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Transfinite Milnor invariants for 3-manifolds

Abstract. In his 1957 paper, John Milnor introduced link invariants which measure the homotopy class of the longitudes of a link relative to the lower central series of the link group. Consequently, these invariants determine the lower central series quotients of the link group. This work has driven decades of research with profound influence. One of Milnor's original problems remained unsolved: to extract similar invariants from the transfinite lower central series of the link group. We reformulate and extend Milnor's invariants in the broader setting of 3-manifolds, with his original invariants as special cases. We present a solution to Milnor's problem for general 3-manifold groups, developing a theory of transfinite invariants and realizing nontrivial values.

Keywords. Transfinite Milnor invariants, 3-manifolds, homology cobordism, transfinite lower central series

Contents

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1. Introduction

In John Milnor's 1954 Ph.D. thesis [23], he introduced link invariants obtained from the lower central series of the fundamental group. Milnor's work vastly extended the classical linking number, and has influenced decades of fundamental research.

Roughly speaking, Milnor's invariants inductively measure whether the fundamental group of the exterior of a given link has the same lower central series quotients as that of the free group [23]. Another key feature of Milnor's invariant, due to Stallings [26], is invariance under link concordance, and more generally under homology cobordism of the link exterior. Invariance under homology cobordism seeds a fundamental connection between Milnor's invariants and the topology of 4-manifolds.

Although seldom noted, the first part of Milnor's paper [23] concerns fundamental groups of exteriors of links in an *arbitrary* 3-*manifold*, while the latter part of Milnor's paper, as well as most subsequent research of others, focuses on the special case of links in S^3.

The following problems posed by Milnor in [23] have remained unsolved for more than 60 years.

Milnor's Problem ([23, p. 52, Problem (b)]). Find a method of attacking the *transfinite* lower central series quotients and extracting information from it.

That is, develop a transfinite lower central series version of Milnor's invariants which contains nonvacuous information.

Recall that the *transfinite lower central series* of a group G consists of subgroups G_κ indexed by ordinals κ and defined by

$$G_\kappa = \begin{cases} G & \text{if } \kappa = 1, \\ [G, G_{\kappa-1}] & \text{if } \kappa > 1 \text{ is a successor ordinal,} \\ \bigcap_{\lambda < \kappa} G_\lambda & \text{if } \kappa \text{ is a limit ordinal.} \end{cases}$$

We acknowledge past progress toward a solution to Milnor's problem. The first viable candidate for a transfinite invariant was given in work of the second author [24]. He presented a reformulation of the original Milnor link invariants by introducing a homotopy-theoretic approach. Papers [17, 24] answered numerous problems from [23], including the realizability and independence of Milnor invariants. But Orr's "transfinite" invariant of links continues to resist computation. (See recent progress by E. D. Farjoun and R. Mikhailov [10].)

J. Levine [20, 21] refined Orr's transfinite invariant by developing the fundamental notion of "algebraic closure of groups". This arose, in part, from harvesting key insights from work of M. Gutiérrez [14] and P. Vogel [28] (see also [19]). With his breakthrough, Levine proved further realization and geometric characterization results. Unfortunately, Levine's refinement resists computation as well.

In particular, it remains open whether the invariants in [20, 21, 24] always vanish for links with vanishing classical Milnor invariants!

Our contribution. In this paper, we develop new families of *transfinite* invariants for closed, orientable 3-*manifolds*. For one family of these invariants we find striking parallels to Milnor's link invariants, leading us to name that family of invariants *Milnor invariants of 3-manifolds*. The Milnor invariants we introduce are indexed by arbitrary ordinal numbers called the *length* of the invariant. This allows one to extend the integer grading in Milnor's original work. Our invariants include classical Milnor invariants as a special case.

We show that our invariants are highly nontrivial even at infinite ordinals. Thus, *we view these invariants as presenting a solution to Milnor's problem within the broad context of oriented closed* 3-*manifolds.*

Indeed, we define four closely related invariants. The invariants we call the Milnor invariants is denoted by $\overline{\mu}_\kappa(M)$, where κ is the length. The invariant $\overline{\mu}_\kappa(M)$ has the following features:

(i) *Determination of lower central series quotients*: $\overline{\mu}_\kappa(M)$ inductively determine the isomorphism classes of the lower central series quotients, as do Milnor's link invariants. Furthermore, this inductive process extends to transfinite ordinals.

(ii) *Homology cobordism invariance*: $\overline{\mu}_\kappa(M)$ is invariant under homology cobordism, as are Milnor's link invariants.

(iii) *Specialization to Milnor's link invariants*: $\overline{\mu}_\kappa(M)$ with finite κ determines Milnor's link invariants, when M is the 0-surgery on a link in S^3.

(iv) *Obstructions to gropes*: like those of links, $\overline{\mu}_\kappa(M)$ is an obstruction to building gropes. Moreover, this extends to the transfinite length case, using an appropriate notion of transfinite gropes.

(v) *Realization*: $\overline{\mu}_\kappa(M)$ lives in a set $\mathcal{R}_\kappa(\Gamma)/\approx$, whose elements are explicitly characterized. Every element in $\mathcal{R}_\kappa(\Gamma)/\approx$ is realized as $\overline{\mu}_\kappa(M)$ for some closed 3-manifold M.

This shows that many *fundamental characterizing properties* of Milnor's link invariants generalize to our 3-manifold Milnor invariants, thereby extending Milnor's theory across all ordinals and 3-manifolds.

Using realizability from (v), we show the aforementioned result that the transfinite theory is *highly nontrivial even at infinite ordinals*—we exhibit infinitely many explicit 3-manifolds M with vanishing $\overline{\mu}_\kappa(M)$ for all finite κ but having nonvanishing, pairwise distinct $\overline{\mu}_\omega(M)$ for the first transfinite ordinal ω.

We also define and study a "universal" transfinite invariant, which generalizes Levine's link invariant [20] over algebraic closures to the case of 3-manifolds. We prove that this universal invariant is highly nontrivial, even for 3-manifolds for which all transfinite Milnor invariants vanish. As mentioned earlier, for links, whether Levine's invariant can be nonzero remains open.

We define two additional invariants central to our paper. The following section describes all four invariants and provides precise statements of our main results as well as applications, (i)–(v).

The new results of this paper, especially the framework of transfinite invariants, opens multiple avenues for future research. We discuss a small portion of these, including potential applications to link concordance and to Whitney towers, at the end of the paper.

2. Statements of main results

In this section, we describe our main results. In Section 2.1, we provide a quick review of homology localization. In Sections 2.2–2.9, we define four invariants for 3-manifolds and present their key features, including 3-manifold Milnor invariants. In Sections 2.10–2.11, we discuss examples exhibiting rich information extracted from these invariants.

Throughout this paper, we consider only compact oriented manifolds unless stated otherwise explicitly. The notation $H_*(-)$ denotes homology with integral coefficients.

2.1. Homology localization of groups

We begin with a brief introduction to the role of *locally finite homology localization* of groups, also known as *algebraic closure*. Readers who are already familiar with this might prefer to skip to the last paragraph (or the last sentence) of this subsection.

The invariance of the original Milnor invariants under concordance and homology cobordism follows from a well known result of Stallings that the lower central quotients $\pi_1(-)/\pi_1(-)_k$ are preserved under homology equivalence of spaces for all $k < \infty$ [26]. (See also [3].) By contrast, the transfinite lower central quotient $\pi_1(-)/\pi_1(-)_\kappa$ is not invariant under homology cobordism (or homology equivalence): for instance, this follows from an example of Hillman [16, pp. 56–57].

To extract information invariant under concordance of links and homology cobordism of 3-manifolds, we follow an approach suggested in work of Vogel [28] and Levine [20, 21], using homology localization of groups.

In general, localization is defined for a given collection Ω of morphisms in a category $\mathcal{C}$. Briefly, a localization designates a functor $E : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $A = 1_{\mathcal{C}}(A) \to E(A)$ such that

 (i) $E(\phi)$ is an equivalence for all morphisms ϕ in Ω, and

(ii) E is universal (initial) among those satisfying (i).

A precise definition will be stated in Section 3. Observe that a homology equivalence $X \to Y$ of spaces gives rise to a group homomorphism $\pi_1(X) \to \pi_1(Y)$ which induces an isomorphism on $H_1(-)$ and an epimorphism on $H_2(-)$. We call a group homomorphism with this homological property *2-connected*. Due to an unpublished manuscript of Vogel [28] and an independent approach of Levine [20, 21], there exists a localization, in the category of groups, for the collection of 2-connected homomorphisms $\phi : A \to B$ with A and B finitely presented. (For those who are familiar with Bousfield's $H\mathbb{Z}$ localization [1, 2], we remark that the key difference between Vogel–Levine and the $H\mathbb{Z}$ case

is the finite presentability of A and B, which turns out to provide a crucial advantage for applications to compact manifolds.)

We observe that Levine's version in [20] differs slightly from what we use here. With applications to link concordance in mind, he adds the additional requirement that the image $\phi(A)$ normally generates B. (This reflects the property that meridians for a link normally generate the link group $\pi_1(S^3 \smallsetminus L)$.) Levine was aware of both notions of localization. The first detailed exposition of what we use can be found in [4]. We denote this homology localization by $G \to \widehat{G}$; see Section 3 for more details.

The following two properties of the homology localization $\widehat{G}$ are essential for our purpose. For brevity, denote the transfinite lower central subgroup $(\widehat{G})_\kappa$ by $\widehat{G}_\kappa$.

(i) A 2-connected homomorphism $G \to \Gamma$ between finitely presented groups induces an isomorphism on $\widehat{G}/\widehat{G}_\kappa \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ for every ordinal κ.

(ii) When G is finitely presented, $\widehat{G}/\widehat{G}_k \cong G/G_k$ for all k finite.

See Section 3, especially Corollary 3.2.

So, $\widehat{G}/\widehat{G}_\kappa$ is a transfinite generalization of the finite lower central quotients G/G_k, which remains invariant under homology cobordism of compact manifolds for every ordinal κ. In this regard, $\widehat{G}/\widehat{G}_\kappa$ is a correct generalization of G/G_k for studies related to homology cobordism, concordance, and disk embedding in dimension 4. From now on, "transfinite lower central quotient" in this paper means $\widehat{G}/\widehat{G}_\kappa$, instead of G/G_κ, where $\widehat{G}$ is the integer coefficient Vogel–Levine homology localization as constructed in [4].

2.2. Definition of the transfinite invariants

Milnor's original work [23] compares the lower central quotients π/π_k of a link group $\pi = \pi_1(S^3 \smallsetminus L)$ with that of the trivial link, namely the free nilpotent quotients F/F_k, inductively on k. We provide a relative theory, comparing the lower central quotients of other 3-manifolds to that of a fixed 3-manifold we choose arbitrarily. For instance, when studying links, we can begin with 0-surgery on a nontrivial link, and compare its lower central series quotients to that of other links. By replacing a 3-manifold group with its localization, we extend this theory throughout the *transfinite* lower central series.

Fix a closed 3-manifold Y, which will play the role analogous to the trivial link in Milnor's work. Denote $\Gamma = \pi_1(Y)$. Suppose M is another closed 3-manifold with $\pi = \pi_1(M)$. Our invariants compare the transfinite lower central quotients with that of Γ.

Indeed, we define and study four invariants of M:

(1) a θ-invariant $\theta_\kappa(M)$ defined as a 3-dimensional homology class,

(2) a reduced version of $\theta_\kappa(M)$ living in a certain "cokernel,"

(3) 3-manifold Milnor invariant $\overline{\mu}_\kappa(M)$, and

(4) a universal θ-invariant $\widehat{\theta}(M)$.

The first three invariants are indexed by arbitrary ordinals κ. In Sections 2.2–2.9, we describe the definitions and state their key features.

We begin with $\theta_\kappa(M)$. Fix an arbitrary ordinal κ, and suppose the 3-manifold group π admits an isomorphism $f : \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$. The goal is to determine whether the next stage quotient $\hat{\pi}/\hat{\pi}_{\kappa+1}$ is isomorphic to $\hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$.

The following definition is motivated from work of the second author [25] and Levine [20, 21]. (See also [15].)

Definition 2.1. Let $\theta_\kappa(M, f) \in H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ be the image of the fundamental class $[M] \in H_3(M)$ under the composition

$$H_3(M) \to H_3(\pi) \to H_3(\hat{\pi}) \to H_3(\hat{\pi}/\hat{\pi}_\kappa) \xrightarrow[\cong]{f_*} H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa).$$

We call κ the *length* of the invariant θ_κ. We will generally write $\theta_\kappa(M)$, in order to avoid an excess of notation, and will write $\theta(M, f)$ when we need to emphasize the choice of f.

The value of $\theta_\kappa(M)$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ depends on the choice of $f : \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$, and could be denoted $\theta_\kappa(M, f)$. We choose to omit the reference to f to simplify notation, but we emphasize to the reader that this indeterminacy is often nontrivial.

If we choose to remove indeterminacy, we can do so by comparing possible choices for f. Doing so, we obtain an invariant of 3-manifolds defined from $\theta_\kappa(M)$ by taking the value of $\theta_\kappa(M)$ in the orbit space $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)/\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ of the action of automorphisms of $\hat{\Gamma}/\hat{\Gamma}_\kappa$, thus providing an alternative definition of $\theta_\kappa(M)$ which is independent of the choice of f. It turns out that both versions (with and without indeterminacy) are useful, as we discuss below. We will refer to these invariants as the θ_κ-*invariants of M* (relative to Γ).

2.3. Invariance under homology cobordism

Theorem A. *The class $\theta_\kappa(M)$ is invariant under homology cobordism. More precisely, if M and N are homology cobordant 3-manifolds with $\pi = \pi_1(M)$ and $G = \pi_1(N)$, then for every ordinal κ, the following hold:*

(1) *There is an isomorphism $\phi : \hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\pi}/\hat{\pi}_\kappa$. Consequently, $\theta_\kappa(N)$ is defined if and only if $\theta_\kappa(M)$ is defined.*

(2) *If $f : \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$ is an isomorphism, then $\theta_\kappa(M, f) = \theta_\kappa(N, f \circ \theta)$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$, where ϕ is the isomorphism in (1).*

(3) *If $\theta_\kappa(M)$ and $\theta_\kappa(N)$ are defined using arbitrary isomorphisms $\hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$ and $\hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$, then $\theta_\kappa(M) = \theta_\kappa(N)$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)/\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_\kappa)$.*

We remark that the isomorphism ϕ in (1) and (2) depends on the choice of a homology cobordism. Statement (3) provides an invariant independent of choice.

The proof of Theorem A is given in Section 4. It is a straightforward consequence of the definition of the invariant and basic properties of homology localization.

2.4. Determination of transfinite lower central quotients

Define the set of homology classes which are realizable by θ_κ to be

$$\mathcal{R}_\kappa(\Gamma) = \left\{ \theta \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \, \middle| \, \begin{array}{l} \theta = \theta_\kappa(M) \text{ for some closed 3-manifold } M \\ \text{equipped with } \widehat{\pi_1(M)/\pi_1(M)_\kappa} \overset{\cong}{\Longrightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa \end{array} \right\}. \quad (2.1)$$

Not all homology classes are necessarily realizable, that is, $\mathcal{R}_\kappa(\Gamma) \neq H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ in general. Nor is $\mathcal{R}_\kappa(\Gamma)$ necessarily a subgroup. See Theorem G below, and Sections 10 and 11.

Nonetheless, one can straightforwardly verify that the projection $\widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1} \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ induces a function $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$. Although $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ is not well defined in the usual way because of the lack of a natural group structure, we can define a notion of *vanishing in the cokernel* as follows:

Definition 2.2. We say that a class $\theta \in \mathcal{R}_\kappa(\Gamma)$ *vanishes in* $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ if θ lies in the image of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$.

That is, the invariant $\theta_\kappa(M)$ vanishes in the cokernel if there is a closed 3-manifold N for which $\theta_{\kappa+1}(N)$ is defined (relative to Γ) and the image of $\theta_{\kappa+1}(N)$ is $\theta_\kappa(M)$ under the quotient induced homomorphism below:

$$\begin{array}{ccc} \theta_{\kappa+1}(N) & \in & \mathcal{R}_{\kappa+1}(\Gamma) \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}) \\ \big\downarrow & & \big\downarrow \\ \theta_\kappa(M) & \in & \mathcal{R}_\kappa(\Gamma) \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \end{array}$$

We now state the second main result.

Theorem B. *Suppose M is a closed 3-manifold and $\pi = \pi_1(M)$ is endowed with an isomorphism $f : \widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Longrightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Then the following are equivalent.*

(1) *There exists a lift $\widehat{\pi}/\widehat{\pi}_{\kappa+1} \overset{\cong}{\Longrightarrow} \widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}$ of f which is an isomorphism.*

(2) *The invariant $\theta_\kappa(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$.*

As stated in Theorem C below, it is possible to remove the restriction in Theorem B that the next stage isomorphism $\widehat{\pi}/\widehat{\pi}_{\kappa+1} \cong \widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}$ is a lift, by taking the value of $\theta_\kappa(M)$ modulo the action of $\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$, which is independent of the choice of $\widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Longrightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. To state the result, we use the following definition: a class θ *vanishes in* $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)/\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)\}$ if it lies in the image of the composition $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma) \to \mathcal{R}_\kappa(\Gamma)/\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$.

Theorem C. *Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ which admits an isomorphism $\widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Longrightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Then the following are equivalent:*

(1) *$\widehat{\pi}/\widehat{\pi}_{\kappa+1}$ is isomorphic to $\widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}$ (via any isomorphism not required to be a lift).*

(2) *The invariant $\theta_\kappa(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)/\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)\}$.*

The proof is straightforward, using Theorem B.

Proof of Theorem C. Suppose $g : \hat{\pi}/\hat{\pi}_{\kappa+1} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ is an isomorphism. Let $g_0 : \hat{\pi}/\hat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$ be the isomorphism induced by g, and consider $\theta_{\kappa+1}(M) = \theta_{\kappa+1}(M, g)$ and $\theta_{\kappa}(M) = \theta_{\kappa}(M, g_0)$. Then $\theta_{\kappa}(M)$ is the image of $\theta_{\kappa+1}(M)$ under $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_{\kappa}(\Gamma)$. This shows (1)$\Rightarrow$(2).

For the converse, suppose the invariant $\theta_{\kappa}(M) = \theta_{\kappa}(M, f)$ vanishes in the cokernel of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_{\kappa}(\Gamma)/\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_{\kappa})$ where $f : \hat{\pi}/\hat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$ is an arbitrary isomorphism. By composing an automorphism on $\hat{\Gamma}/\hat{\Gamma}_{\kappa}$ with f, we may assume that $\theta_{\kappa}(M)$ lies in the image of $\mathcal{R}_{\kappa+1}(\Gamma)$. By Theorem B, there is a lift $\hat{\pi}/\hat{\pi}_{\kappa+1} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ of f. ∎

The notion of vanishing in the cokernel generalizes to an equivalence relation $\sim$ on the set $\mathcal{R}_{\kappa}(\Gamma)$, which we describe below. Recall that if $\theta \in \mathcal{R}_{\kappa}(\Gamma)$, we have $\theta = \theta_{\kappa}(M)$ for some closed 3-manifold M equipped with an isomorphism $f : \widehat{\pi_1(M)/\pi_1(M)_{\kappa}} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$. Let I_{θ} be the image of the composition

$$\mathcal{R}_{\kappa+1}(\pi_1(M)) \to \mathcal{R}_{\kappa}(\pi_1(M)) \xrightarrow[f_*]{\cong} \mathcal{R}_{\kappa}(\Gamma).$$

In Lemma 5.3 we show that $\{I_{\theta} \mid \theta \in \mathcal{R}_{\kappa}(\Gamma)\}$ is a partition of the set $\mathcal{R}_{\kappa}(\Gamma)$. Consider the associated equivalence relation:

Definition 2.3. Define $\sim$ on $\mathcal{R}_{\kappa}(\Gamma)$ by $\theta \sim \theta'$ if $\theta' \in I_{\theta}$.

We prove the following result in Section 5.2.

Corollary D. *Suppose M and N are closed 3-manifolds with $\pi = \pi_1(M)$ and $G = \pi_1(N)$, which are equipped with isomorphisms $\hat{\pi}/\hat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$ and $\hat{G}/\hat{G}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$. Then there is an isomorphism $\hat{\pi}/\hat{\pi}_{\kappa+1} \overset{\cong}{\Rightarrow} \hat{G}/\hat{G}_{\kappa+1}$ which is a lift of the composition $\hat{\pi}/\hat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa} \overset{\cong}{\Rightarrow} \hat{G}/\hat{G}_{\kappa}$ if and only if $\theta_{\kappa}(M) \sim \theta_{\kappa}(N)$ in $\mathcal{R}_{\kappa}(\Gamma)$.*

Note that for a class $\theta \in \mathcal{R}_{\kappa}(\Gamma)$, we have $\theta \sim \theta_{\kappa}(Y)$ if and only if θ vanishes in the cokernel of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_{\kappa}(\Gamma)$. Here $\theta_{\kappa}(Y)$ is defined using the identity map $\widehat{\pi_1(Y)/\pi_1(Y)_{\kappa}} \to \hat{\Gamma}/\hat{\Gamma}_{\kappa}$. So, Corollary D generalizes Theorem B.

2.5. Milnor invariants of 3-manifolds

Now we define Milnor invariants of 3-manifolds. It combines the features of Theorem C and Corollary D in a natural way. Once again, we remind the reader of our hypothesis. We fix a 3-manifold Y and let $\Gamma = \pi_1(Y)$. We assume that M is a 3-manifold with $\pi = \pi_1(M)$, κ is an ordinal, and we have an isomorphism $f : \hat{\pi}/\hat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$. The invariant $\theta_{\kappa}(M) \in \mathcal{R}_{\kappa}(\Gamma)$ was defined in Definition 2.1. Here $\mathcal{R}_{\kappa}(\Gamma) \subset H_3(\hat{\Gamma}/\hat{\Gamma}_{\kappa})$ is the subset of realizable classes defined by (2.1).

The following is a coarser version of the equivalence relation $\sim$ on $\mathcal{R}_{\kappa}(\Gamma)$ in Definition 2.3.

Definition 2.4. Let $\theta, \theta' \in \mathcal{R}_{\kappa}(\Gamma)$. Write $\theta \approx \theta'$ if there is $\gamma \in \mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_{\kappa})$ such that $\gamma_*(\theta') \sim \theta$. That is, choosing a 3-manifold M equipped with an isomorphism $f : \widehat{\pi_1(M)/\pi_1(M)_{\kappa}} \overset{\cong}{\Rightarrow} \hat{\Gamma}/\hat{\Gamma}_{\kappa}$ that satisfies $\theta = \theta_{\kappa}(M)$, we have $\theta \approx \theta'$ if and only if there

is $\gamma \in \mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ such that

$$\gamma_*(\theta') \in \mathrm{Im}\left\{\mathcal{R}_{\kappa+1}(\pi_1(M)) \to \mathcal{R}_\kappa(\pi_1(M)) \xrightarrow[f_*]{\cong} \mathcal{R}_\kappa(\Gamma)\right\}.$$

Since $\sim$ is an equivalence relation and $\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is a group, it follows that $\approx$ is an equivalence relation too.

Definition 2.5. The *Milnor invariant of length κ* for M is defined by

$$\bar{\mu}_\kappa(M) := [\theta_\kappa(M)] \in \mathcal{R}_\kappa(\Gamma)/\approx.$$

Here $[\theta_\kappa(M)]$ is the equivalence class of $\theta_\kappa(M) \in \mathcal{R}_\kappa(\Gamma)$ under $\approx$.

We see that θ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)/\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)\}$ in the sense of Section 2.4 if and only if $\theta \approx \theta_\kappa(Y)$ in $\mathcal{R}_\kappa(\Gamma)$. If $\theta_\kappa(M) \approx \theta_\kappa(Y)$, we say that $\bar{\mu}_\kappa(M)$ *vanishes*, or M has vanishing Milnor invariant of length κ.

Theorem E. *Let M be a 3-manifold such that $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\kappa \cong \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Then $\bar{\mu}_\kappa(M)$ is a well-defined homology cobordism invariant, and the following are equivalent:*

(1) $\bar{\mu}_\kappa(M)$ *vanishes.*
(2) $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_{\kappa+1} \cong \widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}$ *(via any isomorphism not required to be a lift).*
(3) *The invariant $\bar{\mu}_{\kappa+1}(M)$ is defined.*

In addition, for M and N such that $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\kappa \cong \widehat{\pi_1(N)}/\widehat{\pi_1(N)}_\kappa \cong \widehat{\Gamma}/\widehat{\Gamma}_\kappa$, the following two conditions are equivalent:

(4) $\bar{\mu}_\kappa(M) = \bar{\mu}_\kappa(N)$ *in $\mathcal{R}_\kappa(\Gamma)/\approx$.*
(5) $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_{\kappa+1} \cong \widehat{\pi_1(N)}/\widehat{\pi_1(N)}_{\kappa+1}$.

Proof. The equivalence of (1)–(3) is the conclusion of Theorem C.

Suppose (4) holds. Fix an isomorphism $f \colon \widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. By definition, $\theta_\kappa(M, f) \sim \gamma_*(\theta_\kappa(N, g))$ for some $\gamma \in \mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ and $g \colon \widehat{\pi_1(N)}/\widehat{\pi_1(N)}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Then $\gamma_*(\theta_\kappa(N, g)) = \theta_\kappa(N, \gamma \circ g)$. By Corollary D, it follows that (5) holds. Conversely, when (5) holds, let ϕ be the induced isomorphism $\widehat{\pi_1(N)}/\widehat{\pi_1(N)}_\kappa \xrightarrow{\cong} \widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\kappa$. Since ϕ lifts, $\theta_\kappa(M, f) \sim \theta_\kappa(N, f \circ \phi)$ by Corollary D. So (4) holds. ∎

Examples showing the nontriviality of the 3-manifold $\bar{\mu}_\kappa$-invariant of transfinite length are given in Section 2.11 (see Theorem L).

Section 9 explains how classical Milnor invariants are special cases of the above theory associated to finite ordinals (see also [17, 20, 25]).

Section 2.7 below states that the $\bar{\mu}_\kappa$-invariant connects to a notion of transfinite gropes, with details in Section 6.

2.6. A transfinite tower interpretation

Corollary D and Theorem E may be viewed as classifications of towers of transfinite lower central quotients of 3-manifold groups. Briefly, we address the following problem:

classify extensions of length $\kappa + 1$, by 3-manifold groups, of the length κ tower of the transfinite lower central quotients

$$\widehat{\Gamma}/\widehat{\Gamma}_\kappa \dashrightarrow \widehat{\Gamma}/\widehat{\Gamma}_\omega \dashrightarrow \widehat{\Gamma}/\widehat{\Gamma}_2 \longrightarrow \widehat{\Gamma}/\widehat{\Gamma}_1 = \{1\} \tag{2.2}$$
$$\| \qquad \qquad \|$$
$$\Gamma/\Gamma_2 \qquad \Gamma/\Gamma_1$$

of a given 3-manifold group $\Gamma = \pi_1(Y)$.

To be precise, we introduce some abstract terminology:

(i) A *length κ tower* in a category $\mathcal{C}$ is a functor A of the (opposite) category of ordinals $\{\lambda \mid \lambda \leq \kappa\}$, with arrows $\lambda \to \lambda'$ for $\lambda' \leq \lambda$ as morphisms, into $\mathcal{C}$. Denote it by $\{A(\lambda)\}_{\lambda \leq \kappa}$ or $\{A(\lambda)\}$.

(ii) A *κ-equivalence* between two towers $\{A(\lambda)\}$ and $\{A'(\lambda)\}$ is a natural equivalence $\phi = \{\phi_\lambda \colon A(\lambda) \xRightarrow{\cong} A'(\lambda)\}_{\lambda \leq \kappa}$ between the two functors, that is, each ϕ_λ is an equivalence and ϕ_λ is a lift of $\phi_{\lambda'}$ for $\lambda' \leq \lambda \leq \kappa$. Say $\{A(\lambda)\}$ and $\{A'(\lambda)\}$ are *κ-equivalent* if there is a κ-equivalence between them.

(iii) A *length $\kappa + 1$ extension* of a length κ tower $\{A(\lambda)\}_{\lambda \leq \kappa}$ is a length $\kappa + 1$ tower $\{B(\lambda)\}_{\lambda \leq \kappa+1}$ equipped with a κ-equivalence between $\{A(\lambda)\}_{\lambda \leq \kappa}$ and $\{B(\lambda)\}_{\lambda \leq \kappa}$.

(iv) Two length $\kappa + 1$ extensions $\{B(\lambda)\}_{\lambda \leq \kappa+1}$ and $\{B'(\lambda)\}_{\lambda \leq \kappa+1}$ of a tower of length κ, $\{A(\lambda)\}_{\lambda \leq \kappa}$, are *equivalent* if the composition $B(\kappa) \xRightarrow{\cong} A(\kappa) \xRightarrow{\cong} B'(\kappa)$ lifts to an equivalence $B(\kappa + 1) \xRightarrow{\cong} B'(\kappa + 1)$.

In this paper, towers and their extensions will always be transfinite lower central quotient towers $\{\widehat{\pi}/\widehat{\pi}_\lambda\}_{\lambda \leq \kappa}$ of 3-manifold groups π. (In this case, $\{\widehat{\pi}/\widehat{\pi}_\lambda\}_{\lambda \leq \kappa}$ and $\{\widehat{G}/\widehat{G}_\lambda\}_{\lambda \leq \kappa}$ are κ-equivalent if and only if $\widehat{\pi}/\widehat{\pi}_\kappa$ and $\widehat{G}/\widehat{G}_\kappa$ are isomorphic.) We define a *length $\kappa + 1$ extension of* (2.2) *by a 3-manifold group* to be a length $\kappa + 1$ extension of the form $\{\widehat{\pi}/\widehat{\pi}_\lambda\}$ where $\pi = \pi_1(M)$ for some closed 3-manifold M.

For towers of 3-manifold groups, the following two problems are formulated naturally:

(1) Classify length $\kappa + 1$ extensions of a given fixed tower of length κ, modulo equivalence of extensions in the sense of (iv).

(2) Classify length $\kappa + 1$ towers whose length κ subtowers are κ-equivalent to a given fixed tower of length κ, modulo $(\kappa + 1)$-equivalence in the sense of (ii).

The following results are immediate consequences of Corollary D and Theorem E.

Corollary F. *For every ordinal κ, the following hold:*

(1) *The set of classes*

$$\left\{ \begin{matrix} length\ \kappa + 1\ extensions\ of\ (2.2) \\ by\ 3\text{-}manifold\ groups \end{matrix} \right\} \Big/ \begin{matrix} equivalence\ of\ length \\ (\kappa + 1)\text{-}extensions\ of\ (2.2) \end{matrix}$$

is in one-to-one correspondence with $\mathcal{R}_\kappa(\Gamma)/\sim$, via the invariant θ_κ.

(2) *The set of classes*

$$\left\{ \begin{array}{l} \textit{length } \kappa + 1 \textit{ towers of 3-manifold groups} \\ \textit{with length } \kappa \textit{ subtower } \kappa\textit{-equivalent to (2.2)} \end{array} \right\} \big/ (\kappa + 1)\textit{-equivalence}$$

is in one-to-one correspondence with $\mathcal{R}_\kappa(\Gamma)/\approx$, *via the 3-manifold Milnor invariant* $\overline{\mu}_\kappa$.

Remark 2.6. The two classifications in Corollary F (1, 2) are indeed *not* identical. More precisely, the natural surjection from the set of classes in Corollary F (1) onto that in (2), or equivalently the surjection $\mathcal{R}_\kappa(\Gamma)/\sim \to \mathcal{R}_\kappa(\Gamma)/\approx$, is *not* injective in general. In fact, for the first transfinite ordinal ω, Theorem I below presents an explicit 3-manifold example for which $\mathcal{R}_\omega(\Gamma)/\sim$ is an infinite set but $\mathcal{R}_\omega(\Gamma)/\approx$ is a singleton.

2.7. Transfinite gropes and the invariants

In this paper, we also introduce a previously unexplored notion of *transfinite gropes* (see Section 6.2), and relate them to the transfinite Milnor invariants. Once again, this extends well known results concerning classical Milnor invariant of links and the existence of finite (asymmetric) gropes. For instance, Freedman and Teichner [13] and Conant, Schneiderman and Teichner as summarized in [8], as well as work of the first author [5].

In [13], for finite k, a grope corresponding to the kth term of the lower central series is called a grope of class k. Briefly, we extend this to the case of an arbitrary transfinite ordinal κ, to define a notion of a *grope of (transfinite) class* κ. We say that a 4-dimensional cobordism W between two 3-manifolds M and N is a *grope cobordism of class* κ if $H_1(M) \to H_1(W)$ and $H_1(N) \to H_1(W)$ are isomorphisms and the cokernels of $H_2(M) \to H_2(W)$ and $H_2(N) \to H_2(W)$ are generated by homology classes represented by gropes of class κ. See Definitions 6.5, 6.6 and 6.9 for precise descriptions.

Transfinite gropes give another characterization of the equivalent properties in Theorems C and E, as stated below.

Addendum to Theorems C and E. *Suppose M is a closed* 3-*manifold such that the group* $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\kappa$ *is isomorphic to* $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$. *Then the following is equivalent to properties* (1) *and* (2) *in Theorem C, and to properties* (1)–(3) *of Theorem E:*

(0) *There is a grope cobordism of class $\kappa + 1$ between M and another closed* 3-*manifold N satisfying* $\widehat{\pi_1(N)}/\widehat{\pi_1(N)}_{\kappa+1} \cong \widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1}$.

Its proof is given in Section 6.2.

As a key ingredient of the proof, we develop and use a transfinite generalization of a well-known theorem of Stallings and Dwyer [12, 26]. Since we believe that it will be useful for other applications in the future as well, we present the statement here.

Theorem 6.1. *Let $\kappa > 1$ be an arbitrary ordinal. Suppose $f : \pi \to G$ is a group homomorphism inducing an isomorphism $H_1(\pi) \xrightarrow{\cong} H_1(G)$. In addition, if κ is a transfinite ordinal, suppose G is finitely generated. Then f induces an isomorphism $\widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{G}/\widehat{G}_\kappa$*

if and only if f induces an epimorphism

$$H_2(\hat{\pi}) \to H_2(\hat{G})/\mathrm{Ker}\{H_2(\hat{G}) \to H_2(\hat{G}/\hat{G}_\lambda)\}$$

for all ordinals $\lambda < \kappa$.

See also Corollaries 6.3, 6.4 and 6.8 in Section 6.

2.8. Realization of the invariants

Our next result is an algebraic characterization of the classes in $\mathcal{R}_\kappa(\Gamma)$. Denote by $tH_*(-)$ the torsion subgroup of $H_*(-)$.

Theorem G. *Let $\kappa \geq 2$ be an arbitrary ordinal. A class $\theta \in H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ lies in $\mathcal{R}_\kappa(\Gamma)$ if and only if the following two conditions hold:*

(1) *The cap product*

$$\cap \theta : tH^2(\hat{\Gamma}/\hat{\Gamma}_\kappa) \to tH_1(\hat{\Gamma}/\hat{\Gamma}_\kappa) \cong tH_1(\Gamma)$$

is an isomorphism.

(2) *The composition*

$$H^1(\hat{\Gamma}/\hat{\Gamma}_\kappa) \xrightarrow{\cap \theta} H_2(\hat{\Gamma}/\hat{\Gamma}_\kappa) \xrightarrow{\mathrm{pr}} H_2(\hat{\Gamma}/\hat{\Gamma}_\kappa)/\mathrm{Ker}\{H_2(\hat{\Gamma}/\hat{\Gamma}_\kappa) \to H_2(\hat{\Gamma}/\hat{\Gamma}_\lambda)\}$$

is surjective for all ordinals $\lambda < \kappa$.

We remark that the definition of $\mathcal{R}_\kappa(\Gamma)$ given in (2.1) still makes sense even when Γ is not a 3-manifold group. (In this case $\mathcal{R}_\kappa(\Gamma)$ may be empty.) Theorem G holds for any finitely presented group Γ.

Conditions (1) and (2) in Theorem G may be viewed as Poincaré duality imposed properties of the given class θ with respect to the cap product. Also note that if κ is a successor ordinal, "for all ordinals $\lambda < \kappa$" in (2) can be replaced with "for $\lambda = \kappa - 1$."

For a finite ordinal κ, Theorem G is essentially due to Turaev [27]. Our new contribution in Theorem G is to extend his result transfinitely.

The proof of Theorem G is given in Section 7. Among other ingredients, the transfinite generalization of the Stallings–Dwyer theorem [12,26] stated above as Theorem 6.1 plays a key role in the proof of Theorem G.

