Sebastian Opper

Spherical objects, transitivity and auto-equivalences of Kodaira cycles via gentle algebras

Abstract. This paper studies the class of spherical objects over any Kodaira n-cycle of projective lines and provides a parametrization of their isomorphism classes in terms of closed curves on the n-punctured torus without self-intersections. Employing recent results on gentle algebras, we derive a topological model for the bounded derived category of any Kodaira cycle. The groups of triangle auto-equivalences of these categories are computed and are shown to act transitively on isomorphism classes of spherical objects. This answers a question by Polishchuk (2002) and extends earlier results by Burban–Kreußler (2012) and Lekili–Polishchuk (2019). The description of auto-equivalences is further used to establish faithfulness of a mapping class group action defined by Sibilla (2014). The final part describes the closed curves which correspond to vector bundles and simple vector bundles. This leads to an alternative proof of a result by Bodnarchuk–Drozd–Greuel (2012) which states that simple vector bundles on cycles of projective lines are uniquely determined by their multi-degree, rank and determinant. As a by-product, we obtain a closed formula for the cyclic sequence of any simple vector bundle on C_n as introduced by Burban–Drozd–Greuel (2001).

Keywords: Kodaira cycle, cycle of projective lines, simple vector bundle, spherical objects, spherical twists, gentle algebra.

Contents

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1. Introduction

Our main objective in this paper is to study the class of *spherical objects* in the derived category of the singular algebraic curve C_n for any $n \geq 1$ which is known as a "Kodaira cycle of n projective lines", or a "cycle of projective lines" for short. A cycle C_n is a rational curve with n irreducible components and may be thought of as a union of n projective lines glued in a cycle as illustrated in Figure 1. Spherical objects are perfect complexes over C_n and form a class which is stable under auto-equivalences of the derived category $\mathcal{D}^{\mathrm{b}}(C_n)$. They were introduced by Seidel and Thomas [34] who showed that every spherical object induces an auto-equivalence of the bounded derived category which is referred to as a *spherical twist*.
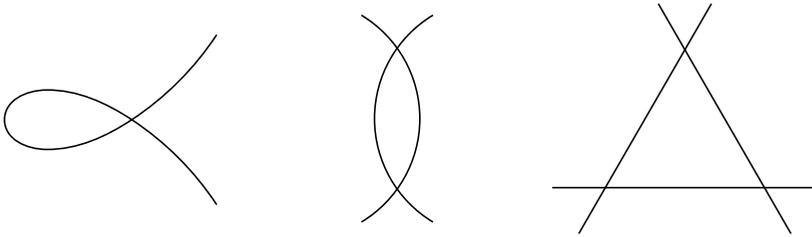

Fig. 1. The curves C_1, C_2 and C_3 (from left to right).

In the work of Polishchuk [32], spherical objects on cycles of projective lines (and more general curves of arithmetic genus one) were used to construct solutions of the classical and the associative Yang–Baxter equation. His construction leads to the question by whether spherical objects can be classified in a meaningful way. Polishchuk asked further whether the group of auto-equivalences of $\mathcal{D}^{\mathrm{b}}(C_n)$ acts transitively on the set of isomorphism classes of spherical objects. An affirmative answer to his question is given in Theorem C.

The problems of classification and transitivity were studied by Burban–Kreußler [15] and Lekili–Polishchuk [25]. Burban and Kreußler established transitivity and provided a classification of spherical objects over C_1 showing that any spherical object is isomorphic to a shift of a skyscraper sheaf of a smooth point or a simple vector bundle. The latter were classified by means of their rank, multi-degree and determinant as shown by Bodnarchuk, Drozd and Greuel [7]. For $n > 1$ however, the class of spherical objects is more complicated as found by Burban and Burban [10] who constructed spherical complexes in $\mathcal{D}^{\mathrm{b}}(C_n)$ which are not quasi-isomorphic to the shift of a sheaf.

Lekili and Polishchuk [25] proved for arbitrary n that the group of auto-equivalences of $\mathcal{D}^{\mathrm{b}}(C_n)$ acts transitively on simple vector bundles. Their proof employed one of their earlier work [23] which showed the equivalence of the derived category of a cycle C_n and the wrapped Fukaya category of an n-punctured torus after Haiden, Katzarkov and Kontsevich [19].

We approach the problem of classification of spherical objects and transitivity from a similar angle. As shown by Burban and Drozd [12], there is an embedding $\mathrm{Perf}(C_n) \hookrightarrow$

$\mathcal{D}^{\mathrm{b}}(\mathrm{Coh}\,\mathbb{X}_n)$ of the category of perfect complexes over C_n into the derived category of a certain non-commutative curve $\mathbb{X}_n$. They further proved that the latter has a tilting object and is equivalent to the derived category of a finite-dimensional algebra Λ_n which belongs to the well-understood class of *gentle algebras*. Although it is difficult to compute the resulting embedding $\mathrm{Perf}(C_n) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ directly, they determined the images of the Picard group and skyscraper sheaves of smooth points.

The results of this paper also rely on works [30, 31] by Plamondon, Schroll and the author which study derived categories of gentle algebras in terms of the geometry of a surface (see also [1, 19, 26] for related work). A result in [30] describes the group of auto-equivalences of such categories as an extension of the mapping class group of the associated surface. As a particular instance of the connection between the derived category of a gentle algebra and its surface model, isomorphism classes of objects in $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ can be fully understood in terms of curves on a torus with n boundary components and marked points on its boundary. The surface model of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ allows us to rephrase many problems about the categories $\mathrm{Perf}(C_n)$ and $\mathcal{D}^{\mathrm{b}}(C_n)$ as considerably simpler problems about curves on a torus.

Throughout this introduction, we assume that $\mathbb{k}$ is an algebraically closed field and that $n \geq 1$ is any natural number.

The first result of this paper provides a geometric interpretation of spherical objects over C_n as closed curves on a punctured torus.

Theorem A (Theorem 7.1). *There exists a bijection between isomorphism classes of spherical objects on C_n, up to shift, and pairs $([\gamma], \lambda)$, where $\lambda \in \mathbb{k} \setminus \{0\}$ and $[\gamma]$ is the homotopy class of an unoriented, non-separating simple loop γ on the n-punctured torus, i.e., γ is embedded into the n-punctured torus in such a way that its complement is connected.*

Theorem A is a special case of a more general correspondence between indecomposable objects of $\mathcal{D}^{\mathrm{b}}(C_n)$ and curves on the n-punctured torus. Using the Verdier localization approach from [24], we extend the topological description of $\mathcal{D}^{\mathrm{b}}(C_n)$ to morphisms (Theorem 8.22), compositions of morphisms (Remark 8.24) and certain mapping cones (Remark 8.26). In fact, the topological model of $\mathcal{D}^{\mathrm{b}}(C_n)$ presented here can be generalized to similar Verdier quotients of derived categories of other gentle algebras which might be of independent interest, see Remark 8.28.

The second result describes the loops of simple vector bundles and provides a geometric interpretation of their ranks. To simplify notation in the next theorem, we identify the torus with the usual quotient $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ and refer to the first coordinate of the product as the *latitudinal coordinate*.

Theorem B (Theorem 9.20). *Under the bijection in Theorem A, a non-separating simple loop γ on a punctured torus corresponds to a simple vector bundle if and only if γ is homotopic to a smooth loop such that the latitudinal coordinate of its derivative is nowhere vanishing. The rank of the vector bundle agrees with the number of full turns of γ in the latitudinal direction.*

In fact, by dropping the assumption on simplicity on vector bundles and curves, we describe all homotopy classes of curves which represent the image of a vector bundle over C_n and provide an easy geometric interpretation of rank and multi-degree on the surface, cf. Theorem 9.20. The representation of vector bundles as curves allows us to give an alternative proof of the result by Bodnarchuk, Drozd and Greuel [7] which states that the isomorphism class of any simple vector bundle over C_n is uniquely determined by its rank, its multi-degree and its determinant (Proposition 9.29).

Our next result gives a positive answer to the question of Polishchuk.

Theorem C (Theorem 7.2). *The group of auto-equivalences of $\mathcal{D}^{\mathrm{b}}(C_n)$ acts transitively on the set of isomorphism classes of spherical objects in* $\mathrm{Perf}(C_n)$.

The proof of Theorem C exploits the relationship between spherical twists in the derived category of a gentle algebra and Dehn twists of its surface.

The following Theorem describes the group of auto-equivalences of any cycle.

Theorem D (Corollary 8.41). *If $n \geq 2$, then the group of auto-equivalences of $\mathcal{D}^{\mathrm{b}}(C_n)$ is an extension of the mapping class group of the n-punctured torus and the group*

$$(\Bbbk^{\times})^n \times \mathbb{Z} \times \mathrm{Pic}^{\emptyset}(C_n),$$

where $\mathrm{Pic}^{\emptyset}(C_n)$ denotes the group of line bundles with vanishing multi-degree. Moreover, $\mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_1))$ is an extension of $\mathrm{PSL}_2(\mathbb{Z})$ and $(\mathbb{Z}_2 \ltimes (\Bbbk^{\times})^n) \times \mathbb{Z} \times \mathrm{Pic}^{\emptyset}(C_1)$.

Theorem D extends earlier results by Burban and Kreußler [14] for the case $n = 1$, see Remark 8.42. Its proof relies on the topological model for $\mathcal{D}^{\mathrm{b}}(C_n)$ and on the relationship between automorphisms of the arc complex of a punctured surface and its extended mapping class group as found by Irmak and McCarthy [22]. Similar techniques were used in [30]. We expect Theorem D to generalize to certain Verdier quotients of arbitrary gentle algebras. For further details, the reader is referred to Remark 8.43.

As an application of Theorem D, we establish faithfulness of an action of the pure mapping class group of the punctured torus on $\mathcal{D}^{\mathrm{b}}(C_n)$, see Theorem 8.44. The group action was constructed by Sibilla [35] who also conjectured its faithfulness.

The final result refines the classification of simple vector bundles obtained by Burban, Drozd and Greuel [13]. They showed that a simple vector bundle is equivalent to the datum of a non-zero scalar and a cyclic integer sequence which satisfies certain strong constraints. The vector bundle can be reconstructed explicitly from this data. However, while the entries in the cyclic sequence can be easily determined from the constraints in [13], it seems that the order in which they appear was unknown apart from the case $n = 1$, see [9]. The final theorem provides a closed formula for this sequence. For simplicity, it is phrased for non-negative multi-degrees.

Theorem E (Corollary 9.9). *Let $\mathcal{L}$ be a simple vector bundle on C_n of rank r and multi-degree $(d_1, \ldots, d_n)$ with $d_i \geq 0$. Then the cyclic sequence of $\mathcal{L}$ is given by the sequence of cardinalities*

$$r\mathbb{Z} \cap (d, d'],$$

where d and d' are any two consecutive entries in the sequence

$$0, d_1, d_1 + d_2, \ldots, \sum_{i=1}^{n} d_i, \sum_{i=1}^{n} d_i + d_1, \ldots, r \cdot \sum_{i=1}^{n} d_i.$$

The results of this paper allow us to give an independent proof of the classification of simple vector bundles on C_n which can be found in Proposition 9.29.

Organization of the paper

Section 2 contains a discussion of cycles of projective lines C_n, their tilted algebras Λ_n and the relevant results from [12] on the embedding $\mathrm{Perf}(C_n) \hookrightarrow \mathcal{D}^b(\Lambda_n)$.

In the subsequent section, we recall a few basic facts about spherical objects and their spherical twists.

Section 4 provides a detailed account of the surface model for the categorical resolution $\mathcal{D}^b(\Lambda_n)$ of C_n. In particular, we explain the relationship between curves and their intersections on the surface model and their connection to indecomposable objects and morphisms between them.

After this preparation, Section 5 contains a first discussion of spherical objects in $\mathcal{D}^b(\Lambda_n)$ but not yet the proof of Theorem A as this requires further results on the mapping class group of the surface model of $\mathcal{D}^b(\Lambda_n)$ and its connection to auto-equivalences of $\mathcal{D}^b(\Lambda_n)$. The necessary background material for these is presented in Section 6 after which Theorem A (classification of spherical objects) and Theorem C (transitivity) are proved in Section 7.

In Section 8, the surface model for $\mathcal{D}^b(\Lambda_n)$ is used to derive a similar model for a Verdier quotient of $\mathcal{D}^b(\Lambda_n)$ by the "boundary objects" which is equivalent to $\mathcal{D}^b(C_n)$. The description of the auto-equivalence group of $\mathcal{D}^b(C_n)$ as an extension of a mapping class group (Theorem D) essentially follows from there by exploiting the relationship between the automorphism group of the *arc complex* and the mapping class group of a surface.

In the final section, we determine the curves on the surface model of $\mathcal{D}^b(C_n)$ which represent vector bundles. The pursued approach avoids further direct computations of the embedding $\mathrm{Perf}(C_n) \hookrightarrow \mathcal{D}^b(\Lambda_n)$ and rather exploits general properties of the class of vector bundles in combination with the surface model of $\mathcal{D}^b(\Lambda_n)$. The latter is rephrased in a suitable combinatorial way akin to the cyclic sequences in [13] which describe (simple) vector bundles.

Conventions and general notation

We fix an algebraically closed field $\Bbbk$ and denote by $\Bbbk^\times$ its group of units. For a ringed space $(X, \mathcal{O}_X)$, we denote by $\mathrm{Coh}\, X$ its category of coherent $\mathcal{O}_X$-modules, by $\mathrm{Perf}(X)$ its category of perfect complexes, and by $\mathcal{D}^b(X)$ its bounded derived category of coherent sheaves. For a finite-dimensional $\Bbbk$-algebra A, we denote by $\mathcal{D}^b(A)$ the bounded derived

category of finite-dimensional left A-modules. For objects X, Y of a triangulated category $\mathcal{T}$, we write $\mathrm{Hom}^\bullet(X, Y) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(X, Y[i])$. Given a natural transformation $\eta \colon \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$ and an object $X \in \mathcal{C}$, we denote by η_X the induced map $\mathcal{F}(X) \to \mathcal{G}(X)$.

For an integer valued variable x and $p, q \in \mathbb{Z}$, we often rewrite the condition $p \leq x \leq q$ as $x \in [p, q]$ and similar for all other types of intervals. For an arrow α in a quiver, we write $s(\alpha)$ (resp. $t(\alpha)$) for the source (resp. the target) of α. Compositions of arrows are to be understood from right to left. We write $\mathbb{Z}_m$ for the cyclic group $\mathbb{Z}/m\mathbb{Z}$ and $\mathfrak{S}_m$ for the symmetric group of m elements. Finally, for every $m \in \mathbb{N} \setminus \{0\}$, $\mathbb{Z}^{\mathbb{Z}_m}$ denotes the set of functions $\mathbb{Z}_m \to \mathbb{Z}$, or, equivalently, the set of cyclic integer sequences of length m.

2. Cycles of projective lines and their categorical resolutions

We recall the definition of a Kodaira cycle of projective lines and the relevant results about their categorical resolutions from [12]. In what follows, let $n \geq 1$.

Definition 2.1. An *n-cycle of projective lines* is a reduced rational projective curve C_n of arithmetic genus 1, that is, a union of n copies of $\mathbb{P}^1$ glued together transversally in a configuration of type $\widetilde{A}_{n-1}$.

By definition, C_n has n irreducible components, henceforth denoted by $\mathbb{P}^1_i$ ($i \in \mathbb{Z}_n$), such that for all $i \in \mathbb{Z}_n$, $\mathbb{P}^1_i$ and $\mathbb{P}^1_{i+1}$ intersect in a nodal singularity. If $n > 1$, $\mathbb{P}^1_i \cong \mathbb{P}^1$ for each $i \in \mathbb{Z}_n$. The curve C_1 is isomorphic to the Weierstraß nodal cubic and is depicted in Figure 1 together with C_2 and C_3.

Burban and Drozd constructed a fully faithful and exact functor

$$\mathrm{Perf}(C_n) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathrm{Coh}\,\mathbb{X}_n),$$

where $\mathbb{X}_n$ is a certain non-commutative curve. They proved that $\mathcal{D}^{\mathrm{b}}(\mathrm{Coh}\,\mathbb{X}_n)$ contains a tilting complex $\mathcal{H}$. The opposite of its endomorphism algebra is isomorphic to the algebra Λ_n which is by definition the quotient of the path algebra of the quiver $Q(n)$ shown in Figure 2 by the ideal generated by the set of paths

$$R := \{b_i a_i, d_i c_i \mid i \in [0, n)\}. \tag{2.1}$$

In other words, the composition of arrows in Figure 2 with different colour vanishes.

For example, Λ_1 is the quotient of the path algebra of the quiver on the left in Figure 3 by the ideal (ba, dc) while the quiver of Λ_3 is shown on the right-hand side.

For every $n \geq 1$, the algebra Λ_n is gentle in the sense of [3] and has global dimension 2. In particular, there is an Auslander–Reiten translation

$$\tau \colon \mathcal{D}^{\mathrm{b}}(\Lambda_n) \to \mathcal{D}^{\mathrm{b}}(\Lambda_n)$$

and $\tau \cong \nu[-1]$, where ν denotes the left derived Nakayama functor.
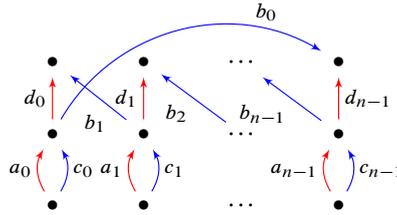
Fig. 2. The underlying quiver $Q(n)$ of the algebra Λ_n. Arrows of different colours compose to zero in Λ_n, i.e., $b_i a_i = 0 = d_i c_i$ for all $i \in \mathbb{Z}_n$.
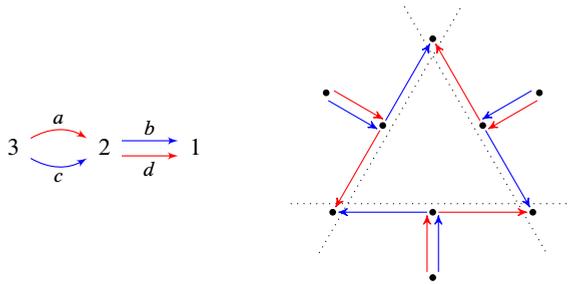

Fig. 3. The quivers $Q(1)$ (left) and $Q(3)$ (right).

It follows from the preceding results that there exists an embedding of triangulated categories

$$\mathbb{F} \colon \operatorname{Perf}(C_n) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda_n).$$

Burban and Drozd provided a characterization of its essential image $\operatorname{Im} \mathbb{F}$.

Theorem 2.2 ([12, Corollaries 3.5 and 6.3]). *The category $\operatorname{Im} \mathbb{F}$ is the full subcategory of τ-invariant objects, i.e., all objects $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that $X \cong \tau X$.*

They further computed the image of the Jacobian $\operatorname{Pic}^0(C_1) \cong \mathbb{k}^\times$ [12, Proposition 7.4], i.e., the line bundles of degree 0, as well as the images of the skyscraper sheaves $\mathbb{k}(x)$ of smooth points $x \in C_1$ [12, Proposition 7.2]. By twisting their tilting object with a line bundle of degree 1, the functor $\mathbb{F}$ identifies isomorphism classes of line bundles of degree 0 with the isomorphism classes of the following family $(\mathcal{O}(\lambda))_{\lambda \in \mathbb{k}^\times}$ of two-term complexes:[1]

$$\mathcal{O}(\lambda) = \cdots \to 0 \to P_1 \xrightarrow{b+\lambda d} P_2 \to 0 \to \cdots,$$

where P_i denotes the indecomposable projective Λ_1-module associated to the vertex i in $Q(1)$. The set $\{\mathbb{k}(x) \mid x \in C_1 \text{ is smooth}\}$ is identified with the set of complexes $\{\mathbb{k}(\lambda) \mid \lambda \in \mathbb{k}^\times\}$, where

$$\mathbb{k}(\lambda) = \cdots \to 0 \to P_2 \xrightarrow{a+\lambda c} P_3 \to 0 \to \cdots.$$

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[1] This observation was communicated to us by Igor Burban.

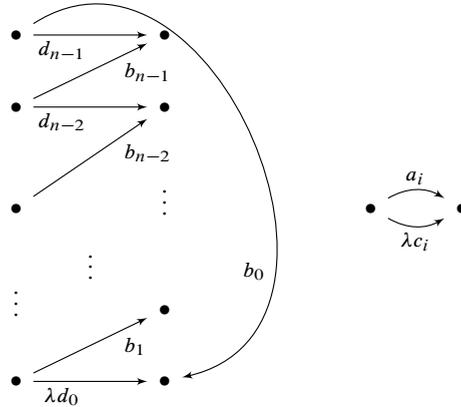**Fig. 4.** The quiver description of the complexes $\mathcal{O}(\lambda)$ (left) and $\Bbbk(\lambda)$ (right).

Fig. 5. The images under $\mathbb{F}$ of $\operatorname{Pic}^{\mathbb{0}}(C_n)$ (left) and $\Bbbk(x)$ of a smooth point $x \in \mathbb{P}^1_i$ (right).

Both $\mathcal{O}(\lambda)$ and $\Bbbk(\lambda)$ are concentrated in degrees -1 and 0 and are quasi-isomorphic to Λ_1-modules. We notice that, up to shift, they are completely encoded in the quivers in Figure 4.

Every vertex of the quivers represents an indecomposable projective Λ_1-module. Arrows connect projective modules in consecutive cohomological degrees and their labels describe the differentials of the complexes.

The complexes $\mathcal{O}(\lambda)$ and $\Bbbk(\lambda)$ have natural generalizations to objects in $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$. Their description in terms of quivers is found in Figure 5.

Every vertex in Figure 5 is the placeholder for an indecomposable projective Λ_n-module and all vertices on the same vertical line represent the direct summands in a fixed cohomological degree which is either -1 or 0 and which increases along arrows. The arrows describe the action of the differential on the complex in which an arrow $\bullet \xrightarrow{\alpha} \bullet$ contributes a direct summand $P_{t(\alpha)}$ in degree -1 and $P_{s(\alpha)}$ in degree 0 and the induced map $\alpha\colon P_{t(\alpha)} \to P_{s(\alpha)}$ as a component to the differential. In other words, the complexes are given by "folding up" the quivers in Figure 5 in the vertical direction. Explicitly, the complex of the quiver on the left-hand side of Figure 5 is given by

$$\cdots \to 0 \to \bigoplus_{i=0}^{n-1} P_{t(d_i)} \xrightarrow{\begin{pmatrix} \lambda d_0 & & & & \\ b_1 & d_1 & & & \\ & b_2 & \ddots & & \\ & & \ddots & d_{n-1} \\ & & & b_0 \end{pmatrix}} \bigoplus_{i=0}^{n-1} P_{s(d_i)} \to 0 \to \cdots .$$

By abuse of notation, we denote the resulting complexes by $\mathcal{O}(\lambda)$ (left in Figure 5) and $\Bbbk(i, \lambda)$ (right in Figure 5), respectively. Again, these complexes are quasi-isomorphic to Λ_n-modules. The complex of $\Bbbk(i, \lambda)$ is obtained from the complex of $\Bbbk(\lambda)$ by replacing a and λc by a_i and λc_i.

It was communicated to us by Igor Burban that the computations in [12] can be generalized to arbitrary $n \in \mathbb{N}$. We denote by $\mathrm{Pic}^{\mathbb{0}}(C_n)$ the set of all line bundles on C_n of multi-degree[2] $\mathbb{0} = (0, \ldots, 0) \in \mathbb{Z}^n$. As in the case $n = 1$, $\mathrm{Pic}^{\mathbb{0}}(C_n) \cong \Bbbk$.

Theorem 2.3 (Burban, Burban–Drozd [12]). (1) *The essential image of* $\mathrm{Pic}^{\mathbb{0}}(C_n)$ *under* $\mathbb{F}$ *consists of the isomorphism classes of the complexes* $\mathcal{O}(\lambda)$, *where* $\lambda \in \Bbbk^{\times}$.

(2) *The essential image of the skyscraper sheaves of smooth points* $x \in \mathbb{P}_i^1$ *under* $\mathbb{F}$ *consists of the isomorphism classes of the complexes* $\Bbbk(i, \lambda)$, *where* $\lambda \in \Bbbk^{\times}$.

The proof of Theorem 2.3 was explained to us by Igor Burban and can be found in the appendix.

3. Serre functors, spherical objects and spherical twists

In this section, we recall the definition of a *spherical object* in a triangulated category and its associated *spherical twist*. Spherical objects and spherical twists were first introduced in [34].

Throughout this section, we fix a $\Bbbk$-linear triangulated category $\mathcal{T}$.

Definition 3.1. Let $X \in \mathcal{T}$ be such that $\mathrm{Hom}_{\mathcal{T}}^{\bullet}(X, Y)$ is finite-dimensional for all $Y \in \mathcal{T}$. Then a *Serre dual* of X is an object $\mathcal{S}(X) \in \mathcal{T}$ such that there exists a $\Bbbk$-linear isomorphism of functors

$$\mathrm{Hom}_{\mathcal{T}}(X, -) \cong \mathrm{Hom}_{\mathcal{T}}(-, \mathcal{S}(X))^{*},$$

where $(-)^{*}$ denotes the duality over the ground field $\Bbbk$.

By Yoneda's lemma, the Serre dual of an object is unique up to unique isomorphism, if it exists, and it follows that the mapping $X \mapsto \mathcal{S}(X)$ is functorial on the full subcategory $\mathcal{C}$ spanned by the objects it is defined on. If the functor $\mathcal{S}: \mathcal{C} \to \mathcal{T}$ restricts to an endo-functor of $\mathcal{C}$ and is essentially surjective, then any such functor is called a *Serre functor* of $\mathcal{C}$ relative to $\mathcal{T}$. For purely formal reasons, $\mathcal{C}$ is triangulated and $\mathcal{S}$ is a $\Bbbk$-linear triangle equivalence which commutes with every $\Bbbk$-linear triangle equivalence of $\mathcal{T}$ or $\mathcal{C}$ up to natural isomorphism, see [33, §I.1].

Example 3.2. The categories $\mathrm{Perf}(C_n)$ and $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ have Serre functors. In the former case, it is well known that the left derived tensor product $- \otimes^{\mathbb{L}} \omega[1]$ is a Serre functor of $\mathrm{Perf}(C_n)$ relative to $\mathcal{D}^{\mathrm{b}}(C_n)$, where ω denotes the canonical sheaf. In fact, ω is trivial,

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[2]Given a vector bundle $\mathcal{E}$ over C_n and a normalization map π, the entry d_i in the multi-degree $(d_i)_{i \in \mathbb{Z}_n}$ of $\mathcal{E}$ is the degree of the restricted pullback bundle $\pi^{*}(\mathcal{E})|_{\pi^{-1}(\mathbb{P}_i^1)}$.

showing that the Serre functor is isomorphic to the shift functor. Since Λ_n has finite global dimension, the left derived functor of the Nakayama functor

$$\nu = (\mathrm{Hom}_{\Lambda_n}(-, {}_{\Lambda_n}\Lambda_n))^*\colon \Lambda_n\,\mathrm{mod} \to \Lambda_n\,\mathrm{mod}$$

is a Serre functor of $\mathcal{D}^{\mathrm{b}}(\Lambda_n) = \mathcal{T} = \mathcal{C}$.

Definition 3.3. Let $d \in \mathbb{N} \setminus \{0\}$. An object $X \in \mathcal{T}$ is *d-Calabi–Yau* if $X[d]$ is a Serre dual of X. A d-Calabi–Yau object $X \in \mathcal{T}$ is *d-spherical* if there exists an isomorphism of graded rings $\mathrm{Hom}^\bullet(X, X) \cong \mathbb{k}[z]/(z^2)$, where $\deg z = d$.

It follows from Example 3.2 that every object $X \in \mathrm{Perf}(C_n)$ is 1-Calabi–Yau. In particular, all of its spherical objects are 1-spherical.

Example 3.4. (1) Every simple vector bundle $\mathcal{L} \in \mathrm{Perf}(C_n)$ is 1-spherical. As a sheaf it has no negative self-extensions. Moreover, $\mathrm{End}(\mathcal{L}, \mathcal{L}) \cong \mathbb{k}$ and by Serre duality,

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}[i]) \cong \mathrm{Hom}(\mathcal{L}, \mathcal{L}[1-i])^* \cong \begin{cases} \mathbb{k} & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

In particular, all line bundles on C_n are spherical.

(2) Let $x \in C_n$ be smooth. By a similar argument as before, the associated skyscraper sheaf is a 1-spherical object in $\mathrm{Perf}(C_n)$.

The following is well known.

Lemma 3.5. *Let $X \in \mathcal{T}$ be spherical. Then X is indecomposable.*

Proof. By assumption, $\mathrm{Hom}(X, X)$ is local and hence X is indecomposable. ∎

Spherical twists. Suppose that $\mathcal{T}$ admits a DG-enhancement. The assumption is satisfied for all triangulated categories which we consider in this paper as shown by Lunts and Schnürer [27]. Under this condition, it was shown by Seidel and Thomas [34] (see also [20]) that every spherical object $X \in \mathcal{T}$ gives rise to an auto-equivalence T_X of $\mathcal{T}$, called a *spherical twist*. By definition, for every $Y \in \mathcal{T}$, the object $T_X(Y)$ sits in a distinguished triangle of the form

$$\mathrm{Hom}^\bullet(X, Y) \otimes_k X \xrightarrow{\ \mathbf{ev}\ } Y \to T_X(Y) \to (\mathrm{Hom}^\bullet(X, Y) \otimes_k X)[1], \qquad (3.1)$$

where **ev** denotes the evaluation map, i.e., the counit of the adjunction between the Hom-functor and the tensor product. Moreover, the morphisms $Y \to T_X(Y)$ define a natural transformation $\mathrm{Id}_{\mathcal{T}} \to T_X$.

