infty$. Then for all $t \ge 0$,

$$\nu_{\mathcal{D}} P_t^{\mathcal{D}} = \lambda(t)\nu_{\mathcal{D}}, \quad \lambda(t) = \mathbb{P}_{\nu_{\mathcal{D}}}(t < \sigma_{\mathcal{D}}).$$

By Theorem 4.1 (a), this implies that

$$\lambda(t) = \mathsf{r}_{\rm sp}(P_t^{\mathcal{D}}|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}) = e^{-\lambda_{\mathcal{D}} t} \quad \text{and} \quad \nu_{\mathcal{D}} = \mu_{\mathcal{D}},$$

which concludes the proof of uniqueness.

Finally, for any $t = n + s$ with $s \in [0, 1[$, by (5.4) we have, for all $f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$,

$$\|e^{(n+s)\lambda_{\mathcal{D}}} P_{n+s}^{\mathcal{D}} f - e^{s\lambda_{\mathcal{D}}} \mu_{\mathcal{D}}(P_s^{\mathcal{D}} f)\varphi\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})} \le C r^n e^{s\lambda_{\mathcal{D}}} \|P_s^{\mathcal{D}} f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}.$$

As $e^{s\lambda_{\mathcal{D}}} \mu_{\mathcal{D}}(P_s^{\mathcal{D}} f) = \mu_{\mathcal{D}}(f)$ and $\sup_{s\in[0,1]} \|P_s^{\mathcal{D}}\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})} \le \sup_{s\in[0,1]} \|P_s \mathsf{W}\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}$ $\le e^{b_1}$ (by the proof in Step 1 of Theorem 3.5 and (**C3**)), we obtain, for all $f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$,

$$\|e^{t\lambda_{\mathcal{D}}} P_t^{\mathcal{D}} f - \mu_{\mathcal{D}}(f)\varphi\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})} \le C' e^{-\delta t} \|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}, \quad \delta := -\log r,\ C' = C e^{b_1} e^{\lambda_{\mathcal{D}}}. \tag{5.5}$$

Thus, for all $f \in b_{\mathsf{W}}\mathcal{B}(\mathcal{D})$ and all measures $\nu \in \mathcal{M}_{b\mathsf{W}}(\mathcal{D})$,

$$\begin{aligned}\left|\mathbb{E}_\nu[f(X_t)\,|\,t<\sigma_{\mathcal{D}}]-\mu_{\mathcal{D}}(f)\right| &= \left|\frac{e^{\lambda_{\mathcal{D}}t}\nu(P_t^{\mathcal{D}}f)}{e^{\lambda_{\mathcal{D}}t}\nu(P_t^{\mathcal{D}}1)}-\mu_{\mathcal{D}}(f)\right|\\ &= \left|\frac{\mu_{\mathcal{D}}(f)\nu(\varphi)+O_1(e^{-\delta t})\nu(\mathsf{W})\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}}{\nu(\varphi)+O_2(e^{-\delta t})\nu(\mathsf{W})\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}}-\mu_{\mathcal{D}}(f)\right|\\ &= \left|\frac{\mu_{\mathcal{D}}(f)+O_1(e^{-\delta t})\frac{\nu(\mathsf{W})}{\nu(\varphi)}\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}}{1+O_2(e^{-\delta t})\frac{\nu(\mathsf{W})}{\nu(\varphi)}\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})}}-\mu_{\mathcal{D}}(f)\right|\\ &\le O_3(t)\frac{\nu(\mathsf{W})}{\nu(\varphi)}\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})},\end{aligned}$$

where for all $k=1,2,3$ and all $t\ge 0$, $|O_k(t)|\le Ce^{-\delta t}$ for some constant C independent of ν and f. That yields

$$\left|\mathbb{E}_\nu[f(X_t)\,|\,t<\sigma_{\mathcal{D}}]-\mu_{\mathcal{D}}(f)\right| \le Ce^{-\delta t}\frac{\nu(\mathsf{W})}{\nu(\varphi)}\|f\|_{b_{\mathsf{W}}\mathcal{B}(\mathcal{D})},\quad \forall f\in b_{\mathsf{W}}\mathcal{B}(\mathcal{D}),\ t>0.$$

Step 3: Conclusion of the proof. We have proved that $\lambda_{\mathcal{D}}\in[0,+\infty)$. Let us now prove that $\lambda_{\mathcal{D}}>0$. If $\lambda_{\mathcal{D}}=0$, then for all $t\ge 0$, $\mu_{\mathcal{D}}(P_t^{\mathcal{D}}1)=\mu_{\mathcal{D}}(1)=1$. This implies that $P_t^{\mathcal{D}}1(\mathsf{x})=1$ for all $\mathsf{x}\in\mathcal{D}$ and $t>0$, due to the fact that the function $1-P_t^{\mathcal{D}}1$ is nonnegative and continuous over $\mathcal{D}$ (by the Feller property of $P_t^{\mathcal{D}}$) and that $\mu_{\mathcal{D}}$ charges all nonempty open subsets of $\mathcal{D}$. That contradicts the second assumption in (**C5**). Thus $\lambda_{\mathcal{D}}>0$.

Now for every $\mathsf{x}\in\mathcal{D}$, by (5.5) with $f=1$,

$$\mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}}=+\infty)=\lim_{t\to+\infty}e^{-\lambda_{\mathcal{D}}t}e^{\lambda_{\mathcal{D}}t}\mathbb{P}_{\mathsf{x}}(t<\sigma_{\mathcal{D}})=0\cdot\varphi(\mathsf{x})\mu_{\mathcal{D}}(1)=0.$$

Next $\mathbb{P}_{\mu_{\mathcal{D}}}(t<\sigma_{\mathcal{D}})=\mu_{\mathcal{D}}(P_t^{\mathcal{D}}1)=e^{-\lambda_{\mathcal{D}}t}$. It remains to prove the independence of $X_{\sigma_{\mathcal{D}}}$ and $\sigma_{\mathcal{D}}$, under $\mathbb{P}_{\mu_{\mathcal{D}}}$. For any $f\in b\mathcal{B}(\partial\mathcal{D})$, letting

$$u(\mathsf{x})=\mathbb{E}_{\mathsf{x}}[f(X_{\sigma_{\mathcal{D}}})]\quad\text{for } \mathsf{x}\in\mathcal{D},$$

by the strong Markov property we have

$$\mathbb{E}_{\mu_{\mathcal{D}}}[f(X_{\sigma_{\mathcal{D}}})1_{t<\sigma_{\mathcal{D}}}]=\mu_{\mathcal{D}}(P_t^{\mathcal{D}}u)=e^{-\lambda_{\mathcal{D}}t}\mu_{\mathcal{D}}(u)=e^{-\lambda_{\mathcal{D}}t}\mathbb{E}_{\mu_{\mathcal{D}}}[f(X_{\sigma_{\mathcal{D}}})],\ t\ge 0,$$

which is the desired independence. This concludes the proof of Theorem 5.3. ∎

Remark 5.4 (On Step 3 in the proof of Theorem 5.3). To prove that $\lambda_{\mathcal{D}}>0$, it is also possible to use the standard result [60, Proposition 2]. In addition, it is also standard that $X_{\sigma_{\mathcal{D}}}$ and $\sigma_{\mathcal{D}}$ are $\mathbb{P}_{\mu_{\mathcal{D}}}$-independent as long as $\mu_{\mathcal{D}}$ is a QSD [60, Proposition 2].

6. Application to hypoelliptic damped Hamiltonian systems

In this section, we apply Theorem 2.2 to hypoelliptic damped Hamiltonian systems on $\mathbb{R}^{2d}$ (see (6.1)) when $\mathcal{D}=\mathsf{O}\times\mathbb{R}^d$ ($\mathsf{O}\subset\mathbb{R}^d$, see more precisely (6.25)). To this end, we first define the setting we consider, and then we check that the assumptions required to apply Theorem 2.2 are satisfied for such processes (namely (**C1**)–(**C5**)): these are the purposes of Sections 6.1 and 6.2 respectively. Finally, in Section 6.3, we state the main result of this section, which is Theorem 6.9.

6.1. Framework and assumptions

Let $d \geq 1$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let $(X_t = (x_t, v_t), t \geq 0)$ be the solution of the following hypoelliptic stochastic differential equation on $\mathbb{R}^{2d}$:

$$\begin{cases} dx_t = v_t dt, \\ dv_t = -\nabla V(x_t)dt - c(x_t, v_t)v_t dt + \Sigma(x_t, v_t)dB_t, \end{cases} \tag{6.1}$$

where $(B_t, t \geq 0)$ is a standard d-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here the state space is $\mathcal{S} = \mathbb{R}^{2d}$. Equation (6.1) describes a system of N particles (in this case $d = 3\mathsf{N}$) moving under interaction forces which are subject to random collisions. The function c is the damping (or friction) coefficient and V is the particle interaction potential function. We refer for instance to [59, 75, 83], and to the review of the literature [56] for the study of such processes in $\mathbb{R}^{2d}$. Let us define the following assumptions on V and c:

(**Av1**) $V : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^1$ and lower bounded on $\mathbb{R}^d$.

(**Ac1**) $c : \mathbb{R}^{2d} \to \mathbb{R}^{d\times d}$ is continuous. In addition, there exist $\eta > 0$ and $L > 0$ such that

$$\forall v \in \mathbb{R}^d, |x| \geq L, \quad \tfrac{1}{2}[c(x, v) + c^T(x, v)] \geq \eta I_{\mathbb{R}^d},$$

and for all $N > 0$,

$$\sup_{|x|\leq N,\, v\in\mathbb{R}^d} \|c(x, v)\|_{\mathrm{HS}} < +\infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm of a matrix and c^T is the transpose of c.

