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Manifolds whose Weyl spectral asymptotics have small but not tiny remainders

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Abstract. A compact, n-dimensional Riemannian manifold M has Weyl spectral asymptotics with remainder $E_M(R)$; i.e., the spectral counting function satisfies $\mathcal{N}(\Delta_M, R) = C(M)R^n + E_M(R)$, with $E_M(R) = o(R^n)$. Generally, one actually has $E_M(R) = O(R^{n-1})$, and one seeks geometrical conditions under which stronger estimates hold on the remainder and also conditions limiting how extra small the remainder can be. Here, we produce n-dimensional manifolds whose Weyl remainders are $o(R^{n-1})$ but not $O(R^{n-1-\alpha})$ for any $\alpha > 0$.

1. Introduction

If M is a compact, n-dimensional Riemannian manifold with Laplace operator Δ_M, then $L^2(M)$ has an orthonormal basis of eigenfunctions $\{u_j\}$, satisfying

$$\Delta_M u_j = -\lambda_j^2 u_j, \quad \lambda_j \nearrow \infty.$$

We define the spectral counting function by

$$\mathcal{N}(\Delta_M, R) = \#\{j : \lambda_j \le R\}.$$

For this, there is the Weyl asymptotic formula

$$\mathcal{N}(\Delta_M, R) = C(M)R^n + E_M(R), \tag{1.1}$$

with $E_M(R) = o(R^n)$, the Weyl remainder. A classical improvement of this estimate is

$$E_M(R) = O(R^{n-1}); \tag{1.2}$$

see [7]. Much work has been done to see when this estimate can be further improved. In [5], it is shown that one can take

$$E_M(R) = o(R^{n-1}) \tag{1.3}$$

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if M has "not too many" closed geodesics. It was shown in [1] that, under certain geometric hypotheses involving no conjugate points, one can improve (1.3) to

$$E_M(R) = O(R^{n-1}/\log R). \tag{1.4}$$

The recent paper [3] obtained such an estimate in much greater generality. Going further, there are various examples for which one has

$$E_M(R) = O(R^{n-1-\alpha}) \tag{1.5}$$

for some $\alpha > 0$. We say M has spectral asymptotics with algebraically small Weyl remainder. The classical example for (1.5) is $M = \mathbb{T}^n$, the flat torus. (The sharp value of α for which (1.5) holds is not known; cf. [2].) In [9], (1.5) is established for Cartesian products of spheres (with at least 2 factors). Other examples are studied in [8, 11].

In this paper, we produce examples of compact Riemannian manifolds M for which the remainder estimate (1.3) holds, but for which the stronger estimate (1.5) fails for each $\alpha > 0$.

Before describing our main results on this, we recall the classical cases for which the estimate (1.3) fails, namely, the n-dimensional unit spheres S^n in $\mathbb{R}^{n+1}$. In such cases, there are exact formulas for the eigenvalues of Δ_{S^n}, and for the dimensions of the eigenspaces, and these dimensions are seen to be sufficiently large that no improvement of (1.2) is possible. For later use, we describe this situation for S^2 in more detail. As is well known, $L^2(S^2)$ has an orthonormal basis $\{Y_k^\ell : k \in \mathbb{Z}^+, \ell \in \mathbb{Z}, |\ell| \le k\}$ (of "spherical harmonics"), satisfying

$$\Delta_{S^2} Y_k^\ell = -k(k+1)Y_k^\ell, \quad XY_k^\ell = \ell Y_k^\ell, \quad k \in \mathbb{Z}^+, \ |\ell| \le k,$$

where

$$X = iZ,$$

and Z is the vector field generating 2π-periodic rotation of $\mathbb{R}^3$ about the x_3-axis. In this case, the $-k(k+1)$-eigenspace of Δ_{S^2} has dimension $2k+1$, foreclosing the possibility of (1.3) holding.

To start with 2D examples, our construction of examples where (1.5) fails will involve taking $M = S^2$ as a manifold but giving M a different metric tensor. The new metric tensor (g_{ij}) will match up with the standard metric (γ_{ij}) of S^2, to infinite order, at the equator $x_3 = 0$ but will differ from (γ_{ij}) off $x_3 = 0$. To show that (1.5) fails for such M, we will show that, for each $\rho \in (1/2, 1)$, the sequence of spaces

$$W_k(S^2) = W_k^\rho(S^2) = \mathrm{Span}\{Y_k^\ell : k - k^\rho \le \ell \le k\}, \quad k \in \mathbb{Z}^+, \tag{1.6}$$

yields *quasimodes* for Δ_M (a concept introduced in [4]). One ingredient in this analysis is to examine how the functions $u \in W_k(S^2)$ concentrate on the equator $x_3 = 0$

as $k \to \infty$. In Section 2, we establish such concentration results. For $\nu \in \mathbb{N},\ s \geq 0$, we obtain

$$u \in W_k(S^2) \Rightarrow \|x_3^\nu u\|_{H^s(S^2)} \leq C\|u\|_{H^{s-\nu\delta/2}(S^2)} \\ \leq Ck^{s-\nu\delta/2}\|u\|_{L^2}, \quad \delta = 1-\rho, \tag{1.7}$$

which is an effective concentration result when $\nu\delta/2 > s$.

We bring in tools from microlocal analysis to establish (1.7) and related estimates. In more detail, with $\rho \in (1/2, 1)$ and $\Lambda = (-\Delta_{S^2} + 1/4)^{1/2} - 1/2$, we set

$$F(\Lambda, X) = \varphi\big((\Lambda - X)\Lambda^{-\rho}\big),$$

where we pick

$$\varphi \in C_0^\infty(\mathbb{R}), \quad \varphi(\tau) = 1 \text{ for } |\tau| \leq 1,\ 0 \text{ for } |\tau| \geq 2.$$

Results of [13] and [14, Chapter 12] imply that $F(\Lambda, X)$ is a pseudodifferential operator (of non-classical type):

$$F(\Lambda, X) \in \mathrm{OPS}^0_{\rho,\delta}, \quad \text{principal symbol } f(x,\xi) = F(|\xi|_x, \langle Y(x), \xi\rangle),$$

leading on the one hand to

$$u = F(\Lambda, X)u, \quad \text{for } u \in W_k(S^2), \tag{1.8}$$

and on the other to

$$x_3^\nu F(\Lambda, X) \in \mathrm{OPS}^{-\nu\delta/2}_{\rho,\delta}, \tag{1.9}$$

from which we deduce (1.7).

