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Isoperimetric inequality for Finsler manifolds with non-negative Ricci curvature

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Abstract. We prove a sharp isoperimetric inequality for measured Finsler manifolds having non-negative Ricci curvature and Euclidean volume growth. We also prove a rigidity result for this inequality, under the additional hypotheses of boundedness of the isoperimetric set and the finite reversibility of the space. As a consequence, we deduce the rigidity of the weighted anisotropic isoperimetric inequality for cones in the Euclidean space, in the irreversible setting.

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1. Introduction

The classical isoperimetric inequality in the Euclidean space states that if E is a (sufficiently regular) subset of $\mathbb{R}^d$, then

$$\mathsf{P}(E) \geq d\,\omega_d^{1/d}\,\mathscr{L}^d(E)^{1-1/d}, \tag{1.1}$$

where $\mathsf{P}(E)$ denotes the perimeter of the set E, ω_d the measure of the unit ball in $\mathbb{R}^d$, and $\mathscr{L}^d$ the Lebesgue measure. Moreover, if the equality is attained in (1.1) by a certain

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set E with positive measure, then the set E coincides with a ball of radius $(\mathscr{L}^d(E)/\omega_d)^{1/d}$. This inequality has been successfully extended in more general settings where the space is not the Euclidean one. Indeed, it turns out that two are the relevant properties of the Euclidean space needed for such generalizations: (1) the fact that $\mathbb{R}^d$ has non-negative Ricci curvature; (2) its Euclidean volume growth, i.e., a constraint on the growth of the measure of large balls.

If (X, g) is an n-dimensional Riemannian manifold, one can consider different measures to the canonical volume Vol_g. In this case, the Ricci tensor has to be replaced with the generalised N-Ricci tensor: if $h\colon X \to (0,\infty)$ is a weight for the volume Vol_g, the generalised N-Ricci tensor, with $N > n$, is defined by

$$\mathrm{Ric}_N := \mathrm{Ric} - (N-n)\,\frac{\nabla^2 h^{1/(N-n)}}{h^{1/(N-n)}}. \tag{1.2}$$

We say that the weighted manifold $(X, g, h\,\mathrm{Vol}_g)$ verifies the so-called curvature-dimension condition $\mathsf{CD}(K,N)$ (see [6]) whenever $\mathrm{Ric}_N \geq Kg$.

In their seminal works, Lott–Villani [30] and Sturm [44, 45] introduced a synthetic definition of $\mathsf{CD}(K,N)$ for complete and separable metric spaces (X, d) endowed with a (locally-finite Borel) reference measure $\mathfrak{m}$ (“metric-measure space", or m.m.s.). This $\mathsf{CD}(K,N)$ condition is formulated using the theory of optimal transport, and it coincides with the Bakry–Émery one in the smooth Riemannian setting (and in particular, in the classical non-weighted one).

A measured Finsler manifold is a triple $(X, F, \mathfrak{m})$, such that X is a differential manifold (possibly with boundary), $\mathfrak{m}$ a Borel measure, and F a Finsler structure, that is, a real-valued function $F\colon TX \to [0,\infty)$ which is convex, positively homogeneous, and $F(v) = 0$ if and only if $v = 0$ (see Section 2.1 for the precise definition). In general, $F(v) \neq F(-v)$. This feature, know as irreversibility, is what prevents to apply the techniques developed for m.m.s.'s to measured Finsler manifolds. Recently, Ohta successfully extended the theory of the curvature-dimension condition for possibly-irreversible Finsler manifolds (see [35, 38, 40]). Namely, a notion of N-Ricci curvature (compatible with the Riemannian one) was introduced and it was proven that a measured Finsler manifold satisfies the $\mathsf{CD}(K,N)$ condition if and only if $\mathrm{Ric}_N \geq K$. More recently, the notion of irreversible metric measure space has been introduced [29].

The isoperimetric problem in the reversible setting has been extensively investigated. E. Milman [32] gave a sharp isoperimetric inequality for weighted Riemannian manifolds satisfying the $\mathsf{CD}(K,N)$ condition (for any $K \in \mathbb{R}$, $N > 1$), with an additional constraint on the diameter. In particular, given $K \in \mathbb{R}$, $N > 1$ and $D \in (0,\infty]$, he gave an explicit description of the so-called *isoperimetric profile* function $\mathcal{I}_{K,N,D}\colon [0,1] \to \mathbb{R}$. The isoperimetric profile has the property that, given a weighted Riemannian manifold satisfying the $\mathsf{CD}(K,N)$ condition with diameter at most D, whose total measure is 1, it holds that $\mathsf{P}(E) \geq \mathcal{I}_{K,N,D}(v)$, for any subset E of measure $v \in [0,1]$; moreover, Milman's result is sharp. Cavalletti and Mondino [14] extended Milman's result to the non-smooth setting finding the same lower bound. Their proof makes use of the localisation method (also known as needle decomposition), a powerful dimensional reduction tool, initially developed by Klartag [27] for Riemannian manifolds, and later extended to $\mathsf{CD}(K,N)$ spaces [14].

In the setting of measured Finsler manifold, less is known. Following the line traced in [14], Ohta [39] extended the localization method to measured Finsler manifolds, obtaining a lower bound for the perimeter for measured Finsler manifolds with finite reversibility constant. The reversibility constant (introduced in [43]) of a Finsler structure F on the manifold X is defined as the least constant (possibly infinite) $\Lambda_F \geq 1$ such that $F(-v) \leq \Lambda_F F(v)$ for all vectors $v \in TX$. Ohta proved [39] that given a measured Finsler manifold $(X, F, \mathfrak{m})$ having finite reversibility constant and $\mathfrak{m}(X) = 1$ satisfying the $\mathsf{CD}(K, N)$ condition, with diameter bounded from above by D, it holds that

$$\mathsf{P}(E) \geq \Lambda_F^{-1} \mathcal{I}_{K,N,D}(\mathfrak{m}(E)), \quad \forall E \subset X,$$

where $\mathcal{I}_{K,N,D}$ is the isoperimetric profile function described by E. Milman. The presence of the factor Λ_F^{-1} suggests that the inequality above is not sharp. Indeed, in the case $N = D = \infty$, this factor can be eliminated obtaining the Barky–Ledoux isoperimetric inequality for Finsler manifolds [41].

Regarding the case $K = 0$, in order to generalize the classical inequality (1.1), we must drop the assumption that the space has measure 1 and consider the case when the measure is infinite. However, it is well known that without an additional condition on the geometry of the space, no non-trivial isoperimetric inequality holds in the case $K = 0$. A way to create an Euclidean-like environment is to impose an additional constraint on the growth of the measure of the balls that mimics the Euclidean one. Letting $B^+(x, r) = \{y : \mathsf{d}(x, y) < r\}$ denote the *forward metric ball* with center $x \in X$ and radius $r > 0$, by the Bishop–Gromov inequality (see (2.7)), the map $r \mapsto \mathfrak{m}(B^+(r, x))/r^N$ is nonincreasing over $(0, \infty)$ for any $x \in X$. The *asymptotic volume ratio* is then naturally defined by

$$\mathsf{AVR}_X = \lim_{r \to \infty} \frac{\mathfrak{m}(B^+(x, r))}{\omega_N r^N}. \tag{1.3}$$

It is easy to see that it is indeed independent of the choice of $x \in X$. When $\mathsf{AVR}_X > 0$, we say that space has *Euclidean volume growth.*

In the Riemannian setting, the isoperimetric inequality for spaces with Euclidean volume growth has been obtained in increasing generality (see, e.g., [1, 10, 23, 25]). The most general result is due to Balogh and Kristály [7], and it is valid for (non-smooth) $\mathsf{CD}(0, N)$ spaces; it was proven exploiting the Brunn–Minkowski inequality for $\mathsf{CD}(0, N)$ spaces.

Theorem 1.1 (Theorem 1.1 in [7]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. satisfying the $\mathsf{CD}(0, N)$ condition for some $N > 1$ and having Euclidean volume growth. Then for every bounded Borel subset $E \subset X$, it holds*

$$\mathfrak{m}^+(E) \geq N\omega_N^{1/N} \mathsf{AVR}_X^{1/N} \mathfrak{m}(E)^{(N-1)/N}. \tag{1.4}$$

Moreover, inequality (1.4) *is sharp.*

In inequality (1.4), $\mathfrak{m}^+$ denotes the Minkowski content. In Appendix B, we will discuss the relation between the Minkowski content and the perimeter; for mildly regular sets, these two notions coincide. Using the Γ-function, one can naturally define the constant ω_N for all $N > 1$.

The rigidity of the inequality has been obtained, under two mild assumptions, by the author and Cavalletti [13]. These assumption are: (1) the essentially-non-branching-ness of the space, that excludes pathological cases; (2) the fact that the set attaining equality in (1.4) is bounded.

Theorem 1.2 (Theorem 1.4 in [13]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying the $\mathsf{CD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Let $E \subset X$ be a bounded Borel set that saturates (1.4).*

Then there exists (a unique) $o \in X$ such that, up to a negligible set, $E = B^+(o, \rho)$, with $\rho = (\mathfrak{m}(E)/\mathsf{AVR}_X\omega_N)^{1/N}$. Moreover, the measure $\mathfrak{m}$ can be represented by the disintegration formula

$$\mathfrak{m} = \int_{\partial B^+(o,\rho)} \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \quad \text{with} \quad \mathfrak{q} \in \mathscr{P}(\partial B^+(o, \rho)),\ \mathfrak{m}_\alpha \in \mathscr{M}_+(X),$$

where $\mathfrak{m}_\alpha$ is concentrated on the geodesic ray from o through α, and $\mathfrak{m}_\alpha$ can be identified (via the unitary speed parametrization of the ray) with $N\omega_N \mathsf{AVR}_X t^{N-1} \mathscr{L}^1 \llcorner_{[0,\infty)}$.

As a consequence of this result, having in mind the fact that "volume cone implies metric cone" [19], we obtain that when the space is $\mathsf{RCD}(0, N)$, we also have a rigidity of the metric, i.e., the space is isometric to a cone over an $\mathsf{RCD}(N-1, N-2)$ space. In the case of non-collapsed RCD spaces, the hypothesis on the boundedness of the set can be lifted [4, 5] (see also [9]).

The scope of the present paper is to extend Theorems 1.1 and 1.2 to the setting of irreversible measurable Finsler manifolds.

1.1. The result

The first result of this paper is the following.

Theorem 1.3. *Let $(X, F, \mathfrak{m})$ be a forward-complete measured Finsler manifold (possibly with boundary) satisfying the $\mathsf{CD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Then for every bounded Borel subset $E \subset X$, it holds*

$$\mathsf{P}(E) \geq N\omega_N^{1/N} \mathsf{AVR}_X^{1/N} \mathfrak{m}(E)^{(N-1)/N}. \tag{1.5}$$

Moreover, inequality (1.5) is sharp.

As we already said, the possible irreversibility of the manifolds does not permit to simply apply Theorem 1.1 to Finsler manifolds. In order to prove Theorem 1.3, we will exploit the Brunn–Minkowski inequality, which holds true also for Finsler manifolds. This strategy was used by Balogh and Kristály [7] for proving Theorem 1.1, and here we introduce no real new idea. Indeed, in the light of [29], it seems that this inequality holds true also for irreversible metric measure spaces; here we confine ourself to the setting of measured Finsler manifolds. To the best of the author knowledge, besides the Barky–Ledoux inequality [39], there is no other isoperimetric inequality for measured Finsler manifolds that does not involve the reversibility constant.

The main result of this paper concerns the rigidity property of the isoperimetric inequality (1.5). To characterise its minima, we will have to additionally require the reversibility constant Λ_F to be finite. This hypothesis is quite expected since in Finsler manifolds with infinite reversibility certain pathological behaviors may arise (e.g., the Sobolev spaces may not be vector spaces [22, 28]).

Theorem 1.4. *Let $(X, F, \mathfrak{m})$ be a forward-complete measured Finsler manifold (possibly with boundary) satisfying the* $\mathsf{CD}(0, N)$ *condition for some $N > 1$, and having reversibility constant $\Lambda_F < \infty$. Assume that ∂X is locally forward convex (see Definition 2.4). Let $E \subset X$ be a bounded Borel set that saturates (1.5).*

Then there exists (a unique) $o \in X$ such that, up to a negligible set, $E = B^+(o, \rho)$, with $\rho = (\mathfrak{m}(E)/\mathsf{AVR}_X \omega_N)^{1/N}$. Moreover, the measure $\mathfrak{m}$ can be represented by the disintegration formula

$$\mathfrak{m} = \int_{\partial B^+(o,\rho)} \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \quad \textit{with} \quad \mathfrak{q} \in \mathscr{P}(\partial B^+(o, \rho)), \ \mathfrak{m}_\alpha \in \mathcal{M}_+(X), \tag{1.6}$$

where $\mathfrak{m}_\alpha$ is concentrated on the geodesic ray from o through α, and $\mathfrak{m}_\alpha$ can be identified (via the unitary speed parametrization of the ray) with $N\omega_N \mathsf{AVR}_X t^{N-1} \mathscr{L}^1 \llcorner_{[0,\infty)}$.

As an application of Theorem 1.4, we deduce the rigidity for the weighted anisotropic isoperimetric problem in Euclidean cones, in the irreversible case (the reversible case was already investigated [13]). We postpone this discussion to Section 1.2; now we briefly present the proof strategy of Theorem 1.4 and the structure of the paper.

The classical approach for proving a rigidity results consists in exploiting properties depending on the saturation of inequalities. In this paper, following the line of [13], we adopt a different approach that starts from the proof of the isoperimetric inequality for non-compact $\mathsf{MCP}(0, N)$ spaces [12]. In [13], it is used the localisation given by the L^1-optimal transport problem between the renormalized measures restricted on the set E and B_R, a large ball of radius R containing E. The localization gives a family of one-dimensional, disjoint transport rays together with a disintegration of the restriction to B_R of reference measure $\mathfrak{m}$. At this point, it is natural to analyze the well-known isoperimetric inequality for the traces of E along the one-dimensional transport rays. As Ohta pointed out in [39], differently from the reversible case, the irreversibility of the space introduces the reversibility constant, obtaining a non-sharp inequality. Indeed, if one tries to prove Theorem 1.3 using the localization of E inside B_R and taking the limit as $R \to \infty$ (as it was done in Theorem 4.3 of [13]), one would obtain a factor Λ_F^{-1} in the lower bound. However, quite surprisingly, when studying the equality case, this factor will disappear.

In order to capture the equality, it is therefore necessary to deal with this limit procedure and to get rid of the reversibility constant. The intuition suggests that, if E saturates inequality (1.5), then for large values of R the one-dimensional traces should be almost optimal. We intend the almost optimality in many respects: for example, the length of each transport ray has to be almost optimal; the disintegration measures has to have density t^{N-1}; the traces of set E has to almost coincide with the interval starting from the starting point of the ray having as length the expected radius of ball saturating the inequality. This last observation will be the key-point for solving the issues arising by the irreversibility.

Indeed, we will see that the transport rays naturally come with a unitary vector field that "points outward" from the set E, and that, in the parametrization of the rays, this vector field points "to the right". When one computes the perimeter in the transport ray, one must compute the measure of the boundary of the trace of E; we divide the boundary in two parts: the part with outward normal vector pointing "to the right" and "to the left", respectively. For the former part, one computes the measure as usual, while for the latter part, one has to take into account the Finsler structure (here appears the reversibility constant). At this point, the almost rigidity of the traces of E is used: the fact that any trace of E almost coincides with the interval $[0,\rho]$ permits us to deduce that the part of the boundary "pointing to the left" contributes little to the perimeter, and therefore we can get rid of the reversibility constant.

Having in mind these estimates, we take the limit as $R \to \infty$. There is no general procedure for taking the limit of a disintegration. However, following the procedure first employed in [13], we will exploit the almost optimality of the traces of E and the densities deduced in Section 6; these properties permit us to obtain a well behaved limit disintegration for the reference measure restricted to the set E, as it is described in Corollary 7.15.

Finally, using the properties of the disintegration, we will deduce that the set E is a ball and the disintegration of the measure in the whole space (see Theorems 8.3 and 8.8, respectively).

The paper is organized as follows. Section 2 recalls a few facts on Finsler manifolds, the curvature-dimension condition, and the localization technique. In Section 3, we prove Theorem 1.3. In Sections 4 and 5, we localize the reference measure and the perimeter and we present the one-dimensional reductions. In Section 6, the one-dimensional estimates are carried out. In Section 7, we deal with the limiting procedure, while Section 8 concludes proof of Theorem 1.4. We added two appendixes to this paper, containing the proof of the facts that the relative perimeter is a measure and that the perimeter is the relaxation of the Minkowski content.

