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Spectral instabilities: variations on a theme loved by Brian Davies

Albrecht Böttcher

Abstract. Minor perturbations to a linear operator may drastically change its spectrum, and hence the difficulty of deciding whether or not a numerically computed quantity is zero causes problems in spectral theory. The purpose of this expository paper is to illustrate such instability phenomena by some examples with Toeplitz-like operators and matrices.

For Brian Davies on his 80th birthday

1. The theme

Instability of the spectrum under tiny perturbations to the operator, though being a very old topic, is one of the favorite topics studied by Brian Davies. See, e.g., [1,19,20] and [22–33]. I love this topic, too. The opportunity to make a contribution to this birthday issue motivated me to embark on the subject once more and to illustrate it by some insights I have gained in my work and which are scattered in several publications. To put it into an analogy with music, I want to refer to the topic as a theme and to consider this paper as a set of variations on the theme. As the instrument I can play best is Toeplitz operators and matrices, the variations will all have a Toeplitz tune.

But let us begin with the theme. Here it is as it appears in Brian's paper [27]. "If $c \in \mathbf{R}$ then the operator $A: \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ defined by

$$(Af)_n = \begin{cases} cf_{n+1} & \text{if } n = 0, \\ f_{n+1} & \text{otherwise} \end{cases}$$

has classical spectrum $\{z : |z| = 1\}$ if $c \neq 0$ and classical spectrum $\{z : |z| \leq 1\}$ if $c = 0$. If c is a very small constructively defined real number and one is not able to determine whether or not $c = 0$, then the spectrum of A cannot be computed even

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approximately even though A is well defined constructively. This implies that there exist straightforward bounded operators whose spectrum will probably never be determined."

The example with the operator A cited by Brian can already be found as Problem 85 in Halmos' Hilbert space problem book [37], where it is attributed to Günter Lumer. In the second edition of the book, by Springer in 1982, it is Problem 102.

2. Variation one

For $c = 1$, the operator A we encounter in the *Theme* is a Laurent operator. A Laurent operator is given by a doubly-infinite matrix $(a_{j-k})_{j,k=-\infty}^{\infty}$ on $\ell^2(\mathbf{Z})$. This matrix induces a bounded operator if and only if the sequence $\{a_k\}_{k=-\infty}^{\infty}$ is the sequence of the Fourier coefficients of a function a in L^∞ over the unit circle $\mathbf{T} = \{z : |z| = 1\}$,

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta \quad (k \in \mathbf{Z}).$$

In that case, we denote the corresponding Laurent matrix and the operator it defines on $\ell^2(\mathbf{Z})$ by $L(a)$ and refer to the function a as the symbol. The Laurent operator $L(a)$ is unitarily similar to multiplication by a on $L^2(\mathbf{T})$, and hence both the spectrum $\sigma(L(a))$ and the essential spectrum $\sigma_{\text{ess}}(L(a))$ are equal to the essential range $\mathcal{R}(a)$ of a. Recall that the essential spectrum is the set of all complex numbers λ for which $L(a) - \lambda I = L(a - \lambda)$ is not Fredholm, that is, not invertible modulo compact operators.

Thus, the operator A in the *Theme* may be written as $A = L(z^{-1}) + (c-1)E_{0,1}$ where $E_{j,k}$ denotes the doubly-infinite matrix whose j, k entry is 1 and all other entries of which are zero. Let us consider the slightly more general operator

$$A_{c,b} = L(z^{-1}) + (c-1)E_{0,1} + bE_{-1,1},$$

which results from $A = A_{c,0}$ by putting a b at the matrix site immediately above c. Alternatively, we may define the operator $A_{c,b}$ by $(A_{c,b} f)_{-1} = f_0 + bf_1$, $(A_{c,b} f)_0 = cf_1$, and $(A_{c,b} f)_n = f_{n+1}$ for $n \notin \{-1, 0\}$. Since $\sigma_{\text{ess}}(L(z^{-1})) = \mathbf{T}$ and $A_{c,b}$ is at most a rank 2 perturbation of $L(z^{-1})$, we conclude that $\mathbf{T}$ is always a subset of the spectrum $\sigma(A_{c,b})$. Now, let $|\lambda| < 1$. Then $L(z^{-1} - \lambda)$ is invertible and we have

$$A_{c,b} - \lambda I = L(z^{-1} - \lambda)\{I + [L(z^{-1} - \lambda)]^{-1}[(c-1)E_{0,1} + bE_{-1,1}]\}.$$

As $[L(z^{-1} - \lambda)]^{-1} = L(z + \lambda z^2 + \lambda^2 z^3 + \cdots)$, a simple computation shows that the operator in braces is invertible if and only if so is the matrix $\begin{pmatrix} 1 & b \\ 0 & \lambda b + c \end{pmatrix}$, that is, if and only if $\lambda b + c \neq 0$. This gives us the part of the spectrum of $A_{c,b}$ contained in the unit disk $\mathbf{D} = \{\lambda : |\lambda| < 1\}$. If $|\lambda| > 1$, then $[L(z^{-1} - \lambda)]^{-1}$ is upper triangular and the operator in braces is always invertible. Consequently, no point outside of $\mathbf{T}$ belongs to the spectrum of $A_{c,b}$. In summary, we have the following:

$$\sigma(A_{c,b}) = \begin{cases} \mathbf{T} & \text{if } b = 0 \text{ and } c \neq 0 \text{ or if } b \neq 0 \text{ and } |c| \geq |b|, \\ \mathbf{T} \cup \{-c/b\} & \text{if } b \neq 0 \text{ and } |c| < |b|, \\ \mathbf{T} \cup \mathbf{D} & \text{if } b = c = 0. \end{cases}$$

3. Variation two

This variation is based on paper [7]. Let $a(z) = \sum_{j=-r}^{r} a_j z^j$ be a Laurent polynomial and consider the Laurent operator $L(a)$ on $\ell^2(\mathbf{Z})$. The matrix of $L(a)$ is banded. We assume that at least one of the coefficients a_r and a_{-r} is nonzero. Let P_n be the projection on $\ell^2(\mathbf{Z})$ defined by $(P_n f)_j = f_j$ for $j \in \{0, 1, \ldots, n-1\}$ and $(P_n f)_j = 0$ otherwise. If A is given by an infinite matrix on $\ell^2(\mathbf{Z})$, then the operator $P_n A P_n$ may be identified with an $n \times n$ matrix in a natural fashion. It is well known (and easily seen by taking, e.g., $a(z) = z$) that the spectrum of $P_n L(a) P_n$ does in general not approximate the spectrum of $L(a)$ as $n \to \infty$.

