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Rotating solutions to the incompressible Euler–Poisson equation with external particle

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Abstract. We consider a two-dimensional, incompressible fluid body, together with self-induced interactions. The body is perturbed by an external particle with small mass. The whole configuration rotates uniformly around the common center of mass. We construct solutions, which are stationary in a rotating coordinate system, using perturbative methods. In addition, we consider a large class of internal motions of the fluid. The angular velocity is related to the position of the external particle and is chosen to satisfy a non-resonance condition.

1. Introduction and previous results

The shape of fluid objects due to the combination of rotational and self-gravitating forces is a classical research field which has been extensively considered for different fluid models. In particular, a detailed description of the historical evolution of the field can be found [11] for the (three-dimensional) incompressible Euler equations. Further results were established by Lichtenstein [31]. For the case of compressible fluids we refer to the works [6, 12, 24, 27–30, 32–34, 39, 40] and references therein. A kinetic model, namely the Vlasov–Poisson equation, has been studied as well; see e.g. [17, 29]. In fact, there is a relation between steady states of the Vlasov–Poisson equation and the compressible Euler equation; see [36] and references therein for an overview of the variational methods used in these problems.

In this paper we consider a two-dimensional, self-interacting, incompressible fluid body modeled by the Euler equations. Furthermore, we study the problem of deformations of the geometry when it is perturbed by some external particle. The fluid body and the external particle are assumed to rotate around their center of mass. This problem (adding a small particle) can be seen as a test of stability of the rotating solutions and also as a simple model of tides. Furthermore, differently from the results reviewed in [11] (excluding the figures studied by Riemann), we construct solutions of the Euler–Poisson equation for which the fluid velocity is in general different from zero in any coordinate system. Recently, in [7], the authors studied the stability of solutions for long times in

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suitable functional spaces close to the equilibrium states of an inviscid, incompressible, and irrotational fluid, subject to the self-gravitational force.

In this work we study a family of interaction potentials including the classical (Newtonian) gravitational forces. The latter can be interpreted as an extremely simplified model for galaxies. However, this does not correspond to a three-dimensional problem restricted to planar geometries. The reason is that the pressure would necessarily act only in the plane which contains the fluid body as well as the external particle. Nevertheless, such a model can be considered in the case of the Vlasov–Poisson equation, assuming that the velocities of the particles are contained only in the same plane as the fluid. In this situation, the tensor describing the pressure is anisotropic and it yields zero forces in the direction perpendicular to the plane but not in the horizontal direction; cf. [35].

Since we consider a two-dimensional fluid body we can apply two tools that cannot be employed in three-dimensional problems. Specifically, we use conformal mappings as well as the Grad–Shafranov method [20,38]. We restrict ourselves to the two-dimensional setting since the corresponding three-dimensional version requires an understanding of some small denominator problem which cannot be tackled with the methods employed in this article.

Besides the problem treated here, a variety of different free-boundary problems arising in fluid mechanics have been studied in the last decades. For instance, the problem of jets and cavities with or without gravity has been studied in [3–5] and the theory of gravity water waves has been developed in several works, cf. [26,41,42]. Let us also highlight the recent survey [23] that covers the mathematical theory of the steady water waves problem. A question that has been discussed in [23, Section 6.2] is the effect on the free boundary of the presence of point vortices. This question is different from the one treated in this article but has some mathematical analogies.

An important difference between the previous free-boundary problems and the one studied in this paper is that the interacting force (e.g. gravity) is due to the fluid itself. Another type of problem that has some similarities with the one considered in this article is that related to the theory of rotating vortex patches. The first rigorous result was shown by Burbea [8] where he constructed rotating vortex patches close to the disk by means of the classical Crandall–Rabinowitz bifurcation approach. A more thorough study of rotating vortex patches can be found in [22,25] and the references therein.

1.1. Setting of the problem

We are concerned with a flat incompressible fluid body with density $\rho = \mathbb{1}_E$. Here, $\mathbb{1}_E$ denotes the indicator function of the set E. The shape of the body $E(t) \subset \mathbb{R}^2$ has a smooth boundary, is simple connected and close to a disk; see below for the precise meaning of this. We also include a particle $X = X(t) \in \mathbb{R}^2$ with small mass m. However, we consider only situations in which the particle and the fluid body are at a positive distance. The velocity field v of the fluid body then satisfies the following free-boundary problem for the Euler–Poisson system:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p - \nabla U_{E(t)} - m \nabla U_{X(t)} & \text{in } E(t), \\ \nabla \cdot v = 0 & \text{in } E(t), \\ n \cdot v = V_N & \text{on } \partial E(t), \end{cases} \tag{1.1}$$

where V_N is the normal velocity of the interface $\partial E(t)$, n the outer unit normal vector of $\partial E(t)$ and ∇ denotes the classical gradient operator in two dimensions, namely, $\nabla = (\partial_{x_1}, \partial_{x_2})$. Here, $U_{E(t)}$ and $U_{X(t)}$ are the gravitational potentials; see below for the precise definitions. Furthermore, $p = p(t, x)$ is the scalar pressure which describes the internal pressure of the body for $x \in E(t)$ and the external pressure of the surrounding space for $x \in \mathbb{R}^2 \backslash E(t)$. We assume the external pressure to be constant on $\mathbb{R}^2 \backslash E(t)$ and without loss of generality we can take this constant to be zero. This reflects that the configuration is surrounded by a uniform medium. Therefore, the continuity of the pressure at the interface that separates the liquid from the exterior implies that

$$p = 0 \quad \text{on } \partial E(t). \tag{1.2}$$

Since there are no external forces acting on the configuration described by the fluid body and the external particle, their common center of mass moves at constant speed. Consequently, we can assume without loss of generality (using a change of the coordinate system) that the center of mass is at zero, i.e.

$$\int_{E(t)} x \, dx + m X(t) = 0.$$

As mentioned in the introduction, we study two cases for the potentials $U_{E(t)}$ and $U_{X(t)}$ in (1.1):

(A) We consider a family of power law potentials, more precisely for $\nu \in (0, 1]$ we define

$$U_{X(t)}(x) := -\frac{1}{|x - X(t)|^{\nu}}, \quad U_{E(t)}(x) := -\int_{E(t)} \frac{dy}{|x - y|^{\nu}}. \tag{1.3}$$

(B) We consider potentials given via the fundamental solution of the two-dimensional Laplace operator, i.e.

$$U_{X(t)}(x) := \ln |x - X(t)|, \quad U_{E(t)}(x) := \int_{E(t)} \ln |x - y| \, dy.$$

Note that in both cases the signs are chosen to yield attractive forces. Furthermore, case (A) with $\nu = 1$ can be interpreted as Newtonian gravitational interactions.

Let us mention here that in case (A) with $\nu = 1$, some care is needed in order to define a solution to (1.1), since the gradient $\nabla U_{E(t)}$ is not well defined due to the onset of a singularity. However, this does not suppose a problem since the pressure gradient ∇p also has a similar singularity with a reverse sign that compensates the singularity of $\nabla U_{E(t)}$. In order to avoid these singular terms, it is convenient to rewrite problem (1.1) subtracting the hydrostatic pressure. To this end, we define $p = P - U_{E(t)} - mU_{X(t)}$, where P is the non-hydrostatic pressure. Then system (1.1) turns into

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla P & \text{in } E(t), \\ \nabla \cdot v = 0 & \text{in } E(t), \\ n \cdot v = V_N & \text{on } \partial E(t), \\ P = U_{E(t)} + mU_{X(t)} & \text{on } \partial E(t), \end{cases} \tag{1.4}$$

where the last equation follows from (1.2). Now, these equations do not contain singular terms.

The solutions to (1.4) studied in this paper are classical solutions, i.e. $v\colon E(t) \to \mathbb{R}^2$ and $\partial E(t)$ are regular. However, the function $P\colon \overline{E(t)} \to \mathbb{R}$ is in general only continuous, i.e. in case (A) the gradient ∇P is not defined on $\partial E(t)$. As we will see in the next section, this condition of continuity of the pressure and the last equation in (1.4) yields an equation for the free boundary.

Furthermore, the solutions constructed in this paper occur as perturbations of solutions to the time-independent equation with $m = 0$, that is,

$$\begin{cases} (v \cdot \nabla) v = -\nabla P & \text{in } E, \\ \nabla \cdot v = 0 & \text{in } E, \\ n \cdot v = 0 & \text{on } \partial E, \\ P = U_E & \text{on } \partial E. \end{cases} \tag{1.5}$$

One particular solution we consider is given by the unit disk $E = \mathbb{D}$, together with a corresponding velocity field v and the non-hydrostatic pressure P.

In addition, we assume that the perturbed fluid body and the external particle solving (1.1) rotate around their center of mass with angular speed of rotation $\Omega_0 > 0$. Furthermore, we look for configurations which are time independent in a rotating frame at angular speed Ω_0; see Figure 1. Changing to such a rotating coordinate system we obtain the equations

$$\begin{cases} (v \cdot \nabla) v + 2\Omega_0 J v - \Omega_0^2 x = -\nabla P & \text{in } E, \\ \nabla \cdot v = 0 & \text{in } E, \\ n \cdot v = 0 & \text{on } \partial E, \\ P = U_E + mU_X & \text{on } \partial E, \\ \Omega_0^2 X = \nabla U_E(X), & \\ |E| = \pi, & \\ \int_E x \, dx + mX = 0. & \end{cases} \tag{1.6}$$

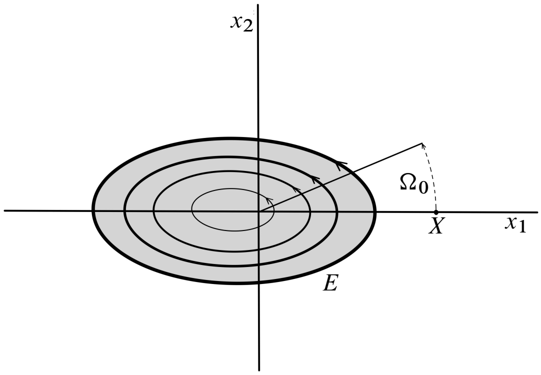

Figure 1. Configuration of the fluid body E and the external particle X. Both rotate around their common center of mass (at the origin) with angular speed Ω_0.

In equations (1.6) we used the matrix J defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.7}$$

which encodes the action of the vector product in the two-dimensional case.

Notice that in this setting, the shape of the body E, the velocity field v and the position of the particle X do not depend on time. Furthermore, we construct solutions $v \neq 0$, which can be interpreted as some type of tidal wave induced by the gravity of the external particle, as well as the velocity of the unperturbed fluid.

We briefly comment on the system of equations (1.6). First, note that the terms $2\Omega_0 J v$ and $-\Omega_0^2 x$ represent the Coriolis and the centrifugal forces, respectively, which appear in the rotating frame of reference. The third equation in (1.6) ensures that the free boundary is stationary, i.e. the fluid inside the body does not move across the boundary. As stated above, the external pressure is assumed to be constant outside the body. The equation $\Omega_0^2 X = \nabla U_E(X)$ follows from Newton's law and ensures that the external particle is at rest. Note that $\nabla U_E(X)$ is now well defined in case (A) also, since we consider only cases with X separated from E. The centrifugal force acting on X balances with the gravitational force of the fluid body. In addition, for definiteness, we assume that the total mass of the fluid is $\pi = |\mathbb{D}|$. The last equation in (1.6) ensures that the center of mass is at the origin. In fact, as we will see in the proof of our main result (see Section 6) this last equation in (1.6) follows from the other equations in (1.6).

Finally, let us mention that equations (1.6) are invariant under rotations around the origin. Hence, we can assume without loss of generality that the particle $X = (a, 0)$ is located on the x_1-axis. In particular, a solution to (1.6) yields a family of solutions by applying rotations.

In this paper we construct solutions to (1.6) obtained as perturbation of solutions to (1.5) with $E = \mathbb{D}$ by means of an implicit function theorem in Hölder spaces. We require a non-resonance condition on Ω_0 and a non-degeneracy condition on the unperturbed velocity field solving (1.5); see Theorem 2.1 and Corollary 2.2.

The paper is organized as follows. In Section 2 we reformulate the problem using Grad–Shafranov, the Bernoulli equation and conformal mappings to derive a reduced system of equations that will be more amenable to mathematical analysis. This new system is solved using an implicit function theorem. To this end, we provide some preliminary results concerning conformal mapping properties and estimates for elliptic equations, as well as suitable representations of the gravitational potentials in Section 3. In Section 4 we prove the Fréchet differentiability of the reduced system of equations with respect to the unknowns of the problem. Furthermore, in Section 5 we prove the invertibility of the Fréchet derivative at the unperturbed solution. Finally, we conclude the article with the proof of the main results in Section 6.

2. Reformulation of the problem and main result

In this section we reduce problem (1.6) to a set of equations that will be studied in the main part of the paper. To this end, we apply in particular conformal mappings, as well as the Grad–Shafranov method.

Conformal mappings. We use conformal mappings, i.e. bijective analytic functions, to parameterize the domain of the fluid. Recall that, by the Riemann mapping theorem, for any simply connected domain $E \subset \mathbb{C}$ one can find a conformal mapping $f\colon \mathbb{D} \to E$. Here, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x_1 + ix_2$. In the case of smooth domains, the mapping extends conformally to $\bar{\mathbb{D}} \to \bar{E}$.

In our study, we consider conformal mappings of the form $f_h\colon \mathbb{D} \to \mathbb{R}^2$, $f_h(z) = z + h(z)$, where h is small such that the domain is close to the disk. Let us mention that under a general smallness condition on some arbitrary analytic function $h\colon \mathbb{D} \to \mathbb{C}$ the mapping f_h is conformal; see Lemma 3.1. We denote the corresponding domain by $E_h = f_h(\mathbb{D})$ to emphasize the dependence on h. Accordingly, we use the notation $U_h = U_{E_h}$. Furthermore, we denote by f_h' the complex derivative, i.e. understanding f_h as a mapping $\mathbb{D} \subset \mathbb{C} \to \mathbb{C}$.

Let us also introduce the so-called Blaschke factors (see [37]), defined by

$$b_{c,d}(z) = d\,\frac{z - c}{1 - \bar{c}z}, \quad c \in \mathbb{D},\ d \in \mathbb{C},\ |d| = 1. \tag{2.1}$$

These factors are the only conformal mappings $\mathbb{D} \to \mathbb{D}$. Choosing c, d accordingly allows us to set $h(0) = 0$ and $h'(0) \in \mathbb{R}$ by replacing f_h by $f_h \circ b_{c,d}$. This defines the conformal mapping f_h and hence also h uniquely.