Remark 3.6. Let $x \in C_n$ be closed and smooth. In this case, the twist functor of $k(x)$ admits a more familiar description. As shown in [34, p. 68, (3.11)], there exists an isomorphism of functors

$$T_{\mathbb{k}(x)}(-) \cong - \otimes^{\mathbb{L}} \mathcal{L}(x),$$

where $\mathcal{L}(x)$ denotes the line bundle associated with the divisor x.

Given a homogeneous basis $f_1, \ldots, f_m$ of $\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}(X[i], Y)[-i]$, the triangle in (3.1) is isomorphic to a distinguished triangle

$$\bigoplus_{i=1}^m X[n_i] \xrightarrow{\oplus_{i=1}^m f_i} Y \to T_X(Y) \to \bigoplus_{i=1}^m X[n_i + 1].$$

This shows that twist functors are compatible with embeddings in the following sense.

Corollary 3.7. *Let $\mathcal{T}'$ be a $\mathbb{k}$-linear triangulated category which admits a DG-enhancement and let $\mathbb{F}: \mathcal{T} \to \mathcal{T}'$ be a $\mathbb{k}$-linear, exact and fully faithful functor. Let $X \in \mathcal{T}$ be spherical and assume that $\mathbb{F}(X)$ is spherical. Then, for all $Y \in \mathcal{T}$,*

$$\mathbb{F} \circ T_X(Y) \cong T_{\mathbb{F}(X)} \circ \mathbb{F}(Y).$$

In particular, if $\mathbb{F}$ is an equivalence and $\mathbb{F}^{-1}$ a quasi-inverse of $\mathbb{F}$, then $\mathbb{F} \circ T_X \circ \mathbb{F}^{-1}(Y) \cong T_{\mathbb{F}(X)}(Y)$.

The previous corollary allows us to analyze the twist functor of any spherical object $X \in \operatorname{Perf}(C_n)$ by means of the twist along $\mathbb{F}(X) \in \mathcal{D}^b(\Lambda_n)$.

The statement of the following lemma is well known.

Lemma 3.8. *If $X \in \mathcal{T}$ is d-spherical, then $T_X(X) \cong X[1 - d]$. If $\operatorname{Hom}^\bullet(X, Y) = 0$, then $T_X(Y) \cong Y$.*

Proof. If $f: X \to X[d]$ is non-zero, then $T_X(X)$ is isomorphic to the mapping cone of $\operatorname{Id}_X[-d] \oplus f: X[-d] \oplus X \to X$. There exists a map of distinguished triangles

$$
\begin{array}{ccccccc}
X[-d] \oplus X & \xrightarrow{(f\ \operatorname{Id}_X)} & X & \longrightarrow & T_X(X) & \longrightarrow & (X[-d] \oplus X)[1] \\
\left(\begin{smallmatrix} \operatorname{Id}_X[-d] & 0 \\ f & \operatorname{Id}_X \end{smallmatrix}\right) \downarrow \wr & & \| & & \downarrow & & \downarrow \\
X[-d] \oplus X & \xrightarrow{(0\ \operatorname{Id}_X)} & X & \longrightarrow & X[1 - d] & \longrightarrow & (X[-d] \oplus X)[1].
\end{array}
$$

It follows from the five lemma that all vertical arrows are isomorphisms. This proves the first assertion. If $\operatorname{Hom}^\bullet(X, Y) = 0$, then $T_X(Y)$ is a mapping cone of the zero map $0 \to Y$ and hence isomorphic to Y. ∎

4. The surface model of the categorical resolution

We recall the relevant results from [31] about the surface model of a gentle algebra which we state in a simplified manner for the algebras Λ_n.

The surface model of the algebra Λ_n consists of a triple $(\mathbb{T}_n, \mathcal{M}, \omega)$, where

- $\mathbb{T}_n$ is a torus with n boundary components and $\mathcal{M} \subset \partial \mathbb{T}_n$ is a set of $2n$ marked points, two on each component, and
- ω is a function which attaches to any oriented loop $\gamma \subseteq \mathbb{T}_n$ an integer $\omega(\gamma)$ which we call the *winding number of γ*.

The triple $(\mathbb{T}_n, \mathcal{M}, \omega)$ captures all essential information about the structure of the triangulated category $\mathcal{D}^b(\Lambda_n)$. Haiden, Katzarkov and Kontsevich showed that the derived category of any (graded) gentle algebra is equivalent to a partially Fukaya category of a compact, oriented surface with marked points which is equipped with a winding number function as above. The topological description of $\mathcal{D}^b(\Lambda_n)$ is an incarnation of this relationship. As we will recall over the course of this section, the geometric model serves as a dictionary to translate algebraic terms such as objects and morphisms to topological notions such as curves and their intersections.

The function ω is derived from the datum of a *line field* η on $\mathbb{T}_n$. For a description of this connection, the reader can consult [1, 30]. However, we do not require any familiarity with the concept of a line field throughout the paper and only use this fact in the proof of Lemma 5.3 and Remark 6.6, where we provide the relevant information.

4.1. Tori, marked points and a lamination

For all integer valued points $(i, j) \in \mathbb{Z}^2 \subseteq \mathbb{R}^2$, let $B_i^j \subseteq \mathbb{R}^2$ denote the open disc with radius $\frac{1}{4}$ and center $(i + \frac{1}{2}, j + \frac{1}{2})$. Denote by $\mathbb{T}_n$ the torus with n removed open discs, i.e., $\mathbb{T}_n$ is the quotient of $\mathbb{R}^2 \setminus B$, where

$$B = \bigsqcup_{(i,j) \in \mathbb{Z}^2} B_i^j,$$

with respect to the equivalence relation generated by $(r, s) \sim (r, s + 1)$ and $(r, s) \sim (r + n, s)$ for all $r, s \in \mathbb{R}$. In particular, $[0, n) \times [0, 1)$ is a fundamental domain of the corresponding quotient map ρ. For convenience, we often refer to a point $x \in \mathbb{T}_n$ by a representative in $\mathbb{R}^2$.

The set of marked points $\mathcal{M}$ is given by

$$\mathcal{M} = \left\{ \left(i + \frac{1}{2}, \frac{3}{4}\right), \left(i + \frac{1}{2}, \frac{1}{4}\right) \;\middle|\; 0 \leq i < n \right\} \subseteq \partial \mathbb{T}_n.$$

Frequently, we also consider the torus $\mathcal{T}^n$ with n punctures and no boundary which we naturally regard as the quotient of $\mathbb{R}^2$ by the same relations $(r, s + 1) \sim (r, s) \sim (r + n, s)$. The images of the centers of the discs B_i^j form a set of n distinct points in the quotient which we regard as interior marked points ("punctures") and we consider $\mathbb{T}_n$ being embedded into $\mathcal{T}^n$ in a natural way.

By design, the set $\mathcal{M} \subset \partial \mathbb{T}_n$ is in bijection with the set of maximal admissible paths of the pair $(Q(n), R)$ which we defined in Figure 2 and (2.1). More precisely, for each $i \in [0, n)$, the bijection identifies the path $d_i a_i$ with the point $(i + \frac{1}{2}, \frac{1}{4})$ and the path $b_i c_i$ with the point $(i - \frac{1}{2}, \frac{3}{4})$.

As a result of its construction, $\mathbb{T}_n$ is further equipped with a collection of *laminates* which are embedded and pairwise disjoint paths L_x, one for each vertex x of $Q = Q(n)$. The laminates are depicted in Figure 6. We denote by $L \subseteq \mathbb{T}_n$ the union of all laminates. The complement $\mathbb{T}_n \setminus L$ is a disjoint union of $2n$ open 6-gons as shown in Figure 7. The boundary of each polygon contains a unique marked point.
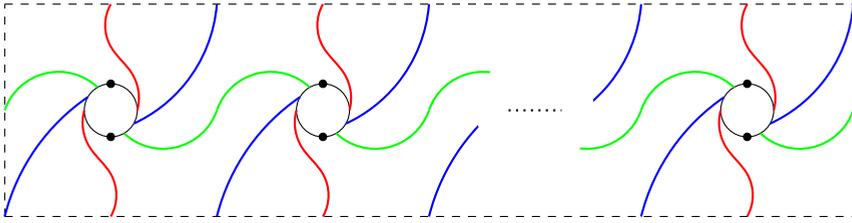
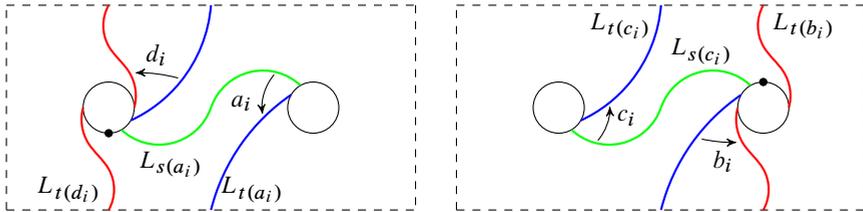**Fig. 6.** The laminates on $\mathbb{T}_n$. Parallel dashed lines are identified.

Fig. 7. The two types of 6-gons occurring in $\mathbb{T}_n \setminus L$. Recall that $L_{t(d_i)} = L_{t(b_{i+1})}$, $L_{s(a_i)} = L_{s(c_i)}$ and $L_{t(a_i)} = L_{t(c_i)} = L_{s(d_i)} = L_{s(b_i)}$.

If Δ denotes a 6-gon in the complement with marked point $\bullet \in \partial\Delta$ which corresponds to a maximal admissible path $\beta\alpha$, then $L_{s(\alpha)}$, $L_{t(\alpha)}$, $L_{t(\beta)}$ are the laminates on its boundary in the clockwise orientation of Δ.

The laminates cut $\partial\mathbb{T}_n$ into segments. Each such boundary segment δ sits in a polygon which contains a marked point $\bullet$ and hence determines a unique admissible path: if p is the maximal admissible path corresponding to $\bullet$ and δ is bounded by the laminates L_{x_s} and L_{x_t} (following the counter-clockwise orientation on the boundary), then we associate with δ the subpath q of p which starts on x_s and ends on x_t. In particular, $q = p$ if and only if δ contains $\bullet$. For the remaining cases, see Figure 7.

4.2. Curves and winding numbers

Certain homotopy classes of curves in $\mathbb{T}_n$ give rise to indecomposable objects in $\mathcal{D}^b(\Lambda_n)$. We consider two types of curves, namely *loops* and *arcs*. The former are closed whereas the latter start and end at marked points. Before we explain how to construct a complex from a curve, we introduce some necessary background material. As indicated above, not all loops give rise to an object and the necessary condition can be expressed in terms of its winding number which we define in Section 4.2.3.

4.2.1. Loops and arcs.

Suppose Σ is a compact, oriented surface, with or without boundary, with a finite subset $\mathcal{M} \subseteq \Sigma$ of marked points. The interior marked points are called *punctures* and their set shall be denoted by $\mathscr{P}$. We will mainly consider two examples in this paper: $\Sigma = \mathbb{T}_n$ or $\Sigma = \mathscr{T}^n$.

A *curve* in Σ is an immersion $\gamma \colon \Omega \to \Sigma$, where $\Omega = [0, 1]$ or $\Omega = S^1$ and $\partial\Omega = \gamma^{-1}(\mathcal{M})$. If $\Omega = [0, 1]$, the curve γ is called an *arc* and if $\Omega = S^1$, it is called a *loop*.

A *homotopy of curves* is a homotopy $H \colon [0, 1] \times \Omega \to \Sigma$ of the underlying maps which is constant on $[0, 1] \times \partial\Omega$ and such that $H^{-1}(\mathscr{P}) \subseteq [0, 1] \times \partial\Omega$. By definition, we assume that a loop is not homotopic to a constant map and that an arc cannot be homotoped into a contractible neighbourhood $U \subset \Sigma$ which contains a single puncture. In particular, there exists a canonical bijection between homotopy classes of loops on $\mathbb{T}_n$ and loops on $\mathscr{T}^n$.

Throughout the paper, we will stick to a few conventions:

- Unless stated otherwise, we assume loops to be primitive: a loop $\gamma \colon S^1 \to \Sigma$ is said to be *primitive* if γ is not homotopic to a loop which factors through a non-trivial covering map of S^1.

- Up to homotopy, we may and will assume that the number of intersections of a curve with all the laminates is minimal in its homotopy class and that all such intersections are double points which lie in the interior.

- Depending on the circumstances, we regard curves as oriented or unoriented objects by choosing (or not choosing) an orientation on its domain. The correct interpretation of the words "arc" and "loop" should be apparent from the context.

4.2.2. Gradings on curves. Let γ be a curve on $\mathbb{T}_n$ and denote by $\gamma \cap L$ the set of intersections of γ with the laminates. Regarding $\gamma \cap L$ as a subset of the domain of γ in a natural way, we can speak of *neighbouring* intersections.

Definition 4.1. A *grading* of γ is a function $g \colon \gamma \cap L \to \mathbb{Z}$ which satisfies the following property for all pairs (p, q) of neighbouring intersections $p, q \in \gamma \cap L$:

If δ denotes the segment of γ between p and q, oriented such that δ starts at p and ends at q, and $\Delta \subseteq \mathbb{T}_n \setminus L$ denotes the open polygon which contains $\delta \setminus \{p, q\}$, then

$$g(q) = \begin{cases} g(p) + 1 & \text{if the marked point in } \Delta \text{ is on the left of } \delta, \\ g(p) - 1 & \text{otherwise.} \end{cases}$$

The notion of grading carries over to homotopy classes of curves in an obvious way. A *graded curve* is a pair (γ, g), where γ is a curve and g is a grading of γ. If g is a grading of γ and $n \in \mathbb{Z}$, then so is its *shift* $g[n]$, defined by $g[n](p) = g(p) - n$ for all $p \in \gamma \cap L$. While all arcs can be graded, there exist loops which do not admit a grading and hence we say that a loop (or generally any curve) is *gradable* if it admits a grading.

If a curve γ is gradable, then any grading g is determined completely by a single value $g(p)$ for any $p \in \gamma \cap L$ and all gradings of γ are shifts of g.

4.2.3. The quiver of a curve and the winding number of a loop. Let $\gamma \subseteq \mathbb{T}_n$ be an oriented curve. Denote by $q_0, \ldots, q_m$ the ordered sequence of elements in $\gamma \cap L$, and for each $i \in [0, m]$ denote by L_{x_i} the laminate which contains q_i. Then, we write $P_i = \Lambda_n x_i$ for the indecomposable projective Λ_n-module associated with x_i and denote by γ_i the

segment of γ between q_i and q_{i+1}. The orientation of γ induces a canonical orientation on γ_i.

This data gives rise to a quiver $Q(\gamma)$ which is of type $\widetilde{A}$ if γ is a loop and which is of type A if γ is an arc. We will see that for every grading g of γ, $Q(\gamma)$ encodes complexes in the same way as the quivers in Figures 4 and 5 encode the complexes $\mathcal{O}(\lambda)$ and $\Bbbk(i, \lambda)$.

The quiver $Q(\gamma)$ has vertices $\{P_0, \ldots, P_m\}$ as defined above and arrows $\sigma_0, \ldots, \sigma_m$ whose start and end points are determined by the segments γ_i in the following way. For each $i \in [0, m)$, σ_i connects P_i with P_{i+1}. More precisely, if $\bullet$ denotes the unique marked point in the polygon $\Delta_i \supseteq \gamma_i$, we have

$$\sigma_i = \begin{cases} P_i \to P_{i+1} & \text{if } \bullet \text{ lies to the left of } \gamma_i \text{ in } \Delta_i, \\ P_i \leftarrow P_{i+1} & \text{if } \bullet \text{ lies to the right of } \gamma_i \text{ in } \Delta_i. \end{cases}$$

Definition 4.2. Assume that γ is a loop. The *winding number* $\omega(\gamma)$ *of* γ is defined as the difference $b - a$, where a denotes the number of arrows σ_i from P_i to P_{i+1} and b denotes the number of arrows σ_j from P_{j+1} to P_j.

A change of orientation of a loop changes the sign of its winding number. On the other hand, one observes that $Q(\gamma)$ does not depend on the chosen orientation as an abstract quiver. Gradable loops can be characterized as follows.

Lemma 4.3. *A loop γ is gradable if and only if $\omega(\gamma) = 0$ if and only if $Q(\gamma)$ has the same number of clockwise and counter-clockwise arrows.*

Note that whether a loop is gradable or not does not depend on the choice of an orientation.

Example 4.4. Every boundary component $B \subseteq \partial \mathbb{T}_n$ determines an embedding $S^1 \to \mathbb{T}_n$ which we regard as a simple loop γ after a small deformation. Then γ inherits a canonical orientation from B (the induced orientation on $\partial \mathbb{T}_n$). The quiver of γ is depicted in Figure 8 from which we determine its winding number as $2 - 4 = -2$. We refer to this winding number as the *winding number of B*.
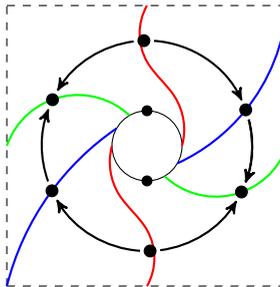

Fig. 8. The quiver of the clockwise boundary loop with 4 clockwise and 2 counter-clockwise arrows.

4.3. The complex of a graded curve

In order to define a complex of a curve, we recall the definition of a local system.

Definition 4.5. Let γ be a curve with domain Ω. A *local system* $\mathcal{V}$ on γ consists of a collection of vector spaces $(\mathcal{V}_z)_{z \in \Omega}$ and isomorphisms $\mathcal{V}_u : \mathcal{V}_{u(0)} \to \mathcal{V}_{u(1)}$ for every path $u : [0, 1] \to \Omega$ which only depends on the homotopy class of u. Moreover, if v is a path with $v(0) = u(1)$, then $\mathcal{V}_{v*u} = \mathcal{V}_v \circ \mathcal{V}_u$, where $v * u$ denotes the concatenation of v with u.

The well-defined number $\dim \mathcal{V} := \dim \mathcal{V}_z$ is the *dimension of* $\mathcal{V}$. After the choice of a base point on Ω, one sees that isomorphism classes of indecomposable local systems (coproducts being defined pointwise) on a loop are in bijection with powers of irreducible polynomials over $\mathbb{k}$ and hence with the polynomials $(X - \lambda)^m$, where $\lambda \in \mathbb{k}^\times$ and $m = \dim \mathcal{V}$. On the other hand, every indecomposable local system on an arc is isomorphic to the constant 1-dimensional local system, i.e., $\mathcal{V}_z = \mathbb{k}$ for all $z \in [0, 1]$ and every isomorphism $\mathcal{V}_{u(0)} \to \mathcal{V}_{u(1)}$ above is the identity.

Let $\mathcal{V}$ be a local system on a graded curve (γ, g). We attach a complex $P_{(\gamma,g)}^\bullet(\mathcal{V})$ to the quiver $Q(\gamma)$ by, roughly speaking, "folding" $Q(\gamma)$ to a complex in the same way as the quivers in Figure 5.

We use the notation of Section 4.2.3. Set $\mathcal{V}_i = \mathcal{V}_{q_i}$. Then, as a graded Λ_n-module,

$$P_{(\gamma,g)}^\bullet(\mathcal{V}) = \bigoplus_i (P_i \otimes \mathcal{V}_i)[-g(q_i)].$$

The differential is described as follows. For each i, let t_i denote the map between P_i and P_{i+1} induced by the unique admissible path α_i determined by the boundary segment in $\partial \mathbb{T}_n \cap \partial \Delta_i$ which is bounded by L_{x_i} and $L_{x_{i+1}}$, cf. the last paragraph of Section 4.1. The segment γ_i of γ which is bounded by L_{x_i} and $L_{x_{i+1}}$ also determines an isomorphism u_i between $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$. The differential is the sum of all maps $t_i \otimes u_i$, shifted appropriately. Lemma 4.3 ensures that the procedure above defines a differential. Note that $t_i : P_{t(\alpha_i)} \to P_{s(\alpha_i)}$ and $(x_i, x_{i+1}) = (t(\alpha_i), s(\alpha_i))$ if and only if the marked point in Δ_i lies to the left of γ_i. Hence, the domain (resp. codomain) of t_i agrees with the start (resp. end) of σ_i in $Q(\gamma)$.

The above construction respects direct sums in the sense that

$$P_{(\gamma,g)}^\bullet(\mathcal{U} \oplus \mathcal{V}) \cong P_{(\gamma,g)}^\bullet(\mathcal{U}) \oplus P_{(\gamma,g)}^\bullet(\mathcal{U}).$$

In fact, $P^\bullet = P_{(\gamma,g)}^\bullet(\mathcal{V})$ is indecomposable if and only if $\mathcal{V}$ is indecomposable and its isomorphism class is invariant under isomorphisms of local systems and change of orientation on γ. If γ is a loop and $\mathcal{V}$ is indecomposable, then $P^\bullet$ is an instance of a *band complex* in the sense of [5, 11]. In particular, $\tau P^\bullet \cong P^\bullet$ as first proved by Bobinski [6]. Moreover, $\dim \mathcal{V}$ is the level of $P^\bullet$ in the corresponding homogeneous tube.

If γ is an arc and $\mathcal{V}$ is indecomposable, then $P^\bullet$ is a so-called *string complex*. As we may assume that $\mathcal{V}$ is constant, we often suppress the additional datum of a local system in this case.

Every homotopy $H : [0, 1] \times \Omega$ between curves $\gamma_0 = H|_{\{0\} \times \Omega}$ and $\gamma_1 = H|_{\{1\} \times \Omega}$, which are in minimal position with L, induces a bijection $\gamma_0 \cap L \cong \gamma_1 \cap L$ and hence

a bijection between the sets gradings on γ_0 and γ_1. Because the domains of γ_0 and γ_1 agree, we further have a canonical bijection between local systems on γ_0 and γ_1. The following statement is found in [31, Theorem 2.12]. The bijection between curves and indecomposable objects was first proved in [19].

Proposition 4.6. *Let $\mathcal{X}$ denote the set of triples $(\gamma, g, \mathcal{V})$, where (γ, g) is a graded curve and $\mathcal{V}$ is an indecomposable local system on γ. Let further $\sim$ denote the equivalence relation on $\mathcal{X}$ such that $(\gamma, g, \mathcal{V}) \sim (\gamma', g', \mathcal{V}')$ if and only if there exists a homotopy between γ and γ' under which g corresponds to g' and $\mathcal{V}$ corresponds to $\mathcal{V}'$. The assignment*

$$(\gamma, g, \mathcal{V}) \to P_{(\gamma, g)}^{\bullet}(\mathcal{V}),$$

induces a bijection from the set of $\sim$-equivalence classes of $\mathcal{X}$ to the set of isomorphism classes of indecomposable objects in $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$.

In summary, Proposition 4.6 states that every indecomposable object $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ is represented by a curve endowed with a grading and an indecomposable local system. To simplify notation, we often refer to such a curve and the local system by γ_X and $\mathcal{V}_X$, respectively, so that $X \cong P_{(\gamma_X, g)}^{\bullet}(\mathcal{V}_X)$ for some unique grading g on γ_X. In this case, we say that X is *represented* by γ_X. Finally, if $Y \in \mathrm{Perf}(C_n)$ is indecomposable, we write γ_Y and $\mathcal{V}_Y$ as short for $\gamma_{\mathbb{F}(Y)}$ and $\mathcal{V}_{\mathbb{F}(Y)}$ and say that Y is represented by γ_Y. As a consequence of Theorem 2.2, we prove in Section 4.3.1 that loops on $\mathbb{T}_n$ (or equivalently, on $\mathcal{T}^n$) represent precisely the indecomposable perfect complexes of C_n.

Example 4.7. Let $\gamma_{\mathrm{Pic}} \colon S^1 \to \mathbb{T}_n$ be the loop defined by $\gamma_{\mathrm{Pic}}(e^{2\pi i t}) := (n \cdot t, \frac{1}{8})$ for all $t \in [0, 1]$. Similarly, for $j \in [0, n)$, let $\gamma_{\mathbb{k}(x)}^j \colon S^1 \to \mathbb{T}_n$ denote the loop defined by

$$\gamma_{\mathbb{k}(x)}^j(e^{2\pi i t}) := (j + 1, t),$$

where $t \in [0, 1]$. The curves γ_{Pic} and $\gamma_{\mathbb{k}(x)}^j$ are displayed in Figure 9. Replacing the vertices of $Q(\gamma_{\mathrm{Pic}})$ by bullet points and assigning the underlying path of t_i as the label of the arrow σ_i, $Q(\gamma_{\mathrm{Pic}})$ becomes the quiver on the left-hand side of Figure 5, whereas $Q(\gamma_{\mathbb{k}(x)}^i)$ is the quiver on the right-hand side. Thus, suppressing the grading we have $P_{\gamma_{\mathrm{Pic}}}^{\bullet}(\mathcal{V}) \cong \mathcal{O}(\lambda)$ and $P_{\gamma_{\mathbb{k}(x)}^i}^{\bullet}(\mathcal{W}) \cong \mathbb{k}(i, \lambda)$ for suitable 1-dimensional local systems $\mathcal{V}$ and $\mathcal{W}$.
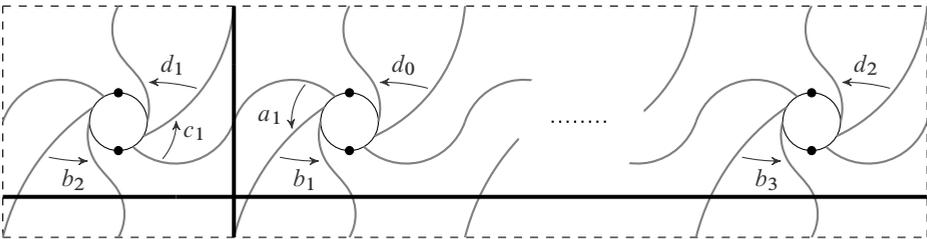

Fig. 9. The loops γ_{Pic} (horizontal) and $\gamma_{\mathbb{k}(x)}^1$ (vertical).

4.3.1. Fractional Calabi–Yau objects: Loops and boundary arcs. One of the main differences between band complexes and string complexes is that the latter are generally not τ-invariant. A criterion which allows us to characterize the τ-invariant indecomposable objects in the derived category of any gentle algebra is provided in [30, Proposition 2.16]. It says that a string complex X satisfies $\tau^m X \cong X[d]$ for some $m > 0$ and some $d \in \mathbb{Z}$ if and only if γ_X is homotopic to a boundary arc on a component with exactly m' marked points and winding number $-d'$ such that (m, d) is an integer multiple of (m', d').

Proposition 4.8. *Let $m \geq 1$, $d \in \mathbb{Z}$ and let $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$. Then the following are true:*

(1) *If $\tau^m X \cong X[d]$, then X is τ-invariant or $\tau^2 X \cong X[2]$.*

(2) *X is τ-invariant if and only if each indecomposable direct summand of X is represented by a loop.*

(3) *We have $\tau^2 X \cong X[2]$ if and only if each indecomposable direct summand of X is represented by a boundary arc.*

Proof. From Example 4.4 and the discussion preceding the proposition, we know that an indecomposable object $Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ satisfies $\tau^p Y \cong Y[q]$ for a pair $(p, q) \in \mathbb{Z}^2$ if and only if there exists $(m, d) \in \{(1, 0), (2, 2)\}$ such that $\tau^m Y \cong Y[d]$. If $(m, d) = (1, 0)$, Y is represented by a loop, and if $(m, d) = (2, 2)$, then Y is represented by a boundary arc. Suppose that X satisfies $\tau^m X \cong X[d]$ and $X \cong \bigoplus_{i=1}^r X_i$ with each X_i indecomposable. Then, for each $j \in [1, r]$, there exists a non-empty subset $J \subset \{1, \ldots, r\}$ such that $j \in J$ and a cyclic permutation $\sigma \colon J \to J$ such that $\tau^m X_i \cong X_{\sigma(i)}[d]$ for all $i \in J$. Since $\sigma^{|J|} = \mathrm{Id}_J$, we conclude that $\tau^{m|J|} X_j \cong X_j[d|J|]$ and that $(m|J|, d|J|)$ is an integer multiple of $(1, 0)$ or $(2, 2)$. Since $|J| \neq 0$, one easily derives that either $\tau X_i \cong X_i$ for all $i \in [1, r]$ or $\tau^2 X_i \cong X_i[2]$ for all $i \in [1, r]$. ∎

Corollary 4.9. *Let $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ be indecomposable. Then X lies in the essential image of the functor $\mathbb{F} \colon \mathrm{Perf}(C_n) \to \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ if and only if γ_X is a loop.*

4.4. Morphisms and intersections

We recall the relationship between morphisms and intersections of curves. A finite set $\{\gamma_1, \ldots, \gamma_m\}$ of curves is in *minimal position* if no three curves from the set intersect in a single point in the interior (only "double-points") and for all (not necessarily distinct) $i, j \in [1, m]$, the number of (self-)intersections of γ_i and γ_j is minimal within their respective homotopy classes.

Convention. For the rest of this paper, we will always assume every instance of a finite set of curves to be in minimal position.