For $n \geq 2r + 1$, let $C_n(a)$ be the $n \times n$ circulant matrix whose first row is

$$(a_0 \; a_{-1} \; \ldots \; a_{-r} \; 0 \; \ldots \; 0 \; a_r \; a_{r-1} \; \ldots \; a_1).$$

The spectrum of $C_n(a)$ can be shown to be $a(\mathbf{T}_n)$, where $\mathbf{T}_n$ is the set of the nth roots of 1, and hence $\sigma(C_n(a))$ approximates $\sigma(L(a)) = a(\mathbf{T})$.

Consider now the operator $L(a) + K$ where the matrix of K has only finitely many nonzero entries. Since $\sigma_{\text{ess}}(L(a)) = a(\mathbf{T})$, we have $\sigma(L(a) + K) = a(\mathbf{T}) \cup X$ with some (possibly empty) set X. The question is whether we can somehow approximate $\sigma(L(a) + K)$ by the eigenvalues of the matrices $C_n(a) + P_n K P_n$. The *Theme* shows that X may contain entire connected components of $\mathbf{C} \setminus a(\mathbf{T})$, and one expects that these cannot be exhausted by approximations. Indeed, if $L(a) + K = L(z^{-1}) + (c-1)E_{0,1}$ is the operator $A_{c,0}$ we met above, then $\sigma(C_n(a) + P_n K P_n)$ can be shown to be the set $\{\lambda : \lambda^n = c\}$, which converges to $\sigma(L(a) + K) = \mathbf{T}$ for $c \neq 0$ but does not converge to $\sigma(L(a) + K) = \mathbf{T} \cup \mathbf{D}$ for $c = 0$.

The following result of [7] tells us that we can find $X \cap G$ by approximations if G is a connected component of $\mathbf{C} \setminus a(\mathbf{T})$ that does not entirely belong to the spectrum of $L(a) + K$. Convergence of plane sets is understood as convergence of compact sets in the Hausdorff metric. We denote by $\overline{G}$ the closure of G and by $\partial G = \overline{G} \setminus G$ the boundary of G.

Theorem 1. *If a connected component G of $\mathbf{C} \setminus a(\mathbf{T})$ is not entirely contained in the spectrum of $L(a) + K$, then*

$$\lim_{n\to\infty} ((\sigma(C_n(a) + P_n K P_n) \cap G) \cup \partial G) = \sigma(L(a) + K) \cap \bar{G}. \tag{1}$$

Equality (1) holds in particular if G is the unbounded component of $\mathbf{C} \setminus a(\mathbf{T})$. In the case of a single-entry perturbation $K = \omega E_{j,k}$ the reasoning of *Variation 1* shows $\sigma(L(a) + \omega E_{j,k})$ is the union of $a(\mathbf{T})$ and the set X of all $\lambda \in \mathbf{C}$ for which $1 + [(a-\lambda)^{-1}]_{k-j}\omega = 0$, where $[(a-\lambda)^{-1}]_{k-j}$ denotes $(k-j)$th Fourier coefficient of $1/(a-\lambda)$. If $[(a-\lambda)^{-1}]_{k-j}$ equals a constant c throughout G and $1 + c\omega = 0$, then all of G is contained in X and Theorem 1 is not applicable. However, if $[(a-\lambda)^{-1}]_{k-j}$ is either identically zero in G or assumes at least two different values in G, which is the same as requiring that $[(a-\lambda)^{-1}]_{k-j}$ is not a nonzero constant in G, then the hypothesis of Theorem 1 is satisfied and hence (1) holds.

4. Variation three

In this variation we follow [7, 8]. Let the symbol a be a Laurent polynomial as above and recall that $E_{j,k}$ stands for the infinite matrix with 1 at site j, k and zeros elsewhere. We here consider the single-entry perturbations $L(a) + \omega E_{j,k}$ where ω is taken from a prescribed compact subset Ω of $\mathbf{C}$ which contains the origin. We are interested in the sets

$$\sigma_\Omega^{(j,k)} C_n(a) = \bigcup_{\omega\in\Omega} \sigma(C_n(a) + \omega P_n E_{j,k} P_n), \quad \sigma_\Omega^{(j,k)} L(a) = \bigcup_{\omega\in\Omega} \sigma(L(a) + \omega E_{j,k}).$$

In [7] we proved the following.

Theorem 2. *Let G be a connected component of $\mathbf{C} \setminus a(\mathbf{T})$ and suppose the $(k-j)$th Fourier coefficient of $1/(a-\lambda)$ is not a nonzero constant throughout G. Then*

$$\lim_{n\to\infty} (\sigma_\Omega^{(j,k)} C_n(a) \cap \bar{G}) = \sigma_\Omega^{(j,k)} L(a) \cap \bar{G}. \tag{2}$$

Note again that equality (2) is in particular true if G is the unbounded component of $\mathbf{C} \setminus a(\mathbf{T})$.

From what was said at the end of *Variation 2* we obtain that $\sigma_\Omega^{(j,k)} L(a)$ is the union of $a(\mathbf{T})$ and the set of all $\lambda \in \mathbf{C}$ for which there is an $\omega \in \Omega$ such that $1 + [(a-\lambda)^{-1}]_{k-j}\omega = 0$. In the special case where Ω is the line segment $[-\varepsilon, \varepsilon] \subset \mathbf{R}$, this implies that $\sigma_{[-\varepsilon,\varepsilon]}^{(j,k)} L(a)$ is the union of $a(\mathbf{T})$ and the set

$$\{\lambda \in \mathbf{C} \setminus a(\mathbf{T}) : [(a-\lambda)^{-1}]_{k-j} \in (-\infty, -1/\varepsilon] \cup [1/\varepsilon, \infty)\}.$$

Consequently, unless $[(a-\lambda)^{-1}]_{k-j}$ is a nonzero constant on a connected component of $\mathbf{C} \setminus a(\mathbf{T})$, the intersection of $\sigma^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ with this open component is either empty or an at most countable union of analytic arcs. In several cases, including the case of tridiagonal matrices and thus symbols of the form $a(z) = z + \alpha^2 z^{-1}$, one can compute everything explicitly. In [8] we explicitly determined $\sigma^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ for all j, k in the case where $a(z) = z + \alpha^2 z^{-1}$ with $\alpha \in [0,1]$. Superimposing these sets, that is, considering $\bigcup_{j-k\neq -1} \sigma^{(j,k)}_{[-4,4]} L(a)$ for $\alpha = 1/3$ we got Figure 1. The case $j-k=-1$ fills the entire interior of the ellipse, by virtue of which we omitted these perturbations in the picture. Clearly, Figure 1 also nicely illustrates Theorem 2.

