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On Trudinger-type inequalities in Musielak–Orlicz–Morrey spaces of an integral form over metric measure spaces

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Abstract. We establish Trudinger-type inequalities for variable Riesz potentials $J_{\alpha(\cdot),\tau} f$ of functions f in Musielak–Orlicz–Morrey spaces of an integral form over metric measure spaces X. As an application and example, we give Trudinger's inequality for double-phase functionals with variable exponents. Finally, we prove the result for Sobolev functions satisfying a Poincaré inequality in X.

1. Introduction

Let G be a bounded open set in $\mathbf{R}^N$. A famous Trudinger inequality in [43] insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability (see also, e.g., [4, 28]). For $0 < \alpha < N$ and a locally integrable function f on $\mathbf{R}^N$, the Riesz potential $U_\alpha f$ of order α is defined by

$$U_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

In [25], Trudinger-type inequalities were studied for $U_\alpha f$ of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\sup_{x\in G} \left(\int_0^{d_G} r^{\nu-N} \varphi_1(r) \left(\int_{B(x,r)} |f(y)|^p \varphi_2(|f(y)|)\, dy \right) \frac{dr}{r} \right)^{1/p} < \infty, \tag{1.1}$$

where $d_G = \sup\{d(x,y) : x,y \in G\}$ and φ_i $(i = 1,2)$ are positive monotone functions on the interval $(0,\infty)$ satisfying the conditions (φ), (i), and (ii). See also, e.g., [6–9,22,24,27] for Trudinger-type inequalities.

In the present paper, we work in metric measure spaces $X = (X, d, \mu)$, where X is a bounded set, d is a metric on X, and μ is a nonnegative complete Borel regular outer measure on X with $\mu(X) < \infty$. We denote by $B(x,r)$ the open ball in X centered at $x \in X$ with radius $r > 0$ and $d_X = \sup\{d(x,y) : x,y \in X\}$. We assume that $d_X < \infty$,

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$\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ satisfies the doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x,2r)) \le c_0\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)(= X)$ and $r > 0$. Otherwise, μ is said to be non-doubling. See, e.g., [31,41] for examples of non-doubling metric measure spaces.

Let $\alpha(\cdot)$ be a measurable function on X such that

$$0 < \alpha^- := \inf_{x\in X} \alpha(x) \le \sup_{x\in X} \alpha(x) =: \alpha^+ < \infty.$$

Following [34, 36] and [11] by Hajłasz and Koskela, we consider the Riesz potential $J_{\alpha(\cdot),\tau} f$ of order $\alpha(\cdot)$ for $\tau \ge 1$ and a locally integrable function f on X by

$$J_{\alpha(\cdot),\tau} f(x) = \sum_{2^i \le 2d_X} \frac{2^{i\alpha(x)}}{\mu(B(x,\tau 2^i))} \int_{B(x,2^i)} f(y)\, d\mu(y),$$

which is better suited to the metric measure case. Trudinger's inequality for $J_{\alpha,1} f$ was studied on $L^p(X)$ in [11, Theorem 5.3] and on $L^{p(\cdot)}(X)$ in [13, Theorem 4.8]. It is known that

$$I_{\alpha(\cdot)} f(x) = \int_X \frac{d(x,y)^{\alpha(x)}}{\mu(B(x,d(x,y)))} f(y)\, d\mu(y) \le C J_{\alpha(\cdot),1} f(x)$$

when μ satisfies the doubling condition. For $I_{\alpha(\cdot)} f$, see, e.g., [11,29,32].

Our main aim is to establish a Trudinger-type inequality for variable Riesz potentials $J_{\alpha(\cdot),\tau} f$ of functions f in Musielak–Orlicz–Morrey spaces of an integral form $\mathcal{L}^{\Phi,\omega,\theta}(X)$ defined by general functions $\Phi(x,t)$ and $\omega(x,r)$ (Theorem 5.1), as an extension of [25, Theorem 5.4] and [11,13]. See Section 2 for the definition of $\mathcal{L}^{\Phi,\omega,\theta}(X)$. We prove Theorem 5.1 by relaxing ($\Phi5;\nu$) in [18,36] by ($\Phi5;\omega$) below. See Remarks 2.4 and 6.3. We refer to [39, Section 8] for the relationship among ($\Phi5;\omega$), ($\Phi5;\nu$) and [12, (A1)] by Harjulehto and Hästö. To obtain Theorem 5.1, we use Hedberg's method [15] and apply the boundedness of the (modified) Hardy–Littlewood maximal function defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,r)} |f(y)|\, d\mu(y)$$

for a locally integrable function f on X and $\lambda \ge 1$.

As a good example, we give a Trudinger-type inequality for double-phase functionals with variable exponents [18]

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)} \big(= t^{p(x)} + (b(x)t)^{q(x)} \big), \quad x \in X,\ t \ge 0,$$

where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions, $p(x) < q(x)$ for $x \in X$, $a(\cdot)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$ and $b(x) = a(x)^{1/q(x)}$ (Corollary 6.5). Thanks to the relaxed condition ($\Phi5;\omega$), we give an improvement of [37, Theorem 5.1] (see Corollary 6.2 and Remark 6.3). For the study on double-phase functional, we refer to, e.g., [2,3,5] by Baroni, Colombo, and Mingione and [21,26].

As an application of our discussions, we study a Trudinger-type inequality for Sobolev functions satisfying a Poincaré inequality in X (Theorem 7.2 and Corollary 7.3), as an extension of [11, Theorem 5.1].

For Sobolev's inequality for Musielak–Orlicz–Morrey spaces, see [34, 37, 38].

Throughout the paper, we let C denote various constants independent of the variables in question and let $C(a, b, \ldots)$ be a constant that depends on $a, b, \ldots$ only.

2. Musielak–Orlicz–Morrey spaces of an integral form

In this section, we define Musielak–Orlicz–Morrey spaces of an integral form. Let us consider a function

$$\Phi(x,t) : X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ3):

(Φ1) $\Phi(\cdot, t)$ is measurable on X for each $t \ge 0$ and $\Phi(x, \cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \Phi(x,1) \le A_1 \quad \text{for all } x \in X;$$

(Φ3) $t \mapsto \Phi(x,t)/t$ is uniformly almost increasing on $(0,\infty)$, namely, there exists a constant $A_2 \ge 1$ such that

$$\Phi(x,t_1)/t_1 \le A_2 \Phi(x,t_2)/t_2 \quad \text{for all } x \in X \text{ whenever } 0 < t_1 < t_2.$$

Remark 2.1. By (Φ2) and (Φ3), we have

$$\Phi(x,t) \le A_1 A_2 t \quad \text{for } 0 \le t \le 1 \quad \text{and} \quad \Phi(x,t) \ge (A_1 A_2)^{-1} t \quad \text{for } t \ge 1. \tag{2.1}$$

Letting $\overline{\phi}(x,t) = \sup_{0<s\le t}(\Phi(x,s)/s)$ and $\overline{\Phi}(x,t) = \int_0^t \overline{\phi}(x,r)\,dr$ for $x \in X$ and $t \ge 0$, then $\overline{\Phi}(x,\cdot)$ is convex and

$$\Phi(x,t/2) \le \overline{\Phi}(x,t) \le A_2 \Phi(x,t) \tag{2.2}$$

for all $x \in X$ and $t \ge 0$. In fact,

$$\overline{\Phi}(x,t) \ge \int_{t/2}^t \overline{\phi}(x,r)\,dr \ge \frac{t}{2}\overline{\phi}\Big(x, \frac{t}{2}\Big) \ge \Phi\Big(x, \frac{t}{2}\Big)$$

and

$$\overline{\Phi}(x,t) = \int_0^t \overline{\phi}(x,r)\,dr \le t\overline{\phi}(x,t) \le A_2 \Phi(x,t)$$

by (Φ3).