1.2. Applications in the Euclidean setting

As a consequence of Theorem 1.4, we present a characterization of minima for the weighted anisotropic isoperimetric problem in Euclidean cones.

The setting is the following: let $\Sigma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin; let $H\colon \mathbb{R}^n \to [0,\infty)$ be a *gauge*, i.e., a non-negative, convex and positively 1-homogeneous function; and let $w\colon \overline{\Sigma} \to (0,\infty)$ be a continuous weight for the Lebesgue measure.

If $E \subset \mathbb{R}^n$ is a set with smooth boundary, we define the *weighted anisotropic perimeter* relative to the cone Σ as

$$\mathsf{P}_{w,H}(E;\Sigma) = \int_{\partial E} H(\nu(x))\, w(x)\, dS$$

(here ν and dS denote the unit outward normal vector and the surface measure, respectively). Under the assumptions that $w^{1/\alpha}$ is concave and w is positively α-homogeneous, it has been proven [11, 33] the following sharp isoperimetric inequality for the weighted anisotropic perimeter:

$$\frac{\mathsf{P}_{w,H}(E;\Sigma)}{w(E\cap\Sigma)^{(N-1)/N}} \ge \frac{\mathsf{P}_{w,H}(W;\Sigma)}{w(W\cap\Sigma)^{(N-1)/N}}, \tag{1.7}$$

where $N = n+\alpha$, W is the Wulff shape associated to H, and the expression $w(A)$ with $A \subset \mathbb{R}^n$ is a short-hand notation $\int_A w\,dx$.

If we take $w = 1$, $\Sigma = \mathbb{R}^n$, and $H = \|\cdot\|_2$, inequality (1.7) becomes the classical sharp isoperimetric inequality.

As observed in [11], Wulff balls W centered at the origin are always minimizers of (1.7). However, in [11] the characterization of the equality case is not present. Many efforts have been done for solving this problem. We now briefly recall the known results. For the unweighted case ($w = 1$), Ciraolo et al. [17] solved the problem under the assumption of H to be a uniformly elliptic positive gauge (i.e., a not necessarily reversible norm). The characterisation in weighted setting has been solved in [16], but only in the isotropic case ($H = \|\cdot\|_2$). The author, together with Cavalletti, solved the weighted problem [13], with the assumption of H to be a norm (i.e., reversible) with strictly convex balls, knowing that the isoperimetric set is bounded. This paper improves [13] by dropping the reversibility assumption.

As it was observed in [11], the assumption that $w^{1/\alpha}$ is concave has a natural interpretation as the $\mathsf{CD}(0, N)$ condition, where $N = n + \alpha$. To be precise, if H is a gauge, then its dual function F is a Finsler structure (with finite reversibility constant), provided that F is smooth outside the origin and F^2 is strictly convex (this two requirements can be equivalently required for the gauge H). One can associate to the triple Σ, H and w the measured Finsler manifold $(\overline{\Sigma}, F, w\mathscr{L}^n)$. One can check that $(\overline{\Sigma}, F, w\mathscr{L}^d)$ satisfies the $\mathsf{CD}(0, d+\alpha)$ condition if $w \in C^\infty$ and $w^{1/\alpha}$ is concave: in Chapter 10 of [40] and in [35], this is done in the case $\Sigma = \mathbb{R}^n$; clearly, the proof extends to the case of convex subsets. The fact that this manifold has finite reversibility, the convexity, and the forward-completeness are trivial checks. The perimeter associated to this space will indeed coincide with $\mathsf{P}_{w,H}$. Moreover, by the homogeneity properties of H and w, one can check that

$$\mathsf{AVR}_\Sigma = \lim_{R\to\infty} \frac{w(B_F^+(0,R)\cap\Sigma)}{\omega_N R^N} = \frac{w(W\cap\Sigma)}{\omega_N} > 0.$$

Indeed, recall that the Wulff shape W of H is the unitary ball of the Finsler structure F, hence the measure scales with power $N = n + \alpha$. Conversely, the perimeter of the rescaled Wulff shape turns out to be the derivative with respect to the scaling factor of the measure, therefore the perimeter of the Wulff shape is N times its measure. This consideration shows that (1.7) follows from (1.5), thus we can apply Theorem 1.4 to $(\Sigma, F, w\mathscr{L}^n)$, obtaining the following result.

Theorem 1.5. *Let $\Sigma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin, and let $H\colon \mathbb{R}^n \to [0,\infty)$ be a gauge. Assume H to have strictly convex balls and to be smooth outside the origin. Consider moreover the α-homogeneous smooth weight $w\colon \overline{\Sigma} \to [0,\infty)$ such that $w^{1/\alpha}$ is concave.*

Then the equality in (1.7) is attained if and only if $E = W \cap \Sigma$, where W is a rescaled Wulff shape.

2. Preliminaries

In this section, we recall the main constructions needed in the paper. In Section 2.1, we review the geometry of measured Finsler manifolds; in Section 2.2, the perimeter in measured Finsler manifolds; in Section 2.3, the curvature-dimension condition; finally, in Section 2.4, the localization method.

2.1. Finsler geometry

We quickly recall the basic notions regarding measured Finsler manifolds. The reader should refer to [40] for more details. We adopt the convention that a manifold may have a boundary, unless otherwise stated. We require the boundary to be Lipschitz.

Definition 2.1. Let X be a connected n-dimensional manifold. We say that a function $F : TX \to [0, \infty)$ is a Finsler structure on X if

(1) (Regularity) F is C^∞ on $TX \setminus 0$, where 0 denotes the null section.
(2) (Positive 1-homogeneity) For all $c > 0$, $v \in TX$, it holds that $F(cv) = cF(v)$.
(3) (Strong convexity) On each tangent space $T_x X$, the function F^2 is strictly convex.

The reader should notice that in general $F(v) \neq F(-v)$. This feature, known as irreversibility, is what precludes us from applying the theory of m.m.s.'s. We define the reversibility constant of a Finsler structure as

$$\Lambda_{X,F} := \sup_{v \in TX : v \neq 0} \frac{F(v)}{F(-v)} \in [1, \infty],$$

or, in other words, $\Lambda_{X,F} \in [1, \infty]$ is the least constant $\Lambda_{X,F} \geq 1$ such that for all $v \in TX$, $F(v) \leq \Lambda_{X,F} F(-v)$. Later we will restrict ourself to the family of Finsler structures with finite reversibility. If no confusion arises, we shall write $\Lambda_F = \Lambda_{X,F}$. If X is compact, then $\Lambda_{X,F} < \infty$.

We define the speed of a C^1 curve η as $F(\dot\eta)$. The notion of speed naturally induces a length functional

$$\mathrm{Length}(\eta) := \int_0^1 F(\dot\eta)\, dt,$$

and thus we have a natural notion of distance between two points given by

$$\mathsf{d}_{X,F}(x, y) := \inf_\eta \{\mathrm{Length}(\eta) : \eta_0 = x, \text{ and } \eta_1 = y\}.$$

Whenever no confusion arises, we shall write $\mathsf{d} = \mathsf{d}_{X,F}$. The distance d satisfies the usual properties of a distance, with the exception of the symmetry:

$$\mathsf{d}(x, y) \leq \mathsf{d}(x, z) + \mathsf{d}(z, y), \quad \forall x, y, z \in X, \quad \text{and} \quad \mathsf{d}(x, y) = 0 \quad \Leftrightarrow \quad x = y.$$

Remark 2.2. We reassure the reader on the fact that the lack of symmetry of the distance does not harm most of the classical theory of metric spaces. Indeed, one can build g_1 and g_2, two Riemannian metric on TX, such that

$$\sqrt{g_1(v, v)} \leq F(v) \leq \sqrt{g_2(v, v)}, \quad \forall v \in TX.$$

Such metrics can be built on local charts and then glued together using a partition of the unity. Furthermore, such metrics can be chosen so that $g_2 \leq f g_1$, for some continuous function $f : X \to [1, \infty)$.

Using these metrics, one can reobtain many classical results for free. In particular, we will make use of the Ascoli–Arzelà theorem, the fact that locally Lipschitz functions (as will be later introduced) are differentiable almost everywhere, and that locally Lipschitz functions with compact support are globally Lipschitz.

We define the forward and backward balls, respectively, as

$$B^+(x,r) := \{y \in X : \mathsf{d}(x,y) < r\} \quad \text{and} \quad B^-(x,r) := \{y \in M : \mathsf{d}(y,x) < r\}.$$

If $A \subset X$, we define its (forward) ε-enlargement to $B^+(A,\varepsilon) := \bigcup_{x\in A} B^+(x,\varepsilon)$. We say that a set A is forward (respectively, backward) bounded, if for some (hence any) $x_0 \in X$, there exists $r > 0$ such that $A \subset B^+(x_0,r)$ (respectively, $A \subset B^-(x_0,r)$). We say that a set is bounded if it both backward and forward bounded. We denote by $\operatorname{diam} A := \sup_{x,y\in A} \mathsf{d}(x,y)$ the diameter of a set; a set has finite diameter if and only if it is bounded.

Definition 2.3. Let (X, F) be a Finsler manifold, possibly with boundary. We say that it is forward-complete if and only if, for all sequences $(x_k)_k \subset X$ satisfying the forward Cauchy condition

$$\forall\varepsilon > 0 : \exists N > 0 : \forall n > m > N : \quad \mathsf{d}(x_m, x_n) \le \varepsilon,$$

then $(x_k)_k$ is converging.

In light of the Hopf–Rinow theorem, forward-completeness of a Finsler manifold is equivalent to the compactness of closed forward balls, and implies that given two points, there exists a minimal geodesic (as defined in the next paragraph) joining these two points. In case of manifolds without boundary, forward-completeness is equivalent also to the definition of the exponential map on the whole tangent bundle.

A curve $\gamma\colon [0,l] \to X$ is called minimal geodesic if it minimizes the length and its speed is constant. We point out that, if $\gamma\colon [0,l] \to X$ is a minimal geodesic, in general the reverse curve $t \mapsto \gamma_{l-t}$ may fail to be a geodesic, due to the possible irreversibility of the manifold. We will denote by $\operatorname{Geo}(X)$ the set of minimal geodesic with domain the interval $[0,1]$. Like in the reversible setting, if $\gamma \in \operatorname{Geo}(X)$ is a minimal geodesic, it holds that

$$\mathsf{d}(\gamma_t, \gamma_s) = (s-t)\mathsf{d}(\gamma_0, \gamma_1), \quad \forall\, 0 \le t \le s \le 1;$$

in this case, the condition $t \le s$ cannot be lifted.

Definition 2.4. Let (X, F) be a Finsler manifold with boundary. We say that ∂X is locally forward convex if and only if, for all points $x, y \in X \setminus \partial X$, and for all minimal geodesic γ connecting x to y, we have that γ does not touch the boundary.

Given a submanifold $Y \subset X$, we can identify the tangent bundle TY as a subset of TX via the standard immersion. With this notation, we can restrict the Finsler structure F to TY; clearly, $F|_{TY}$ satisfies Definition 2.1. Regarding the reversibility constant and the distance, one immediately sees that

$$\Lambda_{Y,F} \le \Lambda_{X,F} \quad \text{and} \quad \mathsf{d}_{X,F}(x,y) \le \mathsf{d}_{Y,F}(x,y), \quad \forall x, y \in Y.$$

We define the dual function $F^*\colon T^*X \to [0,\infty)$ as

$$F^*(\omega) := \sup\{\omega(v) : v \in T_xX, \text{ and } F(v) \le 1\}, \quad \text{if } \omega \in T_x^*X.$$

Notice that, while we have that $\omega(v) \le F^*(\omega)F(v)$, the "reverse" inequality may not hold: $\omega(v) \not\ge -F^*(\omega)F(v)$. We define the Legendre transform $\mathcal{L}\colon T_x^*X \to T_xX$ as $\mathcal{L}(\omega) = v$, where $v \in T_x X$ is the unique vector such that $F(v) = F^*(\omega)$ and $\omega(v) = F(v)^2$ (the uniqueness follows from the fact that F^2 is smooth and strictly convex). Given a differentiable function $f : X \to \mathbb{R}$, we define its gradient as $\nabla f(x) := \mathcal{L}(df(x))$. Please note that, in general, $\nabla(-f) \neq -\nabla f$.

We say that a function $f : X \to \mathbb{R}$ is L-Lipschitz (with $L \geq 0$) if

$$-L\,\mathsf{d}(y,x) \leq f(x) - f(y) \leq L\,\mathsf{d}(x,y), \quad \forall x, y \in X.^{1} \tag{2.1}$$

We point out that the first inequality in (2.1) follows from the second by swapping x with y. The family of L-Lipschitz functions is stable by pointwise limits; the infimum or the supremum of L-Lipschitz functions is still L-Lipschitz. Moreover, by the Ascoli–Arzelà theorem, the family of L-Lipschitz functions forms a compact set in the topology of local uniform convergence. If f is L-Lipschitz, then $-f$ is $(\Lambda_F L)$-Lipschitz. Two examples of 1-Lipschitz functions are given by $f(x) = -\mathsf{d}(o,x)$ and $g(x) = \mathsf{d}(x,o)$, for some fixed o.

We define the slope of a locally Lipschitz function f as

$$|\partial f|(x) := \limsup_{y \to x} \frac{(f(x) - f(y))^+}{d(x,y)}. \tag{2.2}$$

Obviously, if f is L-Lipschitz, then $|\partial f| \leq L$. If $Y \subset X$ is a submanifold, and $f : X \to \mathbb{R}$, then $|\partial_Y f| \leq |\partial_X f|$ in Y, where these two expressions have the meaning of the slope of f as seen as a function defined in Y and X, respectively. If f is differentiable at $x \in X$, the slope can be computed as $|\partial f|(x) = F^*(-df(x)) = F(\nabla(-f)(x))$, hence for locally Lipschitz functions, $|\partial f| = F(\nabla(-f))$ almost everywhere.

Finally, we would like to endow a manifold with a measure. Differently from the Riemannian case, there is no canonical measure induced from the Finsler structure. On the other hand, the theory of m.m.s.'s does not require any strong assumption on the reference measure and, a priori, this measure might have nothing to do with the Hausdorff measure. For this reason, we will only require for the reference measure to have a smooth density when expressed in coordinates. We conclude this section with the definition of measured Finsler manifold.

Definition 2.5. A triple $(X, F, \mathfrak{m})$ is called measured Finsler manifold provided that X is a connected differential manifold (possibly with boundary), F is a Finsler structure on X, and $\mathfrak{m}$ is a positive smooth measure, i.e., given $x_1, \ldots, x_n$ local coordinates, we have that

$$\mathfrak{m} = f\, dx_1 \ldots dx_n, \quad \text{with } f > 0 \text{ and } f \in C^\infty.$$

2.2. Perimeter

Following the line traced in [2, 3, 34], we give the definition of (relative) perimeter for measured Finsler manifold.

Given a Borel subset $E \subset X$ and Ω open, the perimeter of E relative to Ω is denoted by $\mathsf{P}(E;\Omega)$, and is defined as follows:

$$\mathsf{P}(E;\Omega) := \inf\left\{ \liminf_{n\to\infty} \int_\Omega |\partial u_n|\, d\mathfrak{m}\ ;\ u_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), u_n \to \mathbf{1}_E \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\}. \tag{2.3}$$

[1] Please note that we have chosen a sign convention different from [39, 40].

In the unweighted Riemannian setting, if E has smooth boundary, it is a standard fact that $\mathsf{P}(E;\Omega) = \mathscr{H}^{n-1}(\Omega \cap \partial E)$. We say that $E \subset X$ has finite perimeter in X if $\mathsf{P}(E;X) < \infty$. We recall also a few elementary properties of the perimeter functions:

(a) (locality) $\mathsf{P}(E;\Omega) = \mathsf{P}(F;\Omega)$, whenever $\mathfrak{m}((E \triangle F) \cap \Omega) = 0$;

(b) (lower semicontinuity) the map $E \mapsto \mathsf{P}(E,\Omega)$ is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence.

Please note that, due to the possible irreversibility of the Finsler structure, in general the complementation property does not hold. If E is a set of finite perimeter, then the set function $A \to \mathsf{P}(E;A)$ is the restriction to open sets of a finite Borel measure $\mathsf{P}(E;\cdot)$ in X (see Appendix A), defined by

$$\mathsf{P}(E;A) := \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A,\ \Omega \text{ open}\}.$$

Sometimes, for ease of notation, we will write $\mathsf{P}(E)$ instead of $\mathsf{P}(E;X)$.

Given a subset $E \subset X$, we define its (forward) Minkowski content as

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(B^+(E,\varepsilon)) - \mathfrak{m}(E)}{\varepsilon}. \tag{2.4}$$

It can be shown that the perimeter is the relaxation of the Minkowski content with respect to the L^1 distance of sets. The proof of this fact can be found in Appendix B.