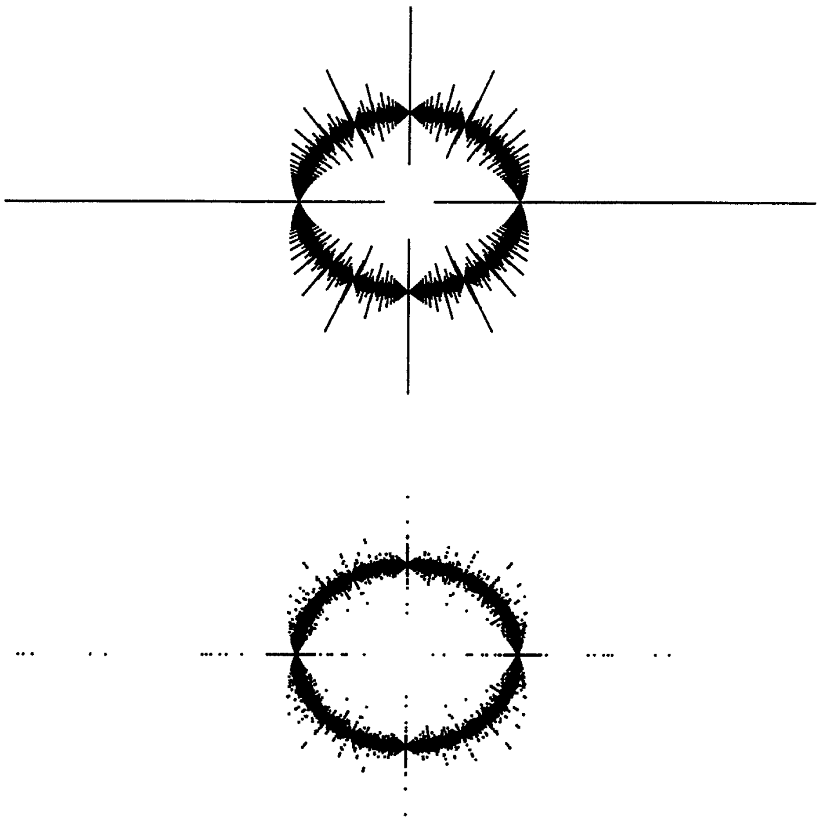

Figure 1. The set $\bigcup_{j-k\neq -1} \sigma^{(j,k)}_{[-4,4]} L(a)$ (top) and eigenvalues of single entry perturbations to $C_{250}(a)$ at random sites j, k with $j-k \neq -1$ by random numbers in $[-4,4]$ (bottom) for the symbol $a(z) = z + z^{-1}/9$. The lower plot superimposes the eigenvalues of 2000 samples.

5. Variation four

We now turn to Toeplitz operators. These are given by the lower-right quarter of Laurent operators. Thus, they act on $\ell^2(\mathbf{Z}_+)$ by a matrix of the form $(a_{j-k})_{j,k=0}^{\infty}$. Such a matrix induces a bounded operator if and only if the numbers a_k $(k \in \mathbf{Z})$ are the Fourier coefficients of a function $a \in L^\infty(\mathbf{T})$, in which case the operator is denoted by $T(a)$ and a is referred to as the symbol of the operator. One can show that $\|T(a)\| = \|a\|_\infty$, with the operator norm on the left and the $L^\infty(\mathbf{T})$ norm on the right.

Let $\mathcal{M}$ be the metric space of all non-empty compact subsets of the plane with the Hausdorff metric. For a sequence $\{M_n\}_{n=1}^{\infty}$ in $\mathcal{M}$, the set $\liminf M_n$ is defined as the set of all λ for which there are $\lambda_n \in M_n$ such that $\lambda_n \to \lambda$, while $\limsup M_n$ is the set of all λ for which there are $n_1 < n_2 < \cdots$ and $\lambda_{n_j} \in M_{n_j}$ such that $\lambda_{n_j} \to \lambda$. Hausdorff himself showed that $\liminf M_n = \limsup M_n =: M$ if and only if M_n converges to M in $\mathcal{M}$, that is, in the metric nowadays named after him; see [41] or Proposition 3.6 of [36].

The spectral theory of Toeplitz operators is incomparably more difficult and richer than its Laurent counterpart. If $a \in C(\mathbf{T})$, then $\sigma_{\mathrm{ess}}(T(a)) = a(\mathbf{T})$ and $\sigma(T(a))$ is the union of $a(\mathbf{T})$ and of all points in the plane that have nonzero winding number with respect to $a(\mathbf{T})$. If a is piecewise continuous, $a \in PC(\mathbf{T})$, the same is true with $a(\mathbf{T})$ replaced by the curve that results from the essential range of a on $\mathbf{T}$ by filling in straight line segments between the endpoints $a(z-0)$ and $a(z+0)$ of each jump. This reveals that small changes of a lead to only small changes in $\sigma_{\mathrm{ess}}(T(a))$ and $\sigma(T(a))$. It was suspected that this is also the case for general $a \in L^\infty(\mathbf{T})$. In other terms, the question was whether the maps $a \mapsto \sigma_{\mathrm{ess}}(T(a))$ and $a \mapsto \sigma(T(a))$ of $L^\infty(\mathbf{T})$ into $\mathcal{M}$ are continuous. This was indeed proved for many classes of symbols, including the algebra $C + H^\infty$, almost periodic symbols, or piecewise quasicontinuous symbols; see [34, 43]. But L^∞ is an abyss!

A bit surprisingly, it turned out that we had not to dive too deep into this abyss. The theme of Chapter 4 of the book [16] is that all evil with Toeplitz operators begins with *SAP*, the C^*-algebra of semi-almost periodic function on $\mathbf{T}$, and the negative answer to the above question found in [13] is just from *SAP*. Here is it.

Theorem 3. *There exist functions a_n and a in SAP such that*

$$\|a_n - a\|_\infty \to 0, \quad \sigma(T(a_n)) = \sigma_{\mathrm{ess}}(T(a_n)) = \mathbf{T}, \quad \sigma(T(a)) = \sigma_{\mathrm{ess}}(T(a)) = \overline{\mathbf{D}}.$$

The C^*-subalgebra *SAP* of $L^\infty(\mathbf{T})$ was introduced by Sarason [46], who also developed a spectral theory for Toeplitz operators with *SAP* symbols. Let $AP(\mathbf{R})$ be the $L^\infty(\mathbf{R})$ closure of the set of all almost periodic polynomials, that is, let $AP(\mathbf{R})$ be the smallest closed subalgebra of $L^\infty(\mathbf{R})$ which contains the functions $e_\lambda(x) = e^{i\lambda x}$

for all $\lambda \in \mathbf{R}$. Let further $C(\overline{\mathbf{R}})$ denote the collection of all functions in $C(\mathbf{R})$ which have finite limits at $\pm\infty$. Then $SAP(\mathbf{R})$ is defined as the smallest closed subalgebra of $L^\infty(\mathbf{R})$ which contains $AP(\mathbf{R}) \cup C(\overline{\mathbf{R}})$. Finally, when x moves along $\mathbf{R}$ from $-\infty$ to $+\infty$, then $(x-i)/(x+i)$ traces out the punctured unit circle $\mathbf{T} \setminus \{1\}$ counter-clockwise starting and terminating at 1, and $SAP = SAP(\mathbf{T})$ is defined as the set of function $a((x-i)/(x+i))$ with a ranging through $SAP(\mathbf{R})$.