We also consider a weight function $\omega(x,r) : X \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(x,\cdot)$ is measurable on $(0,\infty)$ for each $x \in X$;

(ω1) $r \mapsto \omega(x,r)$ is uniformly almost increasing on $(0,\infty)$, namely, there exists a constant $\tilde{c}_1 \ge 1$ such that

$$\omega(x,r_1) \le \tilde{c}_1 \omega(x,r_2)$$

for all $x \in X$ whenever $0 < r_1 < r_2$;

(ω3) there exist a constant $\tilde{c}_3 \ge 1$ such that

$$\omega(x,r) \le \tilde{c}_3$$

for all $x \in X$ and $r > 0$ and

$$\omega(x,d_X) \ge \tilde{c}_3^{-1}$$

for all $x \in X$.

Note that (ω2) in [38], which is the doubling condition on ω, is not needed in this paper.

Let us write that $L_c(t) = \log(c+t)$ for $c > 1$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. Let $f^- := \inf_{x\in X} f(x)$ and $f^+ := \sup_{x\in X} f(x)$ for a measurable function f on X.

Example 2.2. Let $\sigma(\cdot)$ and $\beta_j(\cdot)$, $j = 1,\dots,k$ be measurable functions on X such that $0 < \sigma^- \le \sigma^+ < \infty$ and $-\infty < \beta_j^- \le \beta_j^+ < \infty$ for all $j = 1,\dots,k$. Then,

$$\omega_{\sigma(\cdot),\{\beta_j(\cdot)\}}(x,r) = \begin{cases} r^{\sigma(x)} \prod_{j=1}^k (L_e^{(j)}(1/r))^{\beta_j(x)} & \text{when } 0 < r \le d_X, \\ \omega_{\sigma(\cdot),\{\beta_j(\cdot)\}}(x,d_X) & \text{when } r > d_X \end{cases}$$

satisfies (ω0), (ω1), and (ω3).

Recall that f is a locally integrable function on X if f is an integrable function on all balls B in X. Let $\theta \ge 1$. In connection with (1.1), given $\Phi(x,t)$ and $\omega(x,r)$ as above, we define the $\mathcal{L}^{\Phi,\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} = \inf\left\{\lambda > 0; \sup_{x\in X}\left(\int_0^{2d_X} \frac{\omega(x,r)}{\mu(B(x,\theta r))}\left(\int_{B(x,r)} \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,d\mu(y)\right)\frac{dr}{r}\right) \le 1\right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{\Phi,\omega,\theta}(X)$. The space $\mathcal{L}^{\Phi,\omega,\theta}(X)$ is referred to as a Musielak–Orlicz–Morrey space of an integral form. In the case when $\Phi(x,t) = t^p$, $\mathcal{L}^{\Phi,\omega,\theta}(X)$ is denoted by $\mathcal{L}^{p,\omega,\theta}(X)$ for simplicity.

Remark 2.3. We remark from (ω3) that $2d_X$ in the definition of $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)}$ can be replaced by κd_X with $\kappa > 1$.

We will also consider the following conditions for $\Phi(x,t)$: let $p \ge 1$ be given.

($\Phi3;p$) $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing on $(0,\infty)$, namely, there exists a constant $A_{2,p} \ge 1$ such that

$$t_1^{-p}\Phi(x,t_1) \le A_{2,p} t_2^{-p}\Phi(x,t_2) \quad \text{for all } x \in X \text{ whenever } 0 < t_1 < t_2;$$

($\Phi5;\omega$) for every $\eta > 0$, there exists a constant $B_\eta \ge 1$ such that

$$\Phi(x,t) \le \Phi(y, B_\eta t)$$

whenever $y \in B(x,r)$, $\Phi(x,t) \le \eta\omega(x,r)^{-1}$, and $t \ge 1$.

Note that (Φ4) in [18], which is the doubling condition on Φ, is not needed in this paper.

Remark 2.4. For a measurable set $E \subset \mathbf{R}^N$, $|E|$ denotes its Lebesgue measure. In the Euclidean setting, Maeda, Mizuta, and the authors [18] considered the following condition for $\Phi(x,t)$:

($\Phi5;\nu$) for every $\iota > 0$, there exists a constant $\widetilde{B}_{\iota,\nu} \ge 1$ such that

$$\Phi(x,t) \le \widetilde{B}_{\iota,\nu}\Phi(y,t)$$

whenever $x, y \in \mathbf{R}^N$, $|x-y| \le \iota t^{-\nu}$, and $t \ge 1$.

For the metric measure setting, see ($\Phi5;\nu$) in [33, 36]. Harjulehto and Hästö [12] considered the following condition:

(A1) there exists a constant $0 < \beta < 1$ such that

$$\beta\Phi^{-1}(x,t) \le \Phi^{-1}(y,t)$$

for every $1 \le t \le 1/|B|$, $x, y \in B$ and ball B with $|B| \le 1$.

On the relationship between ($\Phi5;\omega$), ($\Phi5;\nu$), and (A1), see [39, Section 8].

We give two good examples of $\Phi(x,t)$.

Example 2.5. Let $\omega(x,r)$ be as in Example 2.2. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on X such that $1 < p^- \le p^+ < \infty$ and $-\infty < q_j^- \le q_j^+ < \infty$ for all $j = 1,\dots,k$.

Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_e^{(j)}(t))^{q_j(x)}$$

satisfies (Φ1), (Φ2), and (Φ3). It satisfies ($\Phi3;p$) for $1 \le p < p^-$ in general and for $1 \le p \le p^-$ in case $q_j^- \ge 0$ for all $j = 1,\dots,k$.

Moreover, we see that $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies $(\Phi5;\omega)$ if $p(\cdot)$ is log-Hölder continuous, namely,

$$|p(x)-p(y)| \le \frac{C_p}{L_e(1/d(x,y))} \quad (x,y\in X)$$

with a constant $C_p \ge 0$ and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely,

$$|q_j(x)-q_j(y)| \le \frac{C_{q,j}}{L_e^{(j+1)}(1/d(x,y))} \quad (x,y\in X)$$

with constants $C_{q,j} \ge 0$ for each $j=1,\dots,k$. In fact, for $\eta>0$, let $y\in B(x,r)$, $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) \le \eta\omega(x,r)^{-1}$, and $t\ge 1$. Then, we see from (2.1) and $(\omega1)$ that

$$1 \le t \le A_1A_2\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) \le A_1A_2\eta\omega(x,r)^{-1} \le C(\eta)\omega(x,d(x,y))^{-1},$$

so that $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies $(\Phi5;\omega)$.

Example 2.6. The double-phase functional with variable exponents

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)}, \quad x\in X,\ t\ge 0,$$

where $p(x)<q(x)$ for $x\in X$, $a(\cdot)$ is a nonnegative, bounded, and Hölder continuous function of order $\theta\in(0,1]$, was studied in, e.g., [18, 19, 32, 40]. See Section 6.

3. Maximal operator

Recall that

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x,\lambda r))}\int_{B(x,r)} |f(y)|\,d\mu(y).$$

For $\lambda\ge 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C>0$ such that

$$\mu(\{x\in X : M_\lambda f(x) > k\}) \le \frac{C}{k}\int_X |f(y)|\,d\mu(y) \tag{3.1}$$

for all measurable functions $f\in L^1(X)$ and $k>0$. In (3.1), we cannot remove the number λ (Stempak [42]).

The following lemma was given in [35, Theorem 2.4] when $\omega(x,r)=\omega(r)$ and ω satisfies $(\omega2)$ in [35]. In the same manner, Lemma 3.1 can be proved by using $(\omega3)$. Hence, we omit the proof.

Lemma 3.1. *Let* $1\le\theta_1<\theta_2$ *and* $\lambda>\theta_1(\theta_2+1)/(\theta_2-\theta_1)$. *Assume that* X *satisfies* $(M\lambda)$. *Further, suppose that*

$(\omega1')$ $r\mapsto r^{-\varepsilon_1}\omega(x,r)$ *is uniformly almost increasing in* $(0,d_X]$ *for some* $\varepsilon_1>0$.

If $p>1$, *then there is a constant* $C>0$ *such that*

$$\|M_\lambda f\|_{\mathcal{L}^{p,\omega,\theta_2}(X)} \le C\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f\in\mathcal{L}^{p,\omega,\theta_1}(X)$.

