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On the classification of simple amenable C*-algebras with finite decomposition rank, II

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Abstract. We prove that every unital simple separable C*-algebra A with finite decomposition rank which satisfies the UCT has the property that $A \otimes Q$ has generalized tracial rank at most one, where Q is the universal UHF-algebra. Consequently, A is classifiable in the sense of Elliott.

1. Introduction

In a recent development in the Elliott program, the program of classification of amenable C*-algebras, a certain class of finite unital simple separable amenable C*-algebras, denoted by $\mathcal{N}_1$, was shown to be classified by the Elliott invariant [15, 16]. One important feature of this class of C*-algebras is that it exhausts all possible values of the Elliott invariant for unital simple separable C*-algebras which have finite decomposition rank (a property introduced in [20]; see Definition 2.10 below).

The purpose of this note is to show that, in fact, every unital simple separable (non-elementary) C*-algebra which has finite decomposition rank and satisfies the Universal Coefficient Theorem (UCT) is in the class $\mathcal{N}_1$. Since every C*-algebra in $\mathcal{N}_1$ was shown in [15, 16] to be isomorphic to the inductive limit of a sequence of subhomogeneous C*-algebras with no dimension growth, the C*-algebras in $\mathcal{N}_1$ have finite decomposition rank (see Remark 4.7 below). In other words, the class $\mathcal{N}_1$ is precisely the class of all unital simple separable (non-elementary) C*-algebras which have finite decomposition rank and satisfy the UCT, and hence we obtain a classification for all of these C*-algebras.

Theorem 1.1. *Let A be a unital simple separable (non-elementary)* C* *-algebra with finite decomposition rank, and assume that A satisfies the UCT. Then, $A \in \mathcal{N}_1$. (See Definition 2.6 below.) Hence (by* [16, Theorem 29.8]*), if A and B are two (non-elementary) unital simple separable* C* *-algebras with finite decomposition rank which satisfy the UCT, then $A \cong B$ if and only if*

$$\big(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, [1_A]_0, \mathrm{K}_1(A), \mathrm{T}(A), r_A\big) \cong \big(\mathrm{K}_0(B), \mathrm{K}_0(B)_+, [1_B]_0, \mathrm{K}_1(B), \mathrm{T}(B), r_B\big).$$

In fact, we shall obtain (see Theorem 4.4 below) the formally stronger result that every finite unital simple separable (non-elementary) C*-algebra with finite nuclear dimension,

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which satisfies the UCT, and whose tracial states are all quasidiagonal, is in the class $\mathcal{N}_1$. This result, combined with the recent result of [36] that the quasidiagonality hypothesis is redundant, yields that the class $\mathcal{N}_1$ includes all finite unital simple separable C*-algebras with finite nuclear dimension which satisfy the UCT—see Theorem 4.10 below. (The case of infinite unital simple separable C*-algebras with finite nuclear dimension was dealt with over twenty years ago by Kirchberg and Phillips—see Remark 4.11 below.)

In a recent paper [11], two of us (G. A. E. and Z. N.) proved that every unital simple separable (non-elementary) C*-algebra A with finite decomposition rank, satisfying the UCT, and such that $\mathrm{K}_0(A)$ has torsion-free rank one, belongs to $\mathcal{N}_1$. The present paper is a continuation of [11] with, now, a definitive result.

It is perhaps worth mentioning that the mathematical content of this paper (for example, Theorem 4.9) is independent of that of [15, 16]. Please also see Remark 4.6.

2. Preliminaries

Definition 2.1. As usual, let $\mathbb{Q}$ denote the field of rational numbers. Let us use the notation Q for the UHF-algebra with $\mathrm{K}_0(Q) = \mathbb{Q}$ and $[1_Q] = 1$.

Definition 2.2 (N. Brown [3]). Let A be a unital C*-algebra. Denote by $\mathrm{T}(A)$ the tracial state space of A, and denote by $\mathrm{T}_{\mathrm{qd}}(A)$ the subset of the quasidiagonal tracial states—those $\tau \in \mathrm{T}(A)$ with the following property: For any finite subset $\mathcal{F}$ and $\varepsilon > 0$, there exists a unital completely positive map $\varphi : A \to Q$ such that

$$\left|\tau(a) - \mathrm{tr}\left(\varphi(a)\right)\right| < \varepsilon, \quad a \in \mathcal{F},$$
$$\left\|\varphi(a)\varphi(b) - \varphi(ab)\right\| < \varepsilon, \quad a, b \in \mathcal{F},$$

where tr is the unique tracial state of Q.

Definition 2.3. Let F_1 and F_2 be two finite-dimensional C*-algebras and let $\psi_0, \psi_1 : F_1 \to F_2$ be two unital homomorphisms. Consider the corresponding mapping torus,

$$\begin{aligned} C &= \mathrm{C}(F_1, F_2, \psi_0, \psi_1) \\ &= \left\{(f, a) \in \mathrm{C}\big([0,1], F_2\big) \oplus F_1 : f(0) = \psi_0(a) \text{ and } f(1) = \psi_1(a)\right\}. \end{aligned}$$

Denote by $\mathcal{C}$ the class of unital C*-algebras obtained in this way. C*-algebras in the class $\mathcal{C}$ are often called Elliott–Thomsen building blocks. They are also called one-dimensional non-commutative CW complexes.

Denote by $\mathcal{C}_0$ the subclass of $\mathcal{C}$ consisting of those C*-algebras $C \in \mathcal{C}$ such that $\mathrm{K}_1(C) = \{0\}$.

We shall in fact only work with the Q-stabilizations of these algebras, which can be described just by replacing F_1 and F_2 with finite direct sums of copies of Q.

Definition 2.4 ([15, Definition 9.1]). Let A be a (non-elementary) unital simple C*-algebra. We shall say that A has generalized tracial rank at most one if the following property holds.

Let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$, and let $\mathcal{F} \subseteq A$ be a finite subset. There exist a non-zero projection $p \in A$ and a sub-C*-algebra $C \in \mathcal{C}$ with $1_C = p$ such that

$$\|xp - px\| < \varepsilon, \quad x \in \mathcal{F},$$
$$\mathrm{dist}(pxp, C) < \varepsilon, \quad x \in \mathcal{F},$$
$$1 - p \lesssim a.$$

The last condition means that there exists a partial isometry $v \in A$ such that $v^*v = 1 - p$ and $vv^* \in \overline{aAa}$. If A has generalized tracial rank at most one, we will write $\mathrm{gTR}(A) \leq 1$. It was shown in [15] that if $\mathrm{gTR}(A) \leq 1$, then A is quasidiagonal and $\mathcal{Z}$-stable if it is also amenable.

Definition 2.5. Let A and B be unital C*-algebras and let $L : A \to B$ be a contractive completely positive map. Let $\mathcal{G}$ be a finite subset of A and $\delta > 0$. Recall that L is said to be $\mathcal{G}$-δ-multiplicative if $\|L(x)L(y) - L(xy)\| < \delta$ for all $x, y \in \mathcal{G}$. Given a finite subset $\mathcal{P}$ of projections in A, if $\mathcal{G}$ is sufficiently large and δ is sufficiently small, then there is a projection $q \in B$ such that $\|L(p) - q\| < 1/4$. Moreover, for each projection $p \in \mathcal{G}$, if $\delta < 1/4$, then the projection q can be chosen such that

$$\|L(p) - q\| < 2\delta. \tag{2.1}$$

Note that if $q' \in B$ is another projection such that $\|L(p) - q'\| < 1/4$, then q' and q are unitarily equivalent. Recall that $[L(p)]$ often denotes this equivalence class of projections (see e.g. [21]). As usual, when $[L(p)]$ is written, it is understood that $\mathcal{G}$ is sufficiently large and δ is sufficiently small that $[L(p)]$ is well defined.

Definition 2.6 ([15]). Let A be a unital simple separable C*-algebra. Let us say that A has rational generalized tracial rank at most one if $\mathrm{gTR}(A \otimes Q) \leq 1$.

Let us say that A belongs to the class $\mathcal{N}_1$ if, in addition, it is amenable and satisfies the UCT [34] and is Jiang–Su stable, i.e., is invariant under tensoring with the Jiang–Su C*-algebra ([18]; see also [8]). As pointed out above, it follows from [16] (together with [38]) that, instead of the last property, it is equivalent to assume finite decomposition rank (or, by [39], even just finite nuclear dimension); see Definition 2.10 below. (By now, we know (see [4, 29, 35]) that a unital separable simple nuclear C*-algebra is Jiang–Su stable if and only if it has finite nuclear dimension—see "Added November 2, 2021" at the end of this paper.)

The following are the main results of [15, 16].

Theorem 2.7. *Let A and B be two unital* C*-*algebras in $\mathcal{N}_1$. Then, $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$*, i.e., $A \cong B$ if and only if*

$$\big(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, [1_A]_0, \mathrm{K}_1(A), \mathrm{T}(A), r_A\big) \cong \big(\mathrm{K}_0(B), \mathrm{K}_0(B)_+, [1_B]_0, \mathrm{K}_1(B), \mathrm{T}(B), r_B\big).$$

Moreover, any isomorphism between $\mathrm{Ell}(A)$ *and* $\mathrm{Ell}(B)$ *can be lifted to an isomorphism between A and B.*

Proof. The first part of the statement follows from [16, Theorem 29.8].

The second part of the statement needs some explanation. Note that A and B satisfy the assumption of [16, Theorem 29.5], by [15, Corollary 19.3]. Let $\Gamma : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ be an isomorphism. Repeat [16, proof of Theorem 29.5] until the second-last sentence of that proof: namely, "One then obtains a unitary suspended isomorphism which lifts Γ along $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ (see [40])". For the present purpose (note that the lifting statement is not explicitly formulated in [16]), replace the last sentence of that proof by the sentence "It follows from [40, Theorem 7.1] that $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ are isomorphic and the isomorphism lifts Γ". ∎

Theorem 2.8 ([15, Theorem 13.50]). *For any non-zero countable weakly unperforated simple ordered group G_0 with order unit u, any countable abelian group G_1, any non-empty metrizable Choquet simplex T, and any surjective affine map $r : T \to \mathrm{S}_u(G_0)$ ($\mathrm{S}_u(G_0)$ is the state space of G_0—always non-empty), there exists a (unique) unital simple* C^**-algebra C in $\mathcal{N}_1$, which is the inductive limit of a sequence of subhomogeneous* C^**-algebras with two-dimensional spectrum, such that*

$$\mathrm{Ell}(C) = \big(G_0, (G_0)_+, u, G_1, T, r\big).$$

Definition 2.9. Let A and B be C^*-algebras. Recall ([20]) that a completely positive map $\varphi : A \to B$ is said to have order zero if

$$ab = 0 \implies \varphi(a)\varphi(b) = 0, \quad a, b \in A.$$

Definition 2.10 ([20,42]). A C^*-algebra A has nuclear dimension at most n if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)$, $\lambda \in \Lambda$, such that the F_λ are finite-dimensional C^*-algebras, and such that $\psi_\lambda : A \to F_\lambda$ and $\varphi_\lambda : F_\lambda \to A$ are completely positive maps satisfying the following:

(1) $\varphi_\lambda \circ \psi_\lambda \to \mathrm{id}_A$ pointwise (in norm),

(2) $\|\psi_\lambda\| \leq 1$,

(3) for each λ, there is a decomposition $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that each restriction $\varphi_\lambda|_{F_\lambda^{(j)}}$ is a contractive order zero map.

Moreover, if the the map φ_λ can be chosen to be contractive itself, then A is said to have decomposition rank at most n.

Recall that finite nuclear dimension immediately implies nuclearity, which by [5, 17] is equivalent to amenability. The nuclear dimension of a certain C^*-algebra associated with a discrete metric space is related to asymptotical dimension of the underlying space, and the concept of asymptotical dimension has fundamental applications to geometry and topology (see [43, 44]).

The main theorem of this paper is that the class $\mathcal{N}_1$ of C^*-algebras actually contains (and hence coincides with) the class of all (non-elementary) unital simple separable C^*-algebras with finite decomposition rank which also satisfy the UCT. In particular, it follows (on using both Theorems 2.7 and 2.8) that every such C^*-algebra is the inductive limit of a sequence of subhomogeneous C^*-algebras (with no dimension growth).

3. Some existence theorems

Denote by $\mathcal{K}$ the C*-algebra of all compact operators on l^2, an infinite-dimensional separable Hilbert space. Let $\{e_{i,j}\}$ be the canonical system of matrix units for $\mathcal{K}$. We will use the fact that $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ and assume that such an isomorphism has been fixed. Let B be a C*-algebra. We may identify $B \otimes \mathcal{K}$ with $(B \otimes \mathcal{K}) \otimes e_{1,1}$. Let A be another C*-algebra and let $\psi_1, \psi_2, \ldots, \psi_n : A \to B \otimes \mathcal{K}$ be linear maps. For convenience, when there is no confusion, using the identification above, we shall write

$$\Phi := \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_n : A \to B \otimes \mathcal{K}$$

to denote the orthogonal sum:

$$\Phi(a) = \sum_{i=1}^{n} \psi_i(a) \otimes e_{i,i} \quad \text{for all } a \in A. \tag{3.1}$$

Likewise, for projections $p_1, p_2, \ldots, p_m \in B \otimes \mathcal{K}$, we shall write

$$P := p_1 \oplus p_2 \oplus \cdots \oplus p_m \in B \otimes \mathcal{K}$$

if there is no confusion. Therefore, in the case that $\psi_i(A) \subseteq p_i(B \otimes \mathcal{K})p_i$, $1 \le i \le n$, we may view Φ as mapping A to $P(B \otimes \mathcal{K})P$.

We will use this convention repeatedly.