Grad–Shafranov method. In order to construct the velocity field v solving (1.6) we use the Grad–Shafranov method. Roughly speaking, the Grad–Shafranov approach allows us to transform the original problem (1.6) to an elliptic problem for the stream function. These ideas have been very useful for constructing solutions in different problems arising in plasma physics, for instance to prove flexibility and rigidity results in magnetohydrostatics (cf. [13, 14, 21]) or to study boundary value problems (cf. [2]). In the next paragraphs, we will recall the key ideas of this approach.

In this paper we are interested only in two-dimensional vector fields $v = (v_1, v_2)$. However, in order to use classical formulas for fluid mechanics in three dimensions it is convenient to think in those vector fields as three-dimensional fields with zero third component, namely, $\tilde{v} = (v_1, v_2, 0)$. Therefore, the vorticity associated to this vector field $\tilde{v}$ is denoted by $\widetilde{\omega}$, i.e. $\widetilde{\omega} = \widetilde{\nabla} \times \tilde{v}$. Here we use the notation $\widetilde{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) = (\nabla, \partial_{x_3})$. Due to the form of the vector field $\tilde{v}$, it turns out that $\widetilde{\omega} = (0, 0, \omega(x))$ with $x = (x_1, x_2)$. Similarly, the vector angular velocity is denoted by $\widetilde{\Omega} = (0, 0, \Omega_0)$. We denote by $\mathcal{P}\colon \mathbb{R}^3 \to \mathbb{R}^2$ the projector given by

$$\mathcal{P}(y_1, y_2, y_3) = (y_1, y_2), \quad \forall y = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Using the classical formula

$$\tilde{v} \times \widetilde{\omega} = -(\tilde{v} \cdot \widetilde{\nabla})\tilde{v} + \frac{1}{2}\widetilde{\nabla}(|\tilde{v}|^2),$$

as well as

$$\mathcal{P}(\tilde{v} \times \widetilde{\omega}) = -\omega J v,$$

we infer that

$$-vJ\omega = -(v \cdot \nabla)v + \frac{1}{2}\nabla(|v|^2).$$

Recall that J is the matrix defined in (1.7).

Hence, the first three equations in (1.6) can be written as

$$\begin{cases} -(\omega + 2\Omega_0)Jv = \nabla H & \text{in } E_h, \\ \nabla \cdot v = 0 & \text{in } E_h, \\ n_h \cdot v = 0 & \text{on } \partial E_h. \end{cases} \tag{2.2}$$

Here, H is called the Bernoulli head and is defined by

$$H := P + \frac{1}{2}|v|^2 - \frac{\Omega_0^2}{2}|x|^2.$$

The term $2\Omega_0$ can be interpreted as the third component of vorticity of the velocity field $\mathcal{P}(\widetilde{\Omega} \times (x_1, x_2, x_3))$ which occurs in terms of the Coriolis force due to the rotating frame of reference. Applying the operator $\nabla^{\perp}\cdot = (-\partial_{x_2}, \partial_{x_1})\cdot$ to the first equation in (2.2) and using that $\nabla \cdot v = 0$ yields

$$(v \cdot \nabla)(\omega + 2\Omega_0) = 0.$$

Let us remark that this identity holds in general only in two dimensions, which restricts the Grad–Shafranov method to these situations. As a corollary of the above identity we obtain that $\omega + 2\Omega_0$ and thus ω is constant along stream lines (characteristics) of v.

The main object in the Grad–Shafranov approach is the stream function $\psi\colon E_h \to \mathbb{R}$ satisfying $v = \nabla^\perp \psi := J\nabla\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$. Let us mention that, in general, in order to guarantee the existence of a stream function ψ we need to work with a simply connected domain. However, since the boundary conditions (cf. second equation in (2.2)) imply that ψ is a constant in each of the connected components of the boundary of ∂E_h, as well as the fact that the divergence-free condition on v implies that ψ is harmonic, it then follows that the function ψ is well defined for arbitrary domains, not necessarily simply connected. However, during this work E_h is simply connected.

Now, with the stream function at hand, we can write $\omega = \Delta\psi$. Since ψ is also constant along the characteristics of $v = J\nabla\psi$, one might conclude the existence of a function $G\colon \mathbb{R} \to \mathbb{R}$ such that $\Delta\psi = G(\psi)$. Let us remark here that, in general, the existence of G can be concluded only locally when $\nabla\psi \neq 0$. Furthermore, the function G might be multi-valued, a situation that although interesting we will not consider in this paper. In addition, we require $n_h \cdot v = 0$ on ∂E_h, and thus

$$0 = n_h \cdot J\nabla\psi = \tau_h \cdot \nabla\psi,$$

where τ_h is the positively oriented tangential vector on ∂E_h. We integrate along the boundary to get for $x \in \partial E_h$ that $\psi(x) = c_0$ for some constant $c_0 \in \mathbb{R}$. Note that the potential ψ is given up to a constant, so we can choose $c_0 = 0$ by adapting the function G if needed. Thus, the stream function solves the equation

$$\begin{cases} \Delta\psi = G(\psi) & \text{in } E_h, \\ \psi = 0 & \text{on } \partial E_h. \end{cases} \tag{2.3}$$

In the Grad–Shafranov approach, the above reasoning is reversed in the sense that we are given some (regular enough) function G and we construct the stream function (hence also the velocity field) by solving equation (2.3).

Note that $\psi\colon E_h \to \mathbb{R}$ is a function of h, so that we sometimes write ψ_h if we want to emphasize the dependence on h. Let us also remark that the existence and uniqueness of solutions to (2.3) are ensured in general by assuming that G is non-decreasing; see Lemma 3.4.

We now use the conformal mapping in order to reduce equation (2.3) to the domain $\mathbb{D}$. We set $\phi_h := \psi_h \circ f_h$, which is now defined on the disk $\phi_h\colon \mathbb{D} \to \mathbb{R}$. The corresponding equation reads

$$\begin{cases} \Delta\phi_h = |f_h'|^2 G(\phi_h) & \text{in } \mathbb{D}, \\ \phi_h = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{2.4}$$

It must be stressed that the function G only depends on the stream function ψ_h and the conformal mapping, and not on the external mass particle m.

Equation of the free boundary. The equation determining the free boundary can be derived from the fact that the non-hydrostatic pressure P is continuous along the free boundary. Using the stream function $v = \nabla^\perp \psi_h$ we can write

$$\mathcal{P}(\tilde{v} \times (\tilde{\omega} + 2\tilde{\Omega})) = -(G(\psi_h) + 2\Omega_0) J \nabla^\perp \psi_h = (G(\psi_h) + 2\Omega_0)\nabla\psi_h \quad \text{in } E_h.$$

We conclude that

$$\mathcal{P}(\tilde{v} \times (\tilde{\omega} + 2\tilde{\Omega})) = \nabla[F(\psi_h)], \quad F(\psi_h)|_{\partial E_h} = 0,$$

where $F' = G + 2\Omega_0$ is a primitive with $F(0) = 0$. Consequently, in order to ensure equality in the first equation in (2.2), the non-hydrostatic pressure is given (up to a constant λ) by

$$P = F(\psi_h) - \frac{1}{2}|\nabla\psi_h|^2 + \frac{\Omega_0^2}{2}|x|^2 + \lambda \quad \text{in } E_h. \tag{2.5}$$

The condition that P is continuous along the free boundary yields, with $P = U_h + mU_X$ on ∂E_h and $F(\psi_h)|_{\partial E_h} = 0$, the equation

$$\frac{1}{2}|\nabla\psi_h|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_h + mU_X = \lambda \quad \text{on } \partial E_h. \tag{2.6}$$

The evaluation at the boundary $\partial E_h = f_h(\partial\mathbb{D})$ in (2.6) can be performed using the conformal mapping f_h. We now summarize the reduced system that we aim to solve in our study:

$$\begin{cases} \frac{1}{2}\frac{|\nabla\phi_h|^2}{|f_h'|^2} - \frac{\Omega_0^2}{2}|f_h|^2 + U_h \circ f_h + mU_X \circ f_h = \lambda & \text{on } \partial\mathbb{D}, \\ \Delta\phi_h = |f_h'|^2 G(\phi_h) & \text{in } \mathbb{D}, \\ \phi_h = 0 & \text{on } \partial\mathbb{D}, \\ \Omega_0^2 a = \partial_{x_1} U_h(X), \\ |E_h| = \pi. \end{cases} \tag{2.7}$$

Recall that the position of the particle is chosen as $X = (a, 0)$. The unknown triplet is (h, a, λ). As we will see (cf. Corollary 2.2), solutions of (2.7) constructed in this paper yield solutions to (1.6). Let us mention that the fourth equation in (2.7) is the x_1-component of Newton equation for the particle X; see also the fifth equation in (1.6). The other component follows, as we will see in Corollary 2.2, by the symmetry of the domain E with respect to the x_1-axis.

Solution for $m = 0$. In the case when no external particle is present, i.e. $m = 0$, we assume that the fluid body has the shape of a disk $\mathbb{D}$. Furthermore, we consider a velocity field on $\mathbb{D}$ with stream function ϕ_0 solving

$$\begin{cases} \Delta\phi_0 = G(\phi_0) & \text{in } \mathbb{D}, \\ \phi_0 = 0 & \text{on } \partial\mathbb{D}. \end{cases}$$

Note that this coincides with (2.4) for $h = 0$. Observe that due to the rotational invariance $\phi_0 = \phi_0(|x|)$, the equation reduces to the ODE

$$\frac{1}{r}(r\phi_0')' = G(\phi_0(r)), \quad \phi_0(1) = 0.$$

This ODE is complemented by the condition that $\lim_{r\to 0}\phi_0(r)$ exists. Therefore, the velocity field becomes $v(x) = \phi_0'(|x|)Jx/|x|$. It describes a non-uniform rotation with angular speed depending on the distance to the center. Since the velocity field is rotationally symmetric, the velocity in the non-rotating coordinate system is given by $(\phi_0'(|x|)/|x| + \Omega_0)Jx$. Furthermore, note that the function ϕ_0 can be extended to $r > 1$. This is necessary, for instance, when evaluating ϕ_0 on the boundary ∂E_h, which is close to $\partial\mathbb{D}$.

The position of the unperturbed particle is chosen of the form $X_0 = (a_0, 0)$. Since we consider only cases for which the fluid body and the external particle are strictly separated, we assume say $a_0 \geq 2$. Hence, E_h does not contain $X \approx X_0$ for small enough h. The Newton equation for the particle requires that

$$\Omega_0^2 X_0 = \nabla U_0(X_0).$$

Further information on the potentials U_0 of the disk in both cases (A) and (B) is given in Lemmas 3.5 and 3.6. For $a_0 > 1$ we have $U_0'(a_0) > 0$ and furthermore $U_0'(a_0)/a_0 \to 0$ as $a_0 \to \infty$. In addition, $a_0 \mapsto U_0'(a_0)/a_0$ is strictly decreasing for $a_0 > 1$. Hence, there is a one-to-one correspondence between $\Omega_0 \in (0, \sqrt{U_0'(1)}]$ and $a_0 \geq 1$ via

$$\Omega_0 = \sqrt{\frac{U_0'(a_0)}{a_0}}. \tag{2.8}$$

All in all, this defines a map $\Omega_0 \mapsto a_0(\Omega_0)$. Finally, the constant in (2.6) is given by $\lambda_0 = \frac{1}{2}\phi_0'(1)^2 - \frac{1}{2}\Omega_0^2 + U_0(1)$.

2.1. Notation

We will use the following notation throughout the manuscript:

- We use $\mathbb{D}$ to denote the unit disk with boundary $\partial\mathbb{D}$ and $\mathbb{T} = [0, 2\pi]$ the 2π-periodic torus with endpoints identified.
- The Hölder seminorm of a function $u\colon \mathbb{T} \to \mathbb{R}$ or $u\colon \mathbb{D} \to \mathbb{R}$ is defined by

$$[u]_{k,\alpha} = \sup_{x_1 \neq x_2} \frac{|u^{(k)}(x_2) - u^{(k)}(x_1)|}{|x_2 - x_1|^\alpha}, \quad \alpha \in (0,1),$$
$$[u]_{k,0} = \|u^{(k)}\|_\infty, \quad \alpha = 0.$$

- We abbreviate $H^{k,\alpha} := H^{k,\alpha}(\mathbb{D}) := H(\mathbb{D}) \cap C^{k,\alpha}(\bar{\mathbb{D}})$, where $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ and $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. We equip it with the standard Hölder norm $\|\cdot\|_{k,\alpha}$.

- We denote by $H_0^{k,\alpha} \subset H^{k,\alpha}$ the subspace of analytic functions h such that $h(0) = 0$ and $h'(0) \in \mathbb{R}$.
- Furthermore, the Fourier coefficients of a function $g\colon \mathbb{T} \to \mathbb{R}$ are given by

$$\hat{g}_n = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) e^{in\varphi}\, d\varphi.$$

  Recall that $\hat{g}_n = \overline{\hat{g}_{-n}}$, since g is real valued.
- We denote by $C_0^{k,\alpha}(\mathbb{T}) \subset C^{k,\alpha}(\mathbb{T})$ those functions g with zero average, i.e. $\hat{g}_0 = 0$.
- Let us abbreviate by $B_r = B_r(0) \subset H_0^{k,\alpha}$ the ball of radius r around zero.
- We will denote by C a positive generic constant that depends only on fixed parameters including Ω_0 and norms of the function G in (2.7). Note also that this constant might differ from line to line.

2.2. Main result and strategy towards the proof

In order to construct the desired solution, we make use of the implicit function theorem; cf. Lemma 3.3. To do so, let us introduce the following functional spaces:

$$\mathbb{X}^{k+2,\alpha} := H_0^{k+2,\alpha}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}, \quad \mathbb{Z}^{k+1,\alpha} := C^{k+1,\alpha}(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}. \tag{2.9}$$

We define the following function related to system (2.7). Define the map $\mathbb{F}\colon U \times V \to \mathbb{Z}^{k+1,\alpha}$, where $U \subset H_0^{k+2,\alpha}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}$, $V \subset \mathbb{R}$, with $X = (a, 0)$, by

$$\mathbb{F}(h, a, \lambda, m) = \begin{pmatrix} \left[\frac{1}{2}\frac{|\nabla\phi_h|^2}{|f_h'|^2} - \frac{\Omega_0^2}{2}|f_h|^2 + U_h \circ f_h + mU_X \circ f_h - \lambda\right]\Big|_{z=e^{i\varphi}} \\ \Omega_0^2 a - \partial_{x_1} U_h(X) \\ |f_h(\mathbb{D})| - \pi \end{pmatrix}. \tag{2.10}$$

The subset U is a sufficiently small neighborhood of $(0, a_0, \lambda_0)$. In particular, it ensures that h defines a conformal mapping $f_h(z) = z + h(z)$; see Lemma 3.1.

