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Sobolev improvements on sharp Rellich inequalities

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Abstract. There are two Rellich inequalities for the bilaplacian, that is, for $\int (\Delta u)^2 dx$, the one involving $|\nabla u|$ and the other involving $|u|$ at the RHS. In this article, we consider these inequalities with sharp constants and obtain sharp Sobolev-type improvements. More precisely, in our first result, we improve the Rellich inequality with $|\nabla u|$ obtained by Beckner in dimensions $n = 3, 4$ by a sharp Sobolev term, thus complementing existing results for the case $n \geq 5$. In the second theorem, the sharp constant of the Sobolev improvement for the Rellich inequality with $|u|$ is obtained.

Dedicated to E. B. Davies on the occasion of his 80th birthday

1. Introduction

The study of PDEs involving the bilaplacian is often related to functional inequalities for the associated energy, namely, $\int (\Delta u)^2 dx$. Two important such inequalities are the Sobolev inequality and the Rellich inequality.

There are two Rellich inequalities related to the bilaplacian. The first one asserts that for $n \geq 5$ there holds

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} dx, \quad u \in C_c^\infty(\mathbb{R}^n), \tag{1.1}$$

and the constant is the best possible. Inequality (1.1) was proved by Rellich; see [22]. For more results on inequalities of this type and related improvements, we refer to [2–4, 6, 9, 11, 12, 14, 17–20, 23, 25] and references therein.

The second Rellich inequality is valid not only for $n \geq 5$ but also for $n = 3, 4$ and reads

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq c_n \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} dx, \quad u \in C_c^\infty(\mathbb{R}^n), \tag{1.2}$$

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where

$$c_n = \begin{cases} \frac{25}{36}, & n = 3, \\ 3, & n = 4, \\ \frac{n^2}{4}, & n \geq 5 \end{cases} \tag{1.3}$$

is the best possible constant. Inequality (1.2) was proved in [25] in case $n \geq 5$ and then by Beckner for any $n \geq 3$ [8]. An alternative proof for $n \geq 3$ was given by Cazacu [10]. We note that, in cases $n = 3, 4$ there is a breaking of symmetry. For more information on Rellich inequalities in the spirit of (1.2), we refer to [10, 11, 13, 21, 25].

The Sobolev inequality for the bilaplacian in $\mathbb{R}^n$, $n \geq 5$, reads

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \geq S_{2,n} \Big(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}}, \quad u \in C_c^\infty(\mathbb{R}^n). \tag{1.4}$$

The best constant $S_{2,n}$ in (1.4) has been computed in [15] and is given by

$$S_{2,n} = \pi^2 (n^2 - 4n)(n^2 - 4) \Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \Big)^4.$$

The aim of this work is to improve the above Rellich inequalities by adding a Sobolev-type term. In [25], improved versions of (1.1) and (1.2) were obtained for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 5$. More precisely, let $X(r) = (1 - \log r)^{-1}$, $0 < r < 1$, and $D = \sup_\Omega |x|$. In [25, Theorem 1.1], it was shown that for $n \geq 5$ there exist constants C_n and C_n' which depend only on n such that for any $u \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \geq C_n \Big(\int_\Omega X(|x|/D)^{\frac{2(n-2)}{n-4}} |u|^{\frac{2n}{n-4}} dx \Big)^{\frac{n-4}{n}} \tag{1.5}$$

and

$$\int_\Omega (\Delta u)^2 dx - \frac{n^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C_n' \Big(\int_\Omega X(|x|/D)^{\frac{2(n-1)}{n-2}} |\nabla u|^{\frac{2n}{n-2}} dx \Big)^{\frac{n-2}{n}}. \tag{1.6}$$

The present article contains two main results. The first theorem extends inequality (1.6) to dimensions $n = 3, 4$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n = 3$ or $n = 4$, be a bounded domain and let $D = \sup_{x \in \Omega} |x|$. There exists $C > 0$ such that the following statements hold.*

(i) *If $n = 3$, then*

$$\int_\Omega (\Delta u)^2 dx - \frac{25}{36} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C \Big(\int_\Omega |\nabla u|^6 X^4(|x|/D) dx \Big)^{\frac{1}{3}}, \quad u \in C_c^\infty(\Omega).$$

(ii) *If* $n = 4$, *then*

$$\int_\Omega (\Delta u)^2 dx - 3\int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq C\Big(\int_\Omega |\nabla u|^4 dx\Big)^{\frac{1}{2}}, \quad u \in C_c^\infty(\Omega).$$

Moreover, the power X^4 *in case* $n = 3$ *is the best possible.*

It is remarkable that in case $n = 4$ no logarithmic factor is required at the RHS, as opposed to the cases $n = 3$ and $n \geq 5$.

Concerning inequality (1.5), let us first recall what is known for the corresponding Hardy–Sobolev problem. In [1], it was shown that for any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and for any $u \in C_c^\infty(\Omega)$, there holds

$$\int_\Omega |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx \geq (n-2)^{-\frac{2(n-1)}{n}} S_{1,n} \Big(\int_\Omega X^{\frac{2(n-1)}{n-2}}(|x|/D)|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}},$$

where

$$S_{1,n} = \pi n(n-2)\Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\Big)^{\frac{2}{n}}$$

is the best Sobolev constant for the standard Sobolev inequality in $\mathbb{R}^n$. Moreover, the constant $(n-2)^{-\frac{2(n-1)}{n}} S_{1,n}$ is the best possible. Similarly, in the article [7] Sobolev improvements with best constants were obtained to sharp Hardy inequalities in Euclidean and hyperbolic space. We note that by slightly adapting [7, Theorem 5] we obtain that if Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, then

$$\int_\Omega |\nabla u|^2 dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2} dx + \frac{(n-1)(n-3)}{4} \int_\Omega \frac{u^2}{|x|^2} X^2(|x|/D) dx \geq S_{1,n}\Big(\int_\Omega X^{\frac{2(n-1)}{n-2}}(|x|/D)|u|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}} \tag{1.7}$$

for all $u \in C_c^\infty(\Omega)$ and the constant $S_{1,n}$ is sharp.

The second theorem of this article provides an estimate with best Sobolev constant for a slightly modified version of (1.5) which is in the spirit of (1.7).

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 5$, *be a bounded domain and let* $D = \sup_\Omega |x|$. *For any* $u \in C_c^\infty(\Omega)$, *there holds*

$$\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} X^{\frac{2(n-2)}{n-1}} dx \geq S_{2,n}\Big(\int_\Omega X^{\frac{2(n-2)}{n-4}} |u|^{\frac{2n}{n-4}} dx\Big)^{\frac{n-4}{n}};$$

here, $X = X(|x|/D)$. *Moreover, the constant* $S_{2,n}$ *is the best possible.*

The proof of Theorem 1 is in Section 2 and the proof of Theorem 2 is in Section 3.

2. Rellich–Sobolev inequality I

In this section, we will prove Theorem 1. An important tool will be the decomposition of functions in spherical harmonics [24, Section IV.2].

We recall that the eigenvalues of the Laplace–Beltrami operator on the unit sphere S^{n-1} are given by

$$\mu_k = k(k+n-2), \quad k = 0,1,2,\dots .$$

Each μ_k has multiplicity

$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}, \quad k \geq 2,$$

while $d_0 = 1$ and $d_1 = n$.

Let $\{\phi_{kj}\}_{j=1}^{d_k}$ be an orthonormal basis of eigenfunctions for the eigenvalue μ_k. Then, any function $u \in L^2(\mathbb{R}^n)$ can be decomposed as

$$u(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} u_{kj}(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} f_{kj}(r)\phi_{kj}(\omega), \tag{2.1}$$

where $x = r\omega$, $r > 0$, $\omega \in S^{n-1}$, and

$$f_{kj}(r) = \int_{S^{n-1}} u(r\omega)\phi_{kj}(\omega)dS(\omega).$$

We note that each ϕ_{kj} is the restriction on the unit sphere of a harmonic homogeneous polynomial of degree k [24].