(**A**Σ) $\Sigma : \mathbb{R}^{2d} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, uniformly Lipschitz over $\mathbb{R}^{2d}$, and such that for some $\Sigma_0 > 0$ and $\Sigma_\infty > 0$,

$$\forall \mathsf{x} \in \mathbb{R}^{2d}, \quad \Sigma_0 \leq \Sigma(\mathsf{x}) \leq \Sigma_\infty.$$

For some results below, (**Ac1**) can be replaced by a less stringent assumption:

(**Ac0**) $c : \mathbb{R}^{2d} \to \mathbb{R}^{d\times d}$ is continuous and

$$\exists A \geq 0,\ \forall x, v \in \mathbb{R}^d, \quad \tfrac{1}{2}[c(x, v) + c^T(x, v)] \geq -A I_{\mathbb{R}^d}.$$

This will allow us to consider in particular hypoelliptic damped Hamiltonian systems with unbounded v-dependent damping coefficient:

$$\exists \ell_0 > 0,\ \forall x, v \in \mathbb{R}^d, \quad c(x, v) = |v|^{\ell_0}, \tag{6.2}$$

and with fast growing potential, in the sense that there exist $\mathsf{n}_0 > 2$ and $r_0, r > 0$ such that for all $|x| \geq r_0$,

$$V \text{ satisfies } (\mathbf{Av1}), \quad r^{-1}|x|^{\mathsf{n}_0} \leq V(x) \leq r|x|^{\mathsf{n}_0} \quad \text{and} \quad r^{-1}|x|^{\mathsf{n}_0} \leq x \cdot \nabla V(x). \tag{6.3}$$

Notice that when (6.2) holds, assumption (**Ac1**) is not satisfied but (**Ac0**) is satisfied. The condition that $\mathsf{n}_0 > 2$ is justified in Proposition 6.5 (2) below.

Remark 6.1. Condition (6.3) is satisfied for instance for a $\mathcal{C}^1$ function V over $\mathbb{R}^d$ which equals a polynomial function with leading term $a|x|^{2n}$, with $n \geq 2$ and $a > 0$, outside a compact subset of $\mathbb{R}^d$.

When V, c, and Σ satisfy respectively (**Av1**), (**Ac0**), and (**A**Σ), there is a unique weak solution to (6.1) by [83, Lemma 1.1], which is thus a strong Markov process. We will thus always assume at least (**A**Σ), (**Av1**) and (**Ac0**) in what follows.

For $t \geq 0$, we recall that $(P_t, t \geq 0)$ denotes the semigroup of the process $(X_t, t \geq 0)$, that is, $P_t(\mathsf{x}, A) = \mathbb{P}_{\mathsf{x}}(X_t \in A)$, where $A \in \mathcal{B}(\mathbb{R}^{2d})$ and $\mathsf{x} = (x, v) \in \mathbb{R}^{2d}$. In the following, we denote by $(X_t^0(\mathsf{x}), t \geq 0)$ the process $(X_t, t \geq 0)$ when $X_0 = \mathsf{x}$. Let us also denote by

$$\mathcal{L}_0 = \frac{\Sigma(x, v)^2}{2}\Delta_v + v \cdot \nabla_x - \nabla V(x).\nabla_v - c(x, v)v \cdot \nabla_v \tag{6.4}$$

the infinitesimal generator of the diffusion (6.1). Let us recall that $\mathbb{D}_e(\mathcal{L})$ denotes the extended domain of the generator of the semigroup $(P_t, t \geq 0)$ of the process (6.1) (see (2.4) for the definition).

Let us check that the assumptions required to apply Theorem 2.2 are satisfied for the process (6.1) when $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$ ($\mathsf{O} \subset \mathbb{R}^d$, see more precisely (6.25)), by prescribing more assumptions on V, c, and ℓ_0 if necessary.

6.2. *On assumptions* (**C1**)–(**C5**)

6.2.1. *On assumptions* (**C1**) *and* (**C2**).
One has the following result from [83].

Lemma 6.2. *Assume that V, c, and Σ satisfy respectively* (**Av1**), (**Ac0**), *and* (**A**Σ). *Then* (**C1**) *and* (**C2**) *are satisfied for the process* (6.1).

Proof. Let us first prove that $(P_t, t \geq 0)$ satisfies (**C1**). Introduce the process $(X_t^0 = (x_t^0, v_t^0), t \geq 0)$ solving (in the strong sense) the stochastic differential equation over $\mathbb{R}^{2d}$:

$$\begin{cases} dx_t^0 = v_t^0 dt, \\ dv_t^0 = \Sigma(x_t^0, v_t^0) dB_t. \end{cases} \tag{6.5}$$

That is, $(X_t^0, t \geq 0)$ is the process (6.1) when $V = 0$ and $c = 0$. Let $(P_t^0, t \geq 0)$ be the semigroup of process (6.5). Under (**A**Σ), for $t > 0$, P_t^0 has a smooth density $\mathbb{R}^{2d} \ni (\mathsf{x}, \mathsf{y}) \mapsto p_t^0(\mathsf{x}, \mathsf{y})$ with respect to the Lebesgue measure $d\mathsf{y}$ by Hörmander's theorem. Therefore, when $(\mathsf{x}_n)_n$ converges to $\mathsf{x} \in \mathbb{R}^{2d}$ as $n \to \infty$, $p_t^0(\mathsf{x}_n, \mathsf{y}) \to p_t^0(\mathsf{x}, \mathsf{y})$. Since for all n, $\int_{\mathbb{R}^{2d}} p_t^0(\mathsf{x}_n, \mathsf{y})\, d\mathsf{y} = \int_{\mathbb{R}^{2d}} p_t^0(\mathsf{x}, \mathsf{y})\, d\mathsf{y} = 1$, it follows by Scheffé's lemma that $p_t^0(\mathsf{x}_n, \mathsf{y}) \to p_t^0(\mathsf{x}, \mathsf{y})$ in $L^1(\mathbb{R}^{2d}, d\mathsf{y})$ as $n \to \infty$. Hence for any $f \in b\mathcal{B}(\mathbb{R}^{2d})$, $P_t^0 f(\mathsf{x}_n) = \int_{\mathbb{R}^{2d}} f(\mathsf{y}) p_t^0(\mathsf{x}, \mathsf{y})\, d\mathsf{y} \to \int_{\mathbb{R}^{2d}} f(\mathsf{y}) p_t^0(\mathsf{x}, \mathsf{y})\, d\mathsf{y} = P_t^0 f(\mathsf{x})$. That is, P_t^0 is strongly Feller for $t > 0$.

When V and c satisfy (**Av1**) and (**Ac0**), using the same arguments as in the proof of [83, Proposition 1.2] we deduce that, for $t > 0$, P_t is strongly Feller and thus satisfies (**C1**).

Moreover, for any $T > 0$, the mapping

$$\mathbb{R}^{2d} \ni \mathsf{x} \mapsto \mathbb{P}_{\mathsf{x}}(X_{[0,T]} \in \cdot) \in \mathcal{M}_1(\mathcal{C}^0([0,T], \mathbb{R}^{2d}))$$

is continuous with respect to the weak convergence of measures on $\mathcal{C}^0([0,T], \mathbb{R}^{2d})$ (equipped with the uniform convergence topology). Indeed, the weak solutions of (6.1) with starting point x_n converge to the solution of the martingale problem with starting point x, as $\mathsf{x}_n \to \mathsf{x} \in \mathbb{R}^{2d}$, by the continuity of the coefficients in (6.1) and the uniqueness of the weak solution of (6.1) (see [83, Lemma 1.1]). Thus (**C2**) is satisfied for the process (6.1). ∎

6.2.2. On assumption (**C3**). Let us define the following last assumptions on V and c.

(**Av2**) There exists a $\mathcal{C}^1$ function $G : \mathbb{R}^d \to \mathbb{R}^d$ such that G and ∇G are bounded over $\mathbb{R}^d$, and

$$\nabla V(x) \cdot G(x) \to +\infty \quad \text{as } |x| \to +\infty.$$

(**Ac2**) There exists a $\mathcal{C}^2$ lower bounded function $U : \mathbb{R}^d \to \mathbb{R}$ such that

$$\sup_{x,v \in \mathbb{R}^d} |c^T(x,v)G(x) - \nabla U(x)| < +\infty.$$

Remark 6.3. Let us recall some examples of functions V and c satisfying (**Av1**), (**Av2**), (**Ac1**), and (**Ac2**) (see [83, Remark 3.2]):

(1) If the damping coefficient c satisfies (**Ac1**) with

$$\sup_{x,v \in \mathbb{R}^d} \|c(x,v)\|_{\mathrm{HS}} < +\infty,$$

then (**Ac2**) is satisfied with $U = 0$.

(2) Assume that V satisfies (**Av1**).

(a) Assume that $\lim_{|x| \to +\infty} \frac{x \cdot \nabla V(x)}{|x|} = +\infty$. Then (**Av2**) is satisfied with

$$x \mapsto G(x) = \frac{x}{|x|}(1 - \chi),$$

where $\chi : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^\infty$, has compact support, and $\chi = 1$ in a neighborhood of 0 in $\mathbb{R}^d$. In particular, (**Av2**) and (**Ac2**) are satisfied when $V : \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that $\nabla V(x) \cdot x \geq c_0|x|^{2k}$ ($k \in \mathbb{N}^*$, $c_0 > 0$) outside a compact subset of $\mathbb{R}^d$ and $c(x,v) = c_1|x|^{2q}$ ($q \in \mathbb{N}^*$, $c_1 > 0$) on $\mathbb{R}^{2d}$ (indeed, choose G as above and $U(x) = c_1(1-\chi)|x|^{2q+1}/(2q+1)$).

(b) Assume that there exists $r > 0$ such that $\{|x| > r\} \ni x \mapsto \mathsf{e}_V(x) = \nabla V(x)/|\nabla V(x)|$ is $\mathcal{C}^1$, bounded, and with bounded derivatives, and $\lim_{|x| \to +\infty} |\nabla V(x)| = +\infty$.