Our goal is to solve the equation $\mathbb{F}(h, a, \lambda, m) = 0$ via the implicit function theorem. To this end, we study the Fréchet derivative at the point $(0, a_0, \lambda_0, 0)$. We will apply a Fourier decomposition for the first component of $\mathbb{F}$, which is a function on the torus $\mathbb{T}$. As we will see (cf. Lemma 5.7), the corresponding linear operator can be diagonalized and the Fourier multipliers have the form

$$\omega_n = -\frac{1}{2}\Omega_0^2 - \frac{1}{2}\phi_0'(1)^2(|n| + 1) + \phi_0'(1)A'_{|n|}(1)(|n| + 1) + c_{|n|}. \tag{2.11}$$

The coefficients ω_n are visible in a non-resonance condition for Ω_0 in our main result; cf. Theorem 2.1. In the definition of ω_n, the function ϕ_0 is the unperturbed stream function for $m = 0$. The coefficients c_n enter through the interaction potential $h \mapsto (U_h \circ f_h)(e^{i\varphi})$. In case (A), they are given by (note we identify again $\mathbb{R}^2 \simeq \mathbb{C}$)

$$c_n = \frac{1}{2}\int_{\mathbb{D}} \left(\nu \frac{1 - y^{n+1}}{1 - y} - 2(n + 1)y^n\right) \frac{dy}{|1 - y|^\nu}, \tag{2.12}$$

and in case (B) by

$$c_n = \begin{cases} \frac{\pi}{2}\Big(1 - \frac{1}{n}\Big), & n \geq 1, \\ \frac{\pi}{2}, & n = 0. \end{cases} \tag{2.13}$$

Let us note that the integral in (2.12) defines a real quantity. Note also that only the first term in (2.11) depends on Ω_0, whereas all the other terms depend on either the function G or the choice of the interaction.

Finally, the numbers $A_n'(1)$ are computed by means of the functions $A_n\colon (0,1) \to \mathbb{R}$ solving the ODE

$$\frac{1}{r}(rA_n')' - \frac{n^2}{r^2}A_n - G'(\phi_0(r))A_n = r^{|n|}G(\phi_0(r)), \quad A_n(1) = 0. \tag{2.14}$$

They appear in the Fréchet derivative of the stream function $h \mapsto \phi_h$; cf. Section 5.

The main result of this work reads as follows:

Theorem 2.1. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $a_0 \geq 2$. Assume $G \in C^{k+3}(\mathbb{R};\mathbb{R})$ to be non-decreasing. Let $\Omega_0 \geq 0$ be related to $X_0 = (a_0(\Omega_0), 0)$ as stated in (2.8) and let the non-resonance condition*

$$\forall n \in \mathbb{N}\colon \omega_n \neq 0 \tag{2.15}$$

be satisfied for ω_n given in (2.11). Furthermore, we assume for the unperturbed stream function ϕ_0 that

$$\phi_0'(1) \neq 0. \tag{2.16}$$

Then there are $\delta > 0$, $\varepsilon > 0$ such that for any $m \in [0,\delta)$ there is a unique solution $(h,a,\lambda) \in \mathbb{X}^{k+2,\alpha}$ of the equation $\mathbb{F}(h,a,\lambda,m) = 0$ satisfying

$$\|h\|_{k+2,\alpha} + |a - a_0| + |\lambda - \lambda_0| < \varepsilon.$$

Furthermore, the dependence $m \mapsto (h,a,\lambda)(m)$ is continuous.

As a corollary we obtain that a solution to $\mathbb{F}(h,a,\lambda,m) = 0$ yields a solution to our original problem (1.6).

Corollary 2.2. *Under the assumption of Theorem 2.1, the domain $E_h = f_h(\mathbb{D})$ in Theorem 2.1 is symmetric with respect to the x_1-axis. Finally, the corresponding velocity field $v = \nabla^\perp \psi_h$, together with the position of the particle $X = (a,0)$ and the non-hydrostatic pressure P, yields a solution to (1.6).*

Remark 2.3. Let us comment on the non-resonance condition (2.15).

(i) It ensures that the linearized operator can be inverted in order to apply the implicit function theorem. In the case that (2.15) is not satisfied, bifurcations to other shapes might occur.

(ii) As mentioned before, the quantities ω_n in (2.11) contain a term only depending on Ω_0, while the other terms depend only on the choice of the function G and the interaction. Thus, the non-resonance condition (2.15) is a condition on Ω_0. Furthermore, note that this condition (2.15) is needed only for $n \in \mathbb{N}$, since ω_n only depends on $|n|$. Furthermore, as we will see in Lemmas 5.3 and 5.5, the leading-order term on the right-hand side of (2.15) is given by $-\phi_0'(1)^2(|n|+1)$, whereas the other terms are at most of order $\mathcal{O}(\ln n)$ as $n \to \infty$. In particular, condition (2.15) is automatically satisfied for sufficiently large n. Hence, it is possible to verify the condition numerically.

(iii) In the particular case that the fluid has no internal motion in the non-rotating coordinate system for $m = 0$ we have $v(x) = -\Omega_0 J x$ and thus $\phi_0(x) = -\Omega_0(|x|^2 - 1)/2$. This corresponds to the choice $G = -2\Omega_0$. Then we can readily check that solutions to (2.14) have the form

$$A_n(r) = -2\Omega_0 \frac{r^n(r^2-1)}{4n+4}, \quad A_n'(1) = -\frac{\Omega_0}{n+1}.$$

Hence, condition (2.15) reduces to

$$\omega_n = -\frac{|n|}{2}\Omega_0^2 + c_{|n|} \neq 0.$$

Remark 2.4. Let us mention that assumption (2.16) in Theorem 2.1 is also needed to prove the invertibility of the Fréchet derivative in order to apply the implicit function theorem. This condition implies that the function ϕ_0 has no local extremum at the boundary. When perturbing such extrema, saddle points are created generically. Consequently, vortices would appear. Furthermore, let us comment on the assumption that G is non-decreasing. This condition crucially implies that the stream function is well defined and regular enough (see Lemma 3.4). It might be possible to relax this assumption and instead assume that in a neighborhood of an initially chosen solution ϕ_0 to (2.3) for $h = 0$ one can uniquely solve equation (2.3). This could be achieved using an auxiliary implicit function theorem. However, we do not pursue this here.

Remark 2.5. We are assuming in Theorem 2.1 that $m \geq 0$ since it is the most natural setting from the physical point of view. However, the proof of Theorem 2.1 is also valid for the case $m \in (-\delta, \delta)$.

Remark 2.6. In this paper we restricted ourselves to interaction potentials defined in cases (A) and (B). The study of more general interactions would require further modifications, in particular a better understanding of results like Lemma 5.8 on pseudo-differential operators on the torus.

Remark 2.7. Finally, let us mention that the corresponding three-dimensional problem of (1.6) requires a different approach, since conformal mappings and the Grad–Shafranov method are restricted to two-dimensional problems. Furthermore, the study of the eigenvalues of the linearization involves several technical complications due to instabilities.

Let us mention that the presence of the external particle does not allow us to construct axisymmetric configurations, since the interaction with the particle breaks this symmetry.

We conclude this section with the discussion of the particular case of constant $G \equiv K \in \mathbb{R}\backslash\{0\}$. This corresponds to an unperturbed velocity field $v_0(x) = KJx/2$ in the rotating and $V_0(x) = (K + 2\Omega_0)Jx/2$ in the non-rotating frame of reference. In this case, one can do a formal linearization using the ansatz $\partial E_h = T_\eta(\mathbb{T})$, $T_\eta(\theta) = 1 + \varepsilon\eta(\theta)$, $\theta \in \mathbb{T}$ for the free boundary. Here, $\eta \in \mathbb{T} \to \mathbb{R}$ allows us to change the boundary of the fluid body and $\varepsilon = m$ is the mass of the particle. More precisely, one can linearize the system (compare with (2.7))

$$\begin{cases} \frac{1}{2}|\nabla\psi_\eta|^2 - \frac{\Omega_0^2}{2} + U_h + mU_X = \lambda & \text{on } \partial E_\eta, \\ \Delta\psi_\eta = K & \text{in } E_\eta, \\ \psi_h = 0 & \text{on } \partial E_\eta, \\ |E_h| = \pi. \end{cases}$$

The linearization yields the following formula for η in terms of Fourier series:

$$\eta(\theta) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n e^{i\theta n}, \quad \hat{\eta}_n = -\frac{\widehat{S}_n}{\widetilde{\omega}_n}, \quad \widetilde{\omega}_n := \frac{K^2}{4} - \frac{K^2}{4}|n| - \Omega_0^2 + \pi - \frac{\pi}{|n|}.$$

The terms $\widehat{S}_n$ are the Fourier coefficients of the perturbation, that is,

$$\widehat{S}_n = \frac{1}{2\pi}\int_0^{2\pi} U_X(\cos\theta, \sin\theta)e^{-in\theta}\, d\theta.$$

Here, $X = (a_0, 0)$ is the unperturbed position of the external particle; cf. (2.8).

Let us mention that the mass constraint $|E_\eta| = \pi$ imposes $\hat{\eta}_0 = 0$. Furthermore, one obtains $\hat{\eta}_n = \hat{\eta}_{-n} \in \mathbb{R}$. In particular, the function η is invariant under reflection $(x_1, x_2) \to (x_1, -x_2)$. In addition, the condition that the center of mass is at zero yields (after linearizing) $\hat{\eta}_1 = -a_0/2\pi$, which in fact can be shown to match with the above formula $\hat{\eta}_1 = -\widehat{S}_1/\widetilde{\omega}_1 = \widehat{S}_1/\Omega_0^2$. Furthermore, note that the Fourier coefficients $\hat{\eta}_n$ do not depend on the sign of K or Ω_0.

Let us mention that the non-resonance condition (2.15) is equivalent to $\widetilde{\omega}_n \neq 0$.

In Figure 2 we plot the function η for the values $\Omega_0 = 1$, $K = -2, 0.1, 10$ and for an interaction potential U_X as in case (B). In the plot the zero level line is shown. Outside this circle the function is positive, whereas inside it is negative. Let us recall that the particular case $K = -2 = -2\Omega_0$ corresponds to the situation in which the unperturbed fluid body has no internal motion in the non-rotating coordinate system. Furthermore, in Figure 3 we plotted the function η in a situation close to resonance due to the mode $n = 8$. In fact, for $\Omega_0 = 1$, $K = 1$ we have $\omega_8 \approx 10^{-3}$ so that the largest contribution to the Fourier series of η is due to the two coefficients $\hat{\eta}_8 = \hat{\eta}_{-8}$.

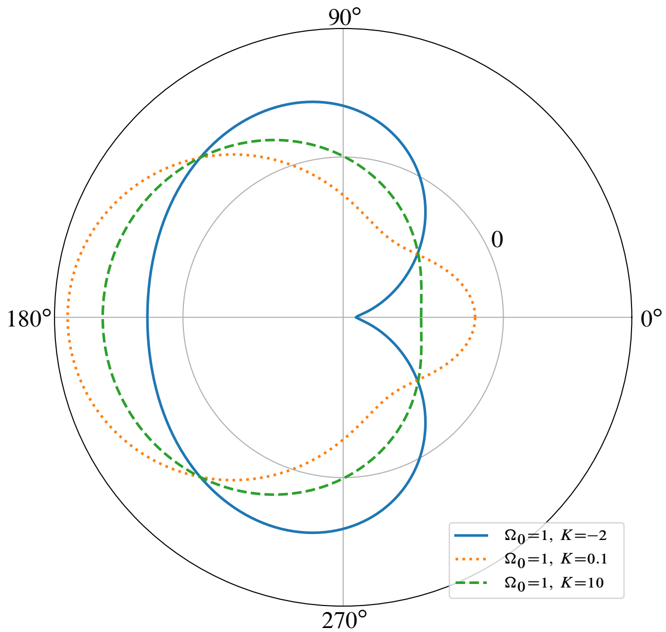

Figure 2. Plot of the function η for $\Omega_0 = 1$ and different values of K. The interaction is given as in case (B). Furthermore, the particle is to the leading order at position $X = (a_0, 0)$, $a_0 = \sqrt{\pi}$; cf. (2.8).

3. Preliminary results

We collect here some auxiliary results that will be used in the subsequent sections. Let us start with a well-known result in complex analysis regarding analytic functions.