Here, we remark that $(\omega1')$ implies $(\omega1)$. Letting $\omega(x, r)$ be as in Example 2.2, then $(\omega1')$ holds for $0 < \varepsilon_1 < \sigma^-$.

Theorem 3.2. *Let* $1 \le \theta_1 < \theta_2$ *and* $\lambda > \theta_1(\theta_2 + 1)/(\theta_2 - \theta_1)$. *Suppose that* $\Phi(x, t)$ *satisfies* $(\Phi3; p)$ *and* $(\Phi5; \omega)$ *for* $p > 1$. *Assume that* X *satisfies* $(M\lambda)$ *and* $(\omega1')$ *holds. Then, there is a constant* $C > 0$ *such that*

$$\|M_\lambda f\|_{\mathcal{L}^{\Phi,\omega,\theta_2}(X)} \le C\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$.

For $p \ge 1$ and $\lambda \ge 1$, set

$$I(f; x, r, \lambda) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} f(y)\, d\mu(y)$$

and

$$J(f; x, r, p, \lambda) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} \Phi(y, f(y))^{1/p}\, d\mu(y).$$

We show the following lemma to prove Theorem 3.2.

Lemma 3.3. *Let* $1 \le \theta < \lambda$. *Suppose that* $\Phi(x, t)$ *satisfies* $(\Phi3; p)$ *and* $(\Phi5; \omega)$ *for* $p \ge 1$. *Then, given* $L \ge 1$, *there exist constants* $C_1 = C(L) \ge 2$ *and* $C_2 > 0$ *such that*

$$\Phi\big(x, I(f; x, r, \lambda)/C_1\big)^{1/p} \le C_2 J(f; x, r, p, \lambda)$$

for all $x \in X$, $0 < r \le d_X$ *and for all nonnegative measurable functions* f *on* X *such that* $f(y) \ge 1$ *or* $f(y) = 0$ *for each* $y \in X$ *and*

$$\sup_{z\in X} \left(\int_0^{2d_X} \frac{\omega(z, t)}{\mu(B(z, \theta t))} \left(\int_{B(z,t)} \Phi(y, f(y))\, d\mu(y) \right) \frac{dt}{t} \right) \le L. \tag{3.2}$$

Proof. Given f as in the statement of the lemma, $x \in X$, and $0 < r \le d_X$, set $I = I(f; x, r, \lambda)$ and $J = J(f; x, r, p, \lambda)$. Taking f, note that (3.2) and $(\omega1)$ imply

$$\begin{aligned} &\frac{\omega(x, r)}{\mu(B(x, \lambda r))} \int_{B(x,r)} \Phi(y, f(y))\, d\mu(y) \\ &\quad \le C \int_r^{\lambda r/\theta} \frac{\omega(x, t)}{\mu(B(x, \theta t))} \left(\int_{B(x,t)} \Phi(y, f(y))\, d\mu(y) \right) \frac{dt}{t} \le C_0 L, \end{aligned}$$

so that

$$J \le C_0^{1/p} L^{1/p} \omega(x, r)^{-1/p}. \tag{3.3}$$

By $(\Phi3; p)$, $\Phi(y, f(y))^{1/p} \ge (A_1 A_{2,p})^{-1/p} f(y)$ for all $y \in X$. Hence,

$$I \le (A_1 A_{2,p})^{1/p} J.$$

Thus, if $J \le 1$, then by (Φ3; p)

$$\Phi(x, I/C_1)^{1/p} \le J\big(A_{2,p}\Phi(x,1)\big)^{1/p} \le (A_1 A_{2,p})^{1/p} J$$

whenever $C_1 \ge (A_1 A_{2,p})^{1/p}$.

Next, suppose $J > 1$. Since $\Phi(x,t)^{1/p} \to \infty$ as $t \to \infty$ by (Φ3; p), there exists $K > 1$ such that

$$\Phi(x, K)^{1/p} = \Phi(x,1)^{1/p} J. \tag{3.4}$$

Let $\eta = A_1 C_0 L$. Since $K > 1$ and

$$\Phi(x, K) \le A_1 J^p \le A_1 C_0 L \omega(x,r)^{-1} = \eta\omega(x,r)^{-1}$$

in view of (3.4) and (3.3), we see from (Φ5; ω) that there is $\beta = \beta(\eta) \ge 1$, independent of f, x, r, such that

$$\Phi(x, K) \le \Phi(y, \beta K)$$

for $y \in B(x,r)$. Hence, with this K, we have by (Φ3; p), (3.4), and (Φ2)

$$\begin{aligned}\int_{B(x,r)} f(y)\,d\mu(y) &\le \beta K \mu(B(x,r)) + A_{2,p}^{1/p}\beta K \int_{B(x,r)} \frac{\Phi(y, f(y))^{1/p}}{\Phi(y,\beta K)^{1/p}}\,d\mu(y)\\ &\le \beta K\mu(B(x,r)) + \frac{A_{2,p}^{1/p}\beta K}{\Phi(x,K)^{1/p}} \int_{B(x,r)} \Phi(y, f(y))^{1/p}\,d\mu(y)\\ &\le \beta K \mu(B(x,\lambda r))\big\{1 + (A_1 A_{2,p})^{1/p}\big\}\end{aligned}$$

as in the proof of [18, Lemma 3.3]. We refer to [20, Lemma 9] for details of the rest of the proof. ∎

Proof of Theorem 3.2. Consider the function

$$\Phi_0(x,t) = \Phi(x,t)^{1/p}.$$

Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \le 1/2$. Let $f_1 = f\chi_{\{x\in X: f(x)\ge 1\}}$, $f_2 = f - f_1$. Applying Lemma 3.3 to f_1 and $L = 1$, there exist constants $C_1 \ge 2$ and $C_2 > 0$ such that

$$\Phi_0\big(x, M_\lambda f_1(x)/C_1\big) \le C_2 M_\lambda[\Phi_0(\cdot, f_1(\cdot))](x),$$

so that

$$\Phi\big(x, M_\lambda f_1(x)/C_1\big) \le C_2^p \big[M_\lambda[\Phi_0(\cdot, f(\cdot))](x)\big]^p \tag{3.5}$$

for all $x \in X$.

On the other hand, since $M_\lambda f_2 \le 1$, we have by (Φ2) and (Φ3)

$$\Phi\big(x, M_\lambda f_2(x)/C_1\big) \le A_1 A_2 \tag{3.6}$$

for all $x \in X$.

Here, note from ($\omega 1'$) and ($\omega 3$) that there exists a constant $C_3 > 0$ such that

$$\int_0^{2d_X} \omega(z,r)\frac{dr}{r} = \int_0^{2d_X} r^{-\varepsilon_1}\omega(z,r)\cdot r^{\varepsilon_1}\frac{dr}{r} \le C\int_0^{2d_X} r^{\varepsilon_1}\frac{dr}{r} \le C_3 \tag{3.7}$$

for all $z \in X$. In view of (2.2), (3.5), (3.6), (3.7), and Lemma 3.1, we find that

$$\begin{aligned}
&\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \bar{\Phi}(x, M_\lambda f(x)/(2C_1))\,d\mu(x)\right)\frac{dr}{r} \\
&\le \frac{A_2}{2}\left\{\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \Phi(x, M_\lambda f_1(x)/C_1)\,d\mu(x)\right)\frac{dr}{r}\right. \\
&\qquad \left. + \int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \Phi(x, M_\lambda f_2(x)/C_1)\,d\mu(x)\right)\frac{dr}{r}\right\} \\
&\le C\left\{\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \left[M_\lambda[\Phi_0(\cdot, f(\cdot))](x)\right]^p\,d\mu(x)\right)\frac{dr}{r}\right. \\
&\qquad \left. + \int_0^{2d_X} \omega(z,r)\frac{dr}{r}\right\} \\
&\le C
\end{aligned}$$

for all $z \in X$. Thus, we conclude the desired result. ∎

4. Lemmas

Let us recall the following lemma from [16, Lemma 5.1].