Lemma 3.1. *Let A be a unital simple separable amenable quasidiagonal* C*-*algebra satisfying the UCT. Assume that $A \cong A \otimes Q$.*

Let a finite subset $\mathcal{G}$ of A and $\varepsilon_1, \varepsilon_2 > 0$ be given. Let $p_1, p_2, \ldots, p_s \in A$ be projections such that $[1], [p_1], [p_2], \ldots, [p_s] \in \mathrm{K}_0(A)$ are $\mathbb{Q}$-linearly independent. (Recall that $\mathrm{K}_0(A) \cong \mathrm{K}_0(A \otimes Q) \cong \mathrm{K}_0(A) \otimes \mathbb{Q}$.) There are a $\mathcal{G}$-ε_1-multiplicative completely positive map $\sigma : A \to Q$ with $\sigma(1)$, a projection satisfying

$$\mathrm{tr}\big(\sigma(1)\big) < \varepsilon_2$$

(where tr *denotes the unique tracial state on Q), and $\delta > 0$, such that, for any $r_1, r_2, \ldots, r_s \in \mathbb{Q}$ with*

$$|r_i| < \delta, \quad i = 1, 2, \ldots, s,$$

there is a $\mathcal{G}$-ε_1-multiplicative completely positive map $\mu : A \to Q$, with $\mu(1) = \sigma(1)$, such that

$$\big[\sigma(p_i)\big] - \big[\mu(p_i)\big] = r_i, \quad i = 1, 2, \ldots, s.$$

Proof. Let us agree that σ and μ (to be constructed below) are also understood to be required to be sufficiently multiplicative on $p_1, p_2, \ldots, p_k$ that the classes $[\sigma(p_i)]$ and $[\mu(p_i)]$ make sense (see Definition 2.5 above) (similarly for other completely positive approximately multiplicative maps, to be introduced below).

Denote by G_0 the subgroup of $\mathrm{K}_0(A)$ generated by $\{[1_A],[p_1],\dots,[p_s]\}$. Since $[1_A]$, $[p_1],\dots,[p_s]$ are $\mathbb{Q}$-linearly independent, for each $i=1,2,\dots,s$, there exists a homomorphism $\alpha_i:\mathrm{K}_0(A)\to\mathbb{Q}\cong\mathrm{K}_0(Q)$ such that

$$\alpha_i\big([p_i]\big)=1,\quad \alpha_i\big([1_A]\big)=0,\quad \text{and}\quad \alpha_i\big([p_j]\big)=0,\ j\neq i. \tag{3.2}$$

We may regard α_i as an element of $\mathrm{KL}(A,Q)$ (see [7]). Since A is a unital simple amenable quasidiagonal C*-algebra, by [1], A is a unital simple strong NF-algebra. It follows from [1] that $A=\overline{\bigcup_{n=1}^{\infty}A_n}$, where $\{A_n\}$ is an increasing sequence of unital, amenable, residually finite-dimensional C*-algebras. It follows from [21, Theorem 5.9] that there are $\mathcal{G}$-ε_1-multiplicative completely positive maps $\sigma_i,\mu_i:A\to Q\otimes\mathcal{K}$ such that $\sigma_i(1_A)$ and $\mu_i(1_A)$ are projections, and

$$[\sigma_i]\big|_{G_0}-[\mu_i]\big|_{G_0}=\alpha_i|_{G_0},\quad i=1,2,\dots,s. \tag{3.3}$$

Since $\alpha_i([1_A])=0$, we have $[\sigma_i(1_A)]=[\mu_i(1_A)]$. Therefore, without loss of generality, we may assume that

$$\sigma_i(1_A)=\mu_i(1_A)=:P_i,\quad i=1,2,\dots,s.$$

Consider the projection

$$P:=\bigoplus_{i=1}^{s}(P_i\oplus P_i),$$

and the unital $\mathcal{G}$-ε_1-multiplicative completely positive map

$$\bigoplus_{i=1}^{s}(\sigma_i\oplus\mu_i):A\to P(Q\otimes\mathcal{K})P,$$

where $\sigma_i\oplus\mu_i$ means the map $a\mapsto\sigma_i(a)\oplus\mu_i(a)$ (see the beginning of this section). Note that $P(Q\otimes\mathcal{K})P\cong Q$. Choose a projection $R\in Q\otimes\mathcal{K}$ with $0<\mathrm{tr}(R)\leq\min\{1,\varepsilon_2\}$ and a rescaling homomorphism

$$S:Q\otimes\mathcal{K}\to Q\otimes\mathcal{K},\ P\mapsto R.$$

Consider the map

$$\sigma:=S\circ\left(\bigoplus_{i=1}^{s}(\sigma_i\oplus\mu_i)\right):A\to Q\otimes\mathcal{K}$$

and the strictly positive number

$$\delta:=\frac{\mathrm{tr}(R)}{\mathrm{tr}(P)}.$$

(Here, tr denotes the tensor product of the traces on Q and $\mathcal{K}$, normalized to be 1 on $1_Q\otimes e_{11}$.) Note that since $\mathrm{tr}(R)\leq 1$, one has

$$\mathrm{tr}\big(\sigma(1)\big)=\mathrm{tr}(R)\leq 1,$$

and so we may regard σ as a map from A to Q (rather than $Q\otimes\mathcal{K}$).

Let us show that σ and δ satisfy the condition of the lemma.

Let $r_1, r_2, \dots, r_s \in \mathbb{Q}$ be given with

$$|r_i| < \delta, \quad i = 1, 2, \dots, s.$$

For each $i = 1, 2, \dots, s$, choose a projection $R_i \in Q \otimes \mathcal{K}$ with $\mathrm{tr}(R_i) = |r_i|$, and choose a rescaling homomorphism

$$S_i : Q \otimes \mathcal{K} \to Q \otimes \mathcal{K}, \quad 1 \otimes e \mapsto R_i,$$

where e is a minimal non-zero projection of $\mathcal{K}$. For each $i = 1, 2, \dots, s$, consider the pair of maps

$$S_i \circ \sigma_i, S_i \circ \mu_i : A \to Q \otimes \mathcal{K}.$$

Then, for each $i = 1, 2, \dots, s$,

$$\begin{aligned} \big[S_i \circ \sigma_i(p_i)\big] - \big[S_i \circ \mu_i(p_i)\big] &= |r_i|, \\ \big[S_i \circ \sigma_i(1)\big] - \big[S_i \circ \mu_i(1)\big] &= 0, \\ \big[S_i \circ \sigma_i(p_j)\big] - \big[S_i \circ \mu_i(p_j)\big] &= 0, \quad j = 1, 2, \dots, s, \ j \neq i. \end{aligned}$$

Consider the direct sum maps

$$\tilde{\sigma} := \Big(\bigoplus_{r_i > 0} S_i \circ \sigma_i\Big) \oplus \Big(\bigoplus_{r_i < 0} S_i \circ \mu_i\Big),$$
$$\tilde{\mu} := \Big(\bigoplus_{r_i > 0} S_i \circ \mu_i\Big) \oplus \Big(\bigoplus_{r_i < 0} S_i \circ \sigma_i\Big).$$

It follows from (3.2) and (3.3) that

$$\big[\tilde{\sigma}(p_i)\big] - \big[\tilde{\mu}(p_i)\big] = r_i, \quad i = 1, 2, \dots, s.$$

Note that

$$\sigma = S \circ \left(\bigoplus_{i=1}^{s} (\sigma_i \oplus \mu_i)\right) = \bigoplus_{i=1}^{s} \big((S \circ \sigma_i) \oplus (S \circ \mu_i)\big). \tag{3.4}$$

For each $i = 1, 2, \dots, s$, since

$$\begin{aligned} \mathrm{tr}\big(S_i(P)\big) &= \mathrm{tr}(P) \cdot \mathrm{tr}\big(S_i(1 \otimes e)\big) = \mathrm{tr}(P) \cdot \mathrm{tr}(R_i) \\ &= \mathrm{tr}(P)|r_i| < \mathrm{tr}(P)\delta = \mathrm{tr}(R) = \mathrm{tr}\big(S(P)\big), \end{aligned}$$

there is a rescaling homomorphism $T_i : Q \otimes \mathcal{K} \to Q \otimes \mathcal{K}$ such that

$$[S] = [S_i] + [T_i] = [S_i \oplus T_i] \quad \text{on } \mathrm{K}_0(Q) \cong \mathbb{Q}.$$

Therefore, by (3.4), on G_0,

$$
\begin{aligned}
[\sigma] &= \sum_{i=1}^{s} \big([(S\circ\sigma_i)] + [(S\circ\mu_i)]\big) \\
&= \sum_{i=1}^{s} \big([S_i\circ\sigma_i] + [T_i\circ\sigma_i]\big) + \sum_{i=1}^{s} \big([S_i\circ\mu_i] + [T_i\circ\mu_i]\big) \\
&= \Big[\Big(\bigoplus_{r_i>0} \big((S_i\circ\sigma_i)\oplus(T_i\circ\sigma_i)\big)\Big) \oplus \Big(\bigoplus_{r_i\le 0} \big((S_i\circ\sigma_i)\oplus(T_i\circ\sigma_i)\big)\Big) \\
&\quad \oplus \Big(\bigoplus_{r_i<0} \big((S_i\circ\mu_i)\oplus(T_i\circ\mu_i)\big)\Big) \oplus \Big(\bigoplus_{r_i\ge 0} \big((S_i\circ\mu_i)\oplus(T_i\circ\mu_i)\big)\Big)\Big] \\
&= [\tilde\sigma]\oplus[\gamma],
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma = \Big(\bigoplus_{r_i>0}(T_i\circ\sigma_i)\Big) \oplus \Big(\bigoplus_{r_i\le 0} \big((S_i\circ\sigma_i)\oplus(T_i\circ\sigma_i)\big)\Big) \\
\oplus \Big(\bigoplus_{r_i<0}(T_i\circ\mu_i)\Big) \oplus \Big(\bigoplus_{r_i\ge 0} \big((S_i\circ\mu_i)\oplus(T_i\circ\mu_i)\big)\Big).
\end{aligned}
$$

Consider the direct sum completely positive map

$$\mu := \tilde\mu \oplus \gamma.$$

We have

$$\big[\sigma(p_i)\big] - \big[\mu(p_i)\big] = \big[\tilde\sigma(p_i)\big] - \big[\tilde\mu(p_i)\big] = r_i, \quad i = 1,2,\dots,s,$$

as desired (with μ regarded as a map from A to Q, as $\mu(1) = \sigma(1)$). ∎

Remark 3.2. The assumption that A is amenable in Lemma 3.1 can be removed. In the proof one can apply [6, Theorem 5.5] in place of [21, Theorem 5.9].

Let $l, r = 1,2,\dots$ be given. In the rest of the paper, we identify $\mathrm{K}_0(Q^l)$ with $\mathbb{Q}^l$ (and $\mathrm{K}_0(Q^r)$ with $\mathbb{Q}^r$) by identifying $[1_{Q^l}]$ with $(\underbrace{1,1,\dots,1}_{l})$ and $([1_{Q^r}]$ with $(\underbrace{1,1,\dots,1}_{r})$), where

$$Q^l = \underbrace{Q\oplus\cdots\oplus Q}_{l}, \quad Q^r = \underbrace{Q\oplus\cdots\oplus Q}_{r}.$$

If $\psi: Q^l \to Q^r$ are unital, then

$$(\psi)_{*0}(\underbrace{1,1,\dots,1}_{l}) = (\underbrace{1,1,\dots,1}_{r}),$$

and therefore

$$(\psi)_{*0}(\underbrace{t,t,\dots,t}_{l}) = (\underbrace{t,t,\dots,t}_{r}), \quad t\in\mathbb{Q}. \tag{3.5}$$

Lemma 3.3. *Let A be a unital simple separable amenable quasidiagonal* C^**-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$.*

Let $\mathcal{G}$ be a finite subset of A, let $\varepsilon_1, \varepsilon_2 > 0$, and let $p_1, p_2, \ldots, p_s \in A$ be projections such that $[1_A], [p_1], [p_2], \ldots, [p_s] \in \mathrm{K}_0(A)$ are $\mathbb{Q}$-linearly independent. There exists $\delta > 0$ satisfying the following condition.

Let $\psi_k : Q^l \to Q^r$, $k = 0, 1$, be unital homomorphisms, where $l, r = 1, 2, \ldots$. Set

$$\mathbb{D} = \big\{x \in \mathbb{Q}^l : (\psi_0)_{*0}(x) = (\psi_1)_{*0}(x)\big\} \subseteq \mathbb{Q}^l.$$

There exists a $\mathcal{G}$-ε_1-multiplicative completely positive map $\Sigma : A \to Q^l$, such that $\Sigma(1_A)$ is a projection, with the following properties:

$$\tau\big(\Sigma(1_A)\big) < \varepsilon_2, \quad \tau \in \mathrm{T}(Q^l),$$
$$\big[\Sigma(1_A)\big], \big[\Sigma(p_j)\big] \in \mathbb{D}, \quad j = 1, 2, \ldots, s,$$

and, for any $r_1, r_2, \ldots, r_s \in \mathbb{Q}^r$ satisfying

$$|r_{i,j}| < \delta,$$

where $r_i = (r_{i,1}, r_{i,2}, \ldots, r_{i,r})$, $i = 1, 2, \ldots, s$, there is a $\mathcal{G}$-ε_1-multiplicative completely positive map $\mu : A \to Q^r$, with $\mu(1_A)$ a projection, such that

$$\big[\psi_0 \circ \Sigma(p_i)\big] - \big[\mu(p_i)\big] = r_i, \quad i = 1, 2, \ldots, s,$$

and

$$\big[\mu(1_A)\big] = \big[\psi_0 \circ \Sigma(1_A)\big].$$

Proof. Put $p_0 = 1_A$ and $\mathcal{P} = \{[1_A], [p_1], \ldots, [p_s]\}$. Applying Lemma 3.1, we obtain a $\mathcal{G}$-ε_1-multiplicative $\sigma : A \to Q$ and $\delta > 0$ satisfying the conclusion of Lemma 3.1 with respect to $\mathcal{G}$, ε_1, ε_2, and $\mathcal{P}$.

Let us show that δ is as desired.