Lemma 3.1. *Consider the analytic function $f_h(z) = z + h(z)$ with $\|h\|_{C^1(\bar{\mathbb{D}})} < 1/\sqrt{2}$. Then $f_h\colon \mathbb{D} \to f_h(\mathbb{D})$ is conformal.*

Proof. We prove that f_h is injective. Define the function $\zeta(\varphi) = f_h(e^{i\varphi})$, $\varphi \in \mathbb{T}$. Let φ_1, $\varphi_2 \in \mathbb{T}$. We can assume $|\varphi_1 - \varphi_2| \le \pi$. If $|\varphi_1 - \varphi_2| \ge \pi/2$ we have

$$|\zeta(\varphi_2) - \zeta(\varphi_1)| \ge |e^{i\varphi_2} - e^{i\varphi_1}| - 2\|h\|_{C(\bar{\mathbb{D}})} = 2\left|\sin\Big(\frac{\varphi_2 - \varphi_1}{2}\Big)\right| - 2\|h\|_{C(\bar{\mathbb{D}})} > 0.$$

On the other hand, if $|\varphi_1 - \varphi_2| < \pi/2$ we estimate

$$\begin{aligned} |\zeta(\varphi_2) - \zeta(\varphi_1)| &= \left|\int_{\varphi_1}^{\varphi_2} f_h'(e^{i\psi}) i e^{i\psi}\, d\psi\right| \ge |e^{i\varphi_2} - e^{i\varphi_1}| - |\varphi_2 - \varphi_1| \|h\|_{C^1(\bar{\mathbb{D}})} \\ &= 2\left|\sin\Big(\frac{\varphi_2 - \varphi_1}{2}\Big)\right| - |\varphi_2 - \varphi_1| \|h\|_{C^1(\bar{\mathbb{D}})} \\ &\ge |\varphi_2 - \varphi_1| \Big(\frac{1}{\sqrt{2}} - \|h\|_{C^1(\bar{\mathbb{D}})}\Big). \end{aligned}$$

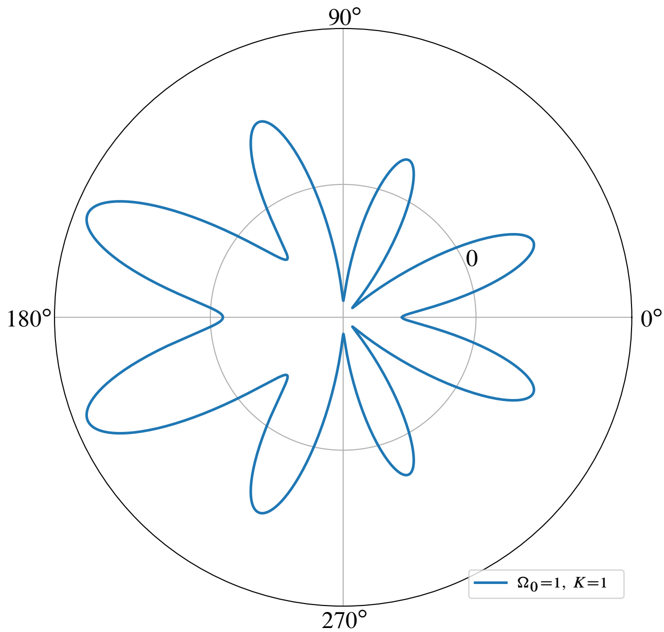

Figure 3. Plot of the function η for $\Omega_0 = 1$ and $K = 1$. In this case there is almost resonance since $\tilde{\omega}_8 \approx 10^{-3}$. In particular, the biggest contribution to the Fourier series of η is due to the Fourier coefficients $\hat{\eta}_8 = \hat{\eta}_{-8}$.

Hence, f_h is one-to-one on the boundary. As a consequence of the Darboux–Picard theorem (see [9, Theorem 9.16]), f_h is injective on $\overline{\mathbb{D}}$. ∎

Lemma 3.2 (Faà di Bruno formula [15]). *For any $n \in \mathbb{N}$ and two functions $f,\ g \in C^n(\mathbb{R};\mathbb{R})$ we have the formula*

$$\frac{d^n}{dx^n}(f \circ g)(x) = \sum_{\substack{\ell_1,\dots,\ell_n \\ 1\cdot\ell_1+\dots+n\cdot\ell_n=n}} n!\Big[\frac{d^{\ell_1+\dots+\ell_n}}{dx^{\ell_1+\dots+\ell_n}} f\Big](g(x)) \prod_{j=1}^{n}\Big(\frac{1}{\ell_j!\,j!}\frac{d^j}{dx^j}g(x)\Big)^{\ell_j}.$$

We recall the following version of the implicit function theorem.

Lemma 3.3 (Implicit function theorem, [16]). *Let X, Y, Z be Banach spaces and $U \subset \mathsf{X}$, $V \subset \mathsf{Y}$ be neighborhoods of x_0, y_0, respectively, where $\mathbb{F}(x_0, y_0) = 0$. Suppose that $\mathbb{F}\colon U \times V \to \mathsf{Z}$ is continuous, continuously differentiable with respect to $x \in U$ and $D_x\mathbb{F}(x_0, y_0) \in \mathcal{L}(\mathsf{X}, \mathsf{Z})$ is invertible. Then there are balls $B_\varepsilon(x_0) \subset U$, $B_\delta(y_0) \subset V$ and a unique map $\xi\colon B_\delta(y_0) \to B_\varepsilon(x_0)$ with $\mathbb{F}(\xi(y), y) = 0$ for all $y \in B_\delta(y_0)$. Furthermore, ξ is continuous.*

Here we denote by $\mathcal{L}(\mathsf{X}, \mathsf{Z})$ the space of bounded linear operators $\mathsf{X} \to \mathsf{Z}$. Furthermore, $D_x\mathbb{F}(x_0, y_0) \in \mathcal{L}(\mathsf{X}, \mathsf{Z})$ is the Fréchet derivative with respect to the first variable, i.e. we have

$$\mathbb{F}(x_0+\xi, y_0) = \mathbb{F}(x_0, y_0) + D_x\mathbb{F}(x_0, y_0)[\xi] + o(\|\xi\|_{\mathsf{X}}), \quad \text{as } \|\xi\|_{\mathsf{X}} \to 0.$$

Let us also give an existence and uniqueness result for equation (2.4). Such elliptic equations have been studied extensively in both Hölder and Sobolev spaces; see e.g. [18, 19].

Lemma 3.4. *Let $h \in B_{1/2} \subset H_0^{k+2,\alpha}$ and assume $G \in C^{k+3}(\mathbb{R};\mathbb{R})$ to be non-decreasing. Then there is a unique solution $\phi_h \in C^{k+2,\alpha}(\bar{\mathbb{D}})$ to (2.4). Furthermore, there exists a constant $C > 0$ independent of h such that*

$$\|\phi_h\|_{C^{k+2,\alpha}(\bar{\mathbb{D}})} \le C. \tag{3.1}$$

Proof. We prove the assertion in terms of $\psi_h = \phi_h \circ f_h^{-1}$. The existence follows from standard methods of The calculus of variations applied to the functional

$$\psi \mapsto \int_{E_h} |\nabla\psi|^2\,dx + \int_{E_h} F(\psi)\,dx,$$

where $F' = G$ is a primitive. Note that F is convex, since G is non-decreasing. The regularity follows via a bootstrapping argument, recalling that $G \in C^{k+3}(\mathbb{R};\mathbb{R})$. Observe that due to $f_h \in H^{k+2,\alpha}$, the boundary ∂E_h is sufficiently regular. The uniqueness can be proved using a comparison principle, since G is non-decreasing.

Estimate (3.1) is a consequence of the maximum principle and Schauder estimates. Indeed, this will be done by separating two cases.

Case 1. We assume that there is $y_0 \in \mathbb{R}$ with $G(y_0) = 0$. Since G is non-decreasing, we can find $N > 0$ sufficiently large such that $G(-N) \le 0 \le G(N)$. We conclude from a comparison principle that $\|\phi_h\|_\infty \le N$. Hence, the right-hand side in (2.4) is uniformly bounded in h. We apply regularity theory in Sobolev spaces to conclude that $\phi_h \in W^{2,2}$ with a bound independent of $h \in B_{1/2}$. Hence, by Sobolev embedding we obtain $\phi_h \in C^\alpha$. Now, the right-hand side in (2.4) is uniformly bounded in C^α. We hence repeatedly apply Schauder estimates to yield the result.

Case 2. If G is always non-zero, we can assume without loss of generality that $G > 0$. In this case, we infer $\phi_h \le 0$ by the maximum principle. Thus, the right-hand side in (2.4) is uniformly bounded. We can now argue as in Case 1. ∎

Next we prove a formula for the unperturbed interaction potential of $\mathbb{D}$, i.e. of the unperturbed solution for $m = 0$, in case (A) with $\nu = 1$ and case (B).

Lemma 3.5. *The following formulas hold in case (A) with $\nu = 1$:*

$$\begin{aligned} U_0(r) &= -\frac{4}{\pi^2}\sum_{k\ge 0} W_{2k}^2\Big(\frac{r}{2k+2} + \frac{r}{2k-1} - \frac{r^{2k}}{2k-1}\Big), && 0 \le r \le 1,\\ U_0(r) &= -\frac{4}{\pi^2}\sum_{k\ge 0} \frac{W_{2k}^2}{2k+2}\frac{1}{r^{2k+1}}, && r \ge 1. \end{aligned} \tag{3.2}$$

Here, $W_\ell = \frac{\pi}{2}\frac{(\ell-1)!!}{\ell!!}$ *is Wallis' formula; see* [1, formula 6.1.49]. *In case* (B) *we have that*

$$U_0(r) = \begin{cases} -\dfrac{\pi}{2}(1-r^2), & r \le 1, \\ \pi \ln r, & r \ge 1. \end{cases} \tag{3.3}$$

Let us recall that $\lim_{\ell\to\infty}\sqrt{\ell}W_\ell = \sqrt{\pi/2}$. Consequently, the series in (3.2) also converges for the critical value $r = 1$.

Proof of Lemma 3.5. To this end, we use a multipole expansion for $x = x(r,\theta,\varphi)$, $y = y(s,\theta',\varphi') \in \mathbb{R}^3$,

$$\begin{aligned} \frac{1}{|x-y|} &= \frac{1}{\sqrt{r^2+s^2-2rs(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'))}} \\ &= \sum_{\ell\ge 0}\sum_{|m|\le\ell}\frac{4\pi}{2\ell+1}\frac{(r\wedge s)^\ell}{(r\vee s)^{\ell+1}}Y_{\ell,m}(\theta,\varphi)Y_{\ell,m}(\theta',\varphi')^*, \end{aligned}$$

where the spherical harmonics are given by

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\varphi}P_\ell^m(\cos\theta).$$

Here, P_ℓ^m are the associated Legendre polynomials. We have for $\theta = \theta' = \pi/2$,

$$U_0(r) = -\sum_{\ell\ge 0}\frac{4\pi c_\ell^2}{2\ell+1}\int_0^1\frac{(r\wedge s)^\ell s\,ds}{(r\vee s)^{\ell+1}}, \qquad c_\ell := Y_{\ell 0}(\pi/2,0) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(0).$$

A computation shows that

$$P_\ell(0) = \begin{cases} (-1)^{\ell/2}\dfrac{(\ell-1)!!}{\ell!!}, & \ell \text{ even}, \\ 0, & \ell \text{ odd}. \end{cases}$$

Rewriting the coefficients of the series in terms of the Wallis' formula and choosing $\ell = 2k$ yields both formulas in (3.2). The formula in (3.3) follows by solving the Poisson equation $\Delta U_0 = \mathbb{1}_{\mathbb{D}}$. ■

The following lemma contains information on the unperturbed potential U_0 in all cases considered.

Lemma 3.6. *In both cases* (A) *and* (B) *the potential* U_0 *satisfies*

(i) $U_0'(r) > 0$ *for* $r > 1$,

(ii) $\frac{d}{dr}[U_0'(r)/r] < 0$ *for* $r > 1$, *and in particular* $r \mapsto U_0'(r)/r$ *is strictly decreasing for* $r \ge 1$. *Furthermore,* $\lim_{r\to\infty} U_0'(r)/r = 0$.

Proof. In case (A) we first use equation (1.3), as well as polar coordinates, to derive the explicit formula

$$U_0(r) = -\int_0^1 \int_0^{2\pi} \frac{s\, ds\, d\varphi}{(r^2 + s^2 - 2rs\cos\varphi)^{\nu/2}}.$$

Then claim (i) follows by differentiating the previous formula and using that $r > 1 \geq s$. Indeed,

$$U_0'(r) = \nu \int_0^1 \int_0^{2\pi} \frac{(r - s\cos\varphi)s\, ds\, d\varphi}{(r^2 + s^2 - 2rs\cos\varphi)^{1+\nu/2}} > 0.$$

Next, let us show claim (ii). To that purpose, we use the change of variable $s \mapsto \eta r$ yielding

$$\frac{U_0'(r)}{r} = \nu r^{-\nu} \int_0^{1/r} \int_0^{2\pi} \frac{(1 - \eta\cos\varphi)\eta\, d\eta\, d\varphi}{(1 + \eta^2 - 2\eta\cos\varphi)^{1+\nu/2}}.$$

Differentiating this formula with respect to r shows that $\frac{d}{dr}[U_0'(r)/r] < 0$ for $r > 1$, since the integrand is non-negative and $\nu > 0$. Furthermore, $U_0'(r)/r \to 0$ as $r \to \infty$ also follows from the previous formula. In case (B) both claims (i) and (ii) are a direct consequence of the explicit formula (3.3). ■

4. Fréchet derivative of the main problem

In this section we prove the Fréchet differentiability of the function $\mathbb{F}$. We separately consider the stream function ϕ_h and the interaction potential $U_h \circ f_h$.

4.1. Fréchet derivative of the stream function

In this subsection we derive the Fréchet differential of the function $h \mapsto \phi_h$.

Lemma 4.1. *Let $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$,*

$$h \mapsto \phi_h \in C^1(B_{\varepsilon_0}; C^{k+2,\alpha}(\bar{\mathbb{D}})).$$

More precisely, the linear operator $D_h\phi_h$ is defined by $g \mapsto D_h\phi_h[g] =: \bar{\phi}$, where

$$\begin{cases} \Delta\bar{\phi} = |f_h'|^2 G'(\phi_h)\bar{\phi} + 2\,\mathrm{Re}[(1 + h')\bar{g'}]G(\phi_h) & \text{in } \mathbb{D}, \\ \bar{\phi} = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{4.1}$$

Proof. First of all, equation (4.1) has a unique solution $\bar{\phi}$, since $G' \geq 0$. We apply Schauder estimates for the Laplacian and absorb the term $|f_h'|^2 G'(\phi_h)\bar{\phi}$ into the left-hand side by choosing $\|h\|_{k+2,\alpha} \leq \varepsilon_0$ sufficiently small. This yields

$$\|\bar{\phi}\|_{k+2,\alpha} \leq C\|g\|_{k+2,\alpha}, \tag{4.2}$$

where $C > 0$ is independent of $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ by Lemma 3.4.

Furthermore, by taking the difference of the equations for ϕ_{h+g} and ϕ_h, given respectively via (2.4), we obtain

$$\left[-\Delta + \int_0^1 G'((1-t)\phi_h + t\phi_{h+g})\,dt\right](\phi_{h+g} - \phi_h) = -\big[|f'_{h+g}|^2 - |f'_h|^2\big]G(\phi_{h+g}).$$

Therefore, using Schauder estimates, we infer that

$$\|\phi_{h+g} - \phi_h\|_{k+2,\alpha} \le C\|g\|_{k+2,\alpha}, \tag{4.3}$$

where $C > 0$ is independent of h.