Assume now that $u \in C_c^\infty(\mathbb{R}^n)$. Since any homogeneous polynomial can be written as a linear combination of harmonic homogeneous polynomials, taking the Taylor expansion of u near the origin, we easily infer that

$$f_{kj}(r) = O(r^k), \quad f'_{kj}(r) = O(r^{k-1}) \quad \text{as } r \to 0 \tag{2.2}$$

for any $k \geq 1$ and any $j = 1,\dots,d_k$.

We note that

$$\mu_k \geq n-1 \quad \forall\, k \geq 1, \tag{2.3}$$

an estimate that will be used several times in what follows.

In what follows, we will use $\sum_{k,j}$ as a shorthand for $\sum_{k=0}^{\infty}\sum_{j=1}^{d_k}$.

For simplicity, we will denote by u_0 (instead of u_{01}) the first (radial) term in the decomposition (2.1) of u into spherical harmonics. We note the relation

$$\int_{\mathbb{R}^n} (\Delta u - \Delta u_0)^2 dx = \sum_{k=1}^{\infty}\sum_{j=1}^{d_k} \int_{\mathbb{R}^n} (\Delta u_{kj})^2 dx. \tag{2.4}$$

Lemma 1. *Let $n \geq 3$. For any $u \in C_c^\infty(\mathbb{R}^n)$, there holds*

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{k,j} \Big\{ \int_0^\infty r^{n-1} f_{kj}''^2 dr + (n-1+2\mu_k) \int_0^\infty r^{n-3} f_{kj}'^2 dr + (2(n-4)\mu_k + \mu_k^2) \int_0^\infty r^{n-5} f_{kj}^2 dr \Big\}, \tag{2.5}$$

$$\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} dx = \sum_{k,j} \Big\{ \int_0^\infty r^{n-3} f_{kj}'^2 dr + \mu_k \int_0^\infty r^{n-5} f_{kj}^2 dr \Big\}. \tag{2.6}$$

Proof. Using the orthonormality of the set $\{\phi_{kj}\}$, we have

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{k,j} \int_{\mathbb{R}^n} (\Delta u_{kj})^2 dx = \sum_{k,j} \int_0^\infty \Big(f_{kj}'' + \frac{n-1}{r} f_{kj}' - \frac{\mu_k}{r^2} f_{kj} \Big)^2 r^{n-1} dr.$$

Equation (2.5) then follows by expanding the square and integrating by parts. Estimates (2.2) ensure that no terms appear from $r = 0$. The proof of (2.6) is similar and is omitted. ∎

For $n \geq 3$, we set

$$\mathbb{I}[u] = \int_{\mathbb{R}^n} (\Delta u)^2 dx - c_n \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} dx,$$

where the constant c_n is given by (1.3).

Lemma 2. *Assume that $n = 3$ or $n = 4$. There exists $c > 0$ such that for any $u \in C_c^\infty(\mathbb{R}^n)$, there holds*

$$\mathbb{I}[u] \geq \mathbb{I}[u_0] + \sum_{j=1}^n \mathbb{I}[u_{1j}] + c \int_{\mathbb{R}^n} \Big(\Delta u - \Delta u_0 - \sum_{j=1}^n \Delta u_{1j} \Big)^2 dx. \tag{2.7}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^n)$. Because of the relation

$$\mathbb{I}[u] = \mathbb{I}[u_0] + \sum_{j=1}^n \mathbb{I}[u_{1j}] + \sum_{k=2}^\infty \sum_{j=1}^{d_k} \mathbb{I}[u_{kj}],$$

inequality (2.7) will follow if we establish the existence of $c > 0$ such that

$$\mathbb{I}[u_{kj}] \geq c \int_{\mathbb{R}^n} (\Delta u_{kj})^2 dx, \quad k \geq 2,\ 1 \leq j \leq d_k. \tag{2.8}$$

Assume first that $n = 3$. Let $\lambda > 0$ be fixed. For $k \geq 2$, we have $\mu_k \geq 6$, and therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\Delta u_{kj})^2 dx \\
&\quad = \int_0^\infty r^2 f_{kj}''^2 dr + (2 + 2\mu_k) \int_0^\infty f_{kj}'^2 dr + (-2\mu_k + \mu_k^2) \int_0^\infty r^{-2} f_{kj}^2 dr \\
&\quad \geq \left(\frac{9}{4} + 2\lambda\mu_k\right) \int_0^\infty f_{kj}'^2 dr + \left(2(1-\lambda)\frac{1}{4}\mu_k - 2\mu_k + \mu_k^2\right) \int_0^\infty r^{-2} f_{kj}^2 dr \\
&\quad \geq \left(\frac{9}{4} + 12\lambda\right) \int_0^\infty f_{kj}'^2 dr + \left(\frac{9}{2} - \frac{\lambda}{2}\right)\mu_k \int_0^\infty r^{-2} f_{kj}^2 dr.
\end{aligned}$$

Choosing $\lambda = 9/50$, we arrive at

$$\int_{\mathbb{R}^3} (\Delta u_{kj})^2 dx \geq \frac{441}{100} \int_{\mathbb{R}^3} \frac{|\nabla u_{kj}|^2}{|x|^2} dx,$$

and (2.8) follows. In case $n = 4$, we argue similarly. We now have $\mu_k \geq 8$; hence,

$$\begin{aligned}
\int_{\mathbb{R}^4} (\Delta u_{kj})^2 dx &= \int_0^\infty r^3 f_{kj}''^2 dr + (3 + 2\mu_k) \int_0^\infty r f_{kj}'^2 dr + \mu_k^2 \int_0^\infty r^{-1} f_{kj}^2 dr \\
&\geq (4 + 2\mu_k) \int_0^\infty r f_{kj}'^2 dr + \mu_k^2 \int_0^\infty r^{-1} f_{kj}^2 dr \\
&\geq 8 \int_{\mathbb{R}^4} \frac{|\nabla u_{kj}|^2}{|x|^2} dx,
\end{aligned}$$

as required. ■

Lemma 3. *Let $n = 3$ or $n = 4$. Then, there exists $c > 0$ such that*

$$\mathbb{I}[u_0] \geq c \left(\int_{B_1} |\nabla u_0|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \tag{2.9}$$

Additionally, for $n = 3$, we have

$$\mathbb{I}[u_{1j}] \geq c \left(\int_{B_1} |\nabla u_{1j}|^6 X^4 dx \right)^{\frac{1}{3}}, \quad j = 1, 2, 3, \tag{2.10}$$

while for $n = 4$

$$\mathbb{I}[u_{1j}] \geq c \left(\int_{B_1} |\nabla u_{1j}|^4 dx \right)^{\frac{1}{2}}, \quad j = 1, 2, 3, 4. \tag{2.11}$$

Here, $X = X(|x|)$.

Proof. From Lemma 1, equation (2.5) and the standard Sobolev inequality, we obtain

$$\mathbb{I}[u_0] \geq \int_0^1 f_0''^2 r^{n-1} dr \geq c\left(\int_0^1 |f_0'|^{\frac{2n}{n-2}} r^{n-1} dr\right)^{\frac{n-2}{n}} = c\left(\int_{B_1} |\nabla u_0|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}$$

as required.