Lemma 4.1. *Let $F(x,t)$ be a positive function on $X \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in X$;*

(F2) *there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le F(x,1) \le K_1 \quad \text{for all } x \in X;$$

(F3) *$t \mapsto t^{-\varepsilon'}F(x,t)$ is uniformly almost increasing for some $\varepsilon' > 0$, namely, there exists a constant $K_2 \ge 1$ such that*

$$t_1^{-\varepsilon'}F(x,t_1) \le K_2 t_2^{-\varepsilon'}F(x,t_2) \quad \text{for all } x \in X \text{ whenever } 0 < t_1 < t_2.$$

Set

$$F^{-1}(x,s) = \sup\{t > 0; F(x,t) < s\}$$

for $x \in X$ and $s > 0$. Then, the following hold.

(1) $F^{-1}(x,\cdot)$ *is nondecreasing.*

(2) $F^{-1}(x,\lambda t) \le (K_2\lambda)^{1/\varepsilon'} F^{-1}(x,t)$ *for all* $x \in X$, $t > 0$, *and* $\lambda \ge 1$.

(3) $F(x, F^{-1}(x,t)) = t$ *for all* $x \in X$ *and* $t > 0$.

(4) $K_2^{-1/\varepsilon'} t \le F^{-1}(x, F(x,t)) \le K_2^{2/\varepsilon'} t$ *for all* $x \in X$ *and* $t > 0$.

(5) $\min\{1, (\frac{s}{K_1K_2})^{1/\varepsilon'}\} \le F^{-1}(x,s) \le \max\{1, (K_1K_2 s)^{1/\varepsilon'}\}$ *for all* $x \in X$ *and* $s > 0$.

Remark 4.2. Note that $F(x,t) = \Phi(x,t)$ is a function satisfying (F1), (F2), and (F3) with $K_1 = A_1, K_2 = A_2$, and $\varepsilon' = 1$.

We consider a function $\zeta(x,r) : X \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ζ0) $\zeta(x,\cdot)$ is measurable on $(0,\infty)$ for each $x \in X$;

(ζ1) there exists a constant $Q_\zeta \ge 1$ such that $\sup_{x\in X, 0<r\le 2d_X} \zeta(x,r) \le Q_\zeta$ and

$$\int_0^{2d_X} \zeta(x,r)\frac{dr}{r} \le Q_\zeta$$

for all $x \in X$.

Lemma 4.3. *Let* $1 \le \theta_1 < \theta_2$ *and* $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$. *Suppose that* $\Phi(x,t)$ *satisfies* ($\Phi3$; p) *and* ($\Phi5$; ω) *for* $p > 1$. *Assume that* X *satisfies* ($M\lambda$) *and* ($\omega1'$) *holds. Let* $0 < \varepsilon \le 1$. *Then, there exists a constant* $C > 0$ *such that*

$$\int_0^{2d_X} \frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \{M_\lambda f(x)\}^\varepsilon \, d\mu(x)\right)\frac{dr}{r} \le C$$

for all $z \in X$ *and* $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$ *with* $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \le 1$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \le 1$. Then, by Theorem 3.2, there exists a constant $C_1 \ge 1$ such that

$$\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \overline{\Phi}(x, M_\lambda f(x)/C_1)\, d\mu(x)\right)\frac{dr}{r} \le 1 \tag{4.1}$$

for all $z \in X$. Let $z \in X$, and set $c_1 = A_1A_2\tilde{c}_3$. Then, we have by Lemma 4.1 (5) and (ω3)

$$\Phi^{-1}\big(z, c_1\omega(z,r)^{-1}\big) \ge \min\{1, (A_1A_2)^{-1}c_1\tilde{c}_3^{-1}\} = 1$$

and by Lemma 4.1 (3)

$$\Phi\big(z, \Phi^{-1}\big(z, c_1\omega(z,r)^{-1}\big)\big) = c_1\omega(z,r)^{-1}$$

for all $z \in X$ and $0 < r \le 2d_X$, so that, by ($\Phi5$; ω), there exists a constant $\beta \ge 1$ such that

$$c_1\omega(z,r)^{-1} = \Phi\big(z, \Phi^{-1}\big(z, c_1\omega(x,r)^{-1}\big)\big) \le \Phi\big(x, \beta\Phi^{-1}\big(z, c_1\omega(x,r)^{-1}\big)\big)$$

whenever $x \in B(z,r)$ and $0 < r \le 2d_X$. Therefore, we find by (Φ3), Lemma 4.1, (ζ1), and (2.2)

$$\begin{aligned}
&\frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\int_{B(z,r)}\{M_\lambda f(x)\}^{\varepsilon}\,d\mu(x)\\
&\le \frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\int_{B(z,r)}\{2C_1\beta\Phi^{-1}(z,c_1\omega(z,r)^{-1})\}^{\varepsilon}\,d\mu(x)\\
&+ A_2\frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\\
&\times\int_{B(z,r)}\{M_\lambda f(x)\}^{\varepsilon}\frac{\{M_\lambda f(x)/(2C_1)\}^{-\varepsilon}\Phi(x,M_\lambda f(x)/(2C_1))}{\{\beta\Phi^{-1}(z,c_1\omega(z,r)^{-1})\}^{-\varepsilon}\Phi(x,\beta\Phi^{-1}(z,c_1\omega(z,r)^{-1}))}\,d\mu(x)\\
&\le (2A_2c_1C_1\beta)^{\varepsilon}\zeta(z,r)\\
&+ A_2^{1+\varepsilon}(2C_1\beta)^{\varepsilon}c_1^{-1+\varepsilon}\frac{\zeta(z,r)\omega(z,r)}{\mu(B(z,\theta_2 r))}\int_{B(z,r)}\Phi(x,M_\lambda f(x)/(2C_1))\,d\mu(x)\\
&\le C\left\{\zeta(z,r)+\frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\int_{B(z,r)}\bar{\Phi}(x,M_\lambda f(x)/C_1)\,d\mu(x)\right\}
\end{aligned}$$

for all $z \in X$ and $0 < r \le 2d_X$, so that

$$\begin{aligned}
&\int_0^{2d_X}\frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\{M_\lambda f(x)\}^{\varepsilon}\,d\mu(x)\right)\frac{dr}{r}\\
&\le C\left\{\int_0^{2d_X}\zeta(z,r)\frac{dr}{r}+\int_0^{2d_X}\frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\bar{\Phi}(x,M_\lambda f(x)/C_1)\,d\mu(x)\right)\frac{dr}{r}\right\}\\
&\le C
\end{aligned}$$

by (ζ1) and (4.1). Hence, we obtain the required result. ■

Let E be a measurable subset of X. To consider Trudinger-type inequalities, we prepare an auxiliary function. For $s_0 = \min\{1, 1/(2d_X)\}$, we consider a function

$$\Gamma(x,s) : E \times [s_0,\infty) \to (0,\infty) \tag{4.2}$$

which satisfies the following conditions:

(Γ1) $s \mapsto \Gamma(x,s)$ is uniformly almost increasing on $[s_0,\infty)$, that is, there exists a constant $c_{\Gamma 1} \ge 1$ such that

$$\Gamma(x,s_1) \le c_{\Gamma 1}\Gamma(x,s_2)$$

for all $x \in E$ and $s_0 \le s_1 < s_2$;

(Γ2) there exists a constant $c_{\Gamma 2} \ge 1$ such that

$$\Gamma(x,2) \le c_{\Gamma 2}\Gamma(x,s_0)$$

for all $x \in E$;

($\Gamma_{\log}$) there exists a constant $c_{\Gamma\ell} \ge 1$ such that

$$\Gamma(x, s^2) \le c_{\Gamma\ell} \Gamma(x, s)$$

for all $x \in E$ and $s \ge 1$.

We recall the following lemma which gives estimates for the function Γ.

Lemma 4.4 (Cf. [23, Lemmas 2.1 and 2.2]). (1) $\Gamma(x, \cdot)$ *has uniform doubling property on* $[s_0, \infty)$*; namely, there exists a constant* $C > 0$ *such that* $\Gamma(x, 2s) \le C\Gamma(x, s)$ *for all* $x \in E$ *and* $s \ge s_0$.