Next we find that for $D_h\phi_h[g] = \bar{\phi}$ and denoting $R := \phi_{h+g} - \phi_h - \bar{\phi}$,

$$\begin{aligned}
\Delta R &= (|f'_{h+g}|^2 - |f'_h|^2 - 2\,\mathrm{Re}[(1+h')\overline{g'}])G(\phi_h) \\
&\quad + (|f'_{h+g}|^2 - |f'_h|^2)G'(\phi_h)\bar{\phi} + |f'_{h+g}|^2(G(\phi_{h+g}) - G(\phi_h) - G'(\phi_h)\bar{\phi}) \\
&= |g'|^2 G(\phi_h) + (|f'_{h+g}|^2 - |f'_h|^2)G'(\phi_h)\bar{\phi} + G'(\phi_h)R \\
&\quad + \int_0^1 G''((1-t)\phi_h + t\phi_{h+g})\,dt\,(\phi_{h+g} - \phi_h)^2.
\end{aligned}$$

Similarly to above, invoking Schauder estimates and bounds (4.2)–(4.3) we obtain (note that $G \in C^{k+3}(\mathbb{R};\mathbb{R})$)

$$\|R\|_{k+2,\alpha} \le C\|g\|^2_{k+2,\alpha}.$$

Here, the constant $C > 0$ is independent of h.

Finally, we need to prove that $h \mapsto D_h\phi_h \in \mathcal{L}(H_0^{k+2,\alpha}; C^{k+2,\alpha}(\overline{\mathbb{D}}))$ is continuous. To this end, one has to consider differences of solutions to (4.1) for $h_1, h_2 \in B_{\varepsilon_0}$. Applying Schauder estimates we find the bound

$$\|D_h\phi_{h_2}[g] - D_h\phi_{h_1}[g]\|_{k+2,\alpha} \le C\|g\|_{k+2,\alpha}(\|h_1 - h_2\|_{k+2,\alpha} + \|\phi_{h_1} - \phi_{h_2}\|_{k+2,\alpha}),$$

which shows the continuity property. ■

4.2. Fréchet derivative of the interaction potential

Here we derive the Fréchet derivative of the mapping $h \mapsto (U_h \circ f_h)(e^{i\varphi})$. We give only the details of the proof of case (A) with $\nu = 1$. The remaining cases can be shown in a similar way (and are in fact simpler since the integrals are less singular). We summarize the corresponding results for case (B) at the end of this section.

First of all, it is convenient to apply a change of variables

$$(U_h \circ f_h)(e^{i\varphi}) = -\int_{\mathbb{D}} \frac{|f'_h(y)|^2}{|f_h(e^{i\varphi}) - f_h(y)|^\nu}\,dy = -\int_{\mathbb{D}} \frac{|f'_h(e^{i\varphi}y)|^2}{|f_h(e^{i\varphi}) - f_h(e^{i\varphi}y)|^\nu}\,dy.$$

We then have the following result:

Proposition 4.2. *Let U_h be defined as in case (A) and let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ we have*

$$h \mapsto (U_h \circ f_h)(e^{i\varphi}) \in C^1(B_{\varepsilon_0}, C^{k+1,\alpha}(\mathbb{T})).$$

More precisely, for $h + g \in B_{\varepsilon_0}$ it holds that

$$D_h(U_h \circ f_h)[g](e^{i\varphi}) = -\int_{\mathbb{D}} \frac{\sigma_h^1[g](\varphi, y)}{d_h(\varphi, y)^\nu}\, dy + \nu \int_{\mathbb{D}} \frac{\sigma_h^2[g](\varphi, y)}{d_h(\varphi, y)^{\nu+2}} |f_h'(e^{i\varphi} y)|^2\, dy,$$

where we define

$$\begin{aligned} d_h(\varphi, y) &:= |f_h(e^{i\varphi}) - f_h(e^{i\varphi} y)|, \\ \sigma_h^1[g](\varphi, y) &:= 2\,\mathrm{Re}\big[(1 + h'(e^{i\varphi} y))\overline{g'(e^{i\varphi} y)}\big], \\ \sigma_h^2[g](\varphi, y) &:= \mathrm{Re}\big[(e^{i\varphi}(1-y) + h(e^{i\varphi}) - h(e^{i\varphi} y))\overline{(g(e^{i\varphi}) - g(e^{i\varphi} y))}\big]. \end{aligned} \tag{4.4}$$

From now on we restrict ourselves to the case $\nu = 1$. In order to prove Proposition 4.2 it is convenient to introduce the following notation:

$$e_h(\varphi, y) = d_h(\varphi, y)^2 = |f_h(e^{i\varphi}) - f_h(e^{i\varphi} y)|^2.$$

Furthermore, we need the following computation with $t \in [0, 1]$:

$$\begin{aligned} \frac{d^2}{dt^2}\Big[-\frac{|f_{h+tg}'(e^{i\varphi} y)|^2}{d_{h+tg}(\varphi, y)}\Big] &= T^0_{h+tg,g}(\varphi, y) + T^1_{h+tg,g}(\varphi, y) + T^2_{h+tg,g}(\varphi, y), \\ T^0_{h+tg,g}(\varphi, y) &:= -\frac{2|g'(e^{i\varphi} y)|^2}{d_{h+tg}(\varphi, y)}, \\ T^1_{h+tg,g}(\varphi, y) &:= \frac{\tau^1_{h+tg,g}(\varphi, y)}{d_{h+tg}(\varphi, y)^3}, \\ \tau^1_{h+tg,g}(\varphi, y) &:= 2\sigma^1_{h+tg}[g](\varphi, y)\sigma^2_{h+tg}[g](\varphi, y) \\ &\quad + |f_{h+tg}'(e^{i\varphi} y)|^2 |g(e^{i\varphi}) - g(e^{i\varphi} y)|^2, \\ T^2_{h+tg,g}(\varphi, y) &:= \frac{\tau^2_{h+tg,g}(\varphi, y)}{d_{h+tg}(\varphi, y)^5}, \\ \tau^2_{h+tg,g}(\varphi, y) &:= -3|f_{h+tg}'(e^{i\varphi} y)|^2 (\sigma^2_{h+tg}(\varphi, y))^2. \end{aligned} \tag{4.5}$$

Lemma 4.3. *Let $k \in \mathbb{N}_0$, $\alpha \in [0,1)$. For $\varepsilon_0 > 0$ sufficiently small and $g, h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ the following estimates hold:*

(i) *For $\ell \in \mathbb{N}_0$, $\ell \le k+1$, $y \in \mathbb{D}$ we have*

$$\sup_{t \in [0,1]} \big[\sigma^2_{h+tg}[g](\cdot, y)\big]_{\ell,\alpha} \le C\|g\|_{k+2,\alpha} |1-y|^{2-\alpha}.$$

(ii) *For* $\ell \in \mathbb{N}_0$, $\ell \le k+1$, $m = 1, 2$, $y \in \mathbb{D}$ *we have*

$$\sup_{t\in[0,1]} \left[\tau^m_{h+tg}[g](\cdot, y)\right]_{\ell,\alpha} \le C\|g\|^2_{k+2,,\alpha}|1-y|^{2m-\alpha}.$$

(iii) *For* $\ell \in \mathbb{N}_0$, $\ell \le k+1$, $y \in \mathbb{D}$ *we have*

$$\sup_{t\in[0,1]} \left[e_{h+tg}[g](\cdot, y)\right]_{\ell,\alpha} \le C|1-y|^{2-\alpha}.$$

Proof. The proof of this lemma is straightforward. Indeed, one can readily check the bounds by means of the following general estimate: for any $u \in H^1(\mathbb{D})$ we have

$$\begin{aligned}
&\frac{1}{|\varphi_2-\varphi_1|^\alpha}\left|[u(e^{i\varphi_1}y) - u(e^{i\varphi_1})] - [u(e^{i\varphi_2}y) - u(e^{i\varphi_2})]\right| \\
&\quad \le (2\|u'\|_\infty)^\alpha(2|1-y|\|u'\|_\infty)^{1-\alpha} \le 2\|u\|_{C^1}|1-y|^{1-\alpha}.
\end{aligned}$$

■

The next lemma is also useful.

Lemma 4.4. *Let* $k \in \mathbb{N}_0$, $\alpha \in [0, 1)$. *There is a sufficiently small* $\varepsilon_0 > 0$ *such that for any* $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ *and* $y \in \mathbb{D}$ *we have*

$$d_h(\varphi, y) \ge c|1-y|. \tag{4.6}$$

Furthermore, for any $q \in \mathbb{N}$, $n \in \mathbb{N}_0$, $n \le k+1$ *and* $y \in \mathbb{D}$ *the estimate*

$$\left[\frac{1}{d_h(\cdot, y)^q}\right]_{n,\alpha} \le \frac{C_{k,q}}{|1-y|^{q+\alpha}}$$

holds. Both $c > 0$ *and* $C_{k,q>} > 0$ *are independent of* h.

Proof. The first assertion follows using the mean-value theorem and choosing $\varepsilon_0 > 0$ sufficiently small. To prove the second assertion we first consider $n = 0$. It then suffices to consider $q = 1$. We have

$$\begin{aligned}
&\frac{1}{|\varphi_1-\varphi_2|^\alpha}\left|\frac{1}{d_h(\varphi_1, y)} - \frac{1}{d_h(\varphi_2, y)}\right| \\
&\quad = \frac{|d_h(\varphi_1, y) - d_h(\varphi_2, y)|}{|\varphi_1-\varphi_2|^\alpha}\frac{1}{d_h(\varphi_1, y)d_h(\varphi_2, y)} \\
&\quad \le \frac{C}{|1-y|^2}\frac{|f_h(e^{i\varphi_1}) - f_h(e^{i\varphi_2}) + f_h(e^{i\varphi_1}y) - f_h(e^{i\varphi_2}y)|}{|\varphi_1-\varphi_2|^\alpha} \\
&\quad \le \frac{C}{|1-y|^{1+\alpha}}.
\end{aligned}$$

We now compute the nth-order derivative, $n \le k+1$. By Faà di Bruno's formula (cf. Lemma 3.2) applied to the composition of the functions $x \mapsto x^{-q/2}$ and $e_h = d_h^2$, the preceding expression is a sum of terms of the form

$$\frac{1}{d_h(\varphi, y)^{2(\ell_1+\cdots+\ell_n)+q}} \prod_{j=1}^{n} \Big(\frac{d^j}{d\varphi^j} e_h(\varphi, y)\Big)^{\ell_j} = \frac{1}{d_h(\varphi, y)^q} \prod_{j=1}^{n} \Big(\frac{1}{d_h(\varphi, y)^2} \frac{d^j}{d\varphi^j} e_h(\varphi, y)\Big)^{\ell_j},$$

where $\ell_1, \ell_2, \ldots, \ell_n \in \mathbb{N}_0$ satisfy $\ell_1 + 2\ell_2 + \cdots + n\ell_n = n$. The supremum norm of each term in the product $j = 1, \ldots, n$ is bounded due to Lemma 4.3 (iii) and (4.6).

We estimate now the seminorm $[\cdot]_\alpha$ of this expression. Note that for products, only one term is estimated in this seminorm, while the other terms are estimated in the supremum norm. For the seminorm we apply Lemma 4.3 (iii) and the case $n = 0$ we discussed above. Hence, the seminorm is bounded up to a constant by $|1 - y|^{-q-\alpha}$. ∎

As a result of the previous lemmas we obtain the following estimates:

Lemma 4.5. *Let $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$. We have for sufficiently small $\varepsilon_0 > 0$, $g, h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ and $m = 0, 1, 2$,*

$$\Big\| \frac{\sigma_h^1[g](\cdot, y)}{d_h(\cdot, y)} \Big\|_{k+1,\alpha} + \Big\| \frac{\sigma_h^2[g](\cdot, y)}{d_h(\cdot, y)^3} \Big\|_{k+1,\alpha} \leq C \|g\|_{k+2,\alpha} |1 - y|^{-1-\alpha},$$

$$\|T^m_{h+tg,g}(\cdot, y)\|_{k+1,\alpha} \leq C \|g\|^2_{k+2,\alpha} |1 - y|^{-1-\alpha}.$$

The constant $C > 0$ is independent of h, g.

Proof. The previous lemmas can be applied without difficulty. Note that in the case of $T^2_{h+tg,g}$ there is a factor d^5_{h+tg} in the denominator. This is compensated by the extra factor in Lemma 4.3 (ii) for $m = 2$. ∎

With the previous lemmas we can give the proof of Proposition 4.2.

Proof of Proposition 4.2. We consider only $\nu = 1$. For the sake of the exposition we divide the proof into two steps.

Step 1. We first show the Fréchet differentiability. We write

$$\begin{aligned} R(\varphi) &:= (U_{h+g} \circ f_{h+g})(e^{i\varphi}) - (U_h \circ f_h)(e^{i\varphi}) - D_h(U_h \circ f_h)[g](e^{i\varphi}) \\ &= \int_{\mathbb{D}} \int_0^1 (1 - t) \frac{d^2}{dt^2} \Big[-\frac{|f'_{h+tg}(e^{i\varphi} y)|^2}{d_{h+tg}(\varphi, y)} \Big] \, dt \, dy \\ &= \int_{\mathbb{D}} \int_0^1 (1 - t) [T^0_{h+tg,g}(\varphi, y) + T^1_{h+tg,g}(\varphi, y) + T^2_{h+tg,g}(\varphi, y)] \, dt \, dy, \end{aligned}$$

recalling (4.5). We apply Lemma 4.5 to get (note that $-1 - \alpha > -2$)

$$\|R\|_{k+1,\alpha} \leq C \|g\|^2_{k+2,\alpha},$$

which implies the Fréchet differentiability.

Step 2. Now we show that $h \mapsto D_h(U_h \circ f_h)[g]$ is continuous. First of all, one can estimate using Lemma 4.5,

$$\|D_h(U_h \circ Y_h)[g]\|_{k+1,\alpha} \le C_k \|g\|_{k+2,\alpha},$$

where $C_k > 0$ is independent of $h, g \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$. We can use these estimates to cut out the singularity in the integral uniformly in $h, g \in B_{\varepsilon_0}$. The remaining integrand is then a smooth function with respect to h. As a consequence it is continuous in h. The above bounds show that this is also uniform in $\|g\|_{k+2,\alpha}$, which ensures these estimates in the operator norm. Hence, $h \mapsto D_h(U_h \circ f_h)[\cdot](e^{i\varphi}) \in \mathcal{L}(H_0^{k+2,\alpha}; C^{k+1,\alpha}(\mathbb{T}))$ is continuous.