2.9. Universal θ-invariant

By generalizing the approach of Levine's work on links [20], we define and study what we call the *universal θ-invariant* of a 3-manifold.

Once again, fix a closed 3-manifold Y and let $\Gamma = \pi_1(Y)$. Now suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ which admits an isomorphism $f : \hat{\pi} \xrightarrow{\cong} \hat{\Gamma}$.

Definition 2.7. Define $\widehat{\theta}(M) \in H_3(\widehat{\Gamma})$ to be the image of the fundamental class $[M] \in H_3(M)$ under the composition

$$H_3(M) \to H_3(\pi) \to H_3(\widehat{\pi}) \xrightarrow[\cong]{f_*} H_3(\widehat{\Gamma}).$$

Also, define the set of realizable classes in $H_3(\widehat{\Gamma})$ by

$$\widehat{\mathcal{R}}(\Gamma) = \left\{ \widehat{\theta}(M) \in H_3(\widehat{\Gamma}) \;\middle|\; \begin{array}{c} M \text{ is a closed 3-manifold equipped with} \\ \text{an isomorphism } \widehat{\pi_1(M)} \xrightarrow{\cong} \widehat{\Gamma} \end{array} \right\}.$$

Note that the value of $\widehat{\theta}(M)$ in the orbit space $\widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma})$ is determined by M, independent of the choice of the isomorphism f.

We remark that if M is equipped with an isomorphism $f: \widehat{\pi} \xrightarrow{\cong} \widehat{\Gamma}$ so that $\widehat{\theta}(M)$ is defined, then f induces an isomorphism $\widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$, and thus the invariant $\theta_\kappa(M)$ is defined for all ordinals κ. Moreover, $\theta_\kappa(M)$ is the image of $\widehat{\theta}(M)$ under $\widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$ induced by the projection $\widehat{\Gamma} \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Since this factors through $\mathcal{R}_{\kappa+1}(\Gamma)$, it follows that $\theta_\kappa(M)$ vanishes in the cokernel of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$, or equivalently $\theta_\kappa(M) \sim \theta_\kappa(Y)$ in $\mathcal{R}_\kappa(\Gamma)$, for every ordinal κ. Consequently, $\overline{\mu}_\kappa(M)$ vanishes for all κ if $\widehat{\theta}(M)$ is defined. It seems to be hard to prove or disprove the converse.

Similarly to the θ_κ-invariants (see Theorem A), $\widehat{\theta}(M) \in \widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma})$ is a homology cobordism invariant. We prove this in Theorem 8.1. Also, we prove a realization theorem characterizing homology classes in $\widehat{\mathcal{R}}(\Gamma)$, which is analogous to Theorem G.

Theorem H. *A homology class $\theta \in H_3(\widehat{\Gamma})$ lies in $\widehat{\mathcal{R}}(\Gamma)$ if and only if the following two conditions hold:*

(1) *The cap product $\cap \theta: t H^2(\widehat{\Gamma}) \to t H_1(\widehat{\Gamma}) \cong t H_1(\Gamma)$ is an isomorphism.*

(2) *The cap product $\cap \theta: H^1(\widehat{\Gamma}) \to H_2(\widehat{\Gamma})$ is surjective.*

We prove Theorem H in Section 8.

We remark that Levine proved a realization theorem for his link invariant which lives in $H_3(\widehat{F})$ [20]: for all $\theta \in H_3(\widehat{F})$, there is a link L for which his invariant is defined and equal to θ. Theorem H says that for general 3-manifolds, not all homology classes in H_3 are necessarily realizable. An example is given in Section 12.

It is an open problem whether Levine's link invariant in [20] is nontrivial. In Theorem J below, for the 3-manifold case, we show that $\widehat{\theta}(M)$ is nontrivial.

2.10. A torus bundle example

This section gives a complete and careful analysis of one example which illustrates the full collection of transfinite invariants considered in this paper. For the underlying 3-manifold, a torus bundle over a circle, the fundamental group of the fiber is a module over the group ring of covering translations, facilitating our computation of the group localization.

We thus compute and analyze the full array of invariants under consideration—θ, $\overline{\mu}$ and $\widehat{\theta}$. Moreover, this example illustrates several fundamental features of the invariants,

including (i) *nontriviality* of θ_ω for the first transfinite ordinal ω, (ii) *nontriviality* of $\widehat{\theta}$ even when all finite length θ and $\overline{\mu}$ vanish, and (iii) *torsion* values of finite length θ and $\overline{\mu}$.

Let Y be the torus bundle over S^1 with monodromy $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, that is, viewing S^1 as the unit circle in the complex plane,

$$Y = S^1 \times S^1 \times [0,1]/\{(z^{-1}, w^{-1}, 0) \sim (z, w, 1)\}.$$

Let $\Gamma = \pi_1(Y)$ be the fundamental group of the torus bundle. In our earlier work [7], we computed the homology localization $\widehat{\Gamma}$. Using this, it is not hard to compute its transfinite lower central quotients and see that $\widehat{\Gamma}$ is transfinitely nilpotent. Indeed, $\widehat{\Gamma}_{\omega+1}$ is trivial. Our computation starts from this.

The first transfinite invariant. For the first transfinite ordinal ω, we compute the homology $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ and its subset of realizable classes $\mathcal{R}_\omega(\Gamma)$. Moreover, we completely determine the two equivalence relations $\sim$ and $\approx$ on $\mathcal{R}_\omega(\Gamma)$, which were defined in Sections 2.4 and 2.5. The computation in particular tells us the following.

Theorem I. *For the torus bundle group Γ, the following hold:*

(1) *The set $\mathcal{R}_\omega(\Gamma)/\sim$ of equivalence classes of realizable values of θ_ω is infinite. Consequently, by Corollary F(1), there are infinitely many distinct equivalence classes of length $\omega + 1$ extensions, by 3-manifolds, of the length ω tower $\{\widehat{\Gamma}/\widehat{\Gamma}_\lambda\}_{\lambda \leq \omega}$ of the torus bundle Y (in the sense of Section 2.6).*

(2) *The set $\mathcal{R}_\omega(\Gamma)/\approx$ is a singleton. Consequently, $\overline{\mu}_\omega(M) \in \mathcal{R}_\omega(\Gamma)/\approx$ vanishes whenever it is defined. Also, for all closed 3-manifold groups π such that the length ω tower $\{\widehat{\pi}/\widehat{\pi}_\lambda\}_{\lambda \leq \omega}$ is ω-equivalent to that of Γ (in the sense of Section 2.6), the length $\omega + 1$ tower $\{\widehat{\pi}/\widehat{\pi}_\lambda\}_{\lambda \leq \omega+1}$ is automatically $(\omega + 1)$-equivalent to that of Γ, by Corollary F(2).*

Theorem I(1) illustrates that the transfinite θ-invariant of length ω provides highly nontrivial information, even when the transfinite Milnor invariant $\overline{\mu}$ of the same length vanishes. Examples with nonvanishing transfinite Milnor invariants will be given in Section 2.11 below (see Theorem L).

The tower interpretations in Theorem I in particular tell us the following: there are 3-manifold groups π such that there is an isomorphism $\widehat{\pi}/\widehat{\pi}_\omega \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\omega$ which does not lift to an isomorphism between $\widehat{\pi}/\widehat{\pi}_{\omega+1}$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$ but $\widehat{\pi}/\widehat{\pi}_{\omega+1}$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$ are isomorphic.

Theorem I is an immediate consequence of Theorem 11.1 and Corollary 11.2, which presents the outcome of our computation for $\widehat{\Gamma}/\widehat{\Gamma}_\omega$. See Section 11 for full details.

The universal invariant. We also carry out computation of the invariant $\widehat{\theta}$ over the homology localization of the torus bundle group. Among consequences of the computation, we have the following.

Theorem J. *For the torus bundle group Γ, the set $\widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma})$ of realizable values of $\widehat{\theta}$ modulo the automorphism action is infinite. This detects the existence of infinitely many distinct homology cobordism classes of closed 3-manifolds M with $\pi = \pi_1(M)$ such that $\widehat{\pi} \cong \widehat{\Gamma}$ and thus $\theta_\kappa(M)$ is defined and vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ for all ordinals κ. In particular, for every ordinal κ, the Milnor invariant $\overline{\mu}_\kappa(M)$ vanishes for these 3-manifolds M.*

This illustrates that the invariant $\widehat{\theta}$ is highly nontrivial for 3-manifolds for which all (transfinite) Milnor type invariants vanish.

This may be compared with the case of Levine's link invariant $\theta(L) \in H_3(\widehat{F})$ where F is a free group [20]. (For 0-surgery on a link, this invariant is equivalent to J. Y. Le Dimet's link concordance invariant defined in [19].) The fundamental question of Levine in [20], which is still left open, asks whether $\theta(L)$ can be nontrivial. Due to Levine's realization result in [20], this is equivalent to whether $H_3(\widehat{F})$ is nontrivial. Our result shows that for general 3-manifold groups, the answer is affirmative, even modulo the automorphism action.

Theorem J is a consequence of Theorems 12.1 and 12.2. Indeed, in Section 12, we provide a complete computation of $\mathcal{R}(\widehat{\Gamma})$ and the action of $\mathrm{Aut}(\widehat{\Gamma})$.

Finite length invariants. The torus bundle example also reveals interesting aspects of the finite length case of the Milnor type invariant. Our computation of the invariant θ_k for finite k proves the following result.

Theorem K. *For every finite k, the set $\mathcal{R}_k(\Gamma)$ of realizable classes is finite, and thus the set $\mathcal{R}_k(\Gamma)/\sim$ of equivalence classes is finite. Moreover,*

$$2 \leq \#(\mathcal{R}_k(\Gamma)/\sim) \leq 7 \cdot 2^{4(k-2)} + 1.$$

Consequently, by Corollary F(1), the number of equivalence classes of length $k+1$ extensions, by 3-manifold groups, of the length k tower $\Gamma/\Gamma_k \to \cdots \to \Gamma/\Gamma_1$ (in the sense of Section 2.6) is between 2 and $7 \cdot 2^{4(k-2)} + 1$ inclusive.

Theorem K is a consequence of Theorem 10.1 and Corollary 10.3, which provide a more detailed description of the structure of $\mathcal{R}_k(\Gamma)$ and related objects.

Remark 2.8. Recall that, for m-component links with vanishing Milnor invariants of length $< k$, the Milnor invariants of length k are integer-valued, and consequently they are either all trivial, or have infinitely many values. (Indeed, the Milnor invariants of length k span a free abelian group of known finite rank; see [25].) However, for the torus bundle case in Theorem K, it turns out that the finite length θ_k invariants live in torsion groups, in fact, finite 2-groups. This leads us to questions related to potential applications to link concordance, and to the higher order Arf invariant conjecture asked by Conant, Schneiderman and Teichner. In the final section of this paper, we discuss these questions, together with other questions arising from our work.

2.11. Modified torus bundle examples

We now consider a family $\{M_r \mid r$ is an odd integer$\}$ of modified torus bundles to show that transfinite Milnor invariants of 3-manifolds are nontrivial in general.

The modified torus bundles are obtained by changing just one entry in the monodromy matrix of the previous example Y: M_r has monodromy $\begin{bmatrix} -1 & r \\ 0 & -1 \end{bmatrix}$, that is,

$$M_r = S^1 \times S^1 \times [0, 1]/\{(z^{-1}, z^r w^{-1}, 0) \sim (z, w, 1)\}.$$

We remark that discussions with Sergei Ivanov and Roman Mikhailov led us to consider this modification. They studied the Bousfield–Kan completion of 3-manifold groups, with $\pi_1(M_r)$ as main examples [18].

Fix an odd integer d, and choose $Y = M_d$ as the basepoint manifold, to which other manifolds M_r are to be compared. Let $\Gamma = \pi_1(M_d)$. We prove the following.

Theorem L. *For any odd integer r, $\overline{\mu}_k(M_r) \in \mathcal{R}_k(\Gamma)/\approx$ is defined and vanishes for every finite k. Moreover, $\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega \cong \widehat{\Gamma}/\widehat{\Gamma}_\omega$ and thus $\overline{\mu}_\omega(M_r)$ is defined. But $\overline{\mu}_\omega(M_r) = \overline{\mu}_\omega(M_s)$ in $\mathcal{R}_\omega(\Gamma)/\approx$ if and only if the rational number $|r/s|$ is a square.*

In particular, the set $\mathcal{R}_\omega(\Gamma)/\approx$ of realizable values of the 3-manifold Milnor invariant is infinite, and there are infinitely many homology cobordism classes of 3-manifolds with the same finite length Milnor invariants but distinct Milnor invariants of length ω.

Indeed, we show that $\mathcal{R}_\omega(\Gamma)/\approx$ is equal to $\mathbb{Z}_{(2)}^\times = \{a/b \in \mathbb{Q} \mid a, b \in 2\mathbb{Z} + 1\}$ modulo the (multiplicative) subgroup $\pm(\mathbb{Z}_{(2)}^\times)^2 = \{\pm\alpha^2 \mid \alpha \in \mathbb{Z}_{(2)}^\times\}$. So, $\mathcal{R}_\omega(\Gamma)/\approx$ can be naturally identified with the set of odd positive integers r with no repeated primes in the factorization. Such an r corresponds to the value of the length ω Milnor invariant $\overline{\mu}_\omega(M_{rd})$ of the 3-manifold M_{rd} (see Theorem 13.1). So, the modified torus bundles explicitly realize nontrivial values of the transfinite Milnor invariant $\overline{\mu}_\omega$ over the group $\Gamma = \pi_1(M_d)$.

The following is a consequence of Theorem L combined with Corollary F (2): there are infinitely many 3-manifold groups π such that the lower central series quotient towers $\{\widehat{\pi}/\widehat{\pi}_\kappa\}_{\kappa \leq \omega}$ of length ω are mutually ω-equivalent (in the sense of Section 2.6), but the length $\omega + 1$ towers $\{\widehat{\pi}/\widehat{\pi}_\kappa\}_{\kappa \leq \omega+1}$ are not pairwise $(\omega + 1)$-equivalent.

3. Homology localization of groups

In this section we review basic facts on the homology localization of groups, and prove some results which will be useful in later sections. All results in this section were known to J. P. Levine. We include these results here for completeness, since group localization and the results herein play an essential role in this paper.

3.1. Preliminaries

We begin with the definition of the homology localization which we use. Recall that a group homomorphism $\pi \to G$ is *2-connected* if it induces an isomorphism on $H_1(-; \mathbb{Z})$

and an epimorphism on $H_2(-; \mathbb{Z})$. Let Ω be the collection of 2-connected homomorphisms between finitely presented groups. A group Γ is *local with respect to* Ω, or simply *local*m if, for every $\alpha: A \to B$ in Ω and every homomorphism $f: A \to \Gamma$, there is a unique homomorphism $g: B \to \Gamma$ satisfying $g \circ \alpha = f$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha\ } & B \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\
\Gamma & &
\end{array}
$$

A *localization with respect to* Ω is defined to be a pair (E, ι) of a functor E from the category of groups to the full subcategory of local groups and a natural transformation $\iota = \{\iota_G: G \to E(G)\}$ satisfying the following: for each homomorphism $f: G \to \Gamma$ with Γ local, there is a unique homomorphism $g: E(G) \to \Gamma$ such that $g \circ \iota_G = f$:

$$
\begin{array}{ccc}
G & \xrightarrow{\ \iota_G\ } & E(G) \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\
\Gamma & &
\end{array}
$$

In this paper, we denote $E(G)$ by $\widehat{G}$.

It is a straightforward exercise that a localization is unique if it exists. The existence of a localization with respect to Ω is due to Vogel and Levine. Indeed, in his unpublished manuscript [28], Vogel developed a general theory of localization of spaces with respect to homology, and its group analogue is the localization we discuss. In [21], Levine developed an alternative approach using certain systems of equations over a group to define a notion of "algebraic closure." He showed that it exists and is equal to the localization with respect to the subset of our Ω consisting of $\alpha: A \to B$ in Ω such that $\alpha(A)$ normally generates B. Although the modified closure with respect to our Ω (that is, omitting the normal closure condition) was known to Levine, this theory first appeared with proof in [4]. For a useful overview on homology localization for geometric topologists, the readers are referred to [6, Section 2].

The following properties of the homology localization $\widehat{\pi}$ are essential for our purpose.

Theorem 3.1 ([4, 20]).

(1) *If $\pi \to G$ is a 2-connected homomorphism between finitely presented groups, then it induces an isomorphism $\widehat{\pi} \xrightarrow{\cong} \widehat{G}$.*

(2) *For a finitely presented group G, there is a sequence*

$$G = P(1) \to P(2) \to \cdots \to P(k) \to \cdots$$

of 2-connected homomorphisms of finitely presented groups $P(k)$ such that the localization $G \to \widehat{G}$ is equal to the colimit homomorphism $G \to \mathrm{colim}_k P(k)$. Consequently, $G \to \widehat{G}$ is 2-connected.

Theorem 3.1 (1) is obtained by a routine standard argument using the universal properties given in the definitions. We omit the details; for instance, see [4, Proposition 6.4],

[20, Proposition 5]. The proof of Theorem 3.1 (2) is not straightforward and uses the actual construction of localization. See [4, Proposition 6.6], [20, Proposition 6].

Corollary 3.2.
(1) *For a finitely presented group G, the homomorphism $G \to \widehat{G}$ induces an isomorphism $G/G_k \to \widehat{G}/\widehat{G}_k$ for each $k < \infty$.*
(2) *A homology equivalence $X \to Y$ between finite CW-complexes X and Y with $\pi = \pi_1(X)$ and $G = \pi_1(Y)$ gives rise to isomorphisms $\widehat{\pi} \overset{\cong}{\to} \widehat{G}$ and $\widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\to} \widehat{G}/\widehat{G}_\kappa$ for each ordinal κ.*

Proof. (1) From Theorem 3.1 (2), it follows that $G \to \widehat{G}$ is 2-connected. By Stallings' theorem [26], $G \to \widehat{G}$ induces an isomorphism $G/G_k \to \widehat{G}/\widehat{G}_k$.

(2) The induced homomorphism $\pi \to G$ is 2-connected, since $K(\pi, 1)$ and $K(G, 1)$ are obtained by attaching cells of dimension ≥ 3 to X and Y. Since X and Y are finite, it follows that $\widehat{\pi} \cong \widehat{G}$ by Theorem 3.1 (1). Therefore $\widehat{\pi}/\widehat{\pi}_\kappa \cong \widehat{G}/\widehat{G}_\kappa$ for every κ. ∎

3.2. Acyclic equations and induced epimorphisms on localizations

J. P. Levine [22] first proved the following result, in much greater generality than stated here. His proof involved group localization determined by closure with respect to contractible equations, not acyclic equations. For this reason, we include a brief proof here. However, this lemma was certainly known to Levine.

Lemma 3.3. *If a group homomorphism $\pi \to G$ induces an epimorphism $H_1(\pi) \to H_1(G)$ and if G is finitely generated, then it induces an epimorphism $\widehat{\pi} \to \widehat{G}$.*

This proof of Lemma 3.3 depends on an equation-based approach to localization. In what follows, we give a quick review of definitions and results we need. Fix a group G. Following the idea of Levine [20] (see also Farjoun–Orr–Shelah [11]) consider a system $S = \{x_i = w_i\}$ of equations of the form

$$x_i = w_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n,$$

where each x_i is a formal variable and $w_i = w_i(x_1, \ldots, x_n)$ is an element of the free product $G * F$ of G and the free group $F = F\langle x_1, \ldots x_n \rangle$ on $x_1, \ldots, x_n$. A *solution* $\{g_i\}_{i=1}^n$ to the system S is defined to be an ordered tuple of n elements $g_i \in G$ such that $g_i = w_i(g_1, \ldots, g_n)$ for $i = 1, \ldots, n$. A group homomorphism $\phi \colon G \to \Gamma$ induces $\phi * \mathrm{id} \colon G * F \to \Gamma * F$, which sends a system S of equations over G to a system $\phi(S) := \{x_i = (\phi * \mathrm{id})(w_i)\}$ over Γ. If $\{g_i\}$ is a solution to S, then $\{\phi(g_i)\}$ is a solution to $\phi(S)$.

Following [4, Definition 4.1], we say an equation $x_i = w_i(x_1, \ldots, x_n)$ is *null-homologous*, or *acyclic*, if w_i lies in the kernel of the projection

$$G * F \to F \to H_1(F) = F/[F, F].$$

A group G is $\mathbb{Z}$-*closed* if every system of acyclic equations over G has a unique solution in G. We remark that these definitions are variations of Levine's notion of contractible equations and algebraic closure in [20].

Theorem 3.4 ([4, 20]).

(1) *A group G is local if and only if G is $\mathbb{Z}$-closed. In particular, every system of acyclic equations over $\widehat{G}$ has a unique solution in $\widehat{G}$.*

(2) *Every element in $\widehat{G}$ is a solution of a system of acyclic equations over G. More precisely, for each $g \in \widehat{G}$, there is a system $S = \{x_i = w_i\}_{i=1}^n$ of acyclic equations over G such that the system $\iota_G(S)$ over $\widehat{G}$ has a solution $\{g_i\}_{i=1}^n$ with $g_1 = g$.*

For the proof of Theorem 3.4 (1), see [4, Theorems 5.2 and 6.1, Corollary 6.3]. For the proof of Theorem 3.4 (2), see [4, Theorem 6.1, Proposition 6.6]. We remark that these proofs follow Levine's approach in [20, Propositions 3 and 6].

Proof of Lemma 3.3. Suppose $f : \pi \to G$ induces an epimorphism $f_* : H_1(\pi) \to H_1(G)$. Fix a finite set $\{a_1, \ldots, a_k\}$ which generates G. We begin by writing equations over G which have $\{a_j\}$ as a solution. Let $F = F\langle y_1, \ldots, y_k \rangle$. For each a_j, since f_* is surjective, $a_j = f(b_j) \cdot c_j$ for some $b_j \in \pi$ and $c_j \in [G, G]$. Write c_j as a product of commutators in the generators $a_i^{\pm 1}$, to choose a word $u_j = u_j(y_1, \ldots, y_k)$ in $[F, F]$ such that $c_j = u_j(a_1, \ldots, a_k)$. Let S_0 be the system $\{y_j = b_j \cdot u_j\}_{j=1}^k$ of acyclic equations over π. Then $\{a_j\}$ is a solution to the system $f(S_0) = \{y_j = f(b_j) \cdot u_j\}$ over G. Applying $\iota_G : G \to \widehat{G}$, it follows that $\{\iota_G(a_j)\}$ is a solution to the system $\iota_G f(S_0)$ over $\widehat{G}$.

Now, to show that $\widehat{f} : \widehat{\pi} \to \widehat{G}$ is surjective, fix $g \in \widehat{G}$. By Theorem 3.4 (2), there is a system $S = \{x_i = w_i\}_{i=1}^n$ of acyclic equations over G, with $w_i \in G * F\langle x_1, \ldots, x_n \rangle$, such that the system $\iota_G(S)$ over $\widehat{G}$ has a solution $\{g_i\}$ with $g_i \in \widehat{G}$, $g_1 = g$. Substitute each occurrence of the generator a_j in the word w_i with $b_j \cdot u_j$, to obtain a new word $v_i = v_i(x_n, \ldots, x_n, y_1, \ldots, y_k)$. Now consider the system $S' = \{x_i = v_i\}_{i=1}^n \cup \{y_j = b_j u_j\}_{j=1}^k$ of $n + k$ equations over the group π. Apply the homomorphism $\iota_\pi : \pi \to \widehat{\pi}$ to obtain the system $\iota_\pi(S')$ over $\widehat{\pi}$. By Theorem 3.4 (2), $\iota_\pi(S')$ has a solution in $\widehat{\pi}$, say $\{r_i\}_{i=1}^n \cup \{s_j\}_{j=1}^k$, that is, $r_i = \iota_\pi v_i(r_1, \ldots, r_n, s_1, \ldots, s_k)$ and $s_j = \iota_\pi b_j \cdot u_j(s_1, \ldots, s_k)$. Now, apply $\widehat{f} : \widehat{\pi} \to \widehat{G}$ to the system $\iota_\pi(S')$. By the functoriality of localization, we have $\widehat{f} \iota_\pi(S') = \iota_G f(S')$, and it has $\{\widehat{f}(r_i)\} \cup \{\widehat{f}(s_j)\}$ as a solution in $\widehat{G}$. The last k equations of $\iota_G f(S')$ form the system $\iota_G f(S_0)$. By the uniqueness of a solution for $\iota_G f(S_0)$, we have $\widehat{f}(s_j) = \iota_G(a_j)$. By the uniqueness of a solution for $f(S')$, it follows that $\widehat{f}(r_i) = g_i$. In particular, $\widehat{f}(r_1) = g_1 = g$. This proves that $\widehat{f} : \widehat{\pi} \to \widehat{G}$ is surjective. ∎

3.3. Transfinite lower central quotients of local groups are local

It is well known that a nilpotent group is local, or equivalently the lower central quotient G/G_k of an arbitrary group G is local for all finite k (see, for instance, [20, p. 573]). But it is no longer true for the ordinary transfinite lower central quotients G/G_κ. For instance, for a free group F of rank > 1, $F/F_\omega \cong F$ and this is not local. However, for local groups the following is true.

Lemma 3.5. *If G is a local group, then G/G_κ is local for every ordinal $\kappa \geq 1$. In particular, $\widehat{G}/\widehat{G}_\kappa$ is local for every group G.*

Proof. To prove this, we will use the equation-based approach. Suppose that $S = \{x_i = w_i(x_1, \ldots, x_n)\}_{i=1}^n$ is a system of acyclic equations over G/G_κ. It suffices to show that S has a unique solution in G/G_κ. For the existence, lift S to a system over G, by replacing each element of G/G_κ which appears in the words w_i with a pre-image in G. Since G is local, there is a solution for the lift, and the image of the solution under the projection $G \to G/G_\kappa$ is a solution for S.

To prove the uniqueness, we proceed by transfinite induction. First, for $\kappa = 1$, $G/G_\kappa = \{e\}$ and thus everything is unique. Suppose $\kappa \geq 2$ and suppose the solution of a system of acyclic equations over G/G_λ is unique for all $\lambda < \kappa$. Suppose $\{x_i = g_i\}$ and $\{x_i = g_i'\}$ are two solutions in G/G_κ for a given system S of acyclic equations.

Suppose that κ is a successor ordinal. Let $p: G/G_\kappa \to G/G_{\kappa-1}$ be the projection. Since $\{x_i = p(g_i)\}$ and $\{x_i = p(g_i')\}$ are solutions of $p(S)$ over $G/G_{\kappa-1}$, $p(g_i) = p(g_i')$ by the uniqueness over $G/G_{\kappa-1}$. So $g_i' = g_i c_i$ for some $c_i \in G_{\kappa-1}/G_\kappa$. Since $G_{\kappa-1}/G_\kappa$ is central in G/G_κ and the image of w_i under $(G/G_\kappa) * F \to F$ lies in $[F, F]$, it follows that

$$g_i' = w_i(g_1', \ldots, g_n') = w_i(g_1 c_1, \ldots, g_n c_n) = w_i(g_1, \ldots, g_n) = g_i.$$

Now, suppose κ is a limit ordinal. For $\lambda < \kappa$, let $p_\lambda: G/G_\kappa \to G/G_\lambda$ be the projection. Since $\{x_i = p_\lambda(g_i)\}$ and $\{x_i = p_\lambda(g_i')\}$ are solutions of $p_\lambda(S)$, it follows that $p_\lambda(g_i) = p_\lambda(g_i')$ by the uniqueness of a solution over G/G_λ. That is, $g_i^{-1} g_i' \in \operatorname{Ker} p = G_\lambda/G_\kappa$ for each $\lambda < \kappa$. Since $G_\kappa = \bigcap_{\lambda < \kappa} G_\lambda$, it follows that $g_i = g_i'$ in G/G_κ. ∎

We remark that when κ is a successor ordinal in the above proof, the existence of a solution can also be shown under an induction hypothesis that $G/G_{\kappa-1}$ is local, without assuming that G is local. Indeed, if $\{x_i = h_i\}$ is a solution for $p(S)$ over $G/G_{\kappa-1}$, then for any choice of $h_i' \in p^{-1}(h_i) \subset G/G_\kappa$, it turns out that the elements $g_i = w_i(h_1', \ldots, h_n')$ form a solution $\{x_i = g_i\}$ for the given S, by a similar argument to the uniqueness proof (see [20, Proposition 1 (c)]). On the other hand, when κ is a limit ordinal, the assumption that G is local is essential for the existence (and necessary—recall the example of $F \cong F/F_\omega$).

3.4. Closure in the completion

For a group G, let $\widetilde{G} = \varprojlim_{k < \infty} G/G_k$ be the nilpotent completion. It is well known that $\widetilde{G}$ is a local group, essentially by Stallings' theorem. Therefore, there is a unique homomorphism $\widehat{G} \to \widetilde{G}$ making the following diagram commutative:

$$\begin{array}{ccc} G & \longrightarrow & \widehat{G} \\ \downarrow & \swarrow & \\ \widetilde{G} & & \end{array}$$

Following Levine's approach in [21], define $\overline{G} = \operatorname{Im}\{\widehat{G} \to \widetilde{G}\}$. We call $\overline{G}$ the *closure in the completion*.

It is straightforward to verify that $\operatorname{Ker}\{\widehat{G} \to \overline{G}\} = \widehat{G}_\omega$, that is, $\overline{G} \cong \widehat{G}/\widehat{G}_\omega$, using Stallings' theorem.

For later use, we will discuss a special case of a metabelian extension. Let G be an abelian group and A be a $\mathbb{Z}G$-module. Denote the semidirect product by $A \rtimes G$. Let $\epsilon \colon \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map, and $I := \operatorname{Ker} \epsilon$ be the augmentation ideal. Then the lower central subgroup $(A \rtimes G)_{k+1}$ is equal to $I^k A$, so that $(A \rtimes G)/(A \rtimes G)_k = (A/I^k A) \rtimes G$. It follows that $\widehat{A \rtimes G} = \widetilde{A} \rtimes G$, where $\widetilde{A} := \varprojlim_{k<\infty} A/I^k A$ is the I-adic completion. Also, $A \rtimes G$ is residually nilpotent if and only if $\bigcap_{k<\infty} I^k A = 0$.

Let $S := \{r \in \mathbb{Z}G \mid \epsilon(r) = 1\}$, and let $S^{-1}A = \{a/s \mid a \in A,\ s \in S\}$ be the classical localization with respect to S. By mutiplying a and s by -1, one sees that $S^{-1}A$ is equal to the localization with respect to a larger subset $\{r \in \mathbb{Z}G \mid \epsilon(r) = \pm 1\}$.

Theorem 3.6. *Suppose $\bigcap_{k<\infty} I^k A = 0$. Then $\overline{A \rtimes G} = S^{-1}A \rtimes G$.*

Theorem 3.6 is due to Levine [22, Proposition 3.2]. Indeed, he gave a proof (of a more general statement) for the localization defined in [20,21], but just by modifying it slightly, his argument applies to the localization we use (which is defined in [4]) as well.

For concreteness and for the reader's convenience, we provide a quick proof.

Proof of Theorem 3.6. By [7, Theorem A.2], $S^{-1}A \rtimes G$ is a local group, since $S^{-1}A$ is the Cohn localization of the $\mathbb{Z}G$-module A and G is abelian and thus local. It follows that there is a unique homomorphism $\widehat{A \rtimes G} \to S^{-1}A \rtimes G$ making the following diagram commutative:

$$
\begin{array}{ccc}
A \rtimes G & \longrightarrow & \widehat{A \rtimes G} \\
\downarrow & \swarrow & \\
S^{-1}A \rtimes G & &
\end{array}
$$

We claim that $\widehat{A \rtimes G} \to S^{-1}A \rtimes G$ is surjective. To show this, it suffices to verify that every $a/s \in S^{-1}A$ lies in the image of $\widehat{A \rtimes G}$. Observe that $x = a/s$ is a solution of the equation $x = w(x)$, where $w(x) = a + (1-s)x$. Write $1 - s = \sum_i n_i g_i$, $n_i \in \mathbb{Z}$, $g_i \in G$. Then, in multiplicative notation, $w(x) = a \prod_i g_i x^{n_i} g_i^{-1}$, a word in $(A \rtimes G) * F\langle x \rangle$. Since $\epsilon(s) = 1$, we have $\sum_i n_i = 0$. That is, the equation $x = w(x)$ over $A \rtimes G$ is acyclic. Therefore, there is a solution $z \in \widehat{A \rtimes G}$ for $x = w(x)$, and z must be sent to $a/s \in S^{-1}A$, since a/s is a solution for $x = w(x)$ in the local group $S^{-1}A \rtimes G$. This proves the claim.

We claim that $A \to \widetilde{A} = \varprojlim_{k<\infty} A/I^k A$ factors through $S^{-1}A$. To show this, it suffices to prove that every $s \in S$ is invertible in $\widetilde{\mathbb{Z}G} = \varprojlim_{k<\infty} \mathbb{Z}G/I^k$. Indeed, this is a known fact verified by an elementary argument as follows. Since $\epsilon(s) = 1$, $1 - s \in I$. So, writing $(1-s)^k = 1 - r_k \cdot s$ with $r_k \in \mathbb{Z}G$, we have $r_k \cdot s \equiv 1 \bmod I^k$. Also, $r_{k+1} \equiv r_k \bmod I^k$, so $(r_k) \in \varprojlim_{k<\infty} \mathbb{Z}G/I^k$ is a multiplicative inverse of the given s.

By the second claim, there is a natural homomorphism $S^{-1}A \rtimes G \to \widetilde{A} \rtimes G = \widehat{A \rtimes G}$. Since $\bigcap I^k A = 0$, the map $A \to \widetilde{A}$ is injective, and thus $S^{-1}A \to \widetilde{A}$ is injective. It follows that $S^{-1}A \rtimes G \to \widehat{A \rtimes G}$ is injective.

Now, consider

$$
\widehat{A \rtimes G} \to S^{-1}A \rtimes G \to \widetilde{A \rtimes G}.
$$

The first arrow is surjective by the first claim, and the second arrow is injective, so $S^{-1}A \rtimes G$ is the image of $\widehat{A \rtimes G}$ in $\widetilde{A \rtimes G}$. That is, $S^{-1}A \rtimes G = \overline{A \rtimes G}$. ∎

4. Invariance under homology cobordism

In this section we give a proof of Theorem A, which says that θ_κ is invariant under homology cobordism. Indeed, it is a straightforward consequence of the definition and the key property of the homology localization. We provide details for concreteness.

Definition 4.1. Two closed 3-manifolds M and N are *homology cobordant* if there is a 4-manifold W such that $\partial W = M \sqcup -N$ and the inclusions induce isomorphisms $H_*(M) \cong H_*(W) \cong H_*(N)$. Such a 4-manifold W is called a *homology cobordism*.

Fix a group Γ and an ordinal κ. Recall that for a closed 3-manifold M with $\pi = \pi_1(M)$ which is equipped with an isomorphism $f \colon \widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$, the invariant $\theta_\kappa(M)$ is defined to be the image of the fundamental class of M under

$$H_3(M) \to H_3(\pi) \to H_3(\widehat{\pi}) \to H_3(\widehat{\pi}/\widehat{\pi}_\kappa) \xrightarrow{f_*} H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa).$$

Proof of Theorem A. Suppose M and N are homology cobordant closed 3-manifolds with $\pi = \pi_1(M)$, $G = \pi_1(N)$. Theorem A (1) asserts that there is an isomorphism $\phi \colon \widehat{G}/\widehat{G}_\kappa \xrightarrow{\cong} \widehat{\pi}/\widehat{\pi}_\kappa$. Let W be a homology cobordism between M and N. Then, by Corollary 3.2 (2), the inclusions of M and N into W induce isomorphisms $\widehat{\pi}/\widehat{\pi}_\kappa \cong \widehat{\pi_1(W)}/\widehat{\pi_1(W)}_\kappa \cong \widehat{G}/\widehat{G}_\kappa$. Let $\phi \colon \widehat{G}/\widehat{G}_\kappa \xrightarrow{\cong} \widehat{\pi}/\widehat{\pi}_\kappa$ be the composition. This is the promised isomorphism.

Suppose $f \colon \widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is an isomorphism. Let $\theta_\kappa(M)$ and $\theta_\kappa(N)$ be the invariants defined using the isomorphisms f and $f \circ \phi$. Theorem A (2) asserts that $\theta_\kappa(M) = \theta_\kappa(N)$ in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$. To show this, consider the commutative diagram

$$
\begin{array}{ccccc}
H_3(M) & \longrightarrow & H_3(\widehat{\pi}/\widehat{\pi}_\kappa) & & \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \cong} & \searrow^{f_*} & \\
H_3(W) & \longrightarrow & H_3(\widehat{\pi_1(W)}/\widehat{\pi_1(W)}_\kappa) & \xrightarrow{\cong} & H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \\
\uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle \cong} & \nearrow_{(f \circ \phi)_*} & \\
H_3(N) & \longrightarrow & H_3(\widehat{G}/\widehat{G}_\kappa) & &
\end{array}
$$

Since the fundamental classes satisfy $i_*[M] - j_*[N] = \partial[W] = 0$ in $H_3(W)$, we have $\theta_\kappa(M) - \theta_\kappa(N) = 0$ in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$.