2.3. Wasserstein distance and the curvature-dimension condition

Given a forward-complete measured Finsler manifold $(X, F, \mathfrak{m})$, by $\mathcal{M}^+(X)$ and $\mathscr{P}(X)$ we denote the space of non-negative Borel measures on X and the space of probability measures, respectively. For $p \in [1,\infty)$, we will consider the space $\mathscr{P}_p(X) \subset \mathscr{P}(X)$ of the measures μ satisfying

$$\int_X (\mathsf{d}(o,x) + \mathsf{d}(x,o))^p\, \mu(dx), \quad \text{for some (hence any) } o \in X,$$

i.e., μ has finite p-th moment. On the space $\mathscr{P}_p(X)$, we define the L^p-Wasserstein distance W_p by setting, for $\mu_0, \mu_1 \in \mathscr{P}_p(X)$,

$$W_p(\mu_0,\mu_1)^p := \inf_\pi \int_{X\times X} \mathsf{d}^p(x,y)\, \pi(dxdy) < \infty. \tag{2.5}$$

The infimum is taken over all probability measure $\pi \in \mathscr{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal, i.e., $(P_1)_\sharp \pi = \mu_0$ and $(P_2)_\sharp \pi = \mu_1$, where P_1 and P_2 denote the projections on the first and second factors. The infimum is attained, and this minimizing problem is called Monge–Kantorovich problem.

We call a geodesic in the Wasserstein space $(\mathscr{P}_p(X), W_p)$ any curve μ: $[0,1] \to \mathscr{P}_p$ such that

$$W_p(\mu_t,\mu_s) = (s-t)\, W_p(\mu_0,\mu_1), \quad \forall\, 0 \le t \le s \le 1.$$

It can be shown that if μ_0 and μ_1 are absolutely continuous, there exists a unique geodesic connecting μ_0 to μ_1.

The $\mathsf{CD}(K,N)$ for condition for m.m.s.'s has been introduced in the seminal works of Sturm [44, 45] and Lott–Villani [30], and later investigated in the realm of measured Finsler manifolds in [36] (see also the survey [37]); here we briefly recall only the basics in the case $K=0$, $1<N<\infty$.

We define the N-Rényi entropy as

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{1-1/N}\,d\mathfrak{m}, \quad \text{where } \mu=\rho\mathfrak{m}+\mu_s \text{ and } \mathfrak{m}\perp\mu_s.$$

Definition 2.6 ($\mathsf{CD}(0,N)$). Let $(X,F,\mathfrak{m})$ be a forward-complete measured Finsler manifold and let $N\in[\dim X,\infty)$. We say that $(X,F,\mathfrak{m})$ satisfies the $\mathsf{CD}(0,N)$ condition if and only if the N'-Rényi entropy is convex along the geodesic of the Wasserstein space for all $N'\geq N$, that is, for all couples of absolutely continuous curves $\mu_0,\mu_1\in\mathscr{P}_2(X)$, it holds that

$$S_N(\mu_t|\mathfrak{m})\leq(1-t)\,S_N(\mu_0|\mathfrak{m})+t\,S_N(\mu_1|\mathfrak{m}),$$

where $(\mu_t)_{t\in[0,1]}$ is the unique geodesic connecting μ_0 to μ_1.

If $(X,g,h\,\mathrm{Vol}_g)$ is a weighted Riemannian manifold, one can introduce the N-Ricci tensor (as defined in (1.2)). It was proven in [30,44,45] that a weighted complete Riemannian manifold without boundary satisfies the $\mathsf{CD}(0,N)$ condition if and only if $\mathrm{Ric}_N\geq 0$.

As in the Riemannian case, a notion of weighted N-Ricci curvature, still denoted by Ric_N, has been introduced. Here we do not give the definition of Ric_N, for it is quite lenghty and useless for our purposes. Ohta [35] proved that a measured Finsler manifold without boundary satisfies the $\mathsf{CD}(0,N)$ condition if and only if $\mathrm{Ric}_N\geq 0$. The possible presence of the boundary in the manifolds the present paper deals with does not harm the results of this paper; indeed, we rely only on the curvature-dimension condition given by Definition 2.6, and never on Ric_N.

Among many consequences of the $\mathsf{CD}(0,N)$ condition, two are of our interest. One is the Brunn–Minkowski inequality (see, e.g., Theorem 18.8 in [40]). Given two measurable subsets A and B of a $\mathsf{CD}(0,N)$ measured Finsler manifold $(X,F,\mathfrak{m})$ and $t\in[0,1]$, we define

$$\begin{aligned} Z_t(A,B) &:= \{\gamma_t : \gamma \text{ is a minimal geodesic such that } \gamma_0\in A \text{ and } \gamma_1\in B\} \\ &= \{z : \exists x\in A, y\in B : \mathsf{d}(x,z)=t\mathsf{d}(x,y) \text{ and } \mathsf{d}(z,y)=(1-t)\mathsf{d}(x,y)\}. \end{aligned}$$

With this notation, we have the Brunn–Minkowski inequality:

$$\mathfrak{m}(Z_t(A,B))^{1/N}\geq(1-t)\,\mathfrak{m}(A)^{1/N}+t\,\mathfrak{m}(B)^{1/N}, \quad t\in[0,1]. \tag{2.6}$$

The other property we are interested in is the Bishop–Gromov inequality, that states

$$\frac{\mathfrak{m}(B^+(x,r))}{r^N}\geq\frac{\mathfrak{m}(B^+(x,R))}{R^N}, \quad \forall\, 0<r\leq R, \tag{2.7}$$

for any fixed point $x\in X$. This inequality guarantees that the definition of asymptotic volume ratio (see (1.3)) is well posed.

2.4. Localization

The localization method, also known as needle decomposition, is now a well-established technique for reducing high-dimensional problems to one-dimensional problems.

In the Euclidean setting, it dates back to Payne and Weinberger [42]. It has been later developed and popularised by Gromov and V. Milman [24], Lovász–Simonovits [31], and Kannan–Lovasz–Simonovits [26]. Klartag [27] reinterpreted the localization method as a measure disintegration adapted to L^1-optimal-transport, and extended it to weighted Riemannian manifolds satisfying $\mathsf{CD}(K,N)$. Cavalletti and Mondino [14] have succeeded to generalise this technique to essentially non-branching m.m.s.'s verifying the $\mathsf{CD}(K,N)$, condition with $N \in (1,\infty)$, and later Otha [39] developed this method for the Finsler setting. Here we only report the case $K = 0$.

In his work, Ohta considered manifolds without boundary. However, his proof also work for manifolds with local forward convex boundary.

Consider a measured Finsler manifold $(X, F, \mathfrak{m})$ satisfying the $\mathsf{CD}(0,N)$ condition and a function $f \in L^1(\mathfrak{m})$ with finite first moment such that

$$\int_X f \, d\mathfrak{m} = 0. \tag{2.8}$$

The function f induces two absolutely continuous measures $\mu_0 = f^+\mathfrak{m}$ and $\mu_1 = f^-\mathfrak{m}$. The well-established theory of L^1-optimal transport, see [46], specifies that the Monge–Kantorovich problem possess a strictly related dual problem, the so-called Kantorovich–Rubinstein dual problem:

$$\max_{\varphi} \int_X f(x)\varphi(x)\,\mathfrak{m}(dx) = \max_{\varphi} \Big\{ \int_X \varphi(x)\,\mu_0(dx) - \int_X \varphi(x)\,\mu_1(dx) \Big\},$$

where the maximum is taken among all possible 1-Lipschitz functions φ. The problem clearly admits a (non-unique) solution φ; we will call φ a *Kantorovich potential* for the problem. Using φ, we can construct the set[2]

$$\Gamma := \{(x,y) \in X \times X : \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},$$

inducing a partial order relation. The maximal chains of this order relation turns out to be the image of curves of maximal slope for φ with unitary speed. To be more precise, we introduce the concept of transport curve: we say that a unitary speed geodesic $\gamma\colon \mathrm{Dom}(\gamma) \subset \mathbb{R} \to X$ is a non-degenerate transport curve, if its domain has at least two points, $d\varphi(\gamma(t))/dt = -1$, and γ cannot be extended to a larger domain.

We distinguish three possible cases, according to the number of non-degenerate transport curves passing through a given point $x \in X$.

- There is no non-degenerate transport curve passing through x. We denote by $\mathcal{D}$ the set of such points. The set $\mathcal{D}$ is generally large.
- There is exactly 1 non-degenerate transport curve passing through x. Such points form the so-called transport set, that will be denoted by $\mathcal{T}$. A fundamental property of $\mathcal{T}$ is that the f is constantly 0 a.e. in $X \setminus \mathcal{T}$.

[2] Please, notice that we use a different sign convention from [39, 40].

- There are 2 or more non-degenerate transport curves passing through x. Such points are called branching points, and the set that they form will be denoted by $\mathcal{A}$. The set $\mathcal{A}$ turns out to be negligible.

All these sets are σ-compact, hence Borel. In the sequel, we will also refer to the sets of forward (respectively, backward) branching points, defined as

$$\mathcal{A}^+ := \{x \in \mathcal{A} : \exists y \neq x \text{ such that } (x, y) \in \Gamma\},$$
$$\mathcal{A}^- := \{x \in \mathcal{A} : \exists y \neq x \text{ such that } (y, x) \in \Gamma\}.$$

On the transport set, we define the relation $\mathcal{R}$ given by

$$\mathcal{R} := (\Gamma \cup \Gamma^{-1}) \cap (\mathcal{T} \times \mathcal{T}).$$

By construction, $\mathcal{R}$ is an equivalence relation on $\mathcal{T}$, and the equivalence classes are precisely the images of the transport curves. One can chose $Q \subset \mathcal{T}$ a Borel section of the equivalence relation $\mathcal{R}$ (this choice is possible as it was shown in Proposition 4.4 of [8]). Define the quotient map $\mathfrak{Q}\colon \mathcal{T} \to Q$ as $\mathfrak{Q}(x) = \alpha$, where α is the unique element of Q such that $(x, \alpha) \in \mathcal{R}$. We shall denote by $(X_\alpha)_{\alpha \in Q}$ the equivalence classes for the relation $\mathcal{R}$, and we will call them *transport rays*.

The existence of a measurable section permits us to construct a measurable parametrization of the transport rays, $g\colon \mathrm{Dom}(g) \subset Q \times [0, +\infty) \to \mathcal{T}$. Fix $\alpha \in Q$ and take γ, a transport curve, such that $\inf(\mathrm{Dom}(\gamma)) = 0$. Then define $g(\alpha, t) := \gamma_t$, whenever $t \in \mathrm{Dom}(\gamma)$. We specify that this parametrization guarantees that $f(g(\alpha, 0)) \geq 0$. By continuity of g with respect to the variable t, we extend g in order to map also the end-points of the rays X_α; the restriction of g to the set $\{(\alpha, t) : t \in (0, |X_\alpha|)\}$ is injective, where $|X_\alpha| := \sup\{t : (\alpha, t) \in \mathrm{Dom}(g)\}$. Notice that $|X_\alpha|$ is not the diameter of X_α, for it may happen that $\mathsf{d}(g(\alpha, |X_\alpha|), g(\alpha, 0)) > |X_\alpha|$.

The transport rays naturally come with the structure of one-dimensional oriented manifold, with the orientation given by $\partial_t g(\alpha, t)$, the velocity of the parametrization. We endow X_α with the Finsler structure given by the restriction of F to X_α; notice that $F(\partial_t g(\alpha, t)) = 1$. As we already pointed out, it holds that

$$\mathsf{d}_{X,F}(x, y) \leq \mathsf{d}_{X_\alpha,F}(x, y), \quad \forall x, y \in X_\alpha;$$

if $(x, y) \in \Gamma$, the inequality above is saturated, hence

$$\mathsf{d}(g(\alpha, t), g(\alpha, s)) = s - t, \quad \forall\, 0 \leq t \leq s \leq |X_\alpha|.$$

Given a finite measure $\mathfrak{q} \in \mathcal{M}^+(Q)$ such that $\mathfrak{q} \ll \mathfrak{Q}_\#(\mathfrak{m}\llcorner_{\mathcal{T}})$, the disintegration theorem, applied to $(\mathcal{T}, \mathcal{B}(\mathcal{T}), \mathfrak{m}\llcorner_{\mathcal{T}})$, gives a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$ consistent with the partition of $\mathcal{T}$ given by the equivalence classes $\{X_\alpha\}_{\alpha \in Q}$ of $\mathcal{R}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha). \tag{2.9}$$

Note that such measure $\mathfrak{q}$ can always be built by taking the push-forward via $\mathfrak{Q}$ of a suitable finite measure mutually absolutely continuous with respect to $\mathfrak{m}_{\mathcal{T}}$. We recall that by disintegration we mean a map $\mathfrak{m}\colon Q \times \mathcal{B}(X) \to \mathbb{R}$, such that

(1) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on X_α,
(2) for all $B \in \mathcal{X}$, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable,
(3) for all $B \in \mathcal{B}(X)$, $\mathfrak{m}(B) = \int_Q \mathfrak{m}_\alpha(B)\, \mathfrak{q}(d\alpha)$.

Remark 2.7. We point out that the disintegration is unique for fixed $\mathfrak{q}$. That means that if there is a family $(\tilde{\mathfrak{m}}_\alpha)_\alpha$ satisfying the conditions above, then for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha = \tilde{\mathfrak{m}}_\alpha$. If we change $\mathfrak{q}$ with a different measure $\hat{\mathfrak{q}}$ such that $\hat{\mathfrak{q}} = \rho\mathfrak{q}$, with $\rho > 0$, then the map $\alpha \mapsto \rho(\alpha)^{-1}\mathfrak{m}_\alpha$ still satisfies the conditions above, with $\hat{\mathfrak{q}}$ in place of $\mathfrak{q}$.

We endow the transport ray X_α with the measure $\mathfrak{m}_\alpha$, making $(X_\alpha, F, \mathfrak{m}_\alpha)$ a one-dimensional oriented measured Finsler manifold.

Differently from the reversible case, it might happen that the transport rays fail to satisfy the $\mathsf{CD}(0, N)$ condition. However, a bound from below on the Ricci curvature can be given in a certain sense. It can be proved that, for a certain non-negative function h_α,

$$\mathfrak{m}_\alpha = (g(\alpha, \cdot))_\#(h_\alpha \mathcal{L}^1_{(0,|X_\alpha|)}).$$

The function h_α satisfies $(h_\alpha^{1/(N-1)})'' \le 0$ in the distributional sense, i.e., the function $h_\alpha^{1/(N-1)}$ is concave. Here we can recognize the $\mathsf{CD}(0, N)$ for weighted Riemannian manifolds, namely, that the space $([0, |X_\alpha|], |\cdot|, h_\alpha \mathcal{L}^1_{[0,|X_\alpha|]})$ satisfies the $\mathsf{CD}(0, N)$ condition. This fact leads us to the following definition.

Definition 2.8. Let $(X, F, \mathfrak{m})$ be a measured Finsler manifold diffeomorphic to an interval, endowed with an orientation given by a vector field v such that $F(v) = 1$. We say that $(X, F, \mathfrak{m})$ satisfies the oriented $\mathsf{CD}(0, N)$ condition ($N > 1$), if the following happens. There exist $g\colon \mathrm{Dom}(g) \subset \mathbb{R} \to X$, a parametrization of X such that $\partial_t g(t) = v(g(t))$, and $h\colon \mathrm{Dom}(g) \to [0, \infty)$, a function such that $g_\#(h\mathcal{L}^1) = \mathfrak{m}$ and $h^{1/(N-1)}$ is concave.

With this definition, clearly holds that the transport rays satisfy the oriented $\mathsf{CD}(0, N)$ condition. For the reader used with the notion of N-Ricci curvature, we point out that the oriented $\mathsf{CD}(0, N)$ condition is equivalent to the fact that $\mathrm{Ric}_N(\partial_t g(\alpha, t)) \ge 0$.

Finally, we point out that, as a consequence of the properties of the optimal transport, we can localize the constraint $\int_X f\, d\mathfrak{m} = 0$, i.e., it holds that $\int_X f\, d\mathfrak{m}_\alpha = 0$, for $\mathfrak{q}$-a.e. $\alpha \in Q$.

We summarize this section in the following theorem.

Theorem 2.9. *Let $(X, F, \mathfrak{m})$ be a measured Finsler manifold satisfying $\mathsf{CD}(0, N)$, for some $N \in (1, \infty)$.*

Let $f \in L^1(\mathfrak{m})$ with $\int_X f\, d\mathfrak{m} = 0$ and

$$\int_X (\mathsf{d}(o, x) + \mathsf{d}(x, o))|f(x)|\, \mathfrak{m}(dx) < \infty, \quad \text{for some (hence any) } o \in X.$$

Then there exist a measurable subset $\mathcal{T} \subset X$ (transport set), a family $\{(X_\alpha, F, \mathfrak{m}_\alpha)\}_{\alpha\in Q}$ of oriented one-dimensional submanifolds of X (transport rays), and a measurable function $g\colon \mathrm{Dom}(g) \subset Q \times [0, \infty)$ such that the following happens.