Assume now that $n = 3$. By Lemma 1 and the improved Hardy–Sobolev inequality of [16], we have

$$\begin{aligned}\mathbb{I}[u_{1j}] &= \int_0^1 f_{1j}''^2 r^2 dr - \frac{1}{4}\int_0^1 f_{1j}'^2 dr + \frac{50}{9}\left(\int_0^1 f_{1j}'^2 dr - \frac{1}{4}\int_0^1 r^{-2} f_{1j}^2 dr\right)\\ &\geq c\left(\int_0^1 |f_{1j}'|^6 X^4 r^2 dr\right)^{\frac{1}{3}} + c\left(\int_0^1 |f_{1j}|^6 X^4 dr\right)^{\frac{1}{3}}\\ &\geq c\left(\int_{B_1} |\nabla u_{1j}|^6 X^4 dx\right)^{\frac{1}{3}}.\end{aligned}$$

In case $n = 4$, we argue similarly applying again Lemma 1 and, now, the standard Sobolev inequality; we obtain

$$\begin{aligned}\mathbb{I}[u_{1j}] &= \int_0^1 f_{1j}''^2 r^3 dr + 6\int_0^1 f_{1j}'^2 r\, dr\\ &\geq c\left(\int_0^1 |f_{1j}'|^4 r^3 dr\right)^{\frac{1}{2}} + c\left(\int_0^1 |f_{1j}|^4 r\, dr\right)^{\frac{1}{2}} \geq c\left(\int_{B_1} |\nabla u_{1j}|^4 dx\right)^{\frac{1}{2}},\end{aligned}$$

as required. ∎

Proof of Theorem 1. We first note that by the standard Sobolev inequality we have

$$\int_\Omega (\Delta u - \Delta u_0 - \sum_{j=1}^n \Delta u_{1j})^2 dx \geq c\left(\int_\Omega |\nabla u - \nabla u_0 - \sum_{j=1}^n \nabla u_{1j}|^{\frac{2n}{n-2}} dx\right)^{\frac{1}{3}}.$$

In case $n = 3$, we apply (2.7), (2.9), and (2.10) and the triangle inequality to obtain

$$\begin{aligned}\mathbb{I}[u] &\geq \mathbb{I}[u_0] + \sum_{j=1}^n \mathbb{I}[u_{1j}] + c\int_{\mathbb{R}^n} \left(\Delta u - \Delta u_0 - \sum_{j=1}^n \Delta u_{1j}\right)^2 dx\\ &\geq c\left(\int_\Omega |\nabla u_0|^6 X^4 dx\right)^{\frac{1}{3}} + c\sum_{j=1}^n \left(\int_{B_1} |\nabla u_{1j}|^6 X^4 dx\right)^{\frac{1}{3}}\\ &\quad + c\left(\int_\Omega |\nabla u - \nabla u_0 - \sum_{j=1}^n \nabla u_{1j}|^6 dx\right)^{\frac{1}{3}} \geq c\left(\int_\Omega |\nabla u|^6 X^4 dx\right)^{\frac{1}{3}}.\end{aligned}$$

In case $n = 4$, we argue similarly, the only difference being that we use (2.11) instead of (2.10).

We next prove the optimality of the power X^4 in (i), that is in case $n = 3$. So, let us assume instead that there exist $\mu < 4$ and $c > 0$ so that

$$\int_\Omega (\Delta u)^2 dx - \frac{25}{36}\int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq c\left(\int_\Omega |\nabla u|^6 X^\mu(|x|/D)dx\right)^{\frac{1}{3}} \tag{2.12}$$

for all $u \in C_c^\infty(\Omega)$. Without loss of generality, we assume that $B_1 \subset \Omega$. We consider small positive numbers ε and δ and define the functions

$$u_{\varepsilon,\delta}(x) = f_{\varepsilon,\delta}(r)\phi_1(\omega) := r^{\frac{1}{2}+\varepsilon} X(r)^{-\frac{1}{2}+\delta}\psi(r)\phi_1(\omega),$$

where $\phi_1(\omega)$ is a normalized eigenfunction for the first non-zero eigenvalue of the Laplace–Beltrami operator on S^2 and $\psi(r)$ is a smooth radially symmetric function supported in B_1 and equal to one near $r = 0$.

Applying Lemma 1, we see that $\int (\Delta u_{\varepsilon,\delta})^2 dx - \frac{25}{36}\int \frac{|\nabla u_{\varepsilon,\delta}|^2}{|x|^2} dx$ is a linear combination of the integrals

$$I_{\varepsilon,\delta}^{(j)} = \int_0^1 r^{-1+2\varepsilon} X^{-1+j+2\delta}\psi^2 dr, \quad 0 \leq j \leq 4,$$

and of integrals that contain at least one derivative of ψ and are, therefore, uniformly bounded. Moreover, simple computations yield that for $j = 3, 4$ the integrals $I_{\varepsilon,\delta}^{(j)}$ are also uniformly bounded for small $\varepsilon, \delta > 0$.

Restricting attention to a small neighborhood of the origin, where $\psi = 1$, we find

$$f'_{\varepsilon,\delta}(r) = r^{-\frac{1}{2}+\varepsilon}\left(\left(\frac{1}{2}+\varepsilon\right)X^{-\frac{1}{2}+\delta} + \left(-\frac{1}{2}+\delta\right)X^{\frac{1}{2}+\delta}\right)$$

and

$$f''_{\varepsilon,\delta}(r) = r^{-\frac{3}{2}+\varepsilon}\left(\left(\varepsilon^2 - \frac{1}{4}\right)X^{-\frac{1}{2}+\delta} + 2\varepsilon\left(-\frac{1}{2}+\delta\right)X^{\frac{1}{2}+\delta} + \left(\delta^2 - \frac{1}{4}\right)X^{\frac{3}{2}+\delta}\right).$$

Hence, we arrive at

$$\begin{aligned}
&\int_{B_1} (\Delta u_{\varepsilon,\delta})^2 dx - \frac{25}{36}\int_{B_1} \frac{|\nabla u_{\varepsilon,\delta}|^2}{|x|^2} dx \\
&\quad = \left(\frac{191}{36}\varepsilon + \frac{173}{36}\varepsilon^2 + \varepsilon^4\right) I_{\varepsilon,\delta}^{(0)} \\
&\qquad - \left(\frac{191}{72} - \frac{191}{36}\delta + \left(\frac{173}{36} - \frac{173}{18}\delta\right)\varepsilon + (2-4\delta)\varepsilon^3\right) I_{\varepsilon,\delta}^{(1)} \\
&\qquad + \left(\frac{209}{144} - \frac{191}{36}\delta + \frac{173}{36}\delta^2 + \left(\frac{1}{2} - 4\delta + 6\delta^2\right)\varepsilon^2\right) I_{\varepsilon,\delta}^{(2)} + O(1).
\end{aligned}$$

It is easy to see that $I_{\varepsilon,0}^{(j)} = \frac{1}{2\varepsilon} + O(1)$, $j = 0, 1, 2$. Hence, rearranging also terms, we obtain

$$\int_{B_1} (\Delta u_{\varepsilon,\delta})^2 dx - \frac{25}{36} \int_{B_1} \frac{|\nabla u_{\varepsilon,\delta}|^2}{|x|^2} dx = \frac{191}{72} \big(2\varepsilon I_{\varepsilon,\delta}^{(0)} - (1-2\delta) I_{\varepsilon,\delta}^{(1)}\big) + \Big(\frac{209}{144} - \frac{191}{36}\delta + \frac{173}{36}\delta^2\Big) I_{\varepsilon,\delta}^{(2)} + O(1).$$

Now, by [5, page 181], we have

$$2\varepsilon I_{\varepsilon,\delta}^{(0)} - (1-2\delta) I_{\varepsilon,\delta}^{(1)} = O(1).$$

Hence, letting $\varepsilon \to 0$, we obtain

$$\int_{B_1} (\Delta u_{\varepsilon,\delta})^2 dx - \frac{25}{36} \int_{B_1} \frac{|\nabla u_{\varepsilon,\delta}|^2}{|x|^2} dx \to \Big(\frac{209}{144} - \frac{191}{36}\delta + \frac{173}{36}\delta^2\Big) I_{0,\delta}^{(2)} + O(1)$$
$$= \frac{209}{144} \int_0^1 r^{-1} X^{1+2\delta} \psi^2 dr + O(1),$$

which is finite for any $\delta > 0$ and diverges to infinity as $\delta \to 0+$.

