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An extension of the Liouville theorem for Fourier multipliers to sub-exponentially growing solutions

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Abstract. We study the equation $m(D)f = 0$ in a large class of sub-exponentially growing functions. Under appropriate restrictions on $m \in C(\mathbb{R}^n)$, we show that every such solution can be analytically continued to a sub-exponentially growing entire function on $\mathbb{C}^n$ if, and only if, $m(\xi) \neq 0$ for $\xi \neq 0$.

1. Introduction

The classical Liouville theorem for the Laplace operator $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on $\mathbb{R}^n$ says that every bounded (polynomially bounded) solution of the equation $\Delta f = 0$ is in fact constant (is a polynomial). Recently, similar results have been obtained for solutions of more general equations of the form $m(D)f = 0$, where $m(D) := \mathcal{F}^{-1} m(\xi) \mathcal{F}$, and

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\, d\xi$$

are the Fourier and the inverse Fourier transforms, see [1–3, 12], and the references therein. Namely, it was shown that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, the implication

$$\begin{aligned} &f \text{ is bounded (polynomially bounded) and } m(D)f = 0 \\ &\implies f \text{ is constant (is a polynomial)} \end{aligned}$$

holds if, and only if, $m(\xi) \neq 0$ for $\xi \neq 0$. Much of this research has been motivated by applications to infinitesimal generators of Lévy processes.

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In this paper, we study solutions of $m(D)f = 0$ that can grow faster than any polynomial. Of course, one cannot expect such solutions to have a simple structure, not even in the case of $\Delta f = 0$ in $\mathbb{R}^2$, see, e.g., [22, Chapter I, Section 2]. We consider sub-exponentially growing solutions whose growth is controlled by a submultiplicative function, cf. (2.1), satisfying the Beurling–Domar condition (2.3), and we show that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, every such solution admits analytic continuation to a sub-exponentially growing entire function on $\mathbb{C}^n$ if, and only if, $m(\xi) \neq 0$ for $\xi \neq 0$, see Corollary 4.4. Results of this type have been obtained for solutions of partial differential equations with constant coefficients by A. Kaneko and G. E. Šilov, see [17, 18, 26], [7, Chapter 10, Section 2, Theorem 2], and Section 5 below.

Keeping in mind applications to infinitesimal generators of Lévy processes, we do not assume that m is the Fourier transform of a distribution with compact support, so our setting is different from that in, e.g., [6] and [16, Chapter XVI].

The paper is organised as follows. In Chapter 2, we consider submultiplicative functions satisfying the Beurling–Domar condition. For every such function g, we introduce an auxiliary function S_g, see (2.11), (2.12), which appears in our main estimates. Chapter 3 contains weighted L^p estimates for entire functions on $\mathbb{C}^n$, which are a key ingredient in the proof of our main results in Chapter 4. Another key ingredient is the Tauberian Theorem 4.1, which is similar to [3, Theorem 7] and [24, Theorem 9.3]. The main difference is that the function f in Theorem 4.1 is not assumed to be polynomially bounded, and hence it might not be a tempered distribution. So, we avoid using the Fourier transform $\hat{f} = \mathcal{F}f$ and its support (and non-quasianalytic-type ultradistributions). Although we are mainly interested in the case $m(\xi) \neq 0$ for $\xi \neq 0$, we also prove a Liouville type result for m with compact zero set $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\}$, see Theorem 4.3. Finally, we discuss in Section 5 A. Kaneko's Liouville-type results for partial differential equations with constant coefficients, cf. [17, 18], which show that the Beurling–Domar condition is in a sense optimal in our setting.

2. Submultiplicative functions and the Beurling–Domar condition

Let $g\colon \mathbb{R}^n \to (0, \infty)$ be a locally bounded, measurable *submultiplicative* function, i.e., a locally bounded measurable function satisfying

$$g(x + y) \leqslant C g(x) g(y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

where the constant $C \in [1, \infty)$ does not depend on x and y. Without loss of generality, we will always assume that $g \geqslant 1$, as otherwise we can replace g with $g + 1$. Also, replacing g with Cg, we can assume that

$$g(x+y) \leqslant g(x)g(y) \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.1}$$

A locally bounded submultiplicative function is exponentially bounded, i.e.,

$$|g(x)| \leqslant Ce^{a|x|} \tag{2.2}$$

for suitable constants $C, a > 0$, see [25, Section 25] or [14, Chapter VII].

We will say that g satisfies the *Beurling–Domar* condition if

$$\sum_{l=1}^{\infty} \frac{\log g(lx)}{l^2} < \infty \quad \text{for all } x \in \mathbb{R}^n. \tag{2.3}$$

If g satisfies the Beurling–Domar condition, then it also satisfies the Gelfand–Raikov–Shilov condition

$$\lim_{l\to\infty} g(lx)^{1/l} = 1 \quad \text{for all } x \in \mathbb{R}^n,$$

while $g(x) = e^{|x|/\log(e+|x|)}$ satisfies the latter but not the former condition, see [10]. It is also easy to see that $g(x) = e^{|x|/\log^{\gamma}(e+|x|)}$ satisfies the Beurling–Domar condition if, and only if, $\gamma > 1$. The function

$$g(x) = e^{a|x|^b}(1+|x|)^s(\log(e+|x|))^t$$

satisfies the Beurling–Domar condition for any $a, s, t \geqslant 0$ and $b \in [0, 1)$, see [10].

Lemma 2.1. *Let $g\colon \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition* (2.3)*. Then, for every $\varepsilon > 0$, there exists $R_\varepsilon \in (0, \infty)$ such that*

$$\int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau < \varepsilon \quad \textit{for all } x \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : |y| = 1\}.$$

Proof. Since $g \geqslant 1$ is locally bounded,

$$0 \leqslant M := \sup_{|y|\leqslant 1} \log g(y) < \infty. \tag{2.4}$$

Take any $x \in \mathbb{S}^{n-1}$. It follows from (2.1) that

$$\log g((l+1)x) - M \leqslant \log g(\tau x) \leqslant \log g(lx) + M \quad \text{for all } \tau \in [l, l+1].$$

Hence,

$$\sum_{l=L}^{\infty} \frac{\log g((l+1)x) - M}{(l+1)^2} \leqslant \sum_{l=L}^{\infty} \int_{l}^{l+1} \frac{\log g(\tau x)}{\tau^2}\, d\tau \leqslant \sum_{l=L}^{\infty} \frac{\log g(lx) + M}{l^2},$$

and this implies for all $L \in \mathbb{N}$ that

$$\sum_{l=L+1}^{\infty} \frac{\log g(lx)}{l^2} - \frac{M}{L} \leqslant \int_{L}^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau \leqslant \sum_{l=L}^{\infty} \frac{\log g(lx)}{l^2} + \frac{M}{L-1}. \tag{2.5}$$

Let

$$\begin{aligned} &\mathbf{e}_j := (\underbrace{0,\dots,0}_{j-1},1,0,\dots,0),\ j = 1,\dots,n, \quad \mathbf{e}_0 := \frac{1}{\sqrt{n}}(1,\dots,1), \\ &Q := \left\{ y = (y_1,\dots,y_n) \in \mathbb{R}^n : \frac{1}{2\sqrt{n}} < y_j < \frac{2}{\sqrt{n}},\ j = 1,\dots,n \right\}. \end{aligned} \tag{2.6}$$

For every $x \in \mathbb{S}^{n-1}$, there exists an orthogonal matrix $A_x \in O(n)$ such that $x = A_x \mathbf{e}_0$. Hence, $\{AQ\}_{A\in O(n)}$ is an open cover of $\mathbb{S}^{n-1}$. Let $\{A_k Q\}_{k=1,\dots,K}$ be a finite subcover. Take an arbitrary $\varepsilon > 0$. It follows from (2.3) and (2.5) that there exists some $R_\varepsilon > 0$ for which

$$\int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(\tau A_k \mathbf{e}_j)}{\tau^2}\, d\tau < \frac{\varepsilon}{2\sqrt{n}}, \quad k = 1,\dots,K,\ j = 1,\dots,n.$$

For any $x \in \mathbb{S}^{n-1}$, there exist $k = 1,\dots,K$ and $a_j \in \left(\frac{1}{2\sqrt{n}}, \frac{2}{\sqrt{n}}\right)$, $j = 1,\dots,n$ such that

$$x = \sum_{j=1}^{n} a_j A_k \mathbf{e}_j.$$

Using (2.1), one gets

$$\begin{aligned} \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau &\leqslant \sum_{j=1}^{n} \int_{R_\varepsilon}^{\infty} \frac{\log g(\tau a_j A_k \mathbf{e}_j)}{\tau^2}\, d\tau = \sum_{j=1}^{n} a_j \int_{a_j R_\varepsilon}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2}\, dr \\ &\leqslant \sum_{j=1}^{n} \frac{2}{\sqrt{n}} \int_{\frac{R_\varepsilon}{2\sqrt{n}}}^{\infty} \frac{\log g(r A_k \mathbf{e}_j)}{r^2}\, dr < \sum_{j=1}^{n} \frac{2}{\sqrt{n}} \cdot \frac{\varepsilon}{2\sqrt{n}} = n\,\frac{\varepsilon}{n} = \varepsilon. \quad \blacksquare \end{aligned}$$

Let

$$I_{g,x}(r) := \int_{\max\{r,1\}}^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau < \infty,$$

$$J_{g,x}(r) := \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x)\, d\tau < \infty,$$

$$S_{g,x}(r) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2}\, d\tau \quad r \geqslant 0,\ x \in \mathbb{S}^{n-1}.$$