Then (**Av2**) is satisfied with

$$x \mapsto G(x) = \mathsf{e}_V(x)(1-\chi),$$

where $\chi : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^\infty$, has compact support, and $\chi = 1$ on $B(0, r+1)$.

Notice that when $d = 1$, the three conditions $\lim_{|x|\to+\infty} \frac{x\cdot\nabla V(x)}{|x|} = +\infty$, (**Ac2**), and $\lim_{|x|\to+\infty} |\nabla V(x)| = +\infty$ are equivalent (under (**Av1**)).

(3) When $d = 1$, the case when there exist $c_1, c_2, w_0 > 0$ such that

$$\forall x, v \in \mathbb{R}, \quad c(x,v) = c_1 x^2 - c_2 \quad \text{and} \quad V(x) = \tfrac{1}{2} w_0^2 x^2, \tag{6.6}$$

corresponds to the noisy Van Der Pol oscillator. Then (**Av1**), (**Av2**), (**Ac1**), and (**Ac2**) are satisfied with $G(x) = x(1-\chi)/|x|$, and $U(x) = [c_1|x|^3/3 - c_2|x|](1-\chi)$, where $\chi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^\infty$, has compact support, and $\chi = 1$ in a neighborhood of 0 (see [83, Section 5.3]).

The Hamiltonian of the process (6.1) is, for $x, v \in \mathbb{R}^d$,

$$\mathsf{H}(x,v) = V(x) + \tfrac{1}{2}|v|^2.$$

Assume that (**A**Σ), (**Av1**), (**Av2**), (**Ac1**), and (**Ac2**) hold. Let us introduce, for $(x,v) \in \mathbb{R}^{2d}$, the modified Hamiltonian [83, (3.3)]

$$\mathsf{F}_1(x,v) = a\,\mathsf{H}(x,v) + v\cdot(b\,G(x) + \nabla\mathsf{w}(x)) + b\,U(x), \tag{6.7}$$

where G, U are as in (**Av2**) and (**Ac2**), $a, b > 0$, and $\mathsf{w} : \mathbb{R}^d \to \mathbb{R}$ is a compactly supported $\mathcal{C}^2$ function. Define, for all $x, v \in \mathbb{R}^d$,

$$\mathsf{W}_1(x,v) = \exp\Big[\mathsf{F}_1(x,v) - \inf_{\mathbb{R}^{2d}} \mathsf{F}_1\Big] \geq 1. \tag{6.8}$$

We now give a concrete upper bound on W_1 which is useful to verify the integrability condition in Theorem 2.2.

Lemma 6.4. *Assume that V satisfies* (**Av1**) *and* (**Av2**). *Then* $\lim_{x\to+\infty} V(x) = +\infty$. *Let c be such that* (**Ac1**) *and* (**Ac2**) *hold with* $\lim_{|x|\to+\infty} U(x)/V(x) = 0$. *Then, for any $\varepsilon > 0$, there exists $R > 0$ such that if $|x| + |v| \geq R$, then $\mathsf{W}_1(x,v) \leq e^{a(1+\varepsilon)\mathsf{H}(x,v)}$.*

Proof. Let us prove that $\lim_{x\to+\infty} V(x) = +\infty$. Assume without loss of generality that $V \geq 0$. Notice that (since G is bounded) there exists $C > 0$ such that $1/|G| \geq C$ over $\mathbb{R}^d$. In addition, in view of (**Av2**), there exists $r_0 > 0$ such that $|G(x)| > 0$ for all $|x| \geq r_0$. Let $R_0 > r_0$ and $C_0 > 0$ be such that $\nabla V(y)\cdot G(y) \geq C_0$ for $|y| \geq R_0$ (see (**Av2**)). Consider for $|x| > R_0$ the curve $\gamma_t(x)$ solving $\dot\gamma_t(x) = -G(\gamma_t(x))/|G(\gamma_t(x))|$ with $\gamma_0(x) = x$ for all $t \in [0, T_0(x)]$, where $T_0(x) = \inf\{t \geq 0;\ \gamma_t(x) = R_0\} \in \mathbb{R}_+^* \cup \{+\infty\}$. Notice that $T_0(x) \geq |x| - R_0$. Consequently, if $T_0(x) < +\infty$ then

$$V(x) \geq \int_0^{T_0(x)} \frac{\nabla V(\gamma_s(x))\cdot G(\gamma_s(x))}{|G(\gamma_s(x))|}\, ds \geq C C_0(|x| - R_0).$$

The case $T_0(x) = +\infty$ is not possible since it would imply that $V(x) \geq C C_0 t$ for all $t \geq 0$. Thus $\lim_{x\to+\infty} V(x) = +\infty$. The proof of the upper bound on W_1 is a consequence of the fact that, for any $a, b > 0$ fixed, $v \cdot (b\, G(x) + \nabla \mathsf{w}(x)) + b\, U(x) = o(a\mathsf{H}(x, v))$ as $|x| + |v| \to +\infty$. ∎

Let us mention that because $\mathsf{W}_1 \in \mathcal{C}^{1,2}(\mathbb{R}^d \times \mathbb{R}^d)$ (i.e. W_1 is $\mathcal{C}^1$ in the variable x and $\mathcal{C}^2$ in v), we have $\mathsf{W}_1 \in \mathbb{D}_e(\mathcal{L})$ and $\mathcal{L}\mathsf{W}_1 = \mathcal{L}_0\mathsf{W}_1$ quasi-everywhere (see (2.4)).

Let us now check that (**C3**) is satisfied for $(P_t, t \geq 0)$ under the above assumptions on V and c. This is the purpose of the next proposition.

Proposition 6.5. *Assume that* Σ *satisfies* (**A**Σ).

(1) *Assume that the functions* V *and* c *satisfy* (**Av1**), (**Av2**), (**Ac1**), *and* (**Ac2**). *Then, for a suitable function* $\mathsf{w} \in C_c^2(\mathbb{R}^d, \mathbb{R})$, *and some constants* $a, b > 0$ (*see* [83, (3.4)–(3.9)] *for explicit conditions*), *assumption* (**C3**) *is satisfied for the process* (6.1) *with the function* W_1 *defined in* (6.8).

(2) *Assume that* c *and* V *satisfy respectively* (6.2) *and* (6.3). *Then assumption* (**C3**) *is satisfied for the process* (6.1) *if*

$$\ell_0 < \mathsf{n}_0 - 2 \tag{6.9}$$

with the continuous bounded *Lyapunov function* $\mathsf{W}_2 : \mathbb{R}^{2d} \to \mathbb{R}$ *defined in* (6.12) *below.*

Let us mention that W_1 (see (6.8)) and W_2 (see (6.12)) in Proposition 6.5 are not unique by construction (see indeed [83, (3.4)–(3.9)] and the proof of Proposition 6.5 (2) below). Moreover, W_1 is not bounded over $\mathbb{R}^{2d}$. Concerning Proposition 6.5 (1), we also refer to [83, Section 5] for other Lyapunov functions in explicit examples like the noisy Van Der Pol oscillator.

Proof of Proposition 6.5. Item (1) is proved in [83, Section 3] (see more precisely (3.9) there). To prove (2), assume that c and V satisfy respectively (6.2) and (6.3). Recall that the Hamiltonian of (6.1) is

$$\mathbb{R}^{2d} \ni (x, v) \mapsto |v|^2/2 + V(x).$$

The infinitesimal generator of the process (6.1) is in this case (see (6.2)–(6.4))

$$\mathcal{L}_0 = \frac{\Sigma(x, v)^2}{2}\Delta_v + v \cdot \nabla_x - [\nabla V(x) + |v|^{\ell_0} v] \cdot \nabla_v.$$

Let us construct a Lyapunov function for such a process. To avoid any problem of regularity at 0 in the upcoming computations, let us actually consider

$$\mathbb{R}^{2d} \ni (x, v) \mapsto \mathsf{H}_2(x, v) = |v|^2/2 + V(x) + \mathsf{k}_0,$$

where $\mathsf{k}_0 > 0$ is such that $V(x) + \mathsf{k}_0 \geq 1$ for all $x \in \mathbb{R}^d$ (see (6.3)). Let

$$a, \alpha, b, \beta > 0.$$

Assume that (recall $\mathsf{n}_0 > 2$)

$$0 < \beta - \alpha \le \frac{1}{2} - \frac{1}{\mathsf{n}_0}, \tag{6.10}$$

so that the function

$$\mathbb{R}^{2d} \ni (x, v) \mapsto \mathsf{F}_2(x, v) = -a\mathsf{H}_2(x, v)^{-\alpha} + b\, x \cdot v\, \mathsf{H}_2(x, v)^{\beta-\alpha-1}$$

is bounded. Indeed, $a\mathsf{H}_2^{-\alpha}$ is a bounded function over $\mathbb{R}^{2d}$. For the other term, set $\lambda = \beta - \alpha$, and use Young's inequality with $q = 2(1-\lambda) > 1$ ($\lambda < 1/2$, see (6.10)) and $p = q/(q-1) = 2(1-\lambda)/(1-2\lambda)$ to get

$$\frac{|x \cdot v|}{\mathsf{H}_2(x, v)^{1-\lambda}} \le \frac{x^p}{p\mathsf{H}_2(x, v)^{1-\lambda}} + \frac{v^q}{q\mathsf{H}_2(x, v)^{1-\lambda}}. \tag{6.11}$$