Now, for $\delta > (4-\mu)/6$, we have

$$\int_{B_1} |\nabla u_{\varepsilon,\delta}|^6 X^\mu dx \geq c \int_0^{1/2} r^{-1+6\varepsilon} X^{\mu-3+6\delta} dr.$$

Letting first $\varepsilon \to 0$ and then $\delta \to \frac{4-\mu}{6}+$ the last integral tends to infinity. Hence, the Rayleigh quotient tends to zero, which implies that the constant c in (2.12) should be zero. This concludes the proof. ∎

3. Rellich–Sobolev inequality II

In this section, we will prove Theorem 2. Throughout the proof, we will make use of spherical coordinates (r, ω), $r > 0$, $\omega \in \mathrm{S}^{n-1}$. We denote by ∇_ω and Δ_ω the gradient and Laplacian on S^{n-1}.

Lemma 4. *Let $\theta \in \mathbb{R}$. For any $v \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, there holds*

$$\int_{\mathbb{R}^n} (\Delta v)^2 |x|^\theta dx$$
$$= \int_0^\infty \int_{\mathrm{S}^{n-1}} v_{rr}^2 r^{n+\theta-1} dS\, dr + (n-1)(1-\theta) \int_0^\infty \int_{\mathrm{S}^{n-1}} v_r^2 r^{n+\theta-3} dS\, dr$$
$$+ \int_0^\infty \int_{\mathrm{S}^{n-1}} (\Delta_\omega v)^2 r^{n+\theta-5} dS\, dr + 2 \int_0^\infty \int_{\mathrm{S}^{n-1}} |\nabla_\omega v_r|^2 r^{n+\theta-3}\, dS\, dr$$
$$- (\theta-2)(n+\theta-4) \int_0^\infty \int_{\mathrm{S}^{n-1}} |\nabla_\omega v|^2 r^{n+\theta-5} dS\, dr.$$

Proof. This follows by writing

$$\Delta v = v_{rr} + \frac{n-1}{r} v_r + \frac{1}{r^2} \Delta_\omega v$$

and integrating by parts; we omit the details. ■

In the next lemma and also later, we will use subscripts R and NR to denote the radial and non-radial component of a given functional.

Lemma 5. *Let $n \geq 5$, $\beta > 0$ and define*

$$A = \frac{1}{\beta^2}\big(2n - 4 - \beta(n - 4 + \beta)\big).$$

Let $u \in C_c^\infty(\mathbb{R}^n)$. Changing variables by $u(r,\omega) = y(t,\omega)$, $t = r^\beta$, we have

$$\frac{\int_{\mathbb{R}^n} (\Delta u)^2 dx}{\big(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-4}} dx\big)^{\frac{n-4}{n}}} = \beta^{\frac{4(n-1)}{n}} \frac{\mathcal{A}_{\mathrm{R}}[y] + \mathcal{A}_{\mathrm{NR}}[y]}{\big(\int_0^\infty \int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta}{\beta}} |y|^{\frac{2n}{n-4}} dS\, dt\big)^{\frac{n-4}{n}}},$$

where

$$\mathcal{A}_{\mathrm{R}}[y] = \int_0^\infty \int_{\mathbb{S}^{n-1}} \Big(t^{\frac{3\beta+n-4}{\beta}} y_{tt}^2 + A t^{\frac{\beta+n-4}{\beta}} y_t^2\Big) dS\, dt,$$

$$\mathcal{A}_{\mathrm{NR}}[y] = \int_0^\infty \int_{\mathbb{S}^{n-1}} \Bigg(\frac{1}{\beta^4} t^{\frac{n-\beta-4}{\beta}} (\Delta_\omega y)^2 + \frac{2}{\beta^2} t^{\frac{\beta+n-4}{\beta}} |\nabla_\omega y_t|^2 + \frac{2(n-4)}{\beta^4} t^{\frac{n-\beta-4}{\beta}} |\nabla_\omega y|^2\Bigg) dS\, dt.$$

Proof. After some lengthy but otherwise elementary computations, we find that

$$\int_0^\infty \Big(u_{rr} + \frac{n-1}{r} u_r\Big)^2 r^{n-1} dr = \beta^3 \int_0^\infty \Big(t^{\frac{3\beta+n-4}{\beta}} y_{tt}^2 + A t^{\frac{\beta+n-4}{\beta}} y_t^2\Big) dt$$

and

$$\int_0^\infty |u|^{\frac{2n}{n-4}} r^{n-1} dr = \frac{1}{\beta} \int_0^\infty |y|^{\frac{2n}{n-4}} t^{\frac{n-\beta}{\beta}} dt.$$

Similar computations involving the non-radial (tangential) derivatives yield the term $\mathcal{A}_{\mathrm{NR}}[y]$. We omit the details. ■

We now consider the Rayleigh quotient for the Rellich–Sobolev inequality (1.5). Changing variables by $u(x) = |x|^{-\frac{n-4}{2}} v(x)$ we obtain (cf. [25, Lemma 2.3 (ii)])

$$\begin{aligned}
&\int_\Omega (\Delta u)^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \\
&= \int_\Omega \Big(|x|^{4-n} (\Delta v)^2 + \frac{n(n-4)}{2} |x|^{2-n} |\nabla v|^2 - n(n-4)|x|^{-n} (x \cdot \nabla v)^2\Big) dx \\
&=: J[v].
\end{aligned}$$

Applying Lemma 4, we find that

$$
\begin{aligned}
J[v] = &\int_0^1 \int_{\mathbb{S}^{n-1}} r^3 v_{rr}^2 dS\,dr + \frac{n^2-4n+6}{2} \int_0^1 \int_{\mathbb{S}^{n-1}} r v_r^2 dS\,dr \\
&+ \int_0^1 \int_{\mathbb{S}^{n-1}} r^{-1} (\Delta_\omega v)^2 dS\,dr + 2\int_0^1 \int_{\mathbb{S}^{n-1}} |\nabla_\omega v_r|^2 r\,dS\,dr \\
&+ \frac{n(n-4)}{2} \int_0^1 \int_{\mathbb{S}^{n-1}} r^{-1} |\nabla_\omega v|^2 dS\,dr.
\end{aligned} \tag{3.1}
$$

In view of (3.1), we set

$$
\begin{aligned}
J_{\mathrm{R}}[v] &= \int_0^1 \int_{\mathbb{S}^{n-1}} r^3 v_{rr}^2 dS\,dr + \frac{n^2-4n+6}{2} \int_0^1 \int_{\mathbb{S}^{n-1}} r v_r^2 dS\,dr, \\
J_{\mathrm{NR}}[v] &= \int_0^1 \int_{\mathbb{S}^{n-1}} r^{-1} (\Delta_\omega v)^2 dS\,dr + 2\int_0^1 \int_{\mathbb{S}^{n-1}} r\,|\nabla_\omega v_r|^2 dS\,dr \\
&\quad + \frac{n(n-4)}{2} \int_0^1 \int_{\mathbb{S}^{n-1}} r^{-1} |\nabla_\omega v|^2 dS\,dr,
\end{aligned}
$$

the radial and non-radial parts of $J[v]$, so that

$$
J[v] = J_{\mathrm{R}}[v] + J_{\mathrm{NR}}[v].
$$

We will change variables once more and for this we define the functions

$$
g(r) = \exp\left(1 - X(r)^{-\frac{n}{2(n-1)}}\right), \quad \alpha(r) = X(r)^{-\frac{3(n-2)}{4(n-1)}} g(r)^{\frac{n-4}{2\beta}}. \tag{3.2}
$$