∎

Let us give the corresponding result to Proposition 4.2 in case (B). To this end, we write

$$\begin{aligned}(U_h \circ f_h)(e^{i\varphi}) &= \int_{\mathbb{D}} |f_h'(y)|^2 \ln|f_h(e^{i\varphi}) - f_h(y)|\, dy \\ &= \int_{\mathbb{D}} |f_h'(e^{i\varphi} y)|^2 \ln|f_h(e^{i\varphi}) - f_h(e^{i\varphi} y)|\, dy.\end{aligned}$$

Proposition 4.6. *Let U_h be defined as in case (B) and let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ we have*

$$h \mapsto (U_h \circ f_h)(e^{i\varphi}) \in C^1(B_{\varepsilon_0}, C^{k+1,\alpha}(\mathbb{T})).$$

More precisely, for $h + g \in B_{\varepsilon_0}$ it holds that

$$D_h(U_h \circ f_h)[g](e^{i\varphi}) = \int_{\mathbb{D}} \sigma_h^1[g](\varphi, y) \ln d_h(\varphi, y)\, dy + \int_{\mathbb{D}} \frac{\sigma_h^2[g](\varphi, y)}{d_h(\varphi, y)^2} |f_h'(e^{i\varphi} y)|^2\, dy,$$

where $\sigma_h^1[g]$, $\sigma_h^2[g]$ and $d_h(\varphi, y)$ are given in (4.4).

4.3. Fréchet derivative of the full problem

Here we compute the Fréchet derivative of the second and third components of $\mathbb{F}$ in (2.10).

For the second component note that the continuous differentiability of the mapping $(h, X) \mapsto \nabla U_h(X)$ involves no complications since $X = (a, 0)$ is assumed to be close to $X_0 = (a_0, 0)$ with $a_0 \ge 2$. Hence, ∇U_h is smooth on a neighborhood of X_0 and

$$\nabla U_h(X) = \nu \int_{E_h} \frac{X - y}{|X - y|^{\nu+2}}\, dy = \nu \int_{\mathbb{D}} \frac{X - f_h(y)}{|X - f_h(y)|^{\nu+2}} |f_h'(y)|^2\, dy$$

in case (A), and

$$\nabla U_h(X) = \int_{\mathbb{D}} \frac{X - f_h(y)}{|X - f_h(y)|^2} |f_h'(y)|^2\, dy$$

in case (B). Here, we identify f_h with a function $\mathbb{D} \to \mathbb{R}^2$. We obtain the following result for the Fréchet derivative (we again identify $X = (a, 0) \in \mathbb{R}^2 \simeq \mathbb{C}$).

Lemma 4.7. *Let U_h be defined as in case (A) or case (B) and $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. The map $(h,a) \mapsto \partial_{x_1} U_h(a,0)$ is continuously differentiable for $|a - a_0| \le \varepsilon_0$, $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$, $\varepsilon_0 > 0$ sufficiently small, with derivative*

$$D_{(h,a)}(\partial_{x_1} U_h(a,0))[g,b] = \partial_{x_1}^2 U_h(a,0)b + W_{h,a}[g].$$

The function $W_{h,a}[g]$ is defined by

$$\begin{aligned} W_{h,a}[g] := \nu \int_{\mathbb{D}} & \frac{-\operatorname{Re}[g(y)]|f_h'(y)|^2 + 2\operatorname{Re}[(1+h'(y))\overline{g'(y)}]\operatorname{Re}[a - f_h(y)]}{|a - f_h(y)|^{\nu+2}}\, dy \\ & - \nu(\nu+2)\int_{\mathbb{D}} \frac{\operatorname{Re}[(a - f_h(y))\overline{g(y)}]\operatorname{Re}[a - f_h(y)]}{|a - f_h(y)|^{\nu+4}}|f_h'(y)|^2\, dy \end{aligned}$$

in case (A), and by

$$\begin{aligned} W_{h,a}[g] := \int_{\mathbb{D}} & \frac{-\operatorname{Re}[g(y)]|f_h'(y)|^2 + 2\operatorname{Re}[(1+h'(y))\overline{g'(y)}]\operatorname{Re}[a - f_h(y)]}{|a - f_h(y)|^{2}}\, dy \\ & - \int_{\mathbb{D}} \frac{2\operatorname{Re}[(a - f_h(y))\overline{g(y)}]\operatorname{Re}[a - f_h(y)]}{|a - f_h(y)|^{4}}|f_h'(y)|^2\, dy \end{aligned}$$

in case (B).

The mass constraint, i.e. the third component of $\mathbb{F}$, leads to the following Fréchet derivative:

Lemma 4.8. *The Fréchet derivative of the map*

$$h \mapsto |E_h| = \int_{\mathbb{D}} |f_h'(x)|^2\, dx$$

is given by

$$g \mapsto 2\operatorname{Re}\int_{\mathbb{D}} (1+h')\overline{g'}\, dx.$$

We omitted the proof of the previous lemmas since they can be easily checked. Finally, the following differentiability result follows from combining Lemma 4.1, Proposition 4.2, respectively Proposition 4.6, and Lemmas 4.7, 4.8.

Proposition 4.9. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There is $\varepsilon_0 > 0$ sufficiently small such that $\mathbb{F} \in C^1(U; \mathbb{Z}^{k+1,\alpha})$, where*

$$U = B_{\varepsilon_0}(0) \times (a_0 - \varepsilon_0, a_0 + \varepsilon_0) \times (\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0) \subset \mathbb{X}^{k+2,\alpha}.$$

5. Invertibility of the linearized operator

In order to apply the implicit function theorem we need to invert the linearized operator at the point $(0, X_0, \lambda_0, 0)$, i.e. the linear operator

$$\mathbb{X}^{k+2,\alpha} \to \mathbb{Z}^{k+1,\alpha} \colon (g,b,\mu) \mapsto D_{(h,a,\lambda)}\mathbb{F}(0,a_0,\lambda_0,0)[g,b,\mu]. \tag{5.1}$$

We recall that the functional spaces are defined in (2.9). It is convenient to write the function $g \in H_0^{k+2,\alpha}$ using power series of the form

$$g(z) = \sum_{n \geq 0} \hat{g}_n z^{n+1}. \tag{5.2}$$

Recall that $g(0) = 0$ and $g'(0) = \hat{g}_0 \in \mathbb{R}$ since $g \in H_0^{k+2,\alpha}$.

Remark 5.1. Let us briefly comment on the form of the power series (5.2).

(i) We choose an index shift in the coefficients of (5.2), in order that the linearized operator (5.1) is diagonalized when using a Fourier decomposition; cf. Lemma 5.7.

(ii) With this choice the coefficient $\hat{g}_0$ corresponds to a rescaling $z \mapsto (1 + \hat{g}_0)z$. Consequently, it appears in the linearization of $h \mapsto |E_h|$; cf. Lemma 5.7. In the Fourier series it appears as the zeroth coefficient.

(iii) Furthermore, infinitesimal translations are given by the conformal mappings T_ε: $z \mapsto z + \varepsilon$ for small $\varepsilon > 0$. In order to satisfy the conditions $T_\varepsilon(0) = 0$ and $T_\varepsilon'(0) \in \mathbb{R}$ we use a Blaschke factor (see (2.1)), yielding the conformal mapping

$$\begin{aligned} z &\mapsto \frac{z - \varepsilon}{1 - \varepsilon z} + \varepsilon = \frac{z - \varepsilon^2 z}{1 - \varepsilon z} = z + h_\varepsilon(z), \\ h_\varepsilon(z) &= \frac{\varepsilon z^2 - \varepsilon^2 z}{1 - \varepsilon z} = \varepsilon z^2 + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.3}$$

as $\varepsilon \to 0$. In particular, infinitesimal translations correspond to the coefficient of z^2, i.e. $\hat{g}_1$ in (5.2). In Fourier series they correspond to the coefficients for $e^{\pm i\varphi}$, which is $\hat{g}_1$ respectively $\overline{\hat{g}_1}$ in the linearization; cf. Lemma 5.7.

The main result of this section is the following proposition.

Proposition 5.2. *Let $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$. The operator (5.1) is an isomorphism under the assumptions of Theorem 2.1.*

For the purpose of proving Proposition 5.2 it is necessary to compute explicitly the form of the linear operator (5.1). This is done in the following subsections, which also contain further needed auxiliary results.

Linearization of the stream function. For the proof we write the operator $\bar{\phi}[g] := D_h\phi_h(0)[g]$, i.e. the Fréchet derivative of ϕ_h at $h = 0$, more explicitly. Due to Lemma 4.1 it solves the equation

$$\Delta\bar{\phi} = G'(\phi_0)\bar{\phi} + 2\,\mathrm{Re}[g']G(\phi_0), \quad \bar{\phi}|_{\partial\mathbb{D}} = 0. \tag{5.4}$$

Hence, using expression (5.2) we find that

$$2\,\mathrm{Re}[g'(re^{i\varphi})] = 2\,\mathrm{Re}\left[\sum_{n \geq 0}(n+1)\hat{g}_n r^n e^{in\varphi}\right] = \sum_{n \in \mathbb{Z}}(|n|+1)\hat{\xi}_n[g]r^{|n|}e^{in\varphi}, \tag{5.5}$$

where the coefficients $\hat{\xi}_n[g]$ are given by

$$\hat{\xi}_n[g] := \begin{cases} \hat{g}_n, & n \geq 1, \\ 2\hat{g}_0, & n = 0, \\ \overline{\hat{g}_n}, & n \leq -1. \end{cases}$$

Recall that $g'(0) = \hat{g}_0 \in \mathbb{R}$ since $g \in H_0^{k+2,\alpha}$. We use a Fourier decomposition to obtain the formula

$$\bar{\phi}[h](r, \varphi) = \sum_{n \in \mathbb{Z}} (|n| + 1)\hat{\xi}_n[g] A_n(r) e^{in\varphi},$$

where $A_n(r)$ solves the ordinary differential equation (see (2.14))

$$\frac{1}{r}(rA_n')' - \frac{n^2}{r^2} A_n - G_1 A_n = r^{|n|} G_0, \quad A_n(1) = 0. \tag{5.6}$$

Above we used the shortcut notation $G_0(r) := G(\phi_0(r))$, $G_1(r) := G'(\phi_0(r))$. Moreover, notice that $A_n = A_{-n}$ by symmetry.

The function $\bar{\phi}$ enters the linearization of $\mathbb{F}$ in the following way:

$$[\nabla\phi_0 \cdot \nabla\bar{\phi}](e^{i\varphi}) = \phi_0'(1)\partial_r\bar{\phi}(1, \varphi) = \phi_0'(1) \sum_{n \in \mathbb{Z}} (|n| + 1)\hat{\xi}_n[g] A_n'(1) e^{in\varphi}. \tag{5.7}$$

Recall that the unperturbed stream function ϕ_0 is radial. Hereafter we provide a crucial result concerning the asymptotics of $A_n'(1)$ as $n \to \infty$.

Lemma 5.3. *Consider the solution $\bar{\phi}$ of (5.4). Then the coefficients A_n have the asymptotics*

$$\lim_{n \to \infty} nA_n'(1) = \frac{G(\phi_0(1))}{2}. \tag{5.8}$$

Remark 5.4. Compare (5.8) with the explicit solutions for $G \equiv -2\Omega_0$ in Remark 2.3 (iii).

Proof of Lemma 5.3. Let $n \geq 1$ throughout the proof. Writing $\tilde{\phi}_n(r, \varphi) = A_n(r)e^{in\varphi}$ we have

$$(\Delta - G_1)\tilde{\phi}_n = r^n e^{in\varphi} G_0, \quad \tilde{\phi}_n(1, \varphi) = 0.$$

Since $G_1 = G' \circ \phi_0 \geq 0$, the operator $\Delta - G_1$ with zero boundary conditions is invertible. Furthermore, since $G_0 \in C^{k+3,\alpha}$ we have $\tilde{\phi}_n \in C^{k+5,\alpha}(\overline{\mathbb{D}})$.

Let us look at the following auxiliary ODE:

$$\frac{1}{r}(rb_n')' - \frac{n^2}{r^2} b_n = r^n G_0, \quad b_n(1) = 0.$$

Note that comparing it with the ODE solved by A_n given in (5.6), only the term G_1 is removed, which is expected to be of lower order for $n \to \infty$. It is convenient to write $b_n(r) = r^n \beta_n(r)$ with

$$\frac{1}{r}(r\beta_n')' + \frac{2n}{r}\beta_n' = G_0, \quad \beta_n(1) = 0.$$

We can find the solution explicitly up to a parameter

$$\beta_n(r) = -\frac{\beta_n'(1)}{2n}\frac{1-r^{2n}}{r^{2n}} - \frac{1}{2n}\int_r^1 G_0(s)s\,ds + \frac{1}{2n}\frac{1}{r^{2n}}\int_r^1 G_0(s)s^{2n+1}\,ds.$$

In order that β_n exists for $r \to 0$ we choose

$$B_n := \beta_n'(1) := \int_0^1 G_0(s)s^{2n+1}\,ds, \tag{5.9}$$

yielding

$$\beta_n(r) = \frac{B_n}{2n} - \frac{1}{2n}\int_r^1 G_0(s)s\,ds - \frac{1}{2n}\frac{1}{r^{2n}}\int_0^r G_0(s)s^{2n+1}\,ds.$$

Observe that $|\beta_n(r)| \le C/n$ for some constant $C > 0$ independent of r and n.