Every finite set of curves can be deformed into minimal position by homotopy, and hence this assumption does not impose any restrictions on the homotopy classes of the curves.

Let (γ_1, γ_2) be a pair of distinct curves and for $i \in \{1, 2\}$, let Ω_i denote the domain of γ_i. The set $\gamma_1 \overrightarrow{\cap} \gamma_2$ of *oriented intersections* consists of all pairs $(s_1, s_2) \in \Omega_1 \times \Omega_2$ such that $p := \gamma_1(s_1) = \gamma_2(s_2)$, and such that if p is a marked point, then locally around p, γ_1 "comes before" γ_2 in the counter-clockwise orientation as shown in Figure 10. A *self-intersection* of a curve γ with domain Ω is a pair $(s_1, s_2) \in \Omega^2$ with $s_1 \neq s_2$ and $\gamma(s_1) = \gamma(s_2)$.
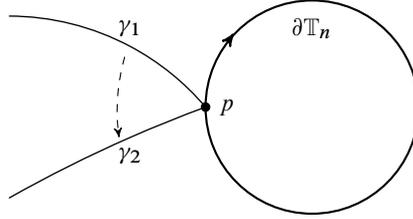

Fig. 10. An oriented boundary intersection p from γ_1 to γ_2.

Every interior intersection of γ_1 and γ_2 contributes elements in both $\gamma_1 \overrightarrow{\cap} \gamma_2$ and $\gamma_2 \overrightarrow{\cap} \gamma_1$, whereas every boundary intersection determines an element in only one of the two sets. Frequently, we do not distinguish between an oriented intersection and its image in the surface.

The following proposition is a special case of [31, Theorem 3.3].

Proposition 4.10. *Let $X, Y \in \mathcal{D}^b(\Lambda_n)$ be indecomposable. Whenever X or Y is τ-invariant, we assume that its associated local system has dimension 1. If X is not isomorphic to a shift of Y or γ_X is an arc, then a homogeneous basis of $\mathrm{Hom}^\bullet(X, Y)$ is in bijection with $\gamma_X \overrightarrow{\cap} \gamma_Y$.*

The remaining case is slightly different. When X is τ-invariant with $\dim \mathcal{V}_X = 1$ and $X \cong Y[m]$ for some $m \in \mathbb{Z}$, then [31, Theorem 3.3] states that there exist a homogeneous basis $\{\phi, \omega\} \cup \mathcal{I}$ of $\mathrm{Hom}^\bullet(X, Y)$ and a bijection $\mathcal{I} \xrightarrow{\sim} \gamma_X \overrightarrow{\cap} \gamma_Y$, where $\phi \in \mathrm{Hom}^m(X, Y)$ is an isomorphism and $\omega \in \mathrm{Hom}^{m+1}(X, Y)$ occurs as the connecting morphism in an Auslander–Reiten triangle

$$Y[m] \to V \to X \xrightarrow{\omega} Y[m + 1].$$

We note that the number of self-intersections of a loop γ in minimal position coincides with the cardinality of $\gamma \overrightarrow{\cap} \gamma'$, where $\gamma' \simeq \gamma$ such that $\{\gamma, \gamma'\}$ is in minimal position. We summarize the situation for τ-invariant objects as follows.

Corollary 4.11. *Let $X, Y \in \mathcal{D}^b(\Lambda_n)$ be τ-invariant indecomposable objects such that $\dim \mathcal{V}_X = 1 = \dim \mathcal{V}_Y$ and $\gamma_X \not\simeq \gamma_Y$. With the notation $\mathrm{hom}^\bullet(U, V) := \dim \mathrm{Hom}^\bullet(U, V)$, we have*

(1) *γ_X and γ_Y have precisely $\mathrm{hom}^\bullet(X, Y)$ intersections;*

(2) *γ_X has precisely $\frac{1}{2} \cdot \mathrm{hom}^\bullet(X, X) - 1$ self-intersections.*

Notation 4.12. If $X, Y \in \mathcal{D}^b(\Lambda_n)$ are indecomposable and $p \in \gamma_X \overset{\rightarrow}{\cap} \gamma_Y$, let

$$f_p \in \mathrm{Hom}^\bullet(X, Y)$$

denote the basis element associated with p.

A remark on degrees: If (γ, g) and (γ', g') are graded and $\dim \mathcal{V} = 1 = \dim \mathcal{V}'$, then every $p \in \gamma \overset{\rightarrow}{\cap} \gamma'$ defines an integer $\deg(p)$, namely the degree of $f_p \colon P^\bullet_{(\gamma,g)}(\mathcal{V}) \to P^\bullet_{(\gamma',g')}(\mathcal{V}')$. Let Δ denote the 6-gon which contains p, where as usual we assume that γ, γ' and all laminates are in minimal position. Let $\bullet \in \partial \Delta$ denote the unique marked point.

   Starting in $\bullet$ and following $\partial \Delta$ counter-clockwise, let us denote by q the first intersection of γ with a laminate and, following $\partial \Delta$ further in the same direction, denote by q' the first intersection of γ' with a laminate after q. The following is a consequence of the constructions in [31, Section 3.3].

Lemma 4.13. *With the notation above,* $\deg(p) = g'(q') - g(q)$.

4.5. Mapping cones and resolutions of crossings

Let Σ be a marked surface. Suppose $\gamma, \gamma' \subseteq \Sigma$ are curves in minimal position and suppose $p \in \gamma \overset{\rightarrow}{\cap} \gamma'$ is an oriented intersection which is not a puncture. We obtain a new curve by resolving p as follows. If $p \in \partial \Sigma$, denote by γ_p the concatenation of γ followed by γ' at the end points which correspond to p. If p lies in the interior, we resolve p by cutting γ and γ' at p and gluing them back together as shown in Figure 11. In this case, the resulting path γ_p is not necessarily a curve but a multi-curve, i.e., a finite set of possibly non-primitive curves. We refer to the individual curves of a multi-curve as its *components*. To simplify the notation in the next theorem, we say that an object $Z \in \mathcal{D}^b(\Lambda_n)$ is represented by a non-primitive loop γ^t, where $t > 1$ and γ is primitive, if $Z \cong P^\bullet_{(\gamma,g)}(\mathcal{U})$ for some local system $\mathcal{U}$ corresponding to a polynomial $X^t - \lambda$, $\lambda \in \Bbbk^\times$. Note that for $p := \mathrm{char}\,\Bbbk$ and $k \geq 0$ such that $t = p^k \cdot m$ with $\gcd(m, p) = 1$, $\mathcal{U}$ splits into m indecomposable local systems of dimension p^k. We further say that a multi-curve $\{\gamma_1, \ldots, \gamma_m\}$ represents an object $X \in \mathcal{D}^b(\Lambda_n)$ if there exists a decomposition $X \cong \bigoplus_{i=1}^m X_i$ such that γ_i represents X_i.

   The following theorem states that the resolved curves represent the mapping cone of a morphism and is taken from [31] with a slight adjustment of the statement due to an inaccuracy in [16] which we comment on below.

Fig. 11. Resolution of an interior intersection from γ to γ'.

Theorem 4.14 (cf. [31, Theorem 4.1]). *Let $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ be indecomposable, let $p \in \gamma_X \overset{\rightarrow}{\cap} \gamma_Y$ and let γ_p denote the resolved multi-curve. Then γ_p represents the mapping cone of $f_p \in \mathrm{Hom}^\bullet(X, Y)$ in the sense above.*

Phrased in our notation it was claimed in [16, Lemma 2.5] that an object $P^\bullet_{(\gamma,g)}(\mathcal{U})$ with $\mathcal{U}$ corresponding to a polynomial $X^{\dim \mathcal{U}} - \lambda$ splits into $\dim \mathcal{U}$ indecomposable direct summands of the form $P^\bullet_{(\gamma,g)}(\mathcal{U}_i)$ with $\dim \mathcal{U}_i = 1$. However, as explained above, this is only true if the characteristic of $\Bbbk$ does not divide $\dim \mathcal{U}$.

4.6. Compositions of morphisms via triangles, bigons and tridents

In this section, we recall the geometric description of compositions of morphisms $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$. Suppose that $\gamma_1, \gamma_2, \gamma_3 \subseteq \mathbb{T}_n$ are curves which have oriented intersections $p \in \gamma_1 \overset{\rightarrow}{\cap} \gamma_2$ and $q \in \gamma_2 \overset{\rightarrow}{\cap} \gamma_3$. Let us denote by $\widetilde{\mathbb{T}}_n$ a fixed universal cover of $\mathbb{T}_n$. By a lift of a loop $\gamma: S^1 \to \mathbb{T}_n$, we mean a lift $\mathbb{R} \to \widetilde{\mathbb{T}}_n$ along the universal cover $\mathbb{R} \to S^1$.

Definition 4.15. Let $\Delta \subseteq \mathbb{R}^2$ be a triangle equipped with the induced orientation and denote by S_1, S_2, S_3 the three segments of $\partial \Delta$ in clockwise order. Moreover, we denote by $D^2 \subseteq \mathbb{R}^2$ the closed unit disc. Let $r \in \gamma_1 \overset{\rightarrow}{\cap} \gamma_3$. Then the triple (p, q, r) is called

(1) an *intersection triangle*, if there exists an orientation-preserving embedding $\varphi: \Delta \to \widetilde{\mathbb{T}}_n$ such that $\varphi|_{S_i}$ is a homeomorphism onto a segment of a lift of γ_i and the corners of Δ are mapped bijectively onto $\{p, q, r\}$;

(2) a *double-bigon*, if all curves are arcs and there exists an orientation-preserving embedding $\varphi: D^2 \to \widetilde{\mathbb{T}}_n$ which maps $S^1 \cap (\mathbb{R} \times \mathbb{R}_{\geq 0})$, $[-1, 1] \times \{0\}$ and $S^1 \cap (\mathbb{R} \times \mathbb{R}_{\leq 0})$ bijectively onto lifts of γ_1, γ_2 and γ_3, respectively; in particular, φ maps p, q and r to the same point in $\{\pm 1\} \subseteq D^2$.

(3) a *trident* if p, q, r correspond to the same point on the boundary and locally the curves are arranged as in Figure 12.

Definition 4.16. In the notation above, define $C(p, q)$ as the set of all $r \in \gamma_1 \overset{\rightarrow}{\cap} \gamma_3$ such that (p, q, r) is an intersection triangle, a double-bigon or a trident.
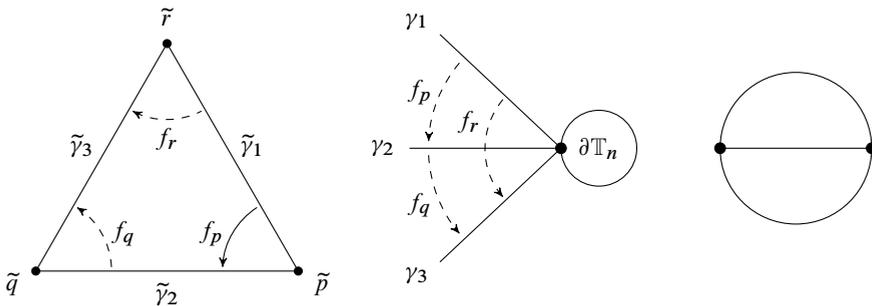

Fig. 12. From left to right: an intersection triangle, a trident and a double-bigon.

Definition 4.17. Let $\gamma, \gamma' \subseteq \mathbb{T}_n$ be curves and let $f: X_\gamma \to X_{\gamma'}$ be a morphism. The *support of f* is the subset $\operatorname{supp} f \subseteq \gamma \overset{\rightarrow}{\cap} \gamma'$ consisting of all $p \in \gamma \overset{\rightarrow}{\cap} \gamma'$ such that f_p appears with non-zero scalar in a decomposition of f with respect to the basis of $\operatorname{Hom}^\bullet(X, Y)$ associated with $\gamma \overset{\rightarrow}{\cap} \gamma'$.

Compositions of morphisms can now be described in the following way.

Proposition 4.18 ([30, Theorem 2.8]). *For curves $\gamma_1, \gamma_2, \gamma_3 \subseteq \mathbb{T}_n$, let $p \in \gamma_1 \overset{\rightarrow}{\cap} \gamma_2$ and $q \in \gamma_2 \overset{\rightarrow}{\cap} \gamma_3$ be intersections and let $f_p: X_{\gamma_1} \to X_{\gamma_2}$, $f_q: X_{\gamma_2} \to X_{\gamma_3}$ denote the corresponding morphisms. If $\gamma_1 \not\simeq \gamma_2$ or γ_1 is an arc, then*

$$\operatorname{supp}(f_q \circ f_p) = C(p, q).$$

For later reference, we want to emphasize the following observation.

Lemma 4.19. *Let f, g be morphisms between indecomposable objects in $\mathcal{D}^b(\Lambda_n)$ such that $g \circ f$ is defined. If f or g is supported only at interior intersections, then so is $g \circ f$.*

Proof. The assumptions imply that no trident or double-bigon appears in the set $C(p, q)$. Note that the point $\tilde{r}$ in Figure 12 cannot be a boundary intersection due to the definition of oriented intersections at the boundary. ∎

5. Spherical objects as simple loops on tori

In this section, we show that isomorphism classes of spherical objects in $\mathcal{D}^b(\Lambda_n)$ are, up to shift, in bijection with a certain set of simple loops on $\mathbb{T}_n$. We call a curve on a marked surface *simple* if all its self-intersections lie on the boundary. In other words, a simple loop has no self-intersections and a simple arc can only intersect itself at its end points.

Lemma 5.1. *Let $Y \in \mathcal{D}^b(\Lambda_n)$. Then Y is spherical if and only if $Y \cong \mathbb{F}(X)$ for some spherical object $X \in \operatorname{Perf}(C_n)$. In particular, every spherical object in $\mathcal{D}^b(\Lambda_n)$ is 1-spherical.*

Proof. This follows from Theorem 2.2 and Proposition 4.8. ∎

Lemma 5.2. *Let $Y \in \mathcal{D}^b(\Lambda_n)$. Then Y is spherical if and only if $Y \cong P^\bullet_{(\gamma, g)}(\mathcal{V})$, where γ is simple and $\dim \mathcal{V} = 1$.*

Proof. By [2, Proposition 5.16], we have $\dim \operatorname{Hom}^\bullet(X, X) \geq 3$ for every τ-invariant indecomposable object $X \in \mathcal{D}^b(\Lambda_n)$ such that $\dim \mathcal{V}_X \geq 2$. Thus, such objects are never spherical and the assertion is a consequence of Corollary 4.11. ∎

It still remains to be discussed which simple loops are gradable and hence correspond to a family of spherical objects.

A simple loop γ on a surface Σ is said to be *non-separating* if its complement $\Sigma \setminus \gamma$ is connected. Otherwise, γ is called *separating*.

Lemma 5.3. *Let $\gamma \subseteq \mathbb{T}_n$ be a separating simple loop. Then $\omega(\gamma) \neq 0$. In particular, γ does not represent an object of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$.*

Proof. This is a consequence of the fact that ω is the winding number function of a line field, e.g., see [30, Lemma 2.4]. Since γ is separating and by additivity of the Euler characteristic, γ bounds a unique subsurface Σ of genus 0. We denote by $B_1, \ldots, B_b$ the boundary components of Σ and assume that γ inherits its orientation from Σ. Since all boundary winding numbers are equal to -2 (see Example 4.4), it follows from the Poincaré–Hopf index theorem [21] that

$$\omega(\gamma) - 2(b-1) = \sum_{i=1}^{b} \omega(B_i) = 2\chi(\Sigma) = 4 - 2b,$$

and hence $\omega(\gamma) = 2$. ∎

Corollary 5.4. *Let $X \in \mathrm{Perf}(C_n)$ be indecomposable. Then X is spherical if and only if γ_X is a simple non-separating loop and $\dim \mathcal{V}_X = 1$.*

We prove in the next section that every non-separating simple loop on $\mathbb{T}_n$ is gradable and hence represents an object of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$.

6. The mapping class group of a torus with boundary

This section discusses the mapping class group of $\mathbb{T}_n$ and its connection to auto-equivalences of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$. We recommend [18] for additional information about mapping class groups.

6.1. Mapping class groups and their connection to auto-equivalences

By an *isotopy*, we mean a smooth path of diffeomorphisms which is constant on marked points. To spell this out, an isotopy is a smooth map $I \colon [0, 1] \times \mathbb{T}_n \to \mathbb{T}_n$ such that for all $t \in [0, 1]$, the map $I(t, -) \colon \mathbb{T}_n \to \mathbb{T}_n$ is a diffeomorphism and such that $I(t, x) = x$ for all $t \in [0, 1]$ and all marked points $x \in \mathbb{T}_n$.

Definition 6.1. The *mapping class group* $\mathcal{MCG}(\mathbb{T}_n)$ is the group consisting of all isotopy classes of orientation-preserving diffeomorphisms $H \colon \mathbb{T}_n \to \mathbb{T}_n$ which preserve the set of marked points. The *pure mapping class group* $\mathcal{PMCG}(\mathbb{T}_n)$ is the subgroup of all those diffeomorphisms which restrict to the identity on $\partial \mathbb{T}_n$.

The mapping class group and the pure mapping class group fit into an exact sequence

$$\mathbf{1} \to \mathcal{PMCG}(\mathbb{T}_n) \to \mathcal{MCG}(\mathbb{T}_n) \xrightarrow{\mid_{\mathcal{M}}} \mathfrak{S}_n \ltimes \mathbb{Z}_2^n \to \mathbf{1}, \tag{6.1}$$

where $\mid_{\mathcal{M}}$ restricts a diffeomorphism to the set of marked points yielding an element in $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$. A transposition in $\mathfrak{S}_n$ is realized by a so-called *half-twist* which permutes
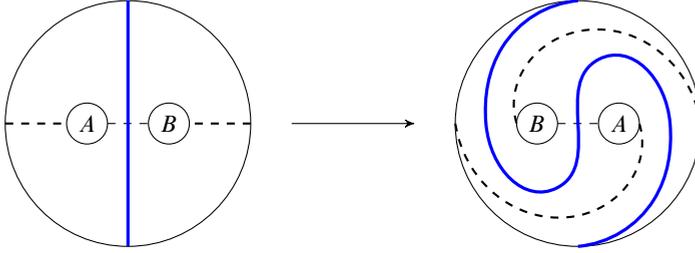
Fig. 13. The action of a half-twist on curves. It acts as the identity outside of a neighbourhood of A and B.
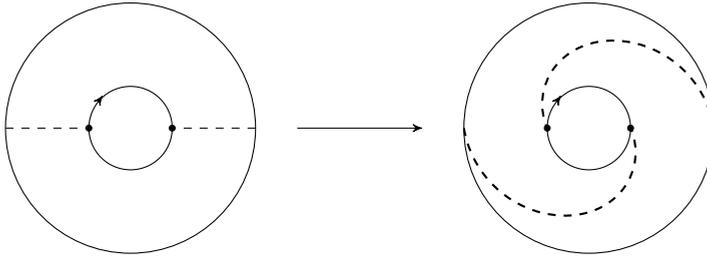

Fig. 14. The action of a counter-clockwise fractional twist on arcs.

a pair of boundary components, see Figure 13. Another example of a mapping class is the fractional twist around a boundary component B. It rotates the surface in a collar neighbourhood of B and acts as the identity outside of this neighbourhood, see Figure 14.

Remark 6.2. Replacing diffeomorphisms by homeomorphisms and smooth maps by continuous maps in Definition 6.1 leads to isomorphic groups. Below, we will switch freely between "homeomorphisms" and "diffeomorphisms" and will not distinguish between continuous and smooth isotopies.

It turns out that there is a close relationship between auto-equivalences of $\mathcal{D}^b(\Lambda_n)$ and mapping classes of its geometric model $\mathbb{T}_n$. The following is a special case of [30, Theorem C].

Proposition 6.3. *There exists a short exact sequence*

$$1 \to (\Bbbk^\times)^{n+1} \times \mathbb{Z} \to \mathrm{Aut}(\mathcal{D}^b(\Lambda_n)) \xrightarrow{\Psi} \mathcal{MCG}(\mathbb{T}_n, \omega) \to 1, \tag{6.2}$$

where $\mathcal{MCG}(\mathbb{T}_n, \omega) \subseteq \mathcal{MCG}(\mathbb{T}_n)$ denotes the subgroup of all mapping classes which preserve the winding numbers of all loops.

The definition of Ψ is recalled in Section 8.3. The $\mathbb{Z}$-component in the kernel of sequence (6.2) acts as the shift functor on $\mathcal{D}^b(\Lambda_n)$, the $(n + 1)$-st power of $\Bbbk^\times$ by multiplying the arrows $c_0, \ldots, c_{n-1}$ and d_0 in the quiver of Λ_n with non-zero scalars. For any auto-equivalence $T \in \mathrm{Aut}(\mathcal{D}^b(\Lambda_n))$, the mapping class $\Psi(T)$ is uniquely determined by the following property.
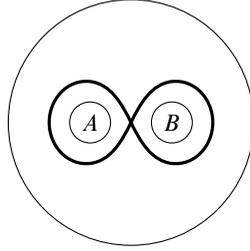
Fig. 15. The loop u.

Proposition 6.4 ([30, Theorem A]). *For all indecomposable objects $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$,*

$$\Psi(T)(\gamma_X) \simeq \gamma_{T(X)}.$$

In other words, the action of $\Psi(T)$ on curves on $\mathbb{T}_n$ mirrors the action of T on indecomposable objects of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$.

Lemma 6.5. *Half-twists and fractional twists are elements of $\mathcal{MCG}(\mathbb{T}_n, \omega)$.*

Proof. Fractional twists act trivially on the homotopy class of any loop and hence preserve all winding numbers. A half-twist, which permutes boundary components A and B, stabilizes an embedded path δ which connects A and B and inverts its orientation. For example, in Figure 13, δ is the horizontal line between A and B which crosses the blue vertical line transversely. A curve γ is affected by the half-twist if and only if γ and δ intersect. Locally around an intersection, a segment of γ (the blue vertical curve in Figure 13) is replaced by the blue thick curve on the right-hand side of Figure 13. As a result, a half-twist preserves all winding numbers if and only if the winding number of the loop u in Figure 15 vanishes. This can be verified by hand or, more conceptually, by the argument explained in Remark 6.6. ∎

Remark 6.6. Let $\gamma \colon [0, 1] \to \mathbb{T}_n$ be an oriented (but not necessarily embedded) arc which cannot be homotoped onto $\partial \mathbb{T}_n$. One can define a gradable loop γ_{loop} as in Figure 15 by concatenating γ with the clockwise simple closed boundary arc based at $\gamma(1)$, followed by the inverse γ^{-1} and the counter-clockwise simple closed boundary arc based at $\gamma(0)$. Using the homological definition of ω, one can give a simple argument for the vanishing of $\omega(\gamma_{\mathrm{loop}})$: there exists a homomorphism $\bar{\omega} \colon H_1(\mathbb{P}(T\mathbb{T}_n), \mathbb{Z}) \to \mathbb{Z}$, where $\mathbb{P}(T\mathbb{T}_n)$ denotes the projectivized tangent bundle. More precisely, $\bar{\omega}$ is the resulting homomorphism of the intersection pairing with the submanifold of $\mathbb{P}(T\Sigma)$ determined by a line field. Further details on the definition of $\bar{w}$ (in the case of compact surfaces) can be found in [26]. The derivative $\dot{\gamma}$ of an immersed loop γ in minimal position defines a simple loop in $\mathbb{P}(T\mathbb{T}_n)$ and hence defines a class $[\dot{\gamma}] \in H_1(\mathbb{P}(T\mathbb{T}_n), \mathbb{Z})$. Then $\omega(\gamma) = \bar{\omega}([\dot{\gamma}])$ and the claim follows from the observation that $[\dot{\gamma_{\mathrm{loop}}}] = [\dot{\delta}_0] - [\dot{\delta}_1]$, where δ_0 and δ_1 denote the clockwise simple loops around the boundary components which contain $\gamma(0)$ and $\gamma(1)$, respectively.

6.2. Dehn twists, Humphries generators and their spherical twists

An important example of a mapping class and the geometric counterpart of spherical twists is the *Dehn twist* about a given simple loop γ.

Let W be a tubular neighbourhood of γ, i.e., a neighbourhood W with an orientation-preserving diffeomorphism $\phi: S^1 \times [-1, 1] \to W$ such that $\phi|_{S^1 \times \{0\}} = \gamma$. Then the *Dehn twist* about γ is the mapping class of the diffeomorphism $D_\gamma: \mathbb{T}_n \to \mathbb{T}_n$ defined on $\mathbb{T}_n \setminus W$ by $D_\gamma|_{\mathbb{T}_n \setminus W} := \mathrm{Id}_{\mathbb{T}_n \setminus W}$ and on W by

$$D_\gamma(\phi(z, t)) := \phi(z \cdot e^{2\pi i(t+1)}, t).$$

While the diffeomorphism D_γ depends on W, its mapping class is well defined. Since the map ϕ above is assumed to be orientation-preserving, the mapping class of D_γ does not depend on the orientation of γ.

In analogy to spherical twists (see Corollary 3.7), the set of Dehn twists is closed under conjugation.

Lemma 6.7. *Let γ be a simple loop on an oriented surface $\mathbb{T}_n$ and let $F: \mathbb{T}_n \to \mathbb{T}_n$ be a diffeomorphism. Then $D_{F \circ \gamma}$ and $F \circ D_\gamma \circ F^{-1}$ define the same mapping classes.*

By classical results of Dehn and Humphries, the group $\mathcal{PMCG}(\mathbb{T}_n)$ is generated by a finite set of Dehn twists. We refer to the Dehn twists about the loops γ_{Pic} and $\gamma_{\Bbbk(x)}^0, \ldots, \gamma_{\Bbbk(x)}^{n-1}$ as the *Humphries generators*.

The following is a special case of a theorem due to Humphries.

Theorem 6.8 (Humphries). *$\mathcal{PMCG}(\mathbb{T}_n)$ is generated by the Humphries generators, i.e., the mapping classes of the Dehn twists about the loops $\gamma_{\mathrm{Pic}}, \gamma_{\Bbbk(x)}^0, \ldots, \gamma_{\Bbbk(x)}^{n-1}$ (see Example 4.7) as well as the Dehn twists about the simple boundary loops.*

Proof. We include a proof in lack of a suitable reference. We regard the surface $\mathscr{T}^n$ as being obtained from $\mathbb{T}_n$ by gluing a once-punctured disc to every boundary component. If we denote by $B_0, \ldots, B_{n-1}$ the components of $\partial \mathbb{T}_n$, then [18, Proposition 3.19] asserts the existence of a short exact sequence

$$1 \to \mathbb{Z}^n \xrightarrow{\iota} \mathcal{PMCG}(\mathbb{T}_n) \xrightarrow{\pi} \mathcal{PMCG}(\mathscr{T}^n) \to 1. \tag{6.3}$$

Here, $\mathcal{PMCG}(\mathscr{T}^n)$ denotes the pure mapping class group of $\mathscr{T}^n$, i.e., the group of self-diffeomorphisms which fix every puncture modulo isotopies which are constant on all punctures. The inclusion ι sends a tuple $(m_i) \in \mathbb{Z}^n$ to the composition of Dehn twists $\prod_{i=0}^{n-1} D_{B_i}^{m_i}$, where D_{B_i} denotes the Dehn twist around the boundary loop of B_i. The projection π sends a diffeomorphism to its radial extension: every self-diffeomorphism φ of the annulus S^1 extends to a self-diffeomorphism $\overline{\varphi}$ of $D^2 \setminus \{(0, 0)\}$ defined by

$$\overline{\varphi}(x) := \|x\| \cdot \varphi\left(\frac{x}{\|x\|}\right).$$

By [18, Corollary 4.15] and the discussion leading to it, $\mathcal{PMCG}(\mathscr{T}^n)$ is generated by the Humphries generators. ∎

We note that every Dehn twist about a boundary loop acts trivially on any homotopy class of loops.[3] In particular, we find the following.

Proposition 6.9. *Every Dehn twist about a boundary loop is an element of $\mathcal{MCG}(\mathbb{T}_n, \omega)$.*

The relevance of the Humphries generators for us and the relationship between spherical twists and Dehn twists becomes apparent in the following application of [30, Theorem C] and Proposition 6.4.

Theorem 6.10. *Let $X \in \mathcal{D}^b(\Lambda_n)$ be spherical. Then $\Psi(T_X) = D_{\gamma_X}$ and for every indecomposable object $Y \in \mathcal{D}^b(\Lambda_n)$,*

$$\gamma_{T_X(Y)} \simeq D_{\gamma_X}(\gamma_Y).$$

The following proposition is a direct consequence of Theorems 6.10 and 2.3 and Corollary 3.7.