The function $(x, v) \mapsto |x \cdot v|\mathsf{H}_2(x, v)^{\lambda-1}$ is thus bounded if $p \le \mathsf{n}_0(1-\lambda)$ (see (6.3)), that is, $2(1-\lambda)/(1-2\lambda) \le \mathsf{n}_0(1-\lambda)$, which reads $2 \le \mathsf{n}_0 - 2\lambda\mathsf{n}_0$, which is precisely (6.10). Then the function

$$\mathbb{R}^{2d} \ni (x, v) \mapsto \mathsf{W}_2(x, v) = \exp\Big[\mathsf{F}_2(x, v) - \inf_{\mathbb{R}^{2d}} \mathsf{F}_2\Big] \tag{6.12}$$

belongs to $\mathcal{C}^{1,2}(\mathbb{R}^d \times \mathbb{R}^d)$ (thus $\mathsf{W}_2 \in \mathbb{D}_e(\mathcal{L})$). For ease of notation, in the following, we will simply denote F_2 (resp. $\mathsf{H}_2, \mathsf{W}_2$) by F (resp. H, W). We have $\frac{\partial_x \mathsf{W}}{\mathsf{W}} = \nabla_x \mathsf{F}$, $\frac{\partial_x \mathsf{W}}{\mathsf{W}} = \nabla_x \mathsf{F}$, and $\frac{\Delta_v \mathsf{W}}{\mathsf{W}} = \Delta_v \mathsf{F} + |\nabla_v \mathsf{F}|^2$. Thus

$$\frac{\mathcal{L}_0 \mathsf{W}}{\mathsf{W}} = \frac{1}{2}\Sigma^2 \Delta_v \mathsf{F} + \frac{1}{2}\Sigma^2 |\nabla_v \mathsf{F}|^2 + v \cdot \nabla_x \mathsf{F} - [\nabla V + |v|^{\ell_0} v]\nabla_v \mathsf{F}, \tag{6.13}$$

where we recall that $\Sigma : \mathbb{R}^{2d} \to \mathbb{R}$ is smooth and bounded, by assumption. We have

$$\begin{aligned}
\nabla_x \mathsf{F} &= a\alpha \nabla V\, \mathsf{H}^{-\alpha-1} + bv\mathsf{H}^{\beta-\alpha-1} - b(\alpha+1-\beta)x \cdot v \nabla V \mathsf{H}^{\beta-\alpha-2},\\
\nabla_v \mathsf{F} &= a\alpha v\mathsf{H}^{-\alpha-1} + bx\mathsf{H}^{\beta-\alpha-1} - b(\alpha+1-\beta)x \cdot v\, v\mathsf{H}^{\beta-\alpha-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
v \cdot \nabla_x \mathsf{F} - [\nabla V + |v|^l v]\nabla_v \mathsf{F} = \mathsf{H}^{-\alpha-1}[&-a\alpha|v|^{\ell_0+2} + b|v|^2\mathsf{H}^\beta - bx \cdot \nabla V \mathsf{H}^\beta\\
&- b|v|^{\ell_0} x \cdot v\mathsf{H}^\beta + b(\alpha+1-\beta)x \cdot v|v|^{\ell_0+2}\mathsf{H}^{\beta-1}].
\end{aligned} \tag{6.14}$$

Moreover,

$$\begin{aligned}
\Delta_v \mathsf{F} = a\alpha\mathsf{H}^{-\alpha-1} &- a\alpha|v|^2(\alpha+1)\mathsf{H}^{-\alpha-2} + b(\beta-\alpha-1)x \cdot v\, \mathsf{H}^{\beta-\alpha-2}\\
&- (d+1)b(\alpha+1-\beta)x \cdot v\mathsf{H}^{\beta-\alpha-2} - b(\alpha+1-\beta)(\beta-\alpha-2)x \cdot v|v|^2\mathsf{H}^{\beta-\alpha-3}.
\end{aligned}$$

The functions $(x, v) \mapsto a\alpha\mathsf{H}^{-\alpha-1}$ and $(x, v) \mapsto |v|^2\mathsf{H}^{-\alpha-2}$ are clearly bounded over $\mathbb{R}^{2d}$. In addition, since H^{-1} is bounded over $\mathbb{R}^{2d}$, there exists $C > 0$ such that

$$|x \cdot v|\mathsf{H}^{\beta-\alpha-2} \le C|x \cdot v|\mathsf{H}^{\beta-\alpha-1} = C|x \cdot v|\mathsf{H}^{\lambda-1},$$

and from the analysis in (6.11) above (see (6.10) and recall that $\lambda = \beta - \alpha$), $(x, v) \mapsto |x \cdot v| \mathsf{H}^{\beta-\alpha-2}$ is bounded over $\mathbb{R}^{2d}$. Finally, $(x, v) \mapsto |x \cdot v|\, |v|^2 |\mathsf{H}^{\beta-\alpha-3}$ is also bounded over $\mathbb{R}^{2d}$ since (see also (6.11) and (6.10)) $|x \cdot v|\, |v|^2 \mathsf{H}^{\beta-\alpha-3} = |x \cdot v| \mathsf{H}^{\lambda-1} \times |v|^2 \mathsf{H}^{-2}$. Consequently,

$$\Sigma^2 \Delta_v \mathsf{F} \text{ is bounded over } \mathbb{R}^{2d}.$$

Similarly, the functions $|v| \mathsf{H}^{-\alpha-1}$, $|x| \mathsf{H}^{\beta-\alpha-1} = |x| \mathsf{H}^{\lambda-1}$ (recall that $\lambda < (\mathsf{n}_0 - 1)/\mathsf{n}_0$ so that $\mathsf{n}_0(1-\lambda) > 1$, see (6.10)), and $|x \cdot v|\, |v|\, |\mathsf{H}^{\beta-\alpha-2} = |x \cdot v| \mathsf{H}^{\lambda-1} \times |v| \mathsf{H}^{-1}$ are bounded, one then deduces that

$$\Sigma^2 |\nabla_v \mathsf{F}|^2 \text{ is bounded over } \mathbb{R}^{2d}.$$

Consequently, from (6.13) and (6.14), for some $C > 0$ independent of x and v,

$$\begin{aligned}\frac{\mathcal{L}_0 \mathsf{W}}{\mathsf{W}} \le C + \mathsf{H}^{-\alpha-1}[&-a\alpha|v|^{\ell_0+2} + b|v|^2 \mathsf{H}^{\beta} - b x \cdot \nabla V \mathsf{H}^{\beta} \\ &- b|v|^{\ell_0} x \cdot v \mathsf{H}^{\beta} + b(\alpha + 1 - \beta) x \cdot v |v|^{\ell_0+2} \mathsf{H}^{\beta-1}].\end{aligned} \tag{6.15}$$

Let us now give a lower bound on the term inside the bracket in (6.15), which we denote by M. Let us assume that

$$\beta < 1.$$

Then $2^{\beta-1}(s^\beta + t^\beta) \le (s+t)^\beta \le s^\beta + t^\beta$ for all $s, t \ge 0$. There exists $r_1 > 0$ such that for all $x, v \in \mathbb{R}$,

$$2^{\beta-1} V_0(x)^\beta + \frac{v^{2\beta}}{2} \le \mathsf{H}(x, v)^\beta \le V_0(x)^\beta + \frac{v^{2\beta}}{2^\beta},$$

where for all $x \in \mathbb{R}^d$ we set

$$V_0(x) = V(x) + \mathsf{k}_0,$$

which satisfies (see (6.3)), for some $r, C > 0$ and all $|x| > r$,

$$C^{-1}|x|^{\mathsf{n}_0} \le V_0(x) \le C|x|^{\mathsf{n}_0} \quad \text{and} \quad C^{-1}|x|^{\mathsf{n}_0} \le x \cdot \nabla V_0(x). \tag{6.16}$$

Therefore, since $b(\alpha + 1 - \beta) > 0$,

$$\begin{aligned}\mathsf{M} \le\ & -a\alpha|v|^{\ell_0+2} + b|v|^2 V_0^\beta + \frac{b}{2^\beta}|v|^{2+2\beta} - 2^{\beta-1} b|x \cdot \nabla V_0| V_0^\beta - \frac{b}{2}|x \cdot \nabla V_0|\, |v|^{2\beta} \\ & + \mathbb{1}_{x\cdot v \le 0} b|x| V_0^\beta |v|^{\ell_0+1} + \frac{\mathbb{1}_{x\cdot v\le 0}}{2^\beta} b|x|\, |v|^{\ell_0+1+2\beta} \\ & + \mathbb{1}_{x\cdot v\ge 0} b(\alpha+1-\beta)\frac{|x| V_0^\beta |v|^{\ell_0+3}}{V_0 + |v|^2/2} + \frac{\mathbb{1}_{x\cdot v \ge 0}}{2^\beta} b(\alpha+1-\beta)\frac{|x|\, |v|^{3+\ell_0+2\beta}}{V_0 + |v|^2/2}.\end{aligned} \tag{6.17}$$

Let us now find conditions such that $-a\alpha|v|^{\ell_0+2}$ and $-2^{\beta-1} b x \cdot \nabla V_0\, V_0^\beta$ are dominant on the right hand side of (6.17). From (6.16), for $|x| \ge r$,

$$C^{-1}|x|^{\mathsf{n}_0 + \mathsf{n}_0\beta} \le |x \cdot \nabla V_0(x)| V_0(x)^\beta = V_0(x)^\beta x \cdot \nabla V_0(x) \tag{6.18}$$

for some $C > 0$ independent of x. Assume that

$$\beta < \ell_0/2, \tag{6.19}$$

so that $|v|^{2+2\beta} = o(|v|^{\ell_0+2})$ as $|v| \to +\infty$. In addition, for $0 < \varepsilon < \lambda_0$, using Young's inequality with $p_\varepsilon = (\ell_0 + 2 - \varepsilon)/2 > 1$ and $q_\varepsilon = p_\varepsilon/(p_\varepsilon - 1) = 1 + 2/(\ell_0 - \varepsilon)$, we obtain

$$|v|^2 V_0^\beta \le p_\varepsilon^{-1}|v|^{\ell_0+2-\varepsilon} + q_\varepsilon^{-1} V_0^{\beta(1+2/(\ell_0-\varepsilon))}.$$