Incidentally, the functions appearing in Theorem 3 can be constructed explicitly. Let $\beta \in AP(\mathbf{R})$ be the 2π-periodic function which increases linearly from 0 to 2π on $[-\pi, 0]$ and decreases linearly from 2π to 0 on $[0, \pi]$, define φ_n and φ in $C(\overline{\mathbf{R}})$ by

$$\varphi_n(x) = \exp\Big(i\Big(1 - \frac{1}{n}\Big)\arctan x\Big), \quad \varphi(x) = \exp(i \arctan x),$$

and put $\alpha_n = e^{-i\beta}\varphi_n$, $\alpha = e^{-i\beta}\varphi$. Then Theorem 3 holds with

$$a_n\Big(\frac{x-i}{x+i}\Big) = \alpha_n(x), \quad a\Big(\frac{x-i}{x+i}\Big) = \alpha(x).$$

If a_n and a are as in Theorem 3, then $\liminf \sigma(T(a_n)) = \limsup \sigma(T(a_n)) \neq \sigma(T(a))$. When choosing $b_n = a_n$ for odd n and $b_n = a$ for even n, we get a uniformly convergent sequence $\{b_n\}$ such that $\liminf \sigma(T(b_n)) \neq \limsup \sigma(T(b_n))$. The following theorem of [13] shows that at the price of leaving SAP we obtain even Toeplitz operators for which all the three sets are different.

Theorem 4. *There exist c_n and c in $L^\infty(\mathbf{T})$ which are continuous on $\mathbf{T} \setminus \{-1, 1\}$ such that $\|c_n - c\|_\infty \to 0$ and*

$$\sigma(T(c)) = \sigma_{\text{ess}}(T(c)) = \overline{\mathbf{D}} \cup (2 + \overline{\mathbf{D}}),$$

$$\sigma(T(c_n)) = \sigma_{\text{ess}}(T(c_n)) = \begin{cases} \mathbf{T} \cup (2 + \mathbf{T}) & \text{if } n \text{ is odd,} \\ \mathbf{T} \cup (2 + \overline{\mathbf{D}}) & \text{if } n \text{ is even.} \end{cases}$$

In particular, $\liminf \sigma(T(c_n))$*,* $\limsup \sigma(T(c_n))$*,* $\sigma(T(c))$ *are three different sets.*

6. Variation five

Let $T_n(a) = (a_{j-k})_{j,k=0}^{n-1}$ denote the principal $n \times n$ truncation of the infinite Toeplitz matrix $T(a)$. The problem of describing the eigenvalue distribution of the matrices $T_n(a)$ as n goes to infinity is a big business for a century. The books [15, 17, 36, 49, 51] are recent treatises of the problem.

In the case where a is a Laurent polynomial as in *Variation 2*, the limiting set

$$\Lambda(a) := \limsup \sigma(T_n(a))$$

was determined by Schmidt and Spitzer [47] in 1960, who also showed $\Lambda(a)$ coincides with $\liminf \sigma(T_n(a))$. The set $\Lambda(a)$ is the union of a finite number of analytic arcs and in general $\Lambda(a)$ is significantly different from $\sigma(T(a))$ although always $\Lambda(a) \subset \sigma(T(a))$. Schmidt and Spitzer expressed the set via formulas, but eventually identifying it remains a challenge. Paper [11] presents a numerical algorithm in the spirit of Beam and Warming [3] that, given a grid parameter $h = 1/N$, reduces testing $O(N^2)$ points in the plane for membership in the limiting set to testing only $O(N)$ points along some one-dimensional curves.

Formulas of the Szegő type describe the asymptotic eigenvalue distribution of $T_n(a)$ in the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} F(a(e^{i\theta}))\, d\theta, \tag{3}$$

where $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ are the eigenvalues of $T_n(a)$ and F, the "test function", can be an arbitrary compactly supported continuous function of $\mathbf{C}$ to $\mathbf{C}$. Formula (3) is true for real-valued $a \in L^\infty(\mathbf{T})$, in which case all involved Toeplitz matrices are Hermitian. We here are interested in the non-Hermitian case. Then the formula is still valid for certain classes of continuous or piecewise continuous symbols a; in addition to the books cited above, see [2, 50, 52]. Notice that (3) implies that, up to $o(n)$ possible outliers, the eigenvalues cluster along the (essential) range $a(\mathbf{T})$ of the symbol, but that (3) does not tell us whether the possible $o(n)$ outliers produce additional pieces of $\Lambda(a)$. Thus, we cannot have recourse to results like (3) when looking for $\Lambda(a)$.

One possibility of determining $\Lambda(a)$ could be to approximate a by a Laurent polynomial a_n (not to be confused with the nth Fourier coefficient of a), surmising that $\Lambda(a_n)$ is close to $\Lambda(a)$. Clearly, this approach fails for piecewise continuous symbols, since a properly piecewise continuous function can never be approximated uniformly by Laurent polynomials as closely as desired. Unexpectedly, this approach does in general also not work for continuous symbols. This is a consequence of the following result, which was established in [14] and shows that, in contrast to the continuity of the spectrum on the space of continuous symbols discussed in *Variation 4*, the asymptotic spectrum $\Lambda(a)$ is discontinuous on the space of continuous symbols.

Theorem 5. *There exist c_n and c in $C(\mathbf{T})$ such that $\|c_n - c\|_\infty \to 0$ but $\Lambda(c_n)$ does not converge to $\Lambda(c)$ in the Hausdorff metric.*

In [14] we proved the theorem with explicitly constructed symbols c_n and c. The function c is given by $c(z) = z^{-1}(33 - (z + z^2)(1 - z^2)^{3/4})$ for $|z| = 1$ and c_n is the nth partial sum of the Fourier series of c. The matrices $T(c)$ and $T(c_n)$ are thus lower Hessenberg matrices. Note that the Fourier series of c converges absolutely, which implies that $c_n \to c$ not only in $C(\mathbf{T})$ but even in the Wiener algebra on $\mathbf{T}$. I also want to remark that c is a continuous function which is piecewise C^∞ but not C^∞. The results of Widom [52] therefore imply that (3) is valid and numerical computations indicate that indeed $\Lambda(c) = c(\mathbf{T})$. The proof of Theorem 5 given in [14] does not require knowledge of the set $\Lambda(c)$. We there prove that $\Lambda(c) \cap \mathbf{D} = \emptyset$ but $0 \in \Lambda(c_n)$ for all odd $n \geq 5$. Pictures showing the evolution of $\Lambda(c_n)$ are in [14] and [15, p. 285].