Proposition 6.11. *Let $x \in \mathbb{P}^1_j \subseteq C_n$ be closed and smooth and let $\mathcal{L} \in \mathrm{Pic}^0(C_n)$, i.e., $\mathcal{L}$ is a line bundle of multi-degree $(0, \ldots, 0)$. Let $Y \in \mathrm{Perf}(C_n)$ be indecomposable. Then $D_{\gamma^j_{\Bbbk(x)}}(\gamma_Y)$ is a representative of $T_{\Bbbk(x)}(Y)$ and $D_{\gamma_{\mathrm{Pic}}}(\gamma_Y)$ is a representative of $T_{\mathcal{L}}(Y)$.*

Corollary 6.12. *We have $\mathcal{MCG}(\mathbb{T}_n) = \mathcal{MCG}(\mathbb{T}_n, \omega)$.*

Proof. Let $H \in \mathcal{MCG}(\mathbb{T}_n)$. By (6.1), there exists $F \in \mathcal{MCG}(\mathbb{T}_n)$, which is a composition of half-twists and fractional twists, such that $H \circ F \in \mathcal{PMCG}(\mathbb{T}_n)$. Since $\mathrm{Im}\,\Psi \subseteq \mathcal{MCG}(\mathbb{T}_n, \omega)$, it follows from Theorems 2.3 and 6.10 that the Dehn twists around the Humphries generators are elements of $\mathcal{MCG}(\mathbb{T}_n, \omega)$. It therefore follows from Theorem 6.8 and Proposition 6.9 that $\mathcal{PMCG}(\mathbb{T}_n) \subseteq \mathcal{MCG}(\mathbb{T}_n, \omega)$. Finally, Lemma 6.5 implies $F \in \mathcal{MCG}(\mathbb{T}_n, \omega)$ and hence $H \in \mathcal{MCG}(\mathbb{T}_n, \omega)$. ∎

With Theorem 6.8 and Proposition 6.11 in mind, we show in Proposition 6.13 that $\mathcal{MCG}(\mathbb{T}_n)$ acts transitively on non-separating simple loops. The proof is essentially the same as in [18, p. 37]. However, we include it as we need a slightly stronger statement than what is proved there.

Proposition 6.13. *The pure mapping class group $\mathcal{PMCG}(\mathbb{T}_n)$ acts transitively on the set of non-separating simple loops.*

Proof. Suppose that γ and γ' are non-separating simple loops on $\mathbb{T}_n$. Then the surfaces Σ and Σ', obtained from $\mathbb{T}_n$ by cutting at γ and γ', respectively, are both connected surfaces of genus 0 with exactly $n + 2$ boundary components. Note that the boundary components $B_1, \ldots, B_n$ of $\mathbb{T}_n$ are canonically identified with boundary components of Σ. We denote by D_1, D_2 (resp. D'_1, D'_2) the remaining two components of $\partial\Sigma$ (resp. $\partial\Sigma'$).

---

[3]By deforming any given loop γ, we may assume that the tubular neighbourhood in Section 6.2 of the boundary loop and γ are disjoint.

Since Σ and Σ' have the same genus and the same number of boundary components, there exists a diffeomorphism $\phi\colon \Sigma \to \Sigma'$. After composing ϕ with any orientation-reversing self-diffeomorphism of Σ or Σ', we may assume that ϕ is orientation-preserving. By a series of half-twists, one can permute any two boundary components on Σ by an orientation-preserving diffeomorphism which restricts to the identity on all other boundary components. In this way, we obtain an orientation-preserving diffeomorphism $\phi\colon \Sigma \to \Sigma'$, which restricts to orientation-preserving diffeomorphisms $B_i \to B_i$ for every $i \in [1, n]$. By Lemma 6.14 below, there exists an orientation-preserving diffeomorphism $\psi\colon \Sigma \to \Sigma$ such that $\phi \circ \psi$ restricts to the identity $B_i \to B_i$ for all $i \in [1, n]$. Identifying D_1 with D_2 and D_1' with D_2', respectively, one sees that $\phi \circ \psi$ gives rise to a homeomorphism $H\colon \mathbb{T}_n \to \mathbb{T}_n$ such that $H(\gamma) = \gamma'$ and which restricts to the identity on B_i for all $i \in [1, n]$. ∎

The following well-known lemma was used in the previous proof.

Lemma 6.14. *Let Σ be a compact oriented surface, B a boundary component of Σ, and let $f\colon B \to B$ be an orientation-preserving diffeomorphism, where B is equipped with the induced orientation. Then, for every closed neighbourhood W of B, f extends to a diffeomorphism $F_W\colon \Sigma \to \Sigma$ which restricts to the identity on the complement of W.*

Proof. The mapping class group of the circle is trivial: given an orientation-preserving self-diffeomorphism $\phi\colon S^1 \to S^1$, consider a lift $\widetilde{\phi}\colon \mathbb{R} \to \mathbb{R}$. In particular, $\widetilde{\phi}(x + 1) = \widetilde{\phi}(x) + 1$ for all $x \in \mathbb{R}$. Then $\widetilde{\phi}$ is orientation-preserving, i.e., strictly increasing, and the convex homotopy

$$H\colon [0, 1] \times \mathbb{R} \to \mathbb{R}, \quad (t, x) \mapsto (1 - t)\widetilde{\phi}(x) + tx,$$

is an isotopy which induces an isotopy from ϕ to the identity. Thus, there exists an isotopy $\psi\colon [0, 1] \times B \to B$ from f to Id_B. For $U \subseteq W$ a collar neighbourhood of B with diffeomorphism $\phi\colon U \to [0, 1] \times B$, define F_W as the extension of the identity of $\Sigma \setminus U$ by the map $\psi \circ \phi\colon U \to U$. ∎

7. The proofs of Theorems A and C

We give proofs of Theorems A and C from the introduction. For the convenience of the reader, we state them again.

Theorem 7.1. *There exists a bijection between isomorphism classes of spherical objects on C_n, up to shift, and pairs $([\gamma], \lambda)$, where $\lambda \in \Bbbk^\times$ and $[\gamma]$ is the homotopy class of an unoriented, non-separating simple loop γ on the n-punctured torus $\mathscr{T}^n$.*

We will see in Section 8 that $\mathscr{T}^n$ is a geometric model for $\mathcal{D}^b(C_n)$. We will derive this model from $\mathbb{T}_n$ which is the geometric model for $\mathcal{D}^b(\Lambda_n)$. From this point of view, the appearance of the punctured torus in Theorem 7.1 is not surprising.

Theorem 7.2. *The group of auto-equivalences of $\mathcal{D}^b(C_n)$ acts transitively on the set of isomorphism classes of spherical objects in* $\operatorname{Perf}(C_n)$.

Proofs of Theorems 7.1 *and* 7.2. We recall that homotopy classes of loops on $\mathbb{T}_n$ are naturally in bijection with homotopy classes of loops on $\mathcal{T}^n$ and so it is sufficient to give a bijection between isomorphism classes of spherical objects in $\operatorname{Perf}(C_n)$ and non-separating simple loops on $\mathbb{T}_n$ instead. According to Corollary 5.4, an indecomposable object $X \in \mathcal{D}^b(C_n)$ is spherical if and only if $\gamma_X \subseteq \mathbb{T}_n$ is a simple non-separating loop and the associated local system $\mathcal{V}_X$ has dimension 1. The parameter $\lambda \in \Bbbk^\times$ in Theorem 7.1 specifies the isomorphism class of $\mathcal{V}_X$ after choosing a base point for γ_X. To finish the proof of Theorem 7.1, it is left to prove that every simple non-separating loop is gradable and hence represents a spherical object. In order to prove Theorem 7.2, it is sufficient to show that for every spherical object $X \in \operatorname{Perf}(C_n)$ there exists an auto-equivalence T of $\mathcal{D}^b(C_n)$ such that $T(X) \cong \Bbbk(z)[m]$, where $m \in \mathbb{Z}$ and $z \in C_n$ is a smooth point. Since $\operatorname{Aut}(C_n)$ acts transitively on its set of smooth points, this will imply Theorem 7.2.

Let $X \in \operatorname{Perf}(C_n)$ be spherical. By Corollary 5.4, $\gamma = \gamma_X$ is a simple non-separating loop on $\mathbb{T}_n$, and by Proposition 6.13, there exists a mapping class $H \in \mathcal{PMCG}(\mathbb{T}_n)$ such that $H(\gamma) \simeq \gamma^0_{\Bbbk(x)}$. Since Dehn twists about boundary loops act trivially on homotopy classes of loops and by Theorem 6.8, we may assume that H is a composition of Humphries generators and their inverses. By Proposition 6.11 and Corollary 3.7, there exists $T \in \operatorname{Aut}(\mathcal{D}^b(C_n))$, which is a composition of spherical twists (and their inverses) by skyscraper sheaves of smooth points and $\mathcal{O}_{C_n}$, with the property that $\mathbb{F}(T(X))$ is represented by $H(\gamma) \simeq \gamma^0_{\Bbbk(x)}$. Thus, $\mathbb{F}(T(X)) \cong \Bbbk(0,\lambda)[m]$ for some $m \in \mathbb{Z}$ and $\lambda \in \Bbbk^\times$. Since $\mathbb{F}$ is an embedding, $T(X)$ is isomorphic to a shift of $\Bbbk(z)$ for a smooth point $z \in \mathbb{P}^1_0 \subseteq C_n$. This finishes the proof of Theorem 7.2.

The proof above and Proposition 6.13 also show that every non-separating simple loop on $\mathbb{T}_n$ represents a spherical object of $\operatorname{Perf}(C_n)$. Namely, γ above is represented by $T^{-1}(\Bbbk(z))$. Together with Corollary 5.4, this completes the proof of Theorem 7.1. ∎

8. A surface model and auto-equivalences of the derived category of a cycle

We describe the group of auto-equivalences of $\mathcal{D}^b(C_n)$. The result is derived by regarding $\mathcal{D}^b(C_n)$ as the Verdier quotient of $\mathcal{D}^b(\Lambda_n)$ at objects which are represented by boundary arcs. The underlying idea to contract boundary arcs by passing to a localization was used in [19] to classify indecomposable objects in the partially wrapped Fukaya categories of a graded marked surface Σ in terms of curves. For our purposes, we use the same approach to provide a topological description of morphisms and their compositions in $\mathcal{D}^b(C_n)$ as in the case of the category $\mathcal{D}^b(\Lambda_n)$. We note that the approach by localization was used by Lekili and Polishchuk [24] to prove equivalences between derived categories of certain stacky nodal curves (which includes the case C_n) and wrapped Fukaya categories of punctured surfaces in the sense of [19]. The topological model for $\mathcal{D}^b(C_n)$ reflects this relationship.

8.1. The category of boundary arcs, the category of loops and perpendicular categories

Definition 8.1. Let $\mathcal{D}_{\mathrm{loop}} \subseteq \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ denote the additive closure of indecomposable objects which are represented by loops and denote by $\mathcal{D}_\partial$ the additive closure of all indecomposable objects which are represented by boundary arcs.

The following is a consequence of Proposition 4.8 and characterizes the categories $\mathcal{D}_\partial$ and $\mathcal{D}_{\mathrm{loop}}$.

Proposition 8.2. *The category $\mathcal{D}_\partial$ coincides with the full subcategory of objects $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that $\tau^2 X \cong X[2]$ and $\mathcal{D}_{\mathrm{loop}}$ is the full subcategory containing all τ-invariant objects. In particular, $\mathcal{D}_{\mathrm{loop}}$ is the essential image of $\mathbb{F}$.*

We recall the notion of perpendicular categories.

Definition 8.3. Given a full subcategory $\mathcal{U} \subseteq \mathcal{D}^{\mathrm{b}}(\Lambda_n)$, its *right perpendicular category* $\mathcal{U}^\perp$ is the full subcategory consisting of all objects $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that $\mathrm{Hom}^\bullet(Y, X) = 0$ for all $Y \in \mathcal{U}$. Similarly, the *left perpendicular category* $^\perp\mathcal{U}$ is defined by interchanging the roles of X and Y.

An application of the five lemma shows that both perpendicular categories are triangulated. Moreover, $X \oplus Y \in \mathcal{U}^\perp$ if and only if $X, Y \in \mathcal{U}^\perp$, i.e., perpendicular categories are *thick* subcategories.

Lemma 8.4. *We have*

$$^\perp\mathcal{D}_{\mathrm{loop}} = \mathcal{D}_\partial = \mathcal{D}_{\mathrm{loop}}{}^\perp \quad and \quad {}^\perp\mathcal{D}_{\mathrm{loop}} = \mathcal{D}_{\mathrm{loop}} = \mathcal{D}_\partial{}^\perp .$$

In particular, the categories $\mathcal{D}_{\mathrm{loop}}$ and $\mathcal{D}_\partial$ are triangulated.

Proof. If $X \in \mathcal{D}_{\mathrm{loop}}$ and $Y \in \mathcal{D}_\partial$, then any pair of direct summands of X and Y is represented by disjoint curves on $\mathbb{T}_n$. Thus, there are no non-zero morphisms between X and Y in any direction. In particular, $^\perp\mathcal{D}_{\mathrm{loop}} \supseteq \mathcal{D}_\partial \subseteq \mathcal{D}_{\mathrm{loop}}{}^\perp$. Any indecomposable object of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ is either contained in $\mathcal{D}_{\mathrm{loop}}$ or is represented by an arc and hence has a non-zero morphism to some object in $\mathcal{D}_\partial$. This proves the equality of $\mathcal{D}_\partial$ and the perpendicular categories. The same type of arguments imply the second equality. ∎

Lemma 8.5. *Let $T \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(\Lambda_n))$. Then T restricts to auto-equivalences of $\mathcal{D}_{\mathrm{loop}}$ and $\mathcal{D}_\partial$, respectively.*

Proof. Since τ lies in the center of $\mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(\Lambda_n))$, Proposition 8.2 implies that both subcategories are stable under triangle equivalences. ∎

8.2. A geometric model of the derived category of C_n via Verdier quotients

8.2.1. Digression: Verdier quotients by thick subcategories.
Let $\mathcal{T}$ be a triangulated category and let $\mathcal{U}$ be a thick subcategory. The *Verdier quotient* of $\mathcal{T}$ by $\mathcal{U}$ is a triangulated category $\mathcal{T}/\mathcal{U}$ and an exact functor $\pi = \pi_\mathcal{U} \colon \mathcal{T} \to \mathcal{T}/\mathcal{U}$ such that $\mathcal{U}$ is in the kernel

of π and π is universal with this property. The objects of $\mathcal{T}/\mathcal{U}$ are the same as the objects of $\mathcal{T}$ and $\pi(X) := X$ for all $X \in \mathcal{T}$. A morphism from an object X to an object Y is an equivalence class of *roofs* (α, f)

$$
\begin{array}{ccc}
 & W & \\
\alpha \swarrow & & \searrow f \\
X & & Y,
\end{array}
$$

where $W \in \mathcal{T}$ and α and f are morphisms in $\mathcal{T}$ such that the mapping cone of α is an object of $\mathcal{U}$.

Two roofs $X \xleftarrow{\alpha} W \xrightarrow{f} Y$ and $X \xleftarrow{\beta} W' \xrightarrow{g} Y$ are *equivalent*, if there exist a roof $X \xleftarrow{u} U \xrightarrow{v} Y$ and maps $w \colon U \to W$ and $w' \colon U \to W'$ which make the following diagram commutative:

$$
\begin{array}{ccccc}
 & & W & & \\
\alpha \swarrow & & \uparrow w & & \searrow f \\
X & \xleftarrow{\;\;u\;\;} & U & \xrightarrow{\;\;v\;\;} & Y. \\
\beta \nwarrow & & \downarrow w' & & \nearrow g \\
 & & W' & &
\end{array}
$$

If $X \xleftarrow{\alpha} W_1 \xrightarrow{f} Y$ and $Y \xleftarrow{\beta} W_2 \xrightarrow{g} Z$ are roofs, we can form the homotopy pullback $W' = W_1 \times_Y W_2$ along f and β. If $u \colon W' \to W_1$ and $v \colon W' \to W_2$ denote the structure maps of the pullback, the composition $(\beta, g) \circ (\alpha, f)$ is the equivalence class of the roof

$$
\begin{array}{ccccccc}
 & & & W' & & & \\
 & & u \swarrow & & \searrow v & & \\
 & W_1 & & & & W_2 & \\
 \alpha \swarrow & & \searrow f & & \beta \swarrow & & \searrow g \\
 X & & & Y & & & Z.
\end{array}
$$

Finally, a triangle in $\mathcal{T}/\mathcal{U}$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{T}$ under π.

We will frequently use the following standard facts:

- A morphism $f \colon X \to Y$ in $\mathcal{T}$ becomes invertible in $\mathcal{T}/\mathcal{U}$ if and only if the mapping cone of f lies in $\mathcal{U}$. In particular, $\pi(U) \cong 0$ for all $U \in \mathcal{U}$.

- Two morphisms $f, g \colon X \to Y$ are identified in $\mathcal{T}/\mathcal{U}$ if and only if $f - g$ factors through $\mathcal{U}$. In particular, the canonical map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\pi(X), \pi(X))$ is an isomorphism if $X \in {}^{\perp}\mathcal{U}$ or $Y \in \mathcal{U}^{\perp}$.

- If $\pi(X) \cong \pi(Y)$, then there exists a morphism $\alpha \in \mathrm{Hom}(X, Y) \cup \mathrm{Hom}(Y, X)$ whose mapping cone is an object of $\mathcal{U}$.

- If $X \in \mathcal{T}$ and $U \in \mathcal{U}$, then the canonical maps $X \to X \oplus U$ and $X \oplus U \to X$ induce mutually inverse isomorphisms in $\mathcal{T}/\mathcal{U}$.

Proofs of the first two statements above can be found in Neeman's book [29, Lemmas 2.1.35 and 2.1.26]. The last two follow easily from the composition law and the fact that every roof $X \xleftarrow{f} W \xrightarrow{g} Y$ is equivalent to the composition $(f, \mathrm{Id}_W) \circ (\mathrm{Id}_W, g)$.

8.2.2. The Verdier quotient by the boundary category. The following proposition serves as the basis for the geometric model. It was first proved in [23].

Proposition 8.6. *There exists a triangle equivalence*

$$\mathscr{G} \colon \mathcal{D}^{\mathrm{b}}(\Lambda_n)/\mathcal{D}_\partial \xrightarrow{\simeq} \mathcal{D}^{\mathrm{b}}(C_n).$$

Proof. Since $\mathcal{D}_\partial$ is triangulated, the quotient is well defined. As shown in [12, Theorems 2 and 6], there exists an exact functor $\mathbb{G} \colon \mathrm{Coh}\,\mathbb{X}_n \to \mathrm{Coh}\,C_n$, where $\mathbb{X}_n$ denotes the non-commutative curve mentioned at the beginning of Section 2. The kernel of $\mathbb{G}$ is a semi-simple abelian category T. Moreover, it is proved in [12] that $\mathbb{G}$ induces an equivalence between the Serre quotient $\mathrm{Coh}\,\mathbb{X}_n/T$ and $\mathrm{Coh}\,C_n$. By a result of Miyachi [28, Theorem 3.2], the derived functor of $\mathbb{G}$ induces an equivalence

$$\mathcal{D}^{\mathrm{b}}(\mathbb{X}_n)/\mathcal{D}^{\mathrm{b}}_T(\mathbb{X}_n) \xrightarrow{\simeq} \mathcal{D}^{\mathrm{b}}(C_n),$$

where $\mathcal{D}^{\mathrm{b}}_T(\mathbb{X}_n)$ denotes the full subcategory of all objects whose cohomologies lie in T. By induction and the use of truncation functors of the standard t-structure in $\mathcal{D}^{\mathrm{b}}(\mathbb{X}_n)$, we see that $\mathcal{D}^{\mathrm{b}}_T(\mathbb{X}_n)$ coincides with the triangulated hull $\mathcal{T}$ of T inside $\mathcal{D}^{\mathrm{b}}(\mathbb{X}_n) \simeq \mathcal{D}^{\mathrm{b}}(\Lambda_n)$. Proposition 12 in [12] and its proof generalize to arbitrary cycles in the obvious way, and we observe that T consists precisely of appropriate shifts of all the complexes

$$P_{t(b_i)} \xrightarrow{b_i} P_{s(b_i)} \xrightarrow{a_i} P_{s(a_i)} \quad \text{and} \quad P_{t(d_i)} \xrightarrow{d_i} P_{s(d_i)} \xrightarrow{c_i} P_{s(c_i)}, \tag{8.1}$$

where $i \in \mathbb{Z}_n$. For each $i \in \mathbb{Z}_n$, these complexes represent the two segments on the boundary component B_i cut out by the marked points. By [31, Theorem 4.1], the concatenation of two arcs at a marked point p represents the mapping cone of the morphism associated with p. It follows that the complexes in (8.1) generate $\mathcal{D}_{\partial_i}$. In particular, $\mathcal{T} = \mathcal{D}_\partial$ which establishes the claimed equivalence. $\blacksquare$

Remark 8.7. The restriction of a Verdier quotient functor to the perpendicular category of its kernel is fully faithful. By Lemma 8.4 and Proposition 8.6, we obtain an embedding $\mathscr{G} \colon \mathcal{D}_{\mathrm{loop}} \hookrightarrow \mathcal{D}^{\mathrm{b}}(C_n)$. In fact, it follows from [12, Theorem 6] that $\mathscr{G} \circ \mathbb{F}$ is isomorphic to the identity functor of $\mathrm{Perf}(C_n)$.

In the subsequent sections, we denote by $\pi \colon \mathcal{D}^{\mathrm{b}}(\Lambda_n) \to \mathcal{D}^{\mathrm{b}}(\Lambda_n)/\mathcal{D}_\partial$ the localization functor.

8.2.3. Indecomposable objects of $\mathcal{D}^b(C_n)$ as curves on the n-punctured torus. We give an interpretation of isomorphism classes on indecomposable objects in $\mathcal{D}^b(C_n)$ as curves on the *n*-punctured torus. Our classification follows already from the results in [19]. However, since some of the intermediate results will be useful in later parts of this paper, we include a slightly more detailed proof of the classification result which avoids the language of A_∞-categories. In what follows, we denote a mapping cone of a morphism f in a triangulated category by C_f.

For our description of indecomposable objects in $\mathcal{D}^b(C_n)$, we need a mild generalization of [31, Theorem 4.1] which states that if p is an intersection of two arcs at a marked point and $f = f_p$, then γ_{C_f} is obtained by concatenation of the original arcs at p.

Lemma 8.8. *Let $Z_1, Z_2 \in \mathcal{D}_\partial$, let $X \in \mathcal{D}^b(\Lambda_n)$ be indecomposable and let $f \colon Z_1 \oplus Z_2 \to X$ be a morphism whose components $f_i \colon Z_i \to X$ correspond to boundary intersections $p_i \in \gamma_{Z_i} \overrightarrow{\cap} \gamma_X$ at distinct end points of γ_X. Then C_f is represented by the concatenation of γ_X with γ_{Z_1} and γ_{Z_2} at p_1 and p_2.*

Proof. Following the explicit description in of [31, Section 3], we see that f_i is of one of the following shapes:

$$
\begin{array}{c}
Z_i \\
\downarrow {\scriptstyle f_i} \\
X
\end{array}
\qquad
\begin{array}{c}
\text{---} \bullet \!\!-\!\! \bullet \!\!-\!\! \cdots \!\!-\!\! \circ \\
{\scriptstyle \alpha}\Big\downarrow {\scriptstyle (\ddagger)} \;\| \qquad \| \\
\text{---} \bullet \!\!-\!\! \bullet \!\!-\!\! \cdots \!\!-\!\! \circ
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
\text{---} \circ \\
\Big\downarrow {\scriptstyle \beta} \\
\text{---} \circ
\end{array}
\tag{8.2}
$$

The rows represent the quivers of Z_i and X respectively with arrows being replaced by unoriented edges for the sake of generality, so that each horizontal line represents one of the maps $t_i \otimes u_i$ in Section 4.3 which constitute the differential. The vertex $\circ$ represents the projective module at the end of a string complex. Downward arrows indicate maps induced by non-trivial admissible paths and double lines represent isomorphisms. Dashed lines and dashed arrows may not be present.
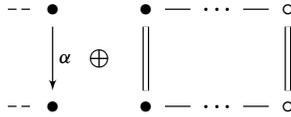
Chain maps as in the left diagram are called *graph maps*, and maps as on the right-hand side are called *singleton single maps* in [2]. For example, the identity map of a string or band complex is an instance of a graph map. If it exists, let α denote the unique downward arrow in the diagram associated with f_i.

If f_i is a singleton single map, then its mapping cone (as a chain complex) is a string complex and its quiver is obtained by joining the quiver of Z_i (the upper row of the diagram) with the quiver of X by the arrow β.

If f_i is a graph map and α exists, then in the leftmost square in (8.2), the upper and the lower horizontal lines are oriented the same way in the original quivers of γ_{Z_i} and γ_X. Let v denote one of the diagonal arrows in $(\ddagger)$, that is,

$$
\begin{array}{c}
\bullet \longleftarrow \bullet \\
{\scriptstyle \alpha}\Big\downarrow \;\searrow{\scriptstyle v} \;\| \\
\bullet \longleftarrow \bullet
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
\bullet \longrightarrow \bullet \\
{\scriptstyle \alpha}\Big\downarrow \;\nearrow{\scriptstyle v} \;\| \\
\bullet \longrightarrow \bullet
\end{array}
\tag{8.3}
$$

If v is a map which renders the corresponding diagram in (8.3) commutative, then v corresponds to an endomorphism $\bar{v}\colon C_{f_i} \to C_{f_i}$ of graded Λ_n-modules (not chain complexes) of degree 0 such that $\bar{v}^2 = 0$. Hence, $1 - \bar{v}$ is an invertible graded map and one can verify that after conjugation with $1 - \bar{v}$, C_{f_i} becomes the direct sum of chain complexes whose representation by quivers is

$$
\begin{array}{ccc}
\bullet & \bullet \!-\! \cdots \!-\! \circ \\
\alpha \downarrow \quad \oplus & \| \| \qquad \| \| \\
\bullet & \bullet \!-\! \cdots \!-\! \circ
\end{array}
$$

The second direct summand is contractible and the first is a string complex which is represented by the concatenation of γ_X and γ_{Z_i} at p_i.

The proof is analogous, if α is not present in which case the horizontal arrows in the upper and lower rows in (8.3) will have mutually opposite orientations.

After this preparation, the actual proof of the assertion is quite short: If p_1 and p_2 correspond to distinct ends of γ_X (even though these might correspond to the *same* marked point in $\mathbb{T}_n$, e.g., if γ_X is closed), then the above transformations of the complex can be performed independently without interference as graph maps at different ends of a string complex cannot overlap other than at the arrow α. In particular, C_f is represented by the concatenation of γ_X with γ_{Z_1} and γ_{Z_2} at p_1 and p_2. ∎

Remark 8.9. In terms of diagrams as in (8.2), the connecting morphism

$$
C_f \to (Z_1 \oplus Z_2)[1]
$$

is given by the projection onto the upper row, and the map $X \to C_f$ is the inclusion of the lower row. If $C_f \not\simeq 0$, both correspond to graph maps or singleton single maps under the transformations in the proof of Lemma 8.8 and are represented by the "obvious" intersections as illustrated in Figure 16.

In order to understand when two indecomposable objects are isomorphic, we need to understand distinguished triangles whose mapping cone is an object of $\mathcal{D}_\partial$, or equivalently, mapping cones of maps $Z \to X$, where $Z \in \mathcal{D}_\partial$ and $X \in \mathcal{D}^b(\Lambda_n)$ is indecomposable. The following lemma serves as a preparation.

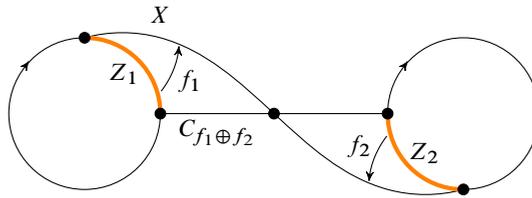**Fig. 16.** The curves of the objects X, Z_1, Z_2 and the mapping cone $C_{f_1 \oplus f_2}$. The intersections $\bullet$ represent the maps f_1, f_2 as well as $X \to C_{f_1 \oplus f_2}$ and $C_{f_1 \oplus f_2} \to (Z_1 \oplus Z_2)[1]$.

Lemma 8.10. *Let $\mathcal{T}$ be a triangulated category and for $i = 1, 2$, let $\beta_i \colon Z_i \to X$ be a morphism in $\mathcal{T}$. If there exists $\rho \colon Z_1 \to Z_2$ such that $\beta_1 = \beta_2 \circ \rho$, then*

$$C_{\beta_1 \oplus \beta_2} \cong Z_1[1] \oplus C_{\beta_2}.$$

Proof. Consider the commutative diagram

$$
\begin{array}{ccccccc}
Z_1 \oplus Z_2 & \xrightarrow{\ (\beta_1\ \beta_2)\ } & X & \longrightarrow & C_{\beta_1 \oplus \beta_2} & \longrightarrow & (Z_1 \oplus Z_2)[1] \\
{\scriptstyle \alpha = \left(\begin{smallmatrix} \mathrm{Id} & 0 \\ \rho & \mathrm{Id} \end{smallmatrix}\right)} \downarrow & & \| & & \downarrow & & \downarrow {\scriptstyle \alpha[1]} \\
Z_1 \oplus Z_2 & \xrightarrow{\ (0\ \beta_2)\ } & X & \xrightarrow{\ (0\ c)^{\mathsf{T}}\ } & Z_1[1] \oplus C_{\beta_2} & \xrightarrow[\left(\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & d \end{smallmatrix}\right)]{} & (Z_1 \oplus Z_2)[1].
\end{array}
$$

The vertical maps in the diagram are invertible and its rows are distinguished triangles. The lower row is the direct sum of a distinguished triangle involving β_2, c and d and a split triangle. ∎

Lemma 8.11. *Let $0 \neq f \colon Z \to X$ be a morphism in $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$, where $Z \in \mathcal{D}_\partial$ and X is indecomposable. Then $C_f \cong Z' \oplus W$, where $Z' \in \mathcal{D}_\partial$ and γ_W is obtained from γ_X by clockwise rotations at its end points. Moreover, the connecting morphism $C_f \to Z[1]$ is a sum of a morphism $Z' \to Z[1]$ and a morphism $W \to Z[1]$ which is supported at a single intersection.*

Proof. Choose a splitting $Z = \bigoplus_a Z^a$ into indecomposable summands. By splitting of summands using Lemma 8.10, we may assume that each component $Z^a \to X$ of f is non-zero. Let $f^a = \sum f_i^a$ denote the corresponding decomposition of $f|_{Z^a}$ such that for all a and all $i \neq j$, f_i^a and f_j^a are non-zero multiples of morphisms associated to distinct intersections $p_i^a, p_j^a \in \gamma_{Z^a} \stackrel{\rightarrow}{\cap} \gamma_X$. For each pair (a, i), there exist an object $Z_i^a \in \mathcal{D}_\partial$ and boundary intersections $q_i^a \in \gamma_{Z^a} \stackrel{\rightarrow}{\cap} \gamma_{Z_i^a}$ and $t_i^a \in \gamma_{Z_i^a} \stackrel{\rightarrow}{\cap} \gamma_X$ such that $p_i^a \in C(q_i^a, t_i^a)$, see Definition 4.16. In particular, if $\alpha_i^a \colon Z^a \to Z_i^a$ and $\beta_i^a \colon Z_i^a \to X$ denote the corresponding morphisms, then f_i^a is a non-zero multiple of $\beta_i^a \circ \alpha_i^a$. The above conditions determine the homotopy class of $\gamma_{Z_i^a}$ as well as q_i^a and t_i^a uniquely. In particular, we observe that for all pairs (a, i) and (a, j), $\gamma_{Z_i^a}$ and $\gamma_{Z_j^a}$ differ by full rotations around the boundary and therefore have identical end points.