Let us now decompose

$$\tilde{\phi}_n(r,\varphi) = \tilde{\phi}_n^1(r,\varphi) + \tilde{\phi}_n^2(r,\varphi), \quad \tilde{\phi}_n^1(r,\varphi) := r^n\beta_n(r)e^{in\varphi}.$$

Accordingly, we get the decomposition

$$A_n(r) = A_n^1(r) + A_n^2(r), \quad A_n^1(r) = r^n\beta_n(r). \tag{5.10}$$

Hence, with the above calculations we have

$$(\Delta - G_1)\tilde{\phi}_n^2 = (\Delta - G_1)(\tilde{\phi}_n - \tilde{\phi}_n^1) = G_1\tilde{\phi}_n^1, \quad \tilde{\phi}_n^2|_{\partial\mathbb{D}} = 0.$$

We can apply regularity estimates in Sobolev spaces to obtain

$$\|\tilde{\phi}_n^2\|_{W^{2,2}(\mathbb{D})} \le C\|g\tilde{\phi}_n^1\|_{L^2(\mathbb{D})} \le \frac{C}{n}\left(\int_0^1 s^{2n+1}\,ds\right)^{1/2} \le \frac{C}{n^{3/2}}.$$

Here we used $|\tilde{\phi}_n^1(r,\varphi)| \le |\beta_n(r)| \le C/n$. Applying the trace theorem (cf. [19]) gives

$$\|\partial_r\tilde{\phi}_n^2(1,\cdot)\|_{L^2(\partial\mathbb{D})} \le \frac{C}{n^{3/2}},$$

and hence $|(A_n^2)'(1)| \le C/n^{3/2}$. In conclusion, combining (5.10) and (5.9) we find that

$$\begin{aligned}
\lim_{n\to\infty} nA_n'(1) &= \lim_{n\to\infty} n(A_n^1)'(1) = \lim_{n\to\infty} n\beta_n'(1) = \lim_{n\to\infty} nB_n \\
&= \lim_{n\to\infty} n\int_0^1 G_0(s)s^{2n+1}\,ds \\
&= \lim_{n\to\infty} n\left(\frac{G_0(1)}{2n+2} - \frac{1}{2n+2}\int_0^1 G_0'(s)s^{2n+2}\,ds\right) \\
&= \frac{G_0(1)}{2} = \frac{G(\phi_0(1))}{2},
\end{aligned}$$

showing the desired result. ∎

Linearization of the interaction potential, case (A). Due to Proposition 4.2 we have for $x = e^{i\varphi}$ (we use here the change of variables $e^{i\varphi} y \mapsto y$),

$$D_h(U_h \circ f_h)|_{h=0}[g](x) = -\int_{\mathbb{D}} \frac{2\,\mathrm{Re}[g'(y)]}{|x-y|^\nu}\,dy + \nu \int_{\mathbb{D}} \frac{\mathrm{Re}[(\overline{x-y})(g(x)-g(y))]}{|x-y|^{2+\nu}}\,dy.$$

As before we use the power series expansion for g given in (5.2). We have

$$D_h(U_h \circ f_h)|_{h=0}[g](x) = \sum_{n=0}^{\infty} \mathrm{Re}\Big[\hat{g}_n \int_{\mathbb{D}} \Big(\nu \frac{x^{n+1}-y^{n+1}}{x-y} - 2(n+1)y^n\Big)\frac{dy}{|x-y|^\nu}\Big].$$

Since $x = e^{i\varphi}$ we can use a rotation to obtain

$$\sum_{n=0}^{\infty} \mathrm{Re}\Big[\hat{g}_n e^{in\varphi} \int_{\mathbb{D}} \Big(\nu \frac{1-y^{n+1}}{1-y} - 2(n+1)y^n\Big)\frac{dy}{|1-y|^\nu}\Big].$$

Recalling the definition of c_n in (2.12), the fact that c_n are real and the definition of $\hat{\xi}_n[g]$ in (5.5), we obtain

$$D_h(U_h \circ f_h)|_{h=0}[g](e^{i\varphi}) = \sum_{n\in\mathbb{Z}} c_n \hat{\xi}_n[g] e^{in\varphi}, \tag{5.11}$$

where we define $c_n = c_{|n|}$ for $n < 0$. The following result will be useful:

Lemma 5.5. *The sequence c_n defined in (2.12) with $\nu = 1$ satisfies*

$$\lim_{n\to\infty} \frac{c_n}{\ln n} = \gamma_0 \quad \textit{with } \gamma_0 := \int_0^\infty \int_0^\infty \frac{e^{-r}\zeta \sin(\zeta)}{(r^2+\zeta^2)^{3/2}}\,dr\,d\zeta. \tag{5.12}$$

For $\nu \in (0,1)$ we have $\sup_{n\geq 0} |c_n| < \infty$.

Proof. First of all, we have

$$c_n = \sum_{k=0}^{n} \frac{\nu}{2} \int_{\mathbb{D}} \frac{y^k}{|1-y|^\nu}\,dy - (n+1)\int_{\mathbb{D}} \frac{y^n}{|1-y|^\nu}\,dy.$$

Let us define

$$\tilde{c}_k := \frac{1}{2}\int_{\mathbb{D}} \frac{y^k}{|1-y|^\nu}\,dy.$$

We show below that

$$\lim_{k\to\infty} k^{2-\nu}\tilde{c}_k = \gamma_0^\nu, \quad \gamma_0^\nu := \nu \int_0^\infty \int_0^\infty \frac{e^{-r}\zeta \sin(\zeta)}{(r^2+\zeta^2)^{(2+\nu)/2}}\,dr\,d\zeta. \tag{5.13}$$

Note that $\gamma_0^1 = \gamma_0$ for $\nu = 1$. With this we infer for $\nu = 1$,

$$\lim_{n\to\infty} \frac{c_n}{H_n} = \gamma_0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Since $H_n = \ln(n)(1+o(1))$ as $n \to \infty$, this implies the asymptotics (5.12) for $\nu = 1$. The claim for $\nu \in (0,1)$ is also a consequence of the above asymptotics.

We now prove (5.13). The term $\tilde{c}_k$ is real valued so that

$$\tilde{c}_k = \frac{1}{2}\int_0^1 \int_0^{2\pi} \frac{r^{k+1}\cos(k\varphi)}{(1+r^2-2r\cos\varphi)^{\nu/2}}\,d\varphi\,dr = I_k^1 + I_k^2. \tag{5.14}$$

The two terms I_k^1 and I_k^2 are defined by splitting the integral (5.14) with respect to r into the regions $(0,1/2)$ and $(1/2,1)$. We can readily check that

$$k^{2-\nu}|I_k^1| \le \frac{Ck^{2-\nu}}{2^{k+1}} \to 0. \tag{5.15}$$

To deal with the I_k^2 we notice that with the change of variables $r = 1-s$,

$$\begin{aligned} k^{2-\nu} I_k^2 &= \frac{k^{2-\nu}}{2}\int_0^{1/2}\int_0^{2\pi} \frac{(1-s)^{k+1}\cos(k\varphi)}{(s^2+4(1-s)\sin^2(\varphi/2))^{\nu/2}}\,d\varphi\,ds \\ &= \frac{k^{1-\nu}}{2}\int_0^{k/2}\int_0^{2\pi} \frac{(1-\frac{r}{k})^{k+1}\cos(k\varphi)}{[(\frac{r}{k})^2+4(1-\frac{r}{k})\sin^2(\frac{\varphi}{2})]^{\nu/2}}\,d\varphi\,dr. \end{aligned}$$

In the second equality we used the change of variables $ks = r$. Furthermore, writing $k\varphi = \psi$ we get

$$\begin{aligned} k^{2-\nu} I_k^2 &= \frac{1}{2}\int_0^{k/2}\int_0^{2k\pi} \frac{(1-\frac{r}{k})^{k+1}\cos(\psi)}{[r^2+4(1-\frac{r}{k})k^2\sin^2(\frac{\psi}{2k})]^{\nu/2}}\,d\psi\,dr \\ &= \int_0^{k/2}\int_0^{k\pi} \frac{(1-\frac{r}{k})^{k+1}\cos(\psi)}{[r^2+4(1-\frac{r}{k})k^2\sin^2(\frac{\psi}{2k})]^{\nu/2}}\,d\psi\,dr, \end{aligned}$$

where we used the symmetry in the last equality. Let us now define the function $\zeta_k\colon (0,k\pi) \to (0,2k)$ by

$$\zeta_k(\psi) = 2k\sin\Big(\frac{\psi}{2k}\Big),$$

which is one-to-one and onto. Furthermore, by a Taylor expansion one can show that $\zeta_k(\psi) \to \psi$ for any $\psi \in (0,k\pi)$ as $k \to \infty$. Consequently, we have for the inverse function $\psi_k(\zeta) \to \zeta$ as $k \to \infty$. We obtain by the change of variables $\psi \mapsto \zeta$,

$$k^{2-\nu} I_k^2 = \int_0^{k/2}\int_0^{2k} \frac{(1-\frac{r}{k})^{k+1}}{[r^2+(1-\frac{r}{k})\zeta^2]^{\nu/2}}\cos(\psi_k(\zeta))\psi_k'(\zeta)\,d\zeta\,dr.$$

We now use an integration by parts in ζ to obtain (note that the boundary terms vanish since $\psi_k(0) = 0$, $\psi_k(2k) = k\pi$)

$$k^{2-\nu} I_k^2 = \nu\int_0^{k/2}\int_0^{2k} \frac{(1-\frac{r}{k})^{k+2}\zeta\sin(\psi_k(\zeta))}{[r^2+(1-\frac{r}{k})\zeta^2]^{(\nu+2)/2}}\,d\zeta\,dr.$$

The integrand converges pointwise to

$$\frac{e^{-r}\zeta\sin(\zeta)}{(r^2+\zeta^2)^{(\nu+2)/2}}$$

as $k\to\infty$. Since

$$\Big(1-\frac{r}{k}\Big)^{k+2}=\exp\Big((k+2)\ln\Big(1-\frac{r}{k}\Big)\Big)\le e^{-r},$$
$$\psi_k(\zeta)=2k\arcsin\Big(\frac{\zeta}{2k}\Big)\le C\zeta,$$

for say $\zeta\in(0,1)$, a majorant is given by

$$\frac{e^{-r}\min(C\zeta^2,\zeta)}{(r^2+\zeta^2/2)^{(2+\nu)/2}}.$$

Hence, we get $k^{2-\nu}I_k^2\to\gamma_0^\nu$. Combining this with (5.15) and (5.14) yields (5.13). ∎

Linearization of the interaction potential, case (B). By Proposition 4.6 we have for $x=e^{i\varphi}$,

$$\begin{aligned}&D_h(U_h\circ f_h)|_{h=0}[g](x)\\&\quad=2\int_{\mathbb{D}}\ln|x-y|\operatorname{Re}[g'(y)]\,dy+\int_{\mathbb{D}}\frac{\operatorname{Re}[(\overline{x-y})(g(x)-g(y))]}{|x-y|^2}\,dy.\end{aligned}$$

Again, we use the power series expansion for g (cf. (5.2)), yielding

$$\begin{aligned}&D_h(U_h\circ f_h)|_{h=0}[g](e^{i\varphi})\\&\quad=\sum_{n=0}^{\infty}2\operatorname{Re}\Big[\hat g_n\int_{\mathbb{D}}\Big((n+1)y^n\ln|x-y|+\frac12\frac{x^{n+1}-y^{n+1}}{x-y}\Big)\,dy\Big].\end{aligned}$$

For $x=e^{i\varphi}$ and applying the change of variables $y\mapsto e^{i\varphi}y$ gives

$$\sum_{n=0}^{\infty}2\operatorname{Re}\Big[\hat g_n e^{in\varphi}\int_{\mathbb{D}}\Big((n+1)y^n\ln|1-y|+\frac12\frac{1-y^{n+1}}{1-y}\Big)\,dy\Big].$$

As we will see below we have (see (2.13))

$$c_n=\int_{\mathbb{D}}\Big((n+1)y^n\ln|1-y|+\frac12\frac{1-y^{n+1}}{1-y}\Big)\,dy=\begin{cases}\frac{\pi}{2}\Big(1-\frac1n\Big), & n\ge 1,\\ \frac{\pi}{2}, & n=0.\end{cases}\tag{5.16}$$

Recalling the definition of $\hat\xi_n[g]$ in (5.5), we have (see (5.11))

$$D_h(U_h\circ f_h)|_{h=0}[g](e^{i\varphi})=\sum_{n\in\mathbb{Z}}c_n\hat\xi_n[g]e^{in\varphi},$$

where we again define $c_n=c_{|n|}$ for $n<0$.

Let us now prove (5.16). For $n = 0$ the integral reduces to $U_0(1) + \pi/2 = \pi/2$; cf. Lemma 3.5. For the other cases let us first observe that

$$\frac{1}{2}\int_{\mathbb{D}} \frac{1-y^{n+1}}{1-y}\,dy = \frac{1}{2}\sum_{k=0}^{n}\int_{\mathbb{D}} y^k\,dy = \frac{1}{2}\sum_{k=0}^{n}\int_0^{2\pi}\int_0^1 r^k e^{ik\varphi} r\,dr\,d\varphi = \frac{\pi}{2}. \tag{5.17}$$

Moreover, we can also write

$$(n+1)\int_{\mathbb{D}} y^n \ln|1-y|\,dy = (n+1)\int_0^1\int_0^{2\pi} r^n e^{in\varphi}\ln|1-re^{i\varphi}| r\,dr\,d\varphi. \tag{5.18}$$

Hence, using the expansion

$$\begin{aligned}\ln|1-re^{i\varphi}| &= \frac{1}{2}(\ln(1-re^{i\varphi}) + \ln(1-re^{-i\varphi}))\\ &= -\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{r^{k+1}e^{i(k+1)\varphi}}{k+1} + \sum_{k=0}^{\infty}\frac{r^{k+1}e^{-i(k+1)\varphi}}{k+1}\right),\end{aligned}$$

and plugging it in (5.18) we find that

$$(n+1)\int_{\mathbb{D}} y^n \ln|1-y|\,dy = -2\pi\frac{n+1}{2n}\int_0^1 r^{2n+1}\,dr = -\frac{\pi}{2n}.$$

Thus, combining (5.17) and (5.18) we infer (5.16).

Remark 5.6. Let us note that in both cases (A) and (B) we have $c_1 = 0$. This holds in general since $(U_{h_\varepsilon}\circ f_{h_\varepsilon})(1) = U_0(1)$, where $h_\varepsilon(z) = \varepsilon z^2 + \mathcal{O}(\varepsilon^2)$ is associated to translations; see formula (5.3) in Remark 5.1. We hence obtain

$$c_1 = D_h(U_h\circ f_h)|_{h=0}[z^2](1) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(U_{h_\varepsilon}\circ f_{h_\varepsilon})(1) = 0.$$

As mentioned in Remark 5.1, the effects of conformal mappings due to translations appear to first order in the Fourier modes $n = \pm 1$ and thus in the coefficient c_1.