7. Intermezzo

Many interesting Toeplitz matrices are nonnormal and hence their eigenvalues are very sensitive to perturbation. I cannot resist to say it with Nick Trefethen and Mark Embree [51, p. 11], who complement their message that the spectrum gives an operator a personality with the words "In the highly nonnormal case, vivid though the image may be, the location of the eigenvalues may be as fragile an indicator of underlying character as the hair color of a Hollywood actor. We shall see that pseudospectra provide equally compelling images that may capture the spirit underneath more robustly."

For $\varepsilon > 0$, the ε-pseudospectrum of an operator or a matrix A is defined as the set

$$\sigma_\varepsilon(A) = \sigma(A) \cup \{\lambda \in \mathbf{C} \setminus \sigma(A) : \|(A - \lambda I)^{-1}\| \geq 1/\varepsilon\},$$

where $\|\cdot\|$ is the operator norm (= spectral norm in the matrix case). One can show (or take as an alternative definition) that

$$\sigma_\varepsilon(A) = \bigcup_{\|E\| \leq \varepsilon} \sigma(A + E).$$

Various examples and pictures can be found in [15, 17, 44, 51].

In contrast to the spectrum, the ε-pseudospectrum is continuous: if B_n, B are bounded Hilbert space operators and $B_n \to B$ in the norm, then $\sigma_\varepsilon(B_n) \to \sigma_\varepsilon(B)$ in the Hausdorff metric; see [51, p. 484] or [39, Theorem 4.4 (v)]. Consequently, letting $A_c = A_{c,0}$ be as in the *Theme* and *Variation 1*, we get $\sigma_\varepsilon(A_c) \to \sigma_\varepsilon(A_0)$ as $c \to 0$ for each $\varepsilon > 0$. Thus, the discontinuity disappears when passing from spectra to pseudospectra. In the special case at hand, even more can be said.

If $|c| \leq 1$ and $|c| < \varepsilon$, then $\sigma_\varepsilon(A_c)$ equals the closed disk of radius $1 + \varepsilon$ centered at the origin, $\sigma_\varepsilon(A_c) = (1 + \varepsilon)\overline{\mathbf{D}}$, and in particular, $\sigma_\varepsilon(A_c) = \sigma_\varepsilon(A_0)$.

This can be proved as follows. We always have

$$\sigma(B) + \varepsilon\overline{\mathbf{D}} \subset \sigma_\varepsilon(B) \subset \overline{W(B)} + \varepsilon\overline{\mathbf{D}},$$

where $W(B)$ is the numerical range of B; see [51, Chapter 17]. Since $\sigma(A_0) = \overline{W(A_0)} = \overline{\mathbf{D}}$, we arrive at the conclusion that $\sigma_\varepsilon(A_0) = (1 + \varepsilon)\overline{\mathbf{D}}$. If $|c| \leq 1$, then $\|A_c\| = 1$, so $\overline{W(A_c)} \subset \overline{\mathbf{D}}$, and we obtain that $\sigma_\varepsilon(A_c) \subset (1 + \varepsilon)\overline{\mathbf{D}}$. Theorem 52.4

of [51] implies that if $|c| < \varepsilon$, then $\bar{\mathbf{D}} \subset (1+\varepsilon-|c|)\bar{\mathbf{D}} = \sigma_{\varepsilon-|c|}(A_0) \subset \sigma_\varepsilon(A_c)$. Finally, taking into account that $\mathbf{T} + \varepsilon\bar{\mathbf{D}} = \sigma(A_c) + \varepsilon\bar{\mathbf{D}} \subset \sigma_\varepsilon(A_c)$, we see that $\sigma_\varepsilon(A_c)$ is all of $(1+\varepsilon)\bar{\mathbf{D}}$ for $|c| \le 1$ and $|c| < \varepsilon$, which completes the proof.

We remark that the restriction to $|c| \le 1$ is essential. Consider the matrix $cE_{0,1}$. The norm $\|(cE_{0,1} - \lambda)^{-1}\|$ equals the norm of the inverse of the 2×2 matrix $\left(\begin{smallmatrix} -\lambda & c \\ 0 & -\lambda \end{smallmatrix}\right)$. This along with an elementary computation shows that

$$\sigma_\delta(cE_{0,1}) = \left\{\lambda : f\left(\frac{|c|}{|\lambda|}\right) \ge \frac{|c|}{\delta}\right\} \quad \text{with } f(y) = y\sqrt{1 + \frac{y^2}{2} + y\sqrt{1 + \frac{y^2}{4}}}\,.$$

Now, let $1 < |c| < \varepsilon$. Thinking of A_c as a perturbation of $cE_{0,1}$ by an operator of norm 1, we deduce from Theorem 52.4 of [51] that

$$\sigma_{\varepsilon-1}(cE_{0,1}) \subset \sigma_\varepsilon(A_c) \subset \sigma_{\varepsilon+1}(cE_{0,1}).$$

Taking $c = 10$ and $\varepsilon = 11$ and using the formula we have just derived with $\delta = \varepsilon \pm 1$, we get after some calculation that

$$\{\lambda : |\lambda| \le 14\} \subset \sigma_{11}(A_{10}) \subset \{\lambda : |\lambda| \le 17\},$$

and thus $\sigma_\varepsilon(A_c)$ is different from both $(1+\varepsilon)\bar{\mathbf{D}} = \{\lambda : |\lambda| \le 12\}$ and $(|c|+\varepsilon)\bar{\mathbf{D}} = \{\lambda : |\lambda| \le 21\}$.

8. Variation six

Let us return to Toeplitz matrices. In contrast to the spectrum of the matrices $T_n(a)$, their pesudospectra behave as nicely as one could ever expect. For example, one can show that if $a \in PC(\mathbf{T})$, then $\sigma_\varepsilon(T_n(a))$ converges in the Hausdorff metric to $\sigma_\varepsilon(T(a))$ for each $\varepsilon > 0$. Such a result was first established by Reichel and Trefethen [44]. They had it for symbols $a \in C(\mathbf{T})$ with absolutely convergent Fourier series. The extension to piecewise continuous symbols was proved in [5]. Note that $T_n(a)$ (when identified with $T_n(a)P_n$ on $\ell^2(\mathbf{Z}_+)$) does not converge to $T(a)$ in the norm; the convergence is only pointwise.