From this, it also follows that $\theta_\kappa(M) = \theta_\kappa(N)$ in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ even when $\theta_\kappa(M)$ and $\theta_\kappa(N)$ are defined using arbitrarily given isomorphisms $\widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ and $\widehat{G}/\widehat{G}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ (not necessarily the above f and $f \circ \phi$), since the orbit of $\theta_\kappa(-)$ under the action of $\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is independent of the choice of the isomorphism. This shows Theorem A (3). ∎

5. Bordism and transfinite lower central quotients

The goal of this section is to prove Theorem B and Corollary D.

5.1. Proof of Theorem B

Recall that Theorem B says that if M is a closed 3-manifold with $\pi = \pi_1(M)$ endowed with an isomorphism $f \colon \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$, the following are equivalent:

(1) There exists a lift $\hat{\pi}/\hat{\pi}_{\kappa+1} \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ of f which is an isomorphism.

(2) The invariant $\theta_\kappa(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$.

In our proof, it is essential to use the fact that $H_3(-)$ is isomorphic to the oriented bordism group $\Omega_3^{SO}(-)$, to obtain a 4-dimensional bordism from condition (2). More specifically, for another closed 3-manifold N with $G = \pi_1(N)$ equipped with $g \colon \hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$, we have $\theta_\kappa(N) = \theta_\kappa(M)$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ if and only if there is a 4-dimensional bordism W between (M, f) and (N, g) over the group $\hat{\Gamma}/\hat{\Gamma}_\kappa$. The core of the proof of Theorem B consists in careful analysis of the relationship of such a bordism W and the fundamental groups involved.

We begin with a general lemma, for which 4-dimensional duality plays a crucial role.

Lemma 5.1. *Suppose W is a 4-dimensional cobordism between two closed 3-manifolds M and N, that is, $\partial W = N \sqcup -M$. Suppose A is an arbitrary abelian group. Let $\partial \colon H_2(W, \partial W; A) \to H_1(\partial W; A)$ be the boundary homomorphism of the long exact sequence of $(W, \partial W)$. If the composition*

$$\mathrm{Im} \hookrightarrow H_1(\partial W; A) = H_1(M; A) \oplus H_1(N; A) \twoheadrightarrow H_1(M; A)$$

of the inclusion and the projection p is injective, then

$$\mathrm{Ker}\{H^2(W; A) \to H^2(M; A)\} \subset \mathrm{Ker}\{H^2(W; A) \to H^2(N; A)\}.$$

Proof. Consider the following diagram:

$$
\begin{array}{ccccccc}
H^2(W; A) & \xrightarrow{k^*} & H^2(\partial W; A) & = & H^2(M; A) \oplus H^2(N; A) & \xrightarrow{i^*} & H^2(M; A) \\
\mathrm{PD}_W \big\uparrow \cong & & \mathrm{PD}_{\partial W} \big\uparrow \cong & & \mathrm{PD}_M \oplus \mathrm{PD}_N \big\uparrow \cong & & \mathrm{PD}_M \big\uparrow \cong \\
H_2(W, \partial W; A) & \xrightarrow{\partial} & H_1(\partial W; A) & = & H_1(M; A) \oplus H_1(N; A) & \xrightarrow{p} & H_1(M; A)
\end{array}
$$

Here i^* and k^* are inclusion-induced, and $\mathrm{PD}_\bullet$ denotes the Poincaré duality isomorphism, that is, $\mathrm{PD}_\bullet^{-1}(c) = c \cap [\bullet]$ where $[\bullet]$ is the fundamental class. The left and middle squares commute since $\partial[W] = [\partial W] = [M] \oplus [N]$. The right square commutes since i^* is equal to the projection onto the first factor.

We have

$$
\begin{aligned}
\mathrm{Ker}\{H^2(W; A) \xrightarrow{i^* k^*} H^2(M; A)\} &= \mathrm{PD}_W(\mathrm{Ker}\, p \circ \partial) \quad \text{by the diagram,} \\
&= \mathrm{PD}_W(\mathrm{Ker}\, \partial) \qquad \text{since } p|_{\mathrm{Im}\,\partial} \text{ is injective.}
\end{aligned}
\tag{5.1}
$$

Apply the same argument to N in place of M to obtain

$$\mathrm{Ker}\{H^2(W; A) \to H^2(N; A)\} = \mathrm{PD}_W(\mathrm{Ker}\, q \circ \partial) \supset \mathrm{PD}_W(\mathrm{Ker}\, \partial) \tag{5.2}$$

where

$$q \colon H_1(\partial W; A) = H_1(M; A) \oplus H_2(N; A) \to H_1(N; A)$$

is the projection onto the second factor. From (5.1) and (5.2), the conclusion follows immediately. ∎

Theorem B will be proven as a consequence of the following result.

Theorem 5.2. *Suppose $\kappa \geq 2$ and M and N are closed 3-manifolds with $\pi = \pi_1(M)$ and $G = \pi_1(N)$ which are endowed with isomorphisms $f \colon \widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ and $g \colon \widehat{G}/\widehat{G}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Define $\theta_\kappa(M)$ and $\theta_\kappa(N)$ using f and g. If $\theta_\kappa(M) = \theta_\kappa(N)$ in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$, then the isomorphism*

$$f^{-1}g \colon \widehat{G}/\widehat{G}_\kappa \xrightarrow{\cong} \widehat{\pi}/\widehat{\pi}_\kappa$$

lifts to an isomorphism

$$\widehat{G}/\widehat{G}_{\kappa+1} \xrightarrow{\cong} \widehat{\pi}/\widehat{\pi}_{\kappa+1}.$$

Proof. Since $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is equal to the oriented bordism group $\Omega_3^{SO}(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$, there exists a 4-dimensional bordism W, over $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$, between M and N. We begin with some claims.

Claim 1. *For any abelian group A, the inclusions $i \colon M \hookrightarrow W$ and $j \colon N \hookrightarrow W$ induce injections $i_* \colon H_1(M; A) \to H_1(W; A)$ and $j_* \colon H_1(N; A) \to H_1(W; A)$.*

To show this, consider the commutative diagram

$$\begin{array}{ccccccc}
H_1(M; A) = H_1(\pi; A) & \xrightarrow[\cong]{} & H_1(\widehat{\pi}; A) & \xrightarrow[\cong]{} & H_1(\widehat{\pi}/\widehat{\pi}_\kappa; A) & \xrightarrow[\cong]{f_*} & H_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa; A) \\
{\scriptstyle i_*}\downarrow & & & & & & \nearrow \\
H_1(W; A) = H_1(\pi_1(W); A) & & & & & &
\end{array}$$

When $A = \mathbb{Z}$, the leftmost horizontal arrow is an isomorphism by Theorem 3.1, and the middle horizontal arrow is an isomorphism too since $\kappa \geq 2$. The rightmost horizontal arrow, f_*, is an isomorphism since so is f. Therefore, the composition $H_1(M; A) \to H_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa; A)$ is an isomorphism for $A = \mathbb{Z}$, and consequently it is an isomorphism for an arbitrary A by the universal coefficient theorem. From this and the above diagram, it follows that i_* is injective. The injectivity of j_* is shown in the same way, using N in place of M. This proves Claim 1.

Claim 2. *For any abelian group A,*

$$\mathrm{Ker}\{H^2(W; A) \xrightarrow{i^*} H^2(M; A)\} = \mathrm{Ker}\{H^2(W; A) \xrightarrow{j^*} H^2(N; A)\}.$$

To show this, use notations of Lemma 5.1. Let $\partial \colon H_2(W, \partial W; A) \to H_1(W; A)$ be the boundary map, and let p and q be the projections of $H_1(W; A) = H_1(M; A) \oplus H_1(N; A)$ onto the first and second factor respectively. By Lemma 5.1, it suffices to show that the restrictions $p|_{\mathrm{Im}\,\partial}$ and $q|_{\mathrm{Im}\,\partial}$ are injective. In our case,

$$\begin{aligned}
\mathrm{Im}\,\partial &= \mathrm{Ker}\{H_1(\partial W; A) \to H_1(W; A)\} \\
&= \{(x, y) \in H_1(M; A) \oplus H_1(N; A) \mid i_*(x) + j_*(y) = 0\}
\end{aligned}$$

where $i_*\colon H_1(M;A) \to H_1(W;A)$ and $j_*\colon H_1(N;A) \to H_1(W;A)$. So, for $(x,y) \in \operatorname{Im} \partial$, if $0 = p(x,y) = x$, then $j_*(y) = -i_*(x) = 0$, and thus $y = 0$ since j_* is injective by Claim 1. This shows that $p|_{\operatorname{Im} \partial}$ is injective. The same argument shows that $q|_{\operatorname{Im} \partial}$ is injective. This completes the proof of Claim 2.

Let $A = \widehat{\pi}_\kappa/\widehat{\pi}_{\kappa+1}$, and realize the short exact sequence

$$0 \to A \to \widehat{\pi}/\widehat{\pi}_{\kappa+1} \to \widehat{\pi}/\widehat{\pi}_\kappa \to 1$$

as a fibration $B(\widehat{\pi}/\widehat{\pi}_{\kappa+1}) \to B(\widehat{\pi}/\widehat{\pi}_\kappa)$ with fiber $B(A)$. We will use the following basic facts from obstruction theory. A map $f\colon X \to B(\widehat{\pi}/\widehat{\pi}_\kappa)$ of a CW-complex X gives an obstruction class $o_X \in H^2(X;A)$ which vanishes if and only if there is a lift $X \to B(\widehat{\pi}/\widehat{\pi}_{\kappa+1})$. In our case, the coefficient system $\{A\}$ is untwisted on $B(\widehat{\pi}/\widehat{\pi}_\kappa)$ since the abelian subgroup $A = \widehat{\pi}_\kappa/\widehat{\pi}_{\kappa+1}$ is central in $\widehat{\pi}/\widehat{\pi}_{\kappa+1}$. So, o_X determines a homotopy class of a map $\phi_X\colon X \to K(A,2)$, which is null-homotopic if and only if f lifts. Conversely, ϕ_X determines o_X. Namely, o_X is the image of id_A under

$$\operatorname{Hom}(A,A) = \operatorname{Hom}(H_2(K(A,2)),A) = H^2(K(A,2);A) \xrightarrow{(\phi_X)^*} H^2(X,A).$$

By the naturality of the obstruction class o_X, ϕ_X is the composition

$$X \xrightarrow{f} B(\widehat{\pi}/\widehat{\pi}_\kappa) \xrightarrow{\phi} K(A,2)$$

where $\phi = \phi_{B(\widehat{\pi}/\widehat{\pi}_\kappa)}$ is the map associated to the identity of $B(\widehat{\pi}/\widehat{\pi}_\kappa)$.

Consider the following specific lifting problem, which is for $X = N$:

$$
\begin{array}{ccccccc}
 & & & & B(\widehat{\pi}/\widehat{\pi}_{\kappa+1}) & & \\
 & & & \nearrow & \downarrow & & \\
N & \xrightarrow{\hspace{1cm}} & B(\widehat{G}/\widehat{G}_\kappa) & \xrightarrow[\simeq]{f^{-1}g} & B(\widehat{\pi}/\widehat{\pi}_\kappa) & \xrightarrow{\phi} & K(A,2) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g}\,\simeq & \nearrow{\scriptstyle f^{-1}} & & & \\
W & \xrightarrow{\hspace{1cm}} & B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) & & & &
\end{array}
$$

Here the bottom row is obtained from W being a bordism over $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$.

Claim 3. *There exists a lift $N \to B(\widehat{\pi}/\widehat{\pi}_{\kappa+1})$.*

To prove this, note that the obstruction o_N is the image of id_A under the composition

$$
\begin{array}{ccccc}
\operatorname{Hom}(A,A) & = & H^2(K(A,2);A) \to H^2(B(\widehat{\pi}/\widehat{\pi}_\kappa);A) & \to & H^2(N;A) \\
 & & \downarrow{\scriptstyle (f^{-1})^*} & & \uparrow{\scriptstyle j^*} \\
 & & H^2(B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa);A) & \to & H^2(W;A)
\end{array}
$$

Thus, o_N vanishes if and only if the image of id_A in $H^2(W; A)$ lies in the kernel of the map $H^2(W; A) \xrightarrow{j^*} H^2(N; A)$. To show that this is the case, consider the following lifting problem for M in place of N:

$$
\begin{array}{ccc}
& & B(\hat{\pi}/\hat{\pi}_{\kappa+1}) \\
& \nearrow & \downarrow \\
M \longrightarrow & B(\hat{\pi}/\hat{\pi}_{\kappa}) & \xrightarrow{\phi} K(A, 2) \\
\Big\downarrow{\scriptstyle i} & \simeq \Big\uparrow{\scriptstyle f^{-1}} & \\
W \longrightarrow & B(\hat{\Gamma}/\hat{\Gamma}_{\kappa}) &
\end{array}
$$

Since $M \to B(\pi) \to B(\hat{\pi}) \to B(\hat{\pi}/\hat{\pi}_{\kappa+1})$ is obviously a lift, the obstruction o_M vanishes. On the other hand, by the above argument applied to this case, o_M vanishes if and only if the image of id_A in $H^2(W; A)$ lies in the kernel of $H^2(W; A) \xrightarrow{i^*} H^2(M; A)$. By Claim 2, it follows that the image of id_A is contained in the kernel of $H^2(W; A) \to H^2(N; A)$ as well, that is, the obstruction o_N vanishes too. This proves Claim 3.

Claim 4. *There is a lift $\alpha \colon \hat{G}/\hat{G}_{\kappa+1} \to \hat{\pi}/\hat{\pi}_{\kappa+1}$ of $f^{-1}g$:*

$$
\begin{array}{ccc}
\hat{G}/\hat{G}_{\kappa+1} & \dashrightarrow^{\alpha} & \hat{\pi}/\hat{\pi}_{\kappa+1} \\
\downarrow & & \downarrow \\
\hat{G}/\hat{G}_{\kappa} & \xrightarrow[f^{-1}g]{\cong} & \hat{\pi}/\hat{\pi}_{\kappa}
\end{array}
$$

To show this, first take the homomorphism $G \to \hat{\pi}/\hat{\pi}_{\kappa+1}$ induced by the lift $N \to B(\hat{\pi}/\hat{\pi}_{\kappa+1})$ in Claim 3. It is a lift of $f^{-1}g \colon \hat{G}/\hat{G}_{\kappa} \xrightarrow{\cong} \hat{\pi}/\hat{\pi}_{\kappa}$. Since $\hat{\pi}/\hat{\pi}_{\kappa+1}$ is local by Lemma 3.5, $G \to \hat{\pi}/\hat{\pi}_{\kappa+1}$ induces a homomorphism $\hat{G} \to \hat{\pi}/\hat{\pi}_{\kappa+1}$. It induces a desired homomorphism $\alpha \colon \hat{G}/\hat{G}_{\kappa+1} \to \hat{\pi}/\hat{\pi}_{\kappa+1}$, since $\hat{G}_{\kappa+1} \subset \hat{G}$ is sent into $(\hat{\pi}/\hat{\pi}_{\kappa+1})_{\kappa+1} = \hat{\pi}_{\kappa+1}/\hat{\pi}_{\kappa+1} = \{e\}$.

Our goal is to show that the lift α in Claim 4 is an isomorphism. For this purpose, exchange the roles of M and N and apply the same argument to obtain a lift of $g^{-1}f \colon \hat{\pi}/\hat{\pi}_{\kappa} \to \hat{G}/\hat{G}_{\kappa}$, and call it $\beta \colon \hat{\pi}/\hat{\pi}_{\kappa+1} \to \hat{G}/\hat{G}_{\kappa+1}$.

Claim 5. *The composition $\alpha\beta \colon \hat{\pi}/\hat{\pi}_{\kappa+1} \to \hat{\pi}/\hat{\pi}_{\kappa+1}$ is an isomorphism.*

To prove this, consider the following diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \hat{\pi}_{\kappa}/\hat{\pi}_{\kappa+1} & \longrightarrow & \hat{\pi}/\hat{\pi}_{\kappa+1} & \longrightarrow & \hat{\pi}/\hat{\pi}_{\kappa} & \longrightarrow & 1 \\
& & \Big\downarrow{\scriptstyle \alpha\beta|_{\hat{\pi}_{\kappa}/\hat{\pi}_{\kappa+1}}} & & \Big\downarrow{\scriptstyle \alpha\beta} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \hat{\pi}_{\kappa}/\hat{\pi}_{\kappa+1} & \longrightarrow & \hat{\pi}/\hat{\pi}_{\kappa+1} & \longrightarrow & \hat{\pi}/\hat{\pi}_{\kappa} & \longrightarrow & 1
\end{array}
$$

Here, the right square commutes since α and β are lifts of $f^{-1}g$ and $g^{-1}f$ and thus $\alpha\beta$ is a lift of the identity. By the five lemma, if the left vertical arrow $\hat{\pi}_\kappa/\hat{\pi}_{\kappa+1} \to \hat{\pi}_\kappa/\hat{\pi}_{\kappa+1}$ is an isomorphism, then $\alpha\beta$ is an isomorphism too. We will indeed show that $\alpha\beta$ restricts to the identity on the (larger) subgroup $\hat{\pi}_2/\hat{\pi}_{\kappa+1}$. Suppose $g \in \hat{\pi}_2/\hat{\pi}_{\kappa+1}$. Write g as a product $g = \prod_i [a_i, b_i]$ of commutators, where $a_i, b_i \in \hat{\pi}/\hat{\pi}_{\kappa+1}$. Since $\alpha\beta$ is a lift of the identity of $\hat{\pi}/\hat{\pi}_\kappa$, we have $\alpha\beta(a_i) = a_i u_i$ and $\alpha\beta(b_i) = b_i v_i$ for some $u_i, v_i \in \hat{\pi}_\kappa/\hat{\pi}_{\kappa+1}$. Since $\hat{\pi}_\kappa/\hat{\pi}_{\kappa+1}$ is central in $\hat{\pi}/\hat{\pi}_{\kappa+1}$, $[a_i u_i, b_i v_i] = [a_i, b_i]$. It follows that

$$\alpha\beta(g) = \prod_i [\alpha\beta(a_i), \alpha\beta(b_i)] = \prod_i [a_i u_i, b_i v_i] = \prod_i [a_i, b_i] = g.$$

This completes the proof of Claim 5.

Now, by Claim 5, α is injective and β is surjective. Exchange the roles of α and β and apply the same argument to conclude that α is surjective and β is injective. Therefore α and β are isomorphisms. This completes the proof of Theorem 5.2. ∎

Proof of Theorem B. Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$, which is endowed with an isomorphism $f: \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$. Suppose f lifts to an isomorphism $\tilde{f}: \hat{\pi}/\hat{\pi}_{\kappa+1} \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$. Then $\theta_{\kappa+1}(M, \tilde{f})$ is sent to $\theta_\kappa(M, f)$ under $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$. Therefore $\theta_\kappa(M) = \theta_{\kappa+1}(M, \tilde{f})$ vanishes in the cokernel of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$.

For the converse, suppose $\theta_\kappa(M) = \theta_\kappa(M, f)$ vanishes in the cokernel of $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$. This means that there is a closed 3-manifold N with $G = \pi_1(N)$ which is endowed with an isomorphism $\tilde{g}: \hat{G}/\hat{G}_{\kappa+1} \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ such that $\theta_{\kappa+1}(N, \tilde{g})$ is sent to $\theta_\kappa(M, f)$ under $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$. Let $g: \hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$ be induced by $\tilde{g}$. Then $\theta_\kappa(N, g) = \theta_\kappa(M, f)$, and thus it follows that $g^{-1}f$ lifts to an isomorphism $\hat{\pi}/\hat{\pi}_{\kappa+1} \xrightarrow{\cong} \hat{G}/\hat{G}_{\kappa+1}$, by Theorem 5.2. Compose this lift with $\tilde{g}: \hat{G}/\hat{G}_{\kappa+1} \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ to obtain an isomorphism $\hat{\pi}/\hat{\pi}_{\kappa+1} \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ which is a lift of f. ∎

5.2. Proof of Corollary D

Recall from Definition 2.3 that the equivalence relation $\sim$ on $\mathcal{R}_\kappa(\Gamma)$ is defined as follows. For $\theta \in \mathcal{R}_\kappa(\Gamma)$, there is a closed 3-manifold M with $\pi = \pi_1(M)$ which is equipped with an isomorphism $f: \hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$ such that $\theta_\kappa(M) = \theta$. Let I_θ be the image of

$$\mathcal{R}_{\kappa+1}(\pi) \to \mathcal{R}_\kappa(\pi) \xrightarrow[f_*]{\approx} \mathcal{R}_\kappa(\Gamma). \tag{5.3}$$

Lemma 5.3. *The set I_θ is well defined, and $I_\phi = I_\theta$ whenever $\phi \in I_\theta$.*

From Lemma 5.3, it follows that the sets I_θ form a partition of $\mathcal{R}_\kappa(\Gamma)$. On $\mathcal{R}_\kappa(\Gamma)$, we write $\theta \sim \phi$ if $I_\theta = I_\phi$.

Proof of Lemma 5.3. Let $\theta \in \mathcal{R}_\kappa(\Gamma)$ and (M, f) be as above, and suppose N is a closed 3-manifold with $G = \pi_1(N)$ equipped with an isomorphism $g: \hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$ such that $\theta_\kappa(N, g)$ lies in the image of the map (5.3). Then there is an isomorphism lift $\widetilde{f^{-1}g}: \hat{G}/\hat{G}_{\kappa+1} \xrightarrow{\cong} \hat{\pi}/\hat{\pi}_{\kappa+1}$ of $f^{-1}g: \hat{G}/\hat{G}_\kappa \xrightarrow{\cong} \hat{\pi}/\hat{\pi}_\kappa$, by Theorem B applied

to $(N, f^{-1}g)$. The induced functions on $\mathcal{R}_*(-)$ form the following commutative diagram, where all horizontal arrows are bijective:

$$
\begin{array}{ccc}
\mathcal{R}_{\kappa+1}(G) & \xrightarrow[\,(f^{-1}g)_*\,]{\approx} & \mathcal{R}_{\kappa+1}(\pi) \\
\downarrow & & \downarrow \\
\mathcal{R}_{\kappa}(G) \xrightarrow[g_*]{\approx} & \mathcal{R}_{\kappa}(\Gamma) & \xleftarrow[f_*]{\approx} \mathcal{R}_{\kappa}(\pi)
\end{array}
$$

Indeed, if P is a closed 3-manifold equipped with $\widetilde{h} \colon \widehat{\pi_1(P)}/\widehat{\pi_1(P)}_{\kappa+1} \overset{\cong}{\Rightarrow} \widehat{G}/\widehat{G}_{\kappa+1}$ which induces $h \colon \widehat{\pi_1(P)}/\widehat{\pi_1(P)}_{\kappa} \overset{\cong}{\Rightarrow} \widehat{G}/\widehat{G}_{\kappa}$, then the images of $\theta_{\kappa+1}(P, \widetilde{h}) \in \mathcal{R}_{\kappa+1}(G)$ under the arrows in the above diagram are given by several θ-invariants of the same P, as shown below:

$$
\begin{array}{ccc}
\theta_{\kappa+1}(P, \widetilde{h}) & \xmapsto{\quad(f^{-1}g)_*\quad} & \theta_{\kappa+1}(P, \widetilde{(f^{-1}g)} \circ \widetilde{h}) \\
\downarrow & & \downarrow \\
\theta_{\kappa}(P, h) \xmapsto{g_*} & \theta_{\kappa}(P, gh) & \xmapsfrom{f_*} \theta_{\kappa}(P, f^{-1}gh)
\end{array}
$$

As an immediate consequence of the commutativity, we have

$$\operatorname{Im}\{\mathcal{R}_{\kappa+1}(G) \to \mathcal{R}_{\kappa}(G) \to \mathcal{R}_{\kappa}(\Gamma)\} = \operatorname{Im}\{\mathcal{R}_{\kappa+1}(\pi) \to \mathcal{R}_{\kappa}(\pi) \to \mathcal{R}_{\kappa}(\Gamma)\}. \quad (5.4)$$

Now, writing $\phi = \theta_{\kappa}(N, g)$, the left and right hand sides of (5.4) are I_ϕ and I_θ, respectively. This shows the assertion $I_\phi = I_\theta$. Also, when $\theta_{\kappa}(N, g) = \theta$, (5.4) shows that I_θ is well defined, independent of the choice of (M, f). ∎

Once we formulate the above setup, it is rather straightforward to obtain Corollary D, which asserts the following: Suppose M and N are closed 3-manifolds with fundamental groups $\pi = \pi_1(M)$ and $G = \pi_1(N)$, which are equipped with isomorphisms $f \colon \widehat{\pi}/\widehat{\pi}_{\kappa} \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_{\kappa}$ and $g \colon \widehat{G}/\widehat{G}_{\kappa} \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_{\kappa}$. Then $f^{-1}g$ lifts to an isomorphism $\widehat{G}/\widehat{G}_{\kappa+1} \overset{\cong}{\Rightarrow} \widehat{\pi}/\widehat{\pi}_{\kappa+1}$ if and only if $\theta_{\kappa}(M) \sim \theta_{\kappa}(N)$ in $\mathcal{R}_{\kappa}(\Gamma)$.

Proof of Corollary D. Let $f_* \colon \mathcal{R}_{\kappa}(\pi) \to \mathcal{R}_{\kappa}(\Gamma)$ be the induced bijection. By definition, $\theta_{\kappa}(N)$ lies in the subset $I_{\theta_{\kappa}(M)}$ of $\mathcal{R}_{\kappa}(\Gamma)$ if and only if $f_*^{-1}\theta_{\kappa}(N) \in \mathcal{R}_{\kappa}(\pi)$ lies in the image of $\mathcal{R}_{\kappa+1}(\pi) \to \mathcal{R}_{\kappa}(\pi)$; in other words, $f_*^{-1}\theta_{\kappa}(N) = 0$ in the cokernel of $\mathcal{R}_{\kappa+1}(\pi) \to \mathcal{R}_{\kappa}(\pi)$. It is the case if and only if $f^{-1}g$ lifts to an isomorphism $\widehat{G}/\widehat{G}_{\kappa+1} \overset{\cong}{\Rightarrow} \widehat{\pi}/\widehat{\pi}_{\kappa+1}$, by applying Theorem B to $(N, f^{-1}g)$. ∎

6. Transfinite Stallings–Dwyer theorem and transfinite gropes

The goal of this section is to provide *transfinite* generalizations of a well known result of Stallings [26] and Dwyer [12], and relate it to a notion of transfinite gropes which we define in this section too. In Section 6.2, we prove the *Addendum to Theorems C and E*, as stated in Section 2.7. The transfinite generalizations of the Stallings–Dwyer theorem will also be used in the proof of realization theorems in Section 7.

6.1. Algebraic statements

Theorem 6.1 (Transfinite Stallings–Dwyer). *Let $\kappa > 1$ be an arbitrary ordinal. Suppose $f: \pi \to G$ is a group homomorphism inducing an isomorphism $H_1(\pi) \xrightarrow{\cong} H_1(G)$. If κ is an infinite ordinal, suppose G is finitely generated. Then f induces an isomorphism $\hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{G}/\hat{G}_\kappa$ if and only if f induces an epimorphism*

$$H_2(\hat{\pi}) \to H_2(\hat{G})/\mathrm{Ker}\{H_2(\hat{G}) \to H_2(\hat{G}/\hat{G}_\lambda)\} \tag{6.1}$$

for all ordinals $\lambda < \kappa$.

Note that if κ is a successor ordinal, then the homomorphism (6.1) is surjective for all $\lambda < \kappa$ if and only if it is surjective for $\lambda = \kappa - 1$. In particular, if κ is finite, then by Corollary 3.2 (1), Theorem 6.1 specializes to the Stallings–Dwyer theorem [12, 26]: *for a homomorphism $f: \pi \to G$ which induces an isomorphism $H_1(\pi) \xrightarrow{\cong} H_1(G)$, f induces an isomorphism $\pi/\pi_k \cong G/G_k$ if and only if f induces an epimorphism*

$$H_2(\pi) \to H_2(G)/\mathrm{Ker}\{H_2(G) \to H_2(G/G_{k-1})\}.$$

Before proving Theorem 6.1 in Section 6.3, we record some consequences. We will use the following notation, which is a transfinite generalization of the notation used in Dwyer [12, p. 178].

Definition 6.2 (Transfinite Dwyer kernel). Suppose G is a group, and $\kappa > 1$ is an ordinal. The *transfinite Dwyer kernel* is defined by

$$\psi_\kappa(G) = \begin{cases} \mathrm{Ker}\{H_2(G) \to H_2(G/G_{\kappa-1})\} & \text{if } \kappa \text{ is a successor ordinal,} \\ \bigcap_{\lambda < \kappa} \psi_\lambda(G) & \text{if } \kappa \text{ is a limit ordinal.} \end{cases}$$

More generally, for a space X with $\pi = \pi_1(X)$, define $\psi_\kappa(X)$ by

$$\psi_\kappa(X) = \begin{cases} \mathrm{Ker}\{H_2(X) \to H_2(\pi) \to H_2(\pi/\pi_{\kappa-1})\} & \text{if } \kappa \text{ is a successor ordinal,} \\ \bigcap_{\lambda < \kappa} \psi_\lambda(X) & \text{if } \kappa \text{ is a limit ordinal.} \end{cases}$$

That is,

$$\psi_\kappa(BG) = \psi_\kappa(G).$$

Corollary 6.3. *Suppose $f: \pi \to G$ induces an isomorphism $H_1(\pi) \to H_1(G)$, $\kappa > 1$, and suppose G is finitely presented if κ is transfinite. If*

$$H_2(\hat{\pi}) \xrightarrow{\hat{f}_*} H_2(\hat{G}) \to H_2(\hat{G})/\psi_\kappa(\hat{G}) \tag{6.2}$$

is surjective, then f induces an isomorphism $\hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{G}/\hat{G}_\kappa$.

Note that Corollary 6.3 assumes the surjectivity of a single homomorphism (6.2), instead of the surjectivity of infinitely many homomorphisms (6.1) in Theorem 6.1, for the limit ordinal case.

Proof of Corollary 6.3. If κ is a successor ordinal, the codomain of (6.2) is equal to that of (6.1), and thus the corollary follows from Theorem 6.1. If κ is a limit ordinal, $H_2(\widehat{G})/\psi_\kappa(\widehat{G})$ surjects onto $H_2(\widehat{G})/\mathrm{Ker}\{H_2(\widehat{G}) \to H_2(\widehat{G}/\widehat{G}_\lambda)\}$ for all $\lambda < \kappa$. From this and Theorem 6.1, the corollary follows.   ∎

In practice, it may be difficult to verify the hypothesis that (6.1) or (6.2) is surjective, since localizations are involved. The following variation does not involve localizations in the hypothesis.

Corollary 6.4. *Suppose* $f : \pi \to G$ *induces an isomorphism* $H_1(\pi) \to H_1(G)$. *Suppose* $\kappa > 1$ *and* G *is finitely presented. If*

$$H_2(\pi) \xrightarrow{f_*} H_2(G) \to H_2(G)/\psi_\kappa(G)$$

is surjective, then f *induces an isomorphism* $\widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{G}/\widehat{G}_\kappa$.

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
H_2(\pi) & \longrightarrow & H_2(G) & \longrightarrow & H_2(G)/\psi_\kappa(G) \\
\downarrow & & \downarrow & & \downarrow \\
H_2(\widehat{\pi}) & \longrightarrow & H_2(\widehat{G}) & \longrightarrow & H_2(\widehat{G})/\psi_\kappa(\widehat{G})
\end{array}
$$

Since G is finitely presented, the middle vertical arrow is surjective by Theorem 3.1 (2), and consequently the right vertical arrow is surjective. It follows that the bottom horizontal composition is surjective if the top horizontal composition is surjective. So Corollary 6.3 implies Corollary 6.4.   ∎

6.2. Transfinite gropes

In this subsection we relate transfinite lower central quotients to a transfinite version of gropes, using the results in Section 6.1. The main statement is Corollary 6.8. This is a transfinite generalization of the finite case approach of Freedman and Teichner [13, Section 2].

We begin with new definitions. In what follows, a *symplectic basis* on a surface of genus g designates a collection of simple closed curves a_i, b_i ($i = 1, \ldots, g$) such that a_i and b_i are transverse and intersect exactly once for all i and $(a_i \cup b_i) \cap (a_j \cup b_j) = \emptyset$ for $i \neq j$.

Definition 6.5 (Transfinite gropes).
(1) Suppose $\Sigma \to X$ is a map of a connected surface Σ with connected or empty boundary to a space X. For a successor ordinal $\kappa > 1$, we say that the map $\Sigma \to X$ *supports a grope of class* κ, or briefly *supports a κ-grope*, if there is a symplectic basis $\{a_i, b_i\}$ on Σ such that a_i bounds a $(\kappa - 1)$-grope in X, in the sense defined below, for each i.
(2) A loop γ in X *bounds a grope of class* λ, that is, *bounds a λ-grope*, if either

(a) $\lambda = 1$,

(b) $\lambda > 1$ is a successor ordinal and there is a map of a surface to X which is bounded by γ and supports a λ-grope, or

(c) λ is a limit ordinal and γ bounds a μ-grope for each $\mu < \lambda$.

Definition 6.6 (The grope class of a second homology class). Let $\kappa > 1$. A homology class $\sigma \in H_2(X)$ is *represented by a κ-grope*, or *is of class κ*, if either

(1) κ is a successor ordinal and σ is represented by a map of a closed surface supporting a κ-grope, or

(2) κ is a limit ordinal and σ is represented by a λ-grope for every $\lambda < \kappa$.

We remark that for finite k, if $\Sigma \to X$ supports a k-grope in our sense, then a map of a k-grope in the sense of [13, Section 2] is obtained by stacking the inductively given surfaces along basis curves, and vice versa.

Proposition 6.7.

(1) *For $\kappa \geq 1$, a loop γ in X bounds a κ-grope if and only if $[\gamma] \in \pi_1(X)_\kappa$.*

(2) *For $\kappa > 1$, a class $\sigma \in H_2(X)$ lies in the transfinite Dwyer kernel $\psi_\kappa(X)$ if and only if σ is represented by a κ-grope.*

For finite κ, Proposition 6.7 (2) appeared earlier in [13, Lemma 2.3].

The following is an immediate consequence of Corollary 6.4 and Proposition 6.7.

Corollary 6.8. *Suppose $\kappa > 1$ is an arbitrary ordinal and $f\colon X \to Y$ is a map of a space X to a finite CW-complex Y which induces an isomorphism $H_1(X) \overset{\cong}{\to} H_1(Y)$. If $\mathrm{Coker}\{H_2(X) \to H_2(Y)\}$ is generated by classes represented by κ-gropes in Y, then f induces an isomorphism*

$$\widehat{\pi_1(X)}/\widehat{\pi_1(X)}_\kappa \overset{\cong}{\longrightarrow} \widehat{\pi_1(Y)}/\widehat{\pi_1(Y)}_\kappa.$$

Proof of Proposition 6.7. From the definitions, (1) follows straightforwardly by transfinite induction. To prove (2), we proceed by transfinite induction too. Since $\pi_1(X)_1 = \pi_1(X)$, every $\sigma \in H_2(X)$ lies in $\psi_2(X)$, and is represented by a 2-grope. So, (2) holds for $\kappa = 2$.

Suppose $\kappa > 2$ and (2) holds for all ordinals less than κ. If κ is a limit ordinal, then by definition, $\sigma \in H_2(X)$ is in $\psi_\kappa(X)$ if and only $\sigma \in \psi_\lambda(X)$ for all $\lambda < \kappa$. By the induction hypothesis, it holds if and only if σ is represented by a λ-grope for all $\lambda < \kappa$. By the definition, it holds if and only if σ is represented by a κ-grope. This shows that (2) holds for κ.