We take up concentration estimates of the eigenfunctions of the Laplace operator Δ_S on S^n for $n \geq 3$ in Section 3. In such a case, $L^2(S^n)$ is an orthogonal direct sum of eigenspaces

$$V_k(S^n) = \{u \in C^\infty(S^n) : \Delta_S u = -\mu_k^2 u\}, \quad \mu_k^2 = k^2 + (n-1)k.$$

Here, we look at the joint spectrum of the commuting operators Δ_S and L, a second-order differential operator that acts like the Laplace operator on $(n-1)$-spheres. (For $n = 2,\ L = -X^2$.) Instead of the pair (Λ, X), we take

$$(\Lambda, \Lambda_0), \quad \Lambda = (-\Delta_S)^{1/2}, \quad \Lambda_0 = -L\Lambda^{-1},$$

and instead of $F(\Lambda, X)$, we use

$$G(\Lambda, \Lambda_0) = \varphi\big((\Lambda - \Lambda_0)\Lambda^{-\rho}\big),$$

as before, with $\rho \in (1/2, 1)$. Instead of (1.6), we take

$$W_k(S^n) = W_k^\rho(S^n) = \bigoplus_j \{\widetilde{V}_{kj}(S^n) : \mu_k^2 - \mu_k^{1+\rho} \le \sigma_j^2 \le \mu_k^2\},$$

where

$$\widetilde{V}_{kj}(S^n) = \{u \in V_k(S^n) : Lu = -\sigma_j^2 u\},$$

so

$$V_k(S^n) = \bigoplus_{j=0}^{k} \widetilde{V}_{kj}(S^n), \quad \sigma_j^2 = j(j+n-2).$$

In place of (1.8)–(1.9), we have

$$u = G(\Lambda, \Lambda_0)u, \quad \text{for } u \in W_k(S^n), \tag{1.10}$$

and (again with $\delta = 1 - \rho$)

$$x_{n+1}^\nu G(\Lambda, \Lambda_0) \in \mathrm{OPS}_{\rho,\delta}^{-\nu\delta/2}. \tag{1.11}$$

To use the spaces $W_k(S^n)$ in our search for manifolds M for which (1.5) fails, it is important to have a good lower bound on their dimensions. We show that

$$\dim W_k(S^n) \ge Ck^{-\delta} \dim V_k(S^n), \tag{1.12}$$

which is clear from (1.6) when $n = 2$. For $n \ge 3$, we get this from the isomorphism

$$\widetilde{V}_{kj}(S^n) \approx V_j(S^{n-1}), \quad 0 \le j \le k,$$

a result that can be restated in terms of an $\mathrm{SO}(n)$-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{\ell=0}^{k} V_\ell(S^{n-1}).$$

This is established in Section 3, with the help of a dimension count, done in Appendix A.

In Section 4, we introduce the following family of n-dimensional Riemannian manifolds. We take M to be S^n, endowed with a metric tensor (g_{ij}) that agrees with the standard metric tensor (γ_{ij}) of S^n to order ν on the equator (for some integer $\nu \ge 2$), i.e.,

$$g_{ij} = \gamma_{ij} + \sigma_{ij}, \quad \sigma_{ij} = O(x_{n+1}^\nu).$$

In such a case, we have from (1.9) and (1.11) that

$$(\Delta_S - \Delta_M)\mathcal{G} = Q \in \mathrm{OPS}_{\rho,\delta}^{2-\nu\delta/2},$$

where $\mathscr{G} = F(\Lambda, X)$ for $n = 2$, $\mathscr{G} = G(\Lambda, \Lambda_0)$ for $n \geq 3$. We deduce from (1.8) and (1.10) that

$$u \in W_k(S^n) \Rightarrow (-\Delta_M - \mu_k^2)u = Qu,$$

and hence,

$$u \in W_k(S^n) \Rightarrow \|(\Lambda_M - \mu_k)u\|_{L^2} \leq Ck^{-\sigma}\|u\|_{L^2}, \tag{1.13}$$

with

$$\Lambda_M = (-\Delta_M)^{1/2}, \quad \sigma = \frac{\nu\delta}{2} - 1.$$

Recall that we take $\rho \in (1/2, 1)$, $\delta = 1 - \rho$. If we pick ν sufficiently large, then $\sigma > 0$. The estimate (1.13) establishes that elements of $W_k(S^n)$ are quasimodes for Δ_M.

Using this set of quasimodes, we establish Proposition 4.2, which shows that, for k sufficiently large,

$$\mu_k^2 = k(k + n - 1),$$

there is an orthonormal set

$$\{\psi_k^\ell : 1 \leq \ell \leq \dim W_k(S^n)\} \subset L^2(M),$$

of eigenfunctions of Δ_M, satisfying

$$\Lambda_M \psi_k^\ell = \mu_{k\ell}\psi_k^\ell, \quad |\mu_{k\ell} - \mu_k| \leq Ck^{-\sigma}. \tag{1.14}$$

Note that there are $\dim W_k(S^n)$ elements in this set, and we have the estimate (1.12).

This puts us in a position to show in Section 5 that, in such a situation, and with the hypothesis on ν strengthened to

$$\frac{\nu}{2} > 1 + \frac{1}{\delta},$$

so $\sigma > \delta$ in (1.14), then, if the remainder estimate holds, we must have $\alpha \leq 1 - \rho$. Taking $\rho \nearrow 1$ ($\delta \searrow 0$), we obtain in Theorem 5.2 the following result.

Theorem A. *If M is an n-dimensional Riemannian manifold as described above and if its metric tensor matches the standard metric tensor on S^n to infinite order at the equator, then the remainder estimate* (1.5) *in the Weyl asymptotic formula* (1.1) *cannot hold for any $\alpha > 0$.*

One can find Riemannian manifolds M of the sort described in Theorem 5.2, having the property that the set of closed geodesics has measure zero, so [5] implies $E_M(R) = o(R^{n-1})$. We present some examples in Appendix B. Going further, it is intriguing to guess that some of these can be shown to satisfy the conditions in [3], yielding an estimate of the form (1.4). We intend to look into this in future work.