The proof given in [5] is based on working with C^*-algebras. It had been known at least since [35] that, for $a \in PC(\mathbf{T})$, the operator $T(a) - \lambda I = T(a-\lambda)$ is invertible if and only if the matrices $T_n(a-\lambda)$ are invertible for all sufficiently large n and the norms of their inverses remain uniformly bounded as $n \to \infty$ This may be written as

$$\limsup_{n\to\infty} \|[T_n(a-\lambda)]^{-1}\| < \infty \iff \|[T(a-\lambda)]^{-1}\| < \infty. \tag{4}$$

The right-hand side of this equivalence is obviously a statement on the spectrum of $T(a)$. The left-hand side may be interpreted as invertibility of the matrix sequence

$\{T_n(a-\lambda)\}$ in a certain algebra of norm-bounded sequences modulo sequences converging to zero in the norm and is thus a statement on the spectrum of the sequence $\{T_n(a)\}$. Elaborating this idea, which includes paying the price of extending (4) to operators of the form

$$\sum_j \prod_k T_n(a_{jk}) - \lambda P_n \quad \text{and} \quad \sum_j \prod_k T(a_{jk}) - \lambda I, \tag{5}$$

one gets a C^*-algebra homomorphism between two unital C^*-algebras which preserves spectra. The point is that such C^*-algebra homomorphisms automatically preserve norms, by virtue of which (4) can be sharpened to the equality

$$\lim_{n\to\infty} \|[T_n(a-\lambda)]^{-1}\| = \|[T(a-\lambda)]^{-1}\|, \tag{6}$$

with the convention that if one side is infinite, then so also is the other. The existence of the limit in (6) is part of the conclusion.

Clearly, with (6) at hand one has everything to prove the convergence of $\sigma_\varepsilon(T_n(a))$ to $\sigma_\varepsilon(T(a))$. Or not? Indeed, not! At some point of the final stage of the proof, one needs the fact that the pseudospectrum $\sigma_\varepsilon(T(a))$ cannot make sudden "jumps" when ε changes continuously, which is equivalent to the question whether the resolvent norm $\|(A-\lambda I)^{-1}\|$ of a Hilbert space operator A can be locally constant. I reported on this question during a Banach semester in Warsaw, and in 1994, Andrzeij Daniluk sent me a proof of what I needed; see [5] or [17, Theorem 3.14]. Actually, the question of whether the resolvent norm can be locally constant has both a prehistory and a posthistory. In this connection, I recommend papers [32, 48].

The bonus of the C^*-algebra approach is that the convergence of $\sigma_\varepsilon(T_n(a))$ to $\sigma_\varepsilon(T(a))$ can be extended to operators like (5). To summarize, we have the following; see [5] or [17, Chapter 3].

Theorem 6. (a) *If $a_{jk} \in PC(\mathbf{T})$ and $\tilde{a}_{jk}(z) := a_{jk}(z^{-1})$, then*

$$\lim_{n\to\infty} \sigma_\varepsilon\Big(\sum_j \prod_k T_n(a_{jk})\Big) = \sigma_\varepsilon\Big(\sum_j \prod_k T(a_{jk})\Big) \cup \sigma_\varepsilon\Big(\sum_j \prod_k T(\tilde{a}_{jk})\Big),$$

(b) *if $a \in PC(\mathbf{T})$ and K is a compact operator, then*

$$\lim_{n\to\infty} \sigma_\varepsilon(T_n(a) + P_n K P_n) = \sigma_\varepsilon(T(a)+K) \cup \sigma_\varepsilon(T(a)),$$

(c) *if $a \in PC(\mathbf{T})$, then*

$$\lim_{n\to\infty} \sigma_\varepsilon(T_n(a)) = \sigma_\varepsilon(T(a)).$$

We remark that $T(\tilde{a})$ is nothing but the transpose of $T(a)$. Note also that always $\sigma(T(a)) \subset \sigma(T(a)+K)$. This is the well-known Coburn–Simonenko theorem. Only recently Steffen Roch [45] proved that, even for continuous symbols, $\sigma_\varepsilon(T(a))$ is not necessarily a subset of $\sigma_\varepsilon(T(a)+K)$. Thus, we cannot omit $\sigma_\varepsilon(T(a))$ in Theorem 6 (b).

To finish this variation, I want to mention that there is an impressive development of the pseudospectral idea to compute or approximate spectra initiated by Anders Hansen [38]. However, embarking on this development would be beyond the frame of a variation, and I instead invite the interested reader to consult [4, 21, 39, 40], for example.

9. Variation seven

The two infinite Toeplitz matrices

$$A = \begin{pmatrix} 0 & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{1} & 0 & \frac{1}{1} & \frac{1}{2} & \ddots \\ \frac{1}{2} & \frac{1}{1} & 0 & \frac{1}{1} & \ddots \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{1} & -\frac{1}{2} & -\frac{1}{3} & \cdots \\ \frac{1}{1} & 0 & -\frac{1}{1} & -\frac{1}{2} & \ddots \\ \frac{1}{2} & \frac{1}{1} & 0 & -\frac{1}{1} & \ddots \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

differ only in a sign change, but they induce very different operators: we have $A = T(a)$ and $B = T(b)$ with

$$a(e^{i\theta}) = -\log(1-e^{i\theta}) - \log(1-e^{-i\theta}), \quad b(e^{i\theta}) = i(\pi-\theta), \quad \theta \in (0,2\pi).$$

As $a(e^{i\theta}) = -2\log|1-e^{i\theta}|$ is not in $L^\infty(\mathbf{T})$, the operator $T(a)$ is unbounded. The function b is bounded, and hence $T(b)$ is a bounded operator with symbol in $PC(\mathbf{T})$. In a sense, $B = T(b)$ is a lucky exception. The symbol of

$$D = \begin{pmatrix} 0 & c\frac{1}{1} & c\frac{1}{2} & c\frac{1}{3} & \cdots \\ \frac{1}{1} & 0 & c\frac{1}{1} & c\frac{1}{2} & \ddots \\ \frac{1}{2} & \frac{1}{1} & 0 & c\frac{1}{1} & \ddots \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is $d(e^{i\theta}) = i(\pi-\theta) - (1+c)\log(1-e^{-i\theta})$, and this is a bounded function if and only if $c = -1$, that is, if and only if $D = B$.