One has, for $r > 1$ and any $\beta \in (0,1)$,

$$\begin{aligned} J_{g,x}(r) &= \frac{1}{r^2} \int_0^r \log g(\tau x)\, d\tau \\ &= \frac{1}{r^2} \int_0^1 \log g(\tau x)\, d\tau + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{r^{2\beta}}\, d\tau + \int_{r^\beta}^{r} \frac{\log g(\tau x)}{r^2}\, d\tau \\ &\leqslant \frac{M}{r^2} + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{\tau^2}\, d\tau + \int_{r^\beta}^{r} \frac{\log g(\tau x)}{\tau^2}\, d\tau \\ &\leqslant \frac{M}{r^2} + \frac{I_{g,x}(1)}{r^{2(1-\beta)}} + I_{g,x}(r^\beta), \end{aligned} \tag{2.7}$$

see (2.4). Further, if $r > 1$, then

$$\begin{aligned} \pi S_{g,x}(r) &= \int_0^{\infty} \frac{\log g(\tau x)}{\tau^2 + r^2}\, d\tau + \int_0^{\infty} \frac{\log g(-\tau x)}{\tau^2 + r^2}\, d\tau \\ &\leqslant \int_0^{r} \frac{\log g(\tau x)}{r^2}\, d\tau + \int_r^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau \\ &\quad + \int_0^{r} \frac{\log g(-\tau x)}{r^2}\, d\tau + \int_r^{\infty} \frac{\log g(-\tau x)}{\tau^2}\, d\tau \\ &= I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r), \end{aligned} \tag{2.8}$$

and, with a similar calculation,

$$\begin{aligned}\pi S_{g,x}(r) &\geqslant \int_0^r \frac{\log g(\tau x)}{2r^2}\,d\tau + \int_r^\infty \frac{\log g(\tau x)}{2\tau^2}\,d\tau \\ &\quad + \int_0^r \frac{\log g(-\tau x)}{2r^2}\,d\tau + \int_r^\infty \frac{\log g(-\tau x)}{2\tau^2}\,d\tau \\ &= \frac{1}{2}\big(I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r)\big). \qquad (2.9)\end{aligned}$$

Since g is locally bounded, it follows from Lemma 2.1 that I_g defined by

$$I_g(r) := \sup_{x\in\mathbb{S}^{n-1}} I_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \int_{\max\{r,1\}}^\infty \frac{\log g(\tau x)}{\tau^2}\,d\tau < \infty,$$

is a decreasing function such that

$$I_g(r) \to 0 \quad \text{as } r \to \infty. \qquad (2.10)$$

Let

$$\begin{aligned} J_g(r) &:= \sup_{x\in\mathbb{S}^{n-1}} J_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \frac{1}{\max\{r,1\}^2}\int_0^r \log g(\tau x)\,d\tau, \\ S_g(r) &:= \sup_{x\in\mathbb{S}^{n-1}} S_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \frac{1}{\pi}\int_{-\infty}^\infty \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2}\,d\tau. \qquad (2.11)\end{aligned}$$

Then, in view of (2.7), (2.8), and (2.9),

$$J_g(r) \leqslant \frac{M}{r^2} + \frac{I_g(1)}{r^{2(1-\beta)}} + I_g(r^\beta),$$

$$\frac{1}{2\pi}\max\{I_g(r), J_g(r)\} \leqslant S_g(r) \leqslant \frac{2}{\pi}(I_g(r) + J_g(r)).$$

Thus, $J_g(r) \to 0$, and

$$S_g(r) \to 0 \quad \text{as } r \to \infty, \qquad (2.12)$$

see (2.10). It is clear that

$$S_g(r) = S_g(1) \text{ for } r \in [0,1] \quad \text{and} \quad S_g \text{ is a decreasing function.} \qquad (2.13)$$

Examples. (1) If $g(x) = (1 + |x|)^s$, $s \geqslant 0$, then we have for all $r \geqslant 1$

$$\begin{aligned}
S_g(r) &= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{s \log(1+|\tau|)}{\tau^2 + r^2}\, d\tau \\
&= \frac{s}{\pi r} \int\limits_{-\infty}^{\infty} \frac{\log(1+r|\lambda|)}{\lambda^2+1}\, d\lambda \\
&\leqslant \frac{s}{\pi r} \int\limits_{-\infty}^{\infty} \frac{\log(1+|\lambda|)}{\lambda^2+1}\, d\lambda + \frac{s\log(1+r)}{\pi r} \int\limits_{-\infty}^{\infty} \frac{1}{\lambda^2+1}\, d\lambda \\
&= \frac{c_1 s}{r} + \frac{s \log(1+r)}{r}, \qquad (2.14)
\end{aligned}$$

where

$$c_1 := \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\log(1+|\lambda|)}{\lambda^2+1}\, d\lambda < \infty.$$

(2) If $g(x) = (\log(e + |x|))^t$, $t \geqslant 0$, then using the obvious inequality

$$u + v \leqslant 2uv, \quad u, v \geqslant 1,$$

yields for $r \geqslant 1$

$$\begin{aligned}
S_g(r) &= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{t \log\log(e+|\tau|)}{\tau^2+r^2}\, d\tau = \frac{t}{\pi r} \int\limits_{-\infty}^{\infty} \frac{\log\log(e+r|\lambda|)}{\lambda^2+1}\, d\lambda \\
&\leqslant \frac{t}{\pi r} \int\limits_{-\infty}^{\infty} \frac{\log\big(\log(e+|\lambda|) + \log(e+r)\big)}{\lambda^2+1}\, d\lambda \\
&\leqslant \frac{t}{\pi r} \int\limits_{-\infty}^{\infty} \frac{\log(2\log(e+|\lambda|))}{\lambda^2+1}\, d\lambda + \frac{t \log\log(e+r)}{\pi r} \int\limits_{-\infty}^{\infty} \frac{1}{\lambda^2+1}\, d\lambda \\
&= \frac{c_2 t}{r} + \frac{t \log\log(e+r)}{r}, \qquad (2.15)
\end{aligned}$$

where

$$c_2 := \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\log(2\log(e+|\lambda|))}{\lambda^2+1}\, d\lambda < \infty.$$

(3) If $g(x) = e^{a|x|^b}$, $a \geqslant 0$, $b \in [0,1)$, then we have for all $r \geqslant 1$

$$S_g(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a|\tau|^b}{\tau^2+r^2}\,d\tau = \frac{ar^{b-1}}{\pi}\int_{-\infty}^{\infty} \frac{|\lambda|^b}{\lambda^2+1}\,d\lambda = \frac{2ar^{b-1}}{\pi}\int_0^{\infty} \frac{t^b}{t^2+1}\,dt$$
$$= \frac{ar^{b-1}}{\pi}\int_0^{\infty} \frac{s^{\frac{b-1}{2}}}{s+1}\,ds = \frac{ar^{b-1}}{\sin\left(\frac{1-b}{2}\pi\right)}, \tag{2.16}$$

see, e.g., [4, Chapter V, Example 2.12].

(4) Finally, let $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma > 1$. Since

$$\frac{\tau(e+\tau)}{\tau^2+r^2} = \frac{1+\frac{e}{\tau}}{1+\frac{r^2}{\tau^2}} \leqslant 1 + \frac{e}{\tau} \leqslant 1 + \frac{e}{r} \quad \text{for } \tau \geqslant r,$$

then for any $\beta \in (0,1)$ and $r \geqslant 1$

$$\begin{aligned}
S_g(r) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|}{(\tau^2+r^2)\log^\gamma(e+|\tau|)}\,d\tau = \frac{2}{\pi}\int_0^{\infty} \frac{\tau}{(\tau^2+r^2)\log^\gamma(e+\tau)}\,d\tau \\
&= \frac{2}{\pi}\left(\int_0^{r^\beta} + \int_{r^\beta}^{r} + \int_r^{\infty}\right)\frac{\tau}{(\tau^2+r^2)\log^\gamma(e+\tau)}\,d\tau \\
&\leqslant \frac{2}{\pi}\int_0^{r^\beta} \frac{\tau}{\tau^2+r^2}\,d\tau + \frac{2}{\pi\log^\gamma(e+r^\beta)}\int_{r^\beta}^{r} \frac{\tau}{\tau^2+r^2}\,d\tau \\
&\quad + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\int_r^{\infty} \frac{1}{(e+\tau)\log^\gamma(e+\tau)}\,d\tau \\
&= \frac{1}{\pi}\log(\tau^2+r^2)\Big|_0^{r^\beta} + \frac{1}{\pi\log^\gamma(e+r^\beta)}\log(\tau^2+r^2)\Big|_{r^\beta}^{r} \\
&\quad + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{1-\gamma}\log^{1-\gamma}(e+\tau)\Big|_r^{\infty} \\
&\leqslant \frac{1}{\pi}\log(1+r^{2(\beta-1)}) + \frac{\log 2}{\pi\log^\gamma(e+r^\beta)} + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r) \\
&\leqslant \frac{r^{2(\beta-1)}}{\pi} + \frac{\log 2}{\pi\log^\gamma(e+r^\beta)} + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r).
\end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{r^{2(\beta-1)} + (\log 2)\log^{-\gamma}(e+r^\beta)}{\log^{-\gamma}(e+r)} = \frac{\log 2}{\beta^\gamma} \quad \text{for all } \beta \in (0,1),$$

one gets, if we take $\beta \in ((\log 2)^{1/\gamma}, 1)$, the following estimate

$$S_g(r) \leqslant \frac{\log^{-\gamma}(e+r)}{\pi} + \frac{2}{\pi}\Big(1+\frac{e}{r}\Big)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r) \tag{2.17}$$

for sufficiently large r.