From (6.19), for $\varepsilon > 0$ small enough, $\beta < (\ell_0 - \varepsilon)/2$ and thus $\mathsf{n}_0\beta(1 + 2/(\ell_0 - \varepsilon)) < \mathsf{n}_0\beta + \mathsf{n}_0$. Thus, for such $\varepsilon > 0$, $|v|^{\ell_0+2-\varepsilon} = o(|v|^{\ell_0+2})$ as $|v| \to +\infty$ and $V_0^{\beta(1+2/(\ell_0-\varepsilon))} = o(|x|^{\mathsf{n}_0+\beta\mathsf{n}_0}) \to +\infty$ as $|x| \to +\infty$ (see (6.16)). For $\varepsilon \ll 1$, using again Young's inequality with $p_\varepsilon = (\ell_0 + 2 - \varepsilon)/(\ell_0 + 1) = 1 + (1 - \varepsilon)/(\ell_0 + 1) > 1$ and $q_\varepsilon = p_\varepsilon/(p_\varepsilon - 1) = (\ell_0 + 2 - \varepsilon)/(1 - \varepsilon) = 1 + (\ell_0 + 1)/(1 - \varepsilon)$ we get

$$|v|^{\ell_0+1}|x|V_0^\beta \le p_\varepsilon^{-1}|v|^{\ell_0+2-\varepsilon} + q_\varepsilon^{-1}|x|V_0^{\beta[1+(\ell_0+1)/(1-\varepsilon)]}.$$

Let us check that $(1 + \mathsf{n}_0\beta)[1 + (\ell_0 + 1)/(1 - \varepsilon)] < \mathsf{n}_0\beta + \mathsf{n}_0$. This is equivalent to $\mathsf{n}_0\beta < (\mathsf{n}_0 - 1)(1 - \varepsilon)/(\ell_0 + 1) - 1$. Notice that from (6.9) we have $(\mathsf{n}_0 - 1)/(\ell_0 + 1) > 1$ and thus $(\mathsf{n}_0 - 1)(1 - \varepsilon)/(\ell_0 + 1) > 1$ for $\varepsilon > 0$ small enough. Then, assume that

$$\mathsf{n}_0\beta < (\mathsf{n}_0 - 1)/(\ell_0 + 1) - 1, \tag{6.20}$$

so that, for $\varepsilon > 0$ small enough, $|x|V_0^{\beta[1+(\ell_0+1)/(1-\varepsilon)]} = o(|x|^{\mathsf{n}_0+\mathsf{n}_0\beta})$. Assume also that

$$\beta < 1/2, \tag{6.21}$$

so that, for $\varepsilon > 0$ small enough, $p_\varepsilon = (\ell_0 + 2 - \varepsilon)/(\ell_0 + 1 + 2\beta) > 1$. Then

$$|x|\,|v|^{\ell_0+1+2\beta} \le p_\varepsilon^{-1}|v|^{\ell_0+2-\varepsilon} + q_\varepsilon^{-1}|x|^{(\ell_0+2-\varepsilon)/(1-\varepsilon-2\beta)}.$$

Assume that

$$\ell_0 + 2 < (\mathsf{n}_0\beta + \mathsf{n}_0)(1 - 2\beta),$$

which is satisfied if $\beta > 0$ is small enough since $\ell_0 + 2 < \mathsf{n}_0$ (see (6.9)). Then for $\varepsilon > 0$ small enough, $(\ell_0 + 2 - \varepsilon)/(1 - \varepsilon - 2\beta) < \mathsf{n}_0\beta + \mathsf{n}_0$ and $|x|^{(\ell_0+2-\varepsilon)/(1-\varepsilon-2\beta)} = o(|x|^{\mathsf{n}_0\beta+\mathsf{n}_0})$.

Moreover, for $\beta > 0$ small enough, we have $\mathsf{n}_0/(1 + \mathsf{n}_0\beta) > 1$, and thus $V_0 + v^2/2 \ge C^{-1}V_0^{1/\mathsf{n}_0+\beta}v^{3/2-2\beta}$ for some $C > 0$. Then, because $|x|V_0^{-1/\mathsf{n}_0}$ is a bounded function (see (6.16)), for some $M > 0$ independent of x and v we have

$$\frac{|x|^{1+\mathsf{n}_0\beta}|v|^{\ell_0+3}}{V_0 + v^2/2} \le M|v|^{\ell_0+3/2+2\beta}. \tag{6.22}$$

If $\beta < 1/4$, then the left hand side of (6.22) is $o(|v|^{\ell_0+2})$ as $|v| \to +\infty$. Finally, since $V_0 + |v|^2/2 \ge C^{-1}V_0^{1/\mathsf{n}_0}|v|^{2(\mathsf{n}_0-1)/\mathsf{n}_0}$, for some $M > 0$ independent of x and v we have

$$\frac{|x|\,|v|^{\ell_0+3+2\beta}}{V_0 + v^2/2} \le M|v|^{\ell_0+1+2\beta+2/\mathsf{n}_0}.$$

Taking $\beta > 0$ such that $2\beta + 2/\mathsf{n}_0 < 1$ (this is possible because $2/\mathsf{n}_0 < 1$ by assumption), the left hand side of the previous inequality is $o(|v|^{\ell_0+2})$ as $|v| \to +\infty$. In conclusion, all the previous estimates together with (6.18) imply that there exists η (depending on ℓ_0 and n_0) such that if

$$0 < \beta < \eta, \tag{6.23}$$

then (recalling also that $0 < \alpha < \beta$ and $\eta \le \ell_0$, see (6.10) and (6.19)), there exists $C > 0$ and a continuous function $\mathbb{R}^d \ni (x, v) \mapsto \mathsf{K}(x, v)$ such that (see (6.15))

$$\frac{\mathcal{L}_0\mathsf{W}}{\mathsf{W}} \le C - \mathsf{K}(x, v), \tag{6.24}$$

with $\mathsf{K}(x, v) \to +\infty$ if $|x| + |v| \to +\infty$. This ends the proof of Proposition 6.5. ∎

Let now O be a nonempty subdomain of $\mathbb{R}^d$ (not necessarily bounded), that is, O is a connected open subset of $\mathbb{R}^d$. As explained in the introduction, we are interested, for applications in statistical physics, in the existence of quasi-stationary distributions for the processes (6.1) in

$$\mathcal{D} = \mathsf{O} \times \mathbb{R}^d. \tag{6.25}$$

Of course, other domains might be considered with our techniques. Recall that σ_D (see (2.1)) is the first exit time from $\mathcal{D}$ for the process (6.1):

$$\begin{aligned} \sigma_{\mathcal{D}}(\mathsf{x}) &= \inf\{t \ge 0;\ X_t(\mathsf{x}) \notin \mathcal{D}\} \\ &= \inf\{t \ge 0;\ x_t(\mathsf{x}) \notin \mathsf{O}\}, \end{aligned} \tag{6.26}$$

where we recall that $(X_t(\mathsf{x}), t \ge 0)$ stands for the process $(X_t, t \ge 0)$ when $X_0 = \mathsf{x} \in \mathbb{R}^{2d}$. Let us now check the other assumptions on $(P_t^{\mathcal{D}}, t \ge 0)$ (the semigroup of the process (6.1) killed when exiting $\mathcal{D}$, see (2.2) and (6.26)) needed to apply Theorem 2.2.

6.2.3. The semigroup $(P_t^{\mathcal{D}}, t \ge 0)$ is topologically irreducible.

Lemma 6.6. *Assume that V, c and Σ satisfy* (**Av1**), (**Ac0**)*, and* (**A**Σ)*. Then $(P_t^{\mathcal{D}}, t \ge 0)$ is topologically irreducible. If the open set $\mathbb{R}^d \setminus \bar{\mathsf{O}}$ is not empty, then for all $\mathsf{x} \in \mathcal{D}$ and $t > 0$,*

$$\mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}} < t) > 0,$$

which implies in particular that $\mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}} < +\infty) > 0$ (thus (**C5**) *is satisfied for the process* (6.1) *when $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$).*

Proof. We will apply the Stroock–Varadhan support theorem.

Step 1: The case when $V = 0$ and $c = 0$. Recall that the process $(X_t^0 = (x_t^0, v_t^0), t \ge 0)$ is the solution (in the strong sense) to the stochastic differential equation (6.5). Denote by $(X_t^0(\mathsf{x}), t \ge 0)$ the process $(X_t^0, t \ge 0)$ when $X_0^0 = \mathsf{x}$. Let $(P_t^{\mathcal{D},0}, t \ge 0)$ denote the semigroup of the process (6.5) killed when exiting $\mathcal{D}$. Denote by $\sigma_{\mathcal{D}}^0$ the first time the process $(X_t^0 = (x_t^0, v_t^0), t \ge 0)$ exits $\mathcal{D}$ (see (6.26)). The stochastic differential equation

(6.5) in the Stratonovich form reads

$$\begin{cases} dx_t^0 = v_t^0 dt, \\ dv_t^0 = -\frac{1}{2}\Sigma(x_t^0, v_t^0)\nabla_v \Sigma(x_t^0, v_t^0) + \Sigma(x_t^0, v_t^0) \circ dB_t. \end{cases}$$

Let $\mathcal{O}_1$ be a nonempty open subset of O and $\mathcal{O}_2$ be a nonempty open subset of $\mathbb{R}^d$. Consider $\mathsf{x}_0 = (x_0, v_0) \in \mathcal{D}$ and $\mathsf{x}_1 = (x_1, v_1) \in \mathcal{O}_1 \times \mathcal{O}_2$. Let $t > 0$ and $\gamma : [0, t] \to \mathsf{O}$ be a $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$ curve such that $\gamma(0) = x_0$, $\dot\gamma(0) = v_0$, $\gamma(t) = x_1$, and $\dot\gamma(t) = v_1$. The construction of such a γ can be done by a local cubic interpolation in time as for instance in [54, Lemma 4.2]. For $s \in [0, t]$, set

$$\mathsf{Y}(s) = \begin{pmatrix} \mathsf{Y}_1(s) \\ \mathsf{Y}_2(s) \end{pmatrix} \quad \text{where} \quad \mathsf{Y}_1(s) = \gamma(s) \text{ and } \mathsf{Y}_2(s) = \dot\gamma(s).$$