Theorem 6 (c) implies that $\sigma_\varepsilon(T_n(b)) \to \sigma_\varepsilon(T(b))$. The matrices $T_n(b)$ and $T(b)$ are skew-symmetric and hence normal. So, their ε-pseudospectra are nothing else than the closed ε-neigborhoods of their spectra. Moreover, since $iT_n(b)$ and $iT(b)$ are Hermitian, we conclude from (3) that even the spectra of $T_n(b)$ converge to the spectrum $\sigma(T(b)) = i[-\pi, \pi]$ and thus the convergence $\sigma_\varepsilon(T_n(b)) \to \sigma_\varepsilon(T(b))$ simply mimics the convergence of the spectra. The eigenvalues of $T_n(b)$ eventually fill $i[-\pi, \pi]$ evenly and densely, and hence the convergence of $\sigma_\varepsilon(T_n(b))$ to $\sigma_\varepsilon(T(b))$ is reasonably fast.

Things change in the non-normal case. Consider the so-called Hilbert Toeplitz matrix

$$T(h) = (1/(j - k + 1/2))_{j,k=0}^{\infty}.$$

The symbol of this Toeplitz matrix is $h(e^{i\theta}) = \pi i e^{-i\theta/2}$, again a function in $PC(\mathbf{T})$. Consequently, the spectrum of $T(h)$ is the closed half-disk bounded by the half-circle $\{\pi i e^{-i\theta/2} : \theta \in [0, 2\pi]\}$ and the line segment $i[-\pi, \pi]$. In [9] it is shown that the convergence $\sigma_\varepsilon(T_n(h)) \to \sigma_\varepsilon(T(h))$ is spectacularly slow. The reason is that the resolvent norm $\|[T_n(h - \lambda)]^{-1}\|$ grows very slowly as n goes to infinity, so that it takes astronomically large n to make this norm reach $1/\varepsilon$. For example, if $\lambda = 1/2$, then this norm grows roughly like $3.8n^{0.30}$. At this rate, the resolvent norm will not exceed 10^5 until $n \approx 10^{15}$. For $\lambda = 0$, $\|[T_n(h - \lambda)]^{-1}\|$ grows roughly like $0.4 \log n + 1.5$ and it will not exceed 10^5 until $n \approx 10^{108572}$.

For rational symbols f, the norm $\|[T_n(f - \lambda)]^{-1}\|$ increases exponentially, which results in fast convergence of the pseudospectra. So, one is expecting that the slow convergence described in the preceding paragraph is caused by the discontinuity of the symbol and that it should not happen for continuous symbols. However, in [12] we showed that the phenomenon also occurs (and is, in a sense, even generic) within the continuous symbols. The Fourier coefficients f_n of a function f in $C^2(\mathbf{T})$ decay as $O(1/n^2)$. Therefore, the two functions

$$(Pf)(z) = \sum_{n=0}^{\infty} f_n z^n, \quad (Qf)(z) = \sum_{n=1}^{\infty} f_{-n} z^{-n} \quad (|z| = 1)$$

are well defined for $f \in C^2(\mathbf{T})$. Here are two results of [12].

Theorem 7. (a) *Given any number $q > 0$, there exists a function $f \in C(\mathbf{T})$ such the $\|[T_n(f - \lambda)]^{-1}\| = O(n^q)$ for some point $\lambda \in \sigma(T(f)) \setminus f(\mathbf{T})$.*

(b) *Let $f \in C^2(\mathbf{T})$ and let $\lambda \in \mathbf{C}$ be a point whose winding number with respect to $f(\mathbf{T})$ is -1 (resp. 1). Then $\|[T_n(f - \lambda)]^{-1}\|$ increases faster than every polynomial,*

$$\lim_{n\to\infty} \|[T_n(f - \lambda)]^{-1}\| n^{-q} = \infty \quad \textit{for each } q > 0,$$

if and only if Pf (resp. Qf) belongs to $C^\infty(\mathbf{T})$.

10. Variation eight

As outlined in [10], a problem in lattice theory leads to the computation of the determinant of the $n \times n$ matrix V_n which results from the pentadiagonal Toeplitz matrix

$$A_n := T_n(|1-z|^4) = T_n(6 - 4(z + z^{-1}) + (z^2 + z^{-2}))$$

by placing ones in the upper-right and lower-left corners. For example,

$$V_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 1 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}.$$

It turns out that $\det V_n = (n+1)^3$. The determinant of A_n is a so-called Fisher–Hartwig determinant, and it had been known for a long time that

$$\det A_n = \frac{1}{12}(n+1)(n+2)^2(n+3) \sim \frac{n^4}{12}. \tag{7}$$

Paper [10] originated from the intriguing question why the perturbations in the corners of A_n lower the growth from n^4 to n^3. To put it into the context of these variations, let $A_n(c)$ denote the matrix A_n with the two corner entries 0 replaced by a number c. In the notation used above,

$$A_n(c) = T_n(|1-z|^4) + c(E_{0,n-1} + E_{n-1,0}).$$

Obviously, $A_n = A_n(0)$ and $V_n = A_n(1)$. How does the growth of $\det A_n(c)$ depend on c?

Corollary 4.4 of [10] deals with much more general situations and in the special case at hand it gives

$$\frac{\det A_n(c)}{\det A_n} = 1 - c^2 + \frac{4}{n}(c + 2c^2) + O\Big(\frac{1}{n^2}\Big).$$

From this and (7), we get after elementary calculations

$$\det A_n(c) = (1-c^2)\frac{n^4}{12} + (1-c^2)\frac{2n^3}{3} + (c+2c^2)\frac{n^3}{3} + O(n^2).$$

Consequently, the growth of $\det A_n(c)$ is as n^4 for $c \neq \pm 1$ and as n^3 if $c = 1$ or $c = -1$. To state things in less precise form but more drastically, we have

$$\lim_{n\to\infty} \frac{\det A_n(c)}{n^3} = \begin{cases} \infty & \text{if } c \neq \pm 1, \\ 1 & \text{if } c = 1, \\ 1/3 & \text{if } c = -1. \end{cases}$$

Note that in [10] we actually consider the limit of $\det(T_n(a)+E_n)/\det T_n(a)$ under the sole assumption that $a \in L^1(\mathbf{T})$, $a \ge 0$ almost everywhere, $\log a \in L^1(\mathbf{T})$ and with $n \times n$ matrices E_n whose nonzero entries are four fixed $m_0 \times m_0$ blocks placed in the four corners of the matrix.

11. Variation nine

In 2004, after many years of work with Toeplitz matrices, I arrived at the question of how to ascertain whether a given matrix is a Toeplitz matrix. This might sound strange at the first glance, but assume our machine has computed and stored an $n \times n$ matrix $X = (x_{jk})_{j,k=1}^n$ with a very large n and we want to know whether it is a Toeplitz matrix. How could we do this?