3. Estimates for entire functions

Let $1 \leqslant p \leqslant \infty$ and let $\omega\colon \mathbb{R}^n \to [0,\infty)$ be a measurable function such that $\omega > 0$ Lebesgue almost everywhere. We set

$$\|f\|_{L^p_\omega} := \|\omega f\|_{L^p} \quad \text{and} \quad L^p_\omega(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{C} \mid f \text{ measurable}, \|f\|_{L^p_\omega} < \infty\}. \tag{3.1}$$

Lemma 3.1. *Let $g\colon \mathbb{R}^n \to [1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let φ be a measurable function such that for almost every $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, $\varphi(z_1, x')$ is analytic in z_1 for $\operatorname{Im} z_1 > 0$ and continuous up to $\mathbb{R}$. Suppose also that $\log|\varphi(z_1, x')| = O(|z_1|)$ for $|z_1|$ large, $\operatorname{Im} z_1 \geqslant 0$, and that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm1}}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Finally, suppose that*

$$k_\varphi := \operatorname{ess\,sup}_{x' \in \mathbb{R}^{n-1}} \Big(\limsup_{0<y_1\to\infty} \frac{\log|\varphi(iy_1, x')|}{y_1}\Big) < \infty.$$

Then

$$\|\varphi(\cdot + iy_1, \cdot)\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)} \leqslant C_g e^{(k_\varphi + S_g(y_1))y_1} \|\varphi\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)}, \quad y_1 > 0, \tag{3.2}$$

see (2.11), (2.12), where the constant $C_g < \infty$ depends only on g.

Proof. Let $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. It follows from (2.1) that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log^+(g^{\mp1}(t,x'))}{1+t^2}\,dt &\leqslant \int_{-\infty}^{\infty} \frac{\log(g(t,x'))}{1+t^2}\,dt \\ &\leqslant \int_{-\infty}^{\infty} \frac{\log(g(t,0)) + \log(g(0,x'))}{1+t^2}\,dt \\ &\leqslant \pi(S_g(1) + \log(g(0,x'))) < +\infty.\end{aligned}$$

Since $g^{\pm1}\varphi \in L^p(\mathbb{R}^n)$, Fubini's theorem implies that

$$g^{\pm1}(\cdot, x')\varphi(\cdot, x') \in L^p(\mathbb{R})$$

for Lebesgue almost all $x' \in \mathbb{R}^{n-1}$. For such $x' \in \mathbb{R}^{n-1}$,

$$\int_{-\infty}^{\infty} \frac{\log^+ |\varphi(t,x')|}{1+t^2}\,dt \leqslant \int_{-\infty}^{\infty} \frac{\log^+ (g^{\pm 1}(t,x')|\varphi(t,x')|)}{1+t^2}\,dt + \int_{-\infty}^{\infty} \frac{\log^+ (g^{\mp 1}(t,x'))}{1+t^2}\,dt < \infty.$$

Then

$$\log|\varphi(x_1+iy_1,x')| \leqslant k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t,x')|}{(t-x_1)^2+y_1^2}\,dt, \quad x_1 \in \mathbb{R},\ y_1 > 0,$$

cf. [20, Chapter III, G, 2], see also [22, Chapter V, Theorems 5 and 7].

Applying (2.1) again, one gets

$$\log g(x) \leqslant \log g(t,x') + \log g(x_1-t,0),$$
$$\log g(t,x') \leqslant \log g(x) + \log g(t-x_1,0) \quad \text{for all } x=(x_1,x') \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

The latter inequality can be rewritten as follows

$$\log g^{-1}(x) \leqslant \log g^{-1}(t,x') + \log g(t-x_1,0).$$

Hence,

$$\log g^{\pm 1}(x) \leqslant \log g^{\pm 1}(t,x') + \log g(\pm(x_1-t),0)$$

for all $x = (x_1,x') \in \mathbb{R}^n$, $t \in \mathbb{R}$, and

$$\begin{aligned}
&\log(|\varphi(x_1+iy_1,x')|g^{\pm 1}(x)) \\
&\leqslant k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t,x')|}{(t-x_1)^2+y_1^2}\,dt + \log g^{\pm 1}(x) \\
&= k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t,x')| + \log g^{\pm 1}(x)}{(t-x_1)^2+y_1^2}\,dt \\
&\leqslant k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log(|\varphi(t,x')|g^{\pm 1}(t,x'))}{(t-x_1)^2+y_1^2}\,dt + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log g(\pm(x_1-t),0)}{(t-x_1)^2+y_1^2}\,dt \\
&= k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log(|\varphi(t,x')|g^{\pm 1}(t,x'))}{(t-x_1)^2+y_1^2}\,dt + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log g(\tau,0)}{\tau^2+y_1^2}\,d\tau.
\end{aligned}$$

If $y_1 \in [0,1]$, then

$$\begin{aligned}\frac{y_1}{\pi}\int_0^\infty \frac{\log g(\tau,0)}{\tau^2+y_1^2}\,d\tau &\leqslant M\frac{y_1}{\pi}\int_0^1 \frac{1}{\tau^2+y_1^2}\,d\tau + \frac{y_1}{\pi}\int_1^\infty \frac{\log g(\tau,0)}{\tau^2+y_1^2}\,d\tau \\ &\leqslant M\frac{y_1}{\pi}\int_{\mathbb{R}} \frac{1}{\tau^2+y_1^2}\,d\tau + \frac{1}{\pi}\int_1^\infty \frac{\log g(\tau,0)}{\tau^2}\,d\tau \\ &\leqslant M + \frac{I_g(1)}{\pi}.\end{aligned}$$

It follows from (2.11) that for $y_1 > 1$,

$$\frac{y_1}{\pi}\int_{-\infty}^\infty \frac{\log g(\tau,0)}{\tau^2+y_1^2}\,d\tau \leqslant y_1 S_g(y_1).$$

So,

$$\begin{aligned}\log\left(|\varphi(x_1+iy_1,x')|g^{\pm 1}(x)\right) \leqslant c_g + (k_\varphi + S_g(y_1))y_1 \\ + \frac{y_1}{\pi}\int_{-\infty}^\infty \frac{\log\left(|\varphi(t,x')|g^{\pm 1}(t,x')\right)}{(t-x_1)^2+y_1^2}\,dt,\end{aligned}$$

where $c_g := M + \frac{I_g(1)}{\pi}$. Using Jensen's inequality, one gets

$$|\varphi(x_1+iy_1,x')|g^{\pm 1}(x) \leqslant C_g e^{(k_\varphi+S_g(y_1))y_1}\frac{y_1}{\pi}\int_{-\infty}^\infty \frac{|\varphi(t,x')|g^{\pm 1}(t,x')}{(t-x_1)^2+y_1^2}\,dt,$$

where

$$C_g := e^{M+\frac{I_g(1)}{\pi}}. \tag{3.3}$$

Estimate (3.2) now follows from Young's convolution inequality and (3.1). ∎

Remark 3.2. Let $n = 1$, $g\colon \mathbb{R} \to [1,\infty)$ be a Hölder continuous submultiplicative function satisfying the Beurling–Domar condition, $g(0) = 1$, and let

$$\begin{aligned}w(x+iy) := \frac{y}{\pi}\int_{-\infty}^\infty \frac{\log g(t)}{(t-x)^2+y^2}\,dt \\ + \frac{i}{\pi}\int_{-\infty}^\infty \left(\frac{x-t}{(t-x)^2+y^2} + \frac{t}{t^2+1}\right)\log g(t)\,dt, \quad x \in \mathbb{R},\ y > 0.\end{aligned}$$

Then $\varphi(z) := e^{w(z)}$ is analytic in z for $\operatorname{Im} z > 0$ and continuous up to $\mathbb{R}$,

$$|\varphi(x)| = e^{\operatorname{Re}(w(x))} = e^{\log g(x)} = g(x), \quad x \in \mathbb{R},$$

see, e.g., [8, Chapter III, Section 1], and

$$|\varphi(iy)| = e^{\operatorname{Re}(w(iy))} = \exp\Big(\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log g(t)}{t^2 + y^2}\,dt\Big) = e^{S_g(y)y}, \quad y \geqslant 1.$$

So,

$$k_\varphi = \limsup_{0<y\to\infty} \frac{\log|\varphi(iy)|}{y} = \limsup_{y\to\infty} S_g(y) = 0$$

see (2.12), and

$$\begin{aligned}\|\varphi(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R})} \geqslant \frac{|\varphi(iy)|}{g(0)} &= |\varphi(iy)| = e^{S_g(y)y} = e^{S_g(y)y}\|1\|_{L^\infty(\mathbb{R})} \\ &= e^{S_g(y)y}\|g^{-1}\varphi\|_{L^\infty(\mathbb{R})} = e^{S_g(y)y}\|\varphi\|_{L^\infty_{g^{-1}}(\mathbb{R})},\end{aligned}$$

which shows that the factor $e^{S_g(y_1)y_1}$ in the right-hand side of (3.2) is optimal in this case.

Clearly,

$$S_{\breve{g}} = S_g, \quad C_{\breve{g}} = C_g, \tag{3.4}$$

where $\breve{g}(x) := g(Ax)$ and $A \in O(n)$ is an arbitrary orthogonal matrix, see (2.11), (3.3) and (2.4).

Theorem 3.3. *Let $g\colon \mathbb{R}^n \to [1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function such that $\log|\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and suppose that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm 1}}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Then, for every multi-index $\alpha \in \mathbb{Z}^n_+$,*

$$\|(\partial^\alpha\varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \leqslant C_\alpha e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|}\|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{3.5}$$

where

$$\kappa_\varphi(\omega) := \sup_{x\in\mathbb{R}^n}\Big(\limsup_{0<t\to\infty} \frac{\log|\varphi(x + it\omega)|}{t}\Big) < \infty, \quad \omega \in \mathbb{S}^{n-1}, \tag{3.6}$$

and the constant $C_\alpha \in (0,\infty)$ depends only on α and g.

Proof (*Cf. the proof of* [21, Lemma 9.29]). Take any $y \in \mathbb{R}^n \setminus \{0\}$. There exists an orthogonal matrix $A \in O(n)$ such that $A\mathbf{e}_1 = \omega := y/|y|$, see (2.6). Define the functions $\breve{\varphi}(z) := \varphi(Az)$, $z \in \mathbb{C}^n$, and $\breve{g}(x) := g(Ax)$, $x \in \mathbb{R}^n$. Then $\breve{\varphi}\colon \mathbb{C}^n \to \mathbb{C}$ is an entire function, and one can apply to it Lemma 3.1 with $\breve{g}$ in place of g, see (3.4).