2. Concentration of spherical harmonics on the equator of S^2

Here, we take $\rho \in (1/2, 1)$, consider the family

$$W_k(S^2) = \mathrm{Span}\{Y_k^\ell : k - k^\rho \le \ell \le k\}, \tag{2.1}$$

and examine how elements of $W_k(S^2)$ concentrate on the equator $x_3 = 0$ of the sphere S^2, as $k \to \infty$. It is convenient to bring in the operator

$$\Lambda = \Big(-\Delta_{S^2} + \frac{1}{4}\Big)^{1/2} - \frac{1}{2},$$

an elliptic, first-order pseudodifferential operator on S^2. (We write $\Lambda \in \mathrm{OPS}^1(S^2)$.) Note that

$$\Lambda Y_k^\ell = k Y_k^\ell, \quad X Y_k^\ell = \ell Y_k^\ell$$

for $k \ge 0$, $|\ell| \le k$. We next set

$$F(\Lambda, X) = \varphi\big((\Lambda - X)\Lambda^{-\rho}\big), \tag{2.2}$$

where we pick

$$\varphi \in C_0^\infty(\mathbb{R}), \quad \mathrm{supp}\,\varphi \subset [-2, 2], \quad \varphi(\tau) = 1 \text{ for } |\tau| \le 1. \tag{2.3}$$

For convenience, we also assume that

$$\varphi \ge 0, \quad \varphi(\tau) \searrow \text{ for } \tau \ge 0. \tag{2.4}$$

Note that

$$Y_k^\ell \in W_k(S^2) \Rightarrow F(\Lambda, X) Y_k^\ell = Y_k^\ell. \tag{2.5}$$

What makes (2.5) effective for concentration estimates comes from the analysis of $F(\Lambda, X)$ as a pseudodifferential operator. Indeed, for $\rho \in (0, 1]$, the function

$$F(\eta) = \varphi\big((\eta_1 - \eta_2)\eta_1^{-\rho}\big)$$

satisfies estimates

$$|D_\eta^\alpha F(\eta)| \le C_\alpha \langle \eta \rangle^{-\rho|\alpha|}, \quad \text{on } S_1 = \{\eta : \eta_1 \ge 1,\ |\eta_2| \le \eta_1\}. \tag{2.6}$$

We establish (2.6) and also some more precise estimates of use for Proposition 2.2. It is convenient to make a linear change of variable,

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_1 - \eta_2$$

and estimate derivatives of

$$\Phi(\xi) = \varphi(\xi_2 \xi_1^{-\rho})$$

on a set of the form

$$S_2 = \{\xi : \xi_1 \geq 1,\ |\xi_2| \leq c\xi_1\}.$$

We have

$$\partial_1 \Phi(\xi) = -\rho \xi_1^{-\rho-1} \xi_2 \varphi'(\xi_2 \xi_1^{-\rho}) = \xi_1^{-1} \varphi_1(\xi_1^{-\rho} \xi_2),$$

where

$$\varphi(\tau) = -\rho\tau\varphi'(\tau), \quad \varphi_1 \in C_0^\infty(\mathbb{R}).$$

Iteratively,

$$\partial_1^\ell \Phi(\xi) = \xi_1^{-\ell} \varphi_\ell(\xi_1^{-\rho} \xi_2), \quad \varphi_\ell \in C_0^\infty(\mathbb{R});$$

hence,

$$\partial_2^m \partial_1^\ell \Phi(\xi) = \xi_1^{-\ell-m\rho} \varphi_\ell^{(m)}(\xi_1^{-\rho} \xi_2), \tag{2.7}$$

yielding

$$\begin{aligned} |\partial_2^m \partial_1^\ell \Phi(\xi)| &\leq C_{\ell m} \langle \xi \rangle^{-\ell-m\rho} \\ &\leq \widetilde{C}_{\ell m} \langle \xi \rangle^{-(\ell+m)\rho}, \quad \text{on } S_2. \end{aligned}$$

This yields (2.6). A more precise estimate, from (2.7), is

$$|\Phi^{(\alpha)}(\xi)| \leq C_\alpha \langle \xi \rangle^{-\rho|\alpha|} |\psi_\alpha(\xi_1^{-\rho} \xi_2)|,$$

on S_2; hence,

$$|F^{(\alpha)}(\eta)| \leq C_\alpha \langle \eta \rangle^{-\rho|\alpha|} |\psi_\alpha((\eta_1 - \eta_2)\eta_1^{-\rho})|, \tag{2.8}$$

on S_1, with $\psi_\alpha \in C_0^\infty(\mathbb{R})$.

As a consequence of these estimates, one has, for $\rho \in (1/2, 1]$,

$$F(\Lambda, X) \in \mathrm{OPS}^0_{\rho,\delta}(S^2), \quad \delta = 1 - \rho, \tag{2.9}$$

with principal symbol

$$f(x, \xi) = F(|\xi|_x, \langle Z(x), \xi \rangle), \quad \bmod S^{-(\rho-\delta)}_{\rho,\delta}. \tag{2.10}$$

The implication that, for $\rho \in (1/2, 1]$,

$$(2.6) \Rightarrow (2.9)\text{–}(2.10)$$

is established in [13], and in [14, Chapter 11, Theorem 1.3], with complements in (1.2)–(1.4) on p. 297, in the broader setting of $F(A_1, \ldots, A_k)$, where A_j are commuting, self-adjoint operators in $\mathrm{OPS}^1(M)$, satisfying the ellipticity condition

$$A_1^2 + \cdots + A_k^2 \text{ is elliptic in } \mathrm{OPS}^2(M).$$

This is also established in [10] for $\rho = 1$, but here we need it for $1/2 < \rho < 1$.

The analysis in [13, 14] involved representing $e^{iy\cdot A}$, for small $y \in \mathbb{R}^k$, as a family of Fourier integral operators,

$$e^{iy\cdot A}u(x) = (2\pi)^{-n/2} \int b(y,x,\xi) e^{i\varphi(y,x,\xi)} \hat{u}(\xi)\, d\xi,$$

modulo smoothing operators, and deducing that if $F \in S^m_\rho(\mathbb{R}^k)$,

$$F(A)u(x) = (2\pi)^{-n/2} \int q(x,\xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi,$$

modulo smoothing, where

$$q(x,\xi)e^{ix\cdot\xi} = F(D_y)\big[b(y,x,\xi)e^{i\varphi(y,x,\xi)}\big]\big|_{y=0},$$

to which a stationary-phase analysis applies, yielding

$$\begin{gathered} q(x,\xi) \sim F(a(x,\xi)) + \sum_{|\alpha|\ge 1} F^{(\alpha)}(a(x,\xi))\psi_\alpha(x,\xi), \\ \psi_\alpha(x,\xi) \in S^{[|\alpha|/2]}_{cl}, \end{gathered} \tag{2.11}$$

where $a(x,\xi) = (a_1(x,\xi),\dots,a_k(x,\xi))$ and $[z]$ denotes the greatest integer $\le z$. Compare [6, Theorem 2.16].