For a given integer $l = 1, 2, \ldots$, consider the map $\Sigma : A \to Q^l$, the sum of l copies of σ,

$$\Sigma = \sigma \oplus \sigma \oplus \cdots \oplus \sigma.$$

Let us show that Σ has the required properties. Let $r = 1, 2, \ldots$ and $\psi_k : Q^l \to Q^r$, $k = 0, 1$, be given (as in the statement of the condition on δ to be verified). Since ψ_0 and ψ_1 are assumed to be unital, $[1_{Q^r}] = (1, 1, \ldots, 1) \in \mathbb{D}$. It then follows that

$$\begin{aligned}\big[\Sigma(p_i)\big] &= \big(\big[\sigma(p_i)\big], \big[\sigma(p_i)\big], \ldots, \big[\sigma(p_i)\big]\big) \\ &= \big[\sigma(p_i)\big](1, 1, \ldots, 1) \in \mathbb{D}, \quad i = 0, 1, 2, \ldots, s,\end{aligned}$$

where $[\sigma(p_i)]$ is regarded as a rational number. In other words,

$$\big[\psi_0 \circ \Sigma(p_i)\big] = \big[\psi_1 \circ \Sigma(p_i)\big], \quad i = 0, 1, 2, \ldots, s.$$

Since $\mathrm{tr}([\sigma(1_A)]) < \varepsilon_2$, one has

$$\tau\big(\big[\Sigma(1_A)\big]\big) < \varepsilon_2, \quad \tau \in \mathrm{T}(Q^l).$$

Let $r_1, r_2, \ldots, r_s \in \mathbb{Q}^r$ be given such that $|r_{i,j}| < \delta$, $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, s$. Let us show that μ exists as required.

Fix $j = 1, 2, \ldots, r$ and let $\mu_j : A \to Q$ (in place of μ) denote the $\mathcal{G}$-ε_1-multiplicative completely positive map given by Lemma 3.1 for the s-tuple $r_{1,j}, r_{2,j}, \ldots, r_{s,j}$. That is, $\mu_j(1_A) = \sigma(1_A)$, and

$$\big[\sigma(p_i)\big] - \big[\mu_j(p_i)\big] = r_{i,j} \in \mathrm{K}_0(Q), \quad i = 1, 2, \ldots, s. \tag{3.6}$$

Define $\mu : A \to Q^r$ by

$$\mu(a) = \big(\underbrace{\mu_1(a), \mu_2(a), \ldots, \mu_r(a)}_{r}\big), \quad a \in A.$$

Then, for each $i = 1, 2, \ldots, s$,

$$\begin{aligned}
&\big[\psi_0 \circ \Sigma(p_i)\big] - \big[\mu(p_i)\big] \\
&= (\psi_0)_{*0}\big(\underbrace{\big[\sigma(p_i)\big], \big[\sigma(p_i)\big], \ldots, \big[\sigma(p_i)\big]}_{l}\big) - \big(\underbrace{\big[\mu_1(p_i)\big], \big[\mu_2(p_i)\big], \ldots, \big[\mu_r(p_i)\big]}_{r}\big) \\
&= \big(\underbrace{\big[\sigma(p_i)\big], \big[\sigma(p_i)\big], \ldots, \big[\sigma(p_i)\big]}_{r}\big) - \big(\underbrace{\big[\mu_1(p_i)\big], \big[\mu_2(p_i)\big], \ldots, \big[\mu_r(p_i)\big]}_{r}\big) \quad \text{(by (3.5))} \\
&= \big(\big(\big[\sigma(p_i)\big] - \big[\mu_1(p_i)\big]\big), \big(\big[\sigma(p_i)\big] - \big[\mu_2(p_i)\big]\big), \ldots, \big(\big[\sigma(p_i)\big] - \big[\mu_r(p_i)\big]\big)\big) \\
&= (r_{i,1}, r_{i,2}, \ldots, r_{i,r}) = r_i \quad \text{(by (3.6))},
\end{aligned}$$

as desired. A similar computation shows that $[\psi_0 \circ \Sigma(1_A)] = [\mu(1_A)]$. ∎

Lemma 3.4. *Let A be a unital simple separable amenable quasidiagonal C^*-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$.*

Let $\mathcal{G} \subseteq A$ be a finite subset, let $\varepsilon_1, \varepsilon_2 > 0$, and let $p_1, p_2, \ldots, p_s \in A$ be projections such that $[1_A], [p_1], [p_2], \ldots, [p_s] \in \mathrm{K}_0(A)$ are $\mathbb{Q}$-linearly independent. Then, there exists $\delta > 0$ satisfying the following condition.

Let $\psi_k : Q^l \to Q^r$, $k = 0, 1$, be unital homomorphisms, where $l, r = 1, 2, \ldots$. Set

$$\mathbb{D} = \big\{x \in \mathbb{Q}^l : (\psi_0)_{*0}(x) = (\psi_1)_{*0}(x)\big\} \subseteq \mathbb{Q}^l.$$

There exists a $\mathcal{G}$-ε_1-multiplicative completely positive map $\Sigma : A \to Q^l$, such that $\Sigma(1_A)$ is a projection, with the following properties:

$$\tau\big(\Sigma(1_A)\big) < \varepsilon_2, \quad \tau \in \mathrm{T}(Q^l),$$
$$\big[\Sigma(1_A)\big], \big[\Sigma(p_i)\big] \in \mathbb{D}, \quad i = 1, 2, \ldots, s,$$

and, for any $r_1, r_2, \ldots, r_s \in \mathbb{Q}^l$ *satisfying*

$$|r_{i,j}| < \delta,$$

where $r_i = (r_{i,1}, r_{i,2}, \ldots, r_{i,l})$, $i = 1, 2, \ldots, s$, *there is a* $\mathcal{G}$-ε_1-*multiplicative completely positive map* $\mu : A \to Q^l$, *with* $\mu(1_A) = \Sigma(1_A)$, *such that*

$$[\Sigma(p_i)] - [\mu(p_i)] = r_i, \quad i = 1, 2, \ldots, s.$$

Proof. This is similar to the proof of Lemma 3.3. Put $[p_0] = [1_A]$, $\mathcal{P} = \{[p_0], [p_1], \ldots, [p_s]\}$. Applying Lemma 3.1, we obtain a $\mathcal{G}$-ε_1-multiplicative $\sigma : A \to Q$ and $\delta > 0$ satisfying the conclusion of Lemma 3.1 with respect to $(\mathcal{G}, \varepsilon_1, \varepsilon_2)$.

Let us show that δ is as desired.

Consider the map $\Sigma : A \to Q^l$, the sum of l copies of σ,

$$\Sigma = \sigma \oplus \sigma \oplus \cdots \oplus \sigma,$$

for a given $l = 1, 2, \ldots$. Then, the same argument as that of Lemma 3.3 shows that

$$[\Sigma(p_i)] \in \mathbb{D}, \quad i = 0, 1, 2, \ldots, s.$$

It is also clear that

$$\tau(\Sigma(1_A)) < \varepsilon_2, \quad \tau \in \mathrm{T}(Q^l).$$

Let $r_1, r_2, \ldots, r_s \in \mathbb{Q}^l$ be given such that $|r_{i,j}| < \delta$, $j = 1, 2, \ldots, l$ and $i = 1, 2, \ldots, s$. Let us show that μ exists as required.

Fix $j = 1, 2, \ldots, l$ and let $\mu_j : A \to Q$ (in place of μ) denote the $\mathcal{G}$-ε_1-multiplicative completely positive map given by Lemma 3.1 for the s-tuple $r_{1,j}, r_{2,j}, \ldots, r_{s,j}$. That is, $\mu_j(1_A) = \sigma(1_A)$, and

$$[\sigma(p_i)] - [\mu_j(p_i)] = r_{i,j} \in \mathrm{K}_0(Q), \quad i = 1, 2, \ldots, s. \tag{3.7}$$

Define $\mu : A \to Q^l$ by

$$\mu(a) = (\mu_1(a), \mu_2(a), \ldots, \mu_l(a)), \quad a \in A.$$

Then, for each $i = 1, 2, \ldots, s$,

$$\begin{aligned}
&[\Sigma(p_i)] - [\mu(p_i)] \\
&\quad = (\underbrace{[\sigma(p_i)], [\sigma(p_i)], \ldots, [\sigma(p_i)]}_{l}) - ([\mu_1(p_i)], [\mu_2(p_i)], \ldots, [\mu_l(p_i)]) \\
&\quad = (([\sigma(p_i)] - [\mu_1(p_i)]), ([\sigma(p_i)] - [\mu_2(p_i)]), \ldots, ([\sigma(p_i)] - [\mu_l(p_i)])) \\
&\quad = (r_{i,1}, r_{i,2}, \ldots, r_{i,l}) = r_i \quad (\text{by } (3.7)),
\end{aligned}$$

as desired. Moreover, since $\mu_j(1_A) = \sigma(1_A)$, we have $\Sigma(1_A) = \mu(1_A)$. ∎

4. The main result

Let us begin by recalling the (stable) uniqueness result used in [11].

Lemma 4.1 ([11, Corollary 2.6]; see also [15, Lemma 4.14] and [22, Definition 5.6 and Theorem 5.9]). *Let A be a unital simple separable amenable C^*-algebra which satisfies the UCT. Assume that $A \cong A \otimes Q$.*

For any $\varepsilon > 0$, any finite subset $\mathcal{F}$ of A, there exist $\delta > 0$, a finite subset $\mathcal{G}$ of A, a finite subset $\mathcal{P}$ of projections in A, and an integer $n \in \mathbb{N}$ with the following property.

For any three completely positive contractions $\varphi, \psi, \xi : A \to Q$ which are $\mathcal{G}$-δ-multiplicative, with $\varphi(1) = \psi(1) = 1_Q - \xi(1)$ a projection, $[\varphi(p)]_0 = [\psi(p)]_0$ in $\mathrm{K}_0(Q)$ for all $p \in \mathcal{P}$, and $\mathrm{tr}(\varphi(1)) = \mathrm{tr}(\psi(1)) < 1/n$, where tr is the unique tracial state of Q, there exists a unitary $u \in Q$ such that

$$\left\| u^*\big(\varphi(a) \oplus \xi(a)\big)u - \psi(a) \oplus \xi(a) \right\| < \varepsilon, \quad a \in \mathcal{F}.$$

The following two existence results are related to [15, Lemma 16.9] (and its proof).

We will use the following known facts: $Q \otimes Q \cong Q$, and any unital endomorphism $\varphi : Q \to Q$ is approximately unitarily equivalent to the identity map.

Lemma 4.2. *Let A be a unital simple separable amenable C^*-algebra which satisfies the UCT. Assume that $A \otimes Q \cong A$.*

For any $\varepsilon > 0$ and any finite subset $\mathcal{F}$ of A, there exist $\delta > 0$, a finite subset $\mathcal{G}$ of A, and a finite subset $\mathcal{P}$ of projections in A with the following property.

Let $\psi, \varphi : A \to Q$ be two unital $\mathcal{G}$-δ-multiplicative completely positive maps such that

$$[\psi]\big|_{\mathcal{P}} = [\varphi]\big|_{\mathcal{P}}.$$

Then, there are a unitary $u \in Q$ and a unital $\mathcal{F}$-ε-multiplicative completely positive map $L : A \to \mathrm{C}([0,1], Q)$ such that

$$\pi_0 \circ L = \psi \quad \text{and} \quad \pi_1 \circ L = \mathrm{Ad}\, u \circ \varphi. \tag{4.1}$$

Moreover, if

$$\left| \mathrm{tr} \circ \psi(h) - \mathrm{tr} \circ \varphi(h) \right| < \varepsilon', \quad h \in \mathcal{H}, \tag{4.2}$$

for a finite set $\mathcal{H} \subseteq A$ and $\varepsilon' > 0$, then L may be chosen such that

$$\left| \mathrm{tr} \circ \pi_t \circ L(h) - \mathrm{tr} \circ \pi_0 \circ L(h) \right| < \varepsilon', \quad h \in \mathcal{H},\ t \in [0,1]. \tag{4.3}$$

Here, $\pi_t : \mathrm{C}([0,1], Q) \to Q$ is the point evaluation at $t \in [0,1]$.

Proof. This is a direct application of the stable uniqueness theorem [11, Corollary 2.6], restated as Lemma 4.1 above. Let $\mathcal{F} \subseteq A$ be a finite subset and let $\varepsilon > 0$ be given. We may assume that $1_A \in \mathcal{F}$ and every element of $\mathcal{F}$ has norm at most one. Write $\mathcal{F}_1 = \{ab : a, b \in \mathcal{F}\}$. Note that $\mathcal{F} \subseteq \mathcal{F}_1$.

Let δ, $\mathcal{G}$, $\mathcal{P}$, and n be as assured by Lemma 4.1 for $\mathcal{F}_1$ and $\varepsilon/4$. We may also assume that $\mathcal{F} \subseteq \mathcal{G}$ and $\delta \leq \varepsilon/4$.