(2) *For* $a > 0$, *there exists a constant* $C \ge 1$ *such that*

$$C^{-1}\Gamma(x, s) \le \Gamma(x, s^a) \le C\Gamma(x, s)$$

for all $x \in E$ *and* $s \ge 1$.

(3) *There exists a constant* $C > 0$ *such that*

$$\Gamma(x, s) \le C s \Gamma(x, s_0)$$

for all $x \in E$ *and* $s \ge s_0$.

We define another useful function with certain properties. We consider a function

$$\gamma(x, \rho) : E \times (0, \infty) \to (0, \infty) \tag{4.3}$$

satisfying the following conditions:

($\gamma 1$) $\gamma(x, \cdot)$ is measurable on $(0, \infty)$ for each $x \in E$;

($\gamma 2$) there exists a constant $B_1 \ge 1$ such that

$$\gamma(x, \rho_1) \le B_1 \gamma(x, \rho_2)$$

for all $x \in E$ whenever $0 < \rho_1/2 < \rho_2 \le \rho_1 \le 2d_X$;

($\gamma 3$) there exists a constant $0 < B_2 \le 1$ such that $\inf_{x \in X, 0 < \rho \le 2d_X} \gamma(x, \rho) \ge B_2$.

Further, we consider the following condition:

($\Gamma\Phi\gamma\alpha\omega$) there exist constants $c_1^* \ge 1$ and $c_2^* \ge 1$ such that

$$\rho^{\alpha(x)} \omega(x, \rho)^{-1} \gamma(x, \rho)^{-1} \Phi^{-1}(x, \gamma(x, \rho)) \le c_1^* \Gamma(x, 1/\rho)$$

for all $x \in E$ whenever $0 < \rho \le 2d_X$ and

$$\int_\delta^{2d_X} \rho^{\alpha(x)} \Phi^{-1}(x, \gamma(x, \rho)) \frac{d\rho}{\rho} \le c_2^* \Gamma(x, 1/\delta)$$

for all $x \in E$ whenever $0 < \delta \le d_X/2$.

Here, note from ($\Gamma\Phi\gamma\alpha\omega$), ($\gamma 3$), and Lemma 4.1 (5) that there exists a constant $c_{\Gamma 3} > 0$ such that

$$\Gamma(x, 2/d_X) \geq c_{\Gamma 3}. \tag{4.4}$$

Now, we state and prove our lemma using the functions Γ from (4.2) and γ from (4.3).

Lemma 4.5. *Let $1 \leq \theta \leq \tau/2$. Suppose that $\Phi(x,t)$ satisfies ($\Phi 5; 1/\gamma$). Assume that ($\Gamma\Phi\gamma\alpha\omega$) holds. Then, there exists a constant $C > 0$ such that*

$$\sum_{2\delta < 2^i \leq 2d_X} \frac{2^{i\alpha(x)}}{\mu(B(x,\tau 2^i))} \int_{B(x,\tau 2^i)} f(y)\, d\mu(y) \leq C\,\Gamma(x, 1/\delta)$$

for all $x \in E$, $0 < \delta < d_X/2$, and nonnegative $f \in \mathcal{L}^{\Phi,\omega,\theta}(X)$ with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} \leq 1$.

Proof. Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta}(X)} \leq 1/2$. Let $x \in E$ and $0 < \delta < d_X/2$. Set $c_1 = A_1 A_2 B_2^{-1}$. Then, we have by ($\gamma 3$), ($\Phi 2$), ($\Phi 3$), and Lemma 4.1

$$\Phi^{-1}(x, c_1\gamma(x,t)) \geq 1$$

and

$$\Phi\big(x, \Phi^{-1}(x, c_1\gamma(x,t))\big) = c_1\gamma(x,t)$$

for all $x \in E$ and $0 < t \leq 2d_X$, so that, by ($\Phi 5; 1/\gamma$), there exists a constant $\beta \geq 1$ such that

$$c_1\gamma(x,t) \leq \Phi\big(y, \beta\Phi^{-1}(x, c_1\gamma(x,t))\big)$$

whenever $y \in B(x,t)$ and $0 < t \leq 2d_X$. Therefore, we find by ($\Phi 3$), Lemma 4.1, and ($\Gamma\Phi\gamma\alpha\omega$)

$$\begin{aligned}
&\frac{t^{\alpha(x)}}{\mu(B(x,\tau t))} \int_{B(x,t)} f(y)\, d\mu(y) \\
&\leq \frac{t^{\alpha(x)}}{\mu(B(x,\tau t))} \int_{B(x,t)} \beta\Phi^{-1}(x, c_1\gamma(x,t))\, d\mu(y) \\
&\quad + A_2 \frac{t^{\alpha(x)}}{\mu(B(x,\tau t))} \\
&\quad \times \int_{B(x,t)} f(y) \frac{f(y)^{-1}\Phi(y, f(y))}{\{\beta\Phi^{-1}(x, c_1\gamma(x,t))\}^{-1}\Phi\big(y, \beta\Phi^{-1}(x, c_1\gamma(x,t))\big)}\, d\mu(y) \\
&\leq C\left\{t^{\alpha(x)}\Phi^{-1}(x, \gamma(x,t)) + \frac{t^{\alpha(x)}\gamma(x,t)^{-1}\Phi^{-1}(x,\gamma(x,t))}{\mu(B(x,\tau t))} \int_{B(x,t)} \Phi(y, f(y))\, d\mu(y)\right\} \\
&\leq C\left\{t^{\alpha(x)}\Phi^{-1}(x, \gamma(x,t)) + \frac{\Gamma(x, 1/t)\omega(x,t)}{\mu(B(x,\tau t))} \int_{B(x,t)} \Phi(y, f(y))\, d\mu(y)\right\}
\end{aligned} \tag{4.5}$$

for all $0 < t \le 2d_X$. It follows from (4.5) and (Γ1) that

$$\begin{aligned}
&\sum_{2\delta<2^i\le 2d_X} \frac{2^{i\alpha(x)}}{\mu(B(x,\tau 2^i))} \int_{B(x,2^i)} f(y)\,d\mu(y) \\
&\le C\Bigg\{ \sum_{2\delta<2^i\le 2d_X} 2^{i\alpha(x)} \Phi^{-1}(x,\gamma(x,2^i)) \\
&\qquad + \sum_{2\delta<2^i\le 2d_X} \frac{\Gamma(x,1/2^i)\omega(x,2^i)}{\mu(B(x,\tau 2^i))} \int_{B(x,2^i)} \Phi(y,f(y))\,d\mu(y)\Bigg\} \\
&\le C\Bigg\{ \sum_{2\delta<2^i\le 2d_X} 2^{i\alpha(x)} \Phi^{-1}(x,\gamma(x,2^i)) \\
&\qquad + \Gamma(x,1/\delta) \sum_{2\delta<2^i\le 2d_X} \frac{\omega(x,2^i)}{\mu(B(x,\tau 2^i))} \int_{B(x,2^i)} \Phi(y,f(y))\,d\mu(y)\Bigg\} \\
&= C(I_1+I_2).
\end{aligned}$$

By (γ2), ($\Gamma\Phi\gamma\alpha\omega$), and Lemma 4.1, we have

$$\begin{aligned}
I_1 &\le C \sum_{2\delta<2^i\le 2d_X} \int_{2^{i-1}}^{2^i} t^{\alpha(x)} \Phi^{-1}(x,\gamma(x,t)) \frac{dt}{t} \\
&\le C \int_{\delta}^{2d_X} t^{\alpha(x)} \Phi^{-1}(x,\gamma(x,t)) \frac{dt}{t} \\
&\le C\,\Gamma(x,1/\delta).
\end{aligned}$$