Returning to the setting of (2.2)–(2.3), we have the following.

Proposition 2.1. *Given $\rho \in (1/2, 1)$, the operator $F(\Lambda, X)$ defined by (2.2)–(2.3) satisfies*

$$F(\Lambda, X) \in \mathrm{OPS}^0_{\rho,\delta}(S^2), \quad \delta = 1-\rho,$$

with principal symbol

$$f(x,\xi) = \varphi\big((1-\langle Z(x),\hat{\xi}\rangle)|\xi|^\delta_x\big), \quad \hat{\xi} = \frac{\xi}{|\xi|_x}. \tag{2.12}$$

To proceed, note that $\langle Z(x),\hat{\xi}\rangle \le |Z(x)|$, and hence,

$$1 - \langle Z(x),\hat{\xi}\rangle \ge Cx_3^2, \tag{2.13}$$

so (2.12) yields

$$|f(x,\xi)| \le \varphi(cx_3^2|\xi|^\delta);$$

hence, for $M \in (0,\infty)$,

$$\big(x_3^2|\xi|^\delta\big)^M |f(x,\xi)| \le C_M.$$

We can estimate the total symbol of $F(\Lambda, X)$ and its derivatives via (2.11) and (2.8). This leads to the following proposition.

Proposition 2.2. *In the setting of Proposition 2.1, we have, for each $\nu \in \mathbb{N}$,*

$$x_3^\nu F(\Lambda, X) \in \mathrm{OPS}^{-\nu\delta/2}_{\rho,\delta}(S^2), \quad \delta = 1 - \rho.$$

In light of the Sobolev mapping property

$$P \in \mathrm{OPS}^m_{\rho,\delta}(M) \Rightarrow P : H^{s+m}(M) \to H^s(M),$$

valid for $0 \le \delta < \rho \le 1$, hence for $\delta = 1 - \rho,\ \rho \in (1/2, 1)$, we have, for $\nu \in \mathbb{N}$,

$$\begin{aligned} u \in W_k(S^2) \Rightarrow \|x_3^\nu u\|_{H^s} &= \|x_3^\nu F(\Lambda, X) u\|_{H^s} \\ &\le C_\nu \|u\|_{H^{s-\nu\delta/2}} \\ &\le C_\nu k^{s-\nu\delta/2} \|u\|_{L^2}, \end{aligned} \tag{2.14}$$

as advertised in (1.7). Note that, by (2.1),

$$\dim W_k(S^2) \ge k^\rho. \tag{2.15}$$

3. Concentration of spherical harmonics on the equator of S^n

The Laplace operator Δ_S on S^n has eigenspaces

$$V_k = \{u \in L^2(S^n) : -\Delta_S u = \mu_k^2 u\}, \quad \mu_k^2 = k^2 + (n-1)k, \tag{3.1}$$

mutually orthogonal spaces of dimension

$$\dim V_k = \binom{k+n-1}{k} + \binom{k+n-2}{k-1}, \tag{3.2}$$

spanning $L^2(S^n)$. See [15, Chapter 8, Section 4]. We want to analyze how certain elements of V_k concentrate on the equator

$$S^{n-1} = \{\omega \in S^n : \omega_{n+1} = 0\},$$

as $k \to \infty$, extending the results of Section 2. To do this, we bring in the second-order differential operator L on S^n, the image of the Laplace operator on $\mathrm{SO}(n)$ under its action on $S^n \subset \mathbb{R}^{n+1}$, via rotation in the $(x_1, \dots, x_n)$-plane, normalized so that, for $u \in C^\infty(S^n)$,

$$Lu\big|_{S^{n-1}} = \Delta_{S^{n-1}}\big(u\big|_{S^{n-1}}\big).$$

The operators Δ_S and L commute and are self-adjoint on $L^2(S^n)$. In case $n = 2$, $L = Z^2$. We can write

$$V_k = \bigoplus_\ell V_{k\ell}, \quad V_{k\ell} = \{u \in V_k : Lu = -\ell^2 u\}. \tag{3.3}$$

(Here, ℓ runs over $\mathbb{R}^+$; it need not be an integer.) We will obtain estimates on how elements of $V_{k\ell}$ concentrate on the equator for ℓ sufficiently close to μ_k.

To proceed, we fix $\rho \in (1/2, 1)$, take $\varphi \in C_0^\infty(\mathbb{R})$, satisfying (2.3)–(2.4), and set

$$\begin{aligned} G(\Lambda, \Lambda_0) &= \varphi(-(\Delta_S - L)\Lambda^{-1}\Lambda^{-\rho}) \\ &= \varphi((\Lambda - \Lambda_0)\Lambda^{-\rho}), \end{aligned} \tag{3.4}$$

where

$$\Lambda = \sqrt{-\Delta_S}, \quad \Lambda_0 = -L\Lambda^{-1} \in \mathrm{OPS}^1(S^n). \tag{3.5}$$

Parallel to (2.9), we have

$$G(\Lambda, \Lambda_0) \in \mathrm{OPS}^0_{\rho,\delta}(S^n), \quad \delta = 1 - \rho. \tag{3.6}$$

Note that

$$\begin{aligned} u \in V_{k\ell} \Rightarrow \Lambda_0 u &= \frac{\ell^2}{\mu_k} u \\ \Rightarrow (\Lambda - \Lambda_0)\Lambda^{-\rho} u &= \frac{\mu_k^2 - \ell^2}{\mu_k^{1+\rho}} u. \end{aligned}$$