Lemma 6. *Let $n \geq 5$, $\beta > 0$ and set $s = \frac{n-4}{2\beta}$. Let $v \in C_c^\infty(B_1 \setminus \{0\})$. Changing variables by*

$$
v(r,\omega) = \alpha(r) w(t,\omega), \quad t = g(r), \tag{3.3}
$$

we have

$$
\begin{aligned}
J_{\mathrm{R}}[v] = &\int_0^1 \int_{\mathbb{S}^{n-1}} \Bigg\{ \left(\frac{n}{2(n-1)}\right)^3 t^{\frac{3\beta+n-4}{\beta}} w_{tt}^2 + t^{\frac{\beta+n-4}{\beta}} G(t) w_t^2 \\
&\qquad + t^{\frac{-\beta+n-4}{\beta}} H(t) w^2 \Bigg\} dS\,dt,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
J_{\mathrm{NR}}[v] = &\frac{2(n-1)}{n} \int_0^1 \int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{8-4n}{n}} (\Delta_\omega w)^2 dS\,dt \\
&+ \frac{n}{n-1} \int_0^1 \int_{\mathbb{S}^{n-1}} t^{\frac{n+\beta-4}{\beta}} X(t)^{\frac{4-2n}{n}} |\nabla_\omega w_t|^2 dS\,dt \\
&+ \int_0^1 \int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} |\nabla_\omega w|^2 K(t) dS\,dt,
\end{aligned} \tag{3.5}
$$

$$
\int_0^1 \int_{\mathbb{S}^{n-1}} r^{-1} X(r)^{\frac{2n-4}{n-4}} |v|^{\frac{2n}{n-4}} dS\,dr = \frac{2(n-1)}{n} \int_0^1 \int_{\mathbb{S}^{n-1}} |w|^{\frac{2n}{n-4}} t^{\frac{n-\beta}{\beta}} dS\,dt, \tag{3.6}
$$

where the functions $G(t)$, $H(t)$, and $K(t)$ are given by

$$G(t) = \frac{n(n^2-4n+8)}{4(n-1)}X(t)^{\frac{4-2n}{n}} - \frac{n^3(2s^2+2s+1)}{8(n-1)^3} + \frac{5n(n-2)(3n-2)}{16(n-1)^3}X(t)^2,$$

$$\begin{aligned} H(t) = &-\frac{s^2n(n^2-4n+8)}{4(n-1)}X(t)^{\frac{4-2n}{n}} + \frac{s(n-2)(n^2-4n+8)}{2(n-1)}X(t)^{\frac{4-n}{n}} \\ &+ \frac{s^4n^3}{8(n-1)^3} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)}X(t)^{\frac{4}{n}} \\ &- \frac{5s^2n(n-2)(3n-2)}{16(n-1)^3}X(t)^2 - \frac{5sn(n-2)(3n-2)}{8(n-1)^3}X(t)^3 \\ &- \frac{9(3n-2)(5n-2)(n^2-4)}{128n(n-1)^3}X(t)^4, \end{aligned}$$

$$\begin{aligned} K(t) = &(n-1)(n-4)X(t)^{\frac{8-4n}{n}} - \frac{n(n-4)^2}{4(n-1)\beta^2}X(t)^{\frac{4-2n}{n}} \\ &+ \frac{(n-2)(n-4)}{(n-1)\beta}X(t)^{\frac{4-n}{n}} + \frac{3(n^2-4)}{4n(n-1)}X(t)^{\frac{4}{n}}. \end{aligned}$$

Proof. To prove (3.4), we set for simplicity

$$J_{\mathrm{R}}^*[v] = \int_0^1 r^3 v_{rr}^2 dr + \frac{n^2-4n+6}{2}\int_0^1 r v_r^2 dr.$$

We first note that r and $t = g(r)$ are also related by the relation

$$X(t) = X(r)^{\frac{n}{2(n-1)}} \tag{3.7}$$

and that

$$dt = \frac{n}{2(n-1)}\frac{g(r)}{r}X(r)^{\frac{n-2}{2(n-1)}}dr.$$

Expressing $J_{\mathrm{R}}^*[v]$ in terms of the function $w(t)$ involves some lengthy computations, of which we include only the main steps.

From (3.3), we have

$$\begin{aligned} v_r &= \alpha g' w_t + \alpha' w, \\ v_{rr} &= \alpha g'^2 w_{tt} + (2\alpha' g' + \alpha g'')w_t + \alpha'' w. \end{aligned}$$

Substituting in $J_{\mathrm{R}}^*[v]$ and expanding, we find that

$$\begin{aligned} J_{\mathrm{R}}^*[v] = &\left(\frac{n}{2(n-1)}\right)^3 \int_0^1 t^{\frac{3\beta+n-4}{\beta}} w_{tt}^2 dt + \int_0^1 B(t)w_t^2 dt + \int_0^1 C(t)w^2 dt \\ &+ \int_0^1 D(t)w_{tt}\, w_t dt + \int_0^1 E(t)w_{tt} w\, dt + \int_0^1 F(t)w_t\, w\, dt, \end{aligned} \tag{3.8}$$

where the functions $B(t), \ldots, F(t)$ will be described below in terms of the variable r. Integrating by parts, we obtain from (3.8) that

$$J_{\mathrm{R}}^*[v] = \left(\frac{n}{2(n-1)}\right)^3 \int_0^1 t^{\frac{3\beta+n-4}{\beta}} w_{tt}^2 dt + \int_0^1 P(t) w_t^2 dt + \int_0^1 Q(t) w^2 dt,$$

where

$$\begin{aligned} P(t) &= B(t) - \frac{1}{2} D_t(t) - E(t), \\ Q(t) &= C(t) + \frac{1}{2} E_{tt}(t) - \frac{1}{2} F_t(t). \end{aligned} \tag{3.9}$$

To compute the functions $P(t)$ and $Q(t)$ it is convenient to regard them as functions of the variable r. To do this, we consider the functions B, C, D, E, and F also as functions of r and indicate this with tildes; we will thus write $B(t) = \widetilde{B}(r)$, etc. Relations (3.9) then take the form

$$\begin{aligned} \widetilde{P}(r) &= \widetilde{B} - \frac{1}{2g'} \widetilde{D}_r - \widetilde{E}, \\ \widetilde{Q}(r) &= \widetilde{C} + \frac{1}{2}\left(\frac{\widetilde{E}_{rr}}{g'^2} - \frac{g'' \widetilde{E}_r}{g'^3}\right) - \frac{1}{2g'} \widetilde{F}_r. \end{aligned} \tag{3.10}$$

After some computations, we eventually find

$$\begin{aligned} \widetilde{B}(r) &= \frac{r^3}{g'}\left(2\alpha' g' + \frac{n-1}{r}\alpha g' + \alpha g''\right)^2 - \frac{n(n-4)}{2} r \alpha^2 g', \\ \widetilde{C}(r) &= \frac{r^3}{g'}\left(\alpha'' + \frac{n-1}{r}\alpha'\right)^2 - \frac{n(n-4)}{2}\frac{r}{g'}\alpha'^2, \\ \widetilde{D}(r) &= 2r^3 \alpha g'\left(2\alpha' g' + \frac{n-1}{r}\alpha g' + \alpha g''\right), \\ \widetilde{E}(r) &= 2r^3 \alpha g'\left(\alpha'' + \frac{n-1}{r}\alpha'\right), \\ \widetilde{F}(r) &= 2r^3\left(2\alpha' + \frac{n-1}{r}\alpha + \alpha \frac{g''}{g'}\right)\left(\alpha'' + \frac{n-1}{r}\alpha'\right) - n(n-4) r \alpha \alpha'. \end{aligned}$$

Substituting in (3.10), we arrive at

$$\begin{aligned} \widetilde{P}(r) &= 2r^3 \alpha'^2 g' - 6r^2 \alpha\alpha' g' + \frac{n^2-4n+6}{2} r \alpha^2 g' - 3r^2 \alpha^2 g'' \\ &\quad - 4r^3 \alpha\alpha'' g' - 2r^3 \alpha\alpha' g'' - r^3 \alpha^2 g''', \\ \widetilde{Q}(r) &= \frac{1}{g'}\left(6r^2 \alpha\alpha''' - \frac{n^2-4n-6}{2} r \alpha\alpha'' - \frac{n^2-4n+6}{2}\alpha\alpha' + r^3 \alpha\alpha^{(4)}\right). \end{aligned}$$