Our goal is to apply Lemma 8.10 to compute C_f. The set $\mathcal{Z} = \{Z_i^a\}$ is ordered by the condition that $Z_i^a \prec Z_j^b$ if and only if there exists a morphism $\rho \colon Z_i^a \to Z_j^b$ associated with a boundary intersection such that β_i^a is a non-zero multiple of $\beta_j^b \circ \rho$. We note that the relation $\prec$ is transitive and reflexive but in general not anti-symmetric as distinct Z_i^a and Z_j^b may be isomorphic. Moreover, Z_i^a and Z_j^b are not necessarily $\prec$-comparable but they are if t_i^a and t_j^b correspond to the same end point of γ_X. Hence, $\mathcal{Z}$ naturally splits into two (possibly empty) subsets $\mathcal{Z} = \mathcal{Z}^+ \sqcup \mathcal{Z}^-$, one for each end point of γ_X, such that any two elements from the same subset are $\prec$-comparable. We denote by $Z^\pm$ any (fixed) $\prec$-maximal element in $\mathcal{Z}^\pm$ and by $z^\pm \colon Z^\pm \to X$ the corresponding map from the set $\{\beta_i^a\}$.

Set $\alpha := \bigoplus_{(a,i)} \alpha_i^a$ and $\beta^\pm := \bigoplus_{Z_i^a \in Z^\pm} \beta_i^a$. Then α, β^+ and β^- are supported only at boundary intersections. We also write $\beta = \beta^+ + \beta^-$ so that $\beta \circ \alpha = f$ after rescaling each α_i^a such that $\beta_i^a \circ \alpha_i^a = f_i^a$. Then $\beta^\pm$ factors through $z^\pm$ and hence β factors through $z^+ \oplus z^-$. Finally, the octahedral axiom shows the existence of a distinguished triangle

$$C_\alpha \to C_f \to C_\beta \xrightarrow{w} C_\alpha[1].$$

Note that the map w can be chosen to be a composition of maps $u \colon C_\beta \to (\bigoplus Z_i^a)[1]$ and $v[1] \colon (\bigoplus Z_i^a)[1] \to C_\alpha[1]$ which appear in distinguished triangles

$$\bigoplus Z_i^a \xrightarrow{\beta} X \to C_\beta \xrightarrow{u} (\bigoplus Z_i^a)[1],$$
$$Z \xrightarrow{\alpha} \bigoplus Z_i^a \xrightarrow{v} C_\alpha \to Z[1].$$

It follows from Remark 8.9 that $w = 0$. Finally, Lemma 8.10 and the fact that $C_\alpha \in \mathcal{D}_\partial$ imply that $C_f \cong C_\beta \oplus C_\alpha \cong C_{z^+ \oplus z^-} \oplus C_\alpha$. Now, the assertion follows from Lemma 8.8 and Remark 8.9. ∎

Remark 8.12. Dual versions of Lemma 8.10 for morphisms $X \to Z_i$ and Lemma 8.11 for morphisms $X \to Z$ are proved in the same way.

The inclusion $\mathbb{T}_n \subseteq \mathcal{T}^n$ induces a surjective map from the set of homotopy classes of curves on $\mathbb{T}_n$ to the set of homotopy classes of curves on $\mathcal{T}^n$. More precisely, we view $\mathcal{T}^n$ as being glued from $\mathbb{T}_n$ and a 1-punctured disc (the puncture being the center) for each boundary component of $\mathbb{T}_n$. Given a curve γ on $\mathbb{T}_n$, we apply the radial contraction of each disc to its puncture in the center. This yields a curve $\overline{\gamma}$ on $\mathcal{T}^n$ with the same domain as γ. If γ is an arc with end points on boundary components B_1 and B_2, then $\overline{\gamma}$ is an arc whose end points are the punctures associated with B_1 and B_2. If γ is a loop, then so is $\overline{\gamma}$. We further observe that if $\gamma_1, \gamma_2 \subseteq \mathbb{T}_n$ are arcs, then $\overline{\gamma_1} \simeq \overline{\gamma_2}$ if and only if γ_1 and γ_2 are obtained from each other through an application of fractional twists.

Proposition 8.13. *There exists a bijection between isomorphism classes of indecomposable objects of $\mathcal{D}^b(C_n)$, up to shift, and homotopy classes of curves on $\mathcal{T}^n$ equipped with indecomposable local systems. More precisely, if $X \in \mathcal{D}^b(\Lambda_n)$ is indecomposable and $(\gamma, \mathcal{V})$ is its associated curve γ with local system $\mathcal{V}$, then $(\overline{\gamma}, \mathcal{V})$ is the pair assigned to $\mathcal{G} \circ \pi(X) \in \mathcal{D}^b(C_n)$.*

Proof. Let $X \in \mathcal{D}^b(\Lambda_n)$ be such that no direct summand of X lies in $\mathcal{D}_\partial$. If $\pi(X)$ is indecomposable and $X \cong X_1 \oplus X_2$, then by additivity of π, $\pi(X) \cong \pi(X_1) \oplus \pi(X_2)$. Consequently, if $\pi(X_1) \ncong 0$, then $X_2 \in \mathcal{D}_\partial$ which shows that X is indecomposable whenever $\pi(X)$ is. Next, suppose that X is indecomposable and $\pi(X) \cong \pi(Y)$, where $Y \in \mathcal{D}^b(\Lambda_n)$. Then Y is isomorphic to the mapping cone of a map $g \colon Z \to X$, where $Z \in \mathcal{D}_\partial$. By Lemma 8.11, $C_g \cong Z' \oplus W$, where $Z' \in \mathcal{D}_\partial$ and W is indecomposable such that γ_W is obtained from γ_X by rotation at end points. By restricting to the case $Y = Y_1 \oplus Y_2$ and using the Krull–Remak–Schmidt property of $\mathcal{D}^b(\Lambda_n)$, we see that

$\pi(X) \cong \pi(W)$ is indecomposable. On the other hand, if $Y \in \mathcal{D}^{b}(\Lambda_n)$ is indecomposable such that γ_X and γ_Y differ by rotations at end points, then $\pi(X) \cong \pi(Y)$ up to shift by Lemma 8.8. ∎

Notation 8.14. By abuse of notation, we will occasionally use γ_X for both the curve $\gamma = \gamma_X \subseteq \mathbb{T}_n$ of an indecomposable object $X \in \mathcal{D}^{b}(\Lambda_n)$ and the curve $\gamma_{\mathcal{G} \circ \pi(X)} \subseteq \mathcal{T}^n$ which represents $\mathcal{G} \circ \pi(X) \in \mathcal{D}^{b}(C_n)$. As explained above, the latter is given by $\bar{\gamma}$.

8.2.4. Morphisms in $\mathcal{D}^{b}(C_n)$ as intersections of curves. We extend the graphical description of morphisms and compositions to the category $\mathcal{D}^{b}(C_n)$. The computations of morphisms in the Verdier quotient of $\mathcal{D}^{b}(\Lambda_n)$ "at the boundary", which we obtain in this subsection, are by no means specific to the algebra Λ_n and can be generalized to arbitrary gentle algebras and similar Verdier quotients, see Remark 8.28.

To begin with, we recall the definition of exceptional cycles in a triangulated category due to Broomhead, Pauksztello and Ploog.

Definition 8.15 ([8]). Let $r \geq 2$. A collection $\{U_i\}_{i \in \mathbb{Z}_r}$ of objects in a triangulated category $\mathcal{T}$ with Serre functor S is an *exceptional r-cycle* if

(1) every object U_i is exceptional, i.e., $\dim \mathrm{Hom}^{\bullet}(U_i, U_i) = 1$;

(2) for each $i \in \mathbb{Z}_r$, there exists $m_i \in \mathbb{Z}$ such that $S U_i \cong U_{i+1}[m_i]$;

(3) $\mathrm{Hom}^{\bullet}(U_i, U_j) = 0$, unless $j = i$ or $j = i + 1$.

As in the case of spherical objects, every exceptional cycle $\mathcal{U} = \{U_i\}$ induces an auto-equivalence $T_{\mathcal{U}}$ [8] described by the evaluation triangle (3.1), where $X = \oplus_{i \in \mathbb{Z}_r} U_i$.

For every component $B \subseteq \partial \mathbb{T}_n$, the segments on B between the two marked points correspond to an exceptional 2-cycle U_B^1, U_B^2 which follows from [30, Section 2.5]. We set $U_B := U_B^1 \oplus U_B^2$ and denote by T_B the associated twist functor. From Corollary 3.7 (which is also true for T_B), we conclude that T_B and $T_{B'}$ commute for any pair of components B, B'.

Corollary 8.16. *The functor $\vartheta := \prod_{B \subseteq \partial \mathbb{T}_n} T_B$, the composition of the twist functors T_B, is well defined up to isomorphism of functors.*

It follows from [31, Corollary 5.2.] and Lemma 8.11 that ϑ coincides with τ^{-1} on the level of objects so that $\Psi(\vartheta) = \Psi(\tau^{-1}) \in \mathcal{MCG}(\mathbb{T}_n)$. In the case of discrete derived categories, such a statement was already proved in [8]. In fact, even the following is true.

Proposition 8.17. *There exists an isomorphism of functors $\vartheta \simeq \tau^{-1}$.*

Proof. The mapping class $\Psi(\vartheta \tau)$ is trivial. By Proposition 6.3, it is therefore a composition of a shift and an equivalence which is induced by an automorphism of Λ_n which multiplies each arrow $\alpha \in \{d_0\} \cup \{c_i\}_{i \in \mathbb{Z}_n}$ in the quiver of Figure 2 with a scalar $\lambda_\alpha \in \Bbbk^{\times}$. We have $\vartheta \tau(Y) \cong Y$ for all $Y \in \mathcal{D}_{\mathrm{loop}} = {}^{\perp}\mathcal{D}_{\vartheta}$. In particular, this applies to $\mathcal{O}(\lambda)$ and all objects $\Bbbk(i, \mu)$ which shows that the shift is trivial and $\lambda_\alpha = 1$ for all $\alpha \in \{d_0\} \cup \{c_i\}_{i \in \mathbb{Z}_n}$. ∎

For each boundary component $B \in \partial \mathbb{T}_n$, the definition of T_B gives rise to a natural transformation $\eta^B : \mathrm{Id}_{\mathcal{D}^{\mathrm{b}}(\Lambda_n)} \to T_B$ given on X by the map $X \to T_B(X)$ in the evaluation triangle. Again, the composition of all η^B is independent from the order of composition and defines a natural transformation $\eta : \mathrm{Id}_{\mathcal{D}^{\mathrm{b}}(\Lambda_n)} \to \vartheta$. That is, if $B_1, \ldots, B_n$ are the components of $\partial \mathbb{T}_n$ and $\eta(i) := \eta^{B_i}$, then η_X is equal to the composition

$$X \xrightarrow{\eta(1)_X} T_{B_1}(X) \xrightarrow{\eta(2)_{T_{B_1}(X)}} T_{B_2} \circ T_{B_1}(X) \to \cdots \to \vartheta(X).$$

Remark 8.18. The natural transformation η gives a different explanation for the natural transformation $\mathrm{Id}_{\mathcal{D}^{\mathrm{b}}(\mathrm{Coh}\,\mathbb{X}_n)} \to \tau^{-1}$ from [12, Corollary 1] in the derived category of the non-commutative curve $\mathbb{X}_n$.

For all $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$, η induces an infinite tower of graded $\mathbb{k}$-linear maps

$$\mathrm{Hom}^{\bullet}(X, Y) \to \mathrm{Hom}^{\bullet}(X, \vartheta(Y)) \to \mathrm{Hom}^{\bullet}(X, \vartheta^2(Y)) \to \cdots$$

which allows us to define its colimit. Moreover, for each number $m \geq 0$, η give rise to a map

$$\mathrm{Hom}^{\bullet}(X, \vartheta^m(Y)) \to \mathrm{Hom}^{\bullet}(\pi(X), \pi(Y)), \quad f \mapsto (\alpha, \vartheta^{-m}(f)), \tag{8.4}$$

where $\alpha : \vartheta^{-m}(X) \to X$ is the map induced by the m-fold power η^m of η. More precisely, $\eta^m : \mathrm{Id} \to \vartheta^m$ is given on X by the composition

$$X \xrightarrow{\eta_X} \vartheta(X) \xrightarrow{\eta_{\vartheta(X)}} \vartheta^2(X) \to \cdots \to \vartheta^m(X).$$

Lemma 8.19. *Let $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}^{\bullet}(X, Y) & \xrightarrow{\ (\eta^m)_Y \circ -\ } & \mathrm{Hom}^{\bullet}(X, \vartheta^m(Y)) \\
{\scriptstyle \mathrm{can}} \downarrow & \swarrow & \\
\mathrm{Hom}^{\bullet}(\pi(X), \pi(Y)) & &
\end{array}
$$

commutes.

Proof. Let $f : X \to Y$. Denote by $\alpha_Y : Y \to \vartheta^m(Y)$ and $\alpha_X : \vartheta^{-m}(X) \to X$ the maps induced by η^m and set $g := \alpha_Y \circ f$. Consider the diagram

$$
\begin{array}{c}
\vartheta^{-m}(X) \\
\end{array}
$$

where each unlabelled arrow represents the map α_X. As η^m is a natural transformation, it follows that $f \circ \alpha_X = \vartheta^{-m}(g)$. Thus, the above diagram commutes and the roofs $(\mathrm{Id}_X, f) = \pi(f)$ and $(\alpha_X, \theta^{-m}((\eta^m)_Y \circ f))$ are equivalent. ∎

Lemma 8.20. *Let $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that no direct summand of Y lies in $\mathcal{D}_\partial$. Then there exists $l \geq 0$ such that for all $m \geq l$, the map $\mathrm{Hom}^\bullet(X, \vartheta^m(Y)) \to \mathrm{Hom}^\bullet(\pi(X), \pi(Y))$ is injective. In particular, for all $m \geq l$, the map $\mathrm{Hom}^\bullet(X, \vartheta^m(Y)) \to \mathrm{Hom}^\bullet(X, \vartheta^{m+1}(Y))$ is injective.*

Proof. Since no direct summand of Y lies in $\mathcal{D}_\partial$, the graphical description of composition shows that a morphism $f \colon X \to Y$ factors through $\mathcal{D}_\partial$ if only if all $p \in \mathrm{supp}\, f$ are interior intersections and p bounds an immersed intersection triangle whose other two corners are end points of γ_X and γ_Y, and whose third side is a boundary arc. As ϑ is given by clockwise rotation of the boundary, we see that γ_X and $\gamma_{\vartheta^m(Y)}$ cannot bound any such triangle for m large enough. For all such m, the first map is injective. Injectivity of the second map follows from Lemma 8.19. ∎

The following proposition describes the morphisms in $\mathcal{D}^{\mathrm{b}}(C_n)$.

Proposition 8.21. *Let $X, Y \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that Y has no direct summands in $\mathcal{D}_\partial$. Then the maps $\mathrm{Hom}^\bullet(X, \vartheta^i(Y)) \to \mathrm{Hom}^\bullet(\pi(X), \pi(Y))$ induce an isomorphism*

$$\underset{i \in \mathbb{N}}{\mathrm{colim}}\, \mathrm{Hom}^\bullet(X, \vartheta^i(Y)) \xrightarrow{\cong} \mathrm{Hom}^\bullet(\pi(X), \pi(Y)).$$

Proof. By Lemmas 8.19 and 8.20, the map exists and is injective. We may assume that X is indecomposable. Let $g \colon \pi(X) \to \pi(Y)$ be a morphism and let (α, f) be a representing roof of g, where $\alpha \colon W \to X$ and $f \colon W \to Y$. We may further assume that W is indecomposable because $\pi(X)$ is indecomposable. Indeed, it follows from Lemma 8.10 and the proof of Lemma 8.11 that $W \cong X' \oplus Z'$, where X' is indecomposable and $Z' \in \mathcal{D}_\partial$. We then can replace (α, f) by the roof $(\alpha \circ \iota, f \circ \iota)$, where $\iota \colon X' \to W$ denotes the natural inclusion.

By definition, the morphism $\alpha \colon W \to X$ sits in a distinguished triangle

$$Z \xrightarrow{u} W \xrightarrow{\alpha} X \to Z[1],$$

where $Z \in \mathcal{D}_\partial$. Since X is indecomposable, Lemma 8.10 and the proof of Lemma 8.11 show that Z must be indecomposable and γ_X is obtained from γ_W by a number of positive rotations at its end points. Let us denote by r and s the necessary numbers of such rotations at each end point. We denote their maximum by m and claim that there exists a map $w \colon \vartheta^{-m}(X) \to W$ such that $\alpha \circ w \colon \vartheta^{-m}(X) \to X$ is induced by η^m. In this case, the roof $(\alpha \circ w, f \circ w)$ is equivalent to (α, f) as can be seen from the commutative diagram

where all unlabelled arrows are induced by η^m. Then $\vartheta^m(f \circ w) \in \mathrm{Hom}(X, \vartheta^m(Y))$ is a preimage of g. Once again, the existence of w can be read off from the surface: α corresponds to a single intersection (Remark 8.9) which forms an intersection triangle with the intersection corresponding to $\eta_{\vartheta^{-m}(X)}$ and a boundary intersection $p \in \gamma_{\vartheta^{-m}(X)} \overrightarrow{\cap} \gamma_W$. We define w as a suitable multiple of f_p. ∎

Proposition 8.21 can be rephrased in the following way.

Theorem 8.22. *Let* $X, Y \in \mathcal{D}^{\mathrm{b}}(C_n) \setminus \mathrm{Perf}(C_n)$ *be indecomposable. Then there exists a bijection between a homogeneous basis of* $\mathrm{Hom}^\bullet(X, Y)$ *and the set consisting of*

(1) *all interior intersections of* γ_X *and* γ_Y, *and*

(2) *paths in the loop quiver*

for each $p \in \gamma_X \overrightarrow{\cap} \gamma_Y$ *which is a puncture.*

Proof. Let $X', Y' \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ such that $\pi(X') \cong X$ and $\pi(Y') \cong Y$. We may assume that the cardinality of $X' \overrightarrow{\cap} Y'$ coincides with the number of intersections of X and Y. In particular, the boundary intersections of $\gamma_{X'}$ and $\gamma_{Y'}$ are in bijection with intersections of γ_X and γ_Y at punctures. Moreover, every interior intersection $p \in \gamma_X \cap \gamma_Y$ corresponds to a unique interior intersection $p' \in \gamma_{X'} \cap \gamma_{Y'}$, and $\pi(f_{p'})$ is by definition an element in the claimed basis element in the claimed basis of $\mathrm{Hom}^\bullet(X, Y)$. Let $p \in \gamma_X \overrightarrow{\cap} \gamma_Y$ be a puncture and let q denote the corresponding boundary intersection of $\gamma_{X'}$ and $\gamma_{Y'}$. Then, for every $m \in 2\mathbb{Z}$, $q \in \gamma_{\vartheta^{-m}(X')} \overrightarrow{\cap} \gamma_{Y'}$ and the paths in the loop quiver of p correspond to roofs

$$X' \xleftarrow{\eta^m} \vartheta^{-m}(X') \xrightarrow{f_q} Y'[\deg q],$$

i.e., the image of $\vartheta^m(f_q)$ under the map $\mathrm{Hom}^\bullet(X', \vartheta^m(Y')) \to \mathrm{Hom}^\bullet(X, Y)$. ∎

Remark 8.23. The morphisms in Theorem 8.22 (2) can be visualized as intersections as follows. Let $\widetilde{\mathcal{T}}$ denote the universal cover of $\mathcal{T}^n$, where all punctures are thought as being removed from the surface. Then, locally around a puncture p, $\mathcal{T}^n$ is homeomorphic to $\mathbb{C}^\times$. Using polar coordinates and the identifications $(-\pi, \pi) \cong \mathbb{R} \cong (0, 1)$, $\widetilde{\mathcal{T}}$ can be identified locally with the open unit disc $\mathbb{D}_{\mathrm{sl}}$ with a slit $[-1, 0] \times \{0\}$ removed. Arcs with end point at p are lifted to non-compact arcs with limit $0 \in \mathbb{D}_{\mathrm{sl}}$. The various morphisms arising from an intersection of γ_X and γ_Y at p then correspond to the intersections at 0 between a fixed lift of γ_X and the different lifts of γ_Y.

Remark 8.24. The isomorphism in Proposition 8.21 is compatible with compositions in the following sense. For $X, Y, Z \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$, consider the tower consisting of the objects

$$\mathrm{Hom}^\bullet(X, \vartheta^i(Y)) \times \mathrm{Hom}(\vartheta^i(Y), \vartheta^i(Z)),$$

and the product of the maps $\mathrm{Hom}^\bullet(X, \vartheta^i(Y)) \to \mathrm{Hom}^\bullet(X, \vartheta^{i+1}(Y))$ with the canonical maps $\mathrm{Hom}(\vartheta^i(Y), \vartheta^i(Z)) \to \mathrm{Hom}(\vartheta^{i+1}(Y), \vartheta^{i+1}(Z))$ induced by ϑ. Each object in the

tower has a composition map to $\mathrm{Hom}^\bullet(X, \vartheta^m(Z))$. As ϑ is an equivalence, the colimit of this tower is naturally isomorphic to

$$\varinjlim_{i \in \mathbb{N}} \mathrm{Hom}^\bullet(X, \vartheta^i(Y)) \times \mathrm{Hom}^\bullet(Y, Z)$$

and has an induced composition map to $\mathrm{Hom}^\bullet(X, Z)$. Under the isomorphism in Proposition 8.21, this coincides with the usual composition map. This observation allows us to interpret compositions in $\mathcal{D}^b(C_n)$ in a similar way as in the category $\mathcal{D}^b(\Lambda_n)$ via tridents, triangles and double-bigons in the universal cover $\widetilde{\mathcal{T}}$ of $\mathcal{T}^n$ whose corners are allowed to converge to a puncture, see Remark 8.23.

Example 8.25. Suppose $\delta_0, \delta_1, \delta_2 \subseteq \mathcal{T}^n$ are arcs which meet at a puncture as in Figure 17. The intersection generates a subalgebra of $\mathrm{End}^\bullet(\bigoplus_{i=1}^3 X_{\delta_i})$ which is isomorphic to the path algebra of the quiver shown on the right-hand side of Figure 17. The degree of a full cycle equals 2, i.e., the inverse winding number around the puncture. The paths in the loop quiver in Theorem 8.22 of the pair $X_{\delta_0}, X_{\delta_2}$ correspond to the paths $(\beta\alpha)(\gamma\beta\alpha)^i$, $i \geq 0$.
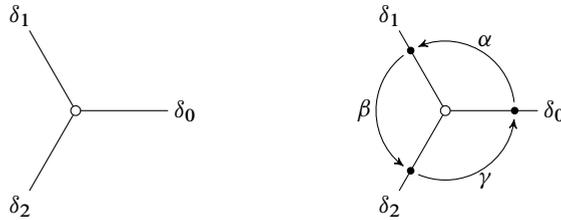

Fig. 17. Example of the quiver associated to the intersections of three arcs at a puncture.

Remark 8.26. Distinguished triangles in $\mathcal{D}^b(\Lambda_n)/\mathcal{D}_\vartheta$ are isomorphic to images of distinguished triangles in $\mathcal{D}^b(\Lambda_n)$. This allows us to determine the curve of the mapping cone of any morphism which corresponds to an intersection. The only new and interesting case appears for an intersection at a puncture. Consider arcs δ_1, δ_2 as in Example 8.25. For $i \geq 0$, let f^i denote the morphism which corresponds to the path $(\alpha\gamma)^i\beta$ in Figure 17. Then C_{f^i} extends δ_1 and δ_2 outside a small neighbourhood of p and winds around the puncture counter-clockwise i times. Figure 18 shows the curves of C_{f^i} for $i = 0, 1, 2$. The general case is analogous.
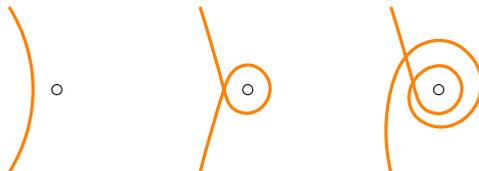

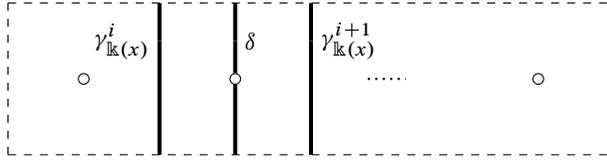**Fig. 18.** The mapping cones of f^0, f^1 and f^2 (from left to right).

Fig. 19. The vertical arc δ represents the structure sheaf of a singular point. White circles indicate punctures.

As an easy application of Theorem 8.22, we can now describe the curves which represent the structure sheaves of singular points.

Corollary 8.27. *Let $z \in C_n$ be singular. Then $\Bbbk(z)$ is represented by a vertical simple arc $\delta \subseteq \mathscr{T}^n$ which lies between two loops $\gamma^i_{\Bbbk(x)}$ and $\gamma^{i+1}_{\Bbbk(x)}$ as shown in Figure 19.*

Proof. Let us denote by γ a representative of $\Bbbk(z)$ in minimal position. Then γ is an arc as $\Bbbk(z)$ is not perfect. Since $\dim \mathrm{Hom}^\bullet(\Bbbk(z), \mathcal{O}_{C_n}) = 1 = \dim \mathrm{Hom}^\bullet(\mathcal{O}_{C_n}, \Bbbk(z))$ and $\mathrm{Hom}^\bullet(\Bbbk(z), \Bbbk(x)) = 0 = \mathrm{Hom}^\bullet(\Bbbk(x), \Bbbk(z))$ for all $x \neq z$, γ must intersect γ_{Pic} exactly once and must be disjoint from any loop $\gamma^i_{\Bbbk(x)}$. The only homotopy classes of arcs which satisfy these constraints are of the claimed shape. ∎

Remark 8.28. Theorem 8.22 and the arguments outlined in Remark 8.24 generalize to the following situation. Suppose A is a gentle algebra and let $\Sigma = \Sigma_A$ denote its marked surface ($\Sigma = \mathbb{T}_n$ in case $A = \Lambda_n$). We assume that Σ is not a cylinder or a disc. A construction of Σ_A and its relation to $\mathcal{D}^b(A)$ are found in [31]. As in the case $A = \Lambda_n$ and for every subset C of boundary components of Σ, one can define the thick subcategory $\mathcal{D}_C$ which is generated by all objects which are represented by a boundary arc on a component of C. In special situations, $\mathcal{D}_C$ can contain τ-invariant objects which correspond to boundary loops, namely if the winding number of the boundary components vanishes. As in the case of the present paper, one can express objects, morphisms, compositions and certain mapping cones in the Verdier quotient $\mathcal{D}^b(A)/\mathcal{D}_C$ through the surface Σ_C which is obtained from Σ by gluing punctured discs to every component in C. The proofs we presented here generalize with minor adaptations in order to account for the special situations described above.

8.3. Auto-equivalences and diffeomorphisms of punctured tori

We study the relationship between auto-equivalences of $\mathcal{D}^b(C_n)$ and diffeomorphisms of $\mathscr{T}^n$. Inspired by the homomorphism $\Psi \colon \mathrm{Aut}(\mathcal{D}^b(\Lambda_n)) \to \mathcal{MCG}(\mathbb{T}_n)$, we construct a surjective homomorphism of groups

$$\Upsilon \colon \mathrm{Aut}(\mathcal{D}^b(C_n)) \to \mathcal{MCG}(\mathscr{T}^n).$$

The group $\mathcal{MCG}(\mathscr{T}^n)$ will be defined below. The underlying idea of the construction exploits the relationship between the groups of automorphisms of the arc complex of a surface.