Then, define the piecewise continuous function $h : [0, t] \to \mathbb{R}^d$ by

$$\mathsf{h}(s) = \frac{1}{\Sigma(\gamma(s), \dot\gamma(s))}\big[\ddot\gamma(s) + \tfrac{1}{2}\Sigma(\gamma(s), \dot\gamma(s))\nabla_v \Sigma(\gamma(s), \dot\gamma(s))\big], \quad s \in [0, t].$$

Clearly, $\mathsf{h} \in L^2([0, t], \mathbb{R}^d)$, $\mathsf{Y}(0) = \mathsf{x}_0$, and for all $s \in [0, t]$,

$$\begin{cases} \dot{\mathsf{Y}}_1(s) = \mathsf{Y}_2(s), \\ \dot{\mathsf{Y}}_2(s) = -\frac{1}{2}\Sigma(\mathsf{Y}_1(s), \mathsf{Y}_2(s))\nabla_v \Sigma(\mathsf{Y}_1(s), \mathsf{Y}_2(s)) + \Sigma(\mathsf{Y}_1(s), \mathsf{Y}_2(s))\,\mathsf{h}(s). \end{cases}$$

By the Stroock–Varadhan support theorem [73] (see also [10, Theorem 4]), for all $\varepsilon > 0$ and $t > 0$,

$$\mathbb{P}_{\mathsf{x}_0}\Big(\sup_{s\in[0,t]} |X_s^0 - \mathsf{Y}(s)| < \varepsilon\Big) > 0.$$

Since for all $s \in [0, t]$, $\mathsf{Y}(s) = (\gamma^T(s), \dot\gamma^T(s))^T \in \mathsf{O} \times \mathbb{R}^d = \mathcal{D}$ and O is open, for $\varepsilon > 0$ small enough, if $\sup_{s\in[0,t]} |X_s^0 - \mathsf{Y}(s)| < \varepsilon$, then $X_s^0 = (x_s, v_s) \in \mathcal{D}$ for all $s \in [0, t]$ (in particular $t < \sigma_{\mathcal{D}}^0$ by continuity of the trajectories), and $x_t \in B(x_1, 2\varepsilon) \subset \mathcal{O}_1$ and $v_t \in B(v_1, 2\varepsilon) \subset \mathcal{O}_2$. Thus

$$P_t^{D,0}(\mathsf{x}_0, \mathcal{O}_1 \times \mathcal{O}_2) = \mathbb{P}_{\mathsf{x}_0}(X_t^0 \in \mathcal{O}_1 \times \mathcal{O}_2, t < \sigma_{\mathcal{D}}^0) \geq \mathbb{P}_{\mathsf{x}_0}\Big(\sup_{s\in[0,t]} |X_s^0 - \mathsf{Y}(s)| < \varepsilon\Big) > 0,$$

which is precisely the topological irreducibility of $(P_t^{\mathcal{D},0}, t \geq 0)$. If the open set $\mathbb{R}^d \setminus \overline{\mathsf{O}}$ is not empty, then choosing $\mathcal{O}_1$ such that $\mathcal{O}_1 \subset \mathbb{R}^d \setminus \overline{\mathsf{O}}$, one deduces with the same arguments as above[4] that for all $\mathsf{x}_0 \in \mathcal{D}$ and $t > 0$,

$$\mathbb{P}_{\mathsf{x}_0}(\sigma_{\mathcal{D}}^0 < +\infty) \geq \mathbb{P}_{\mathsf{x}_0}(\sigma_{\mathcal{D}}^0 < t) \geq \mathbb{P}_{\mathsf{x}_0}(X_t^0 \in \mathcal{O}_1 \times \mathbb{R}^d) > 0.$$

Step 2: The case when $V \neq 0$ and $c \neq 0$. Let us now turn to the case when $V \neq 0$ and $c \neq 0$. Pick $f \in b\mathcal{B}(\mathcal{D})$. Since V and c satisfy (**Av1**) and (**Ac0**), from [83, Lemma 1.1

[4]In this case, let $\gamma : [0, t] \to \mathbb{R}^d$ be a smooth curve such that $\gamma(0) = x_0$, $\dot\gamma(0) = v_0$, $\gamma(t) = x_1$, and $\dot\gamma(t) = v_1$. Such a curve can be easily constructed by a (global) cubic interpolation in time.

and remark on p. 7], for $\mathsf{x} \in \mathbb{R}^{2d}$ and $t > 0$, the law of $(X_s(\mathsf{x}), s \in [0,t])$ is equivalent to the law of $(X_s^0(\mathsf{x}), s \in [0,t])$. In particular, for all $t > 0$ and all $\mathsf{x} \in \mathcal{D}$,

$$\mathbb{P}_{\mathsf{x}}(X_t \in \mathcal{O}_1 \times \mathcal{O}_2,\, t < \sigma_{\mathcal{D}}) > 0 \quad \text{if and only if} \quad \mathbb{P}_{\mathsf{x}}(X_t^0 \in \mathcal{O}_1 \times \mathcal{O}_2,\, t < \sigma_{\mathcal{D}}^0) > 0,$$

and

$$\mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}} < t) > 0 \quad \text{if and only if} \quad \mathbb{P}_{\mathsf{x}}(\sigma_{\mathcal{D}}^0 < t) > 0.$$

This ends the proof of Lemma 6.6. ∎

6.2.4. Weak Feller property of $P_t^{\mathcal{D}}$.

Proposition 6.7. *Assume that V, c, and Σ satisfy respectively assumptions* (**Av1**), (**Ac0**), *and* (**A**Σ). *Assume that O is a $\mathcal{C}^2$ subdomain of $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus \bar{\mathsf{O}}$ is nonempty. Then, for $t > 0$, $P_t^{\mathcal{D}}$ is strongly Feller on $\mathcal{D}$ (and thus weakly Feller on $\mathcal{D}$). Thus, assumption* (**C4**) *is satisfied for $P_t^{\mathcal{D}}$.*

Proof. The proof is divided into several steps.

Step 1: Properties of the process $(X_t^0, t \geq 0)$ (see (6.5)).

Step 1a: Proof of (6.27). In this step, we prove that, for $\mathsf{y} = (x_{\mathsf{y}}, v_{\mathsf{y}}) \in \partial\mathcal{D}$, if

$$\mathsf{n}(x_{\mathsf{y}}) \cdot v_{\mathsf{y}} \geq 0,$$

then almost surely, for all $t > 0$, there exists $u \in (0,t]$ such that

$$x_u^0(\mathsf{y}) \in \mathbb{R}^d \setminus \bar{\mathsf{O}}. \tag{6.27}$$

This has been proved very recently in [54, Proposition 2.8 (i)] for the process (6.1) when Σ is constant (that is, independent of x and v). The proof of (6.27) requires a further analysis when Σ is not constant.

When $v_{\mathsf{y}} \cdot \mathsf{n}(x_{\mathsf{y}}) > 0$, the proof of (6.27) is straightforward. Indeed, because $\partial\mathsf{O}$ is $\mathcal{C}^2$, in a neighborhood U of $x_{\mathsf{y}} \in \partial\mathsf{O}$ in $\mathbb{R}^d$, O is given by $\{\Psi < 0\}$ for some $\mathcal{C}^2$ function $\Psi : \mathbb{R}^d \to \mathbb{R}$ such that $\mathsf{n}(x_{\mathsf{y}}) = \nabla\Psi(x_{\mathsf{y}})$ and $\partial\mathsf{O}$ is given by $\{\Psi = 0\}$. Then, for $t \geq 0$ (sufficiently small, say $t \leq t^*(\mathsf{y})$, so that $x_t^0(\mathsf{y}) \in \mathsf{U}$ for all $t \in [0, t^*(\mathsf{y})]$),

$$\Psi(x_t^0(\mathsf{y})) = \int_0^t \nabla\Psi(x_s^0(\mathsf{y})) \cdot v_s^0(\mathsf{y})\, ds.$$

In addition, since

$$\nabla\Psi(x_0^0(\mathsf{y})) \cdot v_0^0(\mathsf{y}) = v_{\mathsf{y}} \cdot \mathsf{n}(x_{\mathsf{y}}) > 0$$

and because $0 \leq s \mapsto \nabla\Psi(x_s^0(\mathsf{x})) \cdot v_s^0(\mathsf{x})$ is continuous almost surely, one deduces that for all $t > 0$ small enough,

$$\Psi(x_t^0(\mathsf{y})) > 0,$$

which concludes the proof of (6.27) when $v_{\mathsf{y}} \cdot \mathsf{n}(x_{\mathsf{y}}) > 0$.