By (ω1) and $\theta \le \tau/2$, we see that

$$\begin{aligned}
I_2 &\le C\,\Gamma(x,1/\delta) \sum_{2\delta<2^i\le 2d_X} \frac{\omega(x,2^i)}{\mu(B(x,\theta 2^{i+1}))} \int_{B(x,2^i)} \Phi(y,f(y))\,d\mu(y) \\
&\le C\,\Gamma(x,1/\delta) \sum_{2\delta<2^i\le 2d_X} \int_{2^i}^{2^{i+1}} \frac{\omega(x,t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} \Phi(y,f(y))\,d\mu(y) \right) \frac{dt}{t} \\
&\le C\,\Gamma(x,1/\delta) \int_{2\delta}^{4d_X} \frac{\omega(x,t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} \Phi(y,f(y))\,d\mu(y) \right) \frac{dt}{t} \\
&\le C\,\Gamma(x,1/\delta) \int_{0}^{2d_X} \frac{\omega(x,t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} \Phi(y,f(y))\,d\mu(y) \right) \frac{dt}{t} \\
&\le C\,\Gamma(x,1/\delta).
\end{aligned}$$

This completes the proof. ■

5. A Trudinger-type inequality

Before we state our main theorem, we give the assumptions for the function in Trudinger-type inequalities. We consider a function

$$\Psi(x,t) : E \times [0,\infty) \to [0,\infty)$$

with the following properties:

(Ψ1) $\Psi(\cdot,t)$ is measurable on E for each $t \in [0,\infty)$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in E$;

(Ψ2) there is a constant $\tilde{Q}_1 \geq 1$ such that $\Psi(x,t_1) \leq \Psi(x,\tilde{Q}_1 t_2)$ for all $x \in E$ whenever $0 < t_1 < t_2$;

($\Psi\Gamma$) there are constants $\tilde{Q}_2, \tilde{Q}_3 \geq 1$ and $s_0^* \geq s_0$ such that $\Psi(x,\Gamma(x,s)/\tilde{Q}_2) \leq \tilde{Q}_3 s$ for all $x \in E$ and $s \geq s_0^*$.

Note that ($\Gamma\Phi\gamma\alpha\omega$) and ($\Psi\Gamma$) give the relation between Ψ and Φ.

Theorem 5.1. *Let* $1 \leq \lambda \leq \tau$. *Let* $1 \leq \theta_1 < \theta_2$, $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$, *and* $\theta_1 \leq \tau/2$. *Suppose that* $\Phi(x,t)$ *satisfies* ($\Phi3; p$), ($\Phi5; \omega$), *and* ($\Phi5; 1/\gamma$) *for* $p > 1$. *Assume that* X *satisfies* ($M\lambda$) *and* ($\omega 1'$) *holds. Suppose* ($\Gamma\Phi\gamma\alpha\omega$) *holds. Then, for* $\varepsilon > 0$, *there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\int_0^{2d_X} \frac{\zeta(z,r)\left\{\Phi^{-1}\left(z,\omega(z,r)^{-1}\right)\right\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))} \left(\int_{E\cap B(z,r)} \Psi\left(x, \frac{|J_{\alpha(\cdot),\tau} f(x)|}{c_1}\right) d\mu(x) \right) \frac{dr}{r} \leq c_2$$

for all $z \in X$ *and* $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$ *with* $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \leq 1$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \leq 1$. Let $x \in E$ and $\varepsilon > 0$. Then, we may assume $0 < \varepsilon \leq 1$ since we have by Lemma 4.1 (5)

$$\left\{\Phi^{-1}\left(z,\omega(z,r)^{-1}\right)\right\}^{-1} \leq \max\left\{1, A_1 A_2 \tilde{c}_3\right\}$$

for all $z \in X$ and $0 < r \leq 2d_X$. For $0 < \delta \leq d_X/2$, Lemma 4.5 implies

$$\begin{aligned} J_{\alpha(\cdot),\tau} f(x) &\leq \sum_{2^i \leq 2\delta} \frac{2^{i\alpha(x)}}{\mu(B(x,\tau 2^i))} \int_{B(x,2^i)} f(y)\,d\mu(y) + C\Gamma\left(x,\frac{1}{\delta}\right) \\ &\leq C\left\{\delta^{\alpha(x)} M_\lambda f(x) + \Gamma\left(x,\frac{1}{\delta}\right)\right\} \end{aligned}$$

with a constant $C > 0$ independent of x.

If $M_\lambda f(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by (4.4),

$$J_{\alpha(\cdot),\tau} f(x) \leq C\left\{\left(\frac{d_X}{2}\right)^{\alpha(x)-1} + \Gamma\left(x,\frac{2}{d_X}\right)\right\} \leq C\Gamma\left(x,\frac{2}{d_X}\right).$$

By Lemma 4.4 (1), there exists a constant $C > 0$ independent of x such that

$$J_{\alpha(\cdot),\tau} f(x) \le C\Gamma(x, s_0^*) \quad \text{if } M_\lambda f(x) \le 2/d_X. \tag{5.1}$$

Next, suppose $2/d_X < M_\lambda f(x) < \infty$. By Lemma 4.4 (3) and ($\Gamma$1), there exists a constant $m > 0$ such that $\Gamma(x,s)/s \le m\Gamma(x, 2/d_X)$ for $s \ge 2/d_X$. Let

$$\delta = (d_X/2)\left[\frac{\Gamma(x, M_\lambda f(x))}{m\Gamma(x, 2/d_X) M_\lambda f(x)}\right]^{1/\alpha(x)}.$$

Then, by (4.4) and Lemma 4.4 (2),

$$\begin{aligned}\delta^{\alpha(x)} M_\lambda f(x) &= (d_X/2)^{\alpha(x)} \frac{\Gamma(x, M_\lambda f(x))}{m\Gamma(x, 2/d_X)} \le C\Gamma(x, M_\lambda f(x)) \\ &\le C(\varepsilon)\Gamma\big(x, \{M_\lambda f(x)\}^\varepsilon\big).\end{aligned}$$

By the choice of m, $\delta \le d_X/2$. Since $\Gamma(x, 2/d_X) \le C\Gamma(x, M_\lambda f(x))$,

$$\frac{1}{\delta} \le C(M_\lambda f(x))^{1/\alpha(x)}.$$

Hence, using (Γ1) and Lemma 4.4 (1) and (2), we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \le C\Gamma(x, M_\lambda f(x)) \le C(\varepsilon)\Gamma\big(x, \{M_\lambda f(x)\}^\varepsilon\big).$$

Therefore, there exists a constant $C > 0$ independent of x such that

$$J_{\alpha(\cdot),\tau} f(x) \le C\Gamma\big(x, \{M_\lambda f(x)\}^\varepsilon\big) \quad \text{if } 2/d_X < M_\lambda f(x) < \infty. \tag{5.2}$$

By (5.1) and (5.2), there exists a constant $C^* > 0$ such that

$$J_{\alpha(\cdot),\tau} f(x) \le C^*\Gamma\big(x, \max\{s_0^*, \{M_\lambda f(x)\}^\varepsilon\}\big)$$

for a.e. $x \in E$.

Now, let $c_1 = \widetilde{Q}_1\widetilde{Q}_2 C^*$. Then, by ($\Psi$2) and ($\Psi\Gamma$), we have

$$\begin{aligned}\Psi\left(x, \frac{J_{\alpha(\cdot),\tau} f(x)}{c_1}\right) &\le \Psi\Big(x, \Gamma\big(x, \max\{s_0^*, \{M_\lambda f(x)\}^\varepsilon\}\big)/\widetilde{Q}_2\Big) \\ &\le \widetilde{Q}_3 \max\{s_0^*, \{M_\lambda f(x)\}^\varepsilon\} \le \widetilde{Q}_3(s_0^* + \{M_\lambda f(x)\}^\varepsilon)\end{aligned}$$

for a.e. $x \in E$. Thus, we have by Lemma 4.3

$$\begin{aligned}&\int_0^{2d_X} \frac{\zeta(z,r)\big\{\Phi^{-1}\big(z, \omega(z,r)^{-1}\big)\big\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\left(\int_{E\cap B(z,r)} \Psi\left(x, \frac{J_{\alpha(\cdot),\tau} f(x)}{c_1}\right) d\mu(x)\right)\frac{dr}{r} \\ &\le \widetilde{Q}_3 s_0^* \int_0^{2d_X} \zeta(z,r)\big\{\Phi^{-1}\big(z, \omega(z,r)^{-1}\big)\big\}^{-\varepsilon}\frac{dr}{r} \\ &\quad + \widetilde{Q}_3 \int_0^{2d_X} \frac{\zeta(z,r)\big\{\Phi^{-1}\big(z, \omega(z,r)^{-1}\big)\big\}^{-\varepsilon}}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \{M_\lambda f(x)\}^\varepsilon\, d\mu(x)\right)\frac{dr}{r} \\ &\le \widetilde{Q}_3 s_0^* C^{**} + \widetilde{Q}_3 C_M = c_2\end{aligned}$$