Let $\varphi', \psi' : A \to Q \otimes Q$ be defined by $\varphi'(a) = \varphi(a) \otimes 1$ and $\psi'(a) = \psi(a) \otimes 1$ for all $a \in A$. Pick mutually equivalent projections $e_0, e_1, e_2, \dots, e_n \in Q$ satisfying

$$\sum_{i=0}^{n} e_i = 1_Q.$$

Then, consider the maps $\varphi_i, \psi_i : A \to Q \otimes e_i Q e_i$, $i = 0, 1, \dots, n$, which are defined by

$$\varphi_i(a) = \varphi(a) \otimes e_i \quad \text{and} \quad \psi_i(a) = \psi(a) \otimes e_i, \quad a \in A,$$

and consider the finite sequence of maps from A to $Q \otimes Q$

$$\Phi_{n+1} := \varphi' = \varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_n, \quad \Phi_0 := \psi' = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_n,$$

and

$$\Phi_i := \varphi_0 \oplus \cdots \oplus \varphi_{i-1} \oplus \psi_i \oplus \cdots \oplus \psi_n, \quad i = 1, 2, \dots, n.$$

Since e_i is unitarily equivalent to e_0 for all i, one has

$$[\varphi_i]\big|_{\mathcal{P}} = [\psi_j]\big|_{\mathcal{P}}, \quad 0 \leq i, j \leq n,$$

and in particular,

$$[\varphi_i]\big|_{\mathcal{P}} = [\psi_i]\big|_{\mathcal{P}}, \quad i = 0, 1, \dots, n. \tag{4.4}$$

Note that, for each $i = 0, 1, \dots, n$,

$$\Phi_i \sim \psi_i \oplus (\varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_{i-1} \oplus \psi_{i+1} \oplus \psi_{i+2} \oplus \cdots \oplus \psi_n),$$
$$\Phi_{i+1} \sim \varphi_i \oplus (\varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_{i-1} \oplus \psi_{i+1} \oplus \psi_{i+2} \oplus \cdots \oplus \psi_n),$$

where $\sim$ denotes the relation of unitary equivalence. In view of this and (4.4) (identifying $Q \otimes Q$ with Q), applying Lemma 4.1 to $\varphi := \varphi_i$, $\psi := \psi_i$ and

$$\xi := (\varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_{i-1} \oplus \psi_{i+1} \oplus \psi_{i+2} \oplus \cdots \oplus \psi_n),$$

we obtain unitaries $u_i \in Q \otimes Q$, $i = 0, 1, \dots, n$ (with $u_0 = 1_{Q \otimes Q}$ and $\widetilde{\Phi}_0 = \Phi_0 = \psi'$), such that

$$\left\| \widetilde{\Phi}_i(a) - \operatorname{Ad} u_{i+1} \circ \Phi_{i+1}(a) \right\| < \varepsilon/4, \quad a \in \mathcal{F}_1,$$

where $\widetilde{\Phi}_i := \operatorname{Ad} u_i \circ \Phi_i$, $i = 1, 2, \dots, n$.

Consequently,

$$\left\| \widetilde{\Phi}_{i+1}(a) - \widetilde{\Phi}_i(a) \right\| < \varepsilon/4, \quad a \in \mathcal{F}_1, \quad i = 0, 1, \dots, n. \tag{4.5}$$

Put $t_i = i/(n+1)$, $i = 0, 1, \dots, n+1$, and define $L' : A \to \mathrm{C}([0,1], Q \otimes Q)$ by

$$\pi_t \circ L' = (n+1)(t_{i+1} - t)\widetilde{\Phi}_i + (n+1)(t - t_i)\widetilde{\Phi}_{i+1}, \quad t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, n.$$

By construction,

$$\pi_0 \circ L' = \widetilde{\Phi}_0 = \psi' \quad \text{and} \quad \pi_1 \circ L' = \widetilde{\Phi}_{n+1} = \mathrm{Ad}\, u_{n+1} \circ \psi'. \tag{4.6}$$

Since $\widetilde{\Phi}_i$, $i = 0, 1, \ldots, n$, are $\mathcal{G}$-δ-multiplicative (in particular $\mathcal{F}$-$\varepsilon/4$-multiplicative), it follows from (4.5) that L' is $\mathcal{F}$-ε-multiplicative. Let $u' = u_{n+1}$. Then, $\pi_0 \circ L' = \psi'$ and $\pi_1 \circ L' = \mathrm{Ad}\, u' \circ \psi'$.

Choose an isomorphism $s : Q \otimes Q \to Q$. Note that $\mathrm{tr}(s(a)) = \mathrm{tr}(a)$ for all $a \in Q \otimes Q$. Recall that the endomorphism $s \circ j : Q \to Q$ is approximately unitary equivalent to $\mathrm{id}_Q : Q \to Q$, where $j : Q \to Q \otimes Q$ is defined by $j(x) = x \otimes 1$. Thus, there are two unitaries $w_0, w_1 \in Q$ such that

$$\begin{aligned} \left\| \mathrm{Ad}\, w_0 \circ s \circ j\big(\psi(a)\big) - \psi(a)\right\| < \varepsilon/4, \\ \left\| \mathrm{Ad}\, w_1 \circ s \circ j\big(\varphi(a)\big) - \varphi(a)\right\| < \varepsilon/4, \end{aligned} \tag{4.7}$$

for all $a \in \mathcal{F}_1$. Note that $j \circ \psi = \psi'$ and $j \circ \varphi = \varphi'$. Now define $L : A \to \mathrm{C}([0,1], Q)$ by

$$L(t) = \begin{cases} (1-3t)\psi + 3t\, \mathrm{Ad}\, w_0 \circ s \circ \psi', & t \in [0, 1/3], \\ \mathrm{Ad}\, w_0 \circ \mathrm{Ad}\big(s(u')\big) \circ s \circ L'\big(3(t-1/3)\big), & t \in [1/3, 2/3], \\ 3(t-2/3)\, \mathrm{Ad}\, w_0 \circ \mathrm{Ad}\big(s(u')\big) \circ \mathrm{Ad}\, w_1^* \circ \varphi & \\ \qquad + (3-3t)\, \mathrm{Ad}\, w_0 \circ \mathrm{Ad}\big(s(u')\big) \circ \varphi', & t \in [2/3, 1]. \end{cases}$$

Finally let $u = w_0(s(u'))w_1^*$; we have $\pi_0 \circ L = \psi$ and $\pi_1 \circ L = \mathrm{Ad}\, u \circ \varphi$. It follows from (4.7) and the choice of $\mathcal{F}_1$ that L is $\mathcal{F}$-ε-multiplicative (L' is already $\mathcal{F}$-ε multiplicative). Note that $\mathrm{tr}(\Phi_0(a)) = \mathrm{tr}(\psi'(a)) = \mathrm{tr}(\psi(a))$ for all $a \in A$. Suppose that (4.2) holds for some finite subset $\mathcal{H}$ and given ε'. From the definition of Φ_i, we know

$$\left\| \mathrm{tr}\big(\Phi_i(h)\big) - \mathrm{tr}\big(\Phi_0(h)\big)\right\| = \frac{i}{n+1}\left\| \mathrm{tr}\big(\varphi(h)\big) - \mathrm{tr}\big(\psi(h)\big)\right\| < \varepsilon' \quad \text{for all } h \in \mathcal{H}. \tag{4.8}$$

It is then straightforward to verify that L also satisfies (4.3). In fact, if $\xi_0, \xi_1, \xi : A \to Q$ ($= Q \otimes Q$) are three linear maps satisfying $\|\mathrm{tr}(\xi_i(h)) - \mathrm{tr}(\xi(h))\| < \varepsilon'$ ($i = 0, 1$) for all $h \in \mathcal{H}$, then any convex combination $\xi' := t\xi_0 + (1-t)\xi_1$ (where $0 \le t \le 1$) also satisfies

$$\left\| \mathrm{tr}\big(\xi'(h)\big) - \mathrm{tr}\big(\xi(h)\big)\right\| < \varepsilon' \quad \text{for } h \in \mathcal{H}.$$

Note that up to unitary equivalence, $L(t)$ (for $t \in [1/3, 2/3]$) is a convex combination of $s \circ \Phi_i$ and $s \circ \Phi_{i+1}$ (for suitable i). Hence, for $t \in [1/3, 2/3]$,

$$\begin{aligned} \left\| \mathrm{tr}\big(L(t)(h)\big) - \mathrm{tr}\big(L(0)(h)\big)\right\| &= \left\| \mathrm{tr}\big(L(t)(h)\big) - \mathrm{tr}\big(\psi(h)\big)\right\| \\ &= \left\| \mathrm{tr}\big(L(t)(h)\big) - \mathrm{tr}\big(\Phi_0(h)\big)\right\| < \varepsilon' \end{aligned}$$

for $h \in \mathcal{H}$. The inequality also holds for $t \in [0, 1/3]$ (since $\mathrm{tr}(\psi(a)) = \mathrm{tr}(\psi'(a))$ for all $a \in A$) and for $t \in [2/3, 1]$ (since $\mathrm{tr}(\varphi(a)) = \mathrm{tr}(\varphi'(a))$ for all $a \in A$). We obtain (4.3), as desired. ■

Lemma 4.3. *Let A be a unital simple separable amenable* C^**-algebra with* $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$ *which satisfies the UCT. Assume that* $A \otimes Q \cong A$.

For any $\sigma > 0$, $\varepsilon > 0$, *and any finite subset* $\mathcal{F}$ *of* A, *there exist a finite set of projections* $\mathcal{P}$ *in* A *and* $\delta > 0$ *with the following property.*

Denote by $G \subseteq \mathrm{K}_0(A)$ *the subgroup generated by* $\mathcal{P} \cup \{1_A\}$. *Let* $\kappa : G \to \mathrm{K}_0(C)$ *be a positive homomorphism with* $\kappa([1_A]) = [1_C]$, *where* $C = \mathrm{C}([0,1], Q)$, *and let* $\lambda : \mathrm{T}(C) \to \mathrm{T}(A)$ *be a continuous affine map such that*

$$\left|\tau\big(\kappa\big([p]\big)\big) - \lambda(\tau)(p)\right| < \delta, \quad p \in \mathcal{P},\ \tau \in \mathrm{T}(C). \tag{4.9}$$

(In particular, this entails that $\mathrm{T}(A) \neq \emptyset$.*) Then, there is a unital* $\mathcal{F}$*-*ε*-multiplicative completely positive map* $L : A \to C$ *such that*

$$\left|\tau \circ L(a) - \lambda(\tau)(a)\right| < \sigma, \quad a \in \mathcal{F},\ \tau \in \mathrm{T}(C). \tag{4.10}$$

Proof. Let ε, σ, and $\mathcal{F}$ be given. We may assume that every element of $\mathcal{F}$ has norm at most one.

Let δ_1 (in place of δ), $\mathcal{G}$, and $\mathcal{P}$ be as assured by Lemma 4.2 for $\mathcal{F}$ and ε. We may assume that $\mathcal{F} \cup \mathcal{P} \subseteq \mathcal{G}$.

Adjoining 1_A to $\mathcal{P}$, write

$$\mathcal{P} = \{1_A, p_1, p_2, \ldots, p_s\}.$$

Deleting one or more of $p_1, p_2, \ldots, p_s$ (but not 1_A), we may assume that the set

$$\big\{[1_A], [p_1], \ldots, [p_s]\big\}$$

is $\mathbb{Q}$-linearly independent. (Since $A \cong A \otimes Q$, we have $K_0(A) \cong K_0(A) \otimes \mathbb{Q}$.)

Let $\delta_2 > 0$ (in place of δ) be as assured by Lemma 3.1 for $\varepsilon_1 = \delta_1$, $\varepsilon_2 = \sigma/4$, $\mathcal{G}$, and $\{p_1, p_2, \ldots, p_s\}$.

Put $\delta = \min\{\delta_1, \delta_2/8, 1/4\}$, and let us show that $\mathcal{P}$ and δ are as desired.

Let κ and λ be given satisfying (4.9).

Let $\lambda_* : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(C))$ be defined by $\lambda_*(f)(\tau) = f(\lambda(\tau))$ for all $f \in \mathrm{Aff}(\mathrm{T}(A))$ and $\tau \in \mathrm{T}(C)$. Identify $\partial_e(\mathrm{T}(C))$ with $[0,1]$ (that is, identify $\mathrm{tr} \circ \pi_t$ with t, where $\pi_t : C = \mathrm{C}([0,1], Q) \to Q$ is the point evaluation at $t \in [0,1]$), and put $\eta = \min\{\delta, \sigma/12\}$. Choose a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

of the interval $[0,1]$ such that

$$\left|\lambda_*(\hat{g})(t_j) - \lambda_*(\hat{g})(t_{j-1})\right| < \eta, \quad g \in \mathcal{G},\ j = 1, 2, \ldots, n. \tag{4.11}$$

(Here, recall that $\hat{g} \in \mathrm{Aff}(\mathrm{T}(A))$ is given by $\hat{g}(\tau) = \tau(g)$ for any $\tau \in \mathrm{T}(A)$.)

Since $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$, there are unital $\mathcal{G}$-δ-multiplicative completely positive maps $\Psi_j : A \to Q$, $j = 0, 1, 2, \ldots, n$, such that

$$\left| \operatorname{tr} \circ \Psi_j(g) - \lambda_*(\hat{g})(t_j) \right| < \eta, \quad g \in \mathcal{G}. \tag{4.12}$$

It then follows from (2.1), (4.12), and (4.9) that, for each $i = 1, 2, \ldots, s$ and each $j = 1, 2, \ldots, n$,

$$\begin{aligned} &\left| \operatorname{tr}\left(\left[\Psi_j(p_i)\right]\right) - \operatorname{tr}\left(\left[\Psi_0(p_i)\right]\right) \right| \\ &\quad < 4\delta + 2\eta + |\lambda_*(\hat{p}_i)(t_j) - \lambda_*(\hat{p}_i)(t_0)| \\ &\quad < 4\delta + 2\eta + 2\delta + \left| \operatorname{tr} \circ \pi_{t_j}\left(\kappa\left([p_i]\right)\right) - \operatorname{tr} \circ \pi_0\left(\kappa\left([p_i]\right)\right) \right| \\ &\quad = 2\eta + 6\delta \le 8\delta \le \delta_2. \end{aligned} \tag{4.13}$$

(Here, as before, π_t is the point evaluation at $t \in [0, 1]$.) We also have, by (4.11) and (4.12), that

$$\left| \operatorname{tr}\left(\Psi_j(g)\right) - \operatorname{tr}\left(\Psi_{j+1}(g)\right) \right| < 3\eta, \quad g \in \mathcal{G}, \ j = 1, 2, \ldots, n. \tag{4.14}$$

Consider the differences

$$r_{i,j} := \operatorname{tr}\left(\left[\Psi_j(p_i)\right]\right) - \operatorname{tr}\left(\left[\Psi_0(p_i)\right]\right), \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n. \tag{4.15}$$

By (4.13), $r_{i,j} < \delta_2$. Applying Lemma 3.1, we obtain a projection $e \in Q$ with $\operatorname{tr}(e) < \sigma/4$ and $\mathcal{G}$-δ_1-multiplicative unital completely positive maps $\psi_0, \psi_j : A \to eQe$, $j = 1, 2, \ldots, n$, such that

$$\left[\psi_0(p_i)\right] - \left[\psi_j(p_i)\right] = r_{i,j}, \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n. \tag{4.16}$$

Consider the direct sum maps

$$\Phi'_j := \psi_j \oplus \Psi_j : A \to (e \oplus 1)\mathrm{M}_2(Q)(e \oplus 1), \quad j = 0, 1, 2, \ldots, n.$$

Since $\delta \le \delta_1$, these are $\mathcal{G}$-δ_1-multiplicative. By (4.15) and (4.16),

$$\left[\Phi'_j(p_i)\right] = \left[\Phi'_0(p_i)\right], \quad i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, n. \tag{4.17}$$

Define $s : \mathbb{Q} \to \mathbb{Q}$ by $s(x) = x/(1 + \operatorname{tr}(e))$, $x \in \mathbb{Q}$. Choose a (unital) isomorphism

$$S : (e \oplus 1)\mathrm{M}_2(Q)(e \oplus 1) \to Q$$

such that $S_{*0} = s$.