Now, some more computations give

$$g'(r) = \frac{n}{2(n-1)}\frac{g(r)}{r}X(r)^{\frac{n-2}{2(n-1)}},$$

$$g''(r) = \Big(-\frac{n}{2(n-1)}X^{\frac{n-2}{2(n-1)}} + \frac{n^2}{4(n-1)^2}X(r)^{\frac{n-2}{n-1}} + \frac{n(n-2)}{4(n-1)^2}X(r)^{\frac{3n-4}{2(n-1)}}\Big)\frac{g(r)}{r^2},$$

$$\begin{aligned} g'''(r) = \Big(&-\frac{3n(n-2)}{4(n-1)^2}X(r)^{\frac{3n-4}{2(n-1)}} + \frac{3n^2(n-2)}{8(n-1)^3}X^{\frac{2n-3}{n-1}} + \frac{n(n-2)(3n-4)}{8(n-1)^3}X^{\frac{5n-6}{2(n-1)}} \\ &+ \frac{n}{n-1}X(r)^{\frac{n-2}{2(n-1)}} - \frac{3n^2}{4(n-1)^2}X(r)^{\frac{n-2}{n-1}} + \frac{n^3}{8(n-1)^3}X(r)^{\frac{3n-6}{2(n-1)}}\Big)\frac{g(r)}{r^3}. \end{aligned}$$

Moreover,

$$\alpha'(r) = \frac{g(r)^s}{r}\Big(\frac{s}{2(n-1)}X^{\frac{2-n}{4(n-1)}} - \frac{3(n-2)}{4(n-1)}X(r)^{\frac{n+2}{4(n-1)}}\Big),$$

$$\begin{aligned} \alpha''(r) = \frac{g(r)^s}{r^2}\Big(&-\frac{sn}{2(n-1)}X(r)^{\frac{2-n}{4(n-1)}} + \frac{s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}} \\ &+ \frac{3(n-2)}{4(n-1)}X(r)^{\frac{n+2}{4(n-1)}} - \frac{sn(n-2)}{2(n-1)^2}X(r)^{\frac{3n-2}{4(n-1)}} - \frac{3(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}}\Big), \end{aligned}$$

$$\begin{aligned} \alpha'''(r) = \frac{g(r)^s}{r^3}\Big(&\frac{sn}{n-1}X^{\frac{2-n}{4(n-1)}} - \frac{3s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}} \\ &- \frac{3(n-2)}{2(n-1)}X(r)^{\frac{n+2}{4(n-1)}} + \frac{s^3n^3}{8(n-1)^3}X^{\frac{3n-6}{4(n-1)}} + \frac{3sn(n-2)}{2(n-1)^2}X^{\frac{3n-2}{4(n-1)}} \\ &- \frac{3s^2n^2(n-2)}{16(n-1)^3}X(r)^{\frac{5n-6}{4(n-1)}} + \frac{9(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}} \\ &- \frac{sn(n-2)(15n-2)}{32(n-1)^3}X(r)^{\frac{7n-6}{4(n-1)}} - \frac{3(n^2-4)(5n-2)}{64(n-1)^3}X(r)^{\frac{9n-6}{4(n-1)}}\Big) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \alpha^{(4)}(r) = \frac{g(r)^s}{r^4}\Big(&\frac{3sn}{n-1}X(r)^{\frac{2-n}{4(n-1)}} - \frac{11s^2n^2}{4(n-1)^2}X(r)^{\frac{n-2}{4(n-1)}} \\ &- \frac{9(n-2)}{2(n-1)}X(r)^{\frac{n+2}{4(n-1)}} + \frac{3s^3n^3}{4(n-1)^3}X(r)^{\frac{3n-6}{4(n-1)}} \\ &+ \frac{11sn(n-2)}{2(n-1)^2}X(r)^{\frac{3n-2}{4(n-1)}} - \frac{s^4n^4}{16(n-1)^4}X(r)^{\frac{5n-10}{4(n-1)}} \\ &- \frac{9s^2n^2(n-2)}{8(n-1)^3}X(r)^{\frac{5n-6}{4(n-1)}} + \frac{33(n^2-4)}{16(n-1)^2}X(r)^{\frac{5n-2}{4(n-1)}} \\ &- \frac{3sn(n-2)(15n-2)}{16(n-1)^3}X(r)^{\frac{7n-6}{4(n-1)}} + \frac{5s^2n^2(n-2)(3n-2)}{32(n-1)^4}X(r)^{\frac{9n-10}{4(n-1)}} \\ &- \frac{9(5n-2)(n^2-4)}{32(n-1)^3}X(r)^{\frac{9n-6}{4(n-1)}} + \frac{5sn^2(n-2)(3n-2)}{16(n-1)^4}X(r)^{\frac{11n-10}{4(n-1)}} \\ &+ \frac{9(3n-2)(5n-2)(n^2-4)}{256(n-1)^4}X(r)^{\frac{13n-10}{4(n-1)}}\Big). \end{aligned}$$

Combining the above, we eventually arrive at

$$\widetilde{P}(r) = g(r)^{\frac{\beta+n-4}{\beta}}\Big(\frac{n(n^2-4n+8)}{4(n-1)}X(r)^{\frac{2-n}{n-1}} - \frac{n^3(2s^2+2s+1)}{8(n-1)^3} + \frac{5n(n-2)(3n-2)}{16(n-1)^3}X(r)^{\frac{n}{n-1}}\Big)$$

and

$$\begin{aligned}\widetilde{Q}(r) = g(r)^{\frac{-\beta+n-4}{\beta}}\Big(&-\frac{s^2n(n^2-4n+8)}{4(n-1)}X(r)^{\frac{2-n}{n-1}} + \frac{s(n-2)(n^2-4n+8)}{2(n-1)}X(r)^{\frac{4-n}{2(n-1)}}\\ &+\frac{s^4n^3}{8(n-1)^3} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)}X(r)^{\frac{2}{n-1}}\\ &-\frac{5s^2n(n-2)(3n-2)}{16(n-1)^3}X(r)^{\frac{n}{n-1}} - \frac{5sn(n-2)(3n-2)}{8(n-1)^3}X(r)^{\frac{3n}{2(n-1)}}\\ &-\frac{9(3n-2)(5n-2)(n^2-4)}{128n(n-1)^3}X(r)^{\frac{2n}{n-1}}\Big).\end{aligned}$$

Equation (3.4) now follows by recalling (3.7) and noting that

$$P(t) = t^{\frac{\beta+n-4}{\beta}}G(t), \quad Q(t) = t^{\frac{-\beta+n-4}{\beta}}H(t).$$

To prove equation (3.5), we first note that

$$\begin{aligned}\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}(\Delta_\omega v)^2 dS\,dr &= \int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}\alpha(r)^2(\Delta_\omega w)^2\frac{1}{g'(r)}dS\,dt\\ &= \frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}}X(t)^{\frac{8-4n}{n}}(\Delta_\omega w)^2 dS\,dt,\end{aligned}$$

and similarly,

$$\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}|\nabla_\omega v|^2 dS\,dr = \frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}}X(t)^{\frac{8-4n}{n}}|\nabla_\omega w|^2 dS\,dt.$$

For the remaining term in $J_{\mathrm{NR}}[v]$, we compute

$$\begin{aligned}&\int_0^1\int_{\mathbb{S}^{n-1}} r|\nabla_\omega v_r|^2 dS\,dr\\ &= \int_0^1\int_{\mathbb{S}^{n-1}} r\alpha^2 g'|\nabla_\omega w_t|^2 dS\,dt - \int_0^1\int_{\mathbb{S}^{n-1}} |\nabla_\omega w|^2\frac{1}{g'}(\alpha\alpha'' r + \alpha\alpha')dS\,dt.\end{aligned}$$

On the one hand, we have

$$\int_0^1\int_{\mathbb{S}^{n-1}} \alpha^2 g' r|\nabla_\omega w_t|^2 dS\,dt = \frac{n}{2(n-1)}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n+\beta-4}{\beta}}X(t)^{\frac{4-2n}{n}}|\nabla_\omega w_t|^2 dS\,dt,$$

and on the other hand, recalling (3.11),

$$\begin{aligned}
&\int_0^1 \int_{S^{n-1}} |\nabla_\omega w|^2 \frac{1}{g'}(\alpha\alpha'' r + \alpha\alpha')\, dS\, dt \\
&\quad = \frac{n(n-4)^2}{8(n-1)\beta^2} \int_0^1 \int_{S^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4-2n}{n}} |\nabla_\omega w|^2\, dS\, dt \\
&\quad\quad - \frac{(n-2)(n-4)}{2(n-1)\beta} \int_0^1 \int_{S^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4-n}{n}} |\nabla_\omega w|^2\, dS\, dt \\
&\quad\quad - \frac{3(n^2-4)}{8n(n-1)} \int_0^1 \int_{S^{n-1}} t^{\frac{n-\beta-4}{\beta}} X(t)^{\frac{4}{n}} |\nabla_\omega w|^2\, dS\, dt.
\end{aligned}$$

Combining the above, we obtain (3.5). The proof of (3.6) is much simpler and is omitted. ■

To proceed we define

$$G^\#(t) = G(t) - \Big(\frac{n}{2(n-1)}\Big)^3 A, \quad t \in (0,1),$$

where we recall that A has been defined in Lemma 5.