8.3.1. Arc complexes and their automorphisms.

Definition 8.29. Let Σ be a compact, oriented surface with or without boundary and a finite set of marked points. The *arc complex* $\mathcal{A}(\Sigma)$ of Σ is the abstract simplicial complex whose m-simplices are collections of $m + 1$ homotopy classes of simple arcs which are pairwise disjoint away from their end points.

We refer to 0-simplices as *vertices* and to 1-simplices as *edges*.

Remark 8.30. $\mathcal{A}(\Sigma)$ is a *flag complex* which means that $m + 1$ vertices of $\mathcal{A}(\Sigma)$ form an m-simplex if and only if they are pairwise connected by an edge.

Remark 8.31. The inclusion $\mathbb{T}_n \hookrightarrow \mathscr{T}^n$ (as defined in Section 4.1) induces a surjective simplicial map $c \colon \mathcal{A}(\mathbb{T}_n) \to \mathcal{A}(\mathscr{T}^n)$ which maps the homotopy class of a simple arc $\gamma \subseteq \mathbb{T}_n$ to the homotopy class of an arc $\widetilde{\gamma} \subseteq \mathscr{T}^n$ by connecting each end point of γ on a component B with the respective puncture which is enclosed by B.

It is clear that every self-diffeomorphism F of Σ induces a simplicial automorphism $\Phi^\Sigma(F)$ of $\mathcal{A}(\Sigma)$. We write $\mathrm{Aut}(\mathcal{A}(\Sigma))$ for the group of simplicial automorphisms of $\mathcal{A}(\Sigma)$, so that $\Phi^\Sigma(-)$ becomes a group homomorphism

$$\Phi^\Sigma \colon \mathcal{MCG}^\pm(\Sigma) \to \mathrm{Aut}(\mathcal{A}(\Sigma))$$

defined on the *extended mapping class group* of Σ, i.e., the group of isotopy classes of *all* self-diffeomorphisms of Σ which preserve the set of marked points. An isotopy is supposed to leave the boundary and all marked points fixed. The *mapping class group* $\mathcal{MCG}(\Sigma)$ is the subgroup of $\mathcal{MCG}^\pm(\Sigma)$ consisting of all orientation-preserving mapping classes. If $\Sigma = \mathbb{T}_n$, the previous definition reduces to Definition 6.1. We write Φ instead of Φ^Σ if Σ is apparent from the context.

In [30], the map $\Phi = \Phi^{\mathbb{T}_n}$ was used to construct the homomorphism

$$\Psi \colon \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(\Lambda_n)) \to \mathcal{MCG}(\mathbb{T}_n)$$

in Proposition 6.3. Namely, for every $T \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(\Lambda_n))$ one can prove that there exists a unique simplicial automorphism $\overline{T} \in \mathrm{Aut}(\mathcal{A}(\mathbb{T}_n))$ such that $\gamma_{T(X)} \simeq \overline{T}(\gamma_X)$ for every indecomposable object $X \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ which is represented by a simple arc, cf. Theorem 6.10. One then shows that $\overline{T} \in \Phi(\mathcal{MCG}(\mathbb{T}_n))$. A result by Disarlo [17] (for surfaces without punctures) shows that Φ is bijective and $\Psi(T)$ is defined as the unique preimage of $\overline{T}$ under Φ.

Disarlo's theorem has an analogue for punctured surfaces. The following is a special case of a result by Irmak and McCarthy.

Theorem 8.32 ([22]). *For $n \neq 1$ and $\Sigma = \mathscr{T}^n$, $\Phi = \Phi^\Sigma$ is an isomorphism. For $\Sigma = \mathscr{T}^1$, Φ is surjective and induces an isomorphism $\mathrm{PGL}_2(\mathbb{Z}) \cong \mathrm{Aut}(\mathcal{A}(\mathscr{T}^1))$.*

We like to associate an automorphism of $\mathcal{A}(\mathscr{T}^n)$ to every auto-equivalence of $\mathcal{D}^{\mathrm{b}}(C_n)$. This requires us to understand how simple loops and interior intersections are characterized within $\mathcal{D}^{\mathrm{b}}(C_n)$. This will be achieved by means of the singularity category.

8.3.2. The singularity category of C_n and the ideal of interior morphisms. We recall that the *singularity category* $\mathcal{D}_{\mathrm{sg}}(C_n)$ of C_n is by definition the Verdier quotient of $\mathcal{D}^{\mathrm{b}}(C_n)$ with respect to $\mathrm{Perf}(C_n)$. In this sense and due to Proposition 8.6, $\mathcal{D}_{\mathrm{sg}}(C_n)$ is obtained as the localization of the Verdier quotient $\mathcal{D}^{\mathrm{b}}(\Lambda_n)/\mathcal{D}_\partial$ at the essential image of $\mathcal{D}_{\mathrm{loop}}$ under the localization functor. One may also consider another category by first localizing $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ at the subcategory $\mathcal{D}_{\mathrm{loop}}$ and then localizing the resulting quotient at the essential image of the subcategory $\mathcal{D}_\partial$.

Lemma 8.33. *The two localizations above are canonically equivalent.*

Proof. By Proposition 8.2, Lemma 8.4 and Proposition 8.6, both localizations are Verdier quotients of $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ at the triangulated subcategory $\mathcal{D}_{\mathrm{loop}} \oplus \mathcal{D}_\partial$. ∎

Given $X, Y \in \mathcal{D}^{\mathrm{b}}(C_n)$, we denote by $\mathrm{Hom}_{\mathrm{Int}}^\bullet(X, Y)$ the kernel of the canonical map

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(C_n)}^\bullet(X, Y) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(C_n)}^\bullet(X, Y).$$

The set $\mathrm{Hom}_{\mathrm{Int}}(X, Y) \subseteq \mathrm{Hom}(X, Y)$ is defined analogously and we refer to elements in $\mathrm{Hom}_{\mathrm{Int}}(X, Y)$ as *interior morphisms*.

Interior morphisms are stable under triangle equivalences in the following sense.

Lemma 8.34. *Let $T \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$. Then T induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{Int}}^\bullet(X, Y) \to \mathrm{Hom}_{\mathrm{Int}}^\bullet(T(X), T(Y)).$$

Proof. The functor T preserves $\mathrm{Perf}(C_n)$ and induces an equivalence on the singularity category. In particular, it preserves the kernel of

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(C_n)}^\bullet(X, Y) \to \mathrm{Hom}_{\mathcal{D}_{\mathcal{D}_{\mathrm{sg}}(C_n)}}^\bullet(X, Y). \qquad \blacksquare$$

The assertion of the next lemma is analogous to the results of [30, Section 2.9] and provides a topological characterization of interior morphisms.

Lemma 8.35. *Let $X, Y \in \mathcal{D}^{\mathrm{b}}(C_n)$ be indecomposable. Then $\mathrm{Hom}_{\mathrm{Int}}^\bullet(X, Y)$ is the set of morphisms which are supported only at interior intersections of γ_X and γ_Y.*

Proof. Let $U, V \in \mathcal{D}^{\mathrm{b}}(\Lambda_n)$ and let us denote by $\pi_{\mathrm{loop}}, \pi_\partial$ and $\pi_\oplus$ the localization functors at $\mathcal{D}_{\mathrm{loop}}$, $\mathcal{D}_\partial$ and $\mathcal{D}_{\mathrm{loop}} \oplus \mathcal{D}_\partial$, respectively. Since $\mathcal{D}_{\mathrm{loop}}$ and $\mathcal{D}_\partial$ are orthogonal, it is not difficult to see that the kernel of the canonical map $\mathrm{Hom}^\bullet(U, V) \to \mathrm{Hom}^\bullet(\pi_\oplus(U), \pi_\oplus(V))$ splits into the sum of the kernels of the corresponding maps induced by π_{loop} and π_∂. Let $f \colon X \to Y$ be interior and choose a preimage $g \colon X \to \vartheta^i(Y)$ in $\mathcal{D}^{\mathrm{b}}(\Lambda_n)$ of f under map (8.4). Then g lies in the kernel of the map induced by π_{loop} and hence factors through $\mathcal{D}_{\mathrm{loop}}$. By Lemma 4.19, g must be supported at interior intersections. Moreover, by definition of η and by the fact that $\mathcal{D}_{\mathrm{loop}}{}^\perp = \mathcal{D}_\partial$, it follows that g factors through

$$(\eta^m)_{\vartheta^{-m+i}(Y)} \colon \vartheta^{-m+i}(Y) \to \vartheta^i(Y)$$

for all $m \in \mathbb{N}$. This implies that all intersections in $\mathrm{supp}\, g$ correspond to interior intersections of γ_X and γ_Y (as arcs in $\mathscr{T}^n$).

Vice versa, given an interior intersection $p \in \gamma_X \mathbin{\overrightarrow{\cap}} \gamma_Y$, we denote by $f \colon X \to Y$ the corresponding morphism and choose g as before. Then g is supported at a single interior intersection and factors through an object which is represented by the loop $(\gamma_X)_{\mathrm{loop}} \subseteq \mathbb{T}_n$ (see Remark 6.6) and 1-dimensional local system. As $\mathrm{Hom}^\bullet_{\mathrm{Int}}(X, Y)$ is a subvector space, this finishes the proof. ∎

We are prepared to construct the map Υ.

Proposition 8.36. *For $n > 1$, there exists a surjective group homomorphism*

$$\Upsilon \colon \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n)) \to \mathcal{MCG}(\mathcal{T}^n).$$

Moreover, there exists a surjective homomorphism $\mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_1)) \to \mathrm{PSL}_2(\mathbb{Z})$.

Proof. Let $F \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$. Subsequently, we shall call indecomposable, non-perfect objects $X, Y \in \mathcal{D}^{\mathrm{b}}(\mathrm{Coh}\, C_n)$ *disjoint* if their representing homotopy classes of arcs contain a pair of representatives with disjoint interior. If $X = Y$, being disjoint is equivalent to γ_X being simple. Given two indecomposable, non-perfect objects $X, Y \in \mathcal{D}^{\mathrm{b}}(C_n)$, it follows from Lemma 8.35 that the canonical map $\mathrm{Hom}^\bullet(X, Y) \to \mathrm{Hom}^\bullet_{\mathcal{D}_{\mathrm{sg}}(C_n)}(X, Y)$ is injective if and only if X and Y are disjoint. By Lemma 8.34, F maps pairs of disjoint simple objects to disjoint simple objects and hence induces an element $\bar{F} \in \mathrm{Aut}(\mathcal{A}(\mathcal{T}^n))$ by Remark 8.30. Note that $\bar{F}$ is uniquely determined by the property that $\bar{F}$ maps a vertex γ_X corresponding to an object $X \in \mathcal{D}^{\mathrm{b}}(C_n)$ to the vertex $\gamma_{F(X)}$. In particular, $F \mapsto \bar{F}$ defines a group homomorphism $\mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n)) \to \mathrm{Aut}(\mathcal{A}(\mathcal{T}^n))$. If $n \geq 2$, we set $\Upsilon(F) := \Phi^{-1}(\bar{F}) \in \mathcal{MCG}^{\pm}(\mathcal{T}^n)$, where $\Phi = \Phi^{\mathcal{T}^n}$. If $n = 1$, $\Upsilon(F) \in \mathrm{PGL}_2(\mathbb{Z})$ is the element which corresponds to F under the isomorphism $\mathrm{PGL}_2(\mathbb{Z}) \cong \mathrm{Aut}(\mathcal{A}(\mathcal{T}^1))$ from Theorem 8.32.

We claim that $\bar{F}$ lies in the image of $\mathcal{MCG}(\mathcal{T}^n)$ under Φ. This is a consequence of the covariance of F. Namely, let Δ denote the triangulation of $\mathcal{T}^n$ as shown in Figure 20. Then at any puncture p there exist 6 segments of arcs of Δ ending p. Three of these segments correspond to a chain of non-zero morphisms $f \colon U \to V$, $g \colon V \to W$ and $h \colon W \to U[2]$ (see Example 8.25) such that $h \circ g \circ f \neq 0$ and where U, V and W are objects which are represented by pairwise distinct arcs of Δ. As pointed out above, we have $\bar{F}(\gamma_X) = \gamma_{F(X)}$ whenever γ_X is simple. Thus, if $\bar{F}$ was the image of an orientation-reversing homeomorphism, the intersection of the arcs $\gamma_{F(U)}, \gamma_{F(V)}, \gamma_{F(W)}$ and the puncture $F(p)$ would give rise to a non-zero composition $F(U)[2] \to F(W) \to F(V) \to F(U)$. However, there exists no chain of this kind as we can read off from the intersection pattern of the arcs in Figure 20. We conclude that $\bar{F}$ is the image of an orientation-preserving homeomorphism. Note that $\Phi(\mathcal{MCG}(\mathcal{T}^1))$ is the subgroup $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PGL}_2(\mathbb{Z})$.

It remains to prove surjectivity. We recall from sequence (6.3) that there is a surjective group homomorphism $\mathcal{MCG}(\mathbb{T}_n) \to \mathcal{MCG}(\mathcal{T}^n)$ which maps a homeomorphism to its radial extension. In fact, radial extension defines a group homomorphism $\bar{\pi}$ between the corresponding extended mapping class groups. Since $\mathcal{MCG}(\mathcal{T}^n)$ has index 2 inside $\mathcal{MCG}^{\pm}(\mathcal{T}^n)$, it follows that $\bar{\pi}$ and hence the composition $\alpha := \Phi^{\mathcal{T}^n} \circ \bar{\pi} \circ (\Phi^{\mathbb{T}_n})^{-1}$
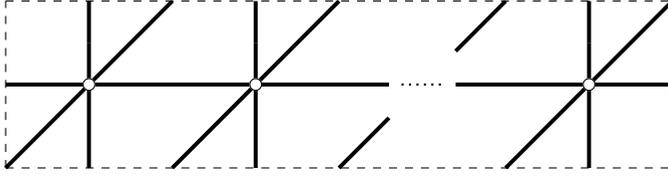
Fig. 20. A special triangulation of the n-punctured torus.

are surjective. The map α can be described as follows. Let $F \in \mathrm{Aut}(\mathcal{A}(\mathbb{T}_n))$ and let $c \colon \mathcal{A}(\mathbb{T}_n) \to \mathcal{A}(\mathcal{T}^n)$ denote the surjective simplicial map from Remark 8.31. Then

$$\alpha(F)(\widetilde{\gamma}) = c(F(\gamma))$$

for every simple arc $\widetilde{\gamma} \subseteq \mathcal{T}^n$ and every preimage $\gamma \subseteq \mathbb{T}_n$ under c. If $n > 1$, it follows that the diagram

$$
\begin{array}{ccc}
\mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(\Lambda_n)) & \longrightarrow & \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n)) \\
\Psi \downarrow & & \downarrow \Phi \circ \Upsilon \\
\mathrm{Aut}(\mathcal{A}(\mathbb{T}_n)) & \xrightarrow{\ \alpha\ } & \mathrm{Aut}(\mathcal{A}(\mathcal{T}^n))
\end{array}
$$

commutes. Thus, Υ is surjective due to Φ being invertible. Finally, if $n = 1$, one replaces $\Phi \circ \Upsilon$ by Υ in the previous diagram. ∎

Corollary 8.37. *Let $X \in \mathcal{D}^{\mathrm{b}}(C_n)$ be indecomposable such that γ_X is a simple arc and let $T \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$. Then*

$$\gamma_{T(X)} \simeq \Upsilon(T)(\gamma_X).$$

In order to describe the kernel of Υ, we will use the following result due to Sibilla.

Proposition 8.38 ([35, Lemma 3.3, Remark 3.4]). *Let $\{x_i\}_{i \in \mathbb{Z}_n}$ be a collection of smooth points such that $x_i \in \mathbb{P}_i^1$ for each $i \in \mathbb{Z}_n$. Suppose that $T \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$ satisfies*

- $T(\mathcal{O}_{C_n}) \cong \mathcal{O}_{C_n}$, *and*
- *for all $i \in \mathbb{Z}_n$, $T(\Bbbk(x_i)) \cong \Bbbk(x_i)$.*

Then there exists an automorphism $f \colon C_n \to C_n$ such that T and the induced equivalence $f^* \colon \mathcal{D}^{\mathrm{b}}(C_n) \to \mathcal{D}^{\mathrm{b}}(C_n)$ are naturally isomorphic.

Remark 8.39. Suppose that $f \colon C_n \to C_n$ is automorphism such that $f^* \in \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$ satisfies the assumptions in Proposition 8.38. Then, as pointed out in [35], $f = \mathrm{Id}$ whenever $n \geq 3$. If $n \leq 2$, then $f^2 = \mathrm{Id}$ and f is induced from a permutation of the preimages of the singular points in a normalization of C_n. Assuming that x_0 and x_1 correspond to $1 \in \mathbb{P}^1$ and the singular points to 0 and ∞, f acts on all irreducible components either as the identity or as the involution $x \mapsto x^{-1}$, where $0^{-1} := \infty$. In particular, the group of all such f is isomorphic to $\mathbb{Z}_2$. If $n = 1$, then $f \in \ker \Upsilon$. If $n = 2$ and $f \neq \mathrm{Id}$, then f^* permutes the skyscrapers of singular points and hence $f \notin \ker \Upsilon$.

Proposition 8.40. *There exists an isomorphism*

$$\ker \Upsilon \cong (\mathcal{N} \ltimes (\Bbbk^\times)^n) \times \mathbb{Z} \times \mathrm{Pic}^0(C_n),$$

where

$$\mathcal{N} \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

If $n = 1$, the structure map of the semi-direct product maps $\bar{1}$ to the automorphism $\lambda \mapsto \lambda^{-1}$.

Proof. First of all, $[1] \in \ker \Upsilon$. By changing coordinates, we may assume that $0, \infty \in \mathbb{P}^1$ are preimages of singular points in $\mathbb{P}_i^1$. Rescaling every irreducible component $\mathbb{P}_i^1$ of C_n by $\lambda_i \in \Bbbk^\times$ defines a faithful action of $(\Bbbk^\times)^n$ on $\mathcal{D}^b(C_n)$. Let Δ denote the triangulation from Figure 20. The shifts of the objects $\Bbbk(x_i)$ are characterized up to isomorphism as those objects $Z \in \mathrm{Perf}(C_n)$ with the following three properties:

- Z is spherical.

- There exists a vertical arc γ_1 and a diagonal arc γ_2 in Δ such that

$$\dim \mathrm{Hom}^\bullet(Z, X_{\gamma_1}) = 1 = \dim \mathrm{Hom}^\bullet(Z, X_{\gamma_2}).$$

- If $\gamma \in \Delta \setminus \{\gamma_1, \gamma_2\}$, then $\mathrm{Hom}^\bullet(Z, X_\gamma) = 0$.

Line bundles in $\mathrm{Pic}^0(C_n)$ such as the structure sheaf are characterized via all vertical arcs in a similar way. Let $f^* \in \mathrm{Aut}(\mathcal{D}^b(C_n))$ be an equivalence which is induced by scaling or the tensor product with a line bundle $\mathcal{L} \in \mathrm{Pic}^0(C_n)$. Then f^* leaves all singular skyscraper sheaves invariant and maps all smooth skyscrapers to such with support in the same component. By Corollaries 8.37 and 8.27, it follows that $\Upsilon(f^*)$ acts trivially on the homotopy classes of the arcs of Δ and hence $\Upsilon(f^*)$ is trivial. Next, suppose $\Upsilon(T)$ is trivial for some equivalence $T \in \mathrm{Aut}(\mathcal{D}^b(C_n))$. Then again, from Corollary 8.37 and the characterization of $\Bbbk(x_i)$ and the Picard group above, we conclude that $T(\Bbbk(x_i)) \cong \Bbbk(y_i)[m_i]$ for some $y_i \in \mathbb{P}_i^1$ and $m_i \in \mathbb{Z}$ for each $i \in \mathbb{Z}_n$ and $T(\mathcal{O}_{C_n})[d] \in \mathrm{Pic}^0(C_n)$ for some $d \in \mathbb{Z}$. As $\mathrm{Hom}^\bullet(\mathcal{O}_{C_n}, \Bbbk(x_i))$ is concentrated in degree 0, we have $m_i = d$ for $i \in \mathbb{Z}_n$. Hence, after composition with $[-d]$, rescaling and composition with $- \otimes^{\mathbb{L}} T(\mathcal{O}_{C_n})^\vee$, we obtain an auto-equivalence T' which satisfies the conditions of Lemma 8.38. Thus, $T' \simeq f^*$ for some automorphism of C_n and the assertion follows from Remark 8.39. ∎

Propositions 8.36 and 8.40 imply the following (Theorem D in the introduction).

Corollary 8.41. *Let $n \geq 1$. There exists a short exact sequence*

$$\mathbf{1} \to (\mathcal{N} \ltimes (\Bbbk^\times)^n) \times \mathbb{Z} \times \mathrm{Pic}^0(C_n) \to \mathrm{Aut}(\mathcal{D}^b(C_n)) \xrightarrow{\Upsilon} G \to \mathbf{1},$$

where $G = \mathcal{MCG}(\mathcal{T}^n)$ and $\mathcal{N}$ is trivial, if $n > 1$, and $\mathcal{N} = \mathbb{Z}_2$ and $G = \mathrm{PSL}_2(\mathbb{Z})$, if $n = 1$.

Remark 8.42. Our results for the case $n = 1$ are in line with [14, Corollary 5.8] which states that there exists a short exact sequence

$$1 \to \mathcal{K} \to \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_1)) \xrightarrow{\varphi} \mathrm{SL}_2(\mathbb{Z}) \to 1,$$

where $\mathcal{K}$ is generated by [2], $\mathrm{Aut}(C_1)$ and tensor products with line bundles of degree 0. The map φ associates to an auto-equivalence T its induced map on the Grothendieck group $\mathcal{K}_0(\mathcal{D}^{\mathrm{b}}(C_1)) \cong \mathbb{Z}^2$. The isomorphism is given by the map $X \mapsto (\deg X, \mathrm{rk}\, X)$. Note that $\mathrm{Aut}(C_1) \cong \mathbb{Z}_2 \ltimes \Bbbk^{\times}$, where the first factor corresponds to the involution $(x \mapsto x^{-1})$ and the second factor to the scaling automorphisms. Let $p\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ denote the projection. Using the explicit isomorphism $\mathrm{Aut}(\mathcal{A}(\mathcal{T}^1)) \cong \mathrm{PGL}_2(\mathbb{Z})$ described in the proof of [22, Theorem 2.1], one sees that $p \circ \varphi = \Upsilon$.

Remark 8.43. We expect Corollary 8.41 to generalize to Verdier quotients $\mathcal{D}$ of other gentle algebras Λ as explained in Remark 8.28. In general, $\mathcal{MCG}(\mathcal{T}^n)$ should be replaced by the stabilizer of a line field in the mapping class group of Σ_C (in the notation of Remark 8.28) and the existence of a map Υ from $\mathrm{Aut}(\mathcal{D})$ into the stabilizer seems to follow in a similar way except when Σ_C is one of finitely many surfaces, where the automorphism group of the arc complex and the mapping class group of Σ_C do not agree. We expect the kernel of Υ to be isomorphic to $\ker \Psi$.

8.4. Faithfulness of group actions

For each collection of smooth points $(x_i)_{i \in \mathbb{Z}_n}$ in C_n such that $x_i \in \mathbb{P}^1_i$, Sibilla [35] defined a group action of a central extension $\mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n)$ of $\mathcal{PMCG}(\mathcal{T}^n)$ on $\mathcal{D}^{\mathrm{b}}(C_n)$. The group $\mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n)$ fits into a short exact sequence

$$1 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n) \to \mathcal{PMCG}(\mathcal{T}^n) \to 1,$$

and is generated by the Dehn twists $\{D_{\gamma_{\mathrm{Pic}}}\} \cup \{D_{\gamma^i_{\Bbbk(x)}} \mid i \in \mathbb{Z}_n\}$ and a central element $t := \iota(1)$. The projection in the short exact sequence above maps the Dehn twists to themselves and t to the identity. The structure homomorphism $\mathfrak{X}\colon \mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n) \to \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n))$ of the group action maps $D_{\gamma_{\mathrm{Pic}}}$ to $T_{\mathcal{O}_{C_n}}$, $D_{\gamma^i_{\Bbbk(x)}}$ to $T_{\Bbbk(x_i)}$ and t to the shift functor.

Sibilla conjectured that the action is faithful. We confirm this in the following theorem.

Theorem 8.44. *The homomorphism $\mathfrak{X}$ is injective. Thus, the group action is faithful.*

Proof. Consider the diagram of short exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n) & \longrightarrow & \mathcal{PMCG}(\mathcal{T}^n) & \longrightarrow & 1 \\
 & & \big\uparrow & & \Big\downarrow{\scriptstyle \mathfrak{X}} & & \big\uparrow & & \\
1 & \longrightarrow & \ker \Upsilon & \longrightarrow & \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n)) & \xrightarrow{\ \Upsilon\ } & \mathcal{MCG}(\mathcal{T}^n) & \longrightarrow & 1,
\end{array}
$$

where the left vertical map is the isomorphism onto the $\mathbb{Z}$-component of $\ker \Upsilon$. The diagram commutes by Proposition 6.11. As the outer vertical maps are injective, so is $\mathcal{X}$. ∎

Remark 8.45. With a little bit more effort and use of gradings on arcs, it is not difficult to see that Υ lifts to a group homomorphism $\widetilde{\Upsilon} \colon \mathrm{Aut}(\mathcal{D}^{\mathrm{b}}(C_n)) \to \mathcal{MCG}_{\mathrm{gr}}(\mathcal{T}^n)$, where $\mathcal{MCG}_{\mathrm{gr}}(\mathcal{T}^n)$ is a central extension of $\mathcal{MCG}(\mathcal{T}^n)$ and one finds that $\widetilde{\Upsilon} \circ \mathcal{X}$ is the inclusion $\mathcal{PMCG}_{\mathrm{gr}}(\mathcal{T}^n) \hookrightarrow \mathcal{MCG}_{\mathrm{gr}}(\mathcal{T}^n)$.

9. Curves of vector bundles and simple vector bundles

In this section, we identify those simple loops on the n-punctured torus which correspond to simple vector bundles on C_n under the bijection in Theorem A. We further characterize the loops which represent vector bundles and provide an easy way to determine the multi-degree and rank of a vector bundle from its representing loop.

9.1. Simple vector bundles on cycles of projective lines

Simple vector bundles on cycles of projective lines were studied by Burban–Drozd–Greuel [13] and Bodnarchuk–Drozd–Greuel [7]. The latter showed that the rank, the multi-degree and the determinant form a complete set of invariants for the isomorphism class of a simple vector bundle. We recall that the *multi-degree* of a vector bundle $\mathcal{E}$ over C_n is the function $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$ whose value $\mathbb{d}(i)$ is the degree of the restricted pullback bundle $(\pi^* \mathcal{E})|_{\pi^{-1}(\mathbb{P}^1_i)}$, where π denotes a normalization map. For any $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$, we refer to the sum $\underline{\mathbb{d}} = \sum_{x \in \mathbb{Z}_n} \mathbb{d}(x)$ as the *total degree* of $\mathbb{d}$.

Theorem 9.1 ([7]). *Let $r \geq 1$ be a natural number, $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$ and $\mathcal{L} \in \mathrm{Pic}^0(C_n) \cong \Bbbk^\times$. Then there exists a simple vector bundle $\mathcal{E}$ on C_n of rank r, multi-degree $\mathbb{d}$ and determinant $\mathcal{L}$ if and only if r and $\underline{\mathbb{d}}$ are coprime. Moreover, in this case, the isomorphism class of $\mathcal{E}$ is uniquely determined by the triple $(r, \mathbb{d}, \mathcal{L})$.*

We provide an alternative proof of Theorem 9.1 in Section 9.4 which is based on the topological model of $\mathcal{D}^{\mathrm{b}}(C_n)$. It is natural to ask how a simple vector bundle with a given set of invariants $(r, \mathbb{d}, \mathcal{L})$ can be constructed, and a short report on the matter is given in the next section.

9.1.1. Vector bundles as glued line bundles and cyclic sequences. Throughout this section, we fix a normalization map $\pi \colon \widetilde{C}_n \to C_n$ and denote by $\widetilde{U}_i$ the preimage of $\mathbb{P}^1_i$ under π.

Let $\mathcal{E}$ be a vector bundle over C_n of rank r and multi-degree $\mathbb{d}$. Then $\pi^*(\mathcal{E})$ decomposes into a direct sum of $n \cdot r$ line bundles over the components $\widetilde{U}_i \cong \mathbb{P}^1$. In other words, there exists a unique multi-set of integers $D_{\mathcal{E}} = \{m_i^j \mid i \in \mathbb{Z}_n, j \in [0, r)\}$ such that

$$\pi^*(\mathcal{E}) \cong \bigoplus_{i \in \mathbb{Z}_n} \mathcal{E}_i, \quad \mathcal{E}_i = \bigoplus_{j=0}^{r-1} \mathcal{O}_{\widetilde{U}_i}(m_i^j).$$

The vector bundle $\mathcal{E}$ then can be thought of as being glued from the vector bundles $\mathcal{E}_i$ by identifying their stalks along the preimages of singular points. The map which identifies the stalks of $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ over the corresponding singular point in $\mathbb{P}_i^1 \cap \mathbb{P}_{i+1}^1$ encodes a matrix M_i over $\Bbbk$. The resulting sequence $(M_i)_{i \in \mathbb{Z}_n}$ describes the vector bundle $\mathcal{E}$.