Consider the composed maps, still $\mathcal{G}$-δ_1-multiplicative, and now unital,

$$\Phi_j := S \circ \Phi'_j : A \to Q, \quad j = 0, 1, 2, \ldots, n.$$

By (4.17),

$$[\Phi_j]|_{\mathcal{P}} = [\Phi_{j-1}]|_{\mathcal{P}}, \quad j = 1, 2, \ldots, n,$$

and by (4.14) and the fact that $\mathrm{tr}(e) < \sigma/4$,

$$\left|\mathrm{tr}\circ\Phi_j(a) - \mathrm{tr}\circ\Phi_{j-1}(a)\right| < 3\eta + \sigma/4 \leq \sigma/2, \quad a \in \mathcal{F},\ j = 1,2,\ldots,n. \tag{4.18}$$

It follows by Lemma 4.2, applied successively for $j=1,2,\ldots,n$ (to the pairs (Φ_0,Φ_1), $(\mathrm{Ad}\,u_1\circ\Phi_1, \mathrm{Ad}\,u_1\circ\Phi_2),\ldots,(\mathrm{Ad}\,u_{n-1}\circ\cdots\circ\mathrm{Ad}\,u_1\circ\Phi_{n-1}, \mathrm{Ad}\,u_{n-1}\circ\cdots\circ\mathrm{Ad}\,u_1\circ\Phi_n)$), that there are, for each j, a unitary $u_j \in Q$ and a unital $\mathcal{F}$-ε-multiplicative completely positive map $L_j : A \to \mathrm{C}([t_{j-1}, t_j], Q)$ such that

$$\pi_0 \circ L_1 = \Phi_0, \quad \pi_{t_1} \circ L_1 = \mathrm{Ad}\,u_1 \circ \Phi_1, \tag{4.19}$$

and

$$\begin{gathered}\pi_{t_{j-1}} \circ L_j = \pi_{t_{j-1}} \circ L_{j-1},\\ \pi_{t_j} \circ L_j = \mathrm{Ad}\,u_j \circ\cdots\circ \mathrm{Ad}\,u_1 \circ \Phi_j,\\ j = 2,3,\ldots,n.\end{gathered} \tag{4.20}$$

Furthermore, applying the "moreover" part of Lemma 4.2, in view of (4.18), we may choose the maps L_j such that

$$\left|\mathrm{tr}\circ\pi_t \circ L_j(a) - \lambda(\mathrm{tr}\circ\pi_t)(a)\right| < \sigma, \quad t \in [t_{j-1}, t_j],\ a \in \mathcal{F},\ j = 1,2,\ldots,n. \tag{4.21}$$

Define $L : A \to \mathrm{C}([0,1], Q)$ by

$$\pi_t \circ L = \pi_t \circ L_j, \quad t \in [t_{j-1}, t_j],\ j = 1,2,\ldots,n.$$

Since L_j, $j = 1,2,\ldots,n$, are $\mathcal{F}$-ε-multiplicative (use (4.19) and (4.20)), we have that L is a unital $\mathcal{F}$-ε-multiplicative completely positive map $A \to \mathrm{C}([0,1], Q)$. (Note that the construction of the map L is different from—is based on—the construction of the map L in the proof of Lemma 4.2.) It follows from (4.21) that L satisfies (4.10), as desired. ■

Theorem 4.4. *Let A be a unital simple separable* C*-*algebra with finite nuclear dimension. Assume that* $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A) \neq \emptyset$ *and that A satisfies the UCT. Then,* $\mathrm{gTR}(A\otimes Q) \leq 1$, *and so (if A is not elementary), $A \in \mathcal{N}_1$.*

Proof. Since A is simple, the assumption $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A) \neq \emptyset$ immediately implies that A is both stably finite and quasidiagonal. Since A is unital, simple, separable, and with finite nuclear dimension, by [39], it is $\mathcal{Z}$-stable. By the definition of $\mathcal{N}_1$, it remains to show that $\mathrm{gTR}(A \otimes Q) \leq 1$. To prove that $\mathrm{gTR}(A \otimes Q) \leq 1$, we may assume that $A \otimes Q \cong A$. With this assumption, by [33], A has stable rank one.

By [8] (see also [15, Corollary 13.51]), together with the assumption $A \cong A \otimes Q$, there is a unital simple C*-algebra $C = \lim_{n\to\infty}(C_n, \iota_n)$, where each C_n is the tensor product of a C*-algebra in $\mathcal{C}_0$ with Q and ι_n is injective, such that

$$\big(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, [1_A]_0, \mathrm{T}(A), r_A\big) \cong \big(\mathrm{K}_0(C), \mathrm{K}_0(C)_+, [1_C]_0, \mathrm{T}(C), r_C\big).$$

Choose an isomorphism Γ as above, and write Γ_{Aff} for the corresponding map from $\mathrm{Aff}(\mathrm{T}(A))$ to $\mathrm{Aff}(\mathrm{T}(C))$.

Let a finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ be given.

Let the finite set $\mathcal{P}$ of projections in A, the finite subset $\mathcal{G}$ of A, and $\delta > 0$ be as assured by Lemma 4.2 for $\mathcal{F}$ and ε. We may assume that $1_A \in \mathcal{P}$. Write $\mathcal{P} = \{1_A, p_1, p_2, \dots, p_s\}$. Deleting some elements (but not 1_A), we may assume that the set

$$\{[1_A], [p_1], [p_2], \dots, [p_s]\} \subseteq \mathrm{K}_0(A)$$

is $\mathbb{Q}$-linearly independent. (Recall $K_0(A) \cong K_0(A) \otimes \mathbb{Q}$ as $A \cong A \otimes Q$.)

We may also assume, without loss of generality, that $\mathcal{F} \cup \mathcal{P} \subseteq \mathcal{G}$, $\delta \le \varepsilon$, and every element of $\mathcal{G}$ has norm at most one.

Let $\sigma > 0$. Let $\delta_1 > 0$ (in place of δ) be as assured by Lemma 3.3 for $\mathcal{G}$, δ (in place of ε_1), and $\sigma/64$ (in place of ε_2). We may assume that $\delta_1 \le 8\delta$.

Let $\delta_3 > 0$ (in place of δ) be as assured by Lemma 3.4 for $\mathcal{G}$, $\delta_1/8$ (in place of ε_1), and $\min\{\delta_1/32, \sigma/256\}$ (in place of ε_2).

Let $\mathcal{P}_1$ (in place of $\mathcal{P}$) and $\delta_2 > 0$ (in place of δ) be as assured by Lemma 4.3 for $\delta_1/8$ (in place of ε), $\min\{\delta_1/32, \sigma/256\}$ (in place of σ), and $\mathcal{G}$ (in place of $\mathcal{F}$). Replacing $\mathcal{P}$ and $\mathcal{P}_1$ by their union, we may assume that $\mathcal{P} = \mathcal{P}_1$.

By [11, Lemma 2.9], there is a unital positive linear map

$$\gamma : \mathrm{Aff}\big(\mathrm{T}(A)\big) \to \mathrm{Aff}\big(\mathrm{T}(C_{n_1})\big)$$

for some $n_1 \ge 1$ such that

$$\big\|(\imath_{n_1,\infty})_{\mathrm{Aff}} \circ \gamma(\hat{f}) - \Gamma_{\mathrm{Aff}}(\hat{f})\big\| < \min\{77\sigma/128, \delta_2, \delta_3/2\}, \quad f \in \mathcal{F} \cup \mathcal{P}. \tag{4.22}$$

We may assume, without loss of generality, that there are projections $p'_1, p'_2, \dots, p'_s \in C_{n_1}$ such that $\Gamma([p_i]) = \imath_{n_1,\infty}([p'_i])$, $i = 1, 2, \dots, s$. To simplify notation, assume that $n_1 = 1$. Let G_0 denote the subgroup of $\mathrm{K}_0(A)$ generated by $\mathcal{P}$. Since G_0 is generated freely by $[1_A]$, p_i, $i = 1, 2, \dots, s$, we can define $\Gamma' : G_0 \to \mathrm{K}_0(C_1)$ by

$$\Gamma'\big([1_A]\big) = [1_{C_1}], \quad \Gamma'\big([p_i]\big) = [p'_i], \quad i = 1, 2, \dots, s. \tag{4.23}$$

Hence,

$$(\imath_{1,\infty})_{*0} \circ \Gamma' = \Gamma|_{G_0}.$$

Since the pair $(\Gamma_{\mathrm{Aff}}, \Gamma|_{\mathrm{K}_0(A)})$ is compatible, as a consequence of (4.22) and (4.23), we have

$$\big\|\widehat{p'_i} - \gamma(\widehat{p_i})\big\|_\infty < \min\{\delta_2, \delta_3/2\}, \quad i = 1, 2, \dots, s. \tag{4.24}$$

Write

$$\begin{aligned} C_1 &= (\psi_0, \psi_1, Q^r, Q^l) \\ &= \big\{(f, a) \in \mathrm{C}\big([0,1], Q^r\big) \oplus Q^l : f(0) = \psi_0(a) \text{ and } f(1) = \psi_1(a)\big\}, \end{aligned}$$

where $\psi_0, \psi_1 : Q^l \to Q^r$ are unital homomorphisms.

Denote by

$$\pi_{\mathrm{e}} : C_1 \to Q^l, \quad (f, a) \mapsto a$$

the canonical quotient map and by $j : C_1 \to \mathrm{C}([0,1], Q^r)$ the canonical map

$$j\big((f,a)\big) = f.$$

Denote by

$$\gamma^* : \mathrm{T}(C_1) \to \mathrm{T}(A)$$

the continuous affine map dual to γ. Denote by $\theta_1, \theta_2, \ldots, \theta_l$ the extreme tracial states of C_1 factoring through $\pi_{\mathrm{e}} : C_1 \to Q^l$.

By the assumption $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$, there exists a unital $\mathcal{G}$-$\min\{\delta_1/8, \delta_3/8\}$-multiplicative completely positive map $\Phi : A \to Q^l$ such that

$$\left|\mathrm{tr}_j \circ \Phi(a) - \gamma^*(\theta_j)(a)\right| < \min\{13\delta_1/32, \delta_3/4, \sigma/32\}, \quad a \in \mathcal{G},\ j = 1, 2, \ldots, l, \tag{4.25}$$

where tr_j is the tracial state supported on the jth direct summand of Q^l. Moreover, since $\mathcal{P} \subseteq \mathcal{G}$, as in (2.1), we also have that

$$\left|\mathrm{tr}_j\left(\left[\Phi(p_i)\right]\right) - \mathrm{tr}_j\left(\Phi(p_i)\right)\right| < \delta_3/4, \quad i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, l. \tag{4.26}$$

Set

$$\mathbb{D} := (\pi_{\mathrm{e}})_{*0}\big(\mathrm{K}_0(C_1)\big) = \ker\big((\psi_0)_{*0} - (\psi_1)_{*0}\big) \subseteq \mathbb{Q}^l.$$

It follows from (4.25) that

$$\left|\tau\big(\Phi(a)\big) - (\pi_{\mathrm{e}})_{\mathrm{Aff}}\big(\gamma(\hat{a})\big)(\tau)\right| < \min\{13\delta_1/32, \delta_3/4, \sigma/32\}, \quad a \in \mathcal{G},\ \tau \in \mathrm{T}(Q^l), \tag{4.27}$$

where

$$(\pi_{\mathrm{e}})_{\mathrm{Aff}} : \mathrm{Aff}\big(\mathrm{T}(C_1)\big) \to \mathrm{Aff}\big(\mathrm{T}(Q^l)\big)$$

is the map induced by π_{e}. By (4.27) for $a \in \mathcal{P} \subseteq \mathcal{G}$, together with (4.26) and (4.24),

$$\left|\tau\big(\left[\Phi(p_i)\right]\big) - \tau \circ (\pi_{\mathrm{e}})_{*0} \circ \Gamma'\big([p_i]\big)\right| < \delta_3, \quad \tau \in \mathrm{T}(Q^l),\ i = 1, 2, \ldots, s.$$

Therefore, applying Lemma 3.4, with $r_i = [\Phi(p_i)] - (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i])$, we obtain $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive maps $\Sigma_1, \mu_1 : A \to Q^l$, with $\Sigma_1(1_A) = \mu_1(1_A)$ a projection, such that

$$\tau\big(\Sigma_1(1_A)\big) < \min\{\delta_1/32, \sigma/256\}, \quad \tau \in \mathrm{T}(Q^l), \tag{4.28}$$

$$\left[\Sigma_1(\mathcal{P})\right] \subseteq \mathbb{D}, \tag{4.29}$$

$$\left[\Sigma_1(p_i)\right] - \left[\mu_1(p_i)\right] = \left[\Phi(p_i)\right] - (\pi_{\mathrm{e}})_{*0} \circ \Gamma'\big([p_i]\big), \quad i = 1, 2, \ldots, s. \tag{4.30}$$