Lemma 7. *Let $v \in C_c^\infty(B_1 \setminus \{0\})$ and let w be defined by (3.3). There holds*

$$\begin{aligned}
J_R[v] &= \Big(\frac{n}{2(n-1)}\Big)^3 \mathcal{A}_R[w] \\
&\quad + \int_0^1 \int_{S^{n-1}} t^{\frac{\beta+n-4}{\beta}} w_t^2 G^\#(t) dS\, dt + \int_0^1 \int_{S^{n-1}} t^{\frac{-\beta+n-4}{\beta}} w^2 H(t) dS\, dt.
\end{aligned}$$

Proof. This is a direct consequence of Lemma 6, equation (3.4). ■

Lemma 8. *Let $n \geq 5$. If*

$$\beta \geq \beta_n := n\Big(\frac{n^2-4n+8}{4n^4-24n^3+83n^2-120n+52}\Big)^{1/2}, \tag{3.12}$$

then the function $G^\#(t)$ is non-negative in $(0,1)$.

Proof. We first note that

$$\begin{aligned}
G^\#(t) &= \frac{n(n^2-4n+8)}{4(n-1)} X(t)^{\frac{4-2n}{n}} - \frac{n^3(n^2-4n+8)}{16(n-1)^3\beta^2} + \frac{5n(n-2)(3n-2)}{16(n-1)^3} X(t)^2 \\
&=: p_1 X(t)^{\frac{4-2n}{n}} + p_2 + p_3 X(t)^2.
\end{aligned}\tag{3.13}$$

Now, it easily follows from (3.13) that $G^\#(t)$ is monotone decreasing in $(0,1]$. Hence, its minimum is equal to

$$p_1 + p_2 + p_3 = \frac{n(4n^4-24n^3+83n^2-120n+52)}{16(n-1)^3} - \frac{n^3(n^2-4n+8)}{16(n-1)^3\beta^2},$$

which is non-negative if $\beta \geq \beta_n$. ■

Lemma 9. *Let $n \geq 5$ and $\beta \geq \beta_n$. For any $w \in C_c^\infty(0,1)$, there holds*

$$\int_0^1 t^{\frac{\beta+n-4}{\beta}} G^\#(t) w_t^2 dt + \int_0^1 t^{\frac{-\beta+n-4}{\beta}} H^\#(t) w^2 dt \geq 0,$$

where

$$\begin{aligned} H^\#(t) = &-\frac{n(n-4)^2(n^2-4n+8)}{16(n-1)\beta^2} X^{\frac{4-2n}{n}} + \frac{(n-2)(n-4)(n^2-4n+8)}{4(n-1)\beta} X^{\frac{4-n}{n}} \\ &+ \frac{n^3(n-4)^2(n^2-4n+8)}{64(n-1)^3\beta^4} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)} X^{\frac{4}{n}} \\ &- \frac{n(n-2)(15n^3-104n^2+256n-152)}{32(n-1)^3\beta^2} X^2 \\ &- \frac{5n(n-2)(n-4)(3n-2)}{16(n-1)^3\beta} X^3 + \frac{45(n-2)^2(3n-2)^2}{n(n-1)^3} X^4. \end{aligned}$$

Proof. Let r_1, r_2 be real numbers to be fixed later. We have

$$\begin{aligned} 0 &\leq \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^\#(t) \left(w_t + \frac{r_1 + r_2 X(t)}{t} w \right)^2 dt \\ &= \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^\#(t) w_t^2 dt + \int_0^1 \Big\{ t^{\frac{-\beta+n-4}{\beta}} G^\#(t)(r_1^2 + 2r_1 r_2 X + r_2^2 X^2) \\ &\qquad - \left(t^{\frac{n-4}{\beta}} G^\#(t)(r_1 + r_2 X(t)) \right)_t \Big\} w^2 dt. \end{aligned}$$

Substituting from (3.13) and carrying out the computations, we arrive at

$$\begin{aligned} 0 \leq &\int_0^1 t^{\frac{\beta+n-4}{\beta}} G^\#(t) w_t^2 dt \\ &+ \int_0^1 t^{\frac{-\beta+n-4}{\beta}} \Big\{ p_1 r_1 \left(r_1 - \frac{n-4}{\beta} \right) X^{\frac{4-2n}{n}} + p_1 \left(2r_1 r_2 - r_2 \frac{n-4}{\beta} + \frac{2n-4}{n} r_1 \right) X^{\frac{4-n}{n}} \\ &\qquad + p_2 r_1 \left(r_1 - \frac{n-4}{\beta} \right) + p_1 r_2 \left(r_2 + \frac{n-4}{n} \right) X^{\frac{4}{n}} + p_2 r_2 \left(2r_1 - \frac{n-4}{\beta} \right) X \\ &\qquad + \left(p_2 r_2^2 - p_2 r_2 + p_3 r_1^2 - p_3 r_1 \frac{n-4}{\beta} \right) X^2 \\ &\qquad + \left(2p_3 r_1 r_2 - 2p_3 r_1 - p_3 r_2 \frac{n-4}{\beta} \right) X^3 \\ &\qquad + (p_3 r_2^2 - 3p_3 r_2) X^4 \Big\} w^2 dt. \end{aligned}$$

We now choose

$$r_1 = \frac{n-4}{2\beta}, \quad r_2 = -\frac{3(n-2)}{2n}.$$

The choice for r_1 minimizes the coefficient of the leading term in the last integral; the parameter r_2 is less important and the choice is made for convenience. Substituting, we obtain

$$
\begin{aligned}
0 \le & \int_0^1 t^{\frac{\beta+n-4}{\beta}} G^{\#}(t) w_t^2 dt \\
& + \int_0^1 t^{\frac{-\beta+n-4}{\beta}} \Bigg\{ -\frac{n(n-4)^2(n^2-4n+8)}{16(n-1)\beta^2} X^{\frac{4-2n}{n}} \\
& \qquad + \frac{(n-2)(n-4)(n^2-4n+8)}{4(n-1)\beta} X^{\frac{4-n}{n}} \\
& \qquad + \frac{n^3(n-4)^2(n^2-4n+8)}{64(n-1)^3\beta^4} + \frac{3(n^2-4)(n^2-4n+8)}{16n(n-1)} X^{\frac{4}{n}} \\
& \qquad - \frac{n(n-2)(15n^3-104n^2+256n-152)}{32(n-1)^3\beta^2} X^2 \\
& \qquad - \frac{5n(n-2)(n-4)(3n-2)}{16(n-1)^3\beta} X^3 \\
& \qquad + \frac{45(n-2)^2(3n-2)^2}{n(n-1)^3} X^4 \Bigg\} w^2 dt,
\end{aligned}
$$

which is the stated inequality. ■

We next define the positive constants

$$
\gamma_1 = \frac{n^6(n-4)^2}{256(n-1)^4}, \quad \gamma_2 = \frac{3n^2(n-2)(5n-6)(n^2-4n+8)}{128(n-1)^4},
$$
$$
\gamma_3 = \frac{9(n-2)(3n-2)(5n-6)(7n-6)}{256(n-1)^4}. \tag{3.14}
$$