We could ask the machine to check whether the entries are constant along the diagonals. To perform this task, we take the $n \times n$ forward-shift matrix U, that is, the matrix with ones on the subdiagonal and zeros elsewhere, and let the machine compute $XU - UX$, which equals

$$\begin{pmatrix} x_{12} & x_{13} & \dots & x_{1,n-1} & 0 \\ x_{22}-x_{11} & x_{23}-x_{12} & \dots & x_{2n}-x_{1,n-1} & -x_{1n} \\ x_{32}-x_{21} & x_{33}-x_{22} & \dots & x_{3n}-x_{2,n-1} & -x_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n2}-x_{n-1,1} & x_{n3}-x_{n-1,2} & \dots & x_{nn}-x_{n-1,n-1} & -x_{n-1,n} \end{pmatrix}.$$

Let $D(X)$ denote the lower-left $(n-1)\times(n-1)$ submatrix of this matrix. Clearly, the original matrix X is Toeplitz if and only if $D(X)$ is the zero matrix. Thus, we arrived at the critical issue of testing whether something is zero. All we can do is to test whether $D(X)$ is small, say whether $\|D(X)\|_2 < \varepsilon$, where $\|A\|_2 = (\sum_{j,k} |a_{jk}|^2)^{1/2}$ denotes the Frobenius norm (= Hilbert–Schmidt norm) of A. Does this imply that X is close to a Toeplitz matrix?

Take, for example, $X = \operatorname{diag}(x_1, x_2, \dots, x_n)$ with $x_j = \exp(2\pi i j/n)$. Then

$$\begin{aligned} \|D(X)\|_2^2 &= |x_1 - x_2|^2 + |x_2 - x_3|^2 + \dots + |x_{n-1} - x_n|^2 \\ &= (n-1)|e^{2\pi i/n} - 1|^2 = 4(n-1)\sin^2\frac{\pi}{n}, \end{aligned}$$

which is small for large n. Let $\mathcal{T}_n$ be the set of all $n \times n$ Toeplitz matrices with entries in $\mathbf{K}$ where either $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$. It easily seen that, for diagonal matrices X,

$$\operatorname{dist}_2^2(X, \mathcal{T}_n) := \min_{T \in \mathcal{T}_n} \|X - T\|_2^2 = \sum_{j=1}^n \Big| x_j - \frac{1}{n}\sum_{k=1}^n x_k \Big|^2.$$

In our case $\sum x_k = 0$, which gives $\text{dist}_2^2(X, \mathcal{T}_n) = \sum |x_j|^2 = n$, and this is large. In [6], the following is proved for general matrices X.

Theorem 8. *We have*

$$\max_{X \notin \mathcal{T}_n} \frac{\text{dist}_2(X, \mathcal{T}_n)}{\|D(X)\|_2} = \frac{1}{2 \sin \frac{\pi}{2n}} \sim \frac{n}{\pi}.$$

Thus, if $\|D(X)\|_2 = \varepsilon$ then $\text{dist}_2(X, \mathcal{T}_n)$ is at most about $n\varepsilon/\pi$. This may be large, but the linear growth prevents $n\varepsilon/\pi$ from becoming an astronomic number if ε and n are appropriately adapted. Moreover, the following result of [6] tells us that the worst-case situation of Theorem 8 is a very rare event for matrices of large sizes.

Theorem 9. *Equip the space* $\mathbf{K}^{n \times n}$ *of the* $n \times n$ *matrices over* $\mathbf{K}$ *with the Frobenius norm and take* X *randomly from the unit sphere of* $\mathbf{K}^{n \times n}$ *with the uniform distribution. Put* $\text{dist}_2(X, \mathcal{T}_n)/\|D(X)\|_2 = 0$ *if* $\|D(X)\|_2 = 0$. *Then*

$$\text{Probability}\Big(\frac{\text{dist}_2(X, \mathcal{T}_n)}{\|D(X)\|_2} > 10\Big) < \frac{13}{n^2} \quad \textit{for} \quad n \geq 10.$$

Thus, although the question on whether we can figure out whether a given matrix is Toeplitz has a negative answer theoretically, these two theorems say that practically and optimistically the answer to this question is in the affirmative.

A Toeplitz-plus-Hankel matrix (T+H matrix for short) is a matrix of the form $(t_{j-k} + h_{j+k})_{j,k=1}^n$. In contrast to the pure Toeplitz or pure Hankel structures, it is not immediately seen whether a given $n \times n$ matrix X (with n being small) is T+H. For example, with unskilled eyes it is not trivial to decide which of the matrices

$$\begin{pmatrix} 2.9 & 2.3 & -1.9 \\ 5.4 & 0.3 & 0.7 \\ 5.2 & -1.2 & 1.9 \end{pmatrix}, \quad \begin{pmatrix} 2.9 & 2.3 & -1.9 \\ 5.4 & 0.4 & 0.7 \\ 5.2 & -1.2 & 1.9 \end{pmatrix}, \quad \begin{pmatrix} 2.9 & 2.3 & -1.9 \\ 5.4 & 0.4 & 0.8 \\ 5.2 & -1.2 & 1.9 \end{pmatrix}$$

are T+H. However, Heinig, and Rost [42] discovered that X is T+H if and only if the central $(n-2) \times (n-2)$ submatrix of $XW - WX$ is zero, where $W = U + U^\top$ is the $n \times n$ matrix with ones on the first superdiagonal and the first subdiagonal and with zeros elsewhere. Thus, let us denote the central $(n-2) \times (n-2)$ submatrix of $XW - WX$ by $F(X)$ and let $\mathcal{T}\mathcal{H}_n$ stand for the space of T+H matrices with $t_{j-k}, h_{j+k} \in \mathbf{R}$. In [6], analogs of Theorems 8 and 9 were proved: there are constants C_1, C_2 such that

$$C_1 n^2 < \max_{X \notin \mathcal{T}\mathcal{H}_n} \frac{\text{dist}_2(X, \mathcal{T}\mathcal{H}_n)}{\|F(X)\|_2} < C_2 n^2,$$

and if X is randomly drawn from the unit sphere of $\mathbf{R}^{n \times n}$ with the uniform distribution, then

$$\text{Probability}\Big(\frac{\text{dist}_2(X, \mathcal{T}\mathcal{H}_n)}{\|F(X)\|_2} > 10\Big) < \frac{79}{n^2} \quad \text{for } n \geq 10.$$

A statistical test was designed in [18]. Suppose $n \geq 20$ and X is from the unit sphere of $\mathbf{R}^{n\times n}$ with the uniform distribution. Compute $\xi = \|F(X)\|_2^2/\|X\|_2^2$. If $\xi < 1.91$ (resp. 0.29), we accept X to be T+H. Then the probability for accepting the matrix as T+H although it is not T+H does not exceed 5% (resp. 1%).

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Albrecht Böttcher
Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Strasse 39, 09107 Chemnitz, Germany; aboettch@mathematik.tu-chemnitz.de