Consider the (unital) direct sum map

$$\Phi' := \Phi \oplus \mu_1 : A \to \big(1 \oplus \Sigma_1(1_A)\big)\mathrm{M}_2(Q^l)\big(1 \oplus \Sigma_1(1_A)\big). \tag{4.31}$$

Note that Φ', like μ_1 and Φ, is $\mathcal{G}$-$\delta_1/8$-multiplicative. It follows from (4.30) that for each $i = 1, 2, \ldots, s$,

$$\begin{aligned}(\psi_0)_{*0}\big([\Phi'(p_i)]\big) &= (\psi_0)_{*0}\big([\mu_1(p_i)] + [\Phi(p_i)]\big)\\ &= (\psi_0)_{*0}\big([\Sigma_1(p_i)] + (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i])\big), \end{aligned}\tag{4.32}$$

$$\begin{aligned}(\psi_1)_{*0}\big([\Phi'(p_i)]\big) &= (\psi_1)_{*0}\big([\mu_1(p_i)] + [\Phi(p_i)]\big)\\ &= (\psi_1)_{*0}\big([\Sigma_1(p_i)] + (\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i])\big). \end{aligned}\tag{4.33}$$

It follows from (4.32) and (4.33), in view of (4.29) and the fact (use (4.23)) that $(\pi_{\mathrm{e}})_{*0} \circ \Gamma'([p_i]) \in (\pi_{\mathrm{e}})_{*0}(\mathrm{K}_0(C_1)) = \mathbb{D}$, that $[\Phi'(p_i)] \in \mathbb{D}$, $i = 1, 2, \ldots, s$, i.e.,

$$(\psi_0)_{*0}\big([\Phi'(p_i)]\big) = (\psi_1)_{*0}\big([\Phi'(p_i)]\big), \quad i = 1, 2, \ldots, s. \tag{4.34}$$

Set $B = \mathrm{C}([0,1], Q^r)$, and (as before) write $\pi_t : B \to Q^r$ for the point evaluation at $t \in [0,1]$. Since $1_A \in \mathcal{P}$, by (4.29), $[\Sigma_1(1_A)] \in \mathbb{D}$, and so there is a projection $e_0 \in B$ such that $\pi_0(e_0) = \psi_0(\Sigma_1(1_A))$ and $\pi_1(e_0) = \psi_1(\Sigma_1(1_A))$. It then follows from (4.28) (applied just for τ factoring through ψ_0—alternatively, for τ factoring through ψ_1) that

$$\tau(e_0) < \min\{\delta_1/32, \sigma/256\}, \quad \tau \in \mathrm{T}(B). \tag{4.35}$$

Let $j^* : \mathrm{T}(B) \to \mathrm{T}(C_1)$ denote the continuous affine map dual to the canonical unital map $j : C_1 \to B$. Let $\gamma_1 : \mathrm{T}(B) \to \mathrm{T}(A)$ be defined by $\gamma_1 := \gamma^* \circ j^*$, and let $\kappa : G_0 \to \mathrm{K}_0(B)$ be defined by $\kappa := j_{*0} \circ \Gamma'$. Then, by (4.23) and (4.24),

$$\begin{aligned}\big|\tau\big(\kappa([p_i])\big) - \gamma_1(\tau)(p_i)\big| &= \big|\tau\big(j_{*0}\big(\Gamma'([p_i])\big)\big) - (\gamma^* \circ j^*)(\tau)(p_i)\big|\\ &= \big|j^*(\tau)\big([p_i']\big) - \gamma(\widehat{p_i})\big(j^*(\tau)\big)\big| < \delta_2 \end{aligned}\tag{4.36}$$

for all $\tau \in \mathrm{T}(B)$, $i = 1, 2, \ldots, s$.

The estimate (4.36) ensures that we can apply Lemma 4.3 with κ and γ_1 (note that $\Gamma'([1_A]) = [1_{C_1}]$ and hence $\kappa([1_A]) = [1_B]$) to obtain a unital $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive map $\Psi' : A \to B$ such that

$$\big|\tau \circ \Psi'(a) - \gamma_1(\tau)(a)\big| < \min\{\delta_1/32, \sigma/256\}, \quad a \in \mathcal{G},\ \tau \in \mathrm{T}(B). \tag{4.37}$$

Amplifying Ψ' slightly (by first identifying Q^r with $Q^r \otimes Q$ and then considering

$$H_0(f)(t) = f(t) \otimes \big(1 + e_0(t)\big)$$

for $t \in [0,1]$), we obtain a unital $\mathcal{G}$-$\delta_1/8$-multiplicative completely positive map $\Psi : A \to (1 \oplus e_0)\mathrm{M}_2(B)(1 \oplus e_0)$ such that (by (4.37) and (4.35))

$$\big|\tau \circ \Psi(a) - \gamma_1(\tau)(a)\big| < 2\min\{\delta_1/32, \sigma/256\} = \min\{\delta_1/16, \sigma/128\}, \tag{4.38}$$

for all $a \in \mathcal{G}$ and $\tau \in \mathrm{T}(B)$. Note that for any element $a \in C_1$,

$$\tau\big(\psi_0\big(\pi_{\mathrm{e}}(a)\big)\big) = \tau\big(\pi_0\big(j(a)\big)\big), \quad \tau\big(\psi_1\big(\pi_{\mathrm{e}}(a)\big)\big) = \tau\big(\pi_1\big(j(a)\big)\big), \quad \tau \in \mathrm{T}(Q^r). \tag{4.39}$$

(Recall that $j : C_1 \to B$ is the canonical map.) Therefore (by (4.39)), for any $a \in \mathcal{G}$ and $\tau \in \mathrm{T}(Q^r)$,

$$\begin{aligned}\big|\tau\big(\psi_0\big(\Phi(a)\big)\big) - \gamma(\hat{a})(\tau \circ \pi_0 \circ j)\big| &= \big|\tau\big(\psi_0\big(\Phi(a)\big)\big) - \gamma(\hat{a})(\tau \circ \psi_0 \circ \pi_{\mathrm{e}})\big| \\ &= \big|\tau \circ \psi_0\big(\Phi(a)\big) - (\pi_{\mathrm{e}})_{\mathrm{Aff}}\big(\gamma(\hat{a})\big)(\tau \circ \psi_0)\big| \\ &< \min\{13\delta_1/32, \sigma/32\} \quad \text{(by (4.27))}. \qquad (4.40)\end{aligned}$$

The same argument shows that

$$\big|\tau\big(\psi_1\big(\Phi(a)\big)\big) - \gamma(\hat{a})(\tau \circ \pi_1 \circ j)\big| < \min\{13\delta_1/32, \sigma/32\}, \quad a \in \mathcal{G},\ \tau \in \mathrm{T}(Q^r). \qquad (4.41)$$

Then, for any $\tau \in \mathrm{T}(Q^r)$ and any $a \in \mathcal{G}$, we have

$$\begin{aligned}&\big|\tau \circ \psi_0 \circ \Phi'(a) - \tau \circ \pi_0 \circ \Psi(a)\big| \\ &= \big|\tau \circ \psi_0\big(\Phi(a) \oplus \mu_1(a)\big) - \tau \circ \pi_0 \circ \Psi(a)\big| \\ &< \big|\tau \circ \psi_0\big(\Phi(a) \oplus \mu_1(a)\big) - \gamma_1(\tau \circ \pi_0)(a)\big| + \min\{\delta_1/16, \sigma/128\} \quad \text{(by (4.38))} \\ &< \big|\tau \circ \psi_0\big(\Phi(a)\big) - \gamma_1(\tau \circ \pi_0)(a)\big| + \min\{3\delta_1/32, 3\sigma/256\} \quad \text{(by (4.28))} \\ &= \big|\tau \circ \psi_0\big(\Phi(a)\big) - \gamma(\hat{a})(\tau \circ \pi_0 \circ j)\big| + \min\{3\delta_1/32, 3\sigma/256\} \\ &< \min\{13\delta_1/32, \sigma/32\} + \min\{3\delta_1/32, 3\sigma/256\} \\ &\le 13\delta_1/32 + 3\delta_1/32 = \delta_1/2 \quad \text{(by (4.40))}. \qquad (4.42)\end{aligned}$$

The same argument, using (4.41) instead of (4.40), shows that

$$\begin{aligned}\big|\tau \circ \psi_1 \circ \Phi'(a) - \tau \circ \pi_1 \circ \Psi(a)\big| &< \min\{13\delta_1/32, \sigma/32\} + \min\{3\delta_1/32, 3\sigma/256\} \\ &\le \delta_1/2, \quad \tau \in \mathrm{T}(Q^r),\ a \in \mathcal{G}. \qquad (4.43)\end{aligned}$$

((4.42) and (4.43)—the σ estimates—will be used later to verify (4.51) and (4.52).)

Noting that Ψ and Φ' are $\delta_1/8$-multiplicative on $\{1_A, p_1, p_2, \dots, p_s\}$, by our convention (see (2.1)), we have, for all $\tau \in \mathrm{T}(Q^r)$,

$$\begin{aligned}\big|\tau\big(\big[\pi_t \circ \Psi(p_i)\big]\big) - \tau\big(\pi_t \circ \Psi(p_i)\big)\big| &< \delta_1/4, \\ \big|\tau\big(\big[\psi_j \circ \Phi'(p_i)\big]\big) - \tau\big(\psi_j \circ \Phi'(p_i)\big)\big| &< \delta_1/4,\end{aligned}$$

$i = 1, 2, \dots, s$, where $t \in [0, 1]$ and $j = 0, 1$.

Combining these inequalities with (4.42) and (4.43), we have

$$\big|\tau\big(\big[\pi_0 \circ \Psi(p_i)\big]\big) - \tau\big(\big[\psi_0 \circ \Phi'(p_i)\big]\big)\big| < \delta_1, \quad i = 1, 2, \dots, s,\ \tau \in \mathrm{T}(Q^r). \qquad (4.44)$$

Therefore (in view of (4.44)), applying Lemma 3.3 with $r_i = [\pi_0 \circ \Psi(p_i)] - [\psi_0 \circ \Phi'(p_i)]$, we obtain $\mathcal{G}$-δ-multiplicative completely positive maps $\Sigma_2 : A \to Q^l$ and $\mu_2 : A \to Q^r$, taking 1_A into projections, such that

$$\big[\psi_k \circ \Sigma_2(1_A)\big] = \big[\mu_2(1_A)\big], \quad k = 0, 1, \qquad (4.45)$$

$$\big[\Sigma_2(\mathcal{P})\big] \subseteq (\pi_{\mathrm{e}})_{*0}\big(\mathrm{K}_0(C_1)\big) = \mathbb{D}, \qquad (4.46)$$

$$\tau\big(\Sigma_2(1_A)\big) < \sigma/64, \quad \tau \in \mathrm{T}(Q^l), \qquad (4.47)$$

and, taking into account (4.46),

$$\begin{aligned}[\psi_0\circ\Sigma_2(p_i)]-[\mu_2(p_i)]&=[\psi_1\circ\Sigma_2(p_i)]-[\mu_2(p_i)]\\&=[\pi_0\circ\Psi(p_i)]-[\psi_0\circ\Phi'(p_i)],\end{aligned}\tag{4.48}$$

where $i=1,2,\ldots,s$. By (4.45), there are unitaries $w_k\in Q^r$, $k=0,1$ such that

$$\psi_k\circ\Sigma_2(1_A)=\operatorname{Ad}w_k\circ\mu_2(1_A),\quad k=0,1.$$

Let $\{w(t)\}_{0\le t\le 1}$ be a continuous path of unitaries in Q^r such that

$$w(0)=w_0\quad\text{and}\quad w(1)=w_1.$$

Consider the four $\mathcal{G}$-δ-multiplicative direct sum maps (note that Φ' and Ψ are $\mathcal{G}$-$\delta_1/8$-multiplicative, and $\delta_1\le 8\delta$), from A to $\mathrm{M}_3(Q^r)$,

$$\begin{aligned}\Phi_0&:=(\psi_0\circ\Phi')\oplus(\psi_0\circ\Sigma_2),&\Phi_1&:=(\psi_1\circ\Phi')\oplus(\psi_1\circ\Sigma_2),\\\Psi_0&:=(\pi_0\circ\Psi)\oplus\operatorname{Ad}w_0\circ\mu_2,&\Psi_1&:=(\pi_1\circ\Psi)\oplus\operatorname{Ad}w_1\circ\mu_2.\end{aligned}\tag{4.49}$$

We then have that for each $i=1,2,\ldots,s$,

$$\begin{aligned}&[\Psi_0(p_i)]-[\Phi_0(p_i)]\\&\quad=\big([(\pi_0\circ\Psi)(p_i)]+[\mu_2(p_i)]\big)-\big([(\psi_0\circ\Phi')(p_i)]+[(\psi_0\circ\Sigma_2)(p_i)]\big)\\&\quad=\big([(\pi_0\circ\Psi)(p_i)]-[(\psi_0\circ\Phi')(p_i)]\big)-\big([(\psi_0\circ\Sigma_2)(p_i)]-[\mu_2(p_i)]\big)\\&\quad=0\quad(\text{by (4.48)}),\end{aligned}$$

and

$$\begin{aligned}&[\Psi_1(p_i)]-[\Phi_1(p_i)]\\&=\big([(\pi_1\circ\Psi)(p_i)]+[\mu_2(p_i)]\big)-\big([(\psi_1\circ\Phi')(p_i)]+[(\psi_1\circ\Sigma_2)(p_i)]\big)\\&=\big([(\pi_1\circ\Psi)(p_i)]-[(\psi_1\circ\Phi')(p_i)]\big)-\big([(\psi_1\circ\Sigma_2)(p_i)]-[\mu_2(p_i)]\big)\\&=\big([(\pi_0\circ\Psi)(p_i)]-[(\psi_1\circ\Phi')(p_i)]\big)-\big([(\psi_1\circ\Sigma_2)(p_i)]-[\mu_2(p_i)]\big)\\&\hspace{20em}(\pi_0\text{ and }\pi_1\text{ are homotopic})\\&=\big([(\pi_0\circ\Psi)(p_i)]-[(\psi_0\circ\Phi')(p_i)]\big)-\big([(\psi_1\circ\Sigma_2)(p_i)]-[\mu_2(p_i)]\big)=0\quad(\text{by (4.34)})\\&=0\quad(\text{by (4.48)}).\end{aligned}$$