If $\kappa > 2$ is a successor ordinal, the finite case argument given in [13, proof of Lemma 2.3] can be carried out. We provide details for the reader's convenience. Let $\pi = \pi_1(X)$. Suppose $\sigma \in H_2(X)$ is represented by a κ-grope, that is, σ is the class of a map $\Sigma \to X$ of a surface admitting a geometrically symplectic basis $\{a_i, b_i\}$ such that each a_i bounds a $(\kappa - 1)$-grope in X. By (1), $[a_i] \in \pi_1(X)_{\kappa-1}$, and so a_i is null-homotopic in $B(\pi/\pi_{\kappa-1})$. By surgery on Σ along the a_i, it follows that the image

of σ in $H_2(B(\pi/\pi_{\kappa-1}))$ is a spherical class, and thus trivial. This shows that σ lies in $\psi_\kappa(X)$. For the converse, suppose a class represented by a map $\Sigma \to X$ of a surface Σ lies in $\psi_\kappa(X)$. Attach 2-cells to X along generators of $\pi_{\kappa-1}$, and attach more cells of dimension ≥ 3, to construct $B(\pi/\pi_{\kappa-1})$. Since Σ is null-homologous in $B(\pi/\pi_{\kappa-1})$ (and since $H_3 = \Omega_3^{SO}$), $\Sigma \to X \hookrightarrow B(\pi/\pi_{\kappa-1})$ extends to a compact 3-manifold R bounded by Σ. We may assume that the center of each cell which we attached to X is a regular value of $R \to B(\pi/\pi_{\kappa-1})$. Remove, from R, tubular neighborhoods of the inverse images of the centers. This gives a bordism over X between $\Sigma \to X$ and a map of a union of tori and spheres. Spheres support a κ-grope by definition. Since the meridian of each torus bounds a disk in $B(\pi/\pi_{\kappa-1})$, the meridian bounds a $(\kappa-1)$-grope in X by (1). By definition, it follows that the tori support a κ-grope. This completes the proof. ∎

As an application, we give a proof of the Addendum to Theorems C and E stated in Section 2.7. We first define a notion used in the statement. Recall that a cobordism W between M and N is an H_1-*cobordism* if inclusions induce isomorphisms $H_1(M) \cong H_1(W) \cong H_1(N)$.

Definition 6.9. Let κ be an ordinal. An H_1-cobordism W between M and N is a *grope cobordism of class κ* if each of $\mathrm{Coker}\{H_2(M) \to H_2(W)\}$ and $\mathrm{Coker}\{H_2(N) \to H_2(W)\}$ is generated by homology classes in $H_2(W)$ represented by κ-gropes.

Now, the Addendum to Theorems C and E says the following: Let Γ be a group and κ be an arbitrarily given ordinal. Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ which is equipped with an isomorphism $\hat{\pi}/\hat{\pi}_\kappa \xrightarrow{\cong} \hat{\Gamma}/\hat{\Gamma}_\kappa$. Then the following are equivalent:

(0) There is a grope cobordism of class $\kappa+1$ between M and another closed 3-manifold N satisfying $\widehat{\pi_1(N)}/\widehat{\pi_1(N)}_{\kappa+1} \cong \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$.

(1) $\hat{\pi}/\hat{\pi}_{\kappa+1}$ is isomorphic to $\hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$.

(2) The invariant $\theta_\kappa(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)/\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_\kappa)\}$.

Proof. We have already shown that (1) and (2) are equivalent in Section 2.4. Suppose (1) holds. Then $M \times [0,1]$ is a grope cobordism of class $\kappa+1$, and thus (0) holds. For the converse, suppose W is a grope cobordism of class $\kappa+1$ given in (0). Since $\mathrm{Coker}\{H_2(M) \to H_2(W)\}$ and $\mathrm{Coker}\{H_2(N) \to H_2(W)\}$ are generated by $(\kappa+1)$-gropes and $H_1(M) \cong H_1(W) \cong H_1(N)$, we have

$$\hat{\pi}/\hat{\pi}_{\kappa+1} \cong \widehat{\pi_1(W)}/\widehat{\pi_1(W)}_{\kappa+1} \cong \widehat{\pi_1(N)}/\widehat{\pi_1(N)}_{\kappa+1}$$

by Corollary 6.8. It follows that (1) holds. ∎

6.3. *Proof of the algebraic statement*

Now, we prove the main algebraic statement of this section.

Proof of Theorem 6.1. First, we assert that the surjectivity of $H_1(\pi) \to H_1(G)$ implies that $\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa$ is surjective. Indeed, if κ is finite, then the assertion is a well known

fact obtained from a standard commutator identity. For the reader's convenience, we describe an outline of the argument. If $a_i \equiv b_i \mod G_2$, then we have

$$[a_1, [a_2, \ldots, [a_{k-1}, a_k] \ldots]] \equiv [b_1, [b_2, \ldots, [b_{k-1}, b_k] \ldots]] \mod G_{k+1}.$$

From this it follows that $\pi_k/\pi_{k+1} \to G_k/G_{k+1}$ is surjective for all finite k. The surjectivity of $\pi/\pi_k \to G/G_k$ is obtained by applying the five lemma, inductively, to the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_{k-1}/\pi_k & \longrightarrow & \pi/\pi_k & \longrightarrow & \pi/\pi_{k-1} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_{k-1}/G_k & \longrightarrow & G/G_k & \longrightarrow & G/G_{k-1} & \longrightarrow & 1
\end{array}
$$

When κ is an infinite ordinal, since G is assumed to be finitely generated, $\hat{\pi} \to \hat{G}$ is surjective if $H_1(\pi) \to H_1(G)$ is surjective, by Lemma 3.3. It follows that $\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa$ is surjective.

Therefore, under the assumption that $\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa$ is surjective, it suffices to prove that the following two conditions are equivalent:

(i)$_\kappa$ $\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa$ is injective, or equivalently is an isomorphism.

(ii)$_\kappa$ $H_2(\hat{\pi}) \to H_2(\hat{G})/K_\lambda(\hat{G})$ is surjective for all $\lambda < \kappa$, where

$$K_\lambda(\hat{G}) := \mathrm{Ker}\{H_2(\hat{G}) \to H_2(\hat{G}/\hat{G}_\lambda)\}.$$

We proceed by transfinite induction on the ordinal κ. For $\kappa = 2$, (i)$_\kappa$ holds since $\hat{\pi}/\hat{\pi}_2 = H_1(\pi) \cong H_1(G) = \hat{G}/\hat{G}_2$, and (ii)$_\kappa$ holds too, since $H_2(\hat{G})/K_1(\hat{G})$ is trivial.

Fix an ordinal $\kappa \geq 3$, and let $f: \hat{\pi} \to \hat{G}$ be a homomorphism which satisfies the hypothesis of Theorem 6.1. Suppose that (i)$_{\kappa'}$ and (ii)$_{\kappa'}$ are equivalent for all $\kappa' < \kappa$.

If κ is a successor ordinal, then we proceed similarly to the original argument of Stallings and Dwyer [12, 26], as described below. First, note that (ii)$_\lambda$ holds for all $\lambda < \kappa$ if and only if (ii)$_{\kappa-1}$ holds, when κ is a successor ordinal. Recall that, for a normal subgroup N of a group Γ, the Lyndon–Hochschild–Serre spectral sequence for the short exact sequence $1 \to N \to \Gamma \to \Gamma/N \to 1$ gives rise to an exact sequence

$$H_2(\Gamma) \to H_2(\Gamma/N) \to H_0(\Gamma/N; H_1(N)) \to H_1(\Gamma) \to H_1(\Gamma/N)$$

which is called Stallings' exact sequence [26]. Apply this to $(\Gamma, N) = (\hat{\pi}, \hat{\pi}_{\kappa-1})$ and $(\hat{G}, \hat{G}_{\kappa-1})$ to obtain the following diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\hat{\pi})/K_{\kappa-1}(\hat{\pi}) & \longrightarrow & H_2(\hat{\pi}/\hat{\pi}_{\kappa-1}) & \longrightarrow & \hat{\pi}_{\kappa-1}/\hat{\pi}_\kappa & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_2(\hat{G})/K_{\kappa-1}(\hat{G}) & \longrightarrow & H_2(\hat{G}/\hat{G}_{\kappa-1}) & \longrightarrow & \hat{G}_{\kappa-1}/\hat{G}_\kappa & \longrightarrow & 0
\end{array}
$$

If (i)$_\kappa$ holds, then (i)$_{\kappa-1}$ holds too. If (ii)$_\kappa$ holds, then (ii)$_{\kappa-1}$ holds too and consequently (i)$_{\kappa-1}$ holds by the induction hypothesis. So, in either case, we may assume that (i)$_{\kappa-1}$

holds. Then the middle vertical arrow of the diagram is an isomorphism. By the snake lemma, it follows that

$$\text{Ker}\{\hat{\pi}_{\kappa-1}/\hat{\pi}_\kappa \to \hat{G}_{\kappa-1}/\hat{G}_\kappa\} \cong \text{Coker}\{H_2(\hat{\pi})/K_{\kappa-1}(\hat{\pi}) \to H_2(\hat{G})/K_{\kappa-1}(\hat{G})\}. \quad (6.3)$$

Since $\hat{\pi}/\hat{\pi}_{\kappa-1} \cong \hat{G}/\hat{G}_{\kappa-1}$ by (i)$_{\kappa-1}$, (i)$_\kappa$ holds if and only if the left hand side of (6.3) is trivial. Also, (ii)$_\kappa$ holds if and only if the right hand side of (6.3) is trivial. It follows that (i)$_\kappa$ and (ii)$_\kappa$ are equivalent.

Now, suppose that κ is a limit ordinal. Suppose (i)$_\kappa$ holds. For each $\lambda < \kappa$, since κ is a limit ordinal, $\lambda + 1 < \kappa$. So (i)$_\kappa$ implies (i)$_{\lambda+1}$. By the induction hypothesis, it follows that (ii)$_{\lambda+1}$ holds. In particular, $H_2(\hat{\pi}) \to H_2(\hat{G})/K_\lambda(\hat{G})$ is surjective. This shows that (ii)$_\kappa$ holds.

For the converse, suppose (ii)$_\kappa$ holds. For each $\lambda < \kappa$, (ii)$_\kappa$ implies (ii)$_\lambda$, and thus (i)$_\lambda$ holds by the induction hypothesis. That is, f induces an isomorphism $\hat{\pi}/\hat{\pi}_\lambda \xrightarrow{\cong} \hat{G}/\hat{G}_\lambda$. Therefore, if $g \in \text{Ker}\{\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa\}$, then $g \in \text{Ker}\{\hat{\pi}/\hat{\pi}_\kappa \to \hat{\pi}/\hat{\pi}_\lambda\}$ for all $\lambda < \kappa$. Since $\hat{\pi}_\kappa = \bigcap_{\lambda < \kappa} \hat{\pi}_\lambda$, it follows that g is trivial. This proves that $\hat{\pi}/\hat{\pi}_\kappa \to \hat{G}/\hat{G}_\kappa$ is injective, and thus (i)$_\kappa$ holds.

This completes the proof of Theorem 6.1. ∎

We remark that the above proof of the equivalence of (i)$_\kappa$ and (ii)$_\kappa$ indeed shows the following statement (just by replacing $\hat{\pi}$ and $\hat{G}$ with P and Z below), which we record as a lemma for later use.

Lemma 6.10. *Suppose $\kappa > 1$ and $f : P \to Z$ is a group homomorphism which induces an epimorphism $P/P_\kappa \to Z/Z_\kappa$ and an isomorphism $H_1(P) \xrightarrow{\cong} H_1(Z)$. Then the following are equivalent:*

(i) *f induces an isomorphism $P/P_\kappa \to Z/Z_\kappa$.*

(ii) *f induces an epimorphism $H_2(P) \to H_2(Z)/K_\lambda(Z)$ for all $\lambda < \kappa$, where*

$$K_\lambda(Z) := \text{Ker}\{H_2(Z) \to H_2(Z/Z_\lambda)\}.$$

7. Realization of transfinite invariants

In this section, we prove Theorem G stated in Section 2.8, which characterizes the realizable classes θ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$. In the proof of Theorem G, we will use the following three lemmas. The first lemma provides a finitely generated approximation of the transfinite lower central quotients of the localization, along the lines of Theorem 3.1 (2).
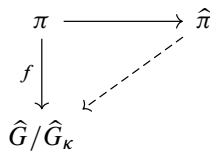
Lemma 7.1. *Suppose G is a finitely presented group, $\kappa > 1$ is an ordinal, and H is a finitely generated subgroup in $\hat{G}/\hat{G}_\kappa$. Then H is contained in a finitely generated subgroup Q in $\hat{G}/\hat{G}_\kappa$ such that the inclusion induces an isomorphism $H_1(Q) \to H_1(\hat{G}/\hat{G}_\kappa)$.*

Proof. Since H is finitely generated, there is a 2-connected homomorphism $P \to \hat{G}$ of a finitely presented group P such that the image of $P \to \hat{G} \to \hat{G}/\hat{G}_\kappa$ contains H,
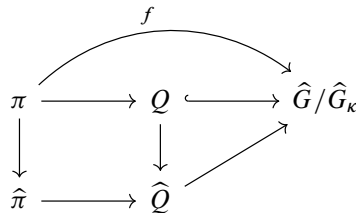
by Theorem 3.1 (2). Let Q be the image of $P \to \widehat{G}/\widehat{G}_\kappa$. Since $P \to Q$ is surjective, $H_1(P) \to H_1(Q)$ is surjective. Since the composition $P \to Q \to \widehat{G}/\widehat{G}_\kappa$ induces an isomorphism on H_1, $H_1(P) \to H_1(Q)$ is injective, and consequently $H_1(P) \cong H_1(Q) \cong H_1(\widehat{G}/\widehat{G}_\kappa)$ under the induced homomorphisms. ∎

Lemma 7.2. *Consider any ordinal κ. Suppose π is finitely generated, G is finitely presented, and $f \colon \pi \to \widehat{G}/\widehat{G}_\kappa$ is a group homomorphism which induces an epimorphism $H_1(\pi) \to H_1(\widehat{G}/\widehat{G}_\kappa) = H_1(G)$. Then f induces an epimorphism $\widehat{\pi} \to \widehat{G}/\widehat{G}_\kappa$.*

Proof. Since $\widehat{G}/\widehat{G}_\kappa$ is trivial for $\kappa = 1$, we may assume that $\kappa \geq 2$. Recall from Lemma 3.5 that the transfinite lower central quotient of a local group is local. So, in our case, $\widehat{G}/\widehat{G}_\kappa$ is local, and thus there is an induced homomorphism $\widehat{\pi} \to \widehat{G}/\widehat{G}_\kappa$ by the universal property of $\widehat{\pi}$:

$$
\begin{array}{ccc}
\pi & \longrightarrow & \widehat{\pi} \\
{\scriptstyle f}\downarrow & \swarrow & \\
\widehat{G}/\widehat{G}_\kappa & &
\end{array}
$$

To show that $\widehat{\pi} \to \widehat{G}/\widehat{G}_\kappa$ is surjective, it suffices to prove that every finitely generated subgroup H in $\widehat{G}/\widehat{G}_\kappa$ is contained in the image of $\widehat{\pi} \to \widehat{G}/\widehat{G}_\kappa$. Since H and π are finitely generated, there is a finitely generated subgroup Q in $\widehat{G}/\widehat{G}_\kappa$ such that the inclusion induces an isomorphism $H_1(Q) \xrightarrow{\cong} H_1(\widehat{G}/\widehat{G}_\kappa)$ and both H and $f(\pi)$ are contained in Q, by Lemma 7.1. Consider the commutative diagram

$$
\begin{array}{ccc}
 & \overset{f}{\curvearrowright} & \\
\pi \longrightarrow & Q \lhook\joinrel\longrightarrow & \widehat{G}/\widehat{G}_\kappa \\
\downarrow & \downarrow & \nearrow \\
\widehat{\pi} \longrightarrow & \widehat{Q} &
\end{array}
$$

Since $H_1(\pi) \to H_1(\widehat{G}/\widehat{G}_\kappa)$ is surjective, it follows that $H_1(\pi) \to H_1(Q)$ is surjective. Therefore, by Lemma 3.3, $\widehat{\pi} \to \widehat{Q}$ is surjective. Since the given subgroup $H \subset \widehat{G}/\widehat{G}_\kappa$ is contained in Q, it follows that H is contained in the image of $\widehat{\pi}$. This completes the proof. ∎

Another key ingredient of our proof of Theorem G is the following "homology surgery" result for 3-manifolds over a *finitely generated* fundamental group, due to Turaev [27]. As in Section 2, we denote the torsion subgroup of $H_*(-)$ by $t H_*(-)$.

Lemma 7.3 (Turaev [27, Lemma 2.2]). *Suppose $g \colon N \to X$ is a map of a closed 3-manifold N to a CW-complex X with finitely generated $\pi_1(X)$ such that the cap product*

$$\cap \, g_*[N] \colon t H^2(X) \to t H_1(X)$$

is an isomorphism. Then (N, g) is bordant, over X, to a pair (M, f) of a closed 3-manifold M and a map $f \colon M \to X$ which induces an isomorphism $f_* \colon H_1(M) \xrightarrow{\cong} H_1(X)$.

Now we are ready to start the proof of Theorem G. Recall from Section 2.4 that the set $\mathcal{R}_\kappa(\Gamma)$ of realizable classes is defined to be the collection of $\theta \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ such that $\theta = \theta_\kappa(M)$ for some closed 3-manifold M with $\pi = \pi_1(M)$ equipped with an isomorphism $\widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Here, Γ is a fixed finitely presented group. Let $\kappa \geq 2$. Theorem G says that $\theta \in \mathcal{R}_\kappa(\Gamma)$ if and only if the following two conditions hold:

(1) The cap product

$$\cap\, \theta : tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \cong tH_1(\Gamma)$$

is an isomorphism.

(2) The composition

$$H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \xrightarrow{\cap\,\theta} H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \xrightarrow{\mathrm{pr}} H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$$

is surjective for all $\lambda < \kappa$, where $K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) = \mathrm{Ker}\{H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\lambda)\}$.

Proof of Theorem G. For the "only if" direction, suppose $\theta \in \mathcal{R}_\kappa(\Gamma)$. Choose a closed 3-manifold M with $\pi = \pi_1(M)$ and an isomorphism $f : \widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ such that $\theta_\kappa(M) = \theta$. That is, $\theta = \phi_*[M]$ where $\phi_* : H_3(M) \to H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is induced by the composition

$$\phi : M \to B\pi \to B\widehat{\pi} \to B(\widehat{\pi}/\widehat{\pi}_\kappa) \xrightarrow[\cong]{f} B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa).$$

Then the following diagram is commutative:

$$
\begin{array}{ccc}
tH^2(M) & \xleftarrow{\ \phi^*\ } & tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \\
{\scriptstyle \cap[M]}\big\downarrow & & \big\downarrow{\scriptstyle \cap\,\theta} \\
tH_1(M) & \xrightarrow{\ \phi_*\ } & tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)
\end{array}
$$

The cap product $\cap\,[M]$ is an isomorphism by Poincaré duality. The bottom arrow ϕ_* is an isomorphism since $H_1(M) = H_1(\pi) = H_1(\widehat{\pi}/\widehat{\pi}_\kappa)$ and $f : \widehat{\pi}/\widehat{\pi}_\kappa \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is an isomorphism. From this it also follows that the top arrow ϕ^* is an isomorphism, since $tH^2(-) = \mathrm{Ext}(H_1(-), \mathbb{Z})$. Therefore, $\cap\,\theta$ is an isomorphism. This shows that (1) holds.

To show that (2) holds, suppose $\lambda < \kappa$ and consider the commutative diagram

$$
\begin{array}{ccc}
H^1(M) & \xleftarrow{\ \phi^*\ } & H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \\
{\scriptstyle \cap[M]}\big\downarrow & & \big\downarrow{\scriptstyle \cap\,\theta}\ \ \ \searrow^{\mathrm{pr}\circ(\cap\,\theta)} \\
H_2(M) & \xrightarrow[\phi_*]{} & H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \xrightarrow[\mathrm{pr}]{} H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \\
\big\downarrow & & \big\uparrow \qquad\qquad\nearrow \\
H_2(\pi) & \longrightarrow\ H_2(\widehat{\pi})\ \longrightarrow & H_2(\widehat{\pi}/\widehat{\pi}_\kappa)
\end{array}
$$

(7.1)

By Poincaré duality, $\cap\,[M]$ is an isomorphism. Since $H^1(-) = \mathrm{Hom}(H_1(-), \mathbb{Z})$ and $H_1(M) = H_1(\widehat{\pi}/\widehat{\pi}_\kappa) \to H_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is an isomorphism, the top arrow ϕ^* is an isomorphism. Also, the assumption that $f : \widehat{\pi}/\widehat{\pi}_\kappa \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is an isomorphism implies that the composition $H_2(\widehat{\pi}) \to H_2(\widehat{\pi}/\widehat{\pi}_\kappa) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is surjective, by applying Lemma 6.10 to the composition $\widehat{\pi} \to \widehat{\pi}/\widehat{\pi}_\kappa \overset{\cong}{\Rightarrow} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Since $H_2(M) \to H_2(\pi)$ and

$H_2(\pi) \to H_2(\widehat{\pi})$ are surjective (see Theorem 3.1 (2) for the latter), it follows that the composition $\mathrm{pr} \circ (\cap \theta)$ in (7.1) is surjective. This proves that (2) holds.

It remains to show the "if" direction. Suppose that (1) and (2) hold for a given class $\theta \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$. Since $H_3 = \Omega_3^{SO}$, there is a map $\psi \colon N \to B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ of a closed 3-manifold N such that $\psi_*[N] = \theta$.

We wish to invoke Turaev's homology surgery for 3-manifolds (Lemma 7.3) to alter (N, ψ). Note that $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is not finitely generated in general, and thus Lemma 7.3 does not apply directly over $B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$. So we proceed as follows, using a finitely generated approximation. Apply Lemma 7.1 to choose a finitely generated subgroup Q in $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$ such that the inclusion induces an isomorphism $H_1(Q) \xrightarrow{\cong} H_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ and $\pi_1(N) \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ factors through Q. Let $\psi' \colon N \to B(\pi_1(N)) \to B(Q)$ be the composition, and consider the commutative diagram

$$
\begin{array}{ccc}
tH^2(Q) & \xrightarrow{\cong} & tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \\
{\scriptstyle \cap \psi'_*[N]} \downarrow & & \cong \downarrow {\scriptstyle \cap \psi_*[N] = \cap \theta} \\
tH_1(Q) & \xleftarrow{\cong} & tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)
\end{array}
$$

The two horizontal arrows and the right vertical arrow $\cap \theta$ are isomorphisms, by our choice of Q, by the fact $tH^2(-) = \mathrm{Ext}(H_1(-), \mathbb{Z})$ and by the hypothesis (1). So $\cap \psi'_*[N]$ is an isomorphism too. Now apply Lemma 7.3 to (N, ψ') to produce a closed 3-manifold M endowed with a map $M \to B(Q)$ which induces an isomorphism on H_1. Let $\phi \colon M \to B(Q) \to B(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ be the composition. It induces an isomorphism $H_1(M) \xrightarrow{\cong} H_1(Q) \cong H_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$. Also, since (M, ϕ) is bordant to (N, ψ), we have $\phi_*[M] = \psi_*[N] = \theta$.

Let $\pi = \pi_1(M)$, and consider $\pi \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ induced by ϕ. It gives rise to a homomorphism $\widehat{\pi} \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ since $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is local by Lemma 3.5. Consider the diagram (7.1) again. Now, the composition $\mathrm{pr} \circ (\cap \theta)$ is surjective by hypothesis (2). Note that this surjection is equal to the composition of the six arrows along the counterclockwise outermost path from $H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ to $H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ in (7.1). So, the map $H_2(\widehat{\pi}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)/K_\lambda(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$, which is the last one applied in the composition, is surjective. By applying Lemma 6.10 to $\widehat{\pi} \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$, it follows that ϕ induces an isomorphism $\widehat{\pi}/\widehat{\pi}_\kappa \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\kappa$. Therefore $\theta = \phi_*[M]$ lies in $\mathcal{R}_\kappa(\Gamma)$. This completes the proof of Theorem G. ∎

8. Universal θ-invariant

We begin by recalling the definition of the universal θ-invariant from Definition 2.7. As before, let Γ be a finitely presented group. Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ equipped with an isomorphism $f \colon \widehat{\pi} \to \widehat{\Gamma}$. Motivated from Levine's link invariant in [20], define $\widehat{\theta}(M) \in H_3(\widehat{\Gamma})$ to be the image of $[M] \in H_3(M)$ under

$$
H_3(M) \to H_3(\pi) \to H_3(\widehat{\pi}) \xrightarrow[\cong]{f_*} H_3(\widehat{\Gamma}).
$$

The value of $\widehat{\theta}(M)$ depends on the choice of f, while its image in $H_3(\widehat{\Gamma})/\mathrm{Aut}(\widehat{\Gamma})$ is independent of the choice of f.

The following is analogous to Theorem A. We omit the proof, since the argument is exactly the same as that of Theorem A.

Theorem 8.1. *The invariant $\widehat{\theta}(M)$ is invariant under homology cobordism in the following sense:*

(1) *If M and N are homology cobordant 3-manifolds with $\pi = \pi_1(M)$ and $G = \pi_1(N)$, then there is an isomorphism $\phi\colon \widehat{G} \xrightarrow{\cong} \widehat{\pi}$, and consequently $\widehat{\theta}(M)$ is defined if and only if $\widehat{\theta}(N)$ is defined.*

(2) *When $\widehat{\theta}(M)$ and $\widehat{\theta}(N)$ are defined using an isomorphism $f\colon \widehat{\pi} \xrightarrow{\cong} \widehat{\Gamma}$ and the composition $f \circ \phi$, we have $\widehat{\theta}(M) = \widehat{\theta}(N)$ in $H_3(\widehat{\Gamma})$.*

(3) *When $\widehat{\theta}(M)$ and $\widehat{\theta}(N)$ are defined using arbitrary isomorphisms $\widehat{\pi} \xrightarrow{\cong} \widehat{\Gamma}$ and $\widehat{G} \xrightarrow{\cong} \widehat{\Gamma}$, we have $\widehat{\theta}(M) = \widehat{\theta}(N)$ in $H_3(\widehat{\Gamma})/\mathrm{Aut}(\widehat{\Gamma})$.*

Let $\widehat{\mathcal{R}}(\Gamma)$ be the collection of classes $\theta \in H_3(\widehat{\Gamma})$ such that there exists a closed 3-manifold M with $\pi = \pi_1(M)$ endowed with an isomorphism $\widehat{\pi} \xrightarrow{\cong} \widehat{\Gamma}$ for which $\widehat{\theta}(M) = \theta$. We now give a proof of Theorem H stated in Section 2.9. For the reader's convenience, we recall the statement: a homology class $\theta \in H_3(\widehat{\Gamma})$ lies in $\widehat{\mathcal{R}}(\Gamma)$ if and only if the following two conditions hold:

(1) The cap product $\cap \theta\colon tH^2(\widehat{\Gamma}) \to tH_1(\widehat{\Gamma}) \cong tH_1(\Gamma)$ is an isomorphism.

(2) The cap product $\cap \theta\colon H^1(\widehat{\Gamma}) \to H_2(\widehat{\Gamma})$ is surjective.

Proof of Theorem H. We will first prove the "only if" part, using an argument almost identical to the proof of Theorem G. Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ and $f\colon \widehat{\pi} \xrightarrow{\cong} \widehat{\Gamma}$ is an isomorphism. Let $\theta = \widehat{\theta}(M) \in H_3(\widehat{\Gamma})$, that is, θ is the image of $[M]$ under the map induced by the composition $\phi\colon M \to B\pi \to B\widehat{\pi} \xrightarrow{f} B\widehat{\Gamma}$. Consider the commutative diagram

$$
\begin{array}{ccc}
tH^2(M) & \xleftarrow{\phi^*} & tH^2(\widehat{\Gamma}) \\
{\scriptstyle \cap[M]}\downarrow & & \downarrow{\scriptstyle \cap\theta} \\
tH_1(M) & \xrightarrow{\phi_*} & tH_1(\widehat{\Gamma})
\end{array}
$$

By Poincaré duality, $\cap[M]$ is an isomorphism. The arrow ϕ_* is an isomorphism since f is an isomorphism. Using $tH^2(-) = \mathrm{Ext}(H_1(-), \mathbb{Z})$, it follows that ϕ^* is an isomorphism. So, by the commutativity, $\cap \theta$ is an isomorphism. This shows that (1) holds.

To show that (2) holds, consider the commutative diagram

$$
\begin{array}{ccc}
H^1(M) & \xleftarrow{\phi^*} & H^1(\widehat{\Gamma}) \\
{\scriptstyle \cap[M]}\downarrow & & \downarrow{\scriptstyle \cap\theta} \\
H_2(M) & \xrightarrow{\phi_*} & H_2(\widehat{\Gamma})
\end{array}
$$

The arrow ϕ^* is an isomorphism since f is an isomorphism, and $\cap [M]$ is an isomorphism by Poincaré duality. Since $H_2(M) \to H_2(\pi)$ and $H_2(\pi) \to H_2(\widehat{\pi})$ are surjective, ϕ_* is surjective. So $\cap \theta$ is surjective, that is, (2) holds.

Now, we will prove the "if" part. Our argument will be different from the proof of Theorem G. Suppose $\theta \in H_3(\widehat{\Gamma})$ is a homology class satisfying (1) and (2). Choose a sequence of 2-connected homomorphisms of finitely presented groups

$$\Gamma = P(1) \to P(2) \to \cdots \to P(\ell) \to \cdots$$

such that $\widehat{\Gamma} = \mathrm{colim}_\ell P(\ell)$, by using Theorem 3.1 (2). Since $H_3(\widehat{\Gamma})$ is the colimit of $H_3(P(\ell))$, the class θ lies in the image of $H_3(P(\ell_0))$ for some ℓ_0. Let $P = P(\ell_0)$ for brevity. Denote $P \to \widehat{\Gamma}$ by ι, and write $\theta = \iota_*(\sigma)$, where $\sigma \in H_3(P)$.

We claim that we may assume that $\iota_*: H_2(P) \to H_2(\widehat{\Gamma})$ is an isomorphism. To prove this, first recall that $H_2(P) \to H_2(\widehat{\Gamma})$ is surjective by the choice of the sequence $\{P(\ell)\}$. Let N be the kernel of $H_2(P) \to H_2(\widehat{\Gamma})$. Since P is finitely presented, $H_2(P)$ is a finitely generated abelian group, and thus N is finitely generated. Since $H_2(\widehat{\Gamma})$ is the colimit of $H_2(P(\ell))$, it follows that the image of N under $H_2(P) \to H_2(P(\ell_1))$ is trivial for some $\ell_1 \geq \ell_0$. Since $H_2(P) \to H_2(P(\ell_1))$ is surjective, we have $H_2(P(\ell_1)) \cong H_2(P)/N \cong H_2(\widehat{\Gamma})$. Replacing P by $P(\ell_1)$, we obtain the claim.

We will use Turaev's homology surgery over the finitely presented group P. Choose a map $\psi: N \to BP$ of a closed 3-manifold N such that $\psi_*[N] = \sigma$, using the fact that $\Omega_3^{SO}(P) = H_3(P)$. Consider the commutative diagram

$$
\begin{array}{ccc}
tH^2(P) & \xleftarrow{\ \iota^*\ } & tH^2(\widehat{\Gamma}) \\
{\scriptstyle \cap\sigma}\downarrow & & \downarrow{\scriptstyle \cap\theta} \\
tH_1(P) & \xrightarrow{\ \iota_*\ } & tH_1(\widehat{\Gamma})
\end{array}
$$

By condition (1), $\cap \theta$ is an isomorphism. The arrows ι_* and ι^* are isomorphisms since $H_1(P) \to H_1(\widehat{\Gamma})$ is an isomorphism by the choice of $\{P(\ell)\}$ and $tH^2(-) = \mathrm{Ext}(H_1(-), \mathbb{Z})$. So, $\cap \sigma$ is an isomorphism. Apply Turaev's Lemma 7.3, to obtain a map $\phi: M \to B(P)$ of a closed 3-manifold M with $\pi = \pi_1(M)$ such that (M, ϕ) is bordant to (N, ψ) over P and $\phi_*: H_1(M) \to H_1(P)$ is an isomorphism. We have $\phi_*[M] = \psi_*[N] = \sigma$ in $H_3(P)$. Consider the diagram

$$
\begin{array}{ccccc}
H^1(M) & \xleftarrow{\ \phi^*\ } & H^1(P) & \xleftarrow{\ \iota^*\ } & H^1(\widehat{\Gamma}) \\
{\scriptstyle \cap[M]}\downarrow & & \downarrow{\scriptstyle \cap\sigma} & & \downarrow{\scriptstyle \cap\theta} \\
H_2(M) & \xrightarrow{\ \phi_*\ } & H_2(P) & \xrightarrow{\ \iota_*\ } & H_2(\widehat{\Gamma})
\end{array}
$$

By condition (2), $\cap \theta$ is surjective. The arrows ι^* and ι_* are isomorphisms by the choice of $\{P(\ell)\}$ and by the claim. The arrow ϕ^* is an isomorphism since ϕ induces an isomorphism on H_1. By Poincaré duality, $\cap [M]$ is an isomorphism. From these facts, it follows that $\phi_*: H_2(M) \to H_2(P)$ is surjective. So, by Theorem 3.1 (1), $\phi_*: \pi = \pi_1(M) \to P$ induces

an isomorphism $\hat{\pi} \overset{\cong}{\Rightarrow} \widehat{P}$. Since ι induces $\widehat{P} \overset{\cong}{\Rightarrow} \widehat{\Gamma}$, it follows that $\iota\phi\colon M \to \widehat{\Gamma}$ induces an isomorphism $\hat{\pi} \overset{\cong}{\Rightarrow} \widehat{\Gamma}$. Since $\phi_*[M] = \sigma$, we have $\hat{\theta}(M) = \iota_*\phi_*[M] = \iota_*\sigma = \theta$. This shows that $\theta \in \widehat{\mathcal{R}}(\Gamma)$. ∎

9. The free group case and Milnor's link invariant

In this section we discuss the case when Γ is a free group, and show that our invariants of finite length applied to the 0-surgery manifold of a link are equivalent to Milnor's link invariants and Orr's homotopy-theoretic reformulation of the Milnor invariant. Most of the results from this section appear in [17, 20, 21, 25]. However, relating prior work to the results herein seems nontrivial. This section will highlight and clarify new perspectives on Milnor's link invariants.

We proceed as follows. Fix a positive integer m, and as the "basepoint" manifold, let Y be the connected sum of m copies of $S^1 \times S^2$. Then $\pi_1(Y) = F$, the free group on m generators. In this case, we have the following useful property.

Lemma 9.1. *For finite $k \geq 2$, $\mathcal{R}_k(F) = H_3(F/F_k)$.*

Proof. Recall that $H_2(F/F_k) = F_k/F_{k+1}$ by Hopf's theorem. Thus the projection induces a zero homomorphism $H_2(F/F_k) \to H_2(F/F_{k-1})$. From this and the fact that $H_1(F) = \mathbb{Z}^m$ is torsion free, it follows that $\mathcal{R}_k(F) = H_3(F/F_k)$, by Theorem G. ∎

So, $\mathcal{R}_k(F)$ is an abelian group, and consequently $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$ is an abelian group too. We remark that the structure of this cokernel was computed in [17, 25]. The cokernel $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\} = \mathrm{Coker}\{H_3(F/F_{k+1}) \to H_3(F/F_k)\}$ is a free abelian group of rank $m\mathcal{R}(m, k) - \mathcal{R}(m, k + 1)$ where

$$\mathcal{R}(m, n) := \frac{1}{n} \sum_{d \mid n} \phi(d) \cdot m^{n/d}$$

and $\phi(d)$ is the Möbius function.

The following is another useful feature of the case of the free group F.

Lemma 9.2. *Suppose π is a group. Then, for finite $k \geq 2$, every isomorphism $f\colon \pi/\pi_k \overset{\cong}{\Rightarrow} F/F_k$ lifts to an isomorphism $\pi/\pi_{k+1} \overset{\cong}{\Rightarrow} F/F_{k+1}$ if and only if there exists an isomorphism $\pi/\pi_{k+1} \overset{\cong}{\Rightarrow} F/F_{k+1}$.*

Proof. The "only if" part is trivial. For the "if" part, observe that a homomorphism $F \to F$ induces an isomorphism $F/F_k \to F/F_k$ if and only if it induces an isomorphism $H_1(F) \to H_1(F)$, by Stallings' theorem [26], since $H_2(F) = 0$. It follows that every automorphism of F/F_k lifts to an automorphism of F/F_{k+1} for $k \geq 2$. The conclusion is a straightforward consequence of this: if $\tilde{g}\colon \pi/\pi_{k+1} \overset{\cong}{\Rightarrow} F/F_{k+1}$ is an isomorphism, then choose an automorphism lift $\tilde{\phi}\colon F/F_{k+1} \to F/F_{k+1}$ of the automorphism $\phi = fg^{-1}$, where $g\colon \pi/\pi_k \overset{\cong}{\Rightarrow} F/F_k$ is the induced isomorphism. Then the composition $\tilde{\phi} \circ \tilde{g}$ is an isomorphism which is a lift of $fg^{-1} \circ g = f$. ∎

Using the results stated in Section 2 and the above lemmas on the free group, we compare the lower central quotients π/π_k of a 3-manifold group $\pi = \pi_1(M)$ with the free nilpotent quotient F/F_k. For the initial case $k = 2$, π/π_k is isomorphic to F/F_k if and only if $H_1(\pi) \cong \mathbb{Z}^m$. The following theorem deals with the induction step.