Linearization of the full problem. We summarize the full linearized operator at $(h_0 \equiv 0, a_0, \lambda_0, m = 0)$ in the following lemma:

Lemma 5.7. *The operator $D_{(h,a,\lambda)}\mathbb{F}(0,a_0,\lambda_0,0)$ has the form*

$$(g,b,\mu) \mapsto \begin{pmatrix}\mathcal{L}g-\mu\\ \Omega_0^2 b - \partial_{x_1}^2 U_0(a_0,0)b - W_{0,a_0}[g]\\ \pi\hat{h}_0\end{pmatrix},$$

$$\mathcal{L}g(\varphi) := 2\omega_0\hat{g}_0 + \sum_{n\geq 1}\omega_n\hat{g}_n e^{in\varphi} + \sum_{n\leq -1}\omega_n\overline{\hat{g}_{|n|}}e^{in\varphi},$$

$$\omega_n = -\frac{1}{2}\phi_0'(1)^2(|n|+1) + \phi_0'(1)A_n'(1)(|n|+1) - \frac{1}{2}\Omega_0^2 + c_{|n|}.$$

Here, $W_{0,a_0}[g]$ is defined in Lemma 4.7 in both cases (A) and (B).

Note that in the last component of the linearized operator we again identify $\mathbb{R}^2 \simeq \mathbb{C}$. Furthermore, the coefficients ω_n have appeared already in (2.11).

Proof of Lemma 5.7. The first component of $\mathbb{F}$ in (2.10) has the linearization at the point $(h = 0, X_0, \lambda_0, m = 0)$

$$\begin{aligned}(g, \mu) \mapsto \phi_0'(1)\partial_r\bar{\phi}(1,\varphi) - \phi_0'(1)^2 \operatorname{Re}[g'(e^{i\varphi})] \\ - \Omega_0^2 \operatorname{Re}[e^{-i\varphi} g(e^{i\varphi})] + D_h(U_h \circ f_h)|_{h=0}[g](e^{i\varphi}) - \mu.\end{aligned}$$

Using (5.2) and (5.5) we obtain

$$\operatorname{Re}[e^{-i\varphi} g(e^{i\varphi})] = \frac{1}{2}\sum_{n\in\mathbb{Z}} \hat{\xi}_n[g]e^{in\varphi}.$$

Using both (5.7) and (5.11) yields the expression for the first component. Applying the definition of $\hat{\xi}_n[g]$ in (5.5) yields the form of the operator $\mathcal{L}g$.

The linearization of the second component $\mathbb{F}$ in (2.10) is a consequence of Lemma 4.7. For the last component, note that the linearization of the mass constraint in Lemma 4.8 becomes $g \mapsto \pi \operatorname{Re}[\hat{g}_0] = \pi \hat{g}_0$, since $g \in H_0^{k+2,\alpha}$. This concludes the proof. ∎

Before providing the proof of Proposition 5.2 we need to show the following result on Fourier multipliers on the torus in Hölder spaces.

Lemma 5.8. *Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Consider a sequence $\beta = (\beta_n)_n$ of the form $\beta_n = \kappa/(|n| + b_n)$, $\beta_0 = 0$, $n \in \mathbb{Z}$ with some real constant $\kappa \neq 0$. Assume that $b_n \neq -|n|$ is a sequence satisfying $\sup_{n\neq 0} |b_n|\,|n|^{-\gamma} \leq C$ for some $0 \leq \gamma \leq 1/2$. Then the periodic pseudo-differential operator*

$$\mathsf{OP}(\beta)\xi(\varphi) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \beta_n \hat{\xi}_n e^{in\varphi}$$

defines a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k+1,\alpha}(\mathbb{T})$.

Proof. Recall that the Hilbert transform $\mathcal{H}$ defined by the Fourier multipliers $-i\operatorname{sgn}(n)$ is a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k,\alpha}(\mathbb{T})$ for all $k \in \mathbb{N}$, $\alpha \in (0,1)$. Since the operator with multiplier $1/in$ corresponds to integration, we conclude that the operator with multiplier $1/|n| = i\operatorname{sgn}(n)/in$ is a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k+1,\alpha}(\mathbb{T})$.

We now write

$$\beta_n = \frac{\kappa}{|n|} - \frac{\kappa}{|n|}\cdot\frac{b_n}{(|n| + b_n)} = \frac{\kappa}{|n|}(1 + r_n).$$

By assumption it holds that $c_1 \leq |1 + b_n/|n||$ for some constant $c_1 > 0$. Hence, we have

$$|r_n| \leq \frac{|b_n|}{c_1|n|} \leq \frac{C}{|n|^{1-\gamma}} \leq \frac{C}{|n|^{1/2}}.$$

Thus, the sequence $r = (r_n)_n$ satisfies the ρ-condition in [10, Theorem 3.1] with $\rho = 1/2$ and hence $\mathtt{OP}(r)$ constitutes a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k,\alpha}(\mathbb{T})$ for all $k \in \mathbb{N}$, $\alpha \in (0,1)$. In the mentioned reference, periodic Besov space $B^s_{\infty,\infty}$ have been used. Recall that $B^s_{\infty,\infty}$ coincides with the classical Hölder space $C^{k,\alpha}(\mathbb{T})$ for $s = k + \alpha \notin \mathbb{N}$. This concludes the proof. ■

Proof of Proposition 5.2. We consider cases (A) and (B) simultaneously, since the proof is the same. Given $(S, Z, M) \in \mathbb{Z}^{k+1,\alpha} = C^{k+1,\alpha}(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}$ we want to solve for $(g, b, \mu) \in H_0^{k+2,\alpha} \times \mathbb{R} \times \mathbb{R}$ the equations

$$\begin{aligned} \mathcal{L}g - \mu &= S, \\ \Omega_0^2 b - \partial_{x_1}^2 U_0(a_0, 0) b - W_{0,a_0}[g] &= Z, \\ \pi \hat{g}_0 &= M. \end{aligned} \tag{5.19}$$

First, we have $\hat{g}_0 = M/\pi$. For the first equation in (5.19) we decompose S in its Fourier coefficients $(\widehat{S}_n)_{n\in\mathbb{Z}}$. Then the first equation in (5.19) becomes

$$\sum_{n\geq 1} \omega_n \hat{g}_n e^{in\varphi} + \sum_{n\leq -1} \omega_n \overline{\hat{g}_{|n|}} e^{in\varphi} = \widehat{S}_0 - \frac{2\omega_0 M}{\pi} + \mu + \sum_{n\geq 1} \widehat{S}_n e^{in\varphi} + \sum_{n\leq -1} \overline{\widehat{S}_{|n|}} e^{in\varphi}.$$

Recall that $\overline{\widehat{S}_{-n}} = \widehat{S}_n$ for $n \geq 0$ since S is a real-valued function. We then choose $\mu = 2\omega_0 M/\pi - \widehat{S}_0$. Since the multipliers ω_n of $\mathcal{L}$ are non-zero by assumption (2.15), we can define $\mathcal{L}^{-1} = \mathtt{OP}(\omega_n^{-1})$. By Lemmas 5.3 and 5.5 we can write

$$\omega_n = \frac{|n| + b_n}{\kappa}, \quad \kappa^{-1} := -\phi_0'(1)^2,$$

with $\sup_{n\neq 0} |b_n| |n|^{-\gamma} \leq C$ for any $\gamma > 0$. Note that by our assumption in Theorem 2.1 we also have $\phi_0'(1) \neq 0$. We can hence apply Lemma 5.8 yielding $F \in C_0^{k+2,\alpha}(\mathbb{T})$ defined by

$$F = \mathtt{OP}(\omega_n^{-1})(S - \widehat{S}_0).$$

Note that F is real valued with $\widehat{F}_n = \widehat{S}_n/\omega_n$ for $n \geq 1$.

The function F is only defined on the torus. We now define the function g from F via

$$g(z) = \hat{g}_0 z + \sum_{n\geq 1} \widehat{F}_n z^{n+1} = \frac{M}{\pi} z + \sum_{n\geq 1} \frac{\widehat{S}_n}{\omega_n} z^{n+1}. \tag{5.20}$$

We need to show that $g \in H_0^{k+2,\alpha}$. To this end, define the function $\widetilde{F} := \frac{1}{2}(I + \mathcal{H})F$, recalling that $\mathcal{H}$ denotes the Hilbert transform. The function $\widetilde{F}$ has the Fourier decomposition

$$\widetilde{F}(\varphi) = \sum_{n\geq 1} \widehat{F}_n e^{in\varphi}, \quad \|\widetilde{F}\|_{C^{k+2,\alpha}(\mathbb{T})} \leq \|F\|_{C^{k+2,\alpha}(\mathbb{T})}.$$

The last inequality follows from the fact that $\mathcal{H}\colon C^{k+2,\alpha}(\mathbb{T}) \to C^{k+2,\alpha}(\mathbb{T})$ is bounded with $\|\mathcal{H}\| = 1$. Since $\widetilde{F}$ contains only Fourier modes $n \geq 0$, there is a unique holomorphic extension in $C^{k+2,\alpha}(\overline{\mathbb{D}})$. This extension has the power series expansion

$$\widetilde{F}(z) = \sum_{n\geq 1} \widehat{F}_n z^{n+1}.$$

Consequently, the function $g(z) := \hat{g}_0 z + \widetilde{F}(z) \in H_0^{k+2,\alpha}$ satisfies (5.20) and hence also (5.19).

Finally, we determine b in (5.19). To this end, we need to solve

$$(\Omega_0^2 - U_0''(a_0))b = Z + W_{0,a_0}[g].$$

At this point $W_{0,a_0}[g]$ is a determined real number. We observe that due to (2.8) and Lemma 3.6,

$$\begin{aligned}\Omega_0^2 - U_0''(a_0) &= \frac{U_0'(a_0)}{a_0} - U_0''(a_0) = -a_0\Big(-\frac{U_0'(a_0)}{a_0^2} + \frac{U_0''(a_0)}{a_0}\Big)\\ &= -a_0 \frac{d}{dr}\Big|_{r=a_0}\Big[\frac{U_0'(r)}{r}\Big] > 0.\end{aligned}$$

Thus, we can invert the above equation in terms of b.

The above arguments show that $D_{(h,a,\lambda)}\mathbb{F}(0, a_0, \lambda_0, 0)$ is one-to-one and onto. Hence, it is an isomorphism which concludes the proof. ∎

6. Proof of Theorem 2.1 and consequences

In this last section we first provide the proof of Theorem 2.1. We also include the details towards Corollary 2.2, which is a direct consequence of the previous main result.

Proof of Theorem 2.1. Due to Proposition 4.9 the function $\mathbb{F}$ is continuously differentiable. Under assumption (2.15) and by Proposition 5.2 we can invert the linearized operator $D_{(h,a,\lambda)}\mathbb{F}(0, a_0, \lambda_0, 0)$. Hence, we can apply the implicit function theorem; see Lemma 3.3. This concludes the proof. ∎

Proof of Corollary 2.2. For the sake of clarity we divide the proof into three steps.

Step 1*: Symmetry.* We first prove the symmetry of the domain E_h. To this end, we show that the function $g(z) := \overline{h(\bar{z})} \in H_0^{k+2,\alpha}$ satisfies $\mathbb{F}(g, a, \lambda, m) = 0$. Note that g induces a conformal map f_g which parameterizes the domain $R(E_h)$, where $R(x_1, x_2) = (x_1, -x_2)$. As a consequence of the uniqueness of solutions to (2.3), the stream function satisfies $\psi_g(x) = \psi_h(Rx)$. Furthermore, we have, recalling $X = (a, 0)$,

$$\begin{aligned}U_g(x) &= U_{R(E_h)}(x) = U_h(Rx),\\ U_X(x) &= U_X(Rx).\end{aligned}$$

Since (h, a, λ, m) is a solution, we obtain from (2.6), which is equivalent to the first component of $\mathbb{F}$, and application of $x \mapsto Rx$,

$$\frac{1}{2}|\nabla^{\perp}\psi_g(x)|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_g(x) + mU_X(x) = \lambda, \quad x \in \partial E_g.$$

The other components of $\mathbb{F}(g, a, \lambda, m) = 0$ follow in the same manner. By the uniqueness statement of the implicit function theorem we have $f_h(z) = f_g(z) = \overline{f_h(\bar{z})}$, i.e. the domain E_h is symmetric.

Step 2: Solution. The symmetry of the domain E_h implies

$$\partial_{x_2} U_h(X) = 0 = \Omega_0^2 X_2.$$

We can now define the non-hydrostatic pressure P as in (2.5) and observe that all equations but the last one in system (1.6) are satisfied for $v = \nabla^{\perp}\psi_h$, $X = (a, 0)$ and P.

Step 3: Center of mass. We now show that the last equation in (1.6) is a consequence of the other equations in (1.6). More precisely, they imply that the center of mass is zero:

$$X_c := \frac{1}{\pi + m}\left(\int_{E_h} x\,dx + mX\right) = 0.$$

Combining the first equation and the fifth equation in (1.6) gives

$$(\pi + m)\Omega_0^2 X_c = \int_{E_h} ((v \cdot \nabla)v + 2\Omega_0 J v + \nabla P)\,dx + m\nabla U_h(X).$$

Since $(v \cdot \nabla)v = \operatorname{div}(v \otimes v)$ and $v \cdot n_h = 0$ on ∂E_h, the first term is zero. Furthermore, due to $v = \nabla^{\perp}\psi_h = J\nabla\psi_h$ and $\psi_h = 0$ on ∂E_h we have

$$\int_{E_h} J v\,dx = -\int_{E_h} \nabla\psi_h\,dx = -\int_{\partial E_h} \psi_h n_h\,dS = 0.$$

We have for the non-hydrostatic pressure,

$$\int_{E_h} \nabla P\,dx = \int_{E_h} P n_h\,dS = \int_{\partial E_h} (U_h + mU_X) n_h\,dS,$$

where we used the fourth equation in (1.6). Furthermore, we have in case (A),

$$\begin{aligned} m\nabla U_h(X) &= -m\int_{E_h} \nabla_y\left[\frac{1}{|X-y|^{\nu}}\right]dy = -m\int_{\partial E_h} \frac{n_h}{|X-y|^{\nu}}\,dS(y) \\ &= -m\int_{\partial E_h} U_X n_h\,dS. \end{aligned}$$

In case (B) we get a corresponding equality. This yields

$$(\pi + m)\Omega_0^2 X_c = \int_{\partial E_h} U_h n_h\,dS.$$

By symmetry of the interaction potential we obtain in case (A),

$$\int_{\partial E_h} U_h n_h \, dS = \int_{E_h} \nabla U_h(x) \, dx = \nu \int_{E_h} \int_{E_h} \frac{x-y}{|x-y|^{\nu+2}} \, dx \, dy = 0.$$

However, this argument holds only for $\nu < 1$ due to the singularity. For $\nu = 1$ we use an approximation. The same conclusion holds in case (B). This implies $X_c = 0$, since $\Omega_0 \neq 0$, which concludes the proof. ∎

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