For any $x \in \mathbb{R}^n$, one has $\varphi(x+iy) = \breve{\varphi}(\tilde{x} + i|y|\mathbf{e}_1) = \breve{\varphi}(\tilde{x}_1 + i|y|, \tilde{x}_2, \dots, \tilde{x}_n)$, where $\tilde{x} := A^{-1}x$. Hence,

$$\begin{aligned}
&\|\varphi(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \\
&\quad = \|\breve{\varphi}(\cdot + i|y|, \cdot)\|_{L^p_{\breve{g}^{\pm 1}}(\mathbb{R}^n)} \leqslant C_{\breve{g}} e^{(k_{\breve{\varphi}} + S_{\breve{g}}(|y|))|y|} \|\breve{\varphi}\|_{L^p_{\breve{g}^{\pm 1}}(\mathbb{R}^n)} \\
&\quad \leqslant C_g e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\breve{\varphi}\|_{L^p_{\breve{g}^{\pm 1}}(\mathbb{R}^n)} = C_g e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)},
\end{aligned}$$

see (3.4), which proves (3.5) for $\alpha = 0$ and $y \neq 0$. This estimate is trivial for $\alpha = 0$ and $y = 0$.

Iterating the standard Cauchy integral formula for one complex variable, one gets

$$\varphi(\zeta) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)} \Big(\prod_{k=1}^n e^{i\theta_k} \Big) d\theta_1 \dots d\theta_n,$$
$$\zeta \in \Delta(z) := \{\eta \in \mathbb{C}^n : |\eta_k - z_k| < 1,\ k = 1, \dots, n\},\ z \in \mathbb{C}^n$$

(cf. [21, Chapter 1, Section 1]), which implies

$$\partial^\alpha \varphi(\zeta) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)^{\alpha_k + 1}} \Big(\prod_{k=1}^n e^{i\theta_k} \Big) d\theta_1 \dots d\theta_n.$$

Hence,

$$\partial^\alpha \varphi(z) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n e^{i\alpha_k \theta_k}} \, d\theta_1 \dots d\theta_n,$$

and

$$|\partial^\alpha \varphi(z)| \leqslant \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})| \, d\theta_1 \dots d\theta_n. \tag{3.7}$$

Since $g \geqslant 1$ is locally bounded,

$$1 \leqslant M_1 := \sup_{|s_k| \leqslant 1, k = 1, \dots, n} g(s) < \infty.$$

Then it follows from (2.1) that

$$g^{\pm 1}(x_1 - \cos\theta_1, \dots, x_n - \cos\theta_n) \leqslant M_1 g^{\pm 1}(x). \tag{3.8}$$

According to the conditions of the theorem, there exists a constant $c_\varphi \in (0, \infty)$ such that one has $\log|\varphi(\zeta)| \leqslant c_\varphi|\zeta|$ for $|\zeta|$ large. Then $\kappa_\varphi(\omega) \leqslant c_\varphi$, see (3.6). Let one set $\varphi_y := \varphi(\cdot + iy)$, $y = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$. Then, similarly to the above inequality, $\kappa_{\varphi_y}(\omega) \leqslant c_\varphi$. Applying (3.5) with $\alpha = 0$ to the function φ_y in place of φ and using (2.13), (3.8), one derives from (3.7)

$$\begin{aligned}
&\|(\partial^\alpha\varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \\
&\leqslant \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \|\varphi(\cdot + iy_1 + e^{i\theta_1}, \dots, \cdot + iy_n + e^{i\theta_n})\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}\, d\theta_1 \dots d\theta_n \\
&\leqslant \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 \|\varphi(\cdot + iy_1 + i\sin\theta_1, \dots, \cdot + iy_n + i\sin\theta_n)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \\
&\qquad \otimes\, d\theta_1 \dots d\theta_n \\
&\leqslant \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}\, d\theta_1 \dots d\theta_n \\
&= \alpha! M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}.
\end{aligned}$$

Applying (3.5) with $\alpha = 0$ again, one gets

$$\begin{aligned}
&\|(\partial^\alpha\varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \\
&\quad \leqslant \alpha! M_1 C_0^2 e^{(c_\varphi + S_g(1))\sqrt{n}} e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}.
\end{aligned}$$

■

Corollary 3.4. *Let $g\colon \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3). Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function such that $\log|\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm 1}}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Then, for every multi-index $\alpha \in \mathbb{Z}^n_+$ and every $\varepsilon > 0$,*

$$\|(\partial^\alpha\varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \leqslant C_{\alpha,\varepsilon} e^{(\kappa_\varphi(y/|y|) + \varepsilon)|y|} \|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{3.9}$$

where κ_φ is defined by (3.6), and the constant $C_{\alpha,\varepsilon} \in (0, \infty)$ depends only on α, ε, and g.

Proof. It follows from (2.12) that, for every $\varepsilon > 0$, there exists some c_ε such that

$$S_g(|y|)|y| \leqslant c_\varepsilon + \varepsilon|y| \quad \text{for all } y \in \mathbb{R}^n.$$

Hence, (3.5) implies (3.9). ■

4. Main results

We will use the notation $\tilde{g}(x) := g(-x)$, $x \in \mathbb{R}^n$. It follows from submultiplicativity of $\tilde{g}$ that $L^1_{\tilde{g}}(\mathbb{R}^n)$ is a convolution algebra.

Taking $y - x$ in place of y in (2.1) and rearranging, one gets

$$\frac{1}{g(x)} \leqslant \frac{g(y-x)}{g(y)}. \tag{4.1}$$

Using (4.1), one can easily show that $f * u \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ for every $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ and $u \in L^1_{\tilde{g}}(\mathbb{R}^n)$. The Fubini–Tonelli theorem implies that

$$f * (v * u) = (f * v) * u \quad \text{for all } f \in L^\infty_{g^{-1}}(\mathbb{R}^n) \text{ and } v, u \in L^1_{\tilde{g}}(\mathbb{R}^n). \tag{4.2}$$

Let $A_{\tilde{g}} := \{c\delta + g \mid c \in \mathbb{C}, g \in L^1_{\tilde{g}}(\mathbb{R}^n)\}$, where δ is the Dirac measure on $\mathbb{R}^n$. This is the algebra $L^1_{\tilde{g}}(\mathbb{R}^n)$ with a unit attached, cf. Rudin [24, 10.3 (d), 11.13 (e)]. Clearly, (4.2) holds for any $v, u \in A_{\tilde{g}}$.

Theorem 4.1. *Let $g\colon \mathbb{R}^n \to [1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3), $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$, and Y be a linear subspace of $L^1_{\tilde{g}}(\mathbb{R}^n)$ such that*

$$f * v = 0 \quad \textit{for every } v \in Y. \tag{4.3}$$

Suppose the set

$$Z(Y) := \bigcap_{v \in Y} \{\xi \in \mathbb{R}^n \mid \hat{v}(\xi) = 0\}$$

is bounded, and $u \in L^1_{\tilde{g}}(\mathbb{R}^n)$ is such that $\hat{u} = 1$ in a neighbourhood of $Z(Y)$. Then $f = f * u$. If $Z(Y) = \emptyset$, then $f = 0$.

Proof. In order to prove the equality $f = f * u$, it is sufficient to show that

$$\langle f, h \rangle = \langle f * u, h \rangle \quad \text{for every } h \in L^1_g(\mathbb{R}^n). \tag{4.4}$$

Since the set of functions h with compactly supported Fourier transforms $\hat{h}$ is dense in $L^1_g(\mathbb{R}^n)$, see [5, Theorem 1.52 and 2.11], it is enough to prove (4.4) for such h. Further,

$$\langle f, h \rangle = (f * \tilde{h})(0).$$

So, we have to show only that

$$f * w = f * u * w$$

for every $w \in L^1_{\tilde{g}}(\mathbb{R}^n)$ with compactly supported Fourier transform $\widehat{w}$. Take any such w and choose $R > 0$ such that the support of $\widehat{w}$ lies in $B_R := \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$. It is clear that $\tilde{g}$ satisfies the Beurling–Domar condition. Then there exists $u_R \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $0 \leqslant \widehat{u_R} \leqslant 1$, $\widehat{u_R}(\xi) = 1$ for $|\xi| \leqslant R$, and $\widehat{u_R}(\xi) = 0$ for $|\xi| \geqslant R+1$, see [5, Lemma 1.24].

If $Z(Y) \neq \emptyset$, let V be an open neighbourhood of $Z(Y)$ such that $\hat{u} = 1$ in V. Similarly to the above, there exists $u_0 \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $0 \leqslant \widehat{u_0} \leqslant 1$, $\widehat{u_0} = 1$ in a neighbourhood $V_0 \subset V$ of $Z(Y)$, and $\widehat{u_0} = 0$ outside V, see [5, Lemma 1.24]. If $Z(Y) = \emptyset$, one can take $u = u_0 = 0$ and $V_0 = \emptyset$ below.

Since Y is a linear subspace, for every $\eta \in B_{R+1} \setminus V_0 \subset \mathbb{R}^n \setminus Z(Y)$, there exists $v_\eta \in Y$ such that $\widehat{v_\eta}(\eta) = 1$. Since $v_\eta \in L^1(\mathbb{R}^n)$, $\widehat{v_\eta}$ is continuous, and there is a neighbourhood V_η of η such that $|\widehat{v_\eta}(\xi) - 1| < 1/2$ for all $\xi \in V_\eta$. Similarly to the above, there exists $u_\eta \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $\operatorname{Re}(\widehat{v_\eta}\widehat{u_\eta}) \geqslant 0$, and $\operatorname{Re}(\widehat{v_\eta}\widehat{u_\eta}) > \frac{1}{2}$ in a neighbourhood $V^0_\eta \subset V_\eta$ of η.