Hence, if we set

$$W_k = \bigoplus_{\ell} \{V_{k\ell} : \mu_k^2 - \ell^2 \le \mu_k^{1+\rho}\}, \tag{3.7}$$

we have

$$u \in W_k \Rightarrow G(\Lambda, \Lambda_0)u = u. \tag{3.8}$$

To apply (3.8) to estimate how elements of W_k concentrate on the equator, we aim to bring in arguments parallel to those provided to prove Propositions 2.1 and 2.2. First, parallel to (2.12), the operator $G(\Lambda, \Lambda_0)$ has principal symbol

$$g(x, \xi) = \varphi((1 - \sigma_{-L}(x, \hat{\xi}))|\xi|_x^\delta), \quad \hat{\xi} = \frac{\xi}{|\xi|_x},$$

with complete symbol expansion derived from (2.11). Next, parallel to (2.13), we have

$$\sigma_{-L}(x, \hat{\xi}) \le 1 - c x_{n+1}^2;$$

hence,

$$1 - \sigma_{-L}(x, \hat{\xi}) \ge c x_{n+1}^2,$$

and therefore,

$$|g(x, \xi)| \le \varphi(c x_{n+1}^2 |\xi|^\delta),$$

so, for $M \in (0, \infty)$,

$$(x_{n+1}^2 |\xi|^\delta)^M |g(x, \xi)| \le C_M.$$

This leads to the following proposition.

Proposition 3.1. *Given $\rho \in (1/2, 1)$, the operator $G(\Lambda, \Lambda_0)$, defined by (3.4)–(3.5), satisfies, for each $\nu \in \mathbb{Z}^+$,*

$$x_{n+1}^{\nu} G(\Lambda, \Lambda_0) \in \mathrm{OPS}_{\rho,\delta}^{-\nu\delta/2}(S^n), \quad \delta = 1 - \rho. \tag{3.9}$$

Having (3.9), we bring in (3.8) to deduce that, for $\nu \in \mathbb{N},\ s \in \mathbb{R}$,

$$\begin{aligned} u \in W_k \Rightarrow \|x_{n+1}^{\nu} u\|_{H^s} &= \|x_{n+1}^{\nu} G(\Lambda, \Lambda_0) u\|_{H^s} \\ &\le C_\nu \|u\|_{H^{s-\nu\delta/2}} \\ &\le C_\nu \mu_k^{s-\nu\delta/2} \|u\|_{L^2}, \end{aligned}$$

parallel to (2.14). As before, this estimate is particularly valuable for $\nu\delta/2 > s$.

At this point, it behooves us to establish a lower estimate on

$$\dim W_k,$$

extending the estimate (2.15), done for $n = 2$. We aim to establish an estimate of the form

$$\dim W_k \ge C(\dim V_k) k^{\rho - 1}. \tag{3.10}$$

To tackle this, it is convenient to refine our notation a bit, relabeling V_k in (3.1) as

$$V_k(S^n) = \{u \in L^2(S^n) : \Delta_S u = -k(k + n - 1)u\}$$

and rewriting (3.3) as

$$V_k(S^n) = \bigoplus_{j=0}^{k} \widetilde{V}_{kj}(S^n), \tag{3.11}$$

where

$$\widetilde{V}_{kj}(S^n) = \{u \in V_k(S^n) : Lu = -j(j + n - 2)u\}. \tag{3.12}$$

We also relabel W_k as $W_k(S^n)$ and, in place of (3.7), write

$$\begin{aligned} W_k(S^n) &= \bigoplus_j \Big\{ \widetilde{V}_{kj}(S^n) : \mu_k^2 - \mu_k^{1+\rho} \le \sigma_j^2 \le \mu_k^2 \Big\}, \\ \mu_k^2 &= k(k + n - 1), \quad \sigma_j^2 = j(j + n - 2). \end{aligned} \tag{3.13}$$

The following proposition is key to our dimension estimate.

Proposition 3.2. *For $0 \le j \le k$, $n \ge 3$,*

$$\widetilde{V}_{kj}(S^n) \approx V_j(S^{n-1}). \tag{3.14}$$

Proof. Note that the natural action of $\mathrm{SO}(n)$ on $L^2(S^n)$ leaves each space $\widetilde{V}_{kj}(S^n)$ in (3.11) invariant. In view of (3.12), we see that, for each $j \in \{0, \dots, k\}$, $\widetilde{V}_{kj}(S^n)$ is either 0 or a direct sum of spaces isomorphic to $V_j(S^{n-1})$. Furthermore, [12, Proposition 2.4] implies that if $\widetilde{V}_{kj}(S^n) \neq 0$, then

$$\mathrm{SO}(n) \text{ acts irreducibly on } \widetilde{V}_{kj}(S^n).$$

Hence, either (3.14) holds or $\widetilde{V}_{kj} = 0$.

At this point, we see that Proposition 3.2 is equivalent to the assertion that there is an $\mathrm{SO}(n)$-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{j=0}^{k} V_j(S^{n-1}), \tag{3.15}$$

and so far, we know that the left-hand side of (3.15) is isomorphic to an $\mathrm{SO}(n)$-invariant linear subspace of the right-hand side. Hence, the proof of Proposition 3.2 is done if we show that

$$\dim V_k(S^n) = \sum_{j=0}^{k} \dim V_j(S^{n-1}).$$

This computation is carried out in Appendix A. ■

To proceed toward a proof of (3.10), we have from (3.14) that

$$\dim W_k(S^n) = \sum_j \{\dim V_j(S^{n-1}) : \mu_k^2 - \mu_k^{1+\rho} \le \sigma_j^2 \le \mu_k^2\}. \tag{3.16}$$

Note that the restriction on j (beyond $0 \le j \le k$) can be written as

$$\mu_k \sqrt{1 - \mu_k^{\rho-1}} \le \sigma_j \le \mu_k,$$

so in light of (3.13), the number of summands in (3.16) is

$$\approx \frac{1}{2}\mu_k^\rho \approx \frac{1}{2}k^\rho$$

for large k. We bring in the asymptotics

$$\dim V_k(S^n) \sim C_n k^{n-1}, \quad \text{as } k \to \infty,$$

which follow from (3.2), and the variant

$$\dim V_j(S^{n-1}) \sim C_{n-1} j^{n-2}.$$

This leads to the estimate

$$\begin{aligned}\dim W_k(S^n) &\geq C k^{n-2} \cdot k^\rho \\ &\geq C \dim V_k(S^n)\, k^{\rho-1},\end{aligned}$$

as asserted in (3.10).

Summarizing the main results of this section, we have the following proposition.