Theorem 9.3. *Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$, equipped with an isomorphism $f: \pi/\pi_k \xrightarrow{\cong} F/F_k$, $k \geq 2$. Then the following are equivalent:*

(1) *The given f lifts to an isomorphism $\pi/\pi_{k+1} \xrightarrow{\cong} F/F_{k+1}$.*

(2) *There is an isomorphism $\pi/\pi_{k+1} \cong F/F_{k+1}$ (which is not necessarily a lift).*

(3) *The invariant $\theta_k(M, f)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$.*

(4) *The invariant $\theta_k(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)/\mathrm{Aut}(F/F_k)\}$.*

(5) *The invariant $\theta_k(M, g)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$ for any isomorphism $g: \pi/\pi_k \xrightarrow{\cong} F/F_k$.*

Proof. (1) and (2) are equivalent by Lemma 9.2; (1) and (3) are equivalent by Theorem B; and (2) and (4) are equivalent by Theorem C. It follows that (2) implies (3) for any isomorphism f. In other words, (2) implies (5). Finally, (5) implies (3) obviously. ∎

Now, we apply the above to links. For an m-component link L in S^3, let M_L be the 0-surgery manifold of L. Note that if L is the trivial link, then M_L is equal to the 3-manifold Y that we use in this section.

In [23], Milnor defined his concordance invariants, which we now call *Milnor's numerical invariants*. These invariants arise as coefficients of the Magnus expansion evaluated on homotopy classes of longitudes of a link. More precisely, for an m-component link L, the Magnus expansion is defined by sending the ιth meridian to $1 + t_\iota$ and extending it multiplicatively, and for a sequence $\iota_1, \ldots, \iota_k$ of integers $\iota_j \in \{1, \ldots, m\}$, Milnor's numerical invariant of length k, $\bar{\mu}_L(\iota_1, \ldots, \iota_k)$, is the coefficient of $t_{\iota_1} \cdots t_{\iota_{k-1}}$ in the Magnus expansion of the ι_kth longitude of L. Milnor's numerical invariants of length k are well defined as integers for L if all Milnor's numerical invariants of length $< k$ vanish for L. One can find details in [23].

Theorem 9.4. *Suppose L is a link with m components. For any finite $k \geq 2$, the following are equivalent:*

$(1)_k$ *There is an isomorphism $\pi_1(S^3 \smallsetminus L)/\pi_1(S^3 \smallsetminus L)_{k+1} \cong F/F_{k+1}$.*

$(2)_k$ *The zero linking longitudes of L lie in $\pi_1(S^3 \smallsetminus L)_k$.*

$(3)_k$ *There is an isomorphism $\pi_1(M_L)/\pi_1(M_L)_k \cong F/F_k$.*

$(4)_k$ *Milnor's numerical invariants of length $k + 1$ are well defined for L as integers.*

If the above $(1)_k$–$(4)_k$ hold, then $(1)_{k+1}$–$(4)_{k+1}$ and the following $(5)_{k+1}$–$(8)_{k+1}$ are equivalent:

$(5)_{k+1}$ *Milnor's numerical invariants of length $k + 1$ vanish for L.*

$(6)_{k+1}$ *For some $f: \pi_1(M_L)/\pi_1(M_L)_k \xrightarrow{\cong} F/F_k$, the invariant $\theta_k(M_L, f)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$.*

$(7)_{k+1}$ For all $f\colon \pi_1(M_L)/\pi_1(M_L)_k \xrightarrow{\cong} F/F_k$, the invariant $\theta_k(M_L, f)$ vanishes in Coker$\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$.

$(8)_{k+1}$ The invariant $\theta_k(M_L)$ vanishes in Coker$\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)/\mathrm{Aut}(F/F_k)\}$.

From Theorem 9.4, it follows that all Milnor invariants of length $k + 1$ are defined without ambiguity if and only if $\theta_k(M_L)$ is defined, and all Milnor invariants of length $k + 1$ vanish if and only if $\theta_k(M_L)$ vanishes in Coker$\{\mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F)\}$.

Proof of Theorem 9.4. The equivalence of $(1)_k$–$(4)_k$ is a folklore consequence of Milnor's theorem [23]:

$$\pi_1(S^3 \smallsetminus L)/\pi_1(S^3 \smallsetminus L)_{k+1} \cong \langle F \mid F_{k+1}, [w_1, x_1], \dots, [w_m, x_m]\rangle$$

where x_i and w_i correspond to a meridian and zero linking longitude of the ith component of L, respectively. Indeed, since F/F_{k+1} is Hopfian, the right hand side, which is a quotient of F/F_{k+1}, is isomorphic to F/F_{k+1} if and only if $[w_i, x_i] \in F_{k+1}$ for all i. A standard application of the Magnus expansion, or the Hall basis theorem, shows that $[w_i, x_i] \in F_{k+1}$ if and only if $w_i \in F_k$. Also, since $\pi_1(M_L)$ is the quotient of $\pi_1(S^3 \smallsetminus L)$ by the normal subgroup generated by the longitudes, we have

$$\pi_1(M_L)/\pi_1(M_L)_k \cong \langle F \mid F_k, w_1, \dots, w_m\rangle$$

by Milnor's theorem. Thus $\pi_1(M_L)/\pi_1(M_L)_k \cong F/F_k$ if and only if $w_i \in F_k$, and this is the case if and only if $\pi_1(S^3 \smallsetminus L)/\pi_1(S^3 \smallsetminus L)_{k+1} \cong F/F_{k+1}$ by the above. Also, it is known that Milnor's invariants of length $k + 1$ are well defined integers without ambiguity if and only if $w_i \in F_k$ [23]. This shows that $(1)_k$–$(4)_k$ are equivalent. Milnor also showed that his invariants of length $k + 1$ vanish if and only if $w_i \in F_{k+1}$ [23]. It follows that $(5)_{k+1}$ is equivalent to $(1)_{k+1}$–$(4)_{k+1}$.

By Theorem 9.3, each of $(6)_{k+1}$–$(8)_{k+1}$ is equivalent to $(3)_{k+1}$. This completes the proof. ∎

In what follows we discuss the relationship of our invariants and the link invariant defined in [25].

Let L be a link for which Milnor's invariants of length $\le k$ vanish. Let E_L be the exterior of L, and $G = \pi_1(E_L) = \pi_1(S^3 \smallsetminus L)$. Let K_k be the mapping cone of the inclusion $\bigvee^m S^1 = B(F) \to B(F/F_k)$, and let $j\colon B(F/F_k) \to K_k$ be the inclusion. By Milnor's result [23] (or by Theorem 9.4), there is an isomorphism $F/F_\ell \xrightarrow{\cong} G/G_\ell$ which takes generators of F to meridians, for $\ell \le k + 1$. When $\ell = k$, this gives rise to a map

$$E_L \to B(G) \to B(G/G_k) \xrightarrow{\simeq} B(F/F_k) \xrightarrow{j} K_k$$

which sends meridians to null-homotopic loops. So this extends to a map $\psi\colon S^3 \to K_k$. Denote the homotopy class of this extension by $\theta_k(L) = [\psi] \in \pi_3(K_k)$. This is the invariant defined and studied in [25].

Recall from the proof of Theorem 9.4 that $G/G_k \xrightarrow{\cong} F/F_k$ induces an isomorphism $f\colon \pi_1(M_L)/\pi_1(M_L)_k \xrightarrow{\cong} F/F_k$. Consider $\theta_k(M_L) = \theta_k(M_L, f)$.

To compare $\theta_k(L)$ with $\theta_k(M_L)$, we will use arguments which are already known to experts.

Let $h\colon \pi_3(K_k) \to H_3(K_k)$ be the Hurewicz homomorphism. Note that the inclusion j induces an isomorphism $j_*\colon \mathcal{R}_k(F) = H_3(F/F_k) \to H_3(K_k)$, since K_k is obtained from $B(F/F_k)$ by attaching 2-cells. We claim that our $\theta_k(M_L)$ and Orr's $\theta_k(L)$ are identical in $H_3(K_k)$. That is, $\theta_k(M_L) = j_*^{-1}h(\theta_k(L))$.

The claim is verified as follows. Attach m 2-handles to $S^3 \times [0, 1]$ along the zero-framing of the link $L \subset S^3 = S^3 \times 1$, to obtain a 4-dimensional cobordism W between S^3 and M_L. Let $\phi\colon M_L \to B(F/F_k)$ be the map induced by the above $f\colon \pi_1(M_L)/\pi_1(M_L)_k \xrightarrow{\cong} F/F_k$. Since ϕ restricts to $\psi\colon E_L \to B(F/F_k)$ and since W is obtained by attaching m dual 2-handles to $M_L \times [0, 1]$ along meridians, $M_L \xrightarrow{\phi} B(F/F_k) \xrightarrow{j} K_k$ extends to a map $W \to K_k$ which restricts to $\psi\colon S^3 \to K_k$. This gives us the following commutative diagram:

$$
\begin{array}{ccccc}
M_L & \longhookrightarrow & W & \longleftarrow\!\!\shortmid & S^3 \\
\phi \downarrow & & & \searrow & \downarrow \psi \\
B(F/F_k) & \xrightarrow{\quad j \quad} & & & K_k
\end{array}
$$

From the diagram, the assertion $j_*(\theta_k(M_L)) = h(\theta_k(L))$ follows.

In addition, $\theta_k(M_L) = 0$ in the cokernel of $H_3(F/F_k) \to H_3(F/F_k)$ if and only if $\theta_k(L) = 0$ in the cokernel of $\pi_3(K_{k+1}) \to \pi_3(K_k)$. It follows immediately from the above and from the known fact that the composition $j_*^{-1}h\colon \pi_3(K_k) \to H_3(F/F_k)$ induces an isomorphism between the cokernels [17, 25].

Consequently, the equivalence of (5)–(7) in Theorem 9.4 subsumes the following result of Orr [25]: for a link L, the Milnor invariants of length $k + 1$ vanish if and only if $\theta_k(L) = 0$ in $\mathrm{Coker}\{\pi_3(K_{k+1}) \to \pi_3(K_k)\}$.

We remark that the same argument shows that Levine's link invariant $\theta(L) \in H_3(\widehat{F})$ defined in [20] can be identified with our final invariant $\widehat{\theta}(M_L)$ of the 0-surgery manifold M_L.

Remark 9.5. Results of this section for general closed 3-manifolds and 0-surgery manifolds of links hold for transfinite ordinals k if one uses $\widehat{G}/\widehat{G}_k$ instead of G/G_k for $G = F$ and $G = \pi_1(M_L)$, as we always do in this paper. More precisely, we have the following:

(1) Lemma 9.2 is true for transfinite k. To prove this, one uses our Theorem 6.1 instead of Stallings' theorem in the above proof of Lemma 9.2 (and use that $H_2(\widehat{F}) = 0$).

(2) Theorem 9.3 is true for transfinite k. To prove this, one uses the transfinite version of Lemma 9.2 instead of Lemma 9.2 in the above proof of Theorem 9.3.

(3) Theorem 9.4 is true for transfinite k if one removes conditions $(1)_k$, $(2)_k$, $(4)_k$ and $(5)_{k+1}$. The proof is the same as in the finite case given above.

On the other hand, for links, we do not know whether the transfinite case of the full version of Theorem 9.4 is true. In particular, the following question seems interesting:

are our invariants of the 0-surgery manifold M_L determined by the homotopy class of the longitudes of L, relative to the transfinite lower central series of the group localization? (See condition $(2)_k$ in Theorem 9.4.)

We also note that in [17], Milnor's link invariants are interpreted as a spanning set for the set of cocycles in $H^3(F/F_k)$, allowing one to compute Milnor's numerical invariants from the Milnor invariants defined and studied in this paper. Explicit formulae for these cocycles are derived and evaluated on an *Igusa Picture* representing the homology class $\theta_k(M_L)$. So, we may also ask: can one read the homotopy class of the longitudes using 3-dimensional cocycles in $H^3(\widehat{F}/\widehat{F}_\kappa)$ for any ordinal κ, thus establishing a numerical formulation for transfinite Milnor invariants of links?

These problems remain open for transfinite ordinals, and possibly hinge on obtaining a deeper computational understanding of the transfinite lower central series of local groups, and especially free local groups.

10. Torus bundle example: invariants of finite length

Let Y be the torus bundle with monodromy $h\colon T^2 \to T^2$ given by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, that is,

$$Y = T^2 \times [0,1]/\{(h(x),0) \sim (x,0)\}. \tag{10.1}$$

Let $\Gamma = \pi_1(Y)$ be the fundamental group. The group Γ is an HNN extension $\mathbb{Z}^2 \rtimes \mathbb{Z}$ of $\pi_1(T^2) = \mathbb{Z}^2$ by $\mathbb{Z} = \langle t \rangle$, which acts on $\mathbb{Z}^2$ by $t(a,b)t^{-1} = (-a,-b)$.

The goal of this section is to study the invariant θ_k of finite length over the torus bundle group Γ. The cases of transfinite length invariants and the final invariant are investigated in Sections 11, 12 and 13. Readers eager to see the transfinite case may wish to skip this section on a first reading.

The following theorem summarizes the result of our computation of finite length invariants. In what follows, $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ denotes the finite cyclic group of order d, and $\mathbb{Z}_d^\times = \{r \in \mathbb{Z}_d \mid \gcd(r,d) = 1\}$ denotes the multiplicative group of units in $\mathbb{Z}_d$.

Theorem 10.1. *For finite $k \geq 2$, the following hold:*

(1) *The third homology is given by $H_3(\Gamma/\Gamma_k) = (\mathbb{Z}_{2^{k-1}})^4$.*

(2) *The set of realizable classes in $H_3(\Gamma/\Gamma_k)$ is given by*

$$\mathcal{R}_k(\Gamma) = \begin{cases} \{(a,b,c,r) \in (\mathbb{Z}_2)^4 \mid ac+b+r = 1\} & \text{for } k = 2, \\ (\mathbb{Z}_{2^{k-1}})^3 \times \mathbb{Z}_{2^{k-1}}^\times & \text{for } 3 \leq k < \infty. \end{cases}$$

(3) *The map $\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)$ induced by the projection $\Gamma/\Gamma_{k+1} \to \Gamma/\Gamma_k$ is given by*

$$(\mathbb{Z}_4)^3 \times \mathbb{Z}_4^\times \to \{(a,b,c,r) \in (\mathbb{Z}_2)^4 \mid ac+b+r = 1\}, \quad (a,b,c,r) \mapsto (0,0,0,r),$$

for $k = 2$, and

$$(\mathbb{Z}_{2^k})^3 \times \mathbb{Z}_{2^k}^\times \to (\mathbb{Z}_{2^{k-1}})^3 \times \mathbb{Z}_{2^{k-1}}^\times, \quad (a,b,c,r) \mapsto (2a,2b,2c,r),$$

for $3 \leq k < \infty$.

(4) *For every automorphism ϕ on Γ/Γ_k, the induced bijection $\phi_*\colon \mathcal{R}_k(\Gamma) \to \mathcal{R}_k(\Gamma)$ sends $\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ onto itself. Consequently, $\theta \in \mathcal{R}_k(\Gamma)$ vanishes in the cokernel of $\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)$ if and only if θ vanishes in the cokernel of $\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)/\mathrm{Aut}(\Gamma/\Gamma_k)$.*

From Theorem 10.1 (4) and Theorems B and C, the following corollary is immediately obtained.

Corollary 10.2. *Let $k \geq 2$ be finite. Suppose M is a closed 3-manifold with $\pi = \pi_1(M)$ and $f\colon \pi/\pi_k \overset{\cong}{\Rightarrow} \Gamma/\Gamma_k$ is an isomorphism. Then f lifts to an isomorphism $f\colon \pi/\pi_{k+1} \overset{\cong}{\Rightarrow} \Gamma/\Gamma_{k+1}$ if and only if there is an isomorphism $\pi/\pi_{k+1} \overset{\cong}{\Rightarrow} \Gamma/\Gamma_{k+1}$ (which is not required to be a lift).*

Using Theorem 10.1 (4), we can also obtain the following estimate of the number of isomorphism classes of the $(k+1)$st lower central quotients of 3-manifold groups with the same kth lower central quotient as that of the torus bundle.

Corollary 10.3. *For each finite $k \geq 2$,*

$$2 \leq \#\left(\left\{\pi/\pi_{k+1} \,\middle|\, \begin{array}{c} \pi = \pi_1(M) \text{ for a closed 3-manifold} \\ M \text{ such that } \pi/\pi_k \cong \Gamma/\Gamma_k \end{array}\right\}\Big/\text{isomorphism}\right)$$

$$\leq 7 \cdot 2^{4(k-2)} + 1.$$

Proof. By Theorem 10.1 (3, 4), there is a class $\theta \in \mathcal{R}_k(\Gamma)$ which does not vanish in the cokernel of $\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)/\mathrm{Aut}(\Gamma/\Gamma_k)$. From this, it follows that there exist at least two isomorphism classes of π/π_{k+1} with $\pi = \pi_1(M)$ for some closed 3-manifold M such that $\pi/\pi_k \cong \Gamma/\Gamma_k$, by Theorem C. This proves the lower bound in the statement.

By Theorem 10.1 (2, 3), we have

$$\#\mathcal{R}_k(\Gamma) = (2^{k-1})^3 \cdot 2^{k-2}, \quad \#\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\} = (2^{k-2})^4.$$

By definition, $\theta \in \mathcal{R}_k(\Gamma)$ is equivalent to $\theta_k(Y)$ if and only if θ lies in the image of $\mathcal{R}_{k+1}(\Gamma)$. So, it follows that

$$\#(\mathcal{R}_k(\Gamma)/\sim) \leq \#\mathcal{R}_k(\Gamma) - \#\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\} + 1 = 7 \cdot 2^{4(k-2)} + 1.$$

By Corollary F (2), the number of isomorphism classes of π/π_{k+1} featuring in the statement is bounded above by $\#(\mathcal{R}_k(\Gamma)/\approx) \leq \#(\mathcal{R}_k(\Gamma)/\sim)$. From this, the desired upper bound is obtained. ∎

Indeed, by Corollary F (1), and by the upper bound of $\#(\mathcal{R}_k(\Gamma)/\sim)$ in the last step of the above proof, it follows that Theorem K holds, which asserts that

$$2 \leq \#\left\{\begin{array}{c} \text{equivalence classes of} \\ \text{length } k+1 \text{ extensions of } \{\Gamma/\Gamma_\lambda\}_{\lambda \leq k} \end{array}\right\} \leq 7 \cdot 2^{4(k-2)} + 1.$$

We remark that the estimates in Corollary 10.3 (and that in Theorem K) are not sharp. Further investigation of the equivalence relation and automorphism action on $\mathcal{R}_k(\Gamma)$ gives us improved bounds. We do not address this here.

The rest of this section is devoted to the proof of Theorem 10.1. We begin with the lower central quotient computation. For $(a, b) \in \mathbb{Z}^2 \subset \Gamma$, we have $[t, (a, b)] = (-2a, -2b)$. By using this equation inductively, it follows that the kth lower central subgroup of Γ is

$$\Gamma_k = 2^{k-1} \mathbb{Z}^2 \subset \mathbb{Z}^2 \subset \Gamma. \tag{10.2}$$

Consequently, the lower central quotient is given by

$$\Gamma / \Gamma_k = (\mathbb{Z}_{2^{k-1}})^2 \rtimes \mathbb{Z},$$

where $\mathbb{Z} = \langle t \rangle$ acts on $(\mathbb{Z}_{2^{k-1}})^2$ by $t(a, b)t^{-1} = (-a, -b)$.

In Section 10.1, we compute the homology of Γ / Γ_k and prove Theorem 10.1 (1). In Section 10.2, we study the cap product structure on Γ / Γ_k and prove Theorem 10.1 (2, 3). In Section 10.3, we study the action of $\mathrm{Aut}(\Gamma / \Gamma_k)$ on $\mathcal{R}_k(\Gamma)$ and prove Theorem 10.1 (4).

10.1. Cell structure of $B(\Gamma / \Gamma_k)$ and homology

To compute homology of Γ / Γ_k, we will use cellular chain complexes. Although spectral sequences provide an alternative approach for HNN extensions, the cellular method turns out to be more efficient for our purpose. We will use the following standard facts:

(1) For the finite cyclic group $\mathbb{Z}_d = \langle g \mid g^d \rangle$ of order d, $B(\mathbb{Z}_d)$ has a cell structure with exactly one i-cell e^i in each dimension $i \geq 0$. The boundary operator of the cellular chain complex $C_\bullet(B(\mathbb{Z}_d); \mathbb{Z}[\mathbb{Z}_d])$ is given by

$$\partial e^{2i+1} = (1 - g)e^{2i}, \quad \partial e^{2i} = (1 + g + \cdots + g^{d-1})e^{2i-1}.$$

(2) Let $G = A \rtimes \mathbb{Z}$ be an HNN extension of an abelian group A determined by an automorphism $h \colon A \to A$, that is, $\mathbb{Z} = \langle t \rangle$ acts on A by $tat^{-1} = h(a)$. For a given cell structure of $B(A)$, we may assume that h is realized by a cellular map $h \colon B(A) \to B(A)$. Then $B(G)$ has an associated cell structure, whose n-cells are of the form $e^p \times \epsilon^q$ with $p + q = n$, $q = 0, 1$, e^p a p-cell of $B(A)$ and ϵ^q ($q = 0, 1$) an abstract q-cell. The boundary operator of $C_\bullet(BG; \mathbb{Z}G)$ is given by

$$\partial(e^p \times \epsilon^0) = (\partial e^p) \times \epsilon^0,$$
$$\partial(e^p \times \epsilon^1) = (\partial e^p) \times \epsilon^1 + (-1)^p (t \cdot e^p \times \epsilon^0 - h(e^p) \times \epsilon^0).$$

Let $d = 2^{k-1}$ and write $\mathbb{Z}_d^2 = (\mathbb{Z}_d)^2$ for brevity. Take the product $B(\mathbb{Z}_d^2) = B(\mathbb{Z}_d) \times B(\mathbb{Z}_d)$ of the cell complex in (1), and construct $B(\Gamma / \Gamma_k) = B(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ using (2). Cells of dimension n in $B(\Gamma / \Gamma_k)$ are of the form $e^i \times e^j \times \epsilon^q$ with $i + j + q = n$, $q = 0, 1$. The negation homomorphism $h(g) = g^{-1}$ on $\mathbb{Z}_d$ induces (the chain homotopy class of) the chain map $C_\bullet(\mathbb{Z}_d; \mathbb{Z}[\mathbb{Z}_d]) \to C_\bullet(\mathbb{Z}_d; \mathbb{Z}[\mathbb{Z}_d])$ given by

$$h(e^{2k-1}) = (-1)^k g^{-1} e^{2k-1}, \quad h(e^{2k}) = (-1)^k e^{2k}.$$

and the monodromy $h: B(\mathbb{Z}_d^2) \to B(\mathbb{Z}_d^2)$ is given by $h(e^i \times e^j) = h(e^i) \times h(e^j)$. Using this together with the above (1), (2) and the product boundary formula, it is straightforward to compute the cellular chain complex $C_\bullet(\Gamma/\Gamma_k; \mathbb{Z}[\Gamma/\Gamma_k])$. Applying the augmentation $\mathbb{Z}[\Gamma/\Gamma_k] \to \mathbb{Z}$, it is seen that $C_\bullet(\Gamma/\Gamma_k) = C_\bullet(\Gamma/\Gamma_k; \mathbb{Z})$ has the following boundary operators in dimension ≤ 4:

$$
\begin{aligned}
\partial_1: e^1 \times e^0 \times \epsilon^0 &\mapsto 0, & \partial_2: e^2 \times e^0 \times \epsilon^0 &\mapsto d \cdot e^1 \times e^0 \times \epsilon^0, \\
e^0 \times e^1 \times \epsilon^0 &\mapsto 0, & e^1 \times e^1 \times \epsilon^0 &\mapsto 0, \\
e^0 \times e^0 \times \epsilon^1 &\mapsto 0, & e^0 \times e^2 \times \epsilon^0 &\mapsto d \cdot e^0 \times e^1 \times \epsilon^0, \\
& & e^1 \times e^0 \times \epsilon^1 &\mapsto -2e^1 \times e^0 \times \epsilon^0, \\
& & e^0 \times e^1 \times \epsilon^1 &\mapsto -2e^0 \times e^1 \times \epsilon^0,
\end{aligned}
$$

$$
\begin{aligned}
\partial_3: e^3 \times e^0 \times \epsilon^0 &\mapsto 0, & \partial_4: e^4 \times e^0 \times \epsilon^0 &\mapsto d \cdot e^3 \times e^0 \times \epsilon^0, \\
e^2 \times e^1 \times \epsilon^0 &\mapsto d \cdot e^1 \times e^1 \times \epsilon^0, & e^3 \times e^1 \times \epsilon^0 &\mapsto 0, \\
e^1 \times e^2 \times \epsilon^0 &\mapsto -d \cdot e^1 \times e^1 \times \epsilon^0, & e^2 \times e^2 \times \epsilon^0 &\mapsto d \cdot e^1 \times e^2 \times \epsilon^0 \\
e^0 \times e^3 \times \epsilon^0 &\mapsto 0, & &\quad + d \cdot e^2 \times e^1 \times \epsilon^0, \\
e^2 \times e^0 \times \epsilon^1 &\mapsto d \cdot e^1 \times e^0 \times \epsilon^1 & e^1 \times e^3 \times \epsilon^0 &\mapsto 0, \\
&\quad + 2 \cdot e^2 \times e^0 \times \epsilon^0, & e^0 \times e^4 \times \epsilon^0 &\mapsto d \cdot e^0 \times e^3 \times \epsilon^0, \\
e^1 \times e^1 \times \epsilon^1 &\mapsto 0, & e^3 \times e^0 \times \epsilon^1 &\mapsto 0, \\
e^0 \times e^2 \times \epsilon^1 &\mapsto d \cdot e^0 \times e^1 \times \epsilon^1 & e^2 \times e^1 \times \epsilon^1 &\mapsto d \cdot e^1 \times e^1 \times \epsilon^1, \\
&\quad + 2 \cdot e^0 \times e^2 \times \epsilon^0, & e^1 \times e^2 \times \epsilon^1 &\mapsto -d \cdot e^1 \times e^1 \times \epsilon^1, \\
& & e^0 \times e^3 \times \epsilon^1 &\mapsto 0.
\end{aligned}
$$

The homology groups $H_i(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ ($i \leq 3$) are immediately obtained from this:

$$
\begin{aligned}
H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) &= \mathbb{Z}_2^2, & (10.3) \\
H_2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) &= \mathbb{Z}_2^2 \times \mathbb{Z}_d, & (10.4) \\
H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) &= \mathbb{Z}_d^4. & (10.5)
\end{aligned}
$$

This shows Theorem 10.1 (1). In addition, the four $\mathbb{Z}_d$ factors of $H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ are respectively generated by

$$
\begin{aligned}
\xi_1 &= e^3 \times e^0 \times \epsilon^0, \\
\xi_2 &= e^2 \times e^1 \times \epsilon^0 + e^1 \times e^2 \times \epsilon^0, \\
\xi_3 &= e^0 \times e^3 \times \epsilon^0, \\
\zeta &= e^1 \times e^1 \times \epsilon^1.
\end{aligned} \tag{10.6}
$$

Here, the basis element $\zeta \in H_3(\Gamma/\Gamma_k)$ is the image of the fundamental class $[Y] \in H_3(Y)$ under $H_3(Y) \to H_3(\Gamma/\Gamma_k)$. In other words, $\theta_k(Y) = \zeta$. To verify this, observe that Y is a subcomplex of $B(\Gamma/\Gamma_k)$ consisting of the cells $e^i \times e^j \times \epsilon^q$ with $i, j, q \in \{0, 1\}$. By computing $H_i(Y)$ using this subcomplex, it is seen that $e^1 \times e^1 \times \epsilon^1$ generates $H_3(Y) = \mathbb{Z}$.

Also, viewing $B(\mathbb{Z}_d^2)$ as a subcomplex of $B(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$, it is seen that the subgroup generated by ξ_1, ξ_2 and ξ_3 is the isomorphic image of $H_3(\mathbb{Z}_d^2)$ under the inclusion-induced map.

The above chain level computation also enables us to compute the projection-induced homomorphism $H_3(\Gamma/\Gamma_{k+1}) \to H_3(\Gamma/\Gamma_k)$. First, consider the projection $\mathbb{Z}_{2d} \to \mathbb{Z}_d$. Abuse notation by denoting the i-cells of $B(\mathbb{Z}_{rd})$ and $B(\mathbb{Z}_d)$ by the same symbol e^i. A routine computation shows that the induced chain map $C_\bullet(\mathbb{Z}_{2d}) \to C_\bullet(\mathbb{Z}_d)$ is given by $e^i \mapsto 2^{\lfloor i/2 \rfloor} \cdot e^i$. (For instance, $e^1 \mapsto e^1$ while $e^2 \mapsto 2e^2$.) From this, it follows that the projection

$$\Gamma/\Gamma_{k+1} = \mathbb{Z}_{2d}^2 \rtimes \mathbb{Z} \to \Gamma/\Gamma_k = \mathbb{Z}_d^2 \rtimes \mathbb{Z}$$

induces the chain map $C_\bullet(\Gamma/\Gamma_{k+1}) \to C_\bullet(\Gamma/\Gamma_k)$ given by

$$e^i \times e^j \times \epsilon^q \mapsto 2^{\lfloor i/2 \rfloor + \lfloor j/2 \rfloor} \cdot e^i \times e^j \times \epsilon^q.$$

Therefore, $H_3(\Gamma/\Gamma_{k+1}) \to H_3(\Gamma/\Gamma_k)$ is the homomorphism

$$\xi_i \mapsto 2 \cdot \xi_i \quad \text{for } i = 1, 2, 3, \quad \zeta \mapsto \zeta. \tag{10.7}$$

10.2. Realizable classes

Now we compute the realizable classes in $H_3(\Gamma/\Gamma_k)$. Fix $\theta \in H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = H_3(\Gamma/\Gamma_k)$ where $d = 2^{k-1}$ with $k \geq 2$ as before. To apply Theorem G, we will investigate the cap product maps

$$\cap \theta : tH^2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) \to tH_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}), \tag{10.8}$$
$$\cap \theta : H^1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) \to H_2(\mathbb{Z}_d^2 \rtimes \mathbb{Z})/K_{k-1}(\Gamma/\Gamma_k). \tag{10.9}$$

Here, $K_{k-1}(\Gamma/\Gamma_k)$ is the kernel of $H_2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = H_2(\Gamma/\Gamma_k) \to H_2(\Gamma/\Gamma_{k-1})$.

Case 1: Suppose $k \geq 3$, that is, $d = 2^{k-1}$ is divisible by 4. Recall that $H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ has the basis $\{\xi_1, \xi_2, \xi_3, \zeta\}$ described in (10.6). Let $\theta \in H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ be a class which is a linear combination of ξ_1, ξ_2 and ξ_3. Since each ξ_i is of the form $\bullet \times \bullet \times \epsilon^0$ in (10.6), ξ_i lies in the image of the inclusion-induced map $i_* : H_3(\mathbb{Z}_d^2) \to H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$. Write $\theta = i_*(z)$ for some $z \in H_3(\mathbb{Z}_d^2)$. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}_2^2 = tH^2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) & \xrightarrow{\cap \theta} & tH_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \mathbb{Z}_2^2 \\
{\scriptstyle i^*} \downarrow & & \uparrow {\scriptstyle i_*} \\
\mathbb{Z}_d^2 = tH^2(\mathbb{Z}_d^2) & \xrightarrow{\cap z} & tH_1(\mathbb{Z}_d^2) = \mathbb{Z}_d^2
\end{array}
$$

Here, $tH_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \mathbb{Z}_2^2$ by (10.3), $H_1(\mathbb{Z}_d^2) = \mathbb{Z}_d^2$ obviously, so that $tH^2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \text{Ext}(H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}_2^2$ and $tH^2(\mathbb{Z}_d^2) = \mathbb{Z}_d^2$. Let $c \in tH^2(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$. Since $2c = 0$ and all order 2 elements in $tH^2(\mathbb{Z}_d^2) = \mathbb{Z}_d^2$ are multiples of $d/2$, $i^*(c)$ is a multiple of $d/2$. So, $i_*(i^*(c) \cap z) = c \cap \theta$ is a multiple of $d/2$, which is a multiple of 2 since $d = 2^{k-1}$ with $k \geq 3$. It follows that $c \cap \theta = 0$, since it lies in $tH_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \mathbb{Z}_2^2$. This shows that

the cap product (10.8) is zero. Also, the cap product (10.9) is zero since $H^1(\mathbb{Z}_d^2) = 0$ and the following diagram commutes:

$$
\begin{array}{ccc}
H^1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) & \xrightarrow{\cap \theta} & H_2(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) \\
\uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i_*} \\
0 = H^1(\mathbb{Z}_d^2) & \xrightarrow{\cap z} & H_2(\mathbb{Z}_d^2)
\end{array}
$$

Now, consider a class of the form $\theta = r\zeta$ with $r \in \mathbb{Z}$. Since ζ is the image of the fundamental class $[Y] \in H_3(\Gamma)$, it is realizable, that is, $\zeta \in \mathcal{R}_k(\Gamma)$. By Theorem G, the cap product (10.8) is an isomorphism for $\theta = \zeta$. From this it follows that (10.8) is an isomorphism for $\theta = r\zeta$ if and only if r is odd, since $tH_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ is a 2-group by (10.4). Also, the cap product (10.9) is surjective for the realizable class $\theta = \zeta \in \mathcal{R}_k(\Gamma)$, by Theorem G. From this it follows that (10.9) is surjective for $\theta = r\zeta$ if r is odd, since $H_2(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$ is a 2-group.

Combine the above conclusions to obtain the following: for an arbitrary class

$$
\theta = a\xi_1 + b\xi_2 + c\xi_3 + r\zeta \in H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z}),
$$

(10.8) is an isomorphism and (10.9) is surjective if and only if r is odd. Applying Theorem G, this proves Theorem 10.1 (2) for $k \geq 3$.

Case 2: Suppose $k = 2$, that is, $d = 2^{k-1} = 2$. In this case, first note that $\Gamma/\Gamma_{k-1} = \Gamma/\Gamma_1$ is trivial by definition, and thus the cap product (10.9) is onto the trivial group. So, it suffices to determine when the cap product (10.8) is an isomorphism.