Since $B_{R+1} \setminus V_0$ is compact, the open cover $\{V^0_\eta\}_{\eta \in B_{R+1} \setminus V_0}$ has a finite subcover. So, there exist functions $v_j \in Y$ and $u_j \in L^1_{\tilde{g}}(\mathbb{R}^n)$, $j = 1, \dots, N$, such that

$$\operatorname{Re}(\sigma) > \frac{1}{2}, \quad \text{where } \sigma := \widehat{u_0} + \sum_{j=1}^{N} \widehat{v_j}\widehat{u_j} + 1 - \widehat{u_R}.$$

Then there exists $\upsilon \in A_{\tilde{g}}$ such that $\hat{\upsilon} = 1/\sigma$, see [5, Theorem 1.53].

Since $\widehat{u_0}(1 - \hat{u}) = 0$ and $(1 - \widehat{u_R})\widehat{w} = 0$, one has

$$\begin{aligned}
\Big(\hat{u} + \sum_{j=1}^{N} \widehat{v_j}\widehat{u_j}\hat{\upsilon}(1-\hat{u})\Big)\widehat{w} &= \big(\hat{u} + (\sigma - (\widehat{u_0} + 1 - \widehat{u_R}))\hat{\upsilon}(1-\hat{u})\big)\widehat{w} \\
&= \big(\hat{u} + (1-\hat{u}) - (\widehat{u_0} + 1 - \widehat{u_R})\hat{\upsilon}(1-\hat{u})\big)\widehat{w} \\
&= \big(1 - (1 - \widehat{u_R})\hat{\upsilon}(1-\hat{u})\big)\widehat{w} \\
&= \widehat{w} - (1 - \widehat{u_R})\widehat{w}\hat{\upsilon}(1-\hat{u}) = \widehat{w}.
\end{aligned}$$

It now follows from (4.2) and (4.3) that

$$\begin{aligned}
f * w &= f * \Big(u + \sum_{j=1}^{N} v_j * u_j * (\upsilon - \upsilon * u)\Big) * w \\
&= f * u * w + f * \Big(\sum_{j=1}^{N} v_j * u_j * (\upsilon - \upsilon * u)\Big) * w \\
&= f * u * w + \sum_{j=1}^{N} (f * v_j) * u_j * (\upsilon - \upsilon * u) * w = f * u * w.
\end{aligned}$$

If $Z(Y) = \emptyset$, one can take $u = 0$, and the equality $f = f * u$ means that $f = 0$. ∎

For a bounded set $E \subset \mathbb{R}^n$, let $\operatorname{conv}(E)$ denote its closed convex hull, and H_E denote its support function:

$$H_E(y) := \sup_{\xi \in E} y \cdot \xi = \sup_{\xi \in \operatorname{conv}(E)} y \cdot \xi, \quad y \in \mathbb{R}^n.$$

Clearly, H_E is positively homogeneous and convex: for all $x, y \in \mathbb{R}^n$ and $\tau \geqslant 0$, we have

$$H_E(\tau y) = \tau H_E(y), \quad H_E(y + x) \leqslant H_E(y) + H_E(x).$$

For every positively homogeneous convex function H,

$$K := \{\xi \in \mathbb{R}^n \mid y \cdot \xi \leqslant H(y) \text{ for all } y \in \mathbb{R}^n\} \tag{4.5}$$

is the unique convex compact set such that $H_K = H$, see, e.g., [15, Theorem 4.3.2].

Theorem 4.2. *Let g, f, and Y satisfy the conditions of Theorem 4.1, and let*

$$\mathcal{H}_Y(y) := H_{Z(Y)}(-y) = \sup_{\xi \in Z(Y)} (-y) \cdot \xi = -\inf_{\xi \in Z(Y)} y \cdot \xi, \quad y \in \mathbb{R}^n.$$

Then f admits analytic continuation to an entire function $f : \mathbb{C}^n \to \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}^n_+$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \leqslant C_\alpha e^{\mathcal{H}_Y(y) + S_g(|y|)|y|} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{4.6}$$

see (2.11), (2.12), *where the constant $C_\alpha \in (0, \infty)$ depends only on α and g.*

Proof. Take any $\varepsilon > 0$. There exists $u \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $\hat{u} = 1$ in a neighbourhood of $Z(Y)$, and $\hat{u} = 0$ outside the $\frac{\varepsilon}{2}$-neighbourhood of $Z(Y)$, see [5, Lemma 1.24]. It follows from the Paley–Wiener–Schwartz theorem, see, e.g., [15, Theorem 7.3.1] that $u = \mathcal{F}^{-1}\hat{u}$ admits analytic continuation to an entire function $u : \mathbb{C}^n \to \mathbb{C}$ satisfying the estimate

$$|u(x + iy)| \leqslant c_\varepsilon e^{\mathcal{H}_Y(y) + \varepsilon|y|/2} \quad \text{for all } x, y \in \mathbb{R}^n$$

with some constant $c_\varepsilon \in (0, \infty)$. So, u satisfies the conditions of Corollary 3.4 with $\tilde{g}$ in place of g, and

$$\|u(\cdot + iy)\|_{L^1_{\tilde{g}}(\mathbb{R}^n)} \leqslant C_{0,\varepsilon/2} e^{\mathcal{H}_Y(y) + \varepsilon|y|} \|u\|_{L^1_{\tilde{g}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n.$$

Since

$$f(x) = \int_{\mathbb{R}^n} u(x - s) f(s)\, ds,$$

see Theorem 4.1, f admits analytic continuation

$$f(x+iy) := \int_{\mathbb{R}^n} u(x+iy-s) f(s)\, ds,$$

see Corollary 3.4, and

$$\begin{aligned} \|f(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} &\leqslant \|u(\cdot+iy)\|_{L^1_{\tilde g}(\mathbb{R}^n)} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \\ &\leqslant C_{0,\varepsilon/2} e^{\mathcal{H}_Y(y)+\varepsilon|y|} \|u\|_{L^1_{\tilde g}(\mathbb{R}^n)} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \\ &=: M_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \end{aligned}$$

see (4.1). Since

$$\frac{|f(x+iy)|}{g(x)} \leqslant M_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)},$$

one has $\log|f(x+iy)| = O(|x+iy|)$ for $|x+iy|$ large, see (2.2), and

$$\begin{aligned} \limsup_{0<t\to\infty} \frac{\log|f(x+it\omega)|}{t} &\leqslant \limsup_{0<t\to\infty} \frac{\log(M_\varepsilon g(x)\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}) + t\mathcal{H}_Y(\omega) + \varepsilon t}{t} \\ &= \mathcal{H}_Y(\omega) + \varepsilon. \end{aligned}$$

Hence,

$$\kappa_f(\omega) := \sup_{x\in\mathbb{R}^n} \left(\limsup_{0<t\to\infty} \frac{\log|f(x+it\omega)|}{t} \right) \leqslant \mathcal{H}_Y(\omega) + \varepsilon$$

for every $\varepsilon > 0$, i.e.,

$$\kappa_f(\omega) \leqslant \mathcal{H}_Y(\omega).$$

So, (4.6) follows from Theorem 3.3. ∎

Theorem 4.3. *Let $g: \mathbb{R}^n \to [1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3), and let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \widetilde{m}(D)\varphi := \mathcal{F}^{-1}(\widetilde{m}\hat{\varphi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_g(\mathbb{R}^n)$. Suppose $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e.,

$$\langle f, \widetilde{m}(D)\varphi\rangle = 0 \quad \textit{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{4.7}$$

If $K := \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$ *is compact, then* f *admits analytic continuation to an entire function* $f : \mathbb{C}^n \to \mathbb{C}$ *such that for every multi-index* $\alpha \in \mathbb{Z}_+^n$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \leqslant C_\alpha e^{H(y)+S_g(|y|)|y|} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{4.8}$$

see (2.11), (2.12), *where* $H(y) := H_K(-y)$, *and the constant* $C_\alpha \in (0, \infty)$ *depends only on* α *and* g.

Conversely, if every $f \in L^\infty(\mathbb{R}^n)$ *satisfying* (4.7) *admits analytic continuation to an entire function* $f : \mathbb{C}^n \to \mathbb{C}$ *such that*

$$\|f(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \leqslant M_\varepsilon e^{H(y)+\varepsilon|y|} \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{4.9}$$

holds for every $\varepsilon > 0$ *with a constant* $M_\varepsilon \in (0, \infty)$ *that depends only on* ε, m, *and* g, *then* $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq K$, *where* K *is the unique convex compact set such that* $H_K(y) = H(-y)$; *cf.* (4.5).

Proof. Denote by $(T_\upsilon \varphi)(x) := \varphi(x - \upsilon)$, $x, \upsilon \in \mathbb{R}^n$ the shift by υ. Since $T_\upsilon \varphi \in C_c^\infty(\mathbb{R}^n)$ for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and all $\upsilon \in \mathbb{R}^n$, it follows from (4.7) that

$$(f * \widetilde{\tilde{m}(D)\varphi})(\upsilon) = \langle f, T_\upsilon \tilde{m}(D)\varphi \rangle = \langle f, \tilde{m}(D)(T_\upsilon \varphi)\rangle = 0 \quad \text{for all } \upsilon \in \mathbb{R}^n.$$

Hence,

$$f * \widetilde{\tilde{m}(D)\varphi} = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

It is easy to see that

$$\begin{aligned} \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid \widehat{\widetilde{\tilde{m}(D)\varphi}}(\eta) = 0\} &= \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid \widehat{\tilde{m}(D)\varphi}(-\eta) = 0\} \\ &= \bigcap_{\varphi \in C_c^\infty(\mathbb{R}^n)} \{\eta \in \mathbb{R}^n \mid m(\eta)\hat{\varphi}(-\eta) = 0\} \\ &= \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = K. \end{aligned}$$

Applying Theorem 4.2 with

$$Y := \{\widetilde{\tilde{m}(D)\varphi} \mid \varphi \in C_c^\infty(\mathbb{R}^n)\} \subset L^1_{\tilde{g}}(\mathbb{R}^n)$$

and $Z(Y) = K$, one gets (4.8).