Let us now prove (6.27) when $v_{\mathsf{y}} \cdot \mathsf{n}(x_{\mathsf{y}}) = 0$. One has, uniformly in $x \in \mathsf{U}$ (recall that $\nabla\Psi$ is $\mathcal{C}^1$),

$$\nabla\Psi(x) = \mathsf{n}(x_{\mathsf{y}}) + O(|x - x_{\mathsf{y}}|),$$

so that, using in addition $v_s^0(\mathsf{y}) \cdot \mathsf{n}(x_0^0(\mathsf{y})) = v_{\mathsf{y}} \cdot \mathsf{n}(x_{\mathsf{y}}) = 0$, for $t \in [0, t^*(\mathsf{y})]$ we have

$$\Psi(x_t^0(\mathsf{y})) = \int_0^t \Big[v_{\mathsf{y}} + \int_0^s \Sigma((x_u^0(\mathsf{y}), v_u^0(\mathsf{y})))\, dB_u \Big] \cdot \big[\mathsf{n}(x_{\mathsf{y}}) + O(\underbrace{|x_s^0(\mathsf{y}) - x_{\mathsf{y}}|}_{=|\int_0^s v_u^0(\mathsf{y})\, du|})\big]\, ds$$

$$= \int_0^t \mathrm{M}_s\, ds + O(t^2) \sup_{s\in[0,t]} |v_s^0(\mathsf{y})|^2, \tag{6.28}$$

where we set

$$\mathrm{M}_s = \int_0^s \Sigma(x_u^0(\mathsf{y}), v_u^0(\mathsf{y}))\, d\omega_u \quad \text{for } s \geq 0,$$

and where $(\omega_u, u \geq 0)$ is a standard one-dimensional Brownian motion ($\omega_u = B_u \cdot \mathsf{n}(x_{\mathsf{y}})$). Thus, to prove (6.27), in view of the previous estimate we have to study the sign of $\int_0^t \mathrm{M}_s\, ds$ (for small $t > 0$). To this end, it is sufficient to show that

$$\limsup_{t\to 0^+} \frac{\int_0^t \mathrm{M}_s\, ds}{\mathsf{L}(T^{-1}(t))} > 0 \quad \text{almost surely,} \tag{6.29}$$

where $\mathsf{L}(r) = \sqrt{2/3}\, r^{3/2} \sqrt{\log\log(1/r)}$ (for $r > 0$), and where for $s \geq 0$,

$$T(s) = \int_0^s \Sigma(x_u^0(\mathsf{y}), v_u^0(\mathsf{y}))^2\, du,$$

which is (almost surely) a strictly increasing and continuously differentiable function $\mathbb{R}^+ \to \mathbb{R}^+$ with $T(s) \geq \Sigma_0^2 s \to +\infty$ as $s \to +\infty$. Notice that because for all $s \geq 0$ we have $\Sigma_0 s \leq T(s) \leq \Sigma_\infty s$, it follows that for all $u \geq 0$,

$$\Sigma_\infty^{-1} u \leq T^{-1}(u) \leq \Sigma_0^{-1} u.$$

Thus for some $C > 0$ we have, for $t \geq 0$ large enough,

$$C^{-1}\mathsf{L}(t) \leq \mathsf{L}(T^{-1}(t)) \leq C\mathsf{L}(t).$$

Consequently, $t^2/\mathsf{L}(T^{-1}(t)) \to 0$ as $t \to 0$ and so, in view of (6.28), (6.29) implies (6.27).

Thus, let us prove (6.29). By assumption on Σ, $(\mathrm{M}_s, s \geq 0)$ is a continuous martingale and $[\mathsf{M}]_s = T(s) \to +\infty$ as $s \to +\infty$. Then, using the Dambis–Dubins–Schwarz theorem, there exists a standard one-dimensional Brownian motion $(\mathrm{V}_t, t \geq 0)$ such that for all $s \geq 0$,

$$\mathrm{M}_s = \mathrm{V}_{[\mathsf{M}]_s}.$$

Since $(T^{-1})'(u) = 1/T'(T^{-1}(u)) = \Sigma(x^0_{T^{-1}(u)}, v^0_{T^{-1}(u)})^{-2}$, we have

$$\int_0^t \mathrm{M}_s\, ds = \int_0^t \mathrm{V}_{T(s)}\, ds = \int_0^{T^{-1}(t)} \mathrm{V}_u\, \Sigma(x^0_{T^{-1}(u)}(\mathsf{y}), v^0_{T^{-1}(u)}(\mathsf{y}))^{-2}\, du. \tag{6.30}$$

Then, setting $\kappa = T^{-1}(t)$ $(t > 0)$, (6.29) is equivalent to

$$\limsup_{\kappa\to 0+} \frac{\int_0^\kappa \mathrm{V}_u \Sigma(x^0_{T^{-1}(u)}(\mathsf{y}), v^0_{T^{-1}(u)}(\mathsf{y}))^{-2}\, du}{\mathsf{L}(\kappa)} > 0 \quad \text{almost surely.} \tag{6.31}$$

Step 1b: Proof of (6.31). By assumption on Σ and since $\nabla\Sigma^{-2} = -2(\nabla\Sigma)\Sigma^{-3}$, uniformly in $\mathsf{x}, \mathsf{z} \in \mathbb{R}^{2d}$, one has

$$\Sigma(\mathsf{x})^{-2} = \Sigma(\mathsf{z})^{-2} + O(|\mathsf{x} - \mathsf{z}|).$$

Thus, using also $x^0_{T^{-1}(u)}(\mathsf{y}) = x_\mathsf{y} + \int_0^{T^{-1}(u)} v^0_s(\mathsf{y})\, ds$ and $T^{-1}(u) \le T^{-1}(t) = \kappa$ (for $0 \le u \le t$), one has

$$\begin{aligned}
\int_0^\kappa \mathrm{V}_u \Sigma(X^0_{T^{-1}(u)})^{-2}\, du &= \Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} \int_0^\kappa \mathrm{V}_u\, du \\
&\qquad + O\Big(\int_0^\kappa |\mathrm{V}_u| [|x^0_{T^{-1}(u)}(\mathsf{y}) - x_\mathsf{y}| + |v^0_{T^{-1}(u)} - v_\mathsf{y}|]\, du\Big) \\
&= \Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} \int_0^\kappa \mathrm{V}_u\, du \\
&\quad + \sup_{u\in[0,\kappa]} |\mathrm{V}_u| O\Big[\int_0^\kappa |x^0_{T^{-1}(u)}(\mathsf{y}) - x_\mathsf{y}|\, du + \kappa \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y}) - v_\mathsf{y}|\Big] \\
&= \Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} \int_0^\kappa \mathrm{V}_u\, du \\
&\quad + \sup_{u\in[0,\kappa]} |\mathrm{V}_u| O\Big[\int_0^\kappa u \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y})|\, du + \kappa \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y}) - v_\mathsf{y}|\Big] \\
&= \Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} \int_0^\kappa \mathrm{V}_u\, du \\
&\quad + \sup_{u\in[0,\kappa]} |\mathrm{V}_u| \Big[\sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y})| O(\kappa^2) + \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y}) - v_\mathsf{y}| O(\kappa)\Big],
\end{aligned}$$

where we have used the fact that $T^{-1}(u) \le \Sigma_0^{-1} u$. Using Watanabe's law of the iterated logarithm [82] for $\int_0^\kappa \mathrm{V}_u\, du$ (see also [50, Theorem 1 (2)]), we see that

$$\limsup_{\kappa\to 0+} \frac{\Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} \int_0^\kappa \mathrm{V}_u\, du}{\mathsf{L}(\kappa)} = \Sigma(x_\mathsf{y}, v_\mathsf{y})^{-2} > 0 \quad \text{almost surely.}$$

Khinchin's law of the iterated logarithm for $\sup_{u\in[0,\kappa]} |\mathrm{V}_u|$ (see e.g. [41]) yields

$$\limsup_{\kappa\to 0+} \frac{\sup_{u\in[0,\kappa]} |\mathrm{V}_u|}{\mathsf{P}(\kappa)} = 1 \quad \text{almost surely,}$$

where $\mathsf{P}(\kappa) = \sqrt{2\kappa}\sqrt{\log\log(1/\kappa)}$. Thus, since $\mathsf{P}(\kappa)/\mathsf{L}(\kappa) \sim \sqrt{3}/\kappa$, one has, almost surely as $\kappa \to 0^+$,

$$\kappa \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y}) - v_\mathsf{y}| \frac{\sup_{u\in[0,\kappa]} |\mathrm{V}_u|}{\mathsf{L}(\kappa)} = \frac{\sup_{u\in[0,\kappa]} |\mathrm{V}_u|}{\mathsf{P}(\kappa)} \frac{\kappa \mathsf{P}(\kappa)}{\mathsf{L}(\kappa)} \sup_{u\in[0,\kappa]} |v^0_u(\mathsf{y}) - v_\mathsf{y}| \to 0,$$

because almost surely, $\sup_{u\in[0,\kappa]}|v^0_u(\mathsf{y})-v_\mathsf{y}|\to 0$ as $\kappa\to 0^+$. This concludes the proof of (6.31) (with more precisely $\limsup_{\kappa\to 0^+}\int_0^\kappa \mathsf{V}_u\Sigma(X^0_{T^{-1}(u)})^{-2}\,du/\mathsf{L}(\kappa)=\Sigma(x_\mathsf{y},v_\mathsf{y})^{-2}$ almost surely), and thus of (6.29) and of (6.27).

Step 1c: Proof of (6.32). Let $\mathsf{x}_n=(x_n,v_n)_n$ be a sequence of elements of $\mathscr{D}$ such that $\mathsf{x}_n\to\mathsf{x}=(x,v)\in\mathscr{D}$ as $n\to\infty$. Let us prove that for $t>0$,

$$\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x}_n)}\to\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x})}\quad\text{in }\mathbb{P}\text{-probability as }n\to\infty. \tag{6.32}$$

First of all, recall that under $(\mathbf{A}\Sigma)$, for $t>0$, P^0_t has a density p^0_t over $\mathbb{R}^{2d}$ with respect to the Lebesgue measure. Then

$$\mathbb{P}_\mathsf{x}(\sigma_{\mathscr{D}}=t)\le\mathbb{P}_\mathsf{x}(x^0_t\in\partial\mathsf{O})=\int_{\partial\mathsf{O}\times\mathbb{R}^d}p^0_t(\mathsf{x},x,v)\,dx\,dv=0, \tag{6.33}$$

since $\partial\mathsf{O}$ has Lebesgue measure 0. Indeed, because $\partial\mathsf{O}$ is $\mathscr{C}^1$, for any $x\in\partial\mathsf{O}$ there exists $\varepsilon_x>0$ such that the open subset $\partial\mathsf{O}\cap B(x,\varepsilon_x)$ of $\partial\mathsf{O}$ has Lebesgue measure 0. Moreover these open subsets of $\partial\mathsf{O}$ clearly cover $\partial\mathsf{O}$ and because $\partial\mathsf{O}$ is Lindelöf (due to the fact that $\mathbb{R}^d$ is Lindelöf and $\partial\mathsf{O}=\bar{\mathsf{O}}\cap(\mathbb{R}^d\setminus\mathsf{O})$ is closed), $\partial\mathsf{O}\subset\bigcup_{i\in\mathbb{N}}B(x_i,\varepsilon_{x_i})\cap\partial\mathsf{O}$. Thus $\partial\mathsf{O}$ has Lebesgue measure 0. This proves (6.33).