Observe that the semidirect product $\mathbb{Z}_2^2 \rtimes \mathbb{Z}$ is equal to the ordinary product $\mathbb{Z}_2^2 \times \mathbb{Z}$ since $-a = a$ in $\mathbb{Z}_2$. This enables us to compute the cap product directly using the standard product cell structures. To prevent confusion with the semidirect product case, denote the i-cell of $B(\mathbb{Z}) = S^1$ by u^i ($i = 0, 1$), while the cells of $B(\mathbb{Z}_2)$ are denoted by e^i as before. It is well known that

$$
\Delta(e^i) = \sum_{p+q=i} (-1)^{pq} e_p \times e_q, \quad \Delta(u^i) = \sum_{p+q=i} u^p \times e^q
$$

are cellular approximations of the diagonal maps $B(\mathbb{Z}_2) \to B(\mathbb{Z}_2) \times B(\mathbb{Z}_2)$ and $B(\mathbb{Z}) \to B(\mathbb{Z}) \times B(\mathbb{Z})$, and thus the chain level cup products of $B(\mathbb{Z}_d)$ and $B(\mathbb{Z})$ defined using them are given by

$$
(e^i)^* \cup (e^j)^* = (-1)^{ij} \cdot (e^{i+j})^*, \quad (u^i)^* \cup (u^j)^* = (u^{i+j})^*.
$$

Here and in what follows, for brevity, we use the convention that $e^i = 0$ for $i < 0$ and $u^i = 0$ for $i \notin \{0, 1\}$. Using this notation, the cap product is given by

$$
(e^i)^* \cap e^j = (-1)^{i(j-i)} \cdot e^{j-i}, \quad (u_i)^* \cap u^j = u^{j-i}.
$$

Therefore, the cap product on the product $B(\mathbb{Z}_2^2 \times \mathbb{Z})$ is as follows:

$$(e^i \times e^j \times u^p)^* \cap (e^k \times e^\ell \times u^q) = (-1)^{jk+pk+pl} \cdot e^{k-i} \times e^{\ell-j} \times u^{q-p}. \quad (10.10)$$

Note that the product cell structure we use here is different from the HNN extension cell structure we used in Case 1. To compute the cap product for the basis elements in (10.6) which are expressed in terms of the cells $e^i \times e^j \times \epsilon^q$, we need to rewrite them in terms of the product cells $e^i \times e^j \times u^q$. The three basis elements ξ_1, ξ_2 and ξ_3 in (10.6) are already in this form, since ϵ^0 is identical with u^0. To make the computation for $\zeta = e^1 \times e^2 \times \epsilon^1$ simpler, consider the projection $\mathbb{Z}^2 \rtimes \mathbb{Z} \to \mathbb{Z}_2^2 \rtimes \mathbb{Z} = \mathbb{Z}_2^2 \times \mathbb{Z}$. It is straightforward to verify that this induces the homotopy class of a chain map

$$C_\bullet(B(\mathbb{Z}^2 \rtimes \mathbb{Z}); \mathbb{Z}[\mathbb{Z}^2 \rtimes \mathbb{Z}]) \to C_\bullet(B(\mathbb{Z}_2^2 \times \mathbb{Z}); \mathbb{Z}[\mathbb{Z}_2^2 \times \mathbb{Z}])$$

which is given by $e^i \times e^j \times \epsilon^q \mapsto e^i \times e^j \times u^q$ in dimension $i + j + q \leq 1$ and by

$$\begin{aligned}
e^1 \times e^0 \times \epsilon^1 &\mapsto e^1 \times e^0 \times u^1 - e^2 \times e^0 \times u^0, \\
e^0 \times e^1 \times \epsilon^1 &\mapsto e^0 \times e^1 \times u^1 - e^0 \times e^2 \times u^0, \\
e^1 \times e^1 \times \epsilon^1 &\mapsto e^1 \times e^1 \times u^1 - e^1 \times e^2 \times u^0 - e^2 \times (g \cdot e^1) \times u^0
\end{aligned}$$

in dimensions 2 and 3. So, applying the augmentation, $\zeta = e^1 \times e^1 \times \epsilon^1$ is expressed in the product complex $C_\bullet(B(\mathbb{Z}_2^2 \times \mathbb{Z}); \mathbb{Z})$ as

$$\zeta = e^1 \times e^1 \times u^1 - e^1 \times e^2 \times u^0 - e^2 \times e^1 \times u^0. \quad (10.11)$$

Now we are ready to compute the cap product (10.8). By a routine computation,

$$\begin{aligned}
H^2(\mathbb{Z}_2^2 \rtimes \mathbb{Z}) &= \mathbb{Z}_2^2 \quad \text{with basis } \{(e^2 \times e^0 \times u^0)^*, (e^0 \times e^2 \times u^0)^*\}, \\
tH_1(\mathbb{Z}_2^2 \rtimes \mathbb{Z}) &= \mathbb{Z}_2^2 \quad \text{with basis } \{e^1 \times e^0 \times u^0, e^0 \times e^1 \times u^0\}.
\end{aligned}$$

Let $\theta = a\xi_1 + b\xi_2 + c\xi_3 + r\zeta \in H_3(\mathbb{Z}_2^2 \rtimes \mathbb{Z})$. From (10.10) and (10.11), it follows that the cap product $\cap \theta$ in (10.8) is given by

$$\begin{bmatrix} a & b-r \\ b-r & c \end{bmatrix}$$

with respect to the above basis. It follows that $\cap \theta$ is an isomorphism if and only if $ac + b + r$ is odd. This completes the proof of Theorem 10.1 (2) for $k = 2$.

Once $\mathcal{R}_k(\Gamma)$ is computed as above, Theorem 10.1 (3) follows immediately from the description of the projection-induced homomorphism in (10.7).

10.3. Automorphism action on the realizable classes

As above, let $d = 2^{k-1}$, and write $\Gamma/\Gamma_k = \mathbb{Z}_d^2 \rtimes \mathbb{Z}$. Suppose $\phi \colon \mathbb{Z}_d^2 \rtimes \mathbb{Z} \to \mathbb{Z}_d^2 \rtimes \mathbb{Z}$ is an automorphism. It induces an automorphism $\phi_* \colon H_3(\Gamma/\Gamma_k) \to H_3(\Gamma/\Gamma_k)$, which restricts to a bijection $\phi_* \colon \mathcal{R}_k(\Gamma) \to \mathcal{R}_k(\Gamma)$. Our goal is to show Theorem 10.1 (4), which says that ϕ_* sends $\text{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ onto itself bijectively.

As the first step, we claim that ϕ sends the subgroup $\mathbb{Z}_d^2 \subset \mathbb{Z}_d^2 \rtimes \mathbb{Z}$ isomorphically onto $\mathbb{Z}_d^2$ itself. To see this, observe that $\mathbb{Z}_d^2$ is the kernel of the horizontal composition in the following diagram:

$$
\begin{array}{ccccc}
\mathbb{Z}_d^2 \rtimes \mathbb{Z} & \longrightarrow & H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \mathbb{Z}_d^2 \times \mathbb{Z} & \longrightarrow & H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}; \mathbb{Q}) = \mathbb{Q} \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi_*} & & \downarrow{\scriptstyle\phi_*} \\
\mathbb{Z}_d^2 \rtimes \mathbb{Z} & \longrightarrow & H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = \mathbb{Z}_d^2 \times \mathbb{Z} & \longrightarrow & H_1(\mathbb{Z}_d^2 \rtimes \mathbb{Z}; \mathbb{Q}) = \mathbb{Q}
\end{array}
$$

Since vertical arrows are automorphisms, the claim follows from the commutativity of the diagram.

Recall from Section 10.1 that the subgroup $\langle \xi_1, \xi_2, \xi_3 \rangle$ is the (isomorphic) image of $H_3(\mathbb{Z}_d^2)$ in $H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z})$. So, by the claim, $\phi_* \langle \xi_1, \xi_2, \xi_3 \rangle$ is equal to $\langle \xi_1, \xi_2, \xi_3 \rangle$. From this it follows that

$$\phi_* \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle = \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle. \tag{10.12}$$

In addition, recall from Section 10.1 that $\zeta \in H_3(\mathbb{Z}_d^2 \rtimes \mathbb{Z}) = H_3(\Gamma/\Gamma_k)$ is the image of the fundamental class $[Y] \in H_3(\Gamma)$. The quotient homomorphism $\Gamma \to \Gamma/\Gamma_k$ factors through Γ/Γ_{k+1}, and $[Y]$ is sent into $\mathcal{R}_{k+1}(\Gamma)$ by definition. From this, it follows that ζ lies in the image of $\mathcal{R}_{k+1}(\Gamma)$. By Theorem 10.1 (3),

$$\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\} = \{\eta + r\zeta \mid \eta \in \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle,\ r \in 2\mathbb{Z} + 1\}.$$

So, $\phi_*(\zeta) = \eta_0 + r_0\zeta$ for some $\eta_0 \in \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle$ and some odd r_0. Now, for an arbitrary $\eta + r\zeta \in \mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ with $\eta \in \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle$ and r odd, we have

$$\phi_*(\eta + r\zeta) = \phi_*(\eta) + r\eta_0 + rr_0\zeta.$$

Here, $\phi_*(\eta) + r\eta_0 \in \langle 2\xi_1, 2\xi_2, 2\xi_3 \rangle$ by (10.12), and rr_0 is obviously odd. This shows that ϕ_* sends $\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ into itself. Since ϕ_* is one-to-one and $\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ is a finite set, it follows that ϕ_* restricts to a bijection of $\mathrm{Im}\{\mathcal{R}_{k+1}(\Gamma) \to \mathcal{R}_k(\Gamma)\}$ onto itself. This completes the proof of Theorem 10.1 (4).

11. Torus bundle example: invariants of transfinite length

In this section, we study transfinite invariants over the torus bundle Y defined in (10.1):

$$Y = T^2 \times [0, 1]/\{(h(x), 0) \sim (x, 0)\}$$

where $h = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. As before, let $\Gamma = \pi_1(Y)$. Denote the ring of integers localized at the prime 2 by

$$\mathbb{Z}_{(2)} = \{a/b \mid a \in \mathbb{Z},\ b \in 2\mathbb{Z} + 1\}.$$

Let $\mathbb{Z}_{(2)}^\times = \{a/b \mid a, b \in 2\mathbb{Z} + 1\}$ be the multiplicative group of units.

The following theorem summarizes our computation of transfinite invariants over the torus bundle.

Theorem 11.1. *For transfinite ordinals κ, the following hold:*

(1) *The third homology of $\widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is given by*

$$H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) = \begin{cases} \mathbb{Z}_{(2)} & \text{for } \kappa = \omega, \\ (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z} & \text{for } \kappa \geq \omega + 1. \end{cases}$$

(2) *The set of realizable classes in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is given by*

$$\mathcal{R}_\kappa(\Gamma) = \begin{cases} \mathbb{Z}_{(2)}^\times & \text{for } \kappa = \omega, \\ (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} & \text{for } \kappa \geq \omega + 1. \end{cases}$$

(3) *The map $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$ induced by $\widehat{\Gamma}/\widehat{\Gamma}_{\kappa+1} \to \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ is given by*

$$(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} \to \mathbb{Z}_{(2)}^\times, \quad (x, \epsilon) \mapsto \epsilon,$$

for $\kappa = \omega$, and

$$(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} \to (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\}, \quad (x, \epsilon) \mapsto (x, \epsilon),$$

for $\kappa \geq \omega + 1$.

(4) *On $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times$, the equivalence relation $\sim$ is given by $r \sim r'$ if and only if $r = \pm r'$. On $\mathcal{R}_\kappa(\Gamma)$ with $\kappa \geq \omega + 1$, $r \sim r'$ for all $r, r' \in \mathcal{R}_\kappa(\Gamma)$.*

(5) *The automorphism group $\text{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ acts on $\mathcal{R}_\omega(\Gamma)$ transitively. Consequently, $\mathcal{R}_\omega(\Gamma)/\approx$ is trivial.*

Combining Theorem 11.1 (4, 5) with Corollary F (1) and Theorem C respectively, the following statements are immediately obtained. Recall that the notion of extensions of a transfinite lower central quotient tower and their equivalence were introduced in Section 2.6.

Corollary 11.2.

(1) *The set*

$$\left\{ \begin{array}{c} \text{length } \omega + 1 \text{ extensions, by 3-manifolds,} \\ \text{of the length } \omega \text{ tower } \widehat{\Gamma}/\widehat{\Gamma}_\omega \to \cdots \to \Gamma/\Gamma_1 = \{1\} \end{array} \right\} \Big/ \begin{array}{c} \text{equivalence of} \\ \text{length } \omega + 1 \text{ extensions} \end{array}$$

is in one-to-one correspondence with the infinite set

$$(\mathbb{Z}_{(2)}^\times)_{>0} := \{a/b \mid a, b \in 2\mathbb{Z} + 1, a, b > 0\}.$$

(2) *If M is a closed 3-manifold with $\pi = \pi_1(M)$ such that $\widehat{\pi}/\widehat{\pi}_\omega \cong \widehat{\Gamma}/\widehat{\Gamma}_\omega$, then $\widehat{\pi}/\widehat{\pi}_\kappa \cong \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ for every ordinal $\kappa \geq \omega + 1$.*

This illustrates that the classification of tower extensions from length ω to $\omega + 1$ may have a completely different nature from the determination of the isomorphism class of the $(\omega + 1)$st lower central quotient for a given ωth lower central quotient. For the case of the torus bundle group Γ, Corollary 11.2 (1) tells us that the former has infinitely many solutions, while the latter has a unique solution by Corollary 11.2 (2). In particular, over Γ, $\bar{\mu}_\kappa(M)$ is trivial for all infinite ordinals κ whenever $\bar{\mu}_\kappa(M)$ is defined.

We remark that our proof of Theorem L in Section 13 presents modified torus bundle groups over which there are infinitely many 3-manifolds M with nontrivial $\bar{\mu}_\omega(M)$.

The remaining part of this section is devoted to the proof of Theorem 11.1. In Section 11.1, we describe the homology localization $\hat{\Gamma}$ and its transfinite lower central series. It turns out that the transfinite lower central series stabilizes at length $\omega + 1$ with $\hat{\Gamma}_{\omega+1} = \{1\}$. In Section 11.2, we study the homology and cap product structure of $\hat{\Gamma}/\hat{\Gamma}_\omega$ and prove Theorem 11.1 $(1, 2)$ for $\kappa = \omega$. In Sections 11.3 and 11.4, we study the homology and the cap product structure of $\hat{\Gamma}$, respectively, and prove Theorem 11.1 $(1, 2)$ for $\kappa \geq \omega + 1$ and Theorem 11.1 (3). In Section 11.5, we study the equivalence relation $\sim$ on $\mathcal{R}_\omega(\Gamma)$ and prove Theorem 11.1 $(4, 5)$.

11.1. Homology localization of the torus bundle group

We start by reviewing the computation of the homology localization $\hat{\Gamma}$ of the torus bundle group Γ from our earlier work [7]. The result expresses $\hat{\Gamma}$ as a colimit of finitely presented groups. (Such a colimit expression of the localization exists for any finitely presented group by Theorem 3.1 (2), but finding an explicit description is nontrivial in general.)

For a positive odd integer ℓ, let

$$\Gamma(\ell) = \langle u, v, t \mid tut^{-1}u, \, tvt^{-1}v, \, [u, v]^{\ell^2}, \, [[u, v], u], \, [[u, v], v], \, [[u, v], t] \rangle. \quad (11.1)$$

It is straightforward to see that $\Gamma(1) = \Gamma$ and the map $\Gamma(\ell) \to \Gamma(r\ell)$ sending t, u, and v to t, u^r, and v^r respectively is a well defined inclusion for all odd r, $\ell \geq 1$. The groups $\Gamma(\ell)$ with these inclusions form a direct system.

Theorem 11.3 ([7, Theorem 3.1]). *The homology localization of Γ is given as*

$$\Gamma \to \hat{\Gamma} = \operatorname*{colim}_{\ell \text{ odd}} \Gamma(\ell).$$

Observe, from the presentation (11.1), that $[u, v] \in \Gamma(\ell)$ generates a finite cyclic subgroup of order ℓ^2, which is normal in $\Gamma(\ell)$, and the quotient of $\Gamma(\ell)$ by this cyclic subgroup is isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where $\mathbb{Z}^2$ is generated by u, v and $\mathbb{Z}$ is generated by t which acts on $\mathbb{Z}^2$ by negation. Note that the restriction of $\Gamma(\ell) \to \Gamma(r\ell)$ to the cyclic subgroup generated by $[u, v]$ is the homomorphism $\mathbb{Z}_{\ell^2} \to \mathbb{Z}_{(r\ell)^2}$ given by $1 \mapsto r^2$, and $\Gamma(\ell) \to \Gamma(r\ell)$ induces a map $\mathbb{Z}^2 \rtimes \mathbb{Z} \to \mathbb{Z}^2 \rtimes \mathbb{Z}$ on the quotients, which is given by $(a, b, c) \mapsto (ra, rb, c)$. So, if we identify $\mathbb{Z}_{\ell^2}$ with

$$\tfrac{1}{\ell^2}\mathbb{Z}/\mathbb{Z} := \left(\tfrac{1}{\ell^2}\mathbb{Z}\right)/\mathbb{Z} = \left\{\tfrac{a}{\ell^2} \mid a \in \mathbb{Z}\right\}/\mathbb{Z},$$

under $[u, v] \mapsto \tfrac{1}{\ell^2}$ and identify $\mathbb{Z}^2 \rtimes \mathbb{Z}$ with $\left(\tfrac{1}{\ell}\mathbb{Z}\right)^2 \rtimes \mathbb{Z}$ under $u \mapsto (\tfrac{1}{\ell}, 0, 0)$, $v \mapsto (0, \tfrac{1}{\ell}, 0)$, $t \mapsto (0, 0, 1)$, then we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tfrac{1}{\ell^2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \Gamma(\ell) & \longrightarrow & \left(\tfrac{1}{\ell}\mathbb{Z}\right)^2 \rtimes \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \tfrac{1}{(r\ell)^2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \Gamma(r\ell) & \longrightarrow & \left(\tfrac{1}{r\ell}\mathbb{Z}\right)^2 \rtimes \mathbb{Z} & \longrightarrow & 1
\end{array} \quad (11.2)$$

Taking colimit, we obtain the central extension

$$1 \to \mathbb{Z}_{(2)}/\mathbb{Z} \to \widehat{\Gamma} \to \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z} \to 1. \tag{11.3}$$

Using this, we can compute the transfinite lower central subgroups of $\widehat{\Gamma}$.

Lemma 11.4. *The first transfinite lower central subgroup $\widehat{\Gamma}_\omega$ is equal to $\mathbb{Z}_{(2)}/\mathbb{Z}$. For $\kappa \geq \omega + 1$, $\widehat{\Gamma}_\kappa$ is trivial.*

Proof. We claim that

$$\Gamma(\ell)_\omega = \tfrac{1}{\ell^2}\mathbb{Z}/\mathbb{Z}.$$

To prove this, we will first verify $\Gamma(\ell)_\omega \subset \frac{1}{\ell^2}\mathbb{Z}/\mathbb{Z}$. Recall from (10.2) that the rth lower central subgroup $(\mathbb{Z}^2 \rtimes \mathbb{Z})_r$ of $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is equal to $(2^{r-1}\mathbb{Z})^2$. So $\mathbb{Z}^2 \rtimes \mathbb{Z}$ is residually nilpotent, that is, $(\mathbb{Z}^2 \rtimes \mathbb{Z})_\omega$ is trivial. From this and (11.2), it follows that $\Gamma(\ell)_\omega \subset \frac{1}{\ell^2}\mathbb{Z}/\mathbb{Z}$.

For the reverse inclusion, first verify that $u^{2^{r-1}} \in \Gamma(\ell)_r$ by induction, using the identity $[t, u^{2^{r-1}}] = u^{-2^r}$. So $[u^{2^{r-1}}, v] = [u, v]^{2^{r-1}}$ lies in $\Gamma(\ell)_{r+1}$. Since $[u, v]$ has order ℓ^2 and ℓ is odd, it implies that $[u, v] \in \Gamma(\ell)_{r+1}$. Since this holds for all r, it follows that $[u, v] \in \Gamma(\ell)_\omega$. In other words, $\frac{1}{\ell^2}\mathbb{Z}/\mathbb{Z} \subset \Gamma(\ell)_\omega$. This shows the claim that $\Gamma(\ell)_\omega = \frac{1}{\ell^2}\mathbb{Z}/\mathbb{Z}$.

From the claim, the promised conclusion $\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}/\mathbb{Z}$ is obtained by taking colimit.

Since $[u, v]$ is central, $\Gamma(\ell)_{\omega+1}$ is trivial, and thus $\Gamma(\ell)_\kappa$ is trivial for all $\kappa \geq \omega + 1$. Take colimit to find that $\widehat{\Gamma}_\kappa$ is trivial for all $\kappa \geq \omega + 1$. ∎

11.2. Third homology and realizable classes for $\kappa = \omega$

The goal of this subsection is to investigate the homology and cap product structure of $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ and prove Theorem 11.1 (1, 2) for $\kappa = \omega$.

We begin with homology computation for $\widehat{\Gamma}/\widehat{\Gamma}_\omega$. By (11.3) and Lemma 11.4, we have $\widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{Z}_{(2)}^2$ by negation. The Lyndon–Hochschild–Serre spectral sequence for the HNN extension

$$1 \to \mathbb{Z}_{(2)}^2 \to \widehat{\Gamma}/\widehat{\Gamma}_\omega \to \mathbb{Z} \to 1$$

gives the Wang exact sequence

$$H_3(\mathbb{Z}_{(2)}^2) \to H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) \to H_2(\mathbb{Z}_{(2)}^2) \xrightarrow{1-t_*} H_2(\mathbb{Z}_{(2)}^2)$$
$$\to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) \to H_1(\mathbb{Z}_{(2)}^2) \xrightarrow{1-t_*} H_1(\mathbb{Z}_{(2)}^2) \tag{11.4}$$

where $t_*\colon H_i(\mathbb{Z}_{(2)}^2) \to H_i(\mathbb{Z}_{(2)}^2)$ is the map induced by negation $(a, b) \mapsto (-a, -b)$ on $\mathbb{Z}_{(2)}^2$. Using the fact that $\mathbb{Z}_{(2)}^2$ is the colimit of $(\frac{1}{d}\mathbb{Z})^2 \cong \mathbb{Z}^2$, it is straightforward to compute the following homology groups of $\mathbb{Z}_{(2)}^2$:

$$H_3(\mathbb{Z}_{(2)}^2) = 0, \quad H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}, \quad H_1(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}^2.$$

Moreover, $1 - t_*$ on $H_2(\mathbb{Z}_{(2)}^2)$ is zero, while $1 - t_*$ on $H_1(\mathbb{Z}_{(2)}^2)$ is multiplication by 2. From this and (11.4), it follows that

$$
\begin{aligned}
H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}, \\
H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}, \\
H_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= \mathbb{Z}_2^2 \times \mathbb{Z}.
\end{aligned}
\tag{11.5}
$$

This shows Theorem 11.1 (1) for $\kappa = \omega$.

Now we investigate cap products to compute the set $\mathcal{R}_\omega(\Gamma)$ of realizable classes. First note that $\theta = 1 \in \mathbb{Z}_{(2)} = H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ is the image of the fundamental class $[Y] \in H_3(\Gamma)$, and thus it lies in $\mathcal{R}_\omega(\Gamma)$ by definition. So, by Theorem G, for $\theta = 1$,

$$
\cap \theta : tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_2^2 \to tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_2^2
\tag{11.6}
$$

is an isomorphism. Also,

$$
\cap \theta : H^1(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z} \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)/\mathrm{Ker}\{H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_k)\}
\tag{11.7}
$$

is surjective for all finite k. That is, $\mathrm{Im}\{\cap 1\} = H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)/\mathrm{Ker}$.

Consider an arbitrary $\theta := a/d \in \mathbb{Z}_{(2)} = H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ with d odd. Then, since the codomain of (11.6) is a finite abelian 2-group, the cap product $\cap \theta = ad \cdot (\cap 1)$ in (11.6) is an isomorphism if and only if a is odd. Moreover, if a is odd, then the cap product in (11.7) satisfies

$$
\mathrm{Im}\{\cap a/d\} = (a/d) \cdot \mathrm{Im}\{\cap 1\} = (a/d) \cdot (H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)/\mathrm{Ker}) = H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)/\mathrm{Ker}
$$

where the last equality holds since a/d is invertible in $H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}$.

So, by applying Theorem G, $\theta = a/d \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ lies in $\mathcal{R}_\omega(\Gamma)$ if and only if a is odd. This proves Theorem 11.1 (2) for $\kappa = \omega$.

11.3. Third homology for $\kappa \geq \omega + 1$

The goal of this subsection is to investigate the homology of $\widehat{\Gamma}$ and prove Theorem 11.1 (1) for $\kappa \geq \omega + 1$.

Recall that $\widehat{\Gamma}/\widehat{\Gamma}_\kappa = \widehat{\Gamma}$ for $\kappa \geq \omega + 1$ by Lemma 11.4, and that $\widehat{\Gamma} = \mathrm{colim}\, \Gamma(\ell)$ by Theorem 11.3, where $\Gamma(\ell)$ is the group defined by (11.1). We restate (11.1) for the reader's convenience:

$$
\Gamma(\ell) = \langle u, v, t \mid tut^{-1}u,\ tvt^{-1}v,\ [u, v]^{\ell^2},\ [[u, v], u],\ [[u, v], v],\ [[u, v], t] \rangle
\tag{11.1}
$$

To understand the homology of $\widehat{\Gamma}$, it is useful to consider an HNN extension described below. Let $A(\ell)$ be the subgroup of $\Gamma(\ell)$ generated by u and v, following [7]. From the presentation (11.1), it is immediately seen that $A(\ell)$ is a normal subgroup of $\Gamma(\ell)$, and $\Gamma(\ell)/A(\ell)$ is the infinite cyclic group generated by t:

$$
1 \to A(\ell) \to \Gamma(\ell) \to \mathbb{Z} \to 1
\tag{11.8}
$$

Note that $\Gamma(\ell) \to \Gamma(r\ell)$ induces an isomorphism

$$\Gamma(\ell)/A(\ell) \stackrel{\cong}{\longrightarrow} \Gamma(r\ell)/A(r\ell) = \mathbb{Z}$$

sending t to t. Let $\mathcal{A} = \operatorname{colim} A(\ell)$, and take the colimit of (11.8) to obtain

$$1 \to \mathcal{A} \to \widehat{\Gamma} \to \mathbb{Z} \to 1.$$

The Lyndon–Hochschild–Serre spectral sequence for this HNN extension gives the following Wang exact sequence:

$$\cdots \to H_i(\mathcal{A}) \xrightarrow{1-t_*} H_i(\mathcal{A}) \to H_i(\widehat{\Gamma}) \to H_{i-1}(\mathcal{A}) \xrightarrow{1-t_*} H_{i-1}(\mathcal{A}) \to \cdots \qquad (11.9)$$

where t_* is induced by the conjugation by t on $\mathcal{A}$.

To compute the homology of $\widehat{\Gamma}$ using (11.9), we first compute the homology of $\mathcal{A}$. From (11.1), it follows that $[u, v] \in A(\ell)$ generates a finite cyclic normal subgroup of order ℓ^2, and $A(\ell)$ is a central extension of this, by the free abelian group of rank 2 generated by u and v. Since $u \mapsto u^r$, $v \mapsto v^r$ and $[u, v] \mapsto [u^r, v^r] = [u, v]^{r^2}$ under $\Gamma(\ell) \to \Gamma(r\ell)$, we have the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_{\ell^2} & \longrightarrow & A(\ell) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1 \\
 & & \downarrow{\cdot r^2} & & \downarrow & & \downarrow{\cdot r} & & \\
1 & \longrightarrow & \mathbb{Z}_{(r\ell)^2} & \longrightarrow & A(r\ell) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1
\end{array}
\qquad (11.10)
$$

Consider the Lyndon–Hochschild–Serre spectral sequence of the top row of (11.10),

$$E_{p,q}^2 = H_p(\mathbb{Z}^2) \otimes H_q(\mathbb{Z}_{\ell^2}) \Longrightarrow H_n(A(\ell)). \qquad (11.11)$$

The E^2 and $E^\infty = E^3$ pages for $q \le 3$ are as follows:

$$
E^2 = \left|
\begin{array}{ccc}
\mathbb{Z}_{\ell^2} & \mathbb{Z}_{\ell^2}^2 & \mathbb{Z}_{\ell^2} \\
0 & 0 & 0 \\
\mathbb{Z}_{\ell^2} & \mathbb{Z}_{\ell^2}^2 & \mathbb{Z}_{\ell^2} \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}
\end{array}
\right.
\qquad
E^\infty = E^3 = \left|
\begin{array}{ccc}
\mathbb{Z}_{\ell^2} & \mathbb{Z}_{\ell^2}^2 & \mathbb{Z}_{\ell^2} \\
0 & 0 & 0 \\
0 & \mathbb{Z}_{\ell^2}^2 & \mathbb{Z}_{\ell^2} \\
\mathbb{Z} & \mathbb{Z}^2 & \ell^2\mathbb{Z}
\end{array}
\right.
\qquad (11.12)
$$

with differential $d_{2,0}^2$.

All entries in (11.12) are immediately obtained from (11.11), except possibly $E_{p,q}^3$ for $(p, q) = (2, 0)$ and $(1, 0)$. To verify these, observe that $E_{0,1}^\infty = E_{0,1}^3$ must vanish since $H_1(A(k)) = \mathbb{Z}^2$ and $E_{1,0}^\infty = E_{1,0}^2 = \mathbb{Z}^2$. From this it follows that the differential $d_{2,0}^2$ is surjective, so its kernel $E_{0,1}^3$ is the subgroup $\ell^2\mathbb{Z}$ of $\mathbb{Z}$.

From the E^∞ page, it follows that $H_2(A(\ell))$ is an extension of $\mathbb{Z}_{\ell^2}^2 = H_1(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{\ell^2})$ by $\ell^2\mathbb{Z} \subset \mathbb{Z} = H_2(\mathbb{Z}^2)$. From (11.10), it follows that $H_1(\mathbb{Z}^2) \to H_1(\mathbb{Z}^2)$ induced by $A(\ell) \to A(r\ell)$ is multiplication by r, while $H_2(\mathbb{Z}^2) \to H_2(\mathbb{Z}^2) = \mathbb{Z}$ and

$H_1(\mathbb{Z}_{\ell^2}) \to H_1(\mathbb{Z}_{(r\ell)^2})$ are multiplication by r^2. So (11.10) gives rise to the diagram

$$
\begin{array}{ccc}
H_1(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{\ell^2}) & & \mathbb{Z} = H_2(\mathbb{Z}^2) \\
\| & & \cup \\
0 \longrightarrow \mathbb{Z}_{\ell^2}^2 \longrightarrow H_2(A(\ell)) \longrightarrow \ell^2\mathbb{Z} \longrightarrow 0 \\
\Big\downarrow {\cdot r^3} \quad \Big\downarrow \quad \cong \Big\downarrow {\cdot r^2} \\
0 \longrightarrow \mathbb{Z}_{(r\ell)^2}^2 \longrightarrow H_2(A(r\ell)) \longrightarrow (r\ell)^2\mathbb{Z} \longrightarrow 0 \\
\| & & \cap \\
H_1(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{(r\ell)^2}) & & \mathbb{Z} = H_2(\mathbb{Z}^2)
\end{array}
\tag{11.13}
$$

The top row of (11.13) for $\ell = 1$ provides an isomorphism $H_2(A(1)) = H_2(\mathbb{Z}^2) \xrightarrow{\cong} \mathbb{Z}$. The colimit map $\mathbb{Z} = 1^2\mathbb{Z} \to \operatorname{colim} \ell^2\mathbb{Z}$ is an isomorphism, since the map $\cdot r^2$ in (11.13) is an isomorphism for all r. On the other hand, since the map $\cdot r^3 \colon \mathbb{Z}_{\ell^2}^2 \to \mathbb{Z}_{(r\ell)^2}^2$ is zero for all large r divisible by ℓ^2, $\operatorname{colim} \mathbb{Z}_{\ell^2}^2$ vanishes. So, by taking the colimit of (11.13), we obtain an isomorphism

$$
\mathbb{Z} = H_2(A(1)) \xrightarrow{\cong} \operatorname*{colim}_{\ell \geq 1 \text{ odd}} H_2(A(\ell)) = H_2(\mathcal{A}).
\tag{11.14}
$$

Moreover, since t on $A(1) = \mathbb{Z}^2$ is the negation, t_* on $H_2(\mathcal{A}) = \mathbb{Z}$ is the identity.

To compute $H_3(\mathcal{A})$, first observe that the E^∞ page in (11.12) tells us that $H_3(A(\ell))$ is an extension of $H_3(\mathbb{Z}_{\ell^2}) = \mathbb{Z}_{\ell^2}$ by $H_2(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{\ell^2}) = \mathbb{Z}_{\ell^2}$. Recall the fact that $H_i(\mathbb{Z}_{\ell^2}) = \mathbb{Z}_{\ell^2} \to H_i(\mathbb{Z}_{(r\ell)^2}) = \mathbb{Z}_{(r\ell)^2}$ induced by the injection $\cdot r^2 \colon \mathbb{Z}_{\ell^2} \to \mathbb{Z}_{(r\ell)^2}$ is multiplication by r^2 for $i = 1, 3$. Also, $\cdot r \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ induces multiplication by r^2 on $H_2(\mathbb{Z}^2)$. From this, it follows that (11.10) gives rise to

$$
\begin{array}{ccc}
H_3(\mathbb{Z}_{\ell^2}) & & H_2(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{\ell^2}) \\
\| & & \| \\
0 \longrightarrow \mathbb{Z}_{\ell^2} \longrightarrow H_3(A(\ell)) \longrightarrow \mathbb{Z}_{\ell^2} \longrightarrow 0 \\
\Big\downarrow {\cdot r^2} \quad \Big\downarrow \quad \Big\downarrow {\cdot r^4} \\
0 \longrightarrow \mathbb{Z}_{(r\ell)^2} \longrightarrow H_3(A(r\ell)) \longrightarrow \mathbb{Z}_{(r\ell)^2} \longrightarrow 0 \\
\| & & \| \\
H_3(\mathbb{Z}_{(r\ell)^2}) & & H_2(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{(r\ell)^2})
\end{array}
\tag{11.15}
$$

Since the vertical map $\cdot r^4$ in (11.15) is trivial if r is divisible by ℓ, the colimit of them is trivial. So, by taking the colimit of (11.15), we obtain an isomorphism

$$
\mathbb{Z}_{(2)}/\mathbb{Z} = \operatorname*{colim}_{\ell \geq 1 \text{ odd}} \mathbb{Z}_{\ell^2} = \operatorname*{colim}_{\ell \geq 1 \text{ odd}} H_3(\mathbb{Z}_{\ell^2}) \xrightarrow{\cong} H_3(\mathcal{A}).
\tag{11.16}
$$

Note that the action of t on $\mathbb{Z}_{\ell^2} \subset A(\ell)$ is trivial, since $t[u, v]t^{-1} = [u^{-1}, v^{-1}] = [u, v]$. It follows that t_* on $H_3(\mathcal{A}) = \mathbb{Z}_{(2)}/\mathbb{Z}$ is the identity.

Now, use (11.14), (11.16) and the fact that $1 - t_* = 0$ on both $H_2(\mathcal{A})$ and $H_3(\mathcal{A})$, to extract the following exact sequence from the Wang sequence (11.9):

$$0 \longrightarrow \underset{\|}{\mathbb{Z}_{(2)}/\mathbb{Z}} \longrightarrow H_3(\widehat{\Gamma}) \longrightarrow \underset{\|}{\mathbb{Z}} \longrightarrow 0 \qquad (11.17)$$
$$H_3(\mathcal{A}) \qquad\qquad\qquad H_2(\mathcal{A})$$

So, $H_3(\widehat{\Gamma})$ is isomorphic to $(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$. To provide a fixed identification, we use a splitting described below. Recall that $\Gamma(1) = \Gamma$, so $H_3(\Gamma(1)) = H_3(\Gamma) = H_3(Y)$ where Y is the torus bundle. Compare (11.17) with the Wang sequence associated to the exact sequence (11.8) for $\ell = 1$ to obtain the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_3(Y) & \overset{\cong}{\longrightarrow} & H_2(A(1)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & H_3(\mathcal{A}) & \longrightarrow & H_3(\widehat{\Gamma}) & \longrightarrow & H_2(\mathcal{A}) & \longrightarrow & 0
\end{array}
$$

From this, it is straightforward to see that the composition of the inverses of the two isomorphisms and $H_3(Y) \to H_3(\widehat{\Gamma})$ is a splitting. It gives an identification

$$H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z} \qquad (11.18)$$

such that $(0, 1) \in (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ represents the image of the fundamental class $[Y]$. This shows Theorem 11.1 (1) for $\kappa \geq \omega + 1$.

For use in the next subsection, we compute $H_2(\widehat{\Gamma})$ here. By taking the abelianization, it is straightforward to see that $H_1(\mathcal{A}) = \mathbb{Z}_{(2)}^2$ and t_* on $H_1(\mathcal{A})$ is the negation. So, from the Wang sequence (11.9), it follows that

$$H_2(\widehat{\Gamma}) = H_2(\mathcal{A}) = H_2(A(1)) = H_2(\mathbb{Z}^2) = \mathbb{Z}. \qquad (11.19)$$

11.4. Realizable classes for $\kappa \geq \omega + 1$

In this subsection, we study the set $\mathcal{R}_\kappa(\Gamma)$ of realizable classes for $\kappa \geq \omega + 1$ to prove Theorem 11.1 (2) for $\kappa \geq \omega + 1$ and Theorem 11.1 (3).

To determine realizable classes in $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) = H_3(\widehat{\Gamma})$ for $\kappa \geq \omega + 1$, consider the cap product

$$\cap\, \theta : t H^2(\widehat{\Gamma}) \to t H_1(\widehat{\Gamma}) = t H_1(\Gamma) = \mathbb{Z}_2^2. \qquad (11.20)$$

If $\theta \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ lies in the subgroup $\mathbb{Z}_{(2)}/\mathbb{Z}$, then $k\theta = 0$ for some odd $k > 0$. Since $t H_1(\widehat{\Gamma})$ is a 2-group, it follows that (11.20) is zero for $\theta \in \mathbb{Z}_{(2)}/\mathbb{Z}$. On the other hand, $\theta = (0, 1) \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ lies in $\mathcal{R}_\kappa(\Gamma)$ since θ is the image of the fundamental class $[Y]$. So (11.20) is an isomorphism for $\theta = (0, 1)$ by Theorem G. Since $t H_1(\widehat{\Gamma})$ is a finite abelian 2-group, it follows that (11.20) is an isomorphism for $\theta = (0, r)$ if and only if $r \in \mathbb{Z}$ is odd. Combining these observations, it follows that (11.20) is an isomorphism for $\theta = (x, r) \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ if and only if $r \in \mathbb{Z}$ is odd.