The function f is zero $\mathfrak{m}$-a.e. in $X \setminus \mathcal{T}$, and $\mathfrak{m}\llcorner_{\mathcal{T}}$ can be disintegrated as follows:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha).$$

Moreover, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the transport ray $(X_\alpha, F, \mathfrak{m}_\alpha)$ is parametrized by the unitary speed geodesic $g(\alpha, \cdot)$, it satisfies the oriented $\mathsf{CD}(0, N)$ condition, and it holds that

$$\int f \, d\mathfrak{m}_\alpha = 0. \tag{2.10}$$

Furthermore, two distinct transport rays can only meet at their extremal points (having measure zero for $\mathfrak{m}_\alpha$).

Remark 2.10. The construction of the needle decomposition depends only on the function φ, rather than the function f. Indeed, given a 1-Lipschitz function φ, one can construct the needle decomposition and reobtain Theorem 2.9, without, of course, equation (2.10), which now makes no sense.

3. Proof of the main inequality

We devote this section to proving Theorem 1.3.

Proof of Theorem 1.3. We will first prove that

$$\mathfrak{m}^+(E) \geq N(\omega_N \,\mathsf{AVR}_X)^{1/N} \mathfrak{m}(E)^{1-1/N}, \quad \forall E \subset X \text{ bounded.}$$

From the inequality above, the thesis will immediately follow by Theorem B.5.

Fix $E \subset X$ bounded and $x_0 \in E$; set $d = \operatorname{diam} E$. Fix $R > 0$ so that $E \subset B^+(x_0, R)$. We claim that $Z_t(E, B^+(x_0, R)) \subset B^+(E, t(d+R))$. Indeed, let $z \in Z_t(E, B^+(x_0, R))$, hence there exist $x \in E$ and $y \in B^+(x_0, R)$ so that $\mathsf{d}(x, z) = t\mathsf{d}(x, y)$. By the triangular inequality, we deduce that

$$\mathsf{d}(x, z) = t\mathsf{d}(x, y) \leq t(\mathsf{d}(x, x_0) + \mathsf{d}(x_0, y)) \leq t(d + R),$$

and thus $z \in B^+(E, t(d+R))$, proving the claim. We are in position to compute the Minkowski content:

$$\begin{aligned}
\mathfrak{m}^+(E) &= \liminf_{\varepsilon \to 0} \frac{\mathfrak{m}(B^+(E, \varepsilon)) - \mathfrak{m}(E)}{\varepsilon} = \liminf_{t \to 0} \frac{\mathfrak{m}(B^+(E, t(d+R))) - \mathfrak{m}(E)}{t(d+R)} \\
&\geq \liminf_{t \to 0} \frac{\mathfrak{m}(Z_t(E, B^+(x_0, R)) - \mathfrak{m}(E)}{t(d+R)} \\
&\geq \liminf_{t \to 0} \frac{((1-t)\mathfrak{m}(E)^{1/N} + t\mathfrak{m}(B^+(x_0, R))^{1/N})^N - \mathfrak{m}(E)}{t(d+R)} \\
&\geq \liminf_{t \to 0} \frac{\mathfrak{m}(E) + N\mathfrak{m}(E)^{1-1/N} t(\mathfrak{m}(B^+(x_0, R))^{1/N} - \mathfrak{m}(E)^{1/N}) + o(t) - \mathfrak{m}(E)}{t(d+R)} \\
&= N\mathfrak{m}(E)^{1-1/N} \frac{\mathfrak{m}(B^+(x_0, R))^{1/N} - \mathfrak{m}(E)^{1/N}}{d+R}.
\end{aligned}$$

By taking the limit as $R \to \infty$, recalling the definition of AVR_X, we conclude the proof. ∎

4. Localization of the measure and the perimeter

From now on, we assume that every Finsler manifold is forward-complete, that it has finite reversibility constant, and local forward convexity. To prove Theorem 1.4, we consider the isoperimetric problem inside a ball with larger and larger radius. In order to apply the needle decomposition given by the localization Theorem 2.9, one also needs in principle the balls to be convex. As in general balls fail to be convex, we will overcome this issue in the following way.

Given a bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix a point $x_0 \in E$ and then consider $R > 0$ such that $E \subset B_R$ (hereinafter, we will adopt the notation $B_R := B^+(x_0, R)$). Consider then the following family of null-average functions:

$$f_R(x) = \chi_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\, \chi_{B_R}.$$

Clearly, f_R falls in the hypothesis of Theorem 2.9. Thus we obtain a measurable subset $\mathcal{T}_R \subset X$ (the transport set) and a family $\{(X_{\alpha,R}, F, \mathfrak{m}_{\alpha,R})\}_{\alpha\in Q_R}$ of transport rays, so that the measure $\mathfrak{m}\llcorner_{\mathcal{T}_R}$ can be disintegrated:

$$\mathfrak{m}\llcorner_{\mathcal{T}_R} = \int_{Q_R} \mathfrak{m}_{\alpha,R}\, \mathfrak{q}_R(d\alpha), \quad \mathfrak{q}_R(Q_R) = \mathfrak{m}(\mathcal{T}_R), \tag{4.1}$$

where $\mathfrak{m}_{\alpha,R}$ are probability densities supported on $X_{\alpha,R}$. Let $g_R(\alpha,\cdot) : [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ be the unit speed parametrization of the transport ray $X_{\alpha,R}$, in the direction given by the natural orientation of the disintegration ray $X_{\alpha,R}$. With this notation, it holds

$$\mathfrak{m}_{\alpha,R} = (g_R(\alpha,\cdot))_\#(h_{\alpha,R}\mathcal{L}^1\llcorner_{[0,|X_{\alpha,R}|]}),$$

for some $\mathsf{CD}(0,N)$ density $h_{\alpha,R}$. The localization of the zero mean implies that (see (2.10))

$$\mathfrak{m}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\, \mathfrak{m}_{\alpha,R}(B_R), \quad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R. \tag{4.2}$$

Unfortunately, the presence of the factor $\mathfrak{m}_{\alpha,R}(B_R)$ in the right-hand side of the equation does make the quantity $\mathfrak{m}_{\alpha,R}$ dependent of α, harming the localization approach. To get rid of this factor, we proceed as follows. We define $T_{\alpha,R}$ to be the unique element of $[0, |X_{\alpha,R}|]$ such that

$$\mathfrak{m}_{\alpha,R}(g_R(\alpha, [0, T_{\alpha,R}])) = \int_0^{T_{\alpha,R}} h_{\alpha,R}(x)\, dx = \mathfrak{m}_{\alpha,R}(B_R).$$

The measurability in α of $\mathfrak{m}_{\alpha,R}$ implies the same measurability for $T_{\alpha,R}$.

Notice that $|X_{\alpha,R}| \le R + \mathrm{diam}(E)$: since $g_R(\alpha,\cdot)$ is the unit speed parametrization of $X_{\alpha,R}$, then

$$\mathsf{d}(g_R(\alpha,0), g_R(\alpha,|X_{\alpha,R}|)) \le \mathsf{d}(g_R(\alpha,0), x_0) + \mathsf{d}(x_0, g_R(\alpha,|X_{\alpha,R}|)) \le \mathrm{diam}(E) + R,$$

and consequently, we deduce $T_{\alpha,R} \le R + \mathrm{diam}(E)$. We restrict $\mathfrak{m}_{\alpha,R}$ to the ray $\widehat{X}_{\alpha,R} := g_R(\alpha, [0, T_{\alpha,R}])$, having the disintegration formula

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha), \quad \widehat{\mathfrak{m}}_{\alpha,R} := \frac{\mathfrak{m}_{\alpha,R}\llcorner_{\widehat{X}_{\alpha,R}}}{\mathfrak{m}_{\alpha,R}(B_R)} \in \mathcal{P}(X),\ \widehat{\mathfrak{q}}_R = \mathfrak{m}_{\cdot,R}(B_R)\,\mathfrak{q}_R, \tag{4.3}$$

where $\widehat{\mathcal{T}}_R := \bigcup_{\alpha \in Q_R} \widehat{X}_{\alpha,R}$. Using (4.1) and the fact that $B_R \subset \mathcal{T}_R$, we get $\hat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$.

The disintegration (4.3) will be a useful localisation only if $(E \cap X_{\alpha,R}) \subset \widehat{X}_{\alpha,R}$; in this case, we have

$$\hat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \quad \hat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R,$$

obtaining a localization constraint independent of α. To prove this inclusion, we will impose that $E \subset B_{R/(4\Lambda_F)}$. Since $g_R(\alpha, \cdot)\colon [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ has unitary speed, we notice that

$$\mathsf{d}(x_0, g_R(\alpha,t)) \le \mathsf{d}(x_0, g_R(\alpha,0)) + \mathsf{d}(g_R(\alpha,0), g_R(\alpha,t)) \le \frac{R}{4\Lambda_F} + t \le \frac{R}{2} + t,$$

where in the second inequality we have used that each starting point of the transport ray has to be inside $E \subset B_{R/(4\Lambda_F)}$, being precisely where $f_R > 0$. The inequality above yields $g_R(\alpha,t) \in B_R$ for all $t < R/2$, hence $((g_R(\alpha,\cdot))^{-1}(B_R) \supset [0, \min\{R/2, |X_{\alpha,R}|\}]$, thus there are no "holes" inside $(g_R(\alpha, \cdot))^{-1}(B_R)$ before $\min\{R/2, |X_{\alpha,R}|\}$, implying that $|\widehat{X}_{\alpha,R}| \ge \min\{R/2, |X_{\alpha,R}|\}$. Fix $x \in E \cap X_{\alpha,R}$ and let $t \in [0, |X_{\alpha,R}|]$ be such that $x = g_R(\alpha,t)$. It holds that

$$t = \mathsf{d}(g_R(\alpha,0), x) \le \mathsf{d}(g_R(\alpha,0), x_0) + \mathsf{d}(x_0, x) \le (\Lambda + 1)\frac{R}{4\Lambda} \le \frac{R}{2},$$

where in the second inequality we used that $g_R(\alpha,0), x \in E \subset B_{R/(4\Lambda)}$. The inequality immediately implies $(g_R(\alpha,\cdot))^{-1}(E) \subset [0, \min\{R/2, |X_{\alpha,R}|\}]$, hence $E \cap X_{R,\alpha} \subset \widehat{X}_{\alpha,R}$, as we desired.

We describe explicitly the measure $\hat{\mathfrak{q}}_R$ in term of a push-forward via the quotient map $\mathfrak{Q}_R$ of the measure $\mathfrak{m}\llcorner_E$:

$$\begin{aligned}\hat{\mathfrak{q}}_R(A) &= \int_{Q_R} \mathbf{1}_A(\alpha)\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\,\hat{\mathfrak{m}}_{\alpha,R}(E)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &= \int_{Q_R} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\,\hat{\mathfrak{m}}_{\alpha,R}(E \cap \mathfrak{Q}_R^{-1}(A))\,\hat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\,\mathfrak{m}(E \cap \mathfrak{Q}_R^{-1}(A)),\end{aligned}$$

hence

$$\hat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E).$$

We study now the relation between the perimeter and the disintegration of the measure (4.3). Let $\Omega \subset X$ be an open set, and consider the relative perimeter $\mathsf{P}(E;\Omega)$. Let $u_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a sequence such that $u_n \to \mathbf{1}_E$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\lim_{n\to\infty}\int_\Omega |\partial u_n|\,d\mathfrak{m} = \mathsf{P}(E;\Omega)$. Using the Fatou lemma, we can compute

$$\begin{aligned}\mathsf{P}(E;\Omega) &= \lim_{n\to\infty}\int_\Omega |\partial u_n|\,d\mathfrak{m} \ge \liminf_{n\to\infty}\int_{\Omega\cap\widehat{\mathcal{T}}_R} |\partial u_n|\,d\mathfrak{m}\\ &= \liminf_{n\to\infty}\int_{Q_R}\int_\Omega |\partial u_n|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha) \ge \int_{Q_R}\liminf_{n\to\infty}\int_\Omega |\partial u_n|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &\ge \int_{Q_R}\liminf_{n\to\infty}\int_{X_{\alpha,R}\cap\Omega} |\partial_{X_{R,\alpha}} u_n|\,\hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha) \ge \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\Omega)\,\hat{\mathfrak{q}}_R(d\alpha),\end{aligned}$$

where $|\partial_{X_{\alpha,R}} u|$ denotes the slope of the restriction of u to the transport ray $\hat{X}_{\alpha,R}$ and $\mathsf{P}_{\hat{X}_{\alpha,R}}$ the perimeter in the submanifold $(\hat{X}_{\alpha,R}, F, \hat{\mathfrak{m}}_{\alpha,R})$.

By arbitrariness of Ω, we deduce the following disintegration inequality:

$$\mathsf{P}(E;\,\cdot\,) \geq \int_{Q_R} \mathsf{P}_{\hat{X}_{\alpha,R}}(E;\,\cdot\,)\,\hat{\mathfrak{q}}_R(d\alpha).$$

Next proposition summarizes this construction.

Proposition 4.1. *Let $(X, F, \mathfrak{m})$ be a $\mathsf{CD}(0, N)$ measured Finsler manifold with $\Lambda_F < \infty$. Given any bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then fix $R > 0$ such that $E \subset B_{R/(4\Lambda_F)}(x_0)$.*

Then there exist a Borel set $\hat{\mathcal{T}}_R \subset X$, with $E \subset \hat{\mathcal{T}}_R$, and a disintegration formula

$$\mathfrak{m}\llcorner_{\hat{\mathcal{T}}_R} = \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}\,\hat{\mathfrak{q}}_R(d\alpha), \quad \hat{\mathfrak{m}}_{\alpha,R}(\hat{X}_{\alpha,R}) = 1, \quad \hat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R), \tag{4.4}$$

such that

$$\hat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \quad \text{for } \hat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R \text{ and } \hat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E), \tag{4.5}$$

Moreover, the transport ray $(\hat{X}_{\alpha,R}, F, \hat{\mathfrak{m}}_{\alpha,R})$ satisfies the oriented $\mathsf{CD}(0, N)$ condition and $|X_\alpha| \leq R + \operatorname{diam}(E)$. Furthermore, it holds true that

$$\mathsf{P}(E;\,\cdot\,) \geq \int_{Q_R} \mathsf{P}_{\hat{X}_{\alpha,R}}(E;\,\cdot\,)\,\hat{\mathfrak{q}}_R(d\alpha). \tag{4.6}$$

The rescaling introduced in Proposition 4.1 will be crucially used to obtain non-trivial limit estimates as $R \to \infty$.

5. One-dimensional analysis

Proposition 4.1 is the first step to obtain, from the optimality of a bounded set E, an almost optimality of $E \cap \hat{X}_{\alpha,R}$. We now have to analyse in details the behaviour of the perimeter in one-dimensional oriented measured Finsler manifolds.

We fix few notation and conventions. A one-dimensional oriented measured Finsler manifold can be identified with the manifold $(I, F, \mathfrak{m})$, where $I \subset \mathbb{R}$ is an interval. Without loss of generality we assume that the orientation is given by the coordinated vector field ∂_t on I. Since we are studying manifolds arising from the localization, we shall consider only Finsler structures that satisfy $F(\partial_t) = 1$. Thus, it is clear that the Finsler structure is completely determined by $F(-\partial_t)$; for this reason, with a slight abuse of notation, we will denote by F, the real-valued function given by $F(-\partial_t)$. With this convention, the reversibility constant turns out to be

$$\Lambda_{I,F} = \sup_{x \in I} \left\{ \max \left\{ F(x), \frac{1}{F(x)} \right\} \right\}.$$

When the interval has finite diameter, we will always assume that $I = [0, D]$. Notice that D in general is not the diameter, for it may happen that $\mathsf{d}(D, 0) > \mathsf{d}(0, D) = D$; however, it holds that $\operatorname{diam}(I, F) \le \Lambda_{I,F} D$.

If $(I, F, \mathfrak{m})$ satisfies the oriented $\mathsf{CD}(0, N)$ condition, then it happens that $\mathfrak{m}$ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^1$ and

$$(h^{1/(N-1)})'' \le 0, \quad \text{in the sense of distributions, where } h = \frac{d\mathfrak{m}}{d\mathscr{L}^1}. \tag{5.1}$$

We stress out that if $(I, F, h\mathscr{L}^1\llcorner_I)$ satisfies the oriented $\mathsf{CD}(0, N)$ condition, then the reversible manifold $(I, |\cdot|, h\mathscr{L}^1\llcorner_I)$ satisfies the $\mathsf{CD}(0, N)$ condition. We will say that the function h itself satisfies the $\mathsf{CD}(0, N)$ condition if (5.1) holds.