Summarizing the calculations in the preceding paragraph, we have

$$[\Phi_i]\big|_{\mathcal{P}}=[\Psi_i]\big|_{\mathcal{P}},\quad i=0,1.\tag{4.50}$$

On the other hand, for any $a\in\mathcal{F}\subseteq\mathcal{G}$ and any $\tau\in\mathrm{T}(Q^r)$, we have

$$\begin{aligned}&\big|\tau\big(\Phi_0(a)\big)-\tau\big(\Psi_0(a)\big)\big|\\&\quad=\big|\tau\big((\psi_0\circ\Phi')(a)\oplus(\psi_0\circ\Sigma_2)(a)\big)-\tau\big((\pi_0\circ\Psi)(a)\oplus\mu_2(a)\big)\big|\\&\quad<\big|\tau\big((\psi_0\circ\Phi')(a)\big)-\tau\big((\pi_0\circ\Psi)(a)\big)\big|+\sigma/32\quad(\text{by (4.47)})\\&\quad<\min\{13\delta_1/16,\sigma/32\}+\min\{3\delta_1/16,3\sigma/256\}+\sigma/32\quad(\text{by (4.42)})\\&\quad\le 5\sigma/64.\end{aligned}\tag{4.51}$$

The same argument, using (4.43) instead of (4.42), also shows that

$$\left|\tau\big(\Phi_1(a)\big) - \tau\big(\Psi_1(a)\big)\right| < 5\sigma/64, \quad a \in \mathcal{F},\ \tau \in \mathrm{T}(Q^r). \tag{4.52}$$

Since $1_A \in \mathcal{P}$, by (4.46), $[\Sigma_2(1_A)] \in \mathbb{D}$, and so there is a projection $e_1 \in B$ such that $\pi_0(e_1) = \psi_0(\Sigma_2(1_A))$ and $\pi_1(e_1) = \psi_1(\Sigma_2(1_A))$. From the construction,

$$\Psi_i(1_A) = \Phi_i(1_A) = 1 \oplus \pi_i(e_0) \oplus \pi_i(e_1), \quad i = 0, 1.$$

It then follows from (4.47) (applied just for τ factoring through ψ_0—alternatively, for τ factoring through ψ_1) that

$$\tau(e_1) < \sigma/64, \quad \tau \in \mathrm{T}(B). \tag{4.53}$$

Set $E_0' = 1 \oplus \pi_0(e_0) \oplus \pi_0(e_1)$, $E_1' = 1 \oplus \pi_1(e_0) \oplus \pi_1(e_1)$, and $D_0 = E_0'\mathrm{M}_3(Q^r)E_0'$, $D_1 = E_1'\mathrm{M}_3(Q^r)E_1'$.

Pick a sufficiently small $r' \in (0, 1/4)$ such that

$$\left\|\Psi(a)\big((1+2r')t - r'\big) - \Psi(a)(t)\right\| < \sigma/64, \quad a \in \mathcal{G},\ t \in \left[\frac{r'}{1+2r'}, \frac{1+r'}{1+2r'}\right]. \tag{4.54}$$

It follows from Lemma 4.2 (in view of (4.50), (4.51), and (4.52)) that there exist unitaries $u_0 \in D_0$ and $u_1 \in D_1$ and unital $\mathcal{F}$-ε-multiplicative completely positive maps $L_0 : A \to \mathrm{C}([-r', 0], D_0)$ and $L_1 : A \to \mathrm{C}([1, 1+r'], D_1)$, such that

$$\pi_{-r'} \circ L_0 = \Phi_0, \quad \pi_0 \circ L_0 = \mathrm{Ad}\, u_0 \circ \Psi_0, \tag{4.55}$$

$$\pi_{1+r'} \circ L_1 = \Phi_1, \quad \pi_1 \circ L_1 = \mathrm{Ad}\, u_1 \circ \Psi_1, \tag{4.56}$$

$$\left|\tau \circ \pi_t \circ L_0(a) - \tau \circ \pi_0 \circ L_0(a)\right| < 5\sigma/32, \quad t \in [-r', 0], \tag{4.57}$$

$$\left|\tau \circ \pi_t \circ L_1(a) - \tau \circ \pi_1 \circ L_1(a)\right| < 5\sigma/32, \quad t \in [1, 1+r'], \tag{4.58}$$

where $a \in \mathcal{F}$, $\tau \in \mathrm{T}(Q^r)$, and (as before) π_t is the point evaluation at $t \in [-r', 1+r']$.

Write $E_3 = 1 \oplus e_0 \oplus e_1 \in \mathrm{M}_3(\mathrm{C}([0,1], Q^r))$ and $B_1 = E_3(\mathrm{M}_3(\mathrm{C}([0,1], Q^r)))E_3$. There exists a unitary $u \in B_1$ such that $u(0) = u_0$ and $u(1) = u_1$. Consider the projection $E_4 \in \mathrm{M}_3(\mathrm{C}([-r', 1+r'], Q^r))$ defined by $E_4|_{[-r,0]} = E_0'$, $E_4|_{[0,1]} = E_3$, and $E_4|_{[1,1+r]} = E_1'$. Set

$$B_2 = E_4\big(\mathrm{M}_3\big(\mathrm{C}\big([-r', 1+r'], Q^r\big)\big)\big)E_4.$$

Define a unital $\mathcal{F}$-ε-multiplicative (note that $\mathcal{F} \subseteq \mathcal{G}$ and $\delta \leq \varepsilon$) completely positive map $L' : A \to B_2$ by

$$L'(a)(t) = \begin{cases} L_0(a)(t), & t \in [-r', 0), \\ \mathrm{Ad}\, u(t) \circ (\pi_t \circ \Psi \oplus \mathrm{Ad}\, w(t) \circ \mu_2)(a), & t \in [0, 1], \\ L_1(a)(t) & t \in (1, 1+r']. \end{cases} \tag{4.59}$$

Note that for any $a \in \mathcal{G}$, and any $\tau \in \mathrm{T}(Q^r)$, by (4.59), if $t \in [0,1]$, then

$$\begin{aligned}
&\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad = \big|\tau\big(\operatorname{Ad} u(t) \circ \big(\pi_t \circ \Psi \oplus \operatorname{Ad} w(t) \circ \mu_2)(a)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad = \big|\tau\big(\pi_t\big(\Psi(a)\big)\big) + \tau\big(\mu_2(a)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad < \big|\big(\pi_t^*(\tau)\big)\big(\Psi(a)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| + \sigma/64 \quad \text{(by (4.45) and (4.47))} \\
&\quad < \min\{\delta_1/16, \sigma/128\} + \sigma/64 \leq 3\sigma/128 \quad \text{(by (4.38))},
\end{aligned} \tag{4.60}$$

where $\pi_t^* : \mathrm{T}(Q^r) \to \mathrm{T}(B)$ is the dual of $\pi_t : B \to Q^r$. Furthermore, if $t \in [-r', 0]$, then for any $a \in \mathcal{F}$, and any $\tau \in \mathrm{T}(Q^r)$,

$$\begin{aligned}
&\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| \\
&= \big|\tau\big(L_0(a)(t)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| \\
&< \big|\tau\big(L_0(a)(0)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| + 5\sigma/32 \quad \text{(by (4.57))} \\
&= \big|\tau\big(\Psi_0(a)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| + 5\sigma/32 \quad \text{(by (4.55))} \\
&= \big|\tau\big((\pi_0 \circ \Psi)(a) \oplus \operatorname{Ad} w_0 \circ \mu_2(a)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| + 5\sigma/32 \\
&< \big|\tau\big((\pi_0 \circ \Psi)(a) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| + \sigma/64 + 5\sigma/32 \quad \text{(by (4.45) and (4.47))} \\
&< \min\{\delta_1/16, \sigma/128\} + \sigma/64 + 5\sigma/32 < 23\sigma/128 \quad \text{(by (4.38))}.
\end{aligned} \tag{4.61}$$

Again, if $t \in [1, 1+r']$, then the same argument shows that for any $a \in \mathcal{F}$, and any $\tau \in \mathrm{T}(Q^r)$,

$$\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_1^*(\tau)\big)(a)\big| < 23\sigma/128. \tag{4.62}$$

Let us modify L' to a unital map from A to B. First, let us renormalize L'. Consider the isomorphism $\eta : \mathbb{Q}^r \to \mathbb{Q}^r$ defined by

$$\eta(x_1, x_2, \ldots, x_r) = \left(\frac{1}{\mathrm{tr}_1(E_3)} x_1, \frac{1}{\mathrm{tr}_2(E_3)} x_2, \ldots, \frac{1}{\mathrm{tr}_r(E_3)} x_r\right),$$

for all $(x_1, x_2, \ldots, x_r) \in \mathbb{Q}^r$, where (as before) tr_k is the tracial state supported on the kth direct summand of Q^r (recall that E_3 has constant rank on $[0,1]$). Then, there is a (unital) isomorphism $\varphi : B_2 \to \mathrm{C}([-r', 1+r'], Q^r)$ such that $\varphi_{*0} = \eta$. Let us replace the map L' by the map $\varphi \circ L'$, and still denote it by L'. Note that it follows from (4.60), (4.35), and (4.53) that for any $t \in [0,1]$, any $a \in \mathcal{F}$, and any $\tau \in \mathrm{T}(Q^r)$,

$$\begin{aligned}
&\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad < 3\sigma/128 + \sup\big\{\tau(e_0) : \tau \in \mathrm{T}(Q^r)\big\} + \sup\big\{\tau(e_1) : \tau \in \mathrm{T}(Q^r)\big\} \\
&\quad < 3\sigma/128 + \min\{\delta_1/16, \sigma/64\} + \sigma/64 \leq 7\sigma/128.
\end{aligned} \tag{4.63}$$

The same argument, using (4.61) and (4.62) instead of (4.60), shows that for any $a \in \mathcal{F}$ and $\tau \in \mathrm{T}(Q^r)$,

$$\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_0^*(\tau)\big)(a)\big| < 27\sigma/128, \quad t \in [-r', 0], \tag{4.64}$$

$$\big|\tau\big(\pi_t\big(L'(a)\big)\big) - \gamma_1\big(\pi_1^*(\tau)\big)(a)\big| < 27\sigma/128, \quad t \in [1, 1+r']. \tag{4.65}$$

Now, put, for $a \in A$,

$$L''(a)(t) = L'(a)\big((1+2r')t - r'\big), \quad t \in [0,1]. \tag{4.66}$$

This perturbation will not change the trace very much, as for any $a \in \mathcal{F}$ and any $\tau \in \mathrm{T}(Q^r)$, if $t \in [0, r'/(1+2r')]$, then

$$\begin{aligned}
&\big|\tau\big(L''(a)(t)\big) - \tau\big(L'(a)(t)\big)\big| \\
&\quad = \big|\tau\big(L'(a)\big((1+2r')t - r'\big)\big) - \tau\big(L'(a)(t)\big)\big| \quad \text{(by (4.66))} \\
&\quad = \big|\tau\big(L_0(a)\big((1+2r')t - r'\big)\big) - \tau((\Psi(a)(t)) + \mathrm{Ad}\, w(t) \circ \mu_2(a))\big| \\
&\quad < \big|\tau\big(L_0(a)\big((1+2r')t - r'\big)\big) - \tau\big(\Psi(a)(t)\big)\big| + \sigma/64 \quad \text{(by (4.45) and (4.47))} \\
&\quad < \big|\tau\big(L_0(a)(0)\big) - \tau\big(\Psi(a)(t)\big)\big| + 5\sigma/32 + \sigma/64 \quad \text{(by (4.57))} \\
&\quad = \big|\tau\big(\Psi_0(a)\big)\big) - \tau\big(\Psi(a)(t)\big)\big| + 11\sigma/64 \quad \text{(by (4.55))} \\
&\quad < \sigma/64 + 11\sigma/64 = 3\sigma/16 \quad \text{(by (4.45) and (4.47))}.
\end{aligned}$$

Furthermore, the same argument, now using (4.58) and (4.56), shows that for any $a \in \mathcal{F}$, $\tau \in \mathrm{T}(Q^r)$, and $t \in [(1+r')/(1+2r'), 1]$,

$$\big|\tau\big(L'(a)\big((1+2r')t - r'\big)\big) - \tau\big(\Psi(a)(t) + \mathrm{Ad}\, w_1 \circ \mu_2(a)\big)\big| < 3\sigma/16,$$

and if $t \in [r'/(1+2r'), (1+r')/(1+2r')]$, then

$$\begin{aligned}
\big|\tau\big(L'(a)\big((1+2r')t - r'\big)\big) - \tau\big(L'(a)(t)\big)\big| &= \big|\tau\big(\Psi(a)\big((1+2r')t - r'\big) - \Psi(a)(t)\big)\big| \\
&< \sigma/64 \quad \text{(by (4.54))}.
\end{aligned}$$

Thus,

$$\big|\tau\big(L''(a)(t)\big) - \tau\big(L'(a)(t)\big)\big| < 3\sigma/16, \quad a \in \mathcal{F},\ \tau \in \mathrm{T}(Q^r),\ t \in [0,1]. \tag{4.67}$$

Hence,

$$\begin{aligned}
&\big|\tau\big(\pi_t\big(L''(a)\big)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad \le \big|\tau\big(L''(a)(t)\big) - \tau\big(L'(a)(t)\big)\big| + \big|\tau\big(L'(a)(t)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| \\
&\quad < 3\sigma/16 + 27\sigma/128 = 51\sigma/128 \quad \text{(by (4.67), (4.63), (4.64), and (4.65))}.
\end{aligned} \tag{4.68}$$

Note that L'' is a unital map from A to B. It is also $\mathcal{F}$-ε-multiplicative since L' is. Consider the order isomorphism $\eta' : \mathbb{Q}^l \to \mathbb{Q}^l$ defined by

$$\eta'(y_1, y_2, \ldots, y_l) = (a_1 y_1, a_2 y_2, \ldots, a_l y_l), \quad (y_1, y_2, \ldots, y_l) \in \mathbb{Q}^l,$$

where

$$a_j = \frac{1}{\mathrm{tr}_j\,\big(1 \oplus \Sigma_1(1_A) \oplus \Sigma_2(1_A)\big)}, \quad j = 1, 2, \ldots, l, \tag{4.69}$$

and (as before) $\mathrm{tr_j}$ is the tracial state supported on the jth direct summand of Q^l. There exists a unital homomorphism

$$\tilde{\varphi} : \big(1 \oplus \Sigma_1(1_A) \oplus \Sigma_2(1_A)\big)\mathrm{M}_3(Q^l)\big(1 \oplus \Sigma_1(1_A) \oplus \Sigma_2(1_A)\big) \to Q^l$$

such that

$$\tilde{\varphi}_{*0} = \eta'.$$

Therefore, by the constructions of L'', L', L_0, and L_1 ((4.66), (4.59), (4.55), and (4.56)), we may assume that

$$\psi_0 \circ \tilde{\varphi} \circ (\Phi' \oplus \Sigma_2) = \pi_0 \circ L'' \quad \text{and} \quad \psi_1 \circ \tilde{\varphi} \circ (\Phi' \oplus \Sigma_2) = \pi_1 \circ L'', \tag{4.70}$$

replacing L'' by $\mathrm{Ad}\ v \circ L''$ for a suitable unitary v if necessary.