Now consider the cap product

$$\cap \, \theta \colon H^1(\widehat{\Gamma}) \to H_2(\widehat{\Gamma})/\mathrm{Ker}\{H_2(\widehat{\Gamma}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)\}. \tag{11.21}$$

By (11.19), $H_2(\widehat{\Gamma}) = H_2(\mathbb{Z}^2) = \mathbb{Z}$. By (11.5), $H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = H_2(\mathbb{Z}^2_{(2)}) = \mathbb{Z}_{(2)}$. Since $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2_{(2)}$ induces the standard inclusion on these H_2, $\mathrm{Ker}\{H_2(\widehat{\Gamma}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)\}$ is trivial, and thus the codomain of (11.21) is equal to $H_2(\widehat{\Gamma}) = \mathbb{Z}$. From this, it follows that the cap product (11.21) is zero for $\theta = (x, 0) \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$, since x is torsion. By Theorem G, the cap product (11.21) is surjective for $\theta = (0, 1)$, since this θ lies in $\mathcal{R}_\kappa(\Gamma)$. So, for a general class $\theta = (x, r) \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$, (11.21) is surjective if and only if $r = \pm 1$. By Theorem G, it is the case if and only if $\theta \in \mathcal{R}_\kappa(\Gamma)$. This proves Theorem 11.1 (2) for $\kappa = \omega + 1$. For $\kappa > \omega + 1$, the computation proceeds along the same lines. The only exception is that we need to replace $\mathrm{Ker}\{H_2(\widehat{\Gamma}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)\}$ in (11.21) by $\mathrm{Ker}\{H_2(\widehat{\Gamma}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\lambda)\}$ with $\lambda < \kappa$. But, since the kernel is already trivial for $\lambda = \omega$ by the above computation, the same argument applies to the case of $\kappa > \omega + 1$ as well. This completes the proof of Theorem 11.1 (2) for $\kappa \geq \omega + 1$.

To compute the map $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ induced by the projection $\widehat{\Gamma} = \widehat{\Gamma}/\widehat{\Gamma}_{\omega+1} \to \widehat{\Gamma}/\widehat{\Gamma}_\omega$, recall from (11.18) that $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ where the $\mathbb{Z}$ factor is identified with $H_2(\mathcal{A}) = H_2(\mathbb{Z}^2)$ via the Wang sequence (11.9). Also, recall from (11.5) that $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = H_2(\mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}$. So, $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}) \times \mathbb{Z} \to H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}$ is given by $(a, r) \mapsto r$, and $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ is the restriction. This shows Theorem 11.1 (3) for $\kappa = \omega$. Since $\widehat{\Gamma}/\widehat{\Gamma}_\kappa = \widehat{\Gamma}$ for $\kappa \geq \omega + 1$, Theorem 11.1 (3) for $\kappa \geq \omega + 1$ is obviously true.

11.5. Equivalence relation and automorphism action for $\kappa = \omega$

In this subsection, we investigate the equivalence relation $\sim$ on $\mathcal{R}_\omega(\Gamma)$ and prove Theorem 11.1 (4, 5).

Recall that $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}^\times_{(2)} \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}$ by Theorem 11.1 (1, 2). Fix $\theta = p/q \in \mathcal{R}_\omega(\Gamma)$, where $p, q \in 2\mathbb{Z} + 1$.

To determine the equivalence class of θ as a subset of $\mathcal{R}_\omega(\Gamma)$, we will use an automorphism of $\widehat{\Gamma}/\widehat{\Gamma}_\omega$, which is equal to $\mathbb{Z}^2_{(2)} \rtimes \mathbb{Z}$ by Lemma 11.4. Define

$$\phi_{p/q} \colon \mathbb{Z}^2_{(2)} \rtimes \mathbb{Z} \to \mathbb{Z}^2_{(2)} \rtimes \mathbb{Z} \quad \text{by} \quad \phi_{p/q}(a, b, r) = ((p/q) \cdot a, b, r)$$

for $a, b \in \mathbb{Z}_{(2)}$, $r \in \mathbb{Z}$. It is straightforward to verify that $\phi_{p/q}$ is an automorphism with inverse $\phi^{-1}_{p/q} = \phi_{q/p}$. We claim that $\phi_{p/q}$ induces $1 \mapsto p/q = \theta$ on $H_3(\mathbb{Z}^2_{(2)} \rtimes \mathbb{Z}) = \mathbb{Z}_{(2)}$. To see this, observe that the restriction of $\phi_{p/q}$ to the subgroup $\mathbb{Z}^2_{(2)}$ induces an automorphism of $H_2(\mathbb{Z}^2_{(2)}) = \mathbb{Z}_{(2)}$ given by $1 \mapsto p/q$. Since $H_3(\mathbb{Z}^2_{(2)} \rtimes \mathbb{Z}) = H_2(\mathbb{Z}^2_{(2)})$ by (11.5), the claim follows from this.

To avoid confusion, for a closed 3-manifold M with $\pi = \pi_1(M)$ equipped with an isomorphism $f \colon \widehat{\pi}/\widehat{\pi}_\omega \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\omega$, denote the invariant $\theta_\omega(M)$ by $\theta_\omega(M, f)$ temporarily. Then, for $M = Y$, the above claim implies that $\theta_\omega(Y, \phi_{p/q}) = \theta$, since $1 \in \mathbb{Z}_{(2)} = H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ represents the image of the fundamental class $[Y]$. So, by definition, the equiv-

alence class $I_\theta = \{\theta' \mid \theta' \sim \theta\}$ of θ in $\mathcal{R}_\omega(\Gamma)$ is equal to the image of the composition

$$\mathcal{R}_{\omega+1}(\Gamma) \longrightarrow \mathcal{R}_\omega(\Gamma) \xrightarrow{\;(\phi_{p/q})_*\;} \mathcal{R}_\omega(\Gamma)$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} \qquad\qquad \mathbb{Z}^\times_{(2)}$$

Here, the projection-induced map $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ is $(a, \pm 1) \mapsto \pm 1$ by Theorem 11.1 (3). From this, it follows that $I_\theta = \{\theta, -\theta\}$. This completes the proof of Theorem 11.1 (4).

In addition, using the above argument, it is straightforward to show Theorem 11.1 (5), which asserts that the action of $\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ on the set $\mathcal{R}_\omega(\Gamma)$ of realizable classes is transitive. Indeed, for an arbitrary $\theta = p/q \in \mathbb{Z}^\times_{(2)} = \mathcal{R}_\omega(\Gamma)$, since the above automorphism $\phi_{p/q}$ on $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ satisfies $\phi_{p/q}(1) = \theta$, it follows that θ and 1 have the same orbit.

12. Torus bundle example: the universal θ-invariant

We continue the study of the localization of the fundamental group Γ of the torus bundle Y defined in (10.1). The goal of this section is to understand the final invariant $\widehat{\theta}$ defined over $\widehat{\Gamma}$ and prove Theorem J, which we state again below for the reader's convenience. Recall that $\widehat{\mathcal{R}}(\Gamma)$ is the set of realizable values of $\widehat{\theta}$. Theorem J says: $\widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma})$ *is infinite. This detects the existence of infinitely many distinct homology cobordism classes of closed* 3-*manifolds M with $\pi = \pi_1(M)$ such that $\widehat{\pi} \cong \widehat{\Gamma}$, and thus $\theta_\kappa(M)$ is defined and vanishes in* $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ *for all ordinals κ. In particular, for every ordinal κ, the Milnor invariant $\overline{\mu}_\kappa(M)$ vanishes for these* 3-*manifolds M.*

We begin with computation of realizable classes in $H_3(\widehat{\Gamma})$. Recall that $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ by Theorem 11.1 (1).

Theorem 12.1. $\widehat{\mathcal{R}}(\Gamma) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} \subset H_3(\widehat{\Gamma})$.

Proof. In the argument used to prove Theorem 11.1 (1) in Section 11.4, we have shown that a homology class $\theta \in H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ lies in $(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\}$ if and only if $\cap\, \theta : t H^2(\widehat{\Gamma}) \to t H_1(\widehat{\Gamma})$ is an isomorphism and $\cap\, \theta : H^1(\widehat{\Gamma}) \to H_2(\widehat{\Gamma})$ is an epimorphism. By Theorem H, it follows that $\widehat{\mathcal{R}}(\Gamma) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\}$. ∎

The next theorem describes the action of $\mathrm{Aut}(\widehat{\Gamma})$ on $H_3(\widehat{\Gamma})$ and $\widehat{\mathcal{R}}(\Gamma)$. To state the result, we use the following notation. For a group G, denote its abelianization by G_{ab}. Recall from Section 11.3 that $\widehat{\Gamma}$ is an HNN extension of a subgroup $\mathcal{A}$ such that $\mathcal{A}_{\mathrm{ab}} = \mathbb{Z}^2_{(2)}$ with basis $\{u, v\}$. We will show in Lemma 12.3 that if $f : \widehat{\Gamma} \to \widehat{\Gamma}$ an automorphism, then f induces an automorphism $f_{\mathcal{A}} \in \mathrm{GL}(2, \mathbb{Z}_{(2)})$ on $\mathcal{A}_{\mathrm{ab}} = \mathbb{Z}^2_{(2)}$ satisfying $\det f_{\mathcal{A}} = \pm 1$, and f induces an automorphism $f_{\mathbb{Z}}$ on the quotient $\widehat{\Gamma}/\mathcal{A} = \mathbb{Z}$. Define

$$\delta_f := \det f_{\mathcal{A}} \in \{1, -1\}, \quad \epsilon_f := f_{\mathbb{Z}}(1) \in \{1, -1\}.$$

One readily sees that $\mathrm{Aut}(\widehat{\Gamma}) \to \{-1, 1\}^2 \cong \mathbb{Z}_2^2$ given by $f \mapsto (\delta_f, \epsilon_f)$ is a surjective group homomorphism onto the Klein 4-group. Indeed, for a given pair $(\delta, \epsilon) \in \{1, -1\}^2$, the automorphism $\Gamma \to \Gamma$ defined by $u \mapsto u^\delta$, $v \mapsto v$, $t \mapsto t^\epsilon$ gives rise to an automorphism $f : \widehat{\Gamma} \to \widehat{\Gamma}$ satisfying $(\delta_f, \epsilon_f) = (\delta, \epsilon)$.

Theorem 12.2. *Suppose f is an automorphism on $\widehat{\Gamma}$. Then the induced automorphism f_* on $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ is given by $f_*(a, n) = (\delta_f \cdot a, \delta_f \cdot \epsilon_f \cdot n)$. Consequently, there are bijections*

$$H_3(\widehat{\Gamma})/\mathrm{Aut}(\widehat{\Gamma}) \approx \{(a, n) \in \mathbb{Z}_{(2)} \times \mathbb{Z} \mid 0 \le a < 1/2,\ n \ge 0\},$$
$$\widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma}) \approx \{a \in \mathbb{Z}_{(2)} \mid 0 \le a < 1/2\}.$$

The first statement says that the natural map $\mathrm{Aut}(\widehat{\Gamma}) \to \mathrm{Aut}(H_3(\widehat{\Gamma}))$ factors through the Klein 4-group $\{1, -1\}^2$, via $f \mapsto (\delta_f, \epsilon_f)$. The two bijections in Theorem 12.2 are immediately obtained from the first statement.

The first sentence of Theorem J, which asserts that $\widehat{\mathcal{R}}(\Gamma)/\mathrm{Aut}(\widehat{\Gamma})$ is infinite, is an immediate consequence of Theorem 12.2. Also, from this, the second statement of Theorem J follows immediately by Theorems 8.1.

The remaining part of this section is devoted to the proof of the first statement of Theorem 12.2.

Recall that $\Gamma(\ell)$ is the subgroup of $\widehat{\Gamma}$ given by (11.1), $\Gamma(1) = \Gamma$, and $\widehat{\Gamma}$ is the colimit of $\Gamma(\ell)$. If $f : \widehat{\Gamma} \to \widehat{\Gamma}$ is an automorphism, then for each odd $\ell \ge 1$, $f(\Gamma(\ell)) \subset \Gamma(r\ell)$ for some odd $r \ge 1$, since $\Gamma(\ell)$ is finitely generated. The restriction $f : \Gamma(\ell) \to \Gamma(r\ell)$ induces isomorphisms on H_1 and H_2, since so does the colimit inclusion $\Gamma(\ell) \to \widehat{\Gamma}$ for every ℓ. So $f : \Gamma(\ell) \to \Gamma(r\ell)$ is 2-connected. Conversely, if $f : \Gamma(\ell) \to \Gamma(r\ell)$ is 2-connected, then it induces an automorphism $f : \widehat{\Gamma} = \widehat{\Gamma(\ell)} \xrightarrow{\cong} \widehat{\Gamma(r\ell)} = \widehat{\Gamma}$, by Theorem 3.1 (1).

This leads us to investigate 2-connected homomorphisms $f : \Gamma(\ell) \to \Gamma(r\ell)$. We begin with a characterization. Recall from the presentation (11.1) that $\Gamma(\ell)$ has generators u, v and t. Let $A(\ell)$ be the subgroup generated by u and v, as done in Section 11.3.

Lemma 12.3. *A homomorphism $f : \Gamma(\ell) \to \Gamma(r\ell)$ is 2-connected if and only if f is given by*

$$\begin{aligned}
f(t) &= t^\epsilon u^p v^q [u, v]^j, \\
f(u) &= u^a v^b [u, v]^m, \\
f(v) &= u^c v^d [u, v]^n,
\end{aligned} \tag{12.1}$$

where $\epsilon, a, b, c, d, j, m, n$ are integers satisfying

$$\begin{aligned}
\epsilon = \pm 1, \quad ad - bc = \pm r^2, \\
2m \equiv aq - bp + ab, \quad 2n \equiv cq - dp + cd \quad \mathrm{mod}\ (r\ell)^2.
\end{aligned} \tag{12.2}$$

Often we will abuse the notation, denoting by f the automorphism of $\widehat{\Gamma}$ induced by a 2-connected map $f : \Gamma(\ell) \to \Gamma(r\ell)$. Note that if f is given by (12.1), then it induces

automorphisms $\frac{1}{r} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ on $H_1(\mathcal{A}) = \mathbb{Z}^2_{(2)}$ and $1 \mapsto \epsilon$ on $\mathbb{Z} = \widehat{\Gamma}/\mathcal{A}$. So, we have

$$\delta_f = \frac{ad - bc}{r^2}, \quad \epsilon_f = \epsilon. \tag{12.3}$$

Proof of Lemma 12.3. Observe that any $g \in \Gamma(\ell)$ can be written as $g = t^\epsilon u^p v^q [u, v]^j$, by using the defining relations in (11.1). Also, $t^\epsilon u^p v^q [u, v]^j$ lies in the subgroup $A(\ell)$ if and only if $\epsilon = 0$. We claim that f sends $A(\ell)$ to $A(r\ell)$. From the claim, it follows that $f(t)$, $f(u)$ and $f(v)$ are of the form of (12.1) for some exponents (without enforcing (12.2) for now).

To show the claim, consider the *first rational derived subgroup* of a group G, which is defined to be the kernel of the natural map $G \to H_1(G) \otimes \mathbb{Q}$. That is, it is the minimal normal subgroup of G such that the quotient is abelian and torsion free. It is straightforward to see that the first rational derived subgroup is characteristic. In our case, for $\Gamma(\ell)$, the first rational derived subgroup is equal to $A(\ell)$. So $f(A(\ell)) \subset A(r\ell)$, as claimed.

Next, we claim that a map of the free group on t, u and v to $\Gamma(r\ell)$ given by (12.1) kills relations of $\Gamma(\ell)$ if and only if $ad - bc \equiv 0 \mod r^2$ and

$$2m \equiv aq - bp + ab, \quad 2n \equiv cq - dp + cd \mod (r\ell)^2.$$

The claim is shown by a routine computation. The map sends the relation $tut^{-1}u$ to

$$t^\epsilon u^p v^q [u, v]^j \cdot u^a v^b [u, v]^m \cdot (t^\epsilon u^p v^q [u, v]^j)^{-1} \cdot u^p v^q [u, v]^\ell = [u, v]^{2m - aq + bp + ab},$$

which is trivial in $\Gamma(r\ell)$ if and only if $(r\ell)^2$ divides $2m - aq + bp + ab$. Similarly $(r\ell)^2$ divides $2n - cq + dp + cd$ if and only if $tvt^{-1}v$ is sent to the identity. Also the relation $[u, v]^{\ell^2}$ of $\Gamma(\ell)$ is sent to $[u^a v^b, u^c v^d]^{\ell^2} = [u, v]^{\ell^2(ad-bc)}$, which is trivial if and only if r^2 divides $ad - bc$, since $[u, v]$ has order $(r\ell)^2$ in $\Gamma(r\ell)$. This proves the claim.

Recall that $H_1(\Gamma(\ell)) = (\mathbb{Z}_2)^2 \times \mathbb{Z}$ where the factors are generated by u, v, and t respectively. So, when f is the homomorphism given by (12.1), $f_*: H_1(\Gamma(\ell)) \to H_1(\Gamma(r\ell))$ is represented by

$$\begin{bmatrix} a & c & p \\ b & d & q \\ 0 & 0 & \epsilon \end{bmatrix}.$$

Therefore, f induces an isomorphism on H_1 if and only if $\epsilon = \pm 1$ and $ad - bc$ is odd.

To investigate the induced map on H_2, first note that $A(\ell)_{ab}$ is equal to $\mathbb{Z}^2$ generated by u and v. We will use the fact that $H_2(\Gamma(\ell))$ can be identified with the subgroup $\ell^2 \mathbb{Z} \subset \mathbb{Z} = H_2(A(\ell)_{ab})$. This can be proven by investigating the Wang sequence for the HNN extension (11.8). An alternative proof is as follows. Recall that $H_2(\Gamma) = H_2(\widehat{\Gamma}) = \mathbb{Z}$ by (11.19). Since $\Gamma = \Gamma(1) \to \Gamma(\ell)$ and $\Gamma(\ell) \to \widehat{\Gamma}$ are 2-connected, it follows that $H_2(\Gamma(\ell))$ is equal to $H_2(\Gamma(1))$, which is equal to $H_2(A(1)) = H_2(\mathbb{Z}^2)$ by (11.19). Note that $\mathbb{Z}^2 = A(1) \to A(\ell)_{ab} = \mathbb{Z}^2$ is scalar multiplication by ℓ. So, $H_2(\Gamma(\ell))$ is the subgroup $\ell^2 \mathbb{Z} \subset \mathbb{Z} = H_2(A(\ell)_{ab})$.

Now, observe that $H_2(A(\ell)_{ab}) = \mathbb{Z} \to H_2(A(r\ell)_{ab}) = \mathbb{Z}$ induced by f given by (12.1) is equal to multiplication by $ad - bc$. From this, it follows that f induces an epimorphism $H_2(\Gamma(\ell)) \to H_2(\Gamma(r\ell))$ if and only if $ad - bc = \pm r^2$. ∎

Using Lemma 12.3, we will investigate the action of $\mathrm{Aut}(\widehat{\Gamma})$ on the torsion part of $H_3(\widehat{\Gamma})$.

Lemma 12.4. *Suppose $f: \widehat{\Gamma} \to \widehat{\Gamma}$ is an automorphism. Then the induced automorphism f_* on the torsion subgroup $t H_3(\widehat{\Gamma}) = \mathbb{Z}_{(2)}/\mathbb{Z}$ is multiplication by $\delta_f \in \{\pm 1\}$.*

Proof. Fix an arbitrary odd $\ell \geq 1$. By Lemma 12.1, the given automorphism f on $\widehat{\Gamma}$ restricts to a 2-connected homomorphism $f|_{\Gamma(\ell)}: \Gamma(\ell) \to \Gamma(r\ell)$ for some odd $r \geq 0$, and $f|_{\Gamma(\ell)}$ is of the form of (12.1). We have

$$f|_{\Gamma(\ell)}([u, v]) = [u^a v^b, u^c v^d] = [u, v]^{ad - bc} = [u, v]^{\delta_f \cdot r^2}.$$

Recall from Section 11.3 that $[u, v] \in \Gamma(\ell)$ generates a subgroup that we identified with $\mathbb{Z}_{\ell^2}$. So, $f|_{\Gamma(\ell)}$ induces $\mathbb{Z}_{\ell^2} \to \mathbb{Z}_{(r\ell)^2}$ given by $1 \mapsto \delta_f \cdot r^2$. It induces the inclusion $H_3(\mathbb{Z}_{\ell^2}) = \mathbb{Z}_{\ell^2} \to H_3(\mathbb{Z}_{(r\ell)^2}) = \mathbb{Z}_{(r\ell)^2}$ given by $1 \mapsto \delta_f \cdot r^2$. This is the map $\frac{1}{\ell^2} \mapsto \delta_f \cdot \frac{1}{\ell^2}$ when $H_3(\mathbb{Z}_{\ell^2}) = \mathbb{Z}_{\ell^2}$ and $H_3(\mathbb{Z}_{(r\ell)^2}) = \mathbb{Z}_{(r\ell)^2}$ are regarded as subgroups of $\mathbb{Z}_{(2)}/\mathbb{Z}$ using (11.16). By (11.17) and (11.16), it follows that the induced map $f_*: t H_3(\widehat{\Gamma}) \to t H_3(\widehat{\Gamma}) = \mathbb{Z}_{(2)}/\mathbb{Z}$ is multiplication by δ_f. ∎

By (11.18), the $\mathbb{Z}$ factor of $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ is generated by the image of the fundamental class $[Y] \in H_3(Y) = H_3(\Gamma)$. The rest of this section is devoted to understanding the action of $\mathrm{Aut}(\widehat{\Gamma})$ on this generator. Since every automorphism of $\widehat{\Gamma}$ is induced by a 2-connected map $f: \Gamma = \Gamma(1) \to \Gamma(r)$, it suffices to investigate $f_*[Y] \in H_3(\Gamma(r))$.

Our strategy is to simplify f given in Lemma 12.3 without altering $f_*[Y]$. We begin with elimination of the $[u, v]^j$ factor in $f(t)$ in the general form (12.1).

Lemma 12.5. *Let $f: \Gamma(1) \to \Gamma(r)$ be a 2-connected map given by (12.1). Let $f': \Gamma(1) \to \Gamma(r)$ be the map*

$$f'(t) = t^\epsilon u^p v^q, \quad f'(u) = u^a v^b [u, v]^m, \quad f'(v) = u^c v^d [u, v]^n. \qquad (12.4)$$

Then f is 2-connected, and f and f' induce the same homomorphism $f_* = f'_*: H_3(\Gamma(1)) \to H_3(\Gamma(r))$.

Proof. By Lemma 12.3, the assignment (12.4) gives a well defined 2-connected homomorphism, since the conditions in (12.2) do not involve the exponent j.

Recall that $B\Gamma(1) = Y = T^2 \times [0, 1]/\{(h(z), 0) \sim (z, 1)\}$ where $h: T^2 \to T^2 = S^1 \times S^1$ is the monodromy $h(\zeta, \xi) = (\zeta^{-1}, \xi^{-1})$. Here S^1 is regarded as the unit circle in $\mathbb{C}$. Use $(1, 1, 0)$ as a basepoint of $B\Gamma(1)$. Choose maps $B\Gamma(1) \to B\Gamma(r)$ realizing f and f', and denote them by f and f', abusing the notation.

Let $T^3 = T^2 \times S^1$, and use $(1, 1, 1) \in T^3$ as a basepoint. Denote by x, y the standard basis of $\pi_1(T^2) = \mathbb{Z}^2$, and denote by s the generator of $\pi_1(S^1) = \mathbb{Z}$, so that x, y and s form a basis of $\pi_1(T^3)$. The element $[u, v] \in \Gamma(r)$ is central by (11.1), and $f(u), f(v) \in \Gamma(r)$ commute since $u, v \in \Gamma(1)$ commute. It follows that there is a map $g: T^3 \to B\Gamma(r)$ which induces $\pi_1(T^3) = \mathbb{Z}^3 \to \Gamma(r)$ given by $x \mapsto f(u)^{-1}, y \mapsto f(u)^{-1}$ and $s \mapsto [u, v]^j$.

By homotopy if necessary, we may assume $g|_{T^2 \times 1}$ is equal to $f'|_{T^2 \times 1}$, since $f = f'$ on u and v. Define $F \colon B\Gamma(1) \to B\Gamma(r)$ to be the composition

$$F \colon B\Gamma(1) = T^2 \times [0,1] / \{(h(z), 0) \sim (z, 1)\}$$

$$\xrightarrow{\quad q \quad} \left(T^2 \times [0, \tfrac{1}{2}] / \{(h(z), 0) \sim (z, \tfrac{1}{2})\}\right) \underset{T^2 \times \frac{1}{2}}{\cup} \left(T^2 \times [\tfrac{1}{2}, 1] / \{(z, \tfrac{1}{2}) \sim (z, 1)\}\right)$$

$$= Y \underset{T^2}{\cup} T^3 \xrightarrow{f' \cup g} B\Gamma(r)$$

where q is the quotient map induced by $(z, t) \mapsto (z, t)$. Observe that the induced map $F \colon \Gamma(1) \to \Gamma(r)$ satisfies $F(u) = f'(u) = f(u)$, $F(v) = f'(v) = f(v)$, and $F(t) = f'(t)g(s) = f(t)$. It follows that f and F are homotopic. Therefore, on H_3, we have

$$f_*[Y] = F_*[Y] = f'_*[Y] + g_*[Y] \in H_3(\Gamma(r)).$$

So it suffices to prove that $g_* \colon H_3(T^3) \to H_3(\Gamma(r))$ is zero. To show this, first observe that $g \colon \pi_1(T^3) \to \Gamma(r)$ sends $\pi_1(T^3) = \mathbb{Z}^3$ to the subgroup $A(r)$. In addition, it induces a morphism of central extensions:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\
& & \Big\downarrow{\cdot j} & & \Big\downarrow{g} & & \Big\downarrow{\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]} & & \\
0 & \longrightarrow & \mathbb{Z}_{r^2} & \longrightarrow & A(r) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0
\end{array}
\qquad (12.5)
$$

Here the top row corresponds to the trivial fibration $S^1 \hookrightarrow T^3 \to T^2$, and the bottom row is the exact sequence in (11.10). The leftmost and rightmost vertical maps are multiplication by j and $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$, by the definition of g and description (12.1) of f. The map g induces a morphism of the spectral sequences. In particular, on $E^2_{2,1}$, g induces a map

$$\mathbb{Z} = H_2(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}) \to H_2(\mathbb{Z}^2) \otimes H_1(\mathbb{Z}_{r^2}) = \mathbb{Z}_{r^2}. \qquad (12.6)$$

This is scalar multiplication by $(ad - bc)j$, by the above descriptions of the vertical maps in (12.5). By Lemma 12.3, $ad - bc = \pm r^2$. It follows that (12.6) is a zero map. Since $E^2_{2,1}$ for $\mathbb{Z}^3$ is equal to $H_3(T^3)$, it follows that $g_* \colon H_3(T^3) \to H_3(A(r))$ is zero. ∎

The next step of our reduction is described by the following lemma.

Lemma 12.6. *Suppose* $f \colon \Gamma = \Gamma(1) \to \Gamma(r)$ *is a 2-connected homomorphism. Then there is a 2-connected homomorphism* $f' \colon \Gamma \to \Gamma(r)$ *such that* $f'(t) = t^{\epsilon_f}$, $\delta_f = \delta_{f'}$ *and* $f_* = f'_*$ *on* $H_3(\Gamma)$.

Proof. Let $\epsilon = \epsilon_f$, and apply Lemmas 12.3 and 12.5 to assume

$$f(t) = t^\epsilon u^p v^q, \quad f(u) = u^a v^b [u,v]^m, \quad f(v) = u^c v^d [u,v]^n. \qquad (12.7)$$

We claim that we may assume that both p and q are even in (12.7). To show this, consider $\phi \colon \Gamma \to \Gamma$ given by $\phi(t) = tu$, $\phi(u) = u$, $\phi(v) = v$. It is a well defined 2-connected

homomorphism by Lemma 12.3. Moreover, it induces the identity on $H_3(\Gamma) = \mathbb{Z}$. This can be seen geometrically, by inspecting the fundamental class $[Y]$ under an appropriate map $xB\Gamma = Y \to Y$ realizing ϕ. Alternatively, use the Wang sequence for the extension $1 \to A(1) \to \Gamma \to \mathbb{Z}$ to identify $H_3(\Gamma)$ with $H_2(A(1)) = H_2(\mathbb{Z}^2) = \mathbb{Z}$, and use the fact that $\phi|_{A(1)}$ is the identity. Now, since $\phi_* = \mathrm{id}$ on H_3, it follows that $f_* = (f \circ \phi)_*$ on $H_3(\Gamma)$. Similarly, define a 2-connected homomorphism $\phi' : \Gamma \to \Gamma$ by $\phi'(t) = tv$, $\phi'(u) = u$, $\phi'(v) = v$. Then $f_* = (f \circ \phi')_*$ on H_3, too. We have

$$(f \circ \phi)(t) = f(tu) = t^\epsilon u^p v^q \cdot u^a v^b [u, v]^m = t^\epsilon u^{p+a} v^{q+b} [u, v]^{m-aq},$$

$$(f \circ \phi')(t) = f(tv) = t^\epsilon u^p v^q \cdot u^c v^d [u, v]^m = t^\epsilon u^{p+c} v^{q+d} [u, v]^{m-cq}.$$

By (12.2), $ad - bc$ is odd. We assume a and d are odd, and b is even, since arguments for other cases are identical. Then composition with ϕ alters the parity of p and preserves the parity of q, and composition with ϕ' alters the parity of q (while the parity of p is left uncontrolled). So, by composition, we may assume that both p and q are even. Note that a, b, c, d and ϵ are left unchanged. Finally, apply Lemma 12.5 to obtain the form of (12.7). This proves the claim.

Now, define $\psi : \Gamma(r) \to \Gamma(r)$ to be $\psi(g) = ugu^{-1}$. Since conjugation induces the identity on H_* (see e.g. [29, p. 191]), we have $(\psi \circ f)_* = f_*$ on H_3. Also, we have

$$(\psi \circ f)(t) = u \cdot t^\epsilon u^p v^q \cdot u^{-1} = t^\epsilon u^{p-2} v^q [u, v]^q,$$

$$(\psi \circ f)(u) = u \cdot u^a v^b [u, v]^m \cdot u^{-1} = u^a v^b [u, v]^{m+b},$$

$$(\psi \circ f)(v) = u \cdot u^c v^d [u, v]^n \cdot u^{-1} = u^c v^d [u, v]^{n+d}.$$

Apply Lemma 12.5, to eliminate $[u, v]^q$ in $(\psi \circ f)(t)$. This changes p to $p - 2$, without altering a, b, c, d, ϵ and q (but m and n are allowed to be altered). Using $\psi'(g) = u^{-1} g u$ in place of ψ, p can also be changed to $p + 2$. Similarly, q can be changed to $q \pm 2$. Applying this repeatedly, we can arrange $p = q = 0$. This gives us a homomorphism $f' : \Gamma \to \Gamma(r)$ of the promised form, which satisfies $f_* = f'_*$. Since ϕ, ϕ', ψ and ψ' used above have $\epsilon_\bullet = 1$ and $\delta_\bullet = 1$, we have $\epsilon_{f'} = \epsilon_f$ and $\delta_{f'} = \delta_f$. ∎

As the final step of our analysis, we investigate the special case of 2-connected homomorphisms in Lemma 12.6. Let $i : \Gamma = \Gamma(1) \to \mathrm{colim}\ \Gamma(\ell) = \widehat{\Gamma}$ be the colimit map, and $i_* : H_3(\Gamma) \to H_3(\widehat{\Gamma})$ be the induced map. Recall that the $\mathbb{Z}$ factor of $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ is generated by the image $i_*[Y]$ of the fundamental class $[Y] \in H_3(\Gamma) = H_3(Y) = \mathbb{Z}$.

Lemma 12.7. *Suppose $f : \Gamma = \Gamma(1) \to \Gamma(r)$ is a 2-connected homomorphism such that $f(t) = t^{\epsilon_f}$. Then the induced map $f_* : H_3(\Gamma) \to H_3(\widehat{\Gamma})$ is given by*

$$f_* = \delta_f \cdot \epsilon_f \cdot i_*.$$

Proof. Consider the subgroup of $\Gamma(\ell)$ generated by u, v and t^2, which corresponds a double cover. Since $[t^2, u] = [t^2, v] = 1$, this subgroup is the internal direct product

$A(\ell) \times 2\mathbb{Z}$, where $A(\ell)$ is generated by u and v and the infinite cyclic group $2\mathbb{Z}$ is generated by t^2. The colimit $\mathcal{A} \times 2\mathbb{Z} = \mathrm{colim}(A(\ell) \times 2\mathbb{Z})$ is an index 2 subgroup of $\widehat{\Gamma}$. Since f sends $A(1)$ to $A(r)$, f lifts to a homomorphism $g \colon A(1) \times 2\mathbb{Z} \to A(r) \times 2\mathbb{Z}$. Compose them with the colimit maps $A(r) \times 2\mathbb{Z} \to \mathrm{colim}(A(\ell) \times 2\mathbb{Z}) = \mathcal{A} \times 2\mathbb{Z}$ and $\Gamma(r) \to \mathrm{colim}_\ell \Gamma(\ell) = \widehat{\Gamma}$, and take H_3, to obtain the diagram

$$
\begin{array}{ccc}
H_3(A(1) \times 2\mathbb{Z}) \xrightarrow{g_*} H_3(A(r) \times 2\mathbb{Z}) \xrightarrow{\mathrm{colim}} H_3(\mathcal{A} \times 2\mathbb{Z}) \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
H_3(\Gamma(1)) \xrightarrow{\quad f_* \quad} H_3(\Gamma(r)) \xrightarrow{\quad \mathrm{colim} \quad} H_3(\widehat{\Gamma})
\end{array}
\tag{12.8}
$$

We will compare the composition of the top row with the homomorphism induced by the colimit $i|_{A(1) \times 2\mathbb{Z}} \colon A(1) \times 2\mathbb{Z} \to \mathcal{A} \times 2\mathbb{Z}$. The key property, which is a consequence of the hypothesis $f(t) = t^{\epsilon_f}$, is that the lift g can be written as a product: $g = (g|_{A(1)}) \times (\epsilon \cdot)$ where $g|_{A(1)}$ is equal to the restriction $f|_{A(1)} \colon A(1) \to A(r)$, and $\epsilon \cdot \colon 2\mathbb{Z} \to 2\mathbb{Z}$ is multiplication by $\epsilon := \epsilon_f$. So, the induced map g_* on H_3 is determined by $g|_{A(1)}$ and ϵ by the Künneth formula. More precisely, since $A(1) = \mathbb{Z}^2$, the composition of the top row of (12.8) is equal to the composition

$$
H_3(A(1) \times 2\mathbb{Z}) = H_2(A(1)) \otimes H_1(2\mathbb{Z}) \xrightarrow{(g|_{A(1)})_* \otimes (\cdot \epsilon)} H_2(A(\ell)) \otimes H_1(2\mathbb{Z})
$$

$$
\xrightarrow{\mathrm{colim} \otimes (\cdot \epsilon)} H_2(\mathcal{A}) \otimes H_1(2\mathbb{Z}) \xrightarrow{\times} H_3(\mathcal{A} \times 2\mathbb{Z}).
$$

By (11.13) and (11.14), $H_2(\mathcal{A})$ is identified with the subgroup $r^2\mathbb{Z} \subset \mathbb{Z} = H_2(A(r)_{\mathrm{ab}})$. In our case, the homomorphism $H_2(A(1)) = \mathbb{Z} \to H_2(A(r)_{\mathrm{ab}}) = \mathbb{Z}$ induced by $g|_{A(1)} = f|_{A(1)}$ is multiplication by the determinant of $A(1) = \mathbb{Z}^2 \to A(r)_{\mathrm{ab}} = \mathbb{Z}^2$, which is equal to $\delta_f \cdot r^2$ by (12.3). From this, it follows that the composition of the top row of (12.8) is equal to $\delta_f \cdot \epsilon \cdot (i|_{A(1) \times 2\mathbb{Z}})_*$.