Given a function $h\colon I \to [0, \infty)$, we shall write

$$\mathfrak{m}_h := h\mathscr{L}^1\llcorner_I.$$

If the interval I is compact, we will assume also that $\int_0^D h = 1$, unless otherwise specified. We also introduce the functions $v_h\colon [0, D] \to [0, 1]$ and $r_h\colon [0, 1] \to [0, D]$ as

$$v_h(r) := \int_0^r h(s)\, ds \quad \text{and} \quad r_h(v) := (v_h)^{-1}(v); \tag{5.2}$$

notice that from the $\mathsf{CD}(0, N)$ condition, $h > 0$ over $(0, D)$, making v_h invertible, and in turn, the definition of r_h well-posed.

We will denote by $\mathsf{P}_{F,h}$ the perimeter in the measured Finsler manifold $(I, F, h\mathscr{L}^1\llcorner_I)$. If $E \subset [0, D]$ is a set of finite perimeter, then it can be decomposed (up to a negligible set) in a family of disjoint intervals

$$E = \bigcup_i (a_i, b_i),$$

and the union is at most countable. In this case, we have that the perimeter is given by the formula

$$\mathsf{P}_{F,h}(E) = \sum_{i:a_i \neq 0} F(a_i)h(a_i) + \sum_{i:b_i \neq D} h(b_i).$$

From the equation above, we immediately deduce a lower bound on the perimeter:

$$\mathsf{P}_{F,h}(E) \ge \Lambda_{I,F}^{-1} \mathsf{P}_{|\cdot|,h}(E) \tag{5.3}$$

5.1. Isoperimetric profile function

The isoperimetric inequality for $\mathsf{CD}(0, N)$ manifolds with bounded diameter is given in terms of the isoperimetric problem in the so-called model spaces. Here we recall the basic notions.

For $N > 1$, $D > 0$, and, $\xi \ge 0$, we consider the model space $([0, D], |\cdot|, h_{N,D}(\xi, \cdot)\mathscr{L}^1)$, where

$$h_{N,D}(\xi, x) := \frac{N}{D^N} \frac{(x + \xi D)^{N-1}}{(\xi + 1)^N - \xi^N}. \tag{5.4}$$

For the model space given by fixed ξ, we can easily compute the functions $v_{N,D}(\xi,\ \cdot) := v_{h_{N,D}(\xi,\cdot)}$ and $r_{N,D}(\xi,\ \cdot) := r_{h_{N,D}(\xi,\cdot)}$:

$$v_{N,D}(\xi, r) = \frac{(r + \xi D)^N - (\xi D)^N}{D^N((1+\xi)^N - \xi^N)},$$
$$r_{N,D}(\xi, v) = D\big((v(1+\xi)^N + (1-v)\xi^N)^{1/N} - \xi\big).$$

The isoperimetric profile function for this model space is given by the formula

$$\begin{aligned}\mathcal{I}_{N,D}(\xi, v) &:= h_{N,D}(\xi, r_{N,D}(\min\{v, 1-v\})) \\ &= \frac{N}{D}\,\frac{(\min\{v,1-v\}(\xi+1)^N + \max\{v,1-v\}\xi^N)^{(N-1)/N}}{(\xi+1)^N - \xi^N}.\end{aligned}$$

The family of one-dimensional measured Finsler manifolds satisfying the $\mathsf{CD}(0,N)$ condition and having $\Lambda_F = 1$ coincides with the of family of weighted Riemannian manifolds. E. Milman [32] gave an explicit lower bound for the perimeter of subset of manifolds in this family with the additional constraint of having diameter bounded by some constant $D > 0$. In other words, Milman proved that given $D \ge D' > 0$ and a $\mathsf{CD}(0,N)$ density $h\colon[0,D'] \to \mathbb{R}$, then for all $E \subset [0,D]$ it holds that $\mathsf{P}_{|\cdot|,h}(E) \ge \mathcal{I}_{N,D}(v)$, where

$$\mathcal{I}_{N,D}(v) := \frac{N}{D}\inf_{\xi\ge 0}\frac{(\min\{v,1-v\}(\xi+1)^N + \max\{v,1-v\}\xi^N)^{(N-1)/N}}{(\xi+1)^N - \xi^N} = \inf_{\xi\ge 0}\mathcal{I}_{N,D}(\xi, v).$$

As immediate consequence, one obtains that if we drop the reversibility hypothesis, the lower bound of the perimeter must be divided by the reversibility constant.

The author and Cavalletti proved, see Lemma 4.1 in [13], an expansion for the isoperimetric profile, as follows.

Lemma 5.1. *Fix $N > 1$. Then, we have the following estimate for $\mathcal{I}_{N,D}$:*

$$\mathcal{I}_{N,D}(w) \ge \frac{N}{D}\, w^{1-1/N}(1 - O(w^{1/N})) = \frac{N}{D}\,(w^{1-1/N} - O(w)), \quad \textit{as } w \to 0.$$

The following corollary incorporates both the irreversible and reversible case.

Corollary 5.2. *Fix $N > 1$. Then for all $D \ge D' > 0$ and for all one-dimensional oriented measured Finsler manifolds $([0,D'], F, h\mathcal{L}^1)$ satisfying the oriented $\mathsf{CD}(0,N)$ condition, it holds that*

$$\begin{aligned}\mathsf{P}_{F,h}(E) &\ge \frac{\mathcal{I}_{N,D}(\mathfrak{m}_h(E))}{\Lambda_F} \ge \frac{N}{\Lambda_F D'}\,\mathfrak{m}_h(E)^{1-1/N}(1 - O(\mathfrak{m}_h(E)^{1/N}) \\ &\ge \frac{N}{\Lambda_F D}\,\mathfrak{m}_h(E)^{1-1/N}(1 - O(\mathfrak{m}_h(E)^{1/N}),\end{aligned} \tag{5.5}$$

for any Borel set $E \subset [0,D']$. If E is of the form $[0, r_h(v)]$, then it holds

$$\mathsf{P}_{F,h}([0,r_h(v)]) = h(r_h(v) \ge \frac{N}{D'}v^{1-1/N}(1-O(v^{1/N})) \ge \frac{N}{D}v^{1-1/N}(1-O(v^{1/N})). \tag{5.6}$$

Remark 5.3. The lower bound in (5.5) is very rough for our purposes. If one attempted to prove the isoperimetric inequality (1.5), the inverse of the reversibility constant would appear in the lower bound.

The only reason why the factor Λ_F^{-1} appears in (5.5) is that the part of the boundary where the external normal vector "points to the left" might be non-empty. Indeed, if E is of the form $[0, b]$, then $\mathsf{P}_{F,h}(E) = \mathsf{P}_{|\cdot|,h}(E)$. We will see that the part of the boundary "pointing to the left" contributes little to the perimeter.

We give the definition of the residual of a set. This object quantifies, in a way that will be detailed in Section 6, how far away is a ray from the expected model space.

Definition 5.4. Let $D \geq D' > 0$ and let $([0, D'], F, h\mathscr{L}^1)$ be a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition. If $E \subset [0, D']$ is a Borel set, we define its D-residual as

$$\mathrm{Res}^D_{F,h}(E) := \frac{D\mathsf{P}_{F,h}(E)}{N(\mathfrak{m}_h(E))^{1-1/N}} - 1. \tag{5.7}$$

If $v \in (0, 1/2)$, we define the D-residual of v as

$$\mathrm{Res}^D_h(v) := \mathrm{Res}^D_{F,h}([0, r_h(v)]) = \frac{Dh(r_h(v))}{Nv^{1-1/N}} - 1. \tag{5.8}$$

Notice that in the definition of $\mathrm{Res}^D_h(v)$ there is no dependence on the Finsler structure F; indeed, the definition of $\mathrm{Res}^D_h(v)$ is given in terms of the perimeter of $[0, r_h(v)]$, and the perimeter of this set in $[0, D']$ does not capture the possible irreversibility of the Finsler structure. Using the residual, inequality (5.5) can be restated as

$$\mathrm{Res}^D_{F,h}(E) \geq \Lambda_F^{-1} - 1 - O(\mathfrak{m}_h(E)^{1/N}). \tag{5.9}$$

On the other hand, whenever the set E is of the form $[0, r]$, we obtain a much refined estimate

$$\mathrm{Res}^D_h(v) = \mathrm{Res}^D_{F,h}([0, r_h(v)]) \geq -O(v^{1/N}). \tag{5.10}$$

5.2. One-dimensional reduction for the optimal region

We are ready to apply the definition of residual to the disintegration rays. In order to ease the notation, we let $\mathsf{P}_{\alpha,R} = \mathsf{P}_{(\hat{X}_{\alpha,R}, F, \hat{\mathfrak{m}}_{\alpha,R})}$. The measure $\hat{\mathfrak{m}}_{\alpha,R}$ will be identified with the ray map $g_R(\alpha, \cdot)$ to $h_{\alpha,R}\mathscr{L}^1$, thus we define

$$\mathrm{Res}_{\alpha,R} := \mathrm{Res}^{R+\mathrm{diam}(E)}_{F,h_{\alpha,R}}(g(\alpha,\cdot)^{-1}(E \cap \hat{X}_{\alpha,R})), \quad \text{for } \alpha \in Q_R,$$
$$\mathrm{Res}_{x,R} := \mathrm{Res}_{\mathfrak{Q}_R(x),R}, \quad \text{for } x \in E.$$

The good rays are those rays having small residual. We quantify their abundance.

Proposition 5.5. *Assume that* $(X, F, \mathfrak{m})$ *is a* $\mathsf{CD}(0, N)$ *measured Finsler manifold, such that* $\mathsf{AVR}_X > 0$. *If* $E \subset X$ *is a bounded set attaining the identity in the inequality* (1.5), *then*

$$\limsup_{R\to\infty} \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} \mathrm{Res}_{\alpha,R}\, \mathfrak{q}_R(d\alpha) \leq 0. \tag{5.11}$$

Proof. In order to check that the function $\alpha \to \mathrm{Res}_{\alpha,R}$ is integrable, it is enough to check that $(\mathrm{Res}_{\alpha,R})^-$ is integrable. This last fact derives from the isoperimetric inequality

$$\mathrm{Res}_{\alpha,R} \geq \Lambda_F^{-1} - 1 - O\Big(\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\Big)^{1/N}\Big),$$

as stated in (5.9). We can now compute the integral in (5.11):

$$\begin{aligned}\int_{Q_R} \mathrm{Res}_{\alpha,R}\ \hat{\mathfrak{q}}_R(d\alpha) &= \int_{Q_R} \Big(\frac{(R+\mathrm{diam}(E))\mathsf{P}_{\alpha,R}(E)}{N}\Big(\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\Big)^{1-1/N} - 1\Big)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &= \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{1/N-1}\,N\,\mathfrak{m}(E)^{1-1/N}}\int_{Q_R}\mathsf{P}_{\alpha,R}(E)\,\hat{\mathfrak{q}}_R(d\alpha) - \mathfrak{m}(B_R)\\ &\leq \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{1/N-1}\,N\,\mathfrak{m}(E)^{1-1/N}}\,\mathsf{P}(E) - \mathfrak{m}(B_R)\\ &\leq \mathfrak{m}(B_R)\,\frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{1/N}}\,(\mathsf{AVR}_X\,\omega_N)^{1/N} - \mathfrak{m}(B_R),\end{aligned}$$

yielding

$$\frac{1}{\mathfrak{m}(B_R)}\int_{Q_R}\mathrm{Res}_{\alpha,R}\ \mathfrak{q}_R(d\alpha) \leq \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{1/N}}\,(\mathsf{AVR}_X\,\omega_N)^{1/N} - 1,$$

and the right-hand side goes to 0, as $R \to \infty$. ∎

Corollary 5.6. *Let $(X, F, \mathfrak{m})$ be a $\mathsf{CD}(0, N)$ measured Finsler manifold, having $\mathsf{AVR}_X > 0$. Let $E \subset X$ be a set saturating the isoperimetric inequality (1.5). Then it holds true that*

$$\limsup_{R\to\infty}\int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\ \mathfrak{m}(dx) \leq 0. \tag{5.12}$$

Proof. A direct computation gives

$$\begin{aligned}\int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\ \mathfrak{m}(dx) &= \int_{Q_R}\int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\ \hat{\mathfrak{m}}_{\alpha,R}(dx)\,\hat{\mathfrak{q}}_R(d\alpha)\\ &= \int_{Q_R}\mathrm{Res}_{\alpha,R}\ \hat{\mathfrak{m}}_{\alpha,R}(E)\,\hat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\int_{Q_R}\mathrm{Res}_{\alpha,R}\ \mathfrak{q}_R(d\alpha) \to 0.\end{aligned}$$ ∎

6. Analysis along the good rays

The last theorem asserts (in a very weak sense) that the residual, in the limit for $R \to \infty$, must be non-positive. Moreover, the measure of the traces of E is $\mathfrak{m}(E)/\mathfrak{m}(B_R)$, hence infinitesimal. For this reason, we now use the residual and the measure of the set to control the density $h\colon [0, D'] \to \mathbb{R}$, proving that in case of small measure and residual, h is close to the model density $x \in [0, D] \mapsto N x^{N-1}/D$. Similarly, we prove that the traces of E are closed to the optimal, i.e., a certain interval of the form $[0, r]$.

Remark 6.1. We will extensively use the Landau "big-O" and "small-o" notations. If several variables appear, but only a few of them are converging, either the "big-O" or "small-o" could in principle depend on the non-converging variables. However, this is not the case.

To be precise, in our setting, the converging variables will be $w \to 0$ and $\delta \to 0$. Conversely, the "free" variables will be the following: (1) D, a bound on the length of the ray; (2) $D' \in (0, D]$, the length of the ray; (3) $([0, D'], F, h)$, a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition (in practice, each transport ray); (4) $E \subset [0, D']$, a set with measure $\mathfrak{m}_h(E) = w$ and residual $\mathrm{Res}^D_{F,h}(E) \le \delta$.

The estimates we will prove are infinitesimal expansions as $w \to 0$ and $\delta \to 0$, and whenever a "big-O" or "small-O" appears, it has to be understood that it is uniform with respect to the "free" variable.

Remark 6.2. An important point to remark is the fact that we consider only the case when E is "on the left", i.e., $E \subset [0, L]$, with the tacit understanding that $L \ll D'$. This is possible because the transport rays come from the optimal transport problem between the bounded set isoperimetric E and the ball B_R.

6.1. Almost rigidity of the set E and of the length of the ray

We start considering the special case when the set E is of the form $E = [0, r]$. In this case, the Finsler structure plays no role, for the outer normal vector on the boundary of E points to the right. For this reason, we omit the proof of the following proposition, because it is exactly what is proven in Propositions 5.3 and 5.4 of [13].

Proposition 6.3. *Fix $N > 1$. Then, for $w \to 0$ and $\delta \to 0$, it holds that*

$$D' \ge D(1 - o(1)), \tag{6.1}$$

$$r_h(w) \le D(w^{1/N}(1 + o(1))), \tag{6.2}$$

$$r_h(w) \ge D(w^{1/N}(1 + o(1))), \tag{6.3}$$

where $D \ge D' > 0$ and $([0, D'], F, h\mathcal{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition such that $\mathrm{Res}^D_h(w) = \mathrm{Res}^D_{F,h}([0, r_h(w)]) \le \delta$.

We now drop the assumption $E = [0, r]$. Up to a negligible set, it holds that $E = \bigcup_{i \in \mathbb{N}}(a_i, b_i)$, where the intervals (a_i, b_i) are far away from each other (i.e., $b_i < a_j$ or $b_j < a_i$, for $i \ne j$). The boundedness of the original set of our isoperimetric problem implies that $E \subset [0, L]$, for some $L > 0$. Define $b(E) := \operatorname{ess\,sup} E \le L$.

In the next proposition, we prove that $b(E)$ is in the essential boundary of E.

Lemma 6.4. *Fix $N > 1$, $L > 0$, and $\Lambda \ge 1$. Then there exist two constants $\bar{w} > 0$ and $\bar{\delta} > 0$ (depending only on N, L, and Λ) such that the following happens. For all $D \ge D' > 0$ with $D \ge 4L\Lambda$, for all $([0, D'], F, h\mathcal{L}^1)$, a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \le \Lambda$, and for all $E \subset [0, L]$ such that $\mathfrak{m}_h(E) \le \bar{w}$ and $\mathrm{Res}^D_{F,h}(E) \le \bar{\delta}$, there exist $a \in [0, b(E))$ and an at-most-countable family of intervals $((a_i, b_i))_i$ such that, up to a negligible set,*

$$E = \bigcup_i (a_i, b_i) \cup (a, b(E)),$$

with $a_i, b_i < a$, $\forall i$. Moreover, h is strictly increasing on $[0, b(E)]$.