For the converse direction, we assume the contrary, that is, that the zero-set $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$ contains some $\gamma \notin K$, see (4.5). Then there exists a $y_0 \in \mathbb{R}^n \setminus \{0\}$ such that $y_0 \cdot \gamma > H_K(y_0) = H(-y_0)$. It is easy to see that $f(x) := e^{ix\cdot\gamma}$ satisfies $m(D)e^{ix\cdot\gamma} = e^{ix\cdot\gamma} m(\gamma) = 0$ for all $x \in \mathbb{R}^n$. Take $\varepsilon < (y_0 \cdot \gamma - H(-y_0))/|y_0|$. Clearly,

$f \in L^\infty(\mathbb{R}^n)$, and

$$\frac{\|f(\cdot - i\tau y_0)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}}{e^{H(-\tau y_0)+\varepsilon|\tau y_0|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}} = \frac{e^{\tau(y_0\cdot\gamma)}}{e^{\tau(H(-y_0)+\varepsilon|y_0|)}} = e^{\tau(y_0\cdot\gamma - H(-y_0)-\varepsilon|y_0|)} \xrightarrow[\tau\to\infty]{} \infty.$$

So, f does not satisfy (4.9). ■

Corollary 4.4. *Let $g\colon \mathbb{R}^n \to [1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling–Domar condition (2.3) and let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \widetilde{m}(D)\varphi := \mathcal{F}^{-1}(\widetilde{m}\hat{\varphi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_g(\mathbb{R}^n)$. Suppose $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e., (4.7) holds. If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \{0\}$, then f admits analytic continuation to an entire function $f\colon \mathbb{C}^n \to \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}^n_+$,

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \leqslant C_\alpha e^{S_g(|y|)|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{4.10}$$

where the constant $C_\alpha \in (0,\infty)$ depends only on α and g. If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$, then $f = 0$.

Conversely, if every $f \in L^\infty(\mathbb{R}^n)$ satisfying (4.7) admits analytic continuation to an entire function $f\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\|f(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \leqslant M_\varepsilon e^{\varepsilon|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n,$$

holds for every $\varepsilon > 0$ with a constant $M_\varepsilon \in (0,\infty)$ that depends only on ε, m, and g, then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq \{0\}$.

Proof. The only part that does not follow immediately from Theorem 4.3 is that $f = 0$ in the case $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$. In this case, one can take the same Y as in the proof of Theorem 4.3, note that $Z(Y) = \emptyset$ and apply Theorem 4.1 to conclude that $f = 0$. (It is instructive to compare this result to [18, Proposition 2.2].) ■

Remark 4.5. The condition that $\widetilde{m}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ to $L^1_g(\mathbb{R}^n)$ is satisfied if m is a linear combination of terms of the form ab, where $a = F\mu$, μ is a finite complex Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \tilde{g}(y)|\mu|(dy) < \infty,$$

and b is the Fourier transform of a compactly supported distribution. Indeed, it is easy to see that $\tilde{b}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into itself, while the convolution operator $\varphi \mapsto \tilde{\mu} * \varphi$ maps $C_c^\infty(\mathbb{R}^n)$ to $L^1_g(\mathbb{R}^n)$.

A particular example is the characteristic exponent of a Lévy process (this is a stochastic process with stationary and independent increments, such that the trajectories are right-continuous with finite left limits, see, e.g., Sato [25])

$$m(\xi) = -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int\limits_{0<|y|<1} (1-e^{iy\cdot\xi} + iy\cdot\xi)\,\nu(dy) + \int\limits_{|y|\geqslant 1} (1-e^{iy\cdot\xi})\nu(dy),$$

where $b\in\mathbb{R}^n$, $Q\in\mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix, and ν is a measure on $\mathbb{R}^n\setminus\{0\}$ such that $\int_{0<|y|<1}|y|^2\nu(dy) + \int_{|y|\geqslant 1} g(y)\,\nu(dy) < \infty$. More generally, one can take

$$m(\xi) = \sum_{|\alpha|=0}^{2s} c_\alpha \frac{i^{|\alpha|}}{\alpha!}\xi^\alpha + \int\limits_{0<|y|<1}\Big[1-e^{iy\cdot\xi} + \sum_{|\alpha|=0}^{2s-1}\frac{i^{|\alpha|}}{\alpha!}y^\alpha\xi^\alpha\Big]\nu(dy) + \int\limits_{|y|\geqslant 1}(1-e^{iy\cdot\xi})\nu(dy)$$

with $s\in\mathbb{N}$, $c_\alpha\in\mathbb{R}$, and a measure ν on $\mathbb{R}^n\setminus\{0\}$ such that $\int_{0<|y|<1}|y|^{2s}\,\nu(dy) + \int_{|y|\geqslant 1} g(y)\,\nu(dy) < \infty$. (As usual, for any $\alpha\in\mathbb{N}_0^n$ and $\xi\in\mathbb{R}^n$, we define $\alpha! := \prod_1^n \alpha_k!$ and $\xi^\alpha := \prod_1^n \xi_k^{\alpha_k}$.) Functions of this type appear naturally in positivity questions related to generalised functions (see, e.g., [9, Chapter II, Section 4] or [28, Chapter 8]). Some authors call the function $-m$ for such an m (under suitable additional conditions on the c_α's) a *conditionally positive definite function*.

Remark 4.6. We are mostly interested in super-polynomially growing weights as polynomially growing ones have been dealt with in our previous paper [3]. Nevertheless, it is instructive to look at the behaviour of the factor $e^{S_g(|y|)|y|}$ for typical super-polynomially, polynomially, and sub-polynomially growing weights.

It follows from (2.17) that if $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma>1$, then there exists a constant C_γ such that

$$\begin{aligned} e^{S_g(|y|)|y|} &\leqslant C_\gamma e^{\frac{1}{\pi}|y|\log^{-\gamma}(e+|y|)(1+\frac{2}{\gamma-1}\log(e+|y|))} \\ &= C_\gamma\big(e^{|y|/\log^\gamma(e+|y|)}\big)^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))} \\ &= C_\gamma(g(y))^{\frac{1}{\pi}(1+\frac{2}{\gamma-1}\log(e+|y|))}. \end{aligned}$$

Similarly, if $g(x) = e^{a|x|^b}$, $a\geqslant 0$, $b\in[0,1)$, then (2.16) implies

$$e^{S_g(|y|)|y|} = e^{a|y|^b(\sin(\frac{1-b}{2}\pi))^{-1}} = (g(y))^{(\sin(\frac{1-b}{2}\pi))^{-1}}. \tag{4.11}$$

If $g(x) = (1 + |x|)^s$, $s \geqslant 0$, then (2.14) implies

$$e^{S_g(|y|)|y|} \leqslant e^{c_1 s + s\log(1+|y|)} = C_s(1 + |y|)^s = C_s g(y). \tag{4.12}$$

Finally, if $g(x) = (\log(e + |x|))^t$, $t \geqslant 0$, then (2.15) implies

$$e^{S_g(|y|)|y|} \leqslant e^{c_2 t + t\log\log(e+|y|)} = C_t(\log(e + |y|))^t = C_t g(y).$$

Remark 4.7. If g is polynomially bounded in Corollary 4.4, then it follows from (4.10) and (4.12) that f is a polynomially bounded entire function on $\mathbb{C}^n$, hence a polynomial, see, e.g., [21, Corollary 1.7]. The fact that f is a polynomial in this case was established in [3, 12].

Remark 4.8. Let $n = 2$, $g(x) := (1 + |x|)^k$, $k \in \mathbb{N}$, $f(x_1, x_2) := (x_1 + ix_2)^k$ (or $f(x_1, x_2) := (x_1 + ix_2)^k + (x_1 - ix_2)^k$ if one prefers to have a real-valued f). Then $f \in L^\infty_{g^{-1}}(\mathbb{R}^2)$, $\Delta f = 0$, $f(x + iy_1\mathbf{e}_1) = (x_1 + iy_1 + ix_2)^k$ for any $y_1 \in \mathbb{R}$, see (2.6), and

$$\frac{\|f(\cdot + iy_1\mathbf{e}_1)\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)}}{g(y_1\mathbf{e}_1)} \geqslant \frac{|y_1|^k}{(1 + |y_1|)^k} \xrightarrow[|y_1|\to\infty]{} 1 = \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)}.$$

So, the factor $e^{S_g(|y|)|y|} \leqslant C_k g(y)$, see (4.12), is optimal in (4.10) in this case.

The case $g(x) = e^{a|x|^b}$, $a > 0$, $b \in [0, 1)$, is perhaps more interesting. Let us take $b = \frac{1}{2}$. Then it follows from (4.11) that $e^{S_g(|y|)|y|} = (g(y))^{\sqrt{2}}$. Let us show that one cannot replace this factor in (4.10) with $(g(y))^{\sqrt{2}(1-\varepsilon)}$, $\varepsilon > 0$. Take any $\varepsilon > 0$. Since

$$\sqrt[4]{1 + \tau^2}\cos\Big(\frac{1}{2}\arctan\frac{1}{\tau}\Big) \xrightarrow[\tau\to 0,\ \tau>0]{} \frac{1}{\sqrt{2}},$$

there exists some $\tau_\varepsilon > 0$ such that

$$\sqrt[4]{1 + \tau_\varepsilon^2}\cos\Big(\frac{1}{2}\arctan\frac{1}{\tau_\varepsilon}\Big) \leqslant \frac{1 + \varepsilon}{\sqrt{2}}.$$

Let us estimate $\operatorname{Re}\sqrt{x_1 + i\kappa x_2}$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $\kappa > 0$ is a constant to be chosen later, and $\sqrt{\cdot}$ is the branch of the square root that is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive on $(0, +\infty)$. If $x_1 \geqslant \tau_\varepsilon\kappa|x_2|$, then

$$\begin{aligned}\operatorname{Re}\sqrt{x_1 + i\kappa x_2} &\leqslant |\sqrt{x_1 + i\kappa x_2}| = \sqrt[4]{x_1^2 + \kappa^2 x_2^2} \\ &\leqslant \sqrt[4]{\Big(1 + \frac{1}{\tau_\varepsilon^2}\Big)x_1^2} \leqslant \Big(1 + \frac{1}{\tau_\varepsilon^2}\Big)^{1/4}\sqrt{x_1} \leqslant \Big(1 + \frac{1}{\tau_\varepsilon^2}\Big)^{1/4}\sqrt{|x|}.\end{aligned}$$