for all $z \in X$ since we have by ($\omega 3$), Lemma 4.1, and ($\zeta 1$)

$$\int_0^{2d_X} \zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}\frac{dr}{r} \le \left(\max\{1, A_1A_2\tilde{c}_3\}\right)^{\varepsilon}\int_0^{2d_X} \zeta(z,r)\frac{dr}{r} \le C^{**}$$

for all $z \in X$. Hence, we obtain the required result. ■

Let $\omega(x,r) = \omega_{\sigma(\cdot),\{\beta_j(\cdot)\}}(x,r)$ be as in Example 2.2, and let $\Phi(x,t) = \Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ be as in Example 2.5. Set $E^{(1)}(t) = e^t - e$, $E^{(j+1)}(t) = \exp(E^j(t)) - e$, and $E_+^{(j)}(t) = \max(E^{(j)}(t), 0)$. Consider $\zeta(z,r) = r^{\varepsilon_1}$ for some $\varepsilon_1 > 0$.

As in the proof of [17], we obtain the following corollaries in view of Theorem 5.1.

Corollary 5.2. *Let* $1 \le \lambda \le \tau$. *Let* $1 \le \theta_1 < \theta_2$, $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$, *and* $\theta_1 \le \tau/2$. *Assume that* X *satisfies* ($M\lambda$). *Suppose* $p(x) = \sigma(x)/\alpha(x)$ *on* E *and* $p^- > 1$. *Assume that there exists an integer* $1 \le j_0 \le k$ *such that*

$$\inf_{x\in E}(p(x) - q_{j_0}(x) - \beta_{j_0}(x) - 1) > 0$$

and

$$\sup_{x\in E}(p(x) - q_j(x) - \beta_j(x) - 1) \le 0$$

for all $j \le j_0 - 1$ *in case* $j_0 \ge 2$. *Then, for* $\varepsilon > 0$, *there exist constants* $c_1, c_2 > 0$ *such that*

$$\sup_{z\in X}\int_0^{2d_X}\frac{r^{\varepsilon}}{\mu(B(z,\theta_2 r))}\left\{\int_{E\cap B(z,r)} E_+^{(j_0)}\left(\left(\frac{|J_{\alpha(\cdot),\tau}f(x)|}{c_1}\right)^{p(x)/(p(x)-q_{j_0}(x)-\beta_{j_0}(x)-1)}\right.\right.$$
$$\left.\left.\times\prod_{j=1}^{k-j_0}\left(L_e^{(j)}\left(\frac{|J_{\alpha(\cdot),\tau}f(x)|}{c_1}\right)\right)^{(q_{j_0+j}(x)+\beta_{j_0+j}(x))/(p(x)-q_{j_0}(x)-\beta_{j_0}(x)-1)}\right)d\mu(x)\right\}\frac{dr}{r}$$
$$\le c_2$$

whenever $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$ *with* $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \le 1$.

Corollary 5.3. *Let* $1 \le \lambda \le \tau$. *Let* $1 \le \theta_1 < \theta_2$, $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$ *and* $\theta_1 \le \tau/2$. *Assume that* X *satisfies* ($M\lambda$). *Suppose* $p(x) = \sigma(x)/\alpha(x)$ *on* E *and* $p^- > 1$. *Assume that*

$$\sup_{x\in E}(p(x) - q_j(x) - \beta_j(x) - 1) \le 0$$

for all $j = 1, \ldots, k$. *Then, for* $\varepsilon > 0$, *there exist constants* $c_1, c_2 > 0$ *such that*

$$\sup_{z\in X}\int_0^{2d_X}\frac{r^{\varepsilon}}{\mu(B(z,\theta_2 r))}\left\{\int_{E\cap B(z,r)} E_+^{(k+1)}\left(\left(\frac{|J_{\alpha(\cdot),\tau}f(x)|}{c_1}\right)^{p(x)/(p(x)-1)}\right)d\mu(x)\right\}\frac{dr}{r}$$
$$\le c_2$$

whenever $f \in \mathcal{L}^{\Phi,\omega,\theta_1}(X)$ *with* $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)} \le 1$.

6. Double-phase functions with variable exponents

Let $\sigma(\cdot)$ be a measurable function on X such that $0 < \sigma^- \le \sigma^+ < \infty$. Set

$$\omega(x,r) = r^{\sigma(x)}.$$

In this section, let us assume that $p(\cdot)$ and $q(\cdot)$ be real-valued measurable functions on X such that

(P1) $1 \le p^- \le p^+ < \infty$,

(Q1) $1 \le q^- \le q^+ < \infty$.

We assume that

(P2) $p(\cdot)$ is log-Hölder continuous, that is,

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/d(x,y))} \quad (x, y \in X)$$

with a constant $C_p \ge 0$, and

(Q2) $q(\cdot)$ is log-Hölder continuous, that is,

$$|q(x) - q(y)| \le \frac{C_q}{L_e(1/d(x,y))} \quad (x, y \in X)$$

with a constant $C_q \ge 0$.

As an example and application, we consider the case where $\Phi(x,t)$ is a double-phase function with variable exponents given by

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)} \big(= t^{p(x)} + (b(x)t)^{q(x)} \big), \quad x \in X,\ t \ge 0,$$

where $p(x) < q(x)$ for $x \in X$, $a(\cdot)$ is nonnegative, bounded, and Hölder continuous of order $\theta \in (0,1]$ and $b(x) = a(x)^{1/q(x)}$ (cf. [1,40]).

This $\Phi(x,t)$ satisfies (Φ1), (Φ2), and ($\Phi 3; p^-$). Set $X_0 = \{x \in X : a(x) > 0\}$.

Let us write

$$E_1 = \{x \in X \setminus X_0 : \sigma(x) = \alpha(x)p(x)\},$$
$$E_2 = \{x \in X_0 : \sigma(x) = \alpha(x)q(x)\}$$

and $E = E_1 \cup E_2$. We define

$$\gamma(x,\rho) = \rho^{-\sigma(x)}(\log(e + 1/\rho))^{-1}$$

for $x \in E$ and $\rho > 0$ and

$$\Gamma(x,s) = \begin{cases} (\log(e+s))^{(p(x)-1)/p(x)}, & x \in E_1, \\ b(x)^{-1}(\log(e+s))^{(q(x)-1)/q(x)}, & x \in E_2, \end{cases}$$

for $s \ge s_0$.

This $\gamma(x,\rho)$ satisfies (γ1), (γ2), and (γ3); $\Gamma(x,s)$ satisfies (Γ1), (Γ2), and ($\Gamma_{\log}$).

Lemma 6.1. (1) *$\Phi(x,t)$ satisfies $(\Phi5;\omega)$ for $\theta \ge \sup_{x\in X_0}\{\sigma(x)(q(x)/p(x)-1)\}$.*
(2) *$\Phi(x,t)$ satisfies $(\Phi5;1/\gamma)$ for $\theta \ge \sup_{x\in X_0}\{\sigma(x)(q(x)/p(x)-1)\}$.*

In fact, for $\eta>0$, let $y\in B(x,r)$, $\Phi(x,t)\le \eta\gamma(x,r)$, and $t\ge 1$. Then, note that

$$\Phi(x,t)\le \eta\gamma(x,r)\le \eta\omega(x,r)^{-1},$$

so that we can show (2) as in (1) [39, Lemma 6.1].

By Theorem 3.2 and Lemma 6.1, we obtain the boundedness of M_λ on $\mathcal{L}^{\Phi,\omega,\theta_1}(X)$, as an extension of [37, Theorem 5.1] in the Euclidean case.