Proof. Taking into account the definition of residual and the isoperimetric inequality (5.9), choosing $\bar{\delta} \leq 1$, we can deduce that

$$\frac{D'}{D} \geq \frac{1+\mathrm{Res}^{D'}_{F,h}(E)}{1+\mathrm{Res}^{D}_{F,h}(E)} \geq \frac{1+\Lambda_F^{-1}-1-O(w^{1/N})}{1+\bar{\delta}} \geq \frac{\Lambda^{-1}}{2} - O(w^{1/N})).$$

If we choose $\bar{w}$ small enough, taking into account the hypothesis $D \geq 4L\Lambda$, we deduce $D' \geq 2L$

Since $E = \bigcup_i (a_i, b_i)$ (up to a negligible set), our aim is to prove that there exists j such that $a_i < a_j$, for all $i \neq j$. In this case, we set $a = a_j$. Suppose, on the contrary, that $\forall j$, $\exists i \neq j$ such that $a_i > a_j$. Hence there exists a sequence $(i_n)_n$ so that $(a_{i_n})_n$ is increasing, thus converging to some $y \in (0, L]$. Recalling that $F \geq \Lambda^{-1}$, we can compute the perimeter:

$$\infty = \sum_{n\in\mathbb{N}} F(a_{i_n})h(a_{i_n}) \leq \mathsf{P}_{F,h}(E) = \frac{N}{D}(\mathfrak{m}_h(E))^{1-1/N}(1+\mathrm{Res}^{D}_{F,h}(E)) < \infty,$$

which is a contradiction.

Finally, we prove that h increases on $[0, b(E)]$. In order to simplify the notation, let $b := b(E)$. Denote by

$$t := \lim_{z\searrow 0} \frac{h(b+z)^{1/(N-1)} - h(b)^{1/(N-1)}}{z}$$

the right-derivative of $h^{1/(N-1)}$ in b (whose existence is guaranteed by concavity). If $t > 0$, then the concavity of $h^{1/(N-1)}$ yields that h is strictly increasing in $[0, b]$. Suppose on the contrary that $t \leq 0$; then it holds that

$$h(x) \leq h(b)\Big(\frac{D'-x}{D'-b}\Big)^{N-1}, \quad \forall x \in [0,b], \quad \text{and} \quad h(x) \leq h(b), \quad \forall x \in [b, D'].$$

We integrate obtaining

$$\begin{aligned}
1 &\leq \int_0^b h(b)\Big(\frac{D'-x}{D'-b}\Big)^{N-1} dx \\
&\quad + \int_b^{D'} h(b)\,dx = \frac{h(b)}{N}\Big(\frac{D'^N-(D'-b)^N}{(D'-b)^{N-1}} + N(D'-b)\Big) \\
&\leq \frac{\mathsf{P}_{F,h}(E)}{N}\Big(\frac{D'^N}{(D'-b)^{N-1}} + ND'\Big) = \frac{\mathsf{P}_{F,h}(E)D'}{N}\Big(\Big(1-\frac{b}{D'}\Big)^{1-N} + N\Big) \\
&= \frac{\mathsf{P}_{F,h}(E)D'}{N}\Big(1+(N-1)\,\frac{b}{D'} + o\Big(\frac{b}{D'}\Big) + N\Big).
\end{aligned} \tag{6.4}$$

The first factor in the right-hand side of the estimate above is controlled just using the definition of residual:

$$\frac{\mathsf{P}_{F,h}(E)D'}{N} \leq \frac{\mathsf{P}_{F,h}(E)D}{N} = \mathfrak{m}_h(E)^{1-1/N}(1+\mathrm{Res}^{D}_{F,h}(E)),$$

and, if $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}^D_{F,h}(E)$ is bounded, then the term above goes to 0. Regarding the second factor, it suffices to prove that b/D' is bounded:

$$\frac{b}{D'} \le \frac{L}{D'} \le \frac{L}{2L} = \frac{1}{2}.$$

If we put together this last two estimates, we deduce that the right-hand side of (6.4) is infinitesimal as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}^D_{F,h}(E) \to 0$, obtaining a contradiction. ■

This proposition guarantees the existence of a right-extremal connected component of the set E; this component is precisely the interval $(a, b(E))$. We will denote by $a(E)$ the number a given by Proposition 6.4. Since our estimates are infinitesimal expansions in the limit as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}^D_{F,h}(E) \to 0$, we will always assume that $\mathfrak{m}_h(E) \le \bar{w}$ and $\mathrm{Res}^D_{F,h}(E) \le \bar{\delta}$, so that the expression $a(E)$ makes sense. For the same reason, we will always assume that h is increasing in the interval $[0, b(E)]$.

We now prove that this component $(a(E), b(E))$ tends to fill the set E, that $b(E)$ converges as expected to $D\,\mathfrak{m}_h(E)^{1/N}$, and that the length of the ray tends to be maximal.

Proposition 6.5. *Fix $N > 1$, $L > 0$, and $\Lambda \ge 1$. Then, for $w \to 0$ and $\delta \to 0$ it holds that*

$$D' \ge D(1 - o(1)), \tag{6.5}$$

$$b(E) \le Dw^{1/N} + Do(w^{1/N}), \tag{6.6}$$

$$b(E) \ge Dw^{1/N} - Do(w^{1/N}), \tag{6.7}$$

$$a(E) \le Do(w^{1/N}), \tag{6.8}$$

where $D \ge 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathscr{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \le \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \le \delta$.

Proof. Part 1. *Inequality* (6.5).

Since h is decreasing on $[0, b(E)]$, we have that $h(r_h(v)) \le h(b(E)) \le \mathsf{P}_{F,h}(E)$, hence $\mathrm{Res}^D_h(v) \le \mathrm{Res}^D_{F,h}(E)$. The thesis follows from estimate (6.1).

Part 2. *Inequality* (6.7).

Since the density h is strictly increasing on $[0, b(E)]$ and $E \subset [0, b(E)]$ (up to a null measure set), it holds that $r_h(w) \le b(E)$ and

$$\mathrm{Res}^D_h(w) = \frac{Dh(r_h(w))}{Nw^{1-1/N}} - 1 \le \frac{Dh(b(E))}{Nw^{1-1/N}} - 1 \le \frac{D\mathsf{P}_{F,h}(E)}{Nw^{1-1/N}} - 1 = \mathrm{Res}^D_{F,h}(E) \le \delta.$$

Estimate (6.3) concludes this part

$$D(w^{1/N} - o(w^{1/N})) \le r_h(w) \le b(E).$$

Part 3. *Inequality* (6.8).

First we prove that $a(E) < r_h(w)$, for w and δ small enough. Suppose on the contrary that $a(E) \ge r_h(w)$, implying that $h(a(E)) \ge h(r_h(w))$, hence $\mathsf{P}_{F,h}(E) \ge \Lambda^{-1}h(a(E)) +$

$h(b(E)) \geq (1+\Lambda^{-1})h(r_h(w))$. We deduce that (compare with (5.10))

$$-O(w^{1/N}) \leq \mathrm{Res}_h^D(w) = \frac{Dh(r_h(w))}{Nw^{1-1/N}} - 1 \leq \frac{D\mathsf{P}_{F,h}(E)}{(1+\Lambda^{-1})Nw^{1-1/N}} - 1$$
$$= \frac{1}{1+\Lambda^{-1}}(\mathrm{Res}_{F,h}^D(E) - \Lambda^{-1}) \leq \frac{\delta - \Lambda^{-1}}{1+\Lambda^{-1}}.$$

If we take the limit as $w \to 0$ and $\delta \to 0$ we obtain a contradiction.

Using the Bishop–Gromov inequality and the isoperimetric inequality (respectively), we get

$$h(a(E)) \geq h(r_h(w))\Big(\frac{a(E)}{r_h(w)}\Big)^{N-1},$$
$$h(b(E)) \geq h(r_h(w)) \geq \frac{N}{D} w^{1-1/N}(1 - O(w^{1/N})).$$

We put together the inequalities above obtaining

$$\frac{N}{D} w^{1-1/N}(1 + \mathrm{Res}_{F,h}^D(E)) = \mathsf{P}_{F,h}(E) \geq h(b(E)) + \Lambda^{-1}h(a(E))$$
$$\geq h(r_h(w)) + \Lambda^{-1}h(a(E)) \geq h(r_h(w))\Big(1 + \Lambda^{-1}\Big(\frac{a(E)}{r_h(w)}\Big)^{N-1}\Big)$$
$$\geq \frac{N}{D} w^{1-1/N}(1 - O(w^{1/N}))\Big(1 + \Lambda^{-1}\Big(\frac{a(E)}{r_h(w)}\Big)^{N-1}\Big),$$

hence

$$\begin{aligned} a(E) &\leq r_h(w)\Lambda^{1/(N-1)}\Big(\frac{1+\mathrm{Res}_{F,h}^D(E)}{1+O(w^{1/N})} - 1\Big)^{1/(N-1)} \\ &\leq r_h(w)\,\Lambda^{1/(N-1)}\big((1+\delta)(1-O(w^{1/N})) - 1\big)^{1/(N-1)} \\ &\leq r_h(w)\,o(1) \leq Dw^{1/N}(1+o(1))o(1) = Do(w^{1/N}), \end{aligned} \tag{6.9}$$

where the estimate (6.2) was taken into account.

Part 4. *Inequality* (6.6).

Since

$$\int_E h = \int_0^{r_h(w)} h,$$

we deduce (taking into account $a(E) \leq r_h(w) \leq b(E)$)

$$\int_{E\cap[0,r_h(w)]} h + \int_{r_h(w)}^{b(E)} h = \int_{E\cap[0,r_h(w)]} h + \int_{[0,r_h(w)]\setminus E} h$$
$$= \int_{E\cap[0,r_h(w)]} h + \int_{[0,a(E)]\setminus E} h,$$

hence

$$(b(E) - r_h(w))\,h(r_h(w)) \leq \int_{r_h(w)}^{b(E)} h = \int_{[0,a(E)]\setminus E} h \leq \int_0^{a(E)} h \leq a(E)\,h(a(E)),$$

yielding

$$b(E) - r_h(w) \leq a(E)\, \frac{h(a(E))}{h(r_h(w))} \leq a(E).$$

Combining the inequality above, the already-proven estimate (6.7), and the estimate (6.2), we reach the conclusion. ∎

6.2. Almost rigidity of the density *h*

In this section we prove that the density h converges uniformly to the density Nx^{N-1}/D^N. The bound from below is easy, and follows from the Bishop–Gromov inequality.

Proposition 6.6. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. Then, for $w \to 0$ and $\delta \to 0$, it holds that*

$$h(x) \geq \frac{N}{D^N}\, x^{N-1}(1 - o(1)), \quad \text{uniformly with respect to } x \in [0, b(E)], \tag{6.10}$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathscr{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the $\mathsf{CD}(0, N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\operatorname{Res}^D_{F,h}(E) \leq \delta$.

Proof. Fix $x \in [0, b(E)]$. The Bishop–Gromov inequality yields

$$h(x) \geq h(b(E))\, \frac{x^{N-1}}{b(E)^{N-1}} \geq h(r_h(w))\, \frac{x^{N-1}}{b(E)^{N-1}}.$$

The first factor is controlled using the isoperimetric inequality (5.6):

$$h(r_h(w)) \geq \frac{N}{D}\, w^{1-1/N}(1 - O(w^{1/N})) = \frac{N}{D}\, w^{1-1/N}(1 - o(1)),$$

whereas the term $b(E)$ is controlled using estimate (6.6):

$$b(E) \leq D w^{1/N}(1 + o(1)).$$

By combining these to estimates, we reach the thesis ∎

The following corollary gives a lower boundary for the residual, under the hypothesis that the (positive part of the) residual is bounded from above, improving inequality (5.9).

Corollary 6.7. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. Then, for $w \to 0$ and $\delta \to 0$, it holds that*

$$\operatorname{Res}^D_{F,h}(E) \geq -o(1), \tag{6.11}$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathscr{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the $\mathsf{CD}(0, N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\operatorname{Res}^D_{F,h}(E) \leq \delta$.

Proof. By a direct computation, recalling estimates(6.10) and (6.7), we obtain

$$\begin{aligned}\operatorname{Res}^D_{F,h}(E) &\geq \frac{Dh(b(E))}{Nw^{1-1/N}} - 1 \geq \frac{b(E)^{N-1}(1 - o(1))}{D^{N-1}w^{1-1/N}} - 1 \\ &\geq \frac{(w^{1/N}(1 - o(1)))^{N-1}}{w^{1-1/N}} - 1 \geq o(1).\end{aligned}$$

∎

In order to prove an upper bound for the density, we present the following, purely technical lemma.

Lemma 6.8. *Fix $N > 1$ and consider the function $f\colon [0,1)\times[0,\infty]\to\mathbb{R}$ given by*

$$f(t,\eta) = \frac{1+\eta-t^N}{1-t}.$$

Define the function g by

$$g(\eta) = \sup\{t-s : f(t,0) \le f(s,\eta)\}. \tag{6.12}$$

Then $\lim_{\eta\to 0} g(\eta) = 0$.

Proof. The proof is by contradiction. Suppose that there exist $\varepsilon > 0$ and three sequences $(\eta_n)_n$, $(t_n)_n$, and $(s_n)_n$ such that $\eta_n \to 0$, $f(t_n,0) \le f(s_n,\eta_n)$, and $t_n - s_n > \varepsilon$. Up to taking a sub-sequence, we can assume that $t_n \to t$ and $s_n \to s$, hence $1 \ge t \ge s+\varepsilon$. The functions $f(\cdot,\eta_n)$ converge to $f(\cdot,0)$, uniformly in the interval $[0,1-\varepsilon/2]$. This implies $f(s_n,\eta_n) \to f(s,0)$, yielding $f(t,0) \le f(s,0)$. Since $t\mapsto f(t,0)$ is strictly increasing, we obtain $t \le s \le t-\varepsilon$, which is a contradiction. ∎

We now obtain an upper bound for h in the interval $[a(E),b(E)]$ going in the opposite direction of the Bishop–Gromov inequality.

Proposition 6.9. *Fix $N > 1$, $L > 0$, and $\Lambda \ge 1$. Then, for $w \to 0$ and $\delta \to 0$, it holds that*

$$h(x) \le h(b(E))\Big(\frac{x}{b(E)} + o(1)\Big)^{N-1}, \quad \textit{uniformly with respect to } x\in[a(E),b(E)], \tag{6.13}$$

where $D \ge 4L\Lambda$, $D' \in (0,D]$, $([0,D'],F,h\mathscr{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0,N)$ condition, with $\Lambda_F \le \Lambda$, and the set $E\subset[0,L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \le \delta$.