Proposition 3.3. *Take $\rho \in (1/2, 1)$, $\delta = 1-\rho$, $n \geq 2$. For $k \geq 1$, there exist linear subspaces $W_k(S^n) \subset V_k(S^n)$ satisfying*

$$\dim W_k(S^n) \geq C k^{-\delta} \dim V_k(S^n), \tag{3.17}$$

$$u \in W_k(S^n) \Rightarrow G(\Lambda, \Lambda_0)u = u, \tag{3.18}$$

with $G(\Lambda, \Lambda_0)$ as in (3.4)–(3.6) and (3.9), and, for $\nu \in \mathbb{N}$, $s \in \mathbb{R}$,

$$u \in W_k(S^n) \Rightarrow \|x_{n+1}^\nu u\|_{H^s} \leq C_\nu \mu_k^{s-\nu\delta/2} \|u\|_{L^2}.$$

(Recall that $\mu_k \sim k$.)

4. Elements of $W_k(S^n)$ as quasimodes for perturbed Laplace operators

As indicated in the introduction, we take the Riemannian manifold M to be S^n, endowed with a metric tensor that is a perturbation of the standard metric tensor of the unit sphere, and investigate how elements of $W_k(S^n)$ yield quasimodes for the Laplace–Beltrami operator Δ_M. We start by examining how Δ_S and Δ_M are related. The metric tensors (g_{ij}) of M and (γ_{ij}) of S^n are related by

$$g_{ij} = \gamma_{ij} + \sigma_{ij}, \quad \sigma_{ij}\big|_{x_{n+1}=0} = 0; \tag{4.1}$$

more precisely, we assume that

$$\sigma_{ij} = O(x_{n+1}^\nu) \tag{4.2}$$

for some $\nu \in \mathbb{N}$ $(\nu \geq 2)$. Now, we compare Laplace operators

$$\begin{aligned}\Delta_S u &= \gamma^{-1/2}\partial_i(\gamma^{1/2}\gamma^{ij}\partial_j u), \\ \Delta_M u &= g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j u).\end{aligned}$$

We obtain

$$\begin{aligned}-\Delta_M &= -\Delta_S + h^{ij}\partial_i\partial_j + h^j\partial_j, \\ h^{ij} &= O(x_{n+1}^\nu), \quad h^j = O(x_{n+1}^{\nu-1}).\end{aligned}$$

Consequently, by Proposition 2.2, with $n = 2$, $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and $F(\Lambda, X)$ as in (2.2)–(2.3),

$$(\Delta_S - \Delta_M) F(\Lambda, X) = Q \in OPS^{2-\nu\delta/2}_{\rho,\delta}.$$

Similarly, by Proposition 3.1, with $n \geq 3$, $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and $G(\Lambda, \Lambda_0)$ as in (3.4)–(3.5),

$$(\Delta_S - \Delta_M) G(\Lambda, \Lambda_0) = Q \in OPS^{2-\nu\delta/2}_{\rho,\delta}.$$

Thanks to (2.5) for $n = 2$, (3.18), for $n \geq 3$, we therefore have (with μ_k as in (3.1))

$$u \in W_k(S^n) \Rightarrow (-\Delta_M - \mu_k^2) u = Qu.$$

Let us set

$$\Lambda_M = (-\Delta_M)^{1/2}, \quad \text{so} \; -\Delta_M - \mu_k^2 = (\Lambda_M + \mu_k)(\Lambda_M - \mu_k).$$

It follows that, for $u \in W_k(S^n)$,

$$\begin{aligned} \|(\Lambda_M - \lambda_k) u\|_{L^2} &\leq \mu_k^{-1} \|(-\Delta_M - \mu_k^2) u\|_{L^2} \\ &\leq C k^{-1} \|Qu\|_{L^2} \\ &\leq c k^{-1} \|u\|_{H^{-(\nu\delta/2-2)}} \\ &\leq C k^{-(\nu\delta/2-1)} \|u\|_{L^2}. \end{aligned}$$

We record our quasimode estimate.

Proposition 4.1. *Take $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and pick ν sufficiently large that*

$$\sigma = \frac{\nu\delta}{2} - 1$$

is positive. Assume that the metric tensor on M satisfies (4.1)–(4.2). *Then,*

$$u \in W_k(S^n) \Rightarrow \|(\Lambda_M - \mu_k) u\|_{L^2} \leq C k^{-\sigma} \|u\|_{L^2}. \tag{4.3}$$

We next show that there is a sequence of actual eigenvalues of Λ_M close to μ_k. To start, it follows directly from (4.3) that there exists $\psi_k^1 \in C^\infty(M)$ such that

$$\|\psi_k^1\|_{L^2(M)} = 1, \quad \Lambda_M \psi_k^1 = \mu_{k1} \psi_k^1, \quad |\mu_{k1} - \mu_k| \leq C k^{-\sigma}.$$

Of course, Λ_M need not leave $W_k(S^n)$ invariant, and we cannot say that ψ_k^1 is in, or even particularly close to, $W_k(S^n)$. Set

$$Z_1 = \operatorname{Span} \psi_k^1.$$

Then,

$$(1 + \Lambda_M)^{-1} : Z_1^\perp \to Z_1^\perp.$$

We have

$$\dim W_k(S^n) \ge 2 \Rightarrow W_k(S^n) \cap Z_1^{\perp} \ne 0,$$

in which case

$$\begin{aligned} &\exists\, \psi_k^2 \in Z_1^{\perp}, \text{ with unit norm, such that} \\ &\Lambda_M \psi_k^2 = \mu_{k2}\psi_k^2, \quad |\mu_{k2} - \mu_k| \le C k^{-\sigma}. \end{aligned}$$

Continue producing an orthonormal set ψ_k^ℓ of smooth elements of $L^2(M)$, satisfying

$$\Lambda_M \psi_k^\ell = \mu_{k\ell}\psi_k^\ell, \quad |\mu_{k\ell} - \mu_k| \le C k^{-\sigma}, \tag{4.4}$$

for $1 \le \ell \le L$, and set

$$Z_L = \operatorname{Span}(\psi_k^1, \dots, \psi_k^L),$$

so $(1+\Lambda_M)^{-1} : Z_L^{\perp} \to Z_L^{\perp}$. We have

$$\dim W_k(S^n) > L \Rightarrow W_k(S^n) \cap Z_L^{\perp} \ne 0,$$

in which case

$$\begin{aligned} &\exists\, \psi_k^{L+1} \in Z_L^{\perp}, \quad \text{with unit norm, such that} \\ &\Lambda_M \psi_k^{L+1} = \mu_{k,L+1}\psi_k^{L+1}, \quad |\mu_{k,L+1} - \mu_k| \le C k^{-\sigma}. \end{aligned}$$

We can do this right up to the point where

$$L = \dim W_k(S^n).$$

This construction leads to the following result on eigenvalues of Δ_M close to $-\mu_k^2$.