Let us turn to the proof of (6.32). For $n\ge 0$, denote $\mathsf{d}_n(t)=\max\{|X^0_s(\mathsf{x}_n)-X^0_s(\mathsf{x})|;\ s\in[0,t]\}$. Using [40, Lemma 3.3] (assumption (A) there is satisfied for the process (6.5), see indeed $(\mathbf{A}\Sigma)$), we have

$$\lim_{n\to\infty}\mathbb{P}(\mathsf{d}_n(t)>r)=0\quad\text{for any }r>0. \tag{6.34}$$

Let $\{n'\}\subset\mathbb{N}$ be a subsequence. By (6.34), $\mathsf{d}_{n'}(t)\to 0$ in $\mathbb{P}$-probability as $n'\to\infty$, and thus there exists a subsequence $\{n''\}\subset\{n'\}$ such that

$$\mathsf{d}_{n''}(t)\to 0\quad\text{a.s. as }n''\to\infty. \tag{6.35}$$

Let us prove that

$$\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x}_{n''})}\to\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x})}\quad\text{a.s. as }n''\to\infty. \tag{6.36}$$

In view of (6.33), we only have to prove (6.36) on the events $\{t<\sigma^0_{\mathscr{D}}(\mathsf{x})\}$ and $\{t>\sigma^0_{\mathscr{D}}(\mathsf{x})\}$.

On the event $\{t<\sigma^0_{\mathscr{D}}(\mathsf{x})\}$, $x^0_s(\mathsf{x})\in\mathsf{O}$ for all $s\in[0,t]$. By (6.35) and since O is open, there exists n''_0 such that for all $n''\ge n''_0$, $x^0_s(\mathsf{x}_{n''})\in\mathsf{O}$ for all $s\in[0,t]$. Therefore, $t<\sigma^0_{\mathscr{D}}(\mathsf{x}_{n''})$ for all $n''\ge n''_0$. We have thus proved that on the event $\{t<\sigma^0_{\mathscr{D}}(\mathsf{x})\}$, $\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x}_{n''})}\to 1=\mathbb{1}_{t<\sigma^0_{\mathscr{D}}(\mathsf{x})}$ as $n''\to\infty$.

Let us now prove (6.36) on the event $\{t>\sigma^0_{\mathscr{D}}(\mathsf{x})\}$. In this case, since $\mathsf{x}\in\mathscr{D}$, it follows that $\mathsf{n}(x^0_{\sigma^0_{\mathscr{D}}(\mathsf{x})})\cdot v^0_{\sigma^0_{\mathscr{D}}(\mathsf{x})}\ge 0$ almost surely. Set $\alpha=(t-\sigma^0_{\mathscr{D}})/2\in(0,t-\sigma^0_{\mathscr{D}})$. Then, from (6.27), and by the strong Markov property of the process (6.1), there exists $u\in(0,\alpha]$ such that $x^0_{\sigma^0_{\mathscr{D}}(\mathsf{x})+u}(\mathsf{x})\in\mathbb{R}^d\setminus\bar{\mathsf{O}}$. By (6.35) and since $\mathbb{R}^d\setminus\bar{\mathsf{O}}$ is open, there exists n''_0 such that $x^0_{\sigma^0_{\mathscr{D}}(\mathsf{x})+u}(\mathsf{x}_{n''})\in\mathbb{R}^d\setminus\bar{\mathsf{O}}$ for all $n''\ge n''_0$. Thus, by continuity of the trajectories of the process (6.5), $\sigma^0_{\mathscr{D}}(\mathsf{x}_{n''})<\sigma^0_{\mathscr{D}}(\mathsf{x})+u<t$ for all $n''\ge n''_0$. We have proved that on the

event $\{t > \sigma^0_{\mathcal{D}}(\mathsf{x})\}$, $1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x}_{n''})} \to 0 = 1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x})}$ as $n'' \to \infty$. This concludes the proof of (6.36).

We now conclude the proof of (6.32). If (6.32) does not hold, there exist $r, \gamma > 0$ and $\{n'\} \subset \mathbb{N}$ such that for all n', $\mathbb{P}(|1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x}_{n'})} - 1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x})}| > r) > \gamma$. However, there exists $\{n''\} \subset \{n'\}$ such that (6.36) holds, a contradiction. The proof of (6.32) is complete.

Step 2: End of the proof of Proposition 6.7. Pick a measurable and bounded function $f : \mathcal{D} \to \mathbb{R}$ (i.e. $f \in b\mathcal{B}(\mathcal{D})$). Extend f by 0 outside $\mathcal{D}$, so that $f \in b\mathcal{B}(\mathbb{R}^{2d})$. From [83, Lemma 1.1], it follows that for $\mathsf{x} \in \mathcal{D}$ and $t > 0$,

$$\mathbb{E}_{\mathsf{x}}[f(X_t)1_{t<\sigma_{\mathcal{D}}}] = \mathbb{E}[f(X^0_t(\mathsf{x}))\, 1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x})}\, \mathsf{M}_t(\mathsf{x})],$$

where

$$\begin{aligned}\mathsf{M}_t = \exp\Big[&-\int_0^t \Sigma(x^0_s, v^0_s)^{-1}\big(c(x^0_s, v^0_s)v^0_s + \nabla V(x^0_s)\big)\, dB_s \\ &- \frac{1}{2}\int_0^t \big|\Sigma(x^0_s, v^0_s)^{-1}[c(x^0_s, v^0_s)v^0_s + \nabla V(x^0_s)]\big|^2\, ds\Big].\end{aligned}$$

Let $(\mathsf{x}_n)_n$ be a sequence of elements of $\mathcal{D}$ such that $\mathsf{x}_n \to \mathsf{x} \in \mathcal{D}$ as $n \to \infty$. Then, from the proof of [83, Proposition 1.2], $f(X^0_t(\mathsf{x}_n)) \to f(X^0_t(\mathsf{x}))$ in $\mathbb{P}$-probability and $\mathsf{M}_t(\mathsf{x}_n) \to \mathsf{M}_t(\mathsf{x})$ in $L^1(\Omega, \mathbb{P})$ as $n \to \infty$. Then, using (6.32), in $\mathbb{P}$-probability,

$$f(X^0_t(\mathsf{x}_n))1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x}_n)} \to 1_{t<\sigma^0_{\mathcal{D}}(\mathsf{x})} f(X^0_t(\mathsf{x})) \quad \text{as } n \to \infty.$$

Thus, $\mathbb{E}_{\mathsf{x}_n}[f(X_t)1_{t<\sigma_{\mathcal{D}}}] \to \mathbb{E}_{\mathsf{x}}[f(X_t)1_{t<\sigma_{\mathcal{D}}}]$ as $n \to \infty$, that is, $P^{\mathcal{D}}_t$ is strongly Feller for $t > 0$. This ends the proof of Proposition 6.7. ∎

Remark 6.8. If $\mathcal{D} = \mathsf{O} \times \mathsf{V}$, where V is a smooth bounded subdomain of $\mathbb{R}^d$, we refer to [13] for the strong Feller property of $P^{\mathcal{D}}_t$.

6.3. Quasi-stationary distributions for hypoelliptic damped Hamiltonian systems (6.1)

With all the previous results (Lemma 6.2, Proposition 6.5, Lemma 6.6, and Proposition 6.7), one deduces from Theorem 2.2 the following theorem on the existence and uniqueness of a quasi-stationary distribution of a process (6.1) in $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$.

Theorem 6.9. *Assume that Σ satisfies* (**A**Σ). *Let O be a $\mathcal{C}^2$ subdomain of $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus \bar{\mathsf{O}}$ is nonempty.*

(1) *Assume that the functions V and c satisfy* (**Av1**), (**Av2**), (**Ac1**), *and* (**Ac2**). *Then there exist parameters $\mathsf{w} \in \mathcal{C}^2_c(\mathbb{R}^d, \mathbb{R})$ and $a, b > 0$ (see* [83, (3.4)–(3.9)] *for explicit conditions on w, a, and b) such that Theorem* 2.2 *is valid for the process* (6.1) *with $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$ and with the Lyapunov function W_1 defined in* (6.8). *We refer to Remark* 6.3 *for concrete examples of functions V and c satisfying these assumptions, and to Lemma* 6.4 *for an upper bound on W_1.*

(2) *Assume that the functions c and V satisfy respectively* (6.2) *and* (6.3)*, and* (6.9) *holds. Then Theorem* 2.2 *is valid for the process* (6.1) *with* $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$ *and with the bounded Lyapunov function* W_2 *defined in* (6.12)*. Let us emphasize that since* W_2 *is bounded* (*see Proposition* 6.5 (2))*, item* (b) *in Theorem* 2.2 *holds, and item* (d) *there is satisfied for any initial distribution* ν *in* $\mathcal{D}$.

In other words, if $(\mathbf{A}\Sigma)$ holds, when $\mathcal{D} = \mathsf{O} \times \mathbb{R}^d$ (where O is as in Theorem 6.9), there exists a unique QSD in $\mathcal{D}$ for the process (6.1) in:

(1) the space $\mathscr{M}_p = \{\nu \in \mathcal{M}_1(\mathcal{D});\ \nu(\mathsf{W}_1^{1/p}) < +\infty\}$ for all $p > 1$ when (**Av1**), (**Av2**), (**Ac1**), and (**Ac2**) hold; in addition, in this case, (2.7) holds for all $\nu \in \mathscr{M}_p$;

(2) the whole space $\mathcal{M}_1(\mathcal{D})$ of probability measures on $\mathcal{D}$ when (6.2), (6.3), and (6.9) hold; moreover, in this case, (2.7) holds for all $\nu \in \mathcal{M}_1(\mathcal{D})$.

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