Theorem 9.2 ([13]). *Let $\mathcal{E}$ be an indecomposable vector bundle. There exist $m \geq 1$ such that $m|r$ and splittings of $\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}$ for which the matrices M_i are of the following shape:*

(1) *All M_i are block matrices with blocks of size $m \times m$ and in each row and column of the block division of M_i there exists a single non-zero block.*

(2) *All non-zero blocks contain the identity matrix except for a single block in M_0 which contains a Jordan block $J_m(\lambda)$ for some $\lambda \in \Bbbk^\times$.*

The scalar λ describes the determinant of $\mathcal{E}$ whereas m specifies the position of $\mathcal{E}$ in the corresponding homogeneous tube in the Auslander–Reiten quiver. In particular, $m = 1$ if $\mathcal{E}$ is simple. Specializing to $m = 1$, we see that the non-zero entries in the matrices M_i determine a cyclic order on $D_{\mathcal{E}}$ by declaring $m_{i+1}^{j'}$ as the successor of m_i^j if $(M_i)_{jj'} \neq 0$. Thus, every vector bundle of rank r which sits at the mouth of a homogeneous tube defines a cyclic sequence $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$.[4]

Burban, Drozd and Greuel found necessary and sufficient conditions for the simplicity of $\mathcal{E}$ in terms of the sequence m. They read as follows.

Theorem 9.3 ([13, Theorem 5.3]). *Let $\mathcal{E}$ be an indecomposable vector bundle over C_n of rank r which sits at the mouth of a homogeneous tube. Let m denote the corresponding cyclic sequence. Then $\mathcal{E}$ is simple if and only if all of the following conditions are satisfied:*

(1) *The rank and the total degree of $\mathcal{E}$ are coprime.*

(2) *The difference between any two entries in m is at most 1.*

(3) *Regarding m as an infinite sequence $(m_i)_{i \in \mathbb{Z}}$ with period $n \cdot r$, for all $t \in \mathbb{Z}$ the sequence*

$$(m_i - m_{i+t})_{i \in \mathbb{Z}},$$

contains no subsequence of the form $1, 0, \ldots, 0, 1$ or $-1, 0, \ldots, 0, -1$.

The conditions in Theorem 9.3 can be used to derive the multi-set $D_{\mathcal{E}}$ and hence the entries of m, see Remark 9.4. However, apart from the case $n = 1$ [9] the cyclic order on $D_{\mathcal{E}}$ and therefore the sequence m seem to be unknown in general. We provide a closed formula for m in Section 9.1.2.

Remark 9.4. Condition (2) in Theorem 9.3 can be used to derive the multi-set $D_{\mathcal{E}}$. Suppose m satisfies the conditions of Theorem 9.3. Let $i \in [0, n)$ and let q denote the minimum in the sequence $\mathrm{m}_i, \mathrm{m}_{i+n}, \mathrm{m}_{i+2n}, \ldots, \mathrm{m}_{i+(r-1) \cdot n}$. If a denotes the number

---

[4]In the same way, every vector bundle defines a cyclic sequence.

of times the number q occurs in the preceding sequence, then the condition (2) implies that

$$a \cdot q + (r - a) \cdot (q + 1) = \sum_{j=0}^{r-1} \mathrm{m}_{i+j \cdot n} = \mathbb{d}(i),$$

from which we deduce that $(r - a) \in [0, r)$ is the residue of $\mathbb{d}(i)$ modulo r and q is the unique integer such that $r \cdot q + (r - a) = \mathbb{d}(i)$.

9.1.2. *The cyclic sequence of a simple vector bundle.*
From now on, we write m_x instead of $\mathrm{m}(x)$ for all $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ and all $x \in \mathbb{Z}_{nr}$. We say that two sequences $\mathrm{m}, \mathrm{m}' \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ are *equivalent* and write $\mathrm{m} \sim \mathrm{m}'$ if they agree up to rotation, i.e., if there exists $t \in n\mathbb{Z}$ such that $\mathrm{m}_x = \mathrm{m}'_{x+t}$ for all $x \in \mathbb{Z}_{nr}$.

Definition 9.5. Let $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$ and let $r \geq 1$ be a natural number such that $\underline{\mathbb{d}}$ and r are coprime. For $i \in \mathbb{Z}$, denote by $S_i = S_i(r, \mathbb{d})$ the sequence of integers defined by $S_0 = 0$ and the recursive formula $S_{i+1} = S_i + \mathbb{d}(i)$. For $i \in \mathbb{N}$ with residue class $x = \bar{i} \in \mathbb{Z}_{nr}$, let

$$\mathrm{m}(r, \mathbb{d})_x := \begin{cases} |r\mathbb{Z} \cap (S_i, S_{i+1}]| & \text{if } S_{i+1} \geq S_i, \\ -|r\mathbb{Z} \cap (S_{i+1}, S_i]| & \text{otherwise.} \end{cases}$$

Define $\mathrm{m}(r, \mathbb{d}) \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ as the cyclic sequence with entries $\mathrm{m}(r, \mathbb{d})_x$ as defined above.

We note that since $\underline{\mathbb{d}}$ and r are coprime, $\mathrm{m}(r, \mathbb{d})$ is well defined. Also note that for each $i \in \mathbb{Z}$,

$$\sum_{j=0}^{r-1} \mathrm{m}(r, \mathbb{d})_{i+j \cdot n} = \mathbb{d}(i).$$

Occasionally, we omit r and $\mathbb{d}$ from the notation and just write m instead if the two are apparent from the context.

It is convenient to think of the cyclic sequence $\mathrm{m}(r, \mathbb{d})$ as a matrix with n columns and r cyclically ordered rows such that we recover the cyclic order by reading the entries from left to right and top to bottom.

Example 9.6. Let $n = 2 = r$, $\mathbb{d}(0) = 2$ and $\mathbb{d}(1) = -1$. Then $\underline{\mathbb{d}} = 1$, $(S_0, \ldots, S_4) = (0, 2, 1, 3, 2)$ and $\mathrm{m} = \mathrm{m}(2, \mathbb{d})$ is given by $(\mathrm{m}_0, \ldots, \mathrm{m}_3) = (1, -1, 1, 0)$. The 2×2 matrix of m is

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The following is an immediate consequence of Definition 9.5.

Lemma 9.7. *Let $r \geq 1$ and $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$ such that $\underline{\mathbb{d}}$ and r are coprime. Let $l \in \mathbb{Z}$, $i \in \mathbb{Z}_n$ and define $\mathbb{e} \in \mathbb{Z}^{\mathbb{Z}_n}$ by*

$$\mathbb{e}(x) = \begin{cases} \mathbb{d}(x) & \text{if } x \neq i, \\ \mathbb{d}(x) + l \cdot r & \text{if } x = i. \end{cases}$$

Then the matrix of $\mathrm{m}(r, \mathbb{e})$ *is obtained from the matrix of* $\mathrm{m}(r, \mathbb{d})$ *by adding the vector* $(l \cdot r, \ldots, l \cdot r)$ *to its* i-*th column.*

Proof. Omitted.   ∎

By the previous lemma, we may always assume that $\mathbb{d}(x) \in [0, r)$ for all $x \in \mathbb{Z}_n$ and hence $\mathrm{m}(r, \mathbb{d})_y \in \{0, 1\}$ for all $y \in \mathbb{Z}_{nr}$.

In the following lemma, we prove that the cyclic sequence $\mathrm{m}(r, \mathbb{d})$ satisfies the conditions of Theorem 9.3.

Proposition 9.8. *Let* $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_n}$ *and* $r \geq 1$ *such that* $\underline{\mathbb{d}}$ *and* r *are coprime. Then* $\mathrm{m} = \mathrm{m}(r, \mathbb{d})$ *satisfies the conditions of Theorem* 9.3.

Proof. By Lemma 9.7, we may assume that $\mathbb{d}(x) \in [0, r)$ for all $x \in \mathbb{Z}_n$. If $i, j \in [0, nr]$ are such that $j - i \in n\mathbb{Z}$, then in the notation of Definition 9.5,

$$S_{i+1} - S_i = \mathbb{d}(i) = \mathbb{d}(j) = S_{j+1} - S_j.$$

Thus, $(S_i, S_{i+1}]$ and $(S_j, S_{j+1}]$ have the same length and the cardinality of $r\mathbb{Z} \cap (S_i, S_{i+1}]$ and $r\mathbb{Z} \cap (S_j, S_{j+1}]$ differ by at most one. This proves condition (2).

Let $i, l \in \mathbb{Z}$ and let $p \geq 1$. The cardinality of $r\mathbb{Z} \cap (S_i, S_{i+p}]$ is $\sum_{j=0}^{p-1} \mathrm{m}_{i+j}$. As before, we regard m as an infinite sequence $(m_i)_{i \in \mathbb{Z}}$. Now, suppose the infinite sequence $(m_i - m_{i+l})_{i \in \mathbb{Z}}$ contains a subsequence of the form $\pm 1, 0, \ldots, 0, \pm 1$. It means that for some $i \in \mathbb{Z}$ and some $p \geq 1$, $r\mathbb{Z} \cap (S_i, S_{i+p}]$ contains two more (resp. two less) elements than $r\mathbb{Z} \cap (S_{i+ln}, S_{i+ln+p}]$. But the intervals $(S_i, S_{i+p}]$ and $(S_{i+ln}, S_{i+ln+p}]$ have the same cardinality which yields a contradiction. This completes the proof of condition (3).   ∎

Corollary 9.9 (Theorem E). *Let* $\mathcal{E}$ *be a simple vector bundle on* C_n *of multi-degree* $\mathbb{d}$ *and rank* r. *If* $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ *satisfies the conditions in Theorem* 9.3, *then* $\mathrm{m} \sim \mathrm{m}(r, \mathbb{d})$.

Proof. This follows from Theorems 9.1 and 9.3 and Proposition 9.8.   ∎

9.2. Loops and their intersections via ribbon graphs

Our goal is to translate certain homotopy classes of loops into a cyclic integer sequence so that for every loop which represents a simple vector bundle of rank r and multi-degree $\mathbb{d}$ this sequence is precisely the sequence $\mathrm{m}(r, \mathbb{d})$ from Corollary 9.9. This is achieved by interpreting loops on $\mathscr{T}^n$ as walks in a *ribbon graph* which we define in terms of an embedded quiver $\Gamma \subseteq \mathscr{T}^n$.

Consider the collection of oriented arcs on $\mathscr{T}^n$ in Figure 21. We label the horizontal arcs by κ_i ($i \in \mathbb{Z}_n$) and denote the vertical arc which intersects κ_i and κ_{i+1} by ε_i. The collection of these arcs cut $\mathscr{T}^n$ into discs. We denote by Γ the dual quiver of this dissection as shown in Figure 22: the set of vertices in Γ is in bijection with the discs, $\Gamma_1 = \{\varepsilon_i, \kappa_i \mid i \in \mathbb{Z}_n\}$ and an arrow $\alpha \in \Gamma_1$, which corresponds to an arc γ, starts (resp. ends) at the vertex which corresponds to the disc which lies on the left-hand (resp. right-hand) side of γ.
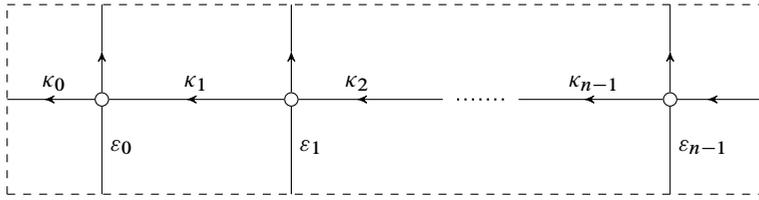
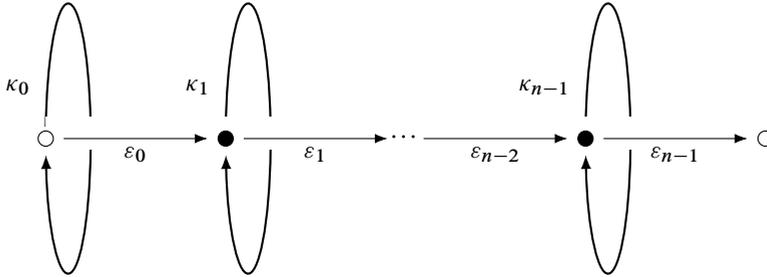**Fig. 21.** The collection of oriented arcs on $\mathscr{T}^n$.

Fig. 22. The quiver Γ. The two copies of the vertex $\circ$ are identified.

By construction, Γ can be embedded into $\mathscr{T}^n$ by mapping a vertex to a point inside of its corresponding disc and the arrow associated with an arc γ to a simple path p_γ which crosses γ exactly once and no other arc of $\{\varepsilon_i, \kappa_i\}$. The orientation of $\mathscr{T}^n$ defines cyclic orders on the set of half-edges emanating from any vertex of Γ and hence turns Γ into a *ribbon graph*. By a *half-edge* we mean any of the two segments of p_γ between its crossing with γ and one of its end points.

Let $\Gamma_1^{-1} := \{\gamma^{-1} \mid \gamma \in \Gamma_1\}$ denote the set of formal inverses of Γ_1. As usual, $s(\gamma^{-1}) := t(\gamma)$ and $t(\gamma^{-1}) := s(\gamma)$. One defines a bijection between the half-edges and elements in $\Gamma_1 \sqcup \Gamma_1^{-1}$ by identifying a segment δ of p_γ, i.e., a half-edge, with $\gamma \in \Gamma_1$, if δ crosses γ from the left, and with γ^{-1} otherwise. Regarding the underlying graph of Γ as a topological space in the usual way, the embedding $\Gamma \hookrightarrow \mathscr{T}^n$ is a deformation retract and induces an isomorphism $\pi_1(\Gamma) \cong \pi_1(\mathscr{T}^n)$ of fundamental groups.

This has a few useful and well-known consequences. First of all, the isomorphism of fundamental groups allows us to regard loops on $\mathscr{T}^n$ as certain words in the alphabet $\Gamma_1 \sqcup \Gamma_1^{-1}$ which we call *admissible walks*. Second, every intersection between loops can be expressed as a common subword of their admissible walks.

Definition 9.10. An *admissible walk* of length l in Γ is a function $W: \mathbb{Z}_l \to \Gamma_1 \sqcup \Gamma_1^{-1}$ such that for all $i \in \mathbb{Z}_l$,

$$W(i) \neq W(i+1)^{-1} \quad \text{and} \quad t(W(i)) = s(W(i+1)),$$

and W is primitive, i.e., it is not of the form $\mathbb{Z}_l \xrightarrow{\varphi} \mathbb{Z}_m \xrightarrow{W'} \Gamma_1 \sqcup \Gamma_1^{-1}$, where l is a proper multiple of m and φ is the canonical surjection.

By definition, we consider two walks W, W' of length l as equivalent if W and W' agree up to rotation and inversion, that is, there exist $\sigma \in \{\pm 1\}$ and $j \in \mathbb{Z}$ such that $W(i) = W'(\sigma \cdot i + j)$ for all $i \in \mathbb{Z}_l$. Usually, we describe a walk W through a sequence $W(i)W(i+1)\cdots W(i-1)$, where $i \in \mathbb{Z}_l$.

The relationship between loops and walks can now be phrased as follows.

Proposition 9.11. *There exists a bijection between homotopy classes of unoriented loops on $\mathscr{T}^n$ and equivalence classes of admissible walks in Γ.*

Given an admissible walk W, we obtain its associated loop simply by regarding W as a closed path in the quiver Γ: if $W = \alpha_1 \dots \alpha_l$ with $\alpha_i \in \Gamma_1 \sqcup \Gamma_1^{-1}$, we start at the vertex $s(\alpha_1)$ and then walk along the arrows $\alpha_1, \dots, \alpha_l$ of Γ by which we mean that if $\alpha_i \in \Gamma_1^{-1}$, then we walk $\alpha_i^{-1} \in \Gamma_1$ backwards from its target to its source. Via the embedding $\Gamma \hookrightarrow \mathscr{T}^n$, we obtain a loop in $\mathscr{T}^n$.

If γ is a loop, we denote by W_γ the equivalence class of walks associated with γ or, by abuse of notation, a representative thereof.

Intersections between loops can be described by maximal common subsequences of their walks. To be precise, suppose $\gamma, \gamma' \subset \mathscr{T}^n$ are loops in minimal position and regard $W = W_\gamma$ and $W' = W_{\gamma'}$ as infinite cyclic sequences with period given by their lengths. Then every intersection $p \in \gamma \overrightarrow{\cap} \gamma'$ corresponds uniquely to one of the following two situations:

(a) After interchanging the roles of W and W' if necessary, there exist decompositions $W = w_1 u c_1 \dots c_l v w_2$ and $W' = w_1' u' c_1 \dots c_l v' w_2'$, where $l \geq 1$ and $c_1, \dots, c_l, u, u'$, $v, v' \in \Gamma_1 \sqcup \Gamma_1^{-1}$, such that $t(u) = t(u')$ but $u \neq u'$ and $s(v) = s(v')$ but $v \neq v'$, and in the cyclic order of half-edges around $t(u) = t(u')$, c_1 lies in between u' and u, and in the counter-clockwise cyclic order of half-edges around $s(v) = s(v')$, c_l lies between v' and v.
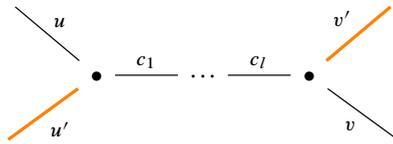
(b) After interchanging the roles of W and W' if necessary, there exist decompositions $W = w_1 u v w_2$ and $W' = w_1 u' v' w_2'$, where $u, u', v, v' \in \Gamma_1 \sqcup \Gamma_1^{-1}$ are pairwise distinct, such that $x := t(u) = t(u') = s(v) = s(v')$ and in the counter-clockwise cyclic order of half-edges around x the order is u, u', v, v'.
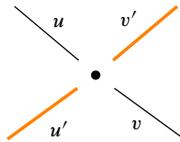
Example 9.12. The loop $\gamma_{\Bbbk(x)}^i$ corresponds to the walk κ_i, and γ_{Pic} corresponds to the walk $\varepsilon_0\varepsilon_1\ldots\varepsilon_{n-1}$. The unique intersection of $\gamma_{\Bbbk(x)}^i$ and γ_{Pic} is an example for the case (b) and corresponds to the decompositions $\ldots\kappa_i\kappa_i\ldots$ and $\ldots\varepsilon_i\varepsilon_{i+1}\ldots$, i.e., $(u, u', v, v') = (\kappa_i, \varepsilon_i, \kappa_i^{-1}, \varepsilon_{i-1}^{-1})$.

9.3. The class of loops which represent vector bundles

In this section, we describe the set of loops in $\mathscr{T}^n$ which represent vector bundles over C_n. The following definitions will play a central role in this section.

Definition 9.13. Let Ω_{Vect}^n denote the set of all homotopy classes of primitive loops on $\mathscr{T}^n$ which are represented by a walk in Γ without any letters of the form ε_i^{-1} ($i \in \mathbb{Z}_n$).

If γ is a loop, we write $\gamma \in \Omega_{\mathrm{Vect}}^n$ to indicate that the homotopy class of γ lies in Ω_{Vect}^n.

Definition 9.14. Let Vect^n denote the additive closure of all indecomposable objects $X \in \mathrm{Perf}(C_n)$ such that $\gamma_X \in \Omega_{\mathrm{Vect}}^n$.

Elements of Ω_{Vect}^n are in bijection with equivalence classes of primitive sequences $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$, where $r \geq 1$. Indeed, for any sequence $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ the corresponding homotopy class $\gamma(\mathrm{m})$ belongs to the walk

$$\cdots \varepsilon_0 \kappa_0^{\mathrm{m}_0} \varepsilon_1 \kappa_1^{\mathrm{m}_1} \cdots \varepsilon_{nr-1} \kappa_{nr-1}^{\mathrm{m}_{nr-1}} \varepsilon_0 \cdots.$$

Note that $\gamma(\mathrm{m})$ is primitive if and only if m is primitive. For a possibly non-primitive loop γ which is a power of a loop in Ω_{Vect}^n, we shall denote by m_γ its associated integer sequence.

Remark 9.15. The set Ω_{Vect}^n admits a simple topological characterization: an immersed loop γ is contained in Ω_{Vect}^n if and only if γ can be oriented in such a way that its derivative has everywhere positive latitudinal coordinate, cf. Theorem B. In other words, the curve γ always travels towards the right in Figure 21.

Definition 9.16. Let $r \geq 1$ and $\mathbb{d} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ such that $\underline{\mathbb{d}}$ and r are coprime. Define $\gamma(r, \mathbb{d}) \in \Omega_{\mathrm{Vect}}^n$ as the homotopy class of $\mathrm{m}(r, \mathbb{d})$.

As the reader probably guesses at this point, $\gamma(r, \mathbb{d})$ contains simple loops and represents the simple vector bundles of rank r and multi-degree $\mathbb{d}$. A proof is given in this section.

Remark 9.17. With $\mathbb{e}$ and $\mathbb{d}$ as in Lemma 9.7, the l-th power of the Dehn twist about $\gamma_{\Bbbk(x)}^i$ transforms the homotopy class $\gamma(r, \mathbb{d})$ into the homotopy class of $\gamma(r, \mathbb{e})$. An analogous statement is true for arbitrary loops $\gamma(\mathrm{m})$.

For γ a power of a loop in Ω_{Vect}^n, we denote by $\mathrm{m}_\gamma \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ the sequence associated with γ and define the *rank* $\mathrm{rk}\,\gamma := \mathrm{rk}\,\mathrm{m}_\gamma$ and the *multi-degree* $\mathbb{d}(\gamma) := \mathbb{d}(\mathrm{m}_\gamma)$. As usual, the *total degree* $\underline{\mathbb{d}}(\gamma)$ of γ is the sum of all entries of $\mathbb{d}(\gamma)$.

The three invariants $\mathrm{rk}\,\gamma$, $\mathbb{d}(\gamma)$ and $\underline{\mathbb{d}}(\gamma)$ have a straightforward interpretation on the surface.

Lemma 9.18. *Let $\gamma \subseteq \mathcal{T}^n$ be a loop. Then $\mathbb{d}(\gamma)(i)$ equals the number of intersections (counted with signs) of γ with the arc κ_i in Figure* 21.

Proof. Omitted.   ■

In a similar way, rank and total degree of a loop γ have a topological interpretation too. One embeds γ into the closed torus $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. Then composing γ with the projection onto the first component of the product determines a self-map of S^1 and hence an integer $r_\gamma \in \mathbb{Z} \cong \pi_1(S^1)$. The number r_γ measures how many times γ wraps around the latitudinal axis of the torus. The projection onto the second component determines an integer d_γ which counts the number of full positive turns of γ around the longitudinal axis.

Comparing the previous construction with the definitions of rank and total degree, we observe the following.

Lemma 9.19. *Let $\gamma \subseteq \mathcal{T}^n$ be a loop. Then r_γ agrees with $\mathrm{rk}\,\gamma$ and d_γ equals $\underline{\mathbb{d}}(\gamma)$.*

Proof. Omitted.   ■

The importance of Ω^n_{Vect}, Vect^n and $\gamma(r, \mathbb{d})$ becomes clear in the main result of this section.

Theorem 9.20. *The following is true:*

(1) *A curve $\gamma \subseteq \mathcal{T}^n$ represents a vector bundle over C_n if and only if $\gamma \in \Omega^n_{\mathrm{Vect}}$. Thus, Vect^n coincides with the additive closure of all vector bundles and their shifts.*

(2) *Let $\mathcal{E}$ be a vector bundle of multi-degree $\mathbb{d}$ over C_n which sits at the k-th level of its homogeneous tube in the Auslander–Reiten quiver. Then $\mathbb{F}(\mathcal{E}) \cong P^\bullet(\mathrm{m})(\mathcal{V})$, where*

$$\dim \mathcal{V} = k \quad and \quad (\mathbb{d}(\mathrm{m}), \mathrm{rk}\,\mathrm{m}) = \left(\mathbb{d}, \frac{\mathrm{rk}\,\mathcal{E}}{k} \right).$$

(3) *Any loop in $\gamma(r, \mathbb{d})$ represents the simple vector bundles of rank r and multi-degree $\mathbb{d}$.*

The proof of Theorem 9.20 can be found at the end of this section and can be summarized as follows.

Outline of the proof: We show that any object in $\mathbb{F}(\mathrm{Vect}^n) \subseteq \mathcal{D}^n(\Lambda_n)$ is a shift of an iterated extension (of degree 1) of objects which correspond to line bundles over C_n. As the category of vector bundles is closed under extensions, this shows the "if"-part in Theorem 9.20 (1). Theorem 9.20 (2) follows along the way due to the additivity of rank and multi-degree. The "only-if"-direction in Theorem 9.20 (1) is implied by a result of Ballard which characterizes shifts of vector bundles inside the derived category by means of their morphism spaces to skyscraper sheaves. At the heart of all our arguments lies the description of certain morphisms in $\mathbb{F}(\mathrm{Vect}^n)$ via intersections which we rephrase in terms of the associated integer sequences. This also enables us to derive simplicity of $\gamma(r, \mathbb{d})$.

The first lemma describes intersections of loops in Ω^n_{Vect} in a convenient combinatorial way which resembles conditions (2) and (3) in Theorem 9.3. As before, we regard a sequence $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ as an infinite sequence with period $n \cdot r$. For such an infinite sequence $\mathrm{m} = (m_i)_{i \in \mathbb{Z}}$ and $l \in \mathbb{Z}$, we write $\mathrm{m}[l]$ for the infinite sequence $(m_{i+l})_{i \in \mathbb{Z}}$.

Lemma 9.21. *Suppose that $\gamma, \gamma' \in \Omega^n_{\text{Vect}}$ are in minimal position with ranks $r = \mathrm{rk}\,\gamma$ and $r' = \mathrm{rk}\,\gamma'$. Let $\mathrm{lcm}(r, r')$ denote the least common multiple of r and r'. Then the set of intersections of γ and γ' is in bijection with the union of the following two sets:*

(1) *The set of subsequences of $\mathrm{m}_\gamma - \mathrm{m}_{\gamma'}[l]$ up to shift by a multiple of $\mathrm{lcm}(r, r')$, of the form*

$$a, 0, \ldots, 0, b,$$

where $a, b > 0$ or $a, b < 0$ and $l \in \mathbb{Z}$.

(2) *The set of triples (x, y, q), where $x \in \mathbb{Z}_{nr}$, $y \in \mathbb{Z}_{nr'}$ and*

$$0 \le q \le |(\mathrm{m}_\gamma)_x - (\mathrm{m}_{\gamma'})_y| - 2.$$

Proof. This is just a paraphrase of the bijection between intersections and maximal common subsequences as described after Proposition 9.11.

Suppose $\mathrm{m}_\gamma - \mathrm{m}_{\gamma'}[l]$ contains a subsequence $a, 0, \ldots, 0, b$ with $a, b > 0$ or $a, b < 0$. Then $W = W_\gamma$ and $W' = W_{\gamma'}$ contain sequences $\kappa_x c \kappa_y$ and $\varepsilon_x c \varepsilon_y$, where

$$c = \kappa_x^u \varepsilon_{x+1} \cdots \varepsilon_{y-1} \kappa_y^v$$

and $u, v \ge 0$. One verifies easily that the common subsequence c corresponds to an intersection.

Next, suppose $|(\mathrm{m}_\gamma)_x - (\mathrm{m}_{\gamma'})_y| \ge 2$. Set $m := (\mathrm{m}_\gamma)_x$ and $m' := (\mathrm{m}_{\gamma'})_y$. Without loss of generality $m > m'$. The corresponding subsequence $\varepsilon_x \kappa_x{}^m \varepsilon_{x+1}$ of W contains $m' - m - 1$ subsequences of the form $\kappa_x c \kappa_x$, where $c = \kappa_x^{m'}$ if $m' \ne 0$ and c is empty otherwise. Since W' contains a subsequence $\varepsilon_x c \varepsilon_{x+1}$, these subsequences give rise to $m' - m - 1$ different intersections.

Finally, we observe that the previous cases cover all possible decompositions of W and W' which correspond to an intersection of γ and γ'. ∎

Corollary 9.22. *Let $r \ge 1$ and $\underline{\mathrm{d}} \in \mathbb{Z}^{\mathbb{Z}_n}$ such that r and $\underline{\mathrm{d}}$ are coprime. Then $\gamma(r, \underline{\mathrm{d}})$ contains a simple loop.*

Proof. This follows from Proposition 9.8 and Lemma 9.21. ∎

Example 9.23. As in Example 9.6, let $n = 2 = r$, $\mathrm{d}(0) = 2$ and $\mathrm{d}(1) = -1$. Then $\mathrm{m} = \mathrm{m}(2, \underline{\mathrm{d}})$ is given by $(\mathrm{m}_0, \ldots, \mathrm{m}_3) = (1, -1, 1, 0)$, and a representative γ of $\gamma(2, \underline{\mathrm{d}})$ is depicted in Figure 23. The intersections of γ with the vertical lines in Figure 23 divide γ into $n \cdot r = 4$ segments. The entries of the sequence m encode the signed intersection number of each segment with the arcs κ_i.
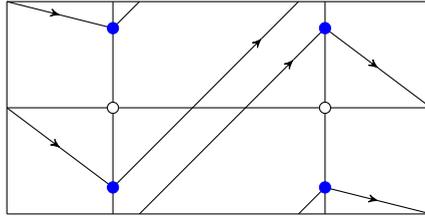
Fig. 23. The loop $\gamma = \gamma(2, \mathbb{d})$. The points $\bullet$ divide γ into 4 segments.