Proposition 4.2. *Keep the setting of Proposition 4.1, including having the metric tensor on M satisfying (4.1)–(4.2). Then, for k sufficiently large, there exists an orthonormal set*

$$\{\psi_k^\ell : 1 \le \ell \le \dim W_k(S^n)\} \subset L^2(M)$$

of eigenfunctions of Δ_M, satisfying (4.4). Furthermore,

$$\begin{aligned} \dim W_k(S^n) &\ge C k^{-\delta} \dim V_k(S^n) \\ &\ge C' k^{n-1-\delta}. \end{aligned}$$

5. Necessary condition for an algebraically small Weyl remainder

As in Section 4, M is a compact, n-dimensional Riemannian manifold, whose metric tensor is a perturbation of that of the standard sphere S^n, satisfying (4.1)–(4.2). We seek a necessary condition that

$$\mathcal{N}(\Delta_M, R) = C(M) R^n + O(R^{n-1-\alpha}) \tag{5.1}$$

for some $\alpha \in (0,1)$. Having this, we deduce a sufficient condition for (5.1) to fail for all $\alpha > 0$. Recall further details of this setup. We pick $\rho \in (1/2, 1)$, $\delta = 1-\rho$, and then take ν in (4.2) sufficiently large that

$$\sigma = \frac{\nu\delta}{2} - 1$$

is positive.

To continue, if (5.1) holds for all (large) R, then, for $b \in [0,1]$,

$$\mathcal{N}(\Delta_M, R+b) - \mathcal{N}(\Delta_M, R-b) = 2nC(M)bR^{n-1} + O(R^{n-1-\alpha}).$$

Let us take $b = cR^{-\sigma}$, so

$$\begin{aligned}&\mathcal{N}(\Delta_M, R+cR^{-\sigma}) - \mathcal{N}(\Delta_M, R-cR^{-\sigma})\\ &\quad = 2ncC(m)R^{n-1-\sigma} + O(R^{n-1-\alpha}).\end{aligned}$$

Proposition 4.2 implies that there exists $c \in (0,\infty)$ such that if $R = \mu_k$, then

$$\mathcal{N}(\Delta_M, R+cR^{-\sigma}) - \mathcal{N}(\Delta_M, R-cR^{-\sigma}) \geq CR^{n-1-\delta}.$$

We deduce that (for $R = \mu_k$)

$$CR^{n-1-\delta} \leq 2ncC(M)R^{n-1-\sigma} + O(R^{n-1-\alpha}). \tag{5.2}$$

At this point, we strengthen our hypothesis on ν, from $\sigma > 0$ to

$$\sigma > \delta, \quad \text{i.e.,}\ \frac{\nu}{2} > \frac{\delta+1}{\delta}. \tag{5.3}$$

With this arranged, we see that (5.2) implies

$$\alpha \leq \delta = 1-\rho.$$

This establishes the following result.

Proposition 5.1. *Let M be a compact, n-dimensional Riemannian manifold. Pick $\rho \in (1/2,1)$, $\delta = 1-\rho$, and assume that $\nu \in \mathbb{N}$ satisfies (5.3). Then, assume that the metric tensor on M satisfies (4.1)–(4.2), i.e., matches the standard metric tensor on S^n to order ν at the equator. In such a case, if the Weyl asymptotic formula (5.1) holds, we must have $\alpha \leq 1-\rho$.*

From here, we have the following conclusion.

Theorem 5.2. *In the setting of Proposition 5.1, if the metric tensor on M matches the standard metric tensor on S^n to infinite order at the equator, then (5.1) cannot hold for any $\alpha > 0$.*

A. Dimension counts

Work in Section 3 makes use of the SO(n)-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{\ell=0}^{k} V_\ell(S^{n-1}), \tag{A.1}$$

where $V_k(S^n)$ denotes the $-k(k+n-1)$-eigenspace of the Laplace operator on S^n, and $V_\ell(S^{n-1})$ is similarly defined. As seen there, results on irreducibility of certain SO(n) actions enable one to establish (A.1) once we have the identity

$$\dim V_k(S^n) = \sum_{\ell=0}^{k} \dim V_\ell(S^{n-1}). \tag{A.2}$$

We establish this here.

In preparation, we recall a standard approach to computing the left-hand side of (A.2), using the isomorphism

$$V_k(S^n) \approx \mathcal{H}_k(\mathbb{R}^{n+1}), \tag{A.3}$$

the space of harmonic polynomials on $\mathbb{R}^{n+1}$, homogeneous of degree k, and the decomposition

$$\mathcal{P}_k(\mathbb{R}^{n+1}) = \mathcal{H}_k(\mathbb{R}^{n+1}) \oplus |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^{n+1}). \tag{A.4}$$

Here and below,

$$\begin{aligned} \mathcal{P}_k(\mathbb{R}^{n+1}) &= \text{space of polynomials on } \mathbb{R}^{n+1}, \text{ homogeneous of degree } k, \\ \mathcal{P}^k(\mathbb{R}^{n+1}) &= \text{space of polynomials on } \mathbb{R}^{n+1}, \text{ of degree} \le k, \\ d_k(n+1) &= \dim \mathcal{P}_k(\mathbb{R}^{n+1}). \end{aligned} \tag{A.5}$$

Note that

$$d_k(n+1) = \dim \mathcal{P}^k(\mathbb{R}^n) = d_k(n) + d_{k-1}(n) + \cdots + d_0(n), \tag{A.6}$$

with a similar result for $d_j(m)$, for other values of j and m.