Proof. Fix $x \in [a(E),b(E)]$. In order to ease the notation, define

$$a := a(E), \quad b := b(E), \quad k := h(x)^{1/(N-1)} \quad \text{and} \quad l := h(b(E))^{1/(N-1)}.$$

The concavity of $h^{1/(N-1)}$ yields

$$\begin{aligned} h(y) &\ge \Big(\frac{y}{x}\Big)^{N-1} k^{N-1}, \quad \forall y\in[a,x],\\ h(y) &\ge \Big(l + (k-l)\,\frac{b-y}{b-x}\Big)^{N-1}, \quad \forall y \in [x,b]. \end{aligned}$$

If we integrate, we obtain

$$\begin{aligned} w &\ge \int_a^x \frac{y^{N-1}}{x^{N-1}}\,k^{N-1}\,dy + \int_x^b \Big(l+(k-l)\,\frac{b-y}{b-x}\Big)^{N-1}\,dy\\ &= \frac{k^{N-1}\,(x^N-a^N)}{Nx^{N-1}} + \frac{b-x}{N}\,\frac{l^N-k^N}{l-k}, \end{aligned}$$

yielding

$$\frac{1-(k/l)^N}{1-k/l} \le \frac{Nw - \frac{k^{N-1}(x^N-a^N)}{x^{N-1}}}{l^{N-1}(b-x)} = \frac{\frac{Nw}{bl^{N-1}} - \frac{k^{N-1}(x^N-a^N)}{b(lx)^{N-1}}}{1-x/b}$$
$$\le \frac{\frac{Nw}{bl^{N-1}} - \frac{x^N-a^N}{b^N}}{1-x/b} = \frac{\frac{Nw}{bl^{N-1}} + \frac{a^N}{b^N} - \frac{x^N}{b^N}}{1-x/b},$$

where in the last inequality we used the Bishop–Gromov inequality, written in the form $k^{N-1}/l^{N-1} \ge x^{N-1}/b^{N-1}$. We now estimate the terms Nw/bl^{N-1} and a^N/b^N. Regarding the former, taking into account (6.7) and the isoperimetric inequality (5.6), we deduce

$$\frac{Nw}{bl^{N-1}} = \frac{Nw}{b(E)\,h(b(E))} \le \frac{Nw}{b(E)\,h(r_h(w))}$$
$$\le \frac{Nw}{Dw^{1/N}(1-o(1))\,\frac{N}{D}w^{1-1/N}(1-O(w^{1/N})} = 1+o(1).$$

Conversely, we estimate the latter term (recall (6.6) and (6.8)) as

$$\frac{a^N}{b^N} = \frac{a(E)^N}{b(E)^N} \le \frac{D^N o(w)}{D^N w(1-o(1))^N} = o(1).$$

Putting all the pieces together, we obtain

$$f\Big(\frac{k}{l},0\Big) = \frac{1-(k/l)^N}{1-k/l} \le \frac{\frac{Nw}{bl^{N-1}} + \frac{a^N}{b^N} - \frac{x^N}{b^N}}{1-x/b} \le \frac{1+o(1)-\frac{x^N}{b^N}}{1-x/b} = f\Big(\frac{x}{b},o(1)\Big),$$

where f is the function of Lemma 6.8. Applying this lemma, we get

$$\frac{k}{l} - \frac{x}{b} \le g(o(1)) = o(1).$$

If we explicit the definitions of k, l, and b, it turns out that the inequality above is precisely the thesis. ∎

6.3. Rescaling the diameter and renormalizing the measure

So far, we have obtained an estimate of the densities h and the set E. The presence of the factor $1/D^N$ in the estimate (6.10) suggests the need of a suitable rescaling to get a non-trivial limit estimate. We rescale the space by $1/b(E)$, and renormalize the measure by $\mathfrak{m}_h(E)$.

Fix $k>0$ and define the rescaling transformation $S_k(x) = x/k$. Given a density $h\colon [0,D'] \to \mathbb{R}$ and $E \subset [0,L]$, we define

$$\nu_{h,E} = (S_{b(E)})_\# \Big(\frac{\mathfrak{m}_h \llcorner E}{\mathfrak{m}_h(E)}\Big) \in \mathscr{P}([0,1]).$$

Clearly, $\nu_{h,E} \ll \mathscr{L}^1$, so we denote by $\tilde{h}_E\colon [0,1] \to \mathbb{R}$ the Radon–Nikodym derivative $d\nu_{h,e}/d\mathscr{L}^1$. The density $\tilde{h}_E$ can be explicitly computed:

$$\tilde{h}_E(t) = \mathbf{1}_E(b(E)t)\,\frac{b(E)}{\mathfrak{m}_h(E)}\,h(b(E)t). \tag{6.14}$$

Notice that, since E could be disconnected, the indicator function in (6.14) prevents $\tilde{h}_E^{1/(N-1)}$ from being concave, i.e., $\tilde{h}_E$ possibly fails the oriented $\mathsf{CD}(0,N)$ condition. However, in the limit, the $\mathsf{CD}(0,N)$ condition reappears, as it is explicated by the following proposition.

Proposition 6.10. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. Then, for $w \to 0$ and $\delta \to 0$, it holds that*

$$\|\tilde{h}_E - Nt^{N-1}\|_{L^\infty(0,1)} \leq o(1),$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathscr{L}^1)$ is a one-dimensional measured Finsler manifold satisfying the $\mathsf{CD}(0,N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof. Fix $t \in [0,1]$. The proof is divided in four parts.

Part 1. *Estimate from below and $t > a(E)/b(E)$.*

Since $t > a(E)/b(E)$, then $t\,b(E) \in E$ (for a.e. t). A direct computation gives

$$\begin{aligned}
\tilde{h}_E(t) &= \frac{b(E)}{w} h(tb(E)) \geq \frac{Nb(E)^N}{D^N w} t^{N-1}(1 - o(1)) \\
&\geq \frac{ND^N w(1+o(1))^N}{D^N w} t^{N-1}(1 - o(1)) = Nt^{N-1} - o(1),
\end{aligned}$$

having used the estimate (6.10), with $x = tb(E)$, in the first inequality, and (6.7) in the second inequality.

Part 2. *Estimate from below and $t \leq a(E)/b(E)$.*

In this case, it may happen that $tb(E) \notin E$, so the best estimate from below is the non-negativity. For this reason, here we exploit the fact that the interval $[0, a(E)/b(E)]$ is "short" and that $t \leq a(E)/b(E)$. A direct computation gives (recall (6.7) and (6.8))

$$\begin{aligned}
\tilde{h}_E(t) &\geq 0 \geq Nt^{N-1} - Nt^{N-1} \geq Nt^{N-1} - N\frac{a(E)^{N-1}}{b(E)^{N-1}} \\
&\geq Nt^{N-1} - N\frac{D^{N-1}o(w^{1-1/N})}{D^{N-1}w^{1-1/N}(1+o(1))^{N-1}} \geq Nt^{N-1} - o(1).
\end{aligned}$$

Part 3. *Estimate from above and $t \geq a(E)/b(E)$.*

We use estimate (6.13), with $x = tb(E)$, deducing

$$\begin{aligned}
\tilde{h}_E(t) &= \frac{b(E)}{w} h(tb(E)) \leq \frac{b(E)}{w} h(b(E))(t+o(1))^{N-1} \leq \frac{b(E)}{w} h(b(E))(t^{N-1}+o(1)) \\
&\leq \frac{Dw^{1/N}(1+o(1))}{w} \mathsf{P}_{F,h}(E)(t^{N-1}+o(1)) \\
&= \frac{Dw^{1/N}(1+o(1))}{w}\frac{N}{D} w^{1-1/N}(1+\mathrm{Res}^D_{F,h}(E))(t^{N-1}+o(1)) \\
&\leq N(1+o(1))(1+\delta)(t^{N-1}+o(1)) = Nt^{N-1}+o(1)
\end{aligned}$$

(in the second inequality we used the uniform continuity of $t \in [0,1] \mapsto t^{N-1}$; in the third one, estimate (6.6)).

Part 4. *Estimate from above and $t \le a(E)/b(E)$.*

Without loss of generality, we can assume that $a(E) \in E$. Using the previous part, we compute

$$\begin{aligned}\tilde h_E(t) &= b(E)\,\frac{\mathbf{1}_E(tb(E))}{\mathfrak{m}_h(E)}\,h(b(E)t) \le \frac{b(E)}{\mathfrak{m}_h(E)}\,h(b(E)t) \le \frac{b(E)}{\mathfrak{m}_h(E)}\,h(a(E))\\ &= \tilde h_E\Big(\frac{a(E)}{b(E)}\Big) \le N\Big(\frac{a(E)}{b(E)}\Big)^{N-1} + o(1) \le o(1) \le Nt^{N-1} + o(1). \qquad\blacksquare\end{aligned}$$

The following theorem summerizes the contents of this section. Notice that the function ω takes as argument the positive part of the residual, and not the residual itself.

Theorem 6.11. *Fix $N > 1$, $L > 0$, and $\Lambda \ge 1$. Then there exists a function $\omega\colon (0,\infty) \times [0,\infty) \to \mathbb{R}$, infinitesimal in 0, such that the following holds. For all $D \ge 4L\Lambda$, $D' \in (0, D)$, for all $([0, D'], F, h\mathscr{L}^1)$ one-dimensional measured Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \le \Lambda$, and for all $E \subset [0, L]$, it holds that*

$$|b(E) - D\,\mathfrak{m}_h(E)^{1/N}| \le D\,\mathfrak{m}_h(E)^{1/N}\,\omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+), \tag{6.15}$$

$$\|\tilde h_E - Nt^{N-1}\|_{L^\infty} \le \omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+), \tag{6.16}$$

$$\mathrm{Res}^D_{F,h}(E) \ge -\omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+), \tag{6.17}$$

where $b(E) = \operatorname{ess\,sup} E$ and $\tilde h_E$ is the density of $\mathfrak{m}_h(E)^{-1}(S_{b(E)})_\#\mathfrak{m}_h\llcorner_E$, with $S_{b(E)}(x) = x/b(E)$.

7. Passage to the limit as $R \to \infty$

We now go back to studying the identity case of the isoperimetric inequality. Fix E a bounded Borel set with positive measure such that

$$\mathsf{P}(E) = N(\omega_N\,\mathsf{AVR}_X)^{1/N}\,\mathfrak{m}(E)^{1-1/N},$$

where $(X, F, \mathfrak{m})$ is a $\mathsf{CD}(0, N)$ measured Finsler manifold having $\mathsf{AVR}_X > 0$. We will use the notation introduced Section 4. Denote by φ_R the 1-Lipschitz Kantorovich potential associated to $f_R = \mathbf{1}_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\mathbf{1}_{B_R}$. If we add a constant to φ_R, we still get a Kantorovich potential, so we can assume that the family φ_R is equibounded on every bounded set. The Ascoli–Arzelà theorem, together with a diagonal argument, implies that, up to subsequences, φ_R converges to a certain 1-Lipschitz function φ_∞, uniformly on every compact set.

We recall the disintegration given by Proposition 4.1:

$$\mathfrak{m}\llcorner_{\hat{\mathcal{T}}_R} = \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}\,\hat{\mathfrak{q}}_R(d\alpha) \quad\text{and}\quad \mathsf{P}(E;\cdot) \ge \int_{Q_R} \mathsf{P}_{\hat X_{\alpha,R}}(E;\cdot)\,\hat{\mathfrak{q}}_R(d\alpha). \tag{7.1}$$

The effort of this section goes in the direction to understand how the properties of the disintegration behave at the limit, and to try to pass to the limit in the disintegration. Throughout this section, we set $\rho = (\frac{\mathfrak{m}(E)}{\omega_N\,\mathsf{AVR}_X})^{1/N}$.

Before going on, using the self-improvement estimate of the residual (6.17), we prove the following proposition.

Proposition 7.1. *Up to taking subsequences, it holds that*

$$\lim_{R\to\infty} \mathrm{Res}_{\mathfrak{Q}_R(x),R} = 0, \quad \mathfrak{m}\llcorner_E\text{-}a.e.. \tag{7.2}$$

Proof. Corollary 5.6 guarantees that

$$\limsup_{R\to\infty} \int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\ \mathfrak{m}(dx) \leq 0,$$

Using estimate (6.17), we estimate the negative part of the residual:

$$(\mathrm{Res}_{\mathfrak{Q}_R(x),R})^- \leq \omega\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, (\mathrm{Res}_{\mathfrak{Q}_R(x),R})^+\Big) = \omega\Big(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, 0\Big),$$

where ω is a function, infinitesimal in $(0,0)$. The L^1-norm of the residual is given by

$$\|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\|_{L^1(E;\mathfrak{m})} = 2\int_E (\mathrm{Res}_{\mathfrak{Q}_R(x),R})^-\,d\mathfrak{m} + \int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\,d\mathfrak{m}.$$

Taking into account the previous inequality and, again, Corollary 5.6, we deduce that $\mathrm{Res}_{\mathfrak{Q}_R(x),R}$, converges to 0 in L^1. By taking a subsequence, we obtain (7.2). ∎

7.1. Passage to the limit of the radius

First of all, we define the *radius* function $r_R\colon \overline{E} \to [0, \operatorname{diam} E]$. Fix $x \in E \cap \hat{\mathcal{T}}_R$ and let

$$E_{x,R} := (g_R(\mathfrak{Q}_R(x),\cdot))^{-1}(E) \subset [0, |\hat{X}_{\mathfrak{Q}_R(x),R}|].$$

Define

$$r_R(x) := \begin{cases} \operatorname{ess\,sup} E_{x,R}, & \text{if } x \in E \cap \hat{\mathcal{T}}_R, \\ 0, & \text{otherwise.} \end{cases} \tag{7.3}$$

Notice that $r_R(x) = b(E_{x,E})$, where the notation $b(E)$ was introduced in Section 6.1.

The radius function is defined on $\overline{E}$ for two motivations: we require a common domain not depending on R and the domain must be compact.

Remark 7.2. The set $E \cap \hat{\mathcal{T}}_R$ has full $\mathfrak{m}\llcorner_E$-measure in $\overline{E}$, hence it does not really matter how r_R is defined outside $E \cap \hat{\mathcal{T}}_R$. This fact is relevant, because we will only take limits in the $\mathfrak{m}\llcorner_E$-a.e. sense.

The next proposition ensures that, in the limit as $R \to \infty$, the function r_R converges to ρ, which is precisely the radius that we expect.

Proposition 7.3. *Up to subsequences, it holds true*

$$\lim_{R\to\infty} r_R = \rho = \Big(\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\Big)^{1/N}, \quad \mathfrak{m}\llcorner_E\text{-}a.e.$$

Proof. By Proposition 7.1, there exist a sequence R_n and a negligible subset $N \subset E$, such that $\lim_{n\to\infty} \mathrm{Res}_{\mathfrak{Q}_{R_n}(x),R_n} = 0$, for all $x \in E \setminus N$.

Define $G := \bigcap_n \widehat{\mathcal{T}}_{R_n} \setminus N$, and notice that $\mathfrak{m}(E \setminus G) = 0$. Fix $n \in \mathbb{N}$ and $x \in G$ and let $\alpha := \mathfrak{Q}_{R_n}(x) \in Q_{R_n}$. Clearly, it holds

$$
\begin{aligned}
|r_{R_n}(x) - \rho| \le{}& \left| r_{R_n}(x) - (R_n + \operatorname{diam} E)\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{1/N} \right| \\
&+ \left| (R_n + \operatorname{diam} E)\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{1/N} - \rho \right|.
\end{aligned}
$$

The second term goes to 0 by definition of AVR, so we focus on the first term. Consider the ray $(\widehat{X}_{\alpha,R_n}, F, \widehat{\mathfrak{m}}_{\alpha,R_n})$. By definition, we have that

$$
\mathrm{Res}^{R_n+\operatorname{diam} E}_{h_{\alpha,R_n}}(E_{x,R_n}) = \mathrm{Res}_{\alpha,R_n}.
$$

We can now use Theorem 6.11 (in particular, estimate (6.15)), obtaining

$$
\begin{aligned}
&\left| r_{R_n}(x) - (R_n + \operatorname{diam} E)\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{1/N} \right| \\
&\quad = \left| r_{R_n}(x) - (R_n + \operatorname{diam} E)(\mathfrak{m}_{h_{\alpha,R_n}}(E_{x,R_n}))^{1/N} \right| \\
&\quad \le (R_n + \operatorname{diam} E)\,\mathfrak{m}_{h_{\alpha,R_n}}(E)^{1/N}\, \omega\big(\mathfrak{m}_{h_{\alpha,R_n}}(E), (\mathrm{Res}^{R_n+\operatorname{diam} E}_{F,h_{\alpha,R_n}}(E_{x,R_n})^+)\big) \\
&\quad = (R_n + \operatorname{diam} E)\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{1/N} \omega\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}, (\mathrm{Res}_{\mathfrak{Q}_R(x),R_n})^+\right).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we conclude the proof. ∎

7.2. Passage to the limit of the rays

Consider now a constant-speed parametrization of the rays inside the set E:

$$
\gamma_s^{x,R} := \begin{cases} g_R(\mathfrak{Q}_R(x), s\, r_R(x)), & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ x, & \text{otherwise,} \end{cases} \tag{7.4}
$$

where $x \in \overline{E}$ and $s \in [0,1]$. Remark 7.2 applies also to the map $x \mapsto \gamma^{x,R}$. A direct consequence of the definition of $\gamma^{x,R}$ and the properties of the disintegration are

$$
\mathsf{d}(\gamma_t^{x,R}, \gamma_s^{x,R}) = \varphi_R(\gamma_t^{x,R}) - \varphi_R(\gamma_s^{x,R}), \quad \forall\, 0 \le t \le s \le 1, \text{ for } \mathfrak{m}\text{-a.e. } x \in E, \tag{7.5}
$$

$$
\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R}) = r_R(x), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E, \tag{7.6}
$$

$$
x \in \gamma^{x,R}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{7.7}
$$

Please notice the order of the quantifiers in (7.5): that equation means that $\exists N \subset E$ negligible such that $\forall t \le s$, $\forall x \in E \setminus N$, (7.5) holds true. In equation (7.7), the expression $x \in \gamma^{x,R}$ means that $\exists t \in [0,1]$ such that $x = \gamma_t^{x,R}$, or, equivalently, $\min_{t\in[0,1]} \mathsf{d}(x, \gamma_t^{x,R}) = 0$.