Lemma 10. *Let $n \ge 5$ and $\beta \ge \beta_n$. Let $v \in C_c^\infty(B_1 \setminus \{0\})$ and let w be defined by (3.3). We then have*

$$
\begin{aligned}
& J_{\mathrm{R}}[v] + \int_0^\infty \int_{S^{n-1}} v^2 r^{-1} \left(\frac{\gamma_1}{\beta^4} X(r)^{\frac{2(n-2)}{n-1}} - \frac{\gamma_2}{\beta^2} X(r)^{\frac{3n-4}{n-1}} + \gamma_3 X(r)^4 \right) dS\, dt \\
& \ge \left(\frac{n}{2(n-1)} \right)^3 \mathcal{A}_{\mathrm{R}}[w].
\end{aligned}
$$

Proof. From Lemmas 7 and 9, we have

$$
J_{\mathrm{R}}[v] \ge \left(\frac{n}{2(n-1)} \right)^3 \mathcal{A}_{\mathrm{R}}[w] + \int_0^1 \int_{S^{n-1}} t^{\frac{n-\beta-4}{\beta}} w^2 (H(t) - H^{\#}(t)) dS\, dt.
$$

But we easily see that

$$\frac{n}{2(n-1)}(H(t)-H^{\#}(t)) = -\frac{\gamma_1}{\beta^4}+\frac{\gamma_2}{\beta^2}X(t)^2-\gamma_3 X(t)^4,$$

hence,

$$J_{\mathrm{R}}[v]+\frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} t^{\frac{n-\beta-4}{\beta}} w^2\Big(\frac{\gamma_1}{\beta^4}-\frac{\gamma_2}{\beta^2}X(t)^2+\gamma_3 X(t)^4\Big)dS\,dt \\ \geq \Big(\frac{n}{2(n-1)}\Big)^3 \mathcal{A}_{\mathrm{R}}[w].$$

We now express the double integral above in terms of the function v using once again (3.3). We note that for any $\sigma \geq 0$, we have

$$\int_0^1 t^{\frac{n-\beta-4}{\beta}} w^2 X(t)^{\sigma} dt = \frac{n}{2(n-1)}\int_0^1 r^{-1}v^2 X(r)^{\frac{\sigma n+4(n-2)}{2(n-1)}}dr.$$

Applying this for $\sigma = 0, 2, 4$, we obtain the required inequality. ∎

Proof of Theorem 2. Let $u \in C_c^\infty(\Omega)$. Without loss of generality, we may assume that $\Omega = B_1$ and that $u \in C_c^\infty(B_1 \setminus \{0\})$. Let $v = |x|^{\frac{n-4}{2}}u$. By the discussion following Lemma 5, the required inequality is written as

$$\frac{J_{\mathrm{R}}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}dS\,dr + J_{\mathrm{NR}}[v]}{\Big(\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr\Big)^{\frac{n-4}{n}}} \geq S_{2,n}.$$

We make the choice

$$\beta = \frac{n}{2(n-1)}.$$

We will prove the following two inequalities where v and w are related by the change of variables (3.3):

$$J_{\mathrm{R}}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}dS\,dr \geq \Big(\frac{n}{2(n-1)}\Big)^3\mathcal{A}_{\mathrm{R}}[w], \tag{3.15}$$

$$J_{\mathrm{NR}}[v] \geq \Big(\frac{n}{2(n-1)}\Big)^3\mathcal{A}_{\mathrm{NR}}[w]. \tag{3.16}$$

We claim that if these are proved then the result will follow. Indeed, by Lemma 6, equation (3.6) the Sobolev terms are related by

$$\int_0^1\int_{\mathbb{S}^{n-1}} r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}dS\,dr = \frac{2(n-1)}{n}\int_0^1\int_{\mathbb{S}^{n-1}} |w|^{\frac{2n}{n-4}}t^{\frac{n-\beta}{\beta}}dS\,dt.$$

Hence, applying Lemma 5, we obtain

$$\frac{J_{\mathrm{R}}[v]+\frac{n^2(n-4)^2}{16}\int_0^1\int_{S^{n-1}} r^{-1}v^2X(r)^{\frac{2(n-2)}{n-1}}\,dS\,dr+J_{\mathrm{NR}}[v]}{\left(\int_0^1\int_{S^{n-1}} r^{-1}X(r)^{\frac{2n-4}{n-4}}|v|^{\frac{2n}{n-4}}\,dS\,dr\right)^{\frac{n-4}{n}}}$$

$$\geq\left(\frac{n}{2(n-1)}\right)^{\frac{4(n-1)}{n}}\frac{\mathcal{A}_{\mathrm{R}}[w]+\mathcal{A}_{\mathrm{NR}}[w]}{\left(\int_0^1\int_{S^{n-1}}|w|^{\frac{2n}{n-4}}t^{\frac{n-\beta}{\beta}}\,dS\,dt\right)^{\frac{n-4}{n}}}$$

$$\geq\left(\frac{n}{2(n-1)\beta}\right)^{\frac{4(n-1)}{n}}S_{2,n}=S_{2,n},$$

and the proof is complete.

Proof of (3.15). For the specific choice of β, we have

$$\frac{\gamma_1}{\beta^4}X(r)^{\frac{2(n-2)}{n-1}}-\frac{\gamma_2}{\beta^2}X(r)^{\frac{3n-4}{n-1}}+\gamma_3X(r)^4$$

$$=\frac{\gamma_1}{\beta^4}X(r)^{\frac{2(n-2)}{n-1}}\left(1-\frac{\gamma_2}{\gamma_1}\beta^2X(r)^{\frac{n}{n-1}}+\frac{\gamma_3}{\gamma_1}\beta^4X(r)^{\frac{2n}{n-1}}\right)$$

$$=\frac{n^2(n-4)^2}{16}X(r)^{\frac{2(n-2)}{n-1}}\left(1-\frac{3(n-2)(5n-6)(n^2-4n+8)}{2n^2(n-1)^2(n-4)^2}X(r)^{\frac{n}{n-1}}\right.$$
$$\left.+\frac{9(n-2)(3n-2)(5n-6)(7n-6)}{16n^2(n-1)^4(n-4)^2}X(r)^{\frac{2n}{n-1}}\right).$$

The function

$$y\mapsto 1-\frac{3(n-2)(5n-6)(n^2-4n+8)}{2n^2(n-1)^2(n-4)^2}y+\frac{9(n-2)(3n-2)(5n-6)(7n-6)}{16n^2(n-1)^4(n-4)^2}y^2$$

is convex and its values at the endpoints $y=0$ and $y=1$ do not exceed one. Noting that

$$n/(2n-2)>\beta_n,$$

the result follows by Lemma 10. ∎

Proof of (3.16). We recall that the functional $\mathcal{A}_{\mathrm{NR}}[w]$ has been defined in Lemma 5 and the functional $J_{\mathrm{NR}}[v]$ is expressed in terms of the function w in Lemma 6.

We observe that the coefficients of the terms involving $(\Delta_\omega w)^2$ in the two sides of (3.16) are equal. The same is true for the coefficients of the terms involving $|\nabla_\omega w_t|^2$. Hence, the result will follow if we establish that

$$K(t)\geq\left(\frac{n}{2(n-1)}\right)^3\cdot\frac{2(n-4)}{\beta^4}=\frac{4(n-1)(n-4)}{n}.$$

Indeed, the first two terms of $K(t)$ are enough for this; that is, there holds

$$(n-1)(n-4)X(t)^{\frac{8-4n}{n}} - \frac{(n-1)(n-4)^2}{n}X(t)^{\frac{4-2n}{n}} - \frac{4(n-1)(n-4)}{n} \geq 0$$

for all $t \in (0,1)$. This completes the proof of the Rellich–Sobolev inequality of Theorem 2.

The sharpness of the constant $S_{2,n}$ in the Rellich–Sobolev inequality follows easily by concentrating near a point $x_0 \in \partial\Omega$ with $|x_0| = D$. ■

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