Using (A.3)–(A.6) yields

$$\begin{aligned} \dim V_k(S^n) &= d_k(n+1) - d_{k-2}(n+1) \\ &= d_k(n) + d_{k-1}(n). \end{aligned} \tag{A.7}$$

Similarly,

$$\sum_{\ell=0}^{k} \dim V_\ell(S^{n-1}) = \sum_{\ell=0}^{k} \{d_\ell(n-1) + d_{\ell-1}(n-1)\}.$$

On the other hand, (A.6) (with n replaced by $n-1$) gives

$$d_k(n) = \sum_{\ell=0}^{k} d_\ell(n-1), \tag{A.8}$$

and similarly, we have

$$d_{k-1}(n) = \sum_{\ell=0}^{k-1} d_\ell(n-1) = \sum_{\ell=0}^{k} d_{\ell-1}(n-1). \tag{A.9}$$

Together, (A.7)–(A.9) yield the desired identity (A.2).

Remark. There is the classical computation

$$d_k(n) = \binom{k+n-1}{k}.$$

In light of this, the identity (A.8) is equivalent to

$$\sum_{\ell=0}^{k} \binom{\ell+m}{\ell} = \binom{k+m+1}{k}$$

(with $m = n-2$), which is sometimes given the whimsical label, the "hockey stick identity."

B. Surfaces satisfying Theorem 5.2 and (1.3)

Our goal here is to exhibit some compact 2D surfaces $M \subset \mathbb{R}^3$ whose metric tensors agree with that of the standard sphere S^2 to infinite order on the "equator" $\gamma = \{(a,b,0) : a^2+b^2 = 1\} \subset M$, so Theorem 5.2 is applicable, and which also have the property that the set of periodic geodesics on M has measure 0, so the remainder estimate $E_M(R) = o(R^{n-1})$ holds (with $n = 2$).

To simplify our arguments, we will impose further restrictions on M. For one, M is invariant under rotation about the z-axis in $\mathbb{R}^3$. We also assume that M is convex, with positive Gauss curvature. In other words, M is obtained by taking a simple closed smooth curve σ in the yz-plane, with positive curvature, and rotating it about the z-axis in the xyz-space $\mathbb{R}^3$. In particular, we assume that σ is invariant under reflection $(y,z) \mapsto (-y,z)$. In addition, we assume that σ is invariant under reflection $(y,z) \mapsto (y,-z)$, so M is invariant under reflection $(x,y,z) \mapsto (x,y,-z)$.

Assume $(\pm 1, 0) \in \sigma$, and that σ is tangent at these points to the unit circle S^1 $\{(a,b) : a^2+b^2 = 1\}$ to infinite order. We also assume that σ is obtained from S^1 by flattening it, in such a way that, for $(y,z) \in \sigma$, y^2+z^2 decreases strictly monotonically as y decreases from 1 to 0. Hence, M is tangent to S^2 to infinite order along the equator $\gamma = M \cap S^2$, and

$$M \setminus \gamma \subset \{(x,y,z) : x^2+y^2+z^2 < 1\}. \tag{B.1}$$

Our task is to study the geodesics on M. To start, the symmetry $z \mapsto -z$ implies that γ is a closed geodesic on M. (So, does the tangency of M and S^2 along γ.) In addition, the geometrical hypothesis of convexity made above implies that

$$\text{each geodesic on } M \text{ intersects } \gamma.$$

If μ is a geodesic other than γ, its intersections with γ are all transverse. The symmetries of M imply that if μ is a geodesic on M, so is $R\mu$, for each rotation R about the z-axis, and so is the "antipodal" geodesic $-\mu$. Now, we can concentrate on unit speed geodesics starting out at $p_0 = (1,0,0) \in \gamma$. If we identify $T_{p_0}M$ with the yz-plane, such geodesics are of the form μ_θ, having initial velocity vector $v_\theta = (\cos\theta, \sin\theta)$. In particular, $\mu_0 = \gamma$, and $\mu_\pi = \tilde{\gamma}$ (γ headed in the opposite direction). We claim that

$$\mu_\theta \text{ is a periodic geodesic for only countably many values of } \theta. \tag{B.2}$$

This will imply that the set of periodic geodesics on M has measure 0.

We take $\theta \in (0,\pi)$ and consider the behavior of $\mu_\theta(t)$, satisfying

$$\mu_\theta(0) = p_0, \quad \mu_\theta'(0) = v_\theta \in T_{p_0}M.$$

This curve leaves γ at angle θ. There is a first time $t_\theta > 0$ at which $\mu_\theta(t)$ intersects γ again:

$$\mu_\theta(t_\theta) = q_\theta \in \gamma, \quad \mu_\theta'(t_\theta) = \tilde{v}_\theta \in T_{q_\theta}M.$$

There is a natural isomorphism $T_{q_\theta}M \approx T_{p_0}M$, given by parallel translation along γ (or equivalently by the action of the rotational symmetries R), yielding

$$\tilde{v}_\theta = (\cos\theta, -\sin\theta),$$

thanks to conservation of angular momentum for the geodesic flow on this surface of revolution. We can write

$$q_\theta = (\cos\omega_\theta, \sin\omega_\theta, 0)$$

for some $\omega_\theta \in (0, 2\pi)$. Keeping in mind the symmetries that are in play here, we see that subsequent points where μ_θ intersects γ are given by

$$q_{\theta,k} = (\cos k\omega_\theta, \sin k\omega_\theta, 0).$$

Hence, we have the following result.

Lemma B.1. *The geodesic μ_θ is periodic if and only if ω_θ is a rational multiple of π.*

Now, if M were simply S^2, we would have $q_\theta = -p_0$; hence, $\omega_\theta = \pi$ for all $\theta \in (0, \pi)$. However, the flattening process, leading to (B.1), forces q_θ, hence ω_θ, to vary with θ. Due to the flattening, the unique shortest geodesics from p_0 to $-p_0$ are $\mu_{\pi/2}$ and $\mu_{-\pi/2}$ (mirror images under $z \mapsto -z$), and the exponential map $\mathrm{Exp}_{p_0} : T_{p_0} \to M$ maps a neighborhood of the ray $\{(0, z) : 0 \le z \le \mathrm{dist}(p_0, -p_0)\}$ diffeomorphically onto its image in M. Hence, $\omega_{\pi/2} = \pi$ and $(d/d\theta)\omega_\theta|_{\theta=\pi/2} \neq 0$. The quantity ω_θ is not a monotone function of θ on all of $(0, \pi)$, since also $q_\theta \to -p_0$ as $\theta \to 0$ and as $\theta \to \pi$, but it is piecewise strictly monotonic. Therefore, we have (B.2), as a consequence of Lemma B.1.

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