If $0 < x_1 < \tau_\varepsilon \kappa |x_2|$, then

$$\begin{aligned}
\operatorname{Re} \sqrt{x_1 + i\kappa x_2} &= |\sqrt{x_1 + i\kappa x_2}| \cos\Big(\frac{1}{2} \arctan \frac{\kappa |x_2|}{x_1}\Big) \\
&\leqslant |\sqrt{\tau_\varepsilon \kappa |x_2| + i\kappa x_2}| \cos\Big(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\Big) \\
&= \kappa^{1/2} |x_2|^{1/2} \sqrt[4]{1 + \tau_\varepsilon^2} \cos\Big(\frac{1}{2} \arctan \frac{1}{\tau_\varepsilon}\Big) \\
&\leqslant \frac{1+\varepsilon}{\sqrt{2}} \kappa^{1/2} |x|^{1/2}.
\end{aligned}$$

Now, take $\kappa_\varepsilon \geqslant 1$ such that

$$\frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} \geqslant \Big(1 + \frac{1}{\tau_\varepsilon^2}\Big)^{1/4}.$$

Then

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leqslant \frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2} \tag{4.13}$$

for $x_1 > 0$. If $x_1 \leqslant 0$, then the argument of $\sqrt{x_1 + i\kappa_\varepsilon x_2}$ belongs to $[\pi/4, \pi/2]$, respectively $[-\pi/2, -\pi/4]$, depending on the sign of x_2. Hence,

$$\operatorname{Re} \sqrt{x_1 + i\kappa_\varepsilon x_2} \leqslant |\sqrt{x_1 + i\kappa_\varepsilon x_2}| \cos \frac{\pi}{4} \leqslant \frac{1}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2},$$

and (4.13) holds for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Since the Taylor series of $\cos w$ contains only even powers of w, $\cos(i\sqrt{z})$ is an analytic function of $z \in \mathbb{C}$. So, $\cos(i\sqrt{x_1 + ix_2})$ is a harmonic function of the variable $x = (x_1, x_2) \in \mathbb{R}^2$. Hence, $f(x_1, x_2) := \cos(i\sqrt{x_1 + i\kappa_\varepsilon x_2})$ is a solution of the elliptic partial differential equation

$$\Big(\partial_{x_1}^2 + \frac{1}{\kappa_\varepsilon^2} \partial_{x_2}^2\Big) f(x_1, x_2) = 0.$$

It follows from (4.13) that

$$|f(x_1, x_2)| \leqslant \frac{1}{2}(1 + e^{\operatorname{Re}\sqrt{x_1 + i\kappa_\varepsilon x_2}}) \leqslant e^{\frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2} |x|^{1/2}}.$$

So, $f \in L^\infty_{g^{-1}}(\mathbb{R}^2)$, where $g(x) = e^{a|x|^{1/2}}$ with $a = \frac{1+\varepsilon}{\sqrt{2}} \kappa_\varepsilon^{1/2}$. Clearly, the analytic continuation of f to $\mathbb{C}^2$ is given by the formula

$$f(x_1 + iy_1, x_2 + iy_2) = \cos(i\sqrt{x_1 + iy_1 + i\kappa_\varepsilon(x_2 + iy_2)}).$$

Finally, see (2.6), letting $(-\infty, 0) \ni y_2 \to -\infty$, we arrive at

$$\frac{\|f(\cdot + iy_2\mathbf{e}_2)\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)}}{(g(y_2\mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} \geqslant \frac{|f(0+iy_2\mathbf{e}_2)|}{g(0)(g(y_2\mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} = \frac{|\cos(i\sqrt{-\kappa_\varepsilon y_2})|}{e^{\sqrt{2}(1-\varepsilon)\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}$$
$$\geqslant \frac{e^{\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2e^{(1-\varepsilon^2)\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} = \frac{1}{2}e^{\varepsilon^2\kappa_\varepsilon^{1/2}|y_2|^{1/2}} \xrightarrow[y_2\to-\infty]{} \infty.$$

5. Concluding remarks

Corollary 4.4 shows that sub-exponentially growing solutions of $m(D)f = 0$ admit analytic continuation to entire functions on $\mathbb{C}^n$. It is well known that no growth restrictions are necessary in the case when $m(D)$ is an elliptic partial differential operator with constant coefficients, and every solution of $m(D)f = 0$ in $\mathbb{R}^n$ admits analytic continuation to an entire function on $\mathbb{C}^n$, see [6, 23].

Remark 5.1. The latter result has a local version similar to Hayman's theorem on harmonic functions, see [13, Theorem 1]: for every elliptic partial differential operator $m(D)$ with constant coefficients there exists a constant $c_m \in (0,1)$ such that every solution of $m(D)f = 0$ in the ball $\{x \in \mathbb{R}^n : |x| < R\}$ of any radius $R > 0$ admits continuation to an analytic function in the ball $\{x \in \mathbb{C}^n : |x| < c_m R\}$. Indeed, let $m_0(D) = \sum_{|\alpha|=N} a_\alpha D^\alpha$ be the principal part of $m(D) = \sum_{|\alpha|\leqslant N} a_\alpha D^\alpha$. There exists $C_m > 0$ such that

$$\sum_{|\alpha|=N} a_\alpha (a+ib)^\alpha = 0, \quad a, b \in \mathbb{R}^n \implies |a| \geqslant C_m|b|,$$

see, for example, [27, Section 7]. Then the same argument as in the proof of [19, Corollary 8.2] shows that f admits continuation to an analytic function in the ball $\{x \in \mathbb{C}^n : |x| < (1 + C_m^{-2})^{-1/2}R\}$. Note that in the case of the Laplacian, one can take $C_m = 1$ and $c_m = (1 + C_m^{-2})^{-1/2} = \frac{1}{\sqrt{2}}$, which is the optimal constant for harmonic functions, see [13].

Let us return to equations in $\mathbb{R}^n$. Below, $m(\xi)$ will always denote a polynomial with $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} \subseteq \{0\}$. For non-elliptic partial differential operators $m(D)$, one needs to place growth restrictions on solutions of $m(D)f = 0$ to make sure that they admit analytic continuation to entire functions on $\mathbb{C}^n$.

We say that a function f defined on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is of *infra-exponential* growth, if for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(z)| \leqslant C_\varepsilon e^{\varepsilon|z|} \quad \text{for all } z \in \mathbb{R}^n \ (z \in \mathbb{C}^n).$$

Let $\mu\colon [0,\infty) \to [0,\infty)$ be an increasing function, which increases to infinity and satisfies

$$\mu(t) \leqslant At + B, \quad t \geqslant 0$$

for some $A, B > 0$, and

$$\int_1^\infty \frac{\mu(t)}{t^2}\,dt < \infty. \tag{5.1}$$

Suppose $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} = \{0\}$. Then, under additional restrictions on μ, every solution f of $m(D)f = 0$ that has growth $O(e^{\varepsilon\mu(|x|)})$ for every $\varepsilon > 0$ admits analytic continuation to an entire function of infra-exponential growth on $\mathbb{C}^n$ see [18]. It is easy to see that (5.1) is equivalent to the Beurling–Domar condition (2.3) for the function $g(x) := e^{\varepsilon\mu(|x|)}$.

One cannot replace $O(e^{\varepsilon\mu(|x|)})$ with $O(e^{\varepsilon|x|})$ in the above result without placing a restriction on the complex zeros of m. If there exists $\delta > 0$ such that $m(\zeta)$ has no complex zeros in

$$|\operatorname{Im}\zeta| < \delta, \quad |\operatorname{Re}\zeta| > \delta^{-1}, \tag{5.2}$$

then every solution of $m(D)f = 0$ that, together with its partial derivatives up to the order of $m(D)$, is of infra-exponential growth on $\mathbb{R}^n$, admits analytic continuation to an entire function of infra-exponential growth on $\mathbb{C}^n$, see [17, 18].

On the other hand, if for every $\delta > 0$, (5.2) contains complex zeros of $m(\zeta)$, then $m(D)f = 0$ has a solution in C^∞ all of whose derivatives are of infra-exponential growth on $\mathbb{R}^n$, but which is not entire infra-exponential in $\mathbb{C}^n$. The proof of the latter result in [17, 18] is not constructive, and the author writes: "Unfortunately we cannot present concrete examples of such solutions." However, it is not difficult to construct, for any $\varepsilon > 0$, a solution in C^∞ all of whose derivatives have growth $O(e^{\varepsilon|x|})$, but which is not real-analytic. Indeed, according to the assumption, there exist complex zeros

$$\zeta_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}^n, \quad k \in \mathbb{N}$$

of $m(\zeta)$ such that

$$|\eta_k| < k^{-1}, \quad |\xi_k| > k. \tag{5.3}$$

Choosing a subsequence, we can assume that $\omega_k := |\xi_k|^{-1}\xi_k$ converge to a point $\omega_0 \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ as $k \to \infty$, and that $|\omega_k - \omega_0| < 1$ for all $k \in \mathbb{N}$. Then

$$\omega_k \cdot \omega_0 = \frac{|\omega_k|^2 + |\omega_0|^2 - |\omega_k - \omega_0|^2}{2} > \frac{1+1-1}{2} = \frac{1}{2}, \quad k \in \mathbb{N}. \tag{5.4}$$

Consider

$$f(x) := \sum_{k>\varepsilon^{-1}} \frac{e^{i\zeta_k\cdot x}}{e^{|\xi_k|^{1/2}}} = \sum_{k>\varepsilon^{-1}} \frac{e^{i\xi_k\cdot x-\eta_k\cdot x}}{e^{|\xi_k|^{1/2}}}, \quad x \in \mathbb{R}^n. \tag{5.5}$$