Remark 9.24 (Construction of simple representatives). For a general homotopy class of a loops on a surface, it is not obvious how to construct a representative in minimal position right away. A simple representative of $\gamma(r, \mathbb{d})$ can be constructed from the sequence $S_i = S_i(r, \mathbb{d})$ (see Definition 9.5) in the following way.

(1) Draw r distinct points in the interior of every arc ε_i which we label by $z_i^0, \ldots, z_i^{r-1}$ while following the orientation of ε_i. If $s \in \mathbb{Z}$, set $z_x^s := z_x^t$, where $t \in [0, r)$ such that $s \equiv t \mod r$.

(2) For each pair $(i, j) \in \mathbb{Z}_n \times [0, nr]$, connect $z_i^{S_j}$ and $z_{i+1}^{S_j+1}$ by the projection of a straight line in $\mathbb{R}^2$.

The resulting loop is a simple representative of $\gamma(r, \mathbb{d})$. The loop in Figure 23 was constructed by means of this procedure.

Lemma 9.25. *Let* $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ *and let* $\mathcal{V}$ *be an indecomposable local system on* $\gamma(\mathrm{m})$. *Then* $P_{\gamma(\mathrm{m})}^\bullet(\mathcal{V})$ *is concentrated in two or three cohomological degrees.*

Proof. Each subsequence $\varepsilon_x \kappa_x^{\mathrm{m}_x} \varepsilon_{x+1}$ corresponds to a subquiver of $Q(\gamma(\mathrm{m}))$ of the form

$$\bullet \xrightarrow{b_i} \bullet \xrightarrow{a_i} \bullet \xleftarrow{c_i} \bullet \xrightarrow{a_i} \cdots \xleftarrow{c_i} \bullet \xrightarrow{d_i} \bullet \qquad (9.1)$$

if $\mathrm{m}_x \geq 0$ and

$$\bullet \xrightarrow{b_i c_i} \bullet \xleftarrow{a_i} \bullet \xrightarrow{c_i} \cdots \xrightarrow{c_i} \bullet \xleftarrow{d_i a_i} \bullet \qquad (9.2)$$

if $\mathrm{m}_x < 0$. ∎

Remark 9.26. The number of arrows with labels a_i (resp. c_i) in (9.1) and (9.2) is $|\mathrm{m}_x|$.

In what follows, for any (not necessarily) primitive sequence m, we write $P^\bullet(\mathrm{m})$ as short for any representative $P_{\gamma(\mathrm{m})}^\bullet(\mathcal{V})$ of $\gamma(\mathrm{m})$, where $\dim \mathcal{V} = 1$, which is concentrated in degrees -1 and 0, or has entries in degrees -1, 0 and 1. It follows from the discussion preceding Theorem 4.14 that if $\mathrm{m} = \mathrm{e}^t$ for some $t > 1$ and a primitive sequence e, then

$$P^\bullet(\mathrm{m}) \cong \bigoplus_{i=1}^m P_{\gamma(\mathrm{e})}^\bullet(\mathcal{V}_i)$$

for indecomposable local systems $\mathcal{V}_1, \ldots, \mathcal{V}_m$ with $\sum_{i=1}^m \dim \mathcal{V}_i = t$.

The next lemma is the key to showing that $P^\bullet(\mathrm{m})$ is an iterated extension of images of line bundles.

Lemma 9.27. *Let $r > 1$, $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ and $\ell \in \mathbb{Z}^{\mathbb{Z}_n}$. Then the following is true:*

(1) *A morphism $f \in \mathrm{Hom}^\bullet(P^\bullet(\ell), P^\bullet(\mathrm{m}))$ which corresponds to a subsequence $a, 0, \ldots,$ $0, b, a, b, > 0$, of $\ell - \mathrm{m}[p]$ for some $p \in \mathbb{Z}$ is of degree 0.*

(2) *If the above sequence has length $n + 1$, then C_f is isomorphic to a complex $P^\bullet(\mathrm{m}')$ for a sequence m' of rank $r - 1$ and multi-degree $\mathbb{d}(\mathrm{m}) - \mathbb{d}(\ell)$. In other words, $P^\bullet(\mathrm{m})$ is an extension of $P^\bullet(\ell)$ and $P^\bullet(\mathrm{m}')$.*

Proof. We may assume that $p = 0$ and that all values of ℓ, m are non-negative. Let $a, 0, \ldots, 0, b$ be a subsequence of $\ell - \mathrm{m}$ with $a, b > 0$. Then, following the proof of Lemma 9.25 (see also Remark 9.26) and the definition of f from [31], f is a graph map as described in the proof of Lemma 8.8. Its diagram has the following shape:

$$
\begin{array}{ccccccccccc}
\bullet & \xleftarrow{c_i} & \bullet & \xrightarrow{a_i} & \bullet & \xleftarrow{c_i} & \bullet & \xrightarrow{a_i} & \cdots & \xleftarrow{c_i} & \bullet & \xrightarrow{a_i} & \bullet \\
& & \| & & \| & & \| & & & & \| & & \\
\bullet & \xrightarrow{b_i} & \bullet & \xrightarrow{a_i} & \bullet & \xleftarrow{c_i} & \bullet & \xrightarrow{a_i} & \cdots & \xleftarrow{c_i} & \bullet & \xleftarrow{d_i} & \bullet
\end{array}
$$

The top line corresponds to the subquiver of $Q(\gamma(\ell))$ associated with the subsequence

$$
\kappa_i^{\mathrm{m}_i+1} \varepsilon_{i+1} \cdots \varepsilon_i \kappa_i^{\mathrm{m}_{i+n}+1}
$$

of $Q(\gamma(\ell))$; the lower line corresponds to the subquiver of $Q(\gamma(\mathrm{m}))$ which is associated with

$$
\varepsilon_i \kappa_i^{\mathrm{m}_i} \varepsilon_{i+1} \cdots \varepsilon_i \kappa_i^{\mathrm{m}_{i+n}} \varepsilon_{i+1}.
$$

By Theorem 4.14, we obtain a loop which represents the mapping cone of f by resolving the corresponding intersection. The reduced walk of the resolved curve is the reduction of

$$
\cdots \varepsilon_i \kappa_i^{\mathrm{m}_i} (\kappa_i^{-\ell_i} \varepsilon_i^{-1} \kappa_{i-1}^{-\ell_i-1} \varepsilon_{i-1}^{-1} \cdots \kappa_{i+1}^{-\ell_{i+1}} \varepsilon_{i+1}^{-1}) \varepsilon_{i+1} \kappa_{i+1}^{\mathrm{m}_{i+1}} \cdots \varepsilon_i \kappa_i^{\mathrm{m}_{i+n}} \cdots .
$$

Since $\ell_{i+s} = \mathrm{m}_{i+s}$ for all $s \in (1, n)$, the reduction is

$$
\cdots \varepsilon_{i-1} \kappa_{i-1}^{\mathrm{m}_i-1} \varepsilon_i \kappa_i^{\mathrm{m}_i-\ell_i+\mathrm{m}_{i+n}} \varepsilon_{i+1} \kappa_i^{\mathrm{m}_{i+n}+1} \cdots .
$$

Thus, the resolved curve is a power of a loop in Ω_{Vect}^n, has rank $r - 1$ and multi-degree $\mathbb{d}(\mathrm{m}) - \mathbb{d}(\ell)$. ∎

Proposition 9.28. *Every object in Vect^n is a direct sum of shifted vector bundles. Moreover, the rank (resp. multi-degree) of a vector bundle coincides with the rank (resp. multi-degree) of its corresponding homotopy class in Ω_{Vect}^n.*

Proof. Let $\mathrm{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ be a primitive cyclic sequence of rank $r \geq 2$. Let $i \in \mathbb{Z}$. Define $\ell \in \mathbb{Z}^{\mathbb{Z}_n}$ by $\ell_i = \max\{\mathrm{m}_i, \mathrm{m}_{i+n}\} + 1$ and $\ell_{i+j} = \mathrm{m}_{i+j}$ for all $j \in (0, n)$. Then by Lemma 9.27, $P^\bullet(\mathrm{m})$ is an extension of $P^\bullet(\ell)$ and $P^\bullet(\mathrm{m}')$ for some m' of rank $r - 1$ and multi-degree $\mathbb{d}(\mathrm{m}) - \mathbb{d}(\ell)$. By induction on r, it follows that $P^\bullet(\mathrm{m})$ is an iterated

extension of objects of the form $P^\bullet(\mathbb{d})$ with $\mathbb{d}$ primitive and rk $\mathbb{d} = 1$. Here we used the fact that every object in the homogeneous tube of a vector bundle is again a vector bundle which in turn is a consequence of the fact that the class of vector bundles is closed under extensions. From Theorem 2.3 and Remark 3.6, we conclude that $P^\bullet(\mathbb{d})$ is the image of a line bundle of multi-degree $\mathbb{d}$. The class of vector bundles is closed under extensions. Thus, the assertion follows from the additivity of rank and multi-degree as well as the fact that $\mathbb{F}$ is a triangulated embedding. ∎

We are finally prepared to prove Theorem 9.20 and hence Theorem B from the introduction.

Proof of Theorem 9.20. By Proposition 9.28, a loop $\gamma \in \Omega^n_{\mathrm{Vect}}$ represents vector bundles of rank rk γ and multi-degree $\mathbb{d}(\gamma)$. By Theorem A, simple loops in Ω^n_{Vect} correspond to simple vector bundles. For every smooth point $x \in C_n$, we consider the family of functors

$$\mathscr{F}_y := - \otimes^{\mathbb{L}} (\mathscr{L}(x) \otimes \mathscr{L}(y)^\vee).$$

By Proposition 6.11, we see that for all indecomposable $Z \in \mathrm{Perf}(C_n)$, $\mathbb{F}(\mathscr{F}_y(Z))$ and $\mathbb{F}(Z)$ are represented by the same loop.

It remains to prove that every vector bundle is represented by a homotopy class of Ω^n_{Vect}. Following [4, Lemma 6.11], a perfect object $X \in \mathrm{Perf}(C_n)$ is isomorphic to the shift of a locally-free sheaf of rank n if and only if there exists $m \in \mathbb{Z}$ such that $\mathrm{Hom}^\bullet(X, \Bbbk(z)) \cong \Bbbk^n[m]$ as graded vector spaces for all closed points $z \in C_n$. The proof of [4, Lemma 6.11] relies on the assumption that C_n is projective and connected but does not require irreducibility. Without loss of generality, we may assume that X sits at the mouth of its homogeneous tube. Suppose now, that the walk W_γ of a loop $\gamma = \gamma_\mathcal{E}$ for a vector bundle $\mathcal{E}$ contains a subsequence of the form $\varepsilon_i \kappa_i^l \varepsilon_{i+1}^{-1}$ or its inverse. This condition is equivalent to $\gamma \notin \Omega^n_{\mathrm{Vect}}$. Using Lemma 4.13, it is not difficult to see that under these conditions the arc of the skyscraper sheaf of a singular point $z \in \mathbb{P}^1_i \cap \mathbb{P}^{i+1}_1$ has two intersections with γ which correspond to morphisms in $\mathrm{Hom}^\bullet(X, \Bbbk(z))$ in different degrees. Hence, such a loop cannot represent a vector bundle of C_n. This finishes the proof. ∎

9.4. An alternative proof for the classification of simple vector bundles

We give an alternative proof of Theorem 9.1. By Theorem 9.20, it is sufficient to prove the following.

Proposition 9.29. *Let* $\mathtt{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$. *If* $\gamma(\mathtt{m})$ *contains a simple loop, then r and $\underline{\mathtt{m}}$ are coprime and* $\mathtt{m} \sim \mathtt{m}(r, \mathbb{d}(\mathtt{m}))$.

Proof. Given a sequence $\mathtt{m} \in \mathbb{Z}^{\mathbb{Z}_{nr}}$ and $q \in \mathbb{Z}_n$, we can define the contraction $C^q(\mathtt{m}) \in \mathbb{Z}^{\mathbb{Z}_{(n-1)r}}$ of $\mathtt{m}$. Regarding $\mathtt{m}$ as element of $\mathrm{Mat}_{n \times r}(\mathbb{Z})$ with cyclic rows and $C^q(-)$ as an operator $C^q : \mathrm{Mat}_{n \times r}(\mathbb{Z}) \to \mathrm{Mat}_{(n-1) \times r}(\mathbb{Z})$, C^q stores the sum of the q-th and the $(q+1)$-th row of $\mathtt{m}$ in the q-th row of $C^q(\mathtt{m})$ and leaves all the other rows untouched while preserving the cyclic order of the rows. On $\mathscr{T}^n$ the matrix $C^q(\mathtt{m})$ corresponds to the

loop $\gamma(\mathrm{m})$ regarded as a loop in $\mathcal{T}^{n-1}$. The operation C^q induces a map $\Omega^n_{\mathrm{Vect}} \to \Omega^{n-1}_{\mathrm{Vect}}$ which preserves simplicity and hence preserves properties (1) and (2) in Lemma 9.8.

Conversely, given a simple loop $\gamma = \gamma(\mathrm{m}) \subseteq \mathcal{T}^{n-1}$, we may assume that γ and $\delta := \gamma^q_{\Bbbk(x)} \subseteq \mathcal{T}^{n-1}$ are in minimal position. In particular, $\gamma \overset{\rightarrow}{\cap} \delta$ contains exactly $r = \mathrm{rk}\,\mathrm{m}$ intersections and is cyclically ordered by the orientation of δ. Choose an intersection $u \in \gamma \overset{\rightarrow}{\cap} \delta$ and denote by $u^+ \in \gamma \overset{\rightarrow}{\cap} \delta$ its successor. Let $p \in \delta \setminus \{u, u^+\}$ be a point on the segment between u to u^+. Let $\varphi \colon \mathcal{T}^{n-1} \to \mathcal{T}^n$ be a homeomorphism which maps all punctures of $\mathcal{T}^{n-1}$ and p to punctures in $\mathcal{T}^n$. Moreover, we require that it maps the set $\{\delta\} \cup \{\varepsilon_i\}_{i \in \mathbb{Z}_{n-1}}$ bijectively onto the set of curves $\{\varepsilon_j\}_{j \in \mathbb{Z}_n}$ in $\mathcal{T}^n$ and maps κ_j to κ_j for all $j \neq q$ as well as κ_q to the concatenation of κ_q and κ_{q+1}. The isotopy class of such a map φ is not unique but appears as part of a $\mathbb{Z}$-family of isotopy classes of homeomorphisms. Namely, every two possible choices for φ as above are related by composition with a homeomorphism

$$D^m_{\gamma^q_{\Bbbk(x)}} \circ D^{-m}_{\gamma^{q+1}_{\Bbbk(x)}} \in \mathcal{PMCG}(\mathcal{T}^n)$$

for some unique $m \in \mathbb{Z}$. For every φ as above, let us denote by $\gamma^\varphi_u \subseteq \mathcal{T}^n$ the image of γ under φ. It follows from Remark 9.17 that we obtain all homotopy classes of simple loops $\gamma' \subseteq \mathcal{T}^n$ in Ω^n_{Vect} such that $C^q(\mathrm{m}(\gamma')) \simeq \mathrm{m}$. Moreover, we observe that the multi-degrees of γ^φ_u and $\gamma^{\varphi'}_{u'}$ agree if and only if φ and φ' are isotopic and $u = u'$. In particular, this is the case if the two loops are homotopic.

By induction, our arguments reduce the uniqueness of sequences m as in Lemma 9.8 to the uniqueness of such sequences in the case $n = 1$. Here the result follows from the well-known fact that homotopy classes of non-separating simple loops on $\mathcal{T}^1$ are in bijection with pairs of coprime integers. The bijection projects a loop onto its associated class in $H_1(\mathcal{T}^1, \mathbb{Z}) \cong \mathbb{Z}^2$. Choosing the classes of γ_{Pic} and $\gamma^0_{\Bbbk(x)}$ as the basis of $H_1(\mathcal{T}^1, \mathbb{Z})$, $\gamma = \gamma(\mathrm{m})$ is mapped to the pair $(\mathrm{rk}\,\mathrm{m}, \mathbb{d}(\mathrm{m}))$. ∎

Appendix A. Images of smooth skyscraper sheaves and the Picard group

This section contains the proof of Theorem 2.3 which was explained to us by Igor Burban. Our main reference is [12]. As before, let $\mathrm{Pic}^0(C_n)$ denote the set of line bundles of multi-degree $(0, \ldots, 0)$ over C_n and let $\mathbb{F} \colon \mathrm{Perf}(C_n) \hookrightarrow \mathcal{D}^b(\Lambda_n)$ denote the embedding. Theorem 2.3 asserts the following.

Theorem A.1. (1) *The essential image of* $\mathrm{Pic}^0(C_n)$ *under* $\mathbb{F}$ *consists of the isomorphism classes of the complexes* $\mathcal{O}(\lambda)$, *where* $\lambda \in \Bbbk^\times$.

(2) *The essential image of the skyscraper sheaves of smooth points* $x \in \mathbb{P}^1_i$ *under* $\mathbb{F}$ *consists of the isomorphism classes of the complexes* $\Bbbk(i, \lambda)$, *where* $\lambda \in \Bbbk^\times$.

The definitions of the complexes $\mathcal{O}(\lambda)$ and $\Bbbk(i, \lambda)$ are found in Figure 5 and in the paragraph succeeding it. The definition of $\mathbb{F}$ is recalled in Section A.1 below.

For the remainder of this section, fix $n \geq 1$ and write $\Lambda = \Lambda_n$, $C = C_n$ as well as $\mathbb{P}^1_i$, $i \in \mathbb{Z}_n$, for the irreducible components of C_n. Let $\pi \colon \widetilde{C} \to C$ be a normalization map.

Then $\tilde{C} = \bigsqcup_{i=0}^{n-1} \tilde{U}_i$, where $\tilde{U}_i = \pi^{-1}(\mathbb{P}_i^1) \cong \mathbb{P}^1$. By changing coordinates, we may assume that $0, \infty \in \mathbb{P}^1$ are the preimages of the singular points in U_i. Set $\tilde{\mathcal{O}} := \pi_*(\mathcal{O}_{\tilde{C}})$. Then $\mathcal{O} := \mathcal{O}_C$ is a subsheaf of $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}$ decomposes as

$$\tilde{\mathcal{O}} = \bigoplus_{i \in \mathbb{Z}_n} \tilde{\mathcal{O}}_i,$$

where $\tilde{\mathcal{O}}_i := \pi_*(\mathcal{O}_{\tilde{C}}|_{\tilde{U}_i})$. We denote by $\mathcal{J}$ the ideal sheaf of the singular locus of C.

A.1. Non-commutative resolutions of cycles

In [12], Burban and Drozd introduced the so-called *Auslander-sheaf* $\mathscr{A}$. It is the sheaf of $\mathcal{O}$-orders on C given by

$$\mathscr{A} = \begin{pmatrix} \mathcal{O} & \tilde{\mathcal{O}} \\ \mathcal{J} & \tilde{\mathcal{O}} \end{pmatrix}.$$

The diagonal embedding of $\mathcal{O}$ defines an action of $\mathcal{O}$ on $\mathscr{A}$ and endows $\mathscr{A}$ with the structure of an $\mathcal{O}$-algebra.

Burban and Drozd proved that the bounded derived category $\mathcal{D}^b(\operatorname{Coh} \mathbb{X})$ of coherent sheaves over the non-commutative ringed space $\mathbb{X} = (C, \mathscr{A})$ admits a tilting object $\mathscr{H}$ defined as follows.

Let $\mathcal{S}$ denote the torsion sheaf of $\mathscr{A}$-modules defined by the cokernel of the canonical inclusion

$$\begin{pmatrix} \mathcal{J} \\ \mathcal{J} \end{pmatrix} \to \begin{pmatrix} \mathcal{O} \\ \mathcal{J} \end{pmatrix}. \tag{A.1}$$

Furthermore, denote by

$$\mathcal{F} := \begin{pmatrix} \mathcal{O} \\ \mathcal{J} \end{pmatrix} \quad \text{and} \quad \mathcal{P} := \begin{pmatrix} \tilde{\mathcal{O}} \\ \tilde{\mathcal{O}} \end{pmatrix}$$

the $\mathscr{A}$-modules corresponding to the columns of $\mathscr{A}$. The tensor product with $\mathcal{F}$ defines a fully faithful functor $\mathbb{J} \colon \operatorname{Coh} C \to \operatorname{Coh} \mathbb{X}$.

By definition, we have $\mathcal{P} = \bigoplus_{i \in \mathbb{Z}_n} \mathcal{P}_i$, where $\mathcal{P}_i := \mathbb{J}(\tilde{\mathcal{O}}_i)$. Moreover, $\mathcal{S} = \bigoplus_{i \in \mathbb{Z}_n} \mathcal{S}_i$, where $\mathcal{S}_i$ is a skyscraper sheaf supported at the singular point corresponding to the image of $0 \in \tilde{U}_i$ under the normalization map π. For any integer $m \in \mathbb{Z}$ and any line bundle $\mathcal{L}$ of multi-degree $(m, \dots, m)$, define

$$\mathcal{P}(m) := \bigoplus_{i \in \mathbb{Z}_n} \mathcal{P}_i \otimes_{\mathcal{O}} \mathcal{L}.$$

It was shown in [12, Lemma 4] that the isomorphism class of $\mathcal{P}(m)$ is independent of the choice of $\mathcal{L}$. In particular, up to isomorphism, the object

$$\mathscr{H} := \mathcal{S}[-1] \oplus \mathcal{P}(-1) \oplus \mathcal{P}$$

is well defined. The object $\mathcal{H}$ is a tilting and Λ is isomorphic to the opposite endomorphism algebra of $\mathcal{H}$. The derived functor of $\mathbb{J}$ restricts to an embedding $\mathrm{Perf}(C) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathbb{X})$ and gives rise to an embedding $\mathbb{B} \colon \mathrm{Perf}(C) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda)$. For $i \in [0, n)$, let $x_i \in U_i$ be smooth. The desired embedding $\mathbb{F} \colon \mathrm{Perf}(C) \to \mathcal{D}^{\mathrm{b}}(\Lambda)$, as cited in the main text of this paper, is given by

$$\mathbb{F} = \mathbb{B} \circ \prod_{i \in \mathbb{Z}_n} T_{\Bbbk(x_i)} \cong \mathbb{B} \circ \otimes^{\mathbb{L}} \mathcal{L}(x_0 + \cdots + x_{n-1}).$$

A.2. Images of skyscraper sheaves of smooth points

For any $i \in \mathbb{Z}_n$, let $x \in U_i$ be smooth corresponding to a point $(\lambda : \mu) \in \mathbb{P}^1$. By our assumption on the chosen coordinates, $\lambda, \mu \neq 0$. Let $\widetilde{x} \in \widetilde{U}_i$ denote the unique preimage of x under π. Then $\pi_*(\Bbbk(\widetilde{x})) \cong \Bbbk(x)$ and if $z_0^i, z_\infty^i \colon \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ correspond to the chosen coordinates on $\widetilde{X}_i$ vanishing at 0 and ∞, respectively, then the $\Bbbk(x)$ is the cokernel of $\alpha_i \colon \widetilde{\mathcal{O}}_i(-1) \to \widetilde{\mathcal{O}}_i$, where

$$\alpha_i := \pi_*(\mu z_0^i(-1) - \lambda z_\infty^i(-1)).$$

Keeping in mind that $\mathbb{J} \circ \pi_*(\mathcal{O}_{\widetilde{U}_i}(m)) = \mathcal{P}_i(m)$, the isomorphism $\Lambda \cong \mathrm{End}(\mathcal{H})^{\mathrm{op}}$ identifies the arrows a_i and c_i in Λ with the maps $\mathbb{J}(\pi_* \circ z_0^i), \mathbb{J}(\pi_* \circ z_\infty^i) \in \mathrm{Hom}(\mathcal{P}_i(-1), \mathcal{P}_i)$. We deduce that there is a short exact sequence

$$0 \to \mathcal{P}_i(-1) \xrightarrow{\mathbb{J}(\alpha_i)} \mathcal{P}_i \to \mathbb{J}(\Bbbk(x)) \to 0.$$

The tilting functor associated with $\mathcal{H}$ sends $\mathcal{P}_i(-1)$ to the indecomposable projective module of $s(a_i)$ and $\mathcal{P}_i$ to the indecomposable projective module of $t(a_i)$ proving that $\mathbb{B}(\Bbbk(x))$ is isomorphic to $\Bbbk(i, \lambda^{-1}\mu)$. Since $\mathrm{Hom}^\bullet(\Bbbk(x), \Bbbk(y)) = 0$ for all $x \neq y$ and $\Bbbk(x)$ is 1-spherical, it follows from Corollary 3.7 and Lemma 3.8 that $\mathbb{F}(\Bbbk(x)) \cong \Bbbk(i, \lambda^{-1}\mu)$.

A.3. Image of the Picard group

The image of $\mathrm{Pic}^0(C)$ under $\mathbb{F}$ coincides with the image of the line bundles of degree $(1, \ldots, 1)$ under $\mathbb{B}$.

Let $\mathcal{L}$ be a line bundle on C of multi-degree $(1, \ldots, 1)$ and set $\mathcal{G} := \mathbb{J}(\mathcal{L}) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}$. We want to compute the dimension of the vector spaces $\mathrm{Ext}^i(\mathcal{S}, \mathcal{G})$, $\mathrm{Ext}^i(\mathcal{P}, \mathcal{G})$ and $\mathrm{Ext}^i(\mathcal{P}(-1), \mathcal{G})$ for $i \geq 0$. Note that by [12, Theorem 2], the global dimension of $\mathrm{Coh}\,\mathbb{X}$ is 2.

Since $\mathcal{S}$ is a torsion sheaf and $\mathcal{G}$ is torsion-free, it follows that $\mathcal{H}om_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) \cong 0$. Thus, $\mathrm{Hom}_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) = 0$. By (A.1), $\mathcal{S}$ has a locally projective resolution of length one. Thus, $\mathcal{E}xt_{\mathscr{A}}^i(\mathcal{S}, \mathcal{G}) = 0$ for all $i \geq 2$ and one deduces from the local computations of $\mathcal{E}xt_{\mathscr{A}}^1(\mathcal{S}_j, \mathcal{G})$ and the local-to-global spectral sequence that $\mathrm{Ext}_{\mathscr{A}}^1(\mathcal{S}_j, \mathcal{G}) \cong \Bbbk$ for all $j \in \mathbb{Z}_n$ and $\mathrm{Ext}_{\mathscr{A}}^i(\mathcal{S}, \mathcal{G}) = 0$ for all $i \geq 2$.

By [12, Corollary 4], there exists an isomorphism of $\mathcal{O}$-modules,

$$\mathcal{I} \otimes \mathcal{L} \cong \mathcal{H}om_{\mathscr{A}}(\mathcal{P}, \mathcal{G}),$$

and by [12, Corollary 3], $\mathrm{Ext}_{\mathscr{A}}^i(\mathscr{P}, \mathscr{G}) = 0$ for all $i \geq 0$. We have a sequence of isomorphisms

$$\mathcal{H}om_{\mathscr{A}}(\mathscr{P}(-1), \mathscr{G}) \cong \mathcal{H}om_{\mathscr{A}}(\mathscr{P}(-1) \otimes_{\mathcal{O}} \mathscr{L}^{\vee}, \mathscr{F}) \cong \mathcal{H}om_{\mathscr{A}}(\mathscr{P}(-2), \mathscr{F}),$$

where $\mathscr{L}^{\vee}$ denotes the dual of $\mathscr{L}$. Using that $\mathscr{P}(-2) \cong \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix}$ and that both $\mathscr{F}$ and $\mathscr{P}(-2)$ are torsion-free, it follows from [12, Proposition 6], that $\mathcal{H}om_{\mathscr{A}}(\mathscr{P}(-1), \mathscr{G}) \cong \widetilde{\mathcal{O}}$ as $\mathcal{O}$-modules. Since $\Bbbk \cong \Gamma(\widetilde{U}_i, \mathcal{O}_{\widetilde{U}_i}) \cong \Gamma(X, \widetilde{\mathcal{O}}_i)$, we deduce $\mathrm{Hom}_{\mathscr{A}}(\mathscr{P}_i(-1), \mathscr{G}) \cong \Bbbk$. Moreover, it follows from [12, Corollary 3], that $\mathrm{Ext}^i(\mathscr{P}(-1), \mathscr{G}) = 0$ for all $i \geq 1$.

We have shown that $\mathrm{Hom}^{\bullet}(\mathscr{H}, \mathscr{G})$, and hence the cohomology of $\mathbb{B}(\mathscr{G})$, is concentrated in degree 0. In particular, $\mathbb{B}(\mathscr{G})$ is quasi-isomorphic to a Λ-module. Since $\mathbb{B}(\mathscr{L})$ is indecomposable, we conclude from the isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(\mathscr{P}_i(-1), \mathscr{G}) \cong \Bbbk \cong \mathrm{Hom}_{\mathscr{A}}(\mathcal{S}_i[-1], \mathscr{G})$$

for all $i \in \mathbb{Z}_n$ and from the classification of τ-invariant indecomposable objects in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ from Section 4.2 that $\mathbb{F}(\mathscr{L}) \cong \mathcal{O}(\lambda)$ for some $\lambda \in \Bbbk^{\times}$.

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