Consequently, the composition of the bottom row of (12.8), which is the induced homomorphism $f_* \colon H_3(\Gamma) \to H_3(\widehat{\Gamma})$, is equal to $\delta_f \cdot \epsilon \cdot i_*$ *on the image of* $H_3(A(1) \times 2\mathbb{Z}) \to H_3(\Gamma(1)) = H_3(Y)$. Since $B(A(1) \times 2\mathbb{Z}) \to B\Gamma(1) = Y$ is a double cover of the 3-manifold Y, the image is the subgroup generated by $2[Y] \in H_3(Y)$. So,

$$
2 \cdot f_*[Y] = 2 \cdot \delta_f \cdot \epsilon \cdot i_*[Y] \quad \text{in } H_3(\widehat{\Gamma}).
$$

Since $H_3(\widehat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ by Theorem 11.1 (1), $2\theta = 0$ implies $\theta = 0$ for every $\theta \in H_3(\widehat{\Gamma})$. It follows that $f_*[Y] = \delta_f \cdot \epsilon \cdot i_*[Y]$. This completes the proof. ∎

Now we are ready to prove Theorem 12.2, which asserts that the action of $f \in \mathrm{Aut}(\widehat{\Gamma})$ on $H_3(\widehat{\Gamma}) = (\mathbb{Z}/\mathbb{Z}_{(2)}) \times \mathbb{Z}$ is given by $f_*(a, n) = (\delta_f \cdot a, \delta_f \cdot \epsilon_f \cdot n)$.

Proof of Theorem 12.2. By Lemma 12.4, the restriction of f_* to $tH_3(\widehat{\Gamma}) = \mathbb{Z}_{(2)}/\mathbb{Z}$ is multiplication by δ_f. So it remains to investigate f_* on the generator $(0, 1) \in (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$. By Lemmas 12.6 and 12.7, we may assume that the map $H_3(\Gamma) \to H_3(\widehat{\Gamma})$ induced by f sends the fundamental class $[Y]$ to $\delta_f \cdot \epsilon_f \cdot i_*[Y]$. Since $(0, 1) \in (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$ is the image of $[Y]$, it follows that $f_*(0, 1) = (0, \delta_f \cdot \epsilon_f)$. ∎

13. Nontrivial transfinite Milnor invariants

The goal of this section is to prove Theorem L stated in Section 2.11, which gives an infinite family of 3-manifolds with vanishing Milnor invariants of finite length, but distinct nontrivial transfinite Milnor invariants of length ω. As mentioned in Section 2.11, we do so by using a family of 3-manifolds, $\{M_d \mid d \in 2\mathbb{Z} + 1\}$: M_d is defined to be the torus bundle $T^2 \times [0, 1]/\{(h_d(z), 0) \sim (z, 1)\}$, with monodromy $h_d = \begin{bmatrix} -1 & d \\ 0 & -1 \end{bmatrix}$. Note that M_d is obtained from the original torus bundle Y studied in the previous sections, by modifying the $(1, 2)$-entry of the monodromy from 0 to d.

Fix an odd integer d. We will use M_d as the basepoint manifold to which other 3-manifolds M_r are compared. That is, let $\Gamma = \pi_1(M_d)$. Our main goal of this section is to prove Theorem L, which asserts the following:

(1) For every odd integer r, $\bar{\mu}_k(M_r)$ is defined and vanishes for all finite k. Moreover, $\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega \cong \widehat{\Gamma}/\widehat{\Gamma}_\omega$, so $\bar{\mu}_\omega(M_r)$ is defined.

(2) But, for odd r and s, $\bar{\mu}_\omega(M_r) = \bar{\mu}_\omega(M_s)$ if and only if $|r/s|$ is a square in $\mathbb{Z}_{(2)}^\times$. In particular, $\bar{\mu}_\omega(M_r)$ is nontrivial if and only if $|r/d|$ is not a square.

(3) Indeed, the set of realizable values of the Milnor invariant of length ω, $\mathcal{R}_\omega(\Gamma)/\approx$, is in 1-1 correspondence with $\mathbb{Z}_{(2)}^\times/\pm(\mathbb{Z}_{(2)}^\times)^2$.

Here $\pm(\mathbb{Z}_{(2)}^\times)^2 := \{\pm\alpha^2 \mid \alpha \in \mathbb{Z}_{(2)}^\times\}$. For every $a/b \in \mathbb{Z}_{(2)}^\times$ with $a, b \in 2\mathbb{Z} + 1$, we have $a/b \equiv |ab| \bmod \pm(\mathbb{Z}_{(2)}^\times)^2$ (multiplicatively), and in the prime factorization of the integer $|ab|$, one can assume that each prime has exponent at most 1, modulo square. So $\mathcal{R}_\omega(\Gamma)/\approx$ is bijective to the set of odd positive integers which have no repeated primes in the factorization.

To show Theorem L, we will compute the realizable classes and the equivalence relations $\sim$ and $\approx$ for the modified torus bundle case. In fact, both the arguments for computation and their outcomes are very close to the original torus bundle $d = 0$ case. However, the modified case has a small but important difference: the action of $\mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ on $\mathcal{R}_\omega(\Gamma)$ turns out to have smaller orbits. See Theorem 13.1 (2) below and compare it with Theorem 11.1 (5). From this the nontriviality of the length ω Milnor invariants will be obtained.

More specifically, we will show the following.

Theorem 13.1. *Let $\Gamma = \pi_1(M_d)$ as above, d odd. Then the following hold:*

(1) *Each of $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$, $H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1})$, $\mathcal{R}_\omega(\Gamma)$, $\mathcal{R}_{\omega+1}(\Gamma)$, the map $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ and the equivalence relation $\sim$ on $\mathcal{R}_\omega(\Gamma)$ is identical with that given in Theorem 11.1: that is,*

$$H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}, \quad H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z},$$
$$\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times, \quad \mathcal{R}_{\omega+1}(\Gamma) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\}.$$

The map $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ is $(x, \epsilon) \mapsto \epsilon$, and on $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times$, $\theta \sim \theta'$ if and only if $\theta = \pm\theta'$.

(2) *The orbits of the action of* $\operatorname{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ *on* $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times$ *is given by:* $\phi(\theta) = \theta'$ *for some* $\phi \in \operatorname{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ *if and only if* θ/θ' *is a square.*

   Consequently, on $\mathcal{R}_\omega(\Gamma)$, $\theta \approx \theta'$ *if and only if* $\theta/\theta' = \pm\alpha^2$ *for some* $\alpha \in \mathbb{Z}_{(2)}^\times$.

(3) *For every odd integer* r, *there is an isomorphism* $f \colon \widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega \xrightarrow{\cong} \widehat{\Gamma}/\widehat{\Gamma}_\omega$ *such that* $\theta_\omega(M_r) = \theta_\omega(M_r, f) = r/d \in \mathbb{Z}_{(2)}^\times = \mathcal{R}_\omega(\Gamma)$. *So* $\bar{\mu}_\omega(M_r) = r/d = rd$ *and* $\bar{\mu}_\omega(M_{rd}) = r$ *in* $\mathbb{Z}_{(2)}^\times/\pm(\mathbb{Z}_{(2)}^\times)^2 = \mathcal{R}_\omega(\Gamma)/\approx$.

Theorem L follows immediately from Theorem 13.1 (2, 3).

The remaining part of this section is devoted to the proof of Theorem 13.1. In Section 13.1, we compute the transfinite lower central quotients $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$. In Section 13.2, we prove Theorem 13.1 (1). In Section 13.3, we prove Theorem 13.1 (2, 3).

13.1. Transfinite lower central series quotients of the localization

The group $\Gamma = \pi_1(M_d)$ is the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z} = \mathbb{Z}^2 \rtimes_{h_d} \mathbb{Z}$, where the generator t of $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by $h_d = \begin{bmatrix} -1 & d \\ 0 & -1 \end{bmatrix}$. In what follows, we will compute $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$.

Recall that the group $\mathcal{A} = \operatorname{colim}_{\ell \text{ odd}} A(\ell)$ was defined at the beginning of Section 11.3. One can write

$$\mathcal{A} = \{x^\alpha y^\beta [x, y]^\gamma \mid \alpha, \beta \in \mathbb{Z}_{(2)}, \ \gamma \in \mathbb{Z}_{(2)}/\mathbb{Z}\}$$

where the group operation is given by

$$x^\alpha y^\beta [x, y]^\gamma \cdot x^\lambda y^\mu [x, y]^\xi = x^{\alpha+\lambda} y^{\beta+\mu} [x, y]^{\gamma+\xi-\beta\lambda}.$$

The group $\mathcal{A}$ has $\mathbb{Z}^2$ as a subgroup, generated by x and y. (Note that $[x, y]$ is trivial in $\mathcal{A}$.) Also, $\mathbb{Z}_{(2)}/\mathbb{Z} = \{[x, y]^\gamma\} \subset \mathcal{A}$ is a central subgroup, and $\mathcal{A}/(\mathbb{Z}_{(2)}/\mathbb{Z}) = \mathbb{Z}_{(2)}^2$.

Recall that, for $d = 0$, we proved that $\widehat{\Gamma} = \Gamma/\widehat{\Gamma}_{\omega+1}$ is equal to the semidirect product $\mathcal{A} \rtimes_{h_0} \mathbb{Z}$, where the generator t of $\mathbb{Z}$ acts on $\mathcal{A}$ by $h_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, that is, $t \cdot x = x^{-1}$, $t \cdot y = y^{-1}$ (see Section 11.3).

We will prove a similar result for the modified torus bundle case.

For this purpose, we need to extend the action of $t = h_d$ on $\mathbb{Z}_2$ to $\mathcal{A}$. Being an extension, $t \cdot x^n = x^{-n}$, $t \cdot y^n = x^{dn} y^{-n}$ must be satisfied for every integer n, but it can be seen that a naive attempt to define $t \cdot x^{1/n} = x^{-1/n}$, $t \cdot y^{1/n} = x^{d/n} y^{-1/n}$ does not give a group homomorphism $t \colon \mathcal{A} \to \mathcal{A}$.

Instead, we use the following lemma, which can be verified by a direct computation. To state it, we need the fact that multiplication by 2, $\mathbb{Z}_{(2)}/\mathbb{Z} \xrightarrow{2 \cdot} \mathbb{Z}_{(2)}/\mathbb{Z}$, is an isomorphism, so $\gamma/2 \in \mathbb{Z}_{(2)}/\mathbb{Z}$ is well defined for every $\gamma \in \mathbb{Z}_{(2)}/\mathbb{Z}$.

Lemma 13.2. *The map* $t \colon \mathcal{A} \to \mathcal{A}$ *defined by*

$$t \cdot (x^\alpha y^\beta [x, y]^\gamma) = x^{-\alpha+d\beta} y^{-\beta} [x, y]^{\gamma+d\beta^2/2}$$

is a group isomorphism which extends $t = h_d \colon \mathbb{Z}^2 \to \mathbb{Z}^2$.

Define a semidirect product $\mathcal{A} \rtimes \mathbb{Z} = \mathcal{A} \rtimes_{h_d} \mathbb{Z}$ by using the action of t in the lemma. The subgroup $\mathbb{Z}_{(2)}/\mathbb{Z}$ is central in $\mathcal{A} \rtimes \mathbb{Z}$, and the quotient $(\mathcal{A} \rtimes \mathbb{Z})/(\mathbb{Z}_{(2)}/\mathbb{Z})$ is the semidirect product $\mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z} = \mathbb{Z}_{(2)}^2 \rtimes_{h_d} \mathbb{Z}$, which is defined using the action of $t = h_d$ on $\mathbb{Z}_{(2)}^2$. In what follows, we omit h_d in the semidirect product notation.

Theorem 13.3. $\widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1} = \mathcal{A} \rtimes \mathbb{Z}$. The natural maps of $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$ into $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ and $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$ are the inclusions.

Indeed, it can also be shown that $\widehat{\Gamma} = \mathcal{A} \rtimes \mathbb{Z}$ and $\widehat{\Gamma}_{\omega+1} = \{1\}$, by modifying the arguments used in [7]. Since we do not use this stronger fact, we will just provide a proof of Theorem 13.3.

Proof of Theorem 13.3. First, we will compute $\widehat{\Gamma}/\widehat{\Gamma}_\omega$ using Theorem 3.6. For this, we need to compute the classical module localization $S^{-1}\mathbb{Z}^2$, where $S = \{s(t) \in \mathbb{Z}[t^{\pm 1}] \mid s(1) = \pm 1\}$. As a $\mathbb{Z}[t^{\pm 1}]$-module, $\mathbb{Z}^2$ is presented by the matrix $tI - h_d = \begin{bmatrix} t+1 & -d \\ 0 & t+1 \end{bmatrix}$. So, $\mathbb{Z}^2$ is annihilated by the determinant, $t^2 + 2t + 1$. Observe that for each $s(t) \in S$, $s(t)s(t^{-1}) \in S$. So, $S^{-1}\mathbb{Z}^2$ is equal to $T^{-1}\mathbb{Z}^2$, where $T = \{\pm s(t)s(t^{-1}) \mid s(t) \in S\}$. An element $p(t) \in T$ satisfies $p(t) = p(t^{-1})$, so $p(t) = a_0 + \sum_{i>0} a_i(t + t^{-1})^i$ with $p(1) = a_0 + \sum_{i>0} 2^i \cdot a_i = \pm 1$. Since $t + t^{-1}$ acts on $\mathbb{Z}^2$ by multiplication by -2, it follows that multiplication by $p(t)$ on $\mathbb{Z}^2$ is equal to multiplication by $a_0 + \sum_{i>0}(-2)^i \cdot a_i$, which is an odd integer. Conversely, an arbitrary odd integer r can be written as $r = \pm 1 + 4k$, so multiplication by r on $\mathbb{Z}^2$ is equal to multiplication by $s(t) := \pm 1 - k(t + t^{-1} - 2)$, which lies in S. It follows that $S^{-1}\mathbb{Z}^2 = (2\mathbb{Z} + 1)^{-1}\mathbb{Z}^2 = \mathbb{Z}_{(2)}^2$.

Also, for the augmentation ideal $I = \{p(t) \in \mathbb{Z}[t^{\pm 1}] \mid p(1) = 0\}$, an element $p(t) \in I^{2k}$ is of the form $p(t) = (t-1)^{2k}q(t) = (t + t^{-1} - 2)^k \cdot t^k q(t)$. Since $t + t^{-1}$ acts on $\mathbb{Z}^2$ by multiplication by -2, it follows that $I^{2k}\mathbb{Z}^2 \subset 4^k\mathbb{Z}^2$, and hence $\bigcap_{k<\infty} I^k\mathbb{Z}^2 = 0$. So $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is residually nilpotent. For later use, note that the same argument proves that $\mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ is residually nilpotent too.

Now, by Theorem 3.6, the closure in the completion is given by

$$\widehat{\Gamma}/\widehat{\Gamma}_\omega = \overline{\mathbb{Z}^2 \rtimes \mathbb{Z}} = S^{-1}\mathbb{Z}^2 \rtimes \mathbb{Z} = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}.$$

This proves the first conclusion.

To compute $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}$, we claim the following:

(1) $H_2(\mathcal{A} \rtimes \mathbb{Z}) = H_2(\mathcal{A}) = H_2(\mathbb{Z}^2) = H_2(\mathbb{Z}^2 \rtimes \mathbb{Z}) = \mathbb{Z}$,

(2) $(\mathcal{A} \rtimes \mathbb{Z})_\omega = \mathbb{Z}_{(2)}/\mathbb{Z} = \{[x, y]^\gamma\}$, $(\mathcal{A} \rtimes \mathbb{Z})_{\omega+1} = \{1\}$.

Here, the equalities between $H_2(-)$ are induced by the inclusions of the groups.

Before proving the claims, we will derive the second conclusion of the theorem. Since $\mathbb{Z}_{(2)}/\mathbb{Z}$ is a central abelian subgroup of $\mathcal{A} \rtimes \mathbb{Z}$ and the quotient $(\mathcal{A} \rtimes \mathbb{Z})/(\mathbb{Z}_{(2)}/\mathbb{Z}) = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ is a local group, $\mathcal{A} \rtimes \mathbb{Z}$ is local, by [7, Theorem A.2, Lemma A.4]. So, the inclusion $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z} \hookrightarrow \mathcal{A} \rtimes \mathbb{Z}$ induces $\widehat{\Gamma} \to \mathcal{A} \rtimes \mathbb{Z}$. We will apply the standard Stallings argument to $\widehat{\Gamma} \to \mathcal{A} \rtimes \mathbb{Z}$. We have already shown that $\widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$. Combining this with claim (2) above, it follows that $\widehat{\Gamma}/\widehat{\Gamma}_\omega \cong (\mathcal{A} \rtimes \mathbb{Z})/(\mathcal{A} \rtimes \mathbb{Z})_\omega$.

Since the composition $H_2(\Gamma) \to H_2(\widehat{\Gamma}) \to H_2(\mathcal{A} \rtimes \mathbb{Z})$ is an isomorphism by the first claim, $H_2(\widehat{\Gamma}) \to H_2(\mathcal{A} \rtimes \mathbb{Z})$ is surjective. So, by Stallings' work [26], $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1} \cong (\mathcal{A} \rtimes \mathbb{Z})/(\mathcal{A} \rtimes \mathbb{Z})_{\omega+1}$. By the second claim, it follows that $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1} \cong \mathcal{A} \rtimes \mathbb{Z}$.

Therefore, to complete the proof, it remains to show the claims. We begin with the first claim, which concerns H_2. In fact, $H_*(\mathcal{A} \rtimes \mathbb{Z})$ can be computed using the Wang sequence

$$\cdots \to H_2(\mathcal{A}) \xrightarrow{1-t_*} H_2(\mathcal{A}) \to H_2(\mathcal{A} \rtimes \mathbb{Z}) \to H_1(\mathcal{A}) \xrightarrow{1-t_*} H_1(\mathcal{A}) \to \cdots \quad (13.1)$$

similarly to Section 11.3. (Our (13.1) here is analogous to (11.9).) By (11.14), $H_2(\mathcal{A}) = H_2(\mathbb{Z}^2) = \mathbb{Z}$. Since h_d has determinant 1, $t_* = \mathrm{id}$ on $H_2(\mathcal{A})$ and thus $1 - t_* = 0$. Also, $H_1(\mathcal{A}) = \mathbb{Z}_{(2)}^2$, and t_* on $H_1(\mathcal{A})$ is given by h_d. So, $1 - t_* = \left[\begin{smallmatrix} 2 & -d \\ 0 & 2 \end{smallmatrix}\right]$ on $H_1(\mathcal{A})$ and this is injective. Therefore, from (13.1), it follows that

$$H_2(\mathcal{A} \rtimes \mathbb{Z}) = H_2(\mathcal{A}) = H_2(\mathbb{Z}^2) = \mathbb{Z}. \quad (13.2)$$

We remark that while our monodromy h_d (d odd) is different from the $d = 0$ case, $H_2(\mathcal{A} \rtimes \mathbb{Z})$ remains the same as in the $d = 0$ case given in (11.19).

For the second claim, we proceed similarly to the proof of Lemma 11.4. We have already shown that $(\mathcal{A} \rtimes \mathbb{Z})/(\mathbb{Z}_{(2)}/\mathbb{Z}) = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ is residually nilpotent. So, $(\mathcal{A} \rtimes \mathbb{Z})_\omega \subset \mathbb{Z}_{(2)}/\mathbb{Z}$. For the reverse inclusion, observe that $[x^\beta, t] = x^{-2\beta}$, so by induction $x^{2^k\beta} \in (\mathcal{A} \rtimes \mathbb{Z})_{k+1}$. Thus $[x, y]^{2^k\beta} = [x^{2^k\beta}, y]$ lies in $(\mathcal{A} \rtimes \mathbb{Z})_{k+2}$ for all $\beta \in \mathbb{Z}_{(2)}$. For every $\gamma \in \mathbb{Z}_{(2)}$, there is $\beta \in \mathbb{Z}_{(2)}$ such that $2^k\beta \equiv \gamma \bmod \mathbb{Z}$, since 2 is invertible in $\mathbb{Z}_{(2)}/\mathbb{Z}$. It follows that $[x, y]^\gamma = [x, y]^{2^k\beta} \in (\mathcal{A} \rtimes \mathbb{Z})_{k+2}$. Since it holds for every k, it follows that $[x, y]^\gamma \in (\mathcal{A} \rtimes \mathbb{Z})_\omega$. This shows that $(\mathcal{A} \rtimes \mathbb{Z})_\omega = \mathbb{Z}_{(2)}/\mathbb{Z}$. Finally, since $[x, y]^\gamma$ is central in $\mathcal{A} \rtimes \mathbb{Z}$, $(\mathcal{A} \rtimes \mathbb{Z})_{\omega+1} = \{1\}$. This completes the proof of the claims. ∎

13.2. Homology and realizable classes

We will prove Theorem 13.1 (1). To compute $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$, we use the Wang sequence for $\widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$, similarly to Section 11.2. Indeed, the Wang sequence was already given in (11.4):

$$0 = H_3(\mathbb{Z}_{(2)}^2) \to H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) \to H_2(\mathbb{Z}_{(2)}^2) \xrightarrow{1-t_*} H_2(\mathbb{Z}_{(2)}^2) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$$
$$\to H_1(\mathbb{Z}_{(2)}^2) \xrightarrow{1-t_*} H_1(\mathbb{Z}_{(2)}^2) \to H_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega) \to \mathbb{Z} \to 0.$$

Here, the difference from Section 11.2 is that t_* is induced by $h_d = \left[\begin{smallmatrix} -1 & d \\ 0 & -1 \end{smallmatrix}\right]$. So, $1 - t_*$ on $H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}$ is zero, and $1 - t_*$ on $H_1(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}^2$ is $\left[\begin{smallmatrix} 2 & -d \\ 0 & 2 \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}, \\ H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}, \\ H_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega) &= \mathbb{Z}_4 \times \mathbb{Z}. \end{aligned} \quad (13.3)$$

Note that $H_i(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ remains the same as that of the original torus bundle in Section 11.2 for $i = 2, 3$, while $H_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ is altered since d is odd (compare 13.3 with (11.5)). But $H_1(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ is still a finite abelian 2-group. Consequently, the analysis of the cap products (11.6) and (11.7) (which uses Theorem G) in Section 11.2 applies to our case without any modification. This shows that $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times$.

To compute $H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1})$, we proceed similarly to Section 11.3. For $\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1} = \mathcal{A} \rtimes \mathbb{Z}$, we have the Wang sequence (11.9)

$$\cdots \to H_i(\mathcal{A}) \xrightarrow{1-t_*} H_i(\mathcal{A}) \to H_i(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) \to H_{i-1}(\mathcal{A}) \xrightarrow{1-t_*} H_{i-1}(\mathcal{A}) \to \cdots$$

where t_* is again induced by $h_d = \left[\begin{smallmatrix} -1 & d \\ 0 & -1 \end{smallmatrix}\right]$. We have $H_3(\mathcal{A}) = H_3(\mathbb{Z}_{(2)}/\mathbb{Z}) = \mathbb{Z}_{(2)}/\mathbb{Z}$ by (11.16). Since the subgroup $\mathbb{Z}_{(2)}/\mathbb{Z} \subset \mathcal{A}$ is generated by $[x, y]^\gamma$ on which our t acts trivially, t_* on $H_3(\mathcal{A})$ is the identity. Also, since $H_2(\mathcal{A}) = H_2(\mathbb{Z}^2) = \mathbb{Z}$ by (11.14), t_* on $H_2(\mathcal{A})$ is the identity too. It follows that $H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \rtimes \mathbb{Z}$, as in (11.18) in Section 11.3.

Also, for $\theta \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1})$, the analysis of the cap products (11.6) and (11.7) in Section 11.4 is carried out for our case

$$\cap \theta \colon tH^2(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) \to tH_1(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) = tH_1(\Gamma),$$
$$\cap \theta \colon H^1(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1})/\mathrm{Ker}\{H_2(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1}) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\omega)\}$$

without modification, using the fact that $tH_1(\Gamma) = \mathbb{Z}_4$ is a finite abelian 2-group. This shows that $\mathcal{R}_{\omega+1}(\Gamma) = \mathbb{Z}_{(2)}/\mathbb{Z} \times \{\pm 1\} \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_{\omega+1})$.

Note that we have shown that $\mathcal{R}_\omega(\Gamma)$ and $\mathcal{R}_{\omega+1}(\Gamma)$ are the same as those in the original torus bundle case ($d = 0$). So, by the argument in the last paragraph, $\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)$ is also the same as in the original torus bundle case.

To complete the proof of Theorem 13.1 (1), it remains to determine the equivalence relation $\sim$ on $\mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}$.

Let $\theta = a/b \in \mathcal{R}_\omega(\Gamma) = \mathbb{Z}_{(2)}^\times$ with a, b odd integers. To compute the equivalence class of θ, we will first find a 3-manifold realizing θ. Recall that M_d is the modified torus bundle, with monodromy $h_d = \left[\begin{smallmatrix} -1 & d \\ 0 & -1 \end{smallmatrix}\right]$, and that $\Gamma = \pi_1(M_d)$. For another odd integer r which will be specified later, consider the 3-manifold M_r. By Theorem 13.3 aplied to r instead of d, we have $\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega = \mathbb{Z}_{(2)}^2 \rtimes_{h_r} \mathbb{Z}$. Because the following observation will also be used later, we state it as a lemma.

Lemma 13.4. *Let $\alpha, \beta \in \mathbb{Z}_{(2)}^\times$. Then $\phi = \phi_{\alpha,\beta} \colon \mathbb{Z}_{(2)}^2 \rtimes_{h_r} \mathbb{Z} \to \mathbb{Z}_{(2)}^2 \rtimes_{h_d} \mathbb{Z}$ given by $\phi(a, b, n) = (\alpha \cdot a, \beta \cdot b, n)$ is a group isomorphism if and only if $d\beta = r\alpha$. In that case, the induced isomorphism*

$$\phi_* \colon H_3(\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega) = \mathbb{Z}_{(2)} \to H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}$$

is multiplication by $\alpha\beta$.

Proof. Since the monodromies are $h_d = \left[\begin{smallmatrix} -1 & d \\ 0 & -1 \end{smallmatrix}\right]$ and $h_r = \left[\begin{smallmatrix} -1 & r \\ 0 & -1 \end{smallmatrix}\right]$ and $\phi = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$ on $\mathbb{Z}_{(2)}^2$, our ϕ is an isomorphism between the semidirect products if and only if the

matrix identity $h_d \phi = \phi h_r$ holds. From this, the first conclusion follows immediately, using the condition $d \neq 0$.

Since $H_3(\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega) = H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}$ by (13.3) and since the restriction $\phi|_{\mathbb{Z}_{(2)}^2}$ is $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$, the induced map ϕ_* on H_3 is multiplication by $\det \phi|_{\mathbb{Z}_{(2)}^2} = \alpha\beta$. ∎

For our purpose, let $r = abd$, $\alpha = 1/b$ and $\beta = a$. By Lemma 13.4, $\phi = \phi_{\alpha,\beta}$ is an isomorphism, and ϕ_* on H_3 is multiplication by a/b. Since the fundamental class $[M_r]$ is equal to $1 \in \mathbb{Z}_{(2)} = (\widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega)$, it follows that the value of the invariant $\theta(M_r) = \theta(M_r, \phi)$ defined using the isomorphism ϕ is equal to the class $\theta = a/b \in \mathcal{R}_\omega(\Gamma)$. Therefore, the equivalence class of θ with respect to $\sim$ is equal to the image of the composition

$$
\begin{array}{ccccc}
\mathcal{R}_{\omega+1}(\pi_1(M_r)) & \longrightarrow & \mathcal{R}_\omega(\pi_1(M_r)) & \xrightarrow[\phi_*]{\cong} & \mathcal{R}_\omega(\Gamma) \\
\| & & \| & & \| \\
(\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} & & \mathbb{Z}_{(2)}^\times & & \mathbb{Z}_{(2)}^\times
\end{array}
$$

by Definition 2.4. Since the first arrow is $(x, \pm 1) \mapsto \pm 1$ and the second arrow is multiplication by $\theta = a/b$, it follows that $\theta \sim \theta'$ if and only if $\theta' = \pm\theta$. The completes the proof of Theorem 13.1 (1).

13.3. Automorphism action and Milnor invariants

Recall that $\Gamma = \pi_1(M_d)$ where d is fixed. We will prove Theorem 13.1 (2).

Suppose $\phi : \widehat{\Gamma}/\widehat{\Gamma}_\omega \to \widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ is an automorphism. Similarly to the proof of Lemma 12.3, we see that ϕ restricts to an automorphism on the subgroup $\mathbb{Z}_{(2)}^2$, since $\mathbb{Z}_{(2)}^2$ is the first rational derived subgroup of $\widehat{\Gamma}/\widehat{\Gamma}_\omega$. Write $\phi|_{\mathbb{Z}_{(2)}^2} = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{Z}_{(2)})$. For the generator t of the $\mathbb{Z}$ factor of $\mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$, we have $\phi(0, t) = (v, t^\epsilon)$ for some $v \in \mathbb{Z}_{(2)}^2$ and $\epsilon \in \{\pm 1\}$, since ϕ is an automorphism on the quotient $(\widehat{\Gamma}/\widehat{\Gamma}_\omega)/\mathbb{Z}_{(2)}^2 = \mathbb{Z}$. Since ϕ is a group homomorphism on the semidirect product with respect to the monodromy h_d, the matrix identity $\phi h_d = h_d^\epsilon \phi$ must be satisfied. By comparing the matrix entries, it implies that $\phi|_{\mathbb{Z}_{(2)}^2} = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \epsilon\alpha \end{smallmatrix}\right]$. (Here one uses the assumption that d is *nonzero*!) From this, it follows that the induced automorphism ϕ_* on $H_3(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = H_2(\mathbb{Z}_{(2)}^2) = \mathbb{Z}_{(2)}$ is equal to multiplication by $\epsilon \cdot \det \phi|_{\mathbb{Z}_{(2)}^2} = \alpha^2$. Note that $\alpha \in \mathbb{Z}_{(2)}^\times$ since $\phi|_{\mathbb{Z}_{(2)}^2}$ is invertible over $\mathbb{Z}_{(2)}$.

Conversely, the above computation also shows that for any square $\alpha^2 \in \mathbb{Z}_{(2)}^\times$, there is an automorphism ϕ on $\widehat{\Gamma}/\widehat{\Gamma}_\omega = \mathbb{Z}_{(2)}^2 \rtimes \mathbb{Z}$ such that ϕ_* on H_3 is multiplication by α^2. For instance, by setting $\beta = 0$ and $\epsilon = 1$, the automorphism ϕ given by $\phi|_{\mathbb{Z}_{(2)}^2} = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right]$ and $\phi(0, t) = (0, t)$ has that property.

From the above, Theorem 13.1 (2) follows immediately: for $\theta, \theta' \in \mathbb{Z}_{(2)}^\times = \mathcal{R}_\omega(\Gamma)$, $\phi(\theta) = \theta'$ for some $\phi \in \mathrm{Aut}(\widehat{\Gamma}/\widehat{\Gamma}_\omega)$ if and only if θ/θ' is a square in $\mathbb{Z}_{(2)}^\times$. By the above

computation of the equivalence relaiton $\sim$ and by Definition 2.4, it also follows that $\theta \approx \theta'$ in $\mathcal{R}_\omega(\Gamma)$ if and only if $\theta/\theta' = \pm\alpha^2$ for some $\alpha \in \mathbb{Z}_{(2)}^\times$.

Finally, we will prove Theorem 13.1 (3). Recall that d is the fixed odd integer. Let r be an arbitrary odd integer. Let $\theta = r/d \in \mathcal{R}_\omega(\widehat{\Gamma}/\widehat{\Gamma}_\omega) = \mathbb{Z}_{(2)}^\times$. Apply Lemma 13.4, for $(\alpha, \beta) = (1, r/d)$, to obtain the isomorphism

$$\phi = \phi_{\alpha,\beta} \colon \widehat{\pi_1(M_r)}/\widehat{\pi_1(M_r)}_\omega \to \widehat{\Gamma}/\widehat{\Gamma}_\omega.$$

Furthermore, Lemma 13.4 says that ϕ_* on H_3 is multiplication by $\alpha\beta = r/d \in \mathbb{Z}_{(2)}^\times$. Since the fundamental class is $[M_r] = 1 \in \mathbb{Z}_{(2)}$, we have $\theta_\omega(M_r) = \theta_\omega(M_r, \phi) = \phi_*(1) = r/d$. This completes the proof of Theorem 13.1, the last theorem of this paper.

14. Questions

We list some questions which naturally arise from this work.

(1) Can one interpret the invariants $\theta_k(M)$ and $\bar{\mu}_k(M)$ of finite length (i.e. $k < \infty$) as Gusarov–Vassiliev finite type invariants in an appropriate sense?

We remark that $\theta_k(M)$ and $\bar{\mu}_k(M)$ are invariant under Habiro–Gusarov clasper surgery, which is now often called Y_k-equivalence. More precisely, the following hold.

Fix a closed 3-manifold group Γ, and let M and M' be two closed 3-manifold which are Y_{k-1}-equivalent. Then $\theta_k(M)$ is defined if and only if $\theta_k(M')$ is defined, and when they are defined, $\theta_k(M) = \theta_k(M')$ in $\mathcal{R}_k(\Gamma)/\mathrm{Aut}(\Gamma/\Gamma_k)$ if M and M' are Y_k-equivalent. The following three questions are relevant.

(2) Can one extract the invariants $\theta_k(M)$ and $\bar{\mu}_k(M)$ of finite length from (some variant of) the Kontsevich integral, or related quantum invariants?

(3) Our results strongly suggest that there should be a notion of transfinite type invariants. Can one interpret the transfinite length invariants $\theta_\kappa(M)$ and $\bar{\mu}_\kappa(M)$ as a finite type invariant? If not, can one generalize the notion of finite type invariants of 3-manifolds to a suitable notion of "transfinite type" invariants, so that the invariants $\theta_\kappa(M)$ of transfinite length can be viewed as invariants of transfinite type?

(4) Can we extend the definition of the Kontsevich integral (of 3-manifolds or links) to a transfinite version?

The following addresses the (non)triviality of the transfinite invariants of a given length.

(5) For every (countable) ordinal κ, is there a closed 3-manifold group Γ for which the sets $\mathcal{R}_\kappa(\Gamma)/\sim$ and $\mathcal{R}_\kappa(\Gamma)/\approx$ have more than one element?

Milnor's original work [23] combined with Orr's result [25] tells us that the answer to (5) is affirmative for finite κ. See also Theorem K in this paper. Theorems I and L show that the answer is affirmative for $\kappa = \omega$.

(6) Is there a countable ordinal ϖ such that if Γ is a 3-manifold group and M is a 3-manifold equipped with an isomorphism $\widehat{\pi_1(M)}/\widehat{\pi_1(M)}_\varpi \to \widehat{\Gamma}/\widehat{\Gamma}_\varpi$ for which $\overline{\mu}_\varpi(M)$ vanishes (over Γ), then $\overline{\mu}_\kappa(M)$ is defined and vanishes (over Γ) for every $\kappa > \varpi$?

(7) Do θ_κ and $\overline{\mu}_\kappa(M)$ (with κ either finite or transfinite) reveal new information on link concordance?

Regarding (7), consider the following. Fix rational numbers $a_1/b_1, \ldots, a_m/b_m \in \mathbb{Q}$. For a given m-component link L, perform Dehn filling on the exterior of the link, with slopes a_i/b_i, to obtain a closed 3-manifold. Call it M_L. Fix a link L_0, and let $Y = M_{L_0}$, $\Gamma = \pi_1(Y)$. To compare a given link L with the link L_0, consider the invariants $\theta_\kappa(M_L)$ and $\overline{\mu}_\kappa(M)$, over the group Γ, as link invariants.

It seems particularly interesting whether θ_κ and $\overline{\mu}_\kappa$ of transfinite length gives a new nontrivial link invariant in this way.

In addition, the finite length case may also have some interesting potential applications. Recall from Section 10 that there are examples for which the finite length invariants θ_k live in finite abelian groups, and thus have torsion values.

(8) Do θ_k and $\overline{\mu}_k$ of finite length give new torsion-valued link concordance invariants?

The following is closely related to (8). In [9] (see also the survey [8] of a series of related papers), Conant, Schneiderman and Teichner proposed a higher order version of the classical Arf invariant for links. It may be viewed as certain 2-torsion-valued information extracted from Whitney towers and gropes in 4-space. A key conjecture in the theory of Whitney towers is whether the higher order Arf invariants are nontrivial.

(9) Are the invariants θ_k and $\overline{\mu}_k$ related to the higher order Arf invariants? More specifically, can one show the conjectural nontriviality of the higher order Arf invariants using these invariants (of certain 3-manifolds associated to links)?

Also, the existence of transfinite Milnor invariants suggest the existence of transfinite Arf-invariants.

(10) Do transfinite Arf invariants of links and 3-manifolds exist?

(11) If so, are these determined by the invariants θ_κ, or some analogue thereof?

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