Corollary 6.2. *Let $1\le\theta_1<\theta_2$ and $\lambda>\theta_1(\theta_2+1)/(\theta_2-\theta_1)$. Assume that X satisfies $(M\lambda)$. If $p^->1$ and $\theta\ge\sup_{x\in X_0}\{\sigma(x)(q(x)/p(x)-1)\}$, then there is a constant $C>0$ such that*

$$\|M_\lambda f\|_{\mathcal{L}^{\Phi,\omega,\theta_2}(X)}\le C\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)}$$

for all $f\in\mathcal{L}^{\Phi,\omega,\theta_1}(X)$.

Remark 6.3. In [37, Theorem 5.1], we considered $(\Phi5;\nu)$ and proved Corollary 6.2 above when $\sup_{x\in X_0}(q(x)-p(x))/\theta\le p^-/\sigma^+$ holds for $X=\mathbf{R}^N$. Hence, we find that $(\Phi5;\omega)$ is better than $(\Phi5;\nu)$.

We recall a lemma which we need in the proof of a Trudinger-type inequality.

Lemma 6.4 (Cf. [19, Lemma 4.9]). *If $\inf_{x\in E_1}p(x)>1$ and $\inf_{x\in E_2}q(x)>1$, then $\Gamma(x,s)$ satisfies $(\Gamma\Phi\gamma\alpha\omega)$.*

If we define

$$\Psi(x,t)=\begin{cases}\exp\left(t^{p(x)/(p(x)-1)}\right), & x\in E_1,\\ \exp\left((b(x)t)^{q(x)/(q(x)-1)}\right), & x\in E_2,\end{cases}$$

for $t>0$, then $\Psi(x,t)$ satisfies $(\Psi1)$, $(\Psi2)$, and $(\Psi\Gamma)$ with $s_0^*=2/d_X$ when $\inf_{x\in E_1}p(x)>1$ and $\inf_{x\in E_2}q(x)>1$.

In view of Lemmas 6.1 and 6.4 and Theorem 5.1, we obtain a Trudinger-type inequality on Musielak–Orlicz–Morrey spaces of an integral form in the framework of double-phase functional with variable exponents.

Corollary 6.5. *Let $1\le\lambda\le\tau$. Let $1\le\theta_1<\theta_2$, $\lambda>\theta_1(\theta_2+1)/(\theta_2-\theta_1)$, and $\theta_1\le\tau/2$. Assume that X satisfies $(M\lambda)$. Suppose $\sup_{x\in X_0}\{\sigma(x)(q(x)/p(x)-1)\}\le\theta$ and $p^->1$. Then, for $\varepsilon>0$, there exist constants $c_1>0$ and $c_2>0$ such that*

$$\sup_{z\in X}\int_0^{2d_X}\frac{r^\varepsilon}{\mu(B(z,\theta_2 r))}\Bigg(\int_{E_1\cap B(z,r)}\left\{\exp\left(c_1|J_{\alpha(\cdot),\tau}f(x)|^{p(x)/(p(x)-1)}\right)-1\right\}d\mu(x)$$
$$+\int_{E_2\cap B(z,r)}\left\{\exp\left(c_1 b(x)|J_{\alpha(\cdot),\tau}f(x)|^{q(x)/(q(x)-1)}\right)-1\right\}d\mu(x)\Bigg)\frac{dr}{r}\le c_2$$

whenever $f\in\mathcal{L}^{\Phi,\omega,\theta_1}(X)$ with $\|f\|_{\mathcal{L}^{\Phi,\omega,\theta_1}(X)}\le 1$.

7. Poincaré inequality

In this section, we assume that μ satisfies the doubling condition. Let $u \in L^1_{loc}(X)$, and let g be a nonnegative measurable function on X. We say that the pair u, g satisfies a Poincaré inequality in X if there exist constants $A_0 > 0$ and $\sigma \geq 1$ such that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u(y) - u_{B(x,r)}|\, d\mu(y) \leq \frac{A_0 r}{\mu(B(x,\sigma r))}\int_{B(x,\sigma r)} g(y)\, d\mu(y)$$

for all $x \in X$ and $r > 0$, where

$$u_{B(x,r)} = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u(y)\, d\mu(y).$$

Remark 7.1. We say that a function $u \in L^{\Phi}(X)$ belongs to Musielak–Orlicz–Hajłasz–Sobolev spaces of an integral form $\mathcal{M}^{1,\Phi,\omega,1}(X)$ if there exists a nonnegative function $g \in \mathcal{L}^{\Phi,\omega,1}(X)$ such that

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \tag{7.1}$$

for μ-almost every $x, y \in X$. Here, we call the function g a Hajłasz gradient of u. For spaces related to Hajłasz spaces, see, e.g., [10, 14, 30]. Integrating both sides in (7.1) over y and x, we obtain the Poincaré inequality.

We show the following result, as an extension of [11, Theorem 5.1], [13, Corollary 5.4], and [30, Theorem 7.7].

Theorem 7.2. *Let $u \in L^1_{loc}(X)$, and let $g \in \mathcal{L}^{\Phi,\omega,1}(X)$ with $\|g\|_{\mathcal{L}^{\Phi,\omega,1}(X)} \leq 1$ be a nonnegative measurable function on X. Assume that the pair u, g satisfies a Poincaré inequality in X. Suppose that $\Phi(x,t)$ satisfies $(\Phi3; p)$, $(\Phi5; \omega)$, and $(\Phi5; 1/\gamma)$ for $p > 1$. Assume that $(\omega1')$ holds and $(\Gamma\Phi\gamma\alpha\omega)$ holds with*

$$\alpha(\cdot) \equiv 1.$$

Then, for $\varepsilon > 0$, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\int_0^{2d_X} \frac{\zeta(z,r)\{\Phi^{-1}(z,\omega(z,r)^{-1})\}^{-\varepsilon}}{\mu(B(z,r))}\left(\int_{E\cap B\cap B(z,r)} \Psi\left(x, \frac{|u(x) - u_B|}{c_1}\right) d\mu(x)\right)\frac{dr}{r} \leq c_2$$

for all balls $B \subset X$ and $z \in X$.

Proof. Since μ is doubling and the pair u, g satisfies a Poincaré inequality in X, we have

$$|u(x) - u_B| \leq C J_{1,1} g(x)$$

for a.e. $x \in B$ (see [11, Theorem 5.2]). Hence, Theorem 5.1 yields this theorem. ∎

Finally, as a corollary, we obtain the double-phase version by Theorem 7.2.

Corollary 7.3. *Let X_0, $p(\cdot), q(\cdot)$, $a(\cdot)$, and $\sigma(\cdot)$ be as in Section 6. Set*

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)} \quad \text{and} \quad \omega(x,r) = r^{\sigma(x)}$$

for $x \in X$, $t \geq 0$, and $r > 0$. Set

$$E_1 = \{x \in X \setminus X_0 : \sigma(x) = p(x)\}$$

and

$$E_2 = \{x \in X_0 : \sigma(x) = q(x)\}.$$

Let $u \in L^1_{loc}(X)$, and let $g \in \mathcal{L}^{\Phi,\omega,1}(X)$ with $\|g\|_{\mathcal{L}^{\Phi,\omega,1}(X)} \leq 1$ be a nonnegative measurable function on X. Assume that the pair u, g satisfies a Poincaré inequality in X. Suppose $\sup_{x\in X_0}\{\sigma(x)(q(x)/p(x) - 1)\} \leq \theta$ and $p^- > 1$. Then, for $\varepsilon > 0$, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\sup_{z\in X} \int_0^{2d_X} \frac{r^\varepsilon}{\mu(B(z,r))} \bigg(\int_{E_1\cap B\cap B(z,r)} \left\{ \exp\left(c_1|u(x) - u_B|^{p(x)/(p(x)-1)}\right) - 1 \right\} d\mu(x) + \int_{E_2\cap B\cap B(z,r)} \left\{ \exp\left(c_1 b(x)|u(x) - u_B|^{q(x)/(q(x)-1)}\right) - 1 \right\} d\mu(x) \bigg) \frac{dr}{r} \leq c_2$$

for all balls $B \subset X$.

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