Define $L : A \to C_1$ by $L(a) = (L''(a), \tilde{\varphi}(\Phi'(a) \oplus \Sigma_2(a)))$, an element of C_1 by (4.70). Since L'' and $\tilde{\varphi} \circ (\Phi' \oplus \Sigma_2)$ are unital and $\mathcal{F}$-ε-multiplicative (since Φ' and Σ_2 are $\mathcal{G}$-δ-multiplicative, $\mathcal{F} \subseteq \mathcal{G}$, and $\delta \leq \varepsilon$), so also is L.

Moreover, for any $a \in \mathcal{F}$, any $\tau \in \mathrm{T}(Q^r)$, and any $t \in (0, 1)$, it follows from (4.68) that

$$\big|\tau\big(\pi_t\big(j\big(L(a)\big)\big)\big) - \gamma_1\big(\pi_t^*(\tau)\big)(a)\big| < 51\sigma/128. \tag{4.71}$$

If $\tau \in \mathrm{T}(Q^l)$, then, for any $a \in \mathcal{F}$,

$$\begin{aligned}
&\big|\tau\big(\pi_{\mathrm{e}}\big(L(a)\big)\big) - \gamma^*\big(\pi_{\mathrm{e}}^*(\tau)\big)(a)\big| \\
&\quad = \big|\tau\big(\tilde{\varphi}\big(\Phi'(a) \oplus \Sigma_2(a)\big)\big) - \gamma^*\big(\pi_{\mathrm{e}}^*(\tau)\big)(a)\big| \\
&\quad < \big|\tau\big(\Phi'(a) \oplus \Sigma_2(a)\big) - \gamma^*\big(\pi_{\mathrm{e}}^*(\tau)\big)(a)\big| + \sigma/32 \quad (\text{by } (4.69), (4.35), \text{and } (4.47)) \\
&\quad < \big|\tau\big(\Phi'(a)\big) - \gamma^*\big(\pi_{\mathrm{e}}^*(\tau)\big)(a)\big| + 3\sigma/64 \quad (\text{by } (4.47)) \\
&\quad < \big|\tau\big(\Phi(a)\big) - \gamma^*\big(\pi_{\mathrm{e}}^*(\tau)\big)(a)\big| + \sigma/8 \quad (\text{by } (4.28)) \\
&\quad < \sigma/32 + \sigma/8 < 51\sigma/128 \quad (\text{by } (4.27)).
\end{aligned}$$

Since each extreme trace of C_1 factors through either the evaluation map π_t or the canonical quotient map π_{e}, by (4.71),

$$\big|\tau\big(L(a)\big) - \gamma^*(\tau)(a)\big| < 51\sigma/128, \quad \tau \in \mathrm{T}(C_1),\ a \in \mathcal{F}. \tag{4.72}$$

Therefore, for any $a \in \mathcal{F}$ and $\tau \in \mathrm{T}(C)$, we have

$$\begin{aligned}
&\big|\tau\big(\imath_{1,\infty}\big(L(a)\big)\big) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)\big| \\
&\quad < \big|\tau\big(\imath_{1,\infty}\big(L(a)\big)\big) - \gamma^*\big((\imath_{1,\infty})_{\mathrm{T}}(\tau)\big)(a)\big| + \big|\gamma^*\big((\imath_{1,\infty})_{\mathrm{T}}(\tau)\big)(a) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)\big| \\
&\quad = \big|\tau\big(\imath_{1,\infty}\big(L(a)\big)\big) - \gamma^*\big((\imath_{1,\infty})_{\mathrm{T}}(\tau)\big)(a)\big| + \big|(\imath_{1,\infty})_{\mathrm{Aff}}(\hat{a})(\tau) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau)\big| \\
&\quad < 51\sigma/128 + 77\sigma/128 = \sigma \quad (\text{by } (4.72) \text{ and } (4.22)),
\end{aligned}$$

where $(\imath_{1,\infty})_{\mathrm{T}} : \mathrm{T}(A) \to \mathrm{T}(C_1)$ is the affine map induced by $\imath_{1,\infty}$.

Since $\mathcal{F}$, ε, and σ are arbitrary, in this way we obtain a sequence of unital completely positive maps $H_n : A \to C$ such that

$$\lim_{n\to\infty} \left\| H_n(ab) - H_n(a)H_n(b) \right\| = 0, \quad a, b \in A,$$

and

$$\lim_{n\to\infty} \sup \left\{ \left| \tau \circ H_n(a) - \Gamma_{\mathrm{Aff}}(\hat{a})(\tau) \right| : \tau \in \mathrm{T}(C) \right\} = 0, \quad a \in A.$$

On using again that the given C*-algebra A has finite nuclear dimension, so that also $A \otimes Q$ does, it follows by [25, Lemma 3.4]—which uses results obtained in [32, 41]—that $\mathrm{gTR}(A \otimes Q) \leq 1$. This together with $\mathcal{Z}$-stability of A (which we established at the very beginning of this proof) says that the given algebra A belongs to the class $\mathcal{N}_1$. ■

Theorem 1.1 follows from the following corollary.

Corollary 4.5. *Let A be a unital simple separable C*-algebra with finite decomposition rank, satisfying the UCT. Then, $\mathrm{gTR}(A \otimes Q) \leq 1$. In particular, A is classifiable.*

Proof. Since A has finite decomposition rank, A is nuclear (see Definition 2.10 above) and quasidiagonal [20, Theorem 5.3]. It follows from [37, Theorem 2.4] that $\mathrm{T}(A) \neq \emptyset$. By [2, Proposition 8.5], $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$. Now, the corollary follows from Theorem 4.4 together with Theorem 2.7. ■

Remark 4.6. We would like to state the following special case of Corollary 4.5. Let A be as in 4.5. Suppose that

$$\begin{aligned} &\big(K_0(A \otimes Q), K_0(A \otimes Q)_+, [1_{A\otimes Q}]_0, \mathrm{T}(A \otimes Q), r_{A\otimes Q}\big) \\ &\quad \cong \big(K_0(C), K_0(C)_+, [1_C]_0, \mathrm{T}(C), r_C\big) \end{aligned} \tag{4.73}$$

for some unital simple $\mathrm{A}\mathbb{T}$-algebra C. Then, $\mathrm{TR}(A \otimes Q) \leq 1$. If this holds for some unital AF-algebra C, then $\mathrm{TR}(A \otimes Q) = 0$.

To see this we note that, in the beginning of the proof of Theorem 4.4, we assume that $A = A \otimes Q$. If C in (4.73) can be chosen to be a unital simple $\mathrm{A}\mathbb{T}$-algebra, then the end of the proof shows that $A \otimes Q$ has tracial rank at most one. In the same way one sees that if C in (4.73) can be chosen to be a unital simple AF-algebra, then $A \otimes Q$ has tracial rank zero.

The preceding (abstract) classification result (Corollary 4.5) depends on (by reducing to) the recent (semi-abstract) classification result of [15, 16]. In fact, there is a more restricted, but still very interesting, setting in which a correspondingly restricted abstract result can be established by reducing to a much earlier result.

Let A be a unital simple separable C*-algebra, satisfying the hypotheses of the preceding corollary (or theorem). Suppose in addition that $\mathrm{S}_{[1_A]}(\mathrm{K}_0(A))$, the state space of $\mathrm{K}_0(A)$, is a Choquet simplex, and that the map $r_A : \mathrm{T}(A) \to \mathrm{S}_{[1_A]}(\mathrm{K}_0(A))$ takes extreme points to extreme points. Without using [15] or [16], the proof of the present Theorem 4.4, above, shows that A is classifiable.

Indeed, by [27], there is a unital simple separable C*-algebra, B, satisfying the UCT, such that $B \otimes Q$ has tracial rank at most one (in the sense of [23]) and such that $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. Since $\mathrm{K}_i(B \otimes Q)$ is torsion-free, by the classification of C*-algebras of tracial rank at most one (see [9, 23]), $C = B \otimes Q$ is an inductive limit of circle algebras (i.e., is $\mathrm{A}\mathbb{T}$). By the first paragraph of this remark, $A \otimes Q$ has tracial rank at most one. Hence, by [24, Corollary 11.9] (see also [28]), A is classifiable.

In particular, the Jiang–Su algebra $\mathcal{Z}$ is the only unital separable simple amenable C*-algebra in the UCT class that has the same Elliott invariant as that of $\mathbb{C}$. The proof just given of this statement does not rely on [15] or [16].

Remark 4.7. It was shown in [13] that any unital simple separable Jiang–Su stable approximately subhomogeneous C*-algebra has decomposition rank at most two. Therefore, it follows from Corollary 4.5 that such a C*-algebra is classifiable. This in particular recovers the classification theorem of [10]. Moreover, by [16] together with the result in [13] mentioned above, every unital simple C*-algebra belonging to the class $\mathcal{N}_1$ has finite decomposition rank.

Remark 4.8. The special case of Corollary 4.5 for C*-algebras for which K_0 separates traces, e.g. the case of unique trace, is known. (See [38, Corollary 5.2] and [26, Theorem 5.4].)

Theorem 4.4 and Corollary 4.5 can also be combined and stated as follows.

Theorem 4.9. *Let A be a unital simple separable amenable (non-zero)* C*-*algebra which satisfies the UCT. Then, the following properties are equivalent:*

(1) $\mathrm{gTR}(A \otimes Q) \leq 1$;

(2) *$A \otimes Q$ has finite nuclear dimension and* $\mathrm{T}(A \otimes Q) = \mathrm{T}_{\mathrm{qd}}(A \otimes Q) \neq \emptyset$;

(3) *the decomposition rank of $A \otimes Q$ is finite.*

Given the fact that every tracial state of a unital simple separable C*-algebra with finite decomposition rank is quasidiagonal [2, Proposition 8.5], it is reasonable to expect that every tracial state of a finite unital simple separable C*-algebra with finite nuclear dimension is also quasidiagonal. Indeed, shortly after the present paper was first announced (and posted on arXiv), Tikuisis, White, and Winter proved that, in fact, every tracial state on a unital simple separable amenable C*-algebra which satisfies the UCT is quasidiagonal [36, Theorem A]. Therefore, we have the following statement.

Theorem 4.10. *Let A be a finite unital simple separable* C*-*algebra with finite nuclear dimension which satisfies the UCT. Then,* $\mathrm{gTR}(A \otimes Q) \leq 1$. *In particular, A is classifiable (and is approximately subhomogeneous (ASH)—see Theorems 2.7 and 2.8).*

Remark 4.11. It was established by Kirchberg and Phillips [19, 31] that purely infinite unital simple separable amenable C*-algebras which satisfy the UCT are classifiable. It has been shown that these C*-algebras have finite nuclear dimension (see [30]). It is also known that every unital simple separable C*-algebra with finite nuclear dimension is either finite or purely infinite (see [14, 39]). Therefore, Theorem 4.10 can now be combined with [19, 31] to obtain the following overall statement.

Corollary 4.12. *The class of all unital simple separable (non-elementary)* C*-*algebras with finite nuclear dimension which satisfy the UCT is classifiable by the Elliott invariant.*

Added November 2, 2021. This paper was originally posted on arXiv in late 2015. Since then, there have been some new developments. Notably, in [4] (also [29, 35]), it was shown that every Jiang–Su stable ($\mathcal{Z}$-stable) unital simple separable amenable C*-algebra has finite nuclear dimension. This yields many examples which come under the aegis of Corollary 4.12. For instance, since by [18], the Jiang–Su algebra, $\mathcal{Z}$, is itself $\mathcal{Z}$-stable, the tensor product of any C*-algebra A with $\mathcal{Z}$ is $\mathcal{Z}$-stable, and so by [4], if A is a unital simple separable amenable C*-algebra and satisfies the UCT, then $A \otimes \mathcal{Z}$ is covered by (4.12). Sometimes, it can be established without tensoring by $\mathcal{Z}$ that a C*-algebra is Jiang–Su stable. For example, this was achieved in [12] for a simple C*-algebra arising from a minimal homeomorphism of an infinite compact metrizable space of mean dimension zero (which includes the cases that the space is finite-dimensional or has a unique invariant measure).

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