Then, for every multi-index α and every $x \in \mathbb{R}^n$,

$$\begin{aligned}|\partial^\alpha f(x)| = \Big| \sum_{k>\varepsilon^{-1}} \frac{(i\zeta_k)^\alpha e^{i\zeta_k\cdot x}}{e^{|\xi_k|^{1/2}}}\Big| &\leqslant \sum_{k>\varepsilon^{-1}} \frac{(|\xi_k|+1)^{|\alpha|}e^{|\eta_k||x|}}{e^{|\xi_k|^{1/2}}} \\ &\leqslant e^{\varepsilon|x|}\sum_{k>\varepsilon^{-1}} \frac{(|\xi_k|+1)^{|\alpha|}}{e^{|\xi_k|^{1/2}}} =: C_\alpha e^{\varepsilon|x|},\end{aligned}$$

see (5.3). Further,

$$m(D)f(x) = \sum_{k>\varepsilon^{-1}} \frac{m(\zeta_k)e^{i\zeta_k\cdot x}}{e^{|\xi_k|^{1/2}}} = 0.$$

On the other hand, f is not real-analytic. Before we prove this, note that formally putting $x - it\omega_0, t > 0$ in place of x in the right-hand side of (5.5), one gets a divergent series. Indeed, its terms can be estimated as follows:

$$\Big|\frac{e^{i\xi_k\cdot x+t\xi_k\cdot\omega_0-\eta_k\cdot x+it\eta_k\cdot\omega_0}}{e^{|\xi_k|^{1/2}}}\Big| = \frac{e^{t|\xi_k|\omega_k\cdot\omega_0-\eta_k\cdot x}}{e^{|\xi_k|^{1/2}}} \geqslant e^{-\varepsilon|x|}\frac{e^{t|\xi_k|/2}}{e^{|\xi_k|^{1/2}}} \to \infty$$

as $k \to \infty$, see (5.3), (5.4).

For any $j > \varepsilon^{-1}$, there exists $\ell_j \in \mathbb{N}$ such that

$$\ell_j \leqslant |\xi_j|^{1/2} < \ell_j + 1. \tag{5.6}$$

It is clear that $\ell_j \to \infty$ as $j \to \infty$, see (5.3). Note that

$$|\arg(\omega_0\cdot\zeta_k)| \leqslant \frac{|\omega_0\cdot\eta_k|}{|\omega_0\cdot\xi_k|} \leqslant \frac{2}{k|\xi_k|}.$$

If $|\xi_k| \geqslant 6\ell_j/\pi k$, then

$$|\arg(\omega_0\cdot\zeta_k)^{\ell_j}| \leqslant \frac{2\ell_j}{k|\xi_k|} \leqslant \frac{\pi}{3},$$

and

$$\operatorname{Re}(\omega_0\cdot\zeta_k)^{\ell_j} \geqslant \frac{1}{2}|\omega_0\cdot\zeta_k|^{\ell_j} \geqslant \frac{1}{2^{\ell_j+1}}|\xi_k|^{\ell_j}.$$

Clearly, $|\xi_j| \geqslant \frac{6\ell_j}{\pi j}$ for sufficiently large j, see (5.6). Hence, one has the following estimate for the directional derivative ∂_{ω_0}:

$$
\begin{aligned}
|((-i\partial_{\omega_0})^{\ell_j} f)(0)| &\geqslant \sum_{k>\varepsilon^{-1}} \frac{\operatorname{Re}(\omega_0\cdot\zeta_k)^{\ell_j}}{e^{|\xi_k|^{1/2}}} \\
&\geqslant -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{|\zeta_k|^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \sum_{k>\varepsilon^{-1},\ |\xi_k|\geqslant\frac{6\ell_j}{\pi k}} \frac{|\xi_k|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_k|^{1/2}}} \\
&\geqslant -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{(|\xi_k|+\frac{1}{k})^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \frac{|\xi_j|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_j|^{1/2}}} \\
&\geqslant -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{1}{e^{|\xi_k|^{1/2}}}\Big(\frac{10\ell_j}{\pi k}\Big)^{\ell_j} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{(\ell_j^2+1)^{1/2}}} \\
&\geqslant -(10\ell_j)^{\ell_j}\sum_{k=1}^{\infty}\frac{1}{e^{|\xi_k|^{1/2}}k^2} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{\ell_j+1}} \\
&= -C(10\ell_j)^{\ell_j} + (2e)^{-(\ell_j+1)}\ell_j^{2\ell_j}.
\end{aligned}
$$

Hence,

$$
|((-i\partial_{\omega_0})^{\ell_j} f)(0)| \geqslant \ell_j^{\frac{3}{2}\ell_j}
$$

for all sufficiently large j, which means that f is not real-analytic in a neighbourhood of 0.

The operator $m(D)$ in the previous example is not hypoelliptic. If $m(D)$ is hypoelliptic, then every solution of $m(D)f = 0$, such that $|f(x)| \leqslant Ae^{a|x|}$, $x \in \mathbb{R}^n$, for some constants $A, a > 0$, admits analytic continuation to an entire function of order one on $\mathbb{C}^n$, see [11, Section 4, Corollary 2]. For elliptic operators, this result can be strengthened: every solution of $m(D)f = 0$, such that $|f(x)| \leqslant Ae^{a|x|^\beta}$, $x \in \mathbb{R}^n$, for $\beta \geqslant 1$ and some constants $A, a > 0$, admits analytic continuation to an entire function of order β on $\mathbb{C}^n$, see [11, Section 4, Corollary 3]. Let us show that for every $\beta > 1$ there exists a semi-elliptic operator $m(D)$, see [16, Theorem 11.1.11], and a C^∞ solution of $m(D)f = 0$, all of whose derivatives have growth $O(e^{a|x|^\beta})$, but which does not admit analytic continuation to an entire function on $\mathbb{C}^n$.

A simple example of such a semi-elliptic operator is $\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}$ with $\ell \in \mathbb{N}$ satisfying $1 + \frac{1}{2\ell} \leqslant \beta$, i.e., $\ell \geqslant \frac{1}{2(\beta-1)}$.

Let

$$f(x_1, x_2) := \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}x_1+kx_2}}{e^{k^{2\ell+1}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

If $x_2 > 0$, then the function $t \mapsto tx_2 - t^{2\ell+1}$ achieves a maximum at $t = (\frac{x_2}{2\ell+1})^{\frac{1}{2\ell}}$, and this maximum is equal to

$$2\ell\Big(\frac{1}{2\ell+1}\Big)^{1+\frac{1}{2\ell}} x_2^{1+\frac{1}{2\ell}} =: c_\ell x_2^{1+\frac{1}{2\ell}}.$$

Hence, for every multi-index α,

$$\begin{aligned}
|\partial^\alpha f(x_1, x_2)| &\leqslant \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{kx_2-k^{2\ell+1}} \\
&= \sum_{k=1}^{[x_2^{\frac{1}{2\ell}}]+1} k^{(2\ell+1)|\alpha|} e^{kx_2-k^{2\ell+1}} + \sum_{k=[x_2^{\frac{1}{2\ell}}]+2}^{\infty} k^{(2\ell+1)|\alpha|} e^{k(x_2-k^{2\ell})} \\
&\leqslant ([x_2^{\frac{1}{2\ell}}]+1)^{(2\ell+1)|\alpha|+1} e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{-k} \\
&\leqslant 2^{(2\ell+1)|\alpha|+1}(x_2^{2|\alpha|+1}+1) e^{c_\ell x_2^{1+\frac{1}{2\ell}}} + c_{\ell,\alpha} \leqslant C_{\ell,\alpha} e^{(c_\ell+1)x_2^{1+\frac{1}{2\ell}}}.
\end{aligned}$$

If $x_2 \leqslant 0$, then

$$|\partial^\alpha f(x_1, x_2)| \leqslant \sum_{k=1}^{\infty} \frac{k^{(2\ell+1)|\alpha|}}{e^{k^{2\ell+1}}} < \sum_{j=1}^{\infty} \frac{j^{|\alpha|}}{e^j} =: C_\alpha < \infty.$$

So, $f \in C^\infty(\mathbb{R}^2)$, and $\partial^\alpha f(x_1, x_2) = O(e^{(c_\ell+1)|x_2|^{1+\frac{1}{2\ell}}}) = O(e^{(c_\ell+1)|x|^{1+\frac{1}{2\ell}}})$. It is easy to see that $(\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}) f(x_1, x_2) = 0$.

The function f admits analytic continuation to the set

$$\Pi_1 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Im} z_1 < 1\}.$$

Indeed, let

$$\begin{aligned}
f(z_1, z_2) = f(x_1 + iy_1, x_2 + iy_2) &= \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}(x_1+iy_1)+k(x_2+iy_2)}}{e^{k^{2\ell+1}}} \\
&= \sum_{k=1}^{\infty} e^{i(ky_2-k^{2\ell+1}x_1)} e^{k^{2\ell+1}(y_1-1)+kx_2}.
\end{aligned}$$

It is easy to see that the last series is uniformly convergent on compact subsets of Π_1. So, f admits analytic continuation to Π_1. On the other hand, $f(iy_1, 0) \to \infty$ as $y_1 \to 1 - 0$. Indeed,

$$f(iy_1, 0) = \sum_{k=1}^{\infty} e^{k^{2\ell+1}(y_1-1)}.$$

Take any $N \in \mathbb{N}$. If $y_1 > 1 - N^{-(2\ell+1)}$, then

$$f(iy_1, 0) > \sum_{k=1}^{\infty} e^{-k^{2\ell+1} N^{-(2\ell+1)}} > \sum_{k=1}^{N} e^{-k^{2\ell+1} N^{-(2\ell+1)}} \geqslant \sum_{k=1}^{N} e^{-1} = \frac{N}{e}.$$

So, $f(iy_1, 0) \to \infty$ as $y_1 \to 1 - 0$.

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