In order to compute the limit behaviour of $\gamma^{x,R}$ as $R \to \infty$, we proceed as follows. Define the set $K := \{\gamma \in \mathrm{Geo}(X) : \gamma_0, \gamma_1 \in \overline{E}\}$; this set is compact by the Ascoli–Arzelá theorem. Define the measure (having mass $\mathfrak{m}(E)$)

$$\tau_R := (\mathrm{Id} \times \gamma^{\cdot,R})_\# \mathfrak{m}\llcorner_E \in \mathcal{M}(\overline{E} \times K).$$

The measures τ_R enjoy the following immediate properties:

$$(P_1)_\# \tau_R = \mathfrak{m}\llcorner_E, \quad \text{and} \quad \gamma = \gamma^{x,R}, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \overline{E} \times K. \tag{7.8}$$

The properties (7.5)–(7.7) can be restated using a more measure-theoretic language:

$$\begin{aligned} &\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)) = 0, \\ &\quad \forall\, 0 \le t \le s \le 1, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \overline{E} \times K, \end{aligned} \tag{7.9}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x) = 0, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \overline{E} \times K, \tag{7.10}$$

$$x \in \gamma, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \overline{E} \times K \tag{7.11}$$

Clearly, the family of measures $(\tau_R)_{R>0}$ is tight, thus, by the Prokhorov theorem, we can extract a sub-sequence such that $\tau_R \rightharpoonup \tau$ weakly, i.e., $\int_{\overline{E}\times K} \psi\, d\tau_R \to \int_{\overline{E}\times K} \psi\, d\tau$, for all $\psi \in C_b(\overline{E} \times K)$.

The next proposition guarantees that the properties (7.9)–(7.11) pass to the limit as $R \to \infty$.

Proposition 7.4. *For τ-a.e. $(x,\gamma) \in \overline{E} \times K$, it holds that*

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)), \quad \forall\, 0 \le t \le s \le 1, \tag{7.12}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{7.13}$$

$$x \in \gamma. \tag{7.14}$$

Proof. Fix $t \le s$ and integrate (7.9) in $\overline{E} \times K$, obtaining

$$\begin{aligned} 0 &= \int_{\overline{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)))\, \tau_R(dx\, d\gamma) \\ &= \int_{\overline{E}\times K} L^{t,s}_{\varphi_R}(\gamma)\, \tau_R(dx\, d\gamma), \end{aligned}$$

having set $L^{t,s}_\psi(\gamma) := \mathsf{d}(e_t(\gamma), e_s(\gamma)) - \psi(e_t(\gamma)) + \psi(e_s(\gamma))$. The map $L^{t,s}_{\varphi_R} : K \to \mathbb{R}$ is clearly continuous and converges uniformly (recall that $\varphi_R \to \varphi_\infty$ uniformly on every compact) to $L^{t,s}_{\varphi_\infty}$. Therefore, we can take the limit in the equation above, obtaining

$$\begin{aligned} 0 &= \int_{\overline{E}\times K} L^{t,s}_{\varphi_\infty}(\gamma)\, \tau(dx\, d\gamma) \\ &= \int_{\overline{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_\infty(e_t(\gamma)) + \varphi_\infty(e_s(\gamma)))\, \tau(dx\, d\gamma). \end{aligned}$$

The 1-lipschitzianity of φ_∞, yields $L^{t,s}_{\varphi_\infty}(\gamma) \ge 0$, $\forall \gamma \in K$, hence

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \text{for } \tau\text{-a.e. } (x,\gamma) \in \overline{E} \times K.$$

In order to conclude, fix a countable dense subset $P \subset [0,1]$, and find a τ-negligible set $N \subset \overline{E} \times K$ such that the equation above is true outside N for all $t \le s$ in P. By density of P and the uniform continuity in the variables t and s, the equation above holds true also for t and s not in P, thus obtaining (7.12).

Now we prove (7.13). The idea is similar, but now we need to be more careful, for the function r_R fails to be continuous. We integrate equation (7.10), obtaining

$$0 = \int_{\overline{E}\times X} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma).$$

We are in position to apply Lusin's and Egorov's theorems. Fix $\varepsilon > 0$ and find a compact $M \subset E$, such that: (1) the restrictions $r_R|_M$ are continuous; (2) the restricted maps $r_R|_M$ converge uniformly to ρ; and (3) $\mathfrak{m}(E \setminus M) \le \varepsilon$. We now compute the limit

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{\overline{E}\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma) \\ &\ge \liminf_{R\to\infty} \int_{M\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma) \\ &\ge \int_{M\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - \rho|\, \tau(dx\, d\gamma) \ge 0. \end{aligned}$$

Therefore,

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \qquad \text{for } \tau\text{-a.e. } (x,\gamma) \in M \times K.$$

This means that the equation above holds true except for a set of measure at most ε, and by letting $\varepsilon \to 0$, we conclude.

Finally, we prove (7.14). In this case, consider the continuous, non-negative function $L(x,\gamma) := \inf_{t\in[0,1]} \mathsf{d}(x, e_t(\gamma))$. Equation (7.11) implies

$$0 = \int_{\overline{E}\times K} L(x,\gamma)\, \tau_R(dx\, d\gamma).$$

This equation passes to the limit as $R \to \infty$, so the conclusion immediately follows. ∎

7.3. Disintegration of the measure and the perimeter

Having in mind the disintegration formula (7.1), we define the map $\overline{E} \ni x \mapsto \mu_{x,R} \in \mathscr{P}(\overline{E})$ as follows:

$$\mu_{x,R} := \begin{cases} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\, (\hat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R}) \llcorner_E, & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ \delta_x, & \text{otherwise.} \end{cases}$$

A direct computation (recall (4.4)–(4.5)) gives

$$\begin{aligned} \mathfrak{m}(A \cap E) &= \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}(A\cap E)\, \hat{\mathfrak{q}}_R(d\alpha) \\ &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{Q_R} \hat{\mathfrak{m}}_{\alpha,R}(A\cap E)\, (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)(d\alpha) \\ &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_X \hat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R}(A\cap E)\, \mathfrak{m}\llcorner_E(dx) = \int_X \mu_{x,R}(A)\, \mathfrak{m}\llcorner_E(dx), \end{aligned}$$

thus the following disintegration formula holds:

$$\mathfrak{m}\llcorner_E = \int_{\overline{E}} \mu_{x,R}\, \mathfrak{m}\llcorner_E(dx). \tag{7.15}$$

Remark 7.5. We briefly discuss the measurablity of the integrand function in (7.15). It holds that the map $x \mapsto \mu_{x,R}(A)$ is measurable and the formula (7.15) holds. Indeed, the map $x \mapsto \mu_{x,R}(A)$ is (up to excluding the a negligible set) the composition of the maps $Q_R \ni \alpha \mapsto \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \hat{\mathfrak{m}}_{\alpha,R}(A \cap E)$ and the projection $\mathfrak{Q}_R$. The first map is $\hat{\mathfrak{q}}_R$-measurable, whereas the map $\mathfrak{Q}_R$ is $\mathfrak{m}$-measurable, with respect to the σ-algebra of Q_R, thus the composition is measurable.

Since $\hat{\mathfrak{m}}_{\alpha,R} = (g_R(\alpha,\cdot))_\#(h_{\alpha,R}\mathcal{L}^1\llcorner_{[0,|\hat{X}_{\alpha,R}|]})$, we can compute explicitly the measure $\mu_{x,R}$ (recall that by (7.3) $r_R(x) = \operatorname{ess\,sup} E_{x,R}$, for $\mathfrak{m}\llcorner_E$-a.e. x):

$$\begin{aligned}\mu_{x,R} &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(g_R(\mathfrak{Q}_R(x),\cdot))_\#\left((g_R(\mathfrak{Q}_R(x),\cdot))^{-1}(E)h_{\mathfrak{Q}_R(x),R}\mathcal{L}^1\llcorner_{[0,r_R(r)]}\right)\\ &= (g_R(\mathfrak{Q}_R(x),\cdot))_\#\Big(\mathbf{1}_{E_{x,R}}\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}h_{\mathfrak{Q}_R(x),R}\mathcal{L}^1\llcorner_{[0,r_R(x)]}\Big)\\ &= (\gamma^{x,R})_\#(\tilde{h}^{x,R}_E\mathcal{L}^1\llcorner_{[0,1]}), \quad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \overline{E},\end{aligned}$$

where

$$\tilde{h}^{x,R}_E(t) = \mathbf{1}_{E_{x,R}}(r_R(x)t)\, r_R(x)\,\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\, h_{\mathfrak{Q}_R(x),R}(r_R(x)t).$$

Having in mind (4.6), we can perform a similar operation for the perimeter. Indeed, in the natural parametrization of the rays, if we consider only the "right extremal" of $E_{x,R}$ and the fact that $F(\partial_t) = 1$, it holds that

$$h_{R,\mathfrak{Q}_R(x)}(r_R(x))\delta_{r_R(x)} \le \mathsf{P}_{F,h_{R,\mathfrak{Q}_R(x)}}(E_{x,R};\cdot).$$

This observation, naturally leads to the definition

$$p_{x,R} := \begin{cases}\min\left\{\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}h_{R,\mathfrak{Q}_R(x)}(r_R(x)),\,\frac{N}{\rho}\right\}\delta_{g_R(\mathfrak{Q}_R(x),r_R(x))}, & \text{if } x \in E \cap \hat{\mathcal{T}}_R,\\ \frac{N}{\rho}\,\delta_x, & \text{if } x \in \overline{E}\setminus(E\cap\mathcal{T}_R).\end{cases}$$

Using the maps $\gamma^{x,R}$ and $\tilde{h}_{x,R}$, we rewrite $p_{x,R}$ as

$$p_{x,R} = \begin{cases}\min\left\{\dfrac{\tilde{h}_{x,R}(1)}{\mathsf{d}(\gamma^{x,R}_0,\gamma^{x,R}_1)},\,\dfrac{N}{\rho}\right\}\delta_{\gamma^{x,R}_1}, & \text{if } x \in E\cap\hat{\mathcal{T}}_R,\\ \dfrac{N}{\rho}\delta_x, & \text{if } x \in \overline{E}\setminus(E\cap\mathcal{T}_R).\end{cases}$$

By definition of $p_{x,R}$, we have that

$$p_{x,R} \le \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;\cdot), \quad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \overline{E},$$

deducing the following "disintegration" formula (equations (4.6) and (4.5) are taken into account):

$$\begin{aligned}\mathsf{P}(E;A) &\ge \int_{Q_R}\mathsf{P}_{X_{\alpha,R}}(E;A)\,\hat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\int_{\overline{E}}\mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;A)\,\mathfrak{m}\llcorner_E(dx)\\ &\ge \int_{\overline{E}} p_{x,R}(A)\,\mathfrak{m}\llcorner_E(dx), \quad \forall A \subset \overline{E} \text{ Borel}.\end{aligned} \tag{7.16}$$

Define now the compact set $F := e_{(0,1)}(K) = \{\gamma_t : \gamma \in K, t \in [0,1]\}$ and let $S \subset \mathcal{M}^+(F)$ be the subset of the non-negative measures on F with mass at most N/ρ. The sets $\mathcal{P}(F)$ and S are naturally endowed with the weak topology of measures. Since K and F are compact Hausdorff spaces, the Riesz–Markov representation theorem implies that the weak topology on $\mathcal{P}(F)$ and S coincides with the weak* topology induced by the duality against continuous functions $C(K)$ and $C(F)$, respectively. The weak* convergence can be metrized on bounded sets, if the primal space is separable; here we chose as a metric

$$d(\mu,\nu) = \sum_{k=1}^{\infty} \frac{1}{2^k \, \|f_k\|_\infty} \Big| \int_X f_k \, d\mu - \int_X f_k \, d\nu \Big|, \tag{7.17}$$

where $\{f_k\}_k$ is a dense set in $C(X)$. We endow the spaces $\mathcal{P}(F)$ and S with the distance defined in (7.17), making them two compact metric spaces.

Define now the map $G_R\colon \overline{E} \times K \to \mathcal{P}(F) \times S$ as

$$G_R(x,\gamma) := \Big(\gamma_\#(\tilde{h}_E^{x,R} \mathscr{L}^1 \llcorner_{[0,1]}), \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho}\Big\} \delta_{e_1(\gamma)}\Big).$$

Clearly, the function G_R is measurable with respect to the variable x and continuous with respect to the variable γ. Define the measure (having mass $\mathfrak{m}(E)$)

$$\sigma_R := (\mathrm{Id} \times G_R)_\# \tau_R \in \mathcal{M}^+(\overline{E} \times K \times \mathcal{P}(F) \times S).$$

In order to ease the notation, we set $Z = \overline{E} \times K \times \mathcal{P}(F) \times S$.

Proposition 7.6. *The measure σ_R enjoys the following properties. For all $\psi \in C_b^0(\overline{E})$,*

$$\int_E \psi \, d\mathfrak{m} = \int_Z \int_E \psi(y) \, \mu(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp), \tag{7.18}$$

and for all $\psi \in C_b^0(\overline{E})$, $\psi \geq 0$,

$$\int_{\overline{E}} \psi(y) \, \mathsf{P}(E, dy) \geq \int_Z \int_{\overline{E}} \psi(y) \, p(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp). \tag{7.19}$$

Proof. Fix a test function $\psi \in C_b^0(\overline{E})$. Notice that for σ_R-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $\mu = \mu_{x,R}$, because

$$\mu = \gamma_\#(\tilde{h}_E^{x,R} \mathscr{L}^1 \llcorner_{[0,1]}) = (\gamma^{x,R})_\#(\tilde{h}_E^{x,R} \mathscr{L}^1 \llcorner_{[0,1]}) = \mu_{x,R}, \quad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p) \in Z,$$

and we used the fact that $\gamma = \gamma_{x,R}$ for τ_R-a.e. $(x,\gamma) \in \overline{E} \times K$. We conclude the proof of (7.18) by a direct computation:

$$\begin{aligned}\int_E \psi \, d\mathfrak{m} &= \int_E \int_E \psi(y) \, \mu_{x,R} \, \mathfrak{m}(dx) = \int_Z \int_E \psi(y) \, \mu_{x,R}(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp) \\ &= \int_Z \int_E \psi(y) \, \mu(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp).\end{aligned}$$

Now fix an open set $\Omega \subset X$ and compute, having in mind (7.16),

$$\begin{aligned}\mathsf{P}(E;\Omega) &\geq \int_E \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R},\gamma_1^{x,R})},\frac{N}{\rho}\Big\}\delta_{\gamma_1^{x,R}}(\Omega)\,d\mathfrak{m}(dx)\\ &= \int_Z \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma^{x,R}),e_1(\gamma^{x,R}))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma^{x,R})}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp)\\ &= \int_Z \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma)}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp).\end{aligned}$$

Taking into account that

$$p = \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N(\omega_N\,\mathsf{AVR}_X)^{1/N}}{\mathfrak{m}(E)^{1/N}}\Big\}\delta_{e_1(\gamma)}(\Omega),\quad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p)\in Z,$$

we continue this chain of inequalities:

$$\begin{aligned}\mathsf{P}(E;\Omega) &\geq \int_Z \min\Big\{\frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma),e_1(\gamma))},\frac{N}{\rho}\Big\}\delta_{e_1(\gamma)}(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp)\\ &= \int_Z p(\Omega)\,d\sigma_R(dx\,d\gamma\,d\mu\,dp).\end{aligned}$$

Since

$$\mathsf{P}(E;A) = \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A \text{ is open}\}$$

for any Borel set A, we can conclude. ∎

Before taking the limit as $R \to \infty$, we state a useful lemma.

Lemma 7.7. *Let X be a Polish space, let Y and Z be two compact metric spaces, and let $\mathfrak{m}$ be a finite Radon measure on X. Consider a sequence of functions $f_n\colon X\times Y\to Z$ and $f\colon X\times Y\to Z$ such that f and f_n are Borel-measurable in the first variable and continuous in the second. Suppose that for $\mathfrak{m}$-a.e. $x\in X$, the sequence $f_n(x,\cdot)$ converges uniformly to $f(x,\cdot)$. Consider a sequence of measures $\mu_n\in\mathcal{M}^+(X\times Y)$ such that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{M}^+(X\times Y)$ and $(\pi_X)_\#\mu_n=\mathfrak{m}$.*

Then we have that

$$(\mathrm{Id}\times f_n)_\#\mu_n \rightharpoonup (\mathrm{Id}\times f)_\#\mu, \quad \textit{weakly in } \mathcal{M}(X\times Y\times Z).$$

Proof. In order to ease the notation, set $\nu_n=(\mathrm{Id}\times f_n)_\#\mu_n$ and $\nu=(\mathrm{Id}\times f)_\#\mu$. Fix $\varepsilon>0$. We make use an extension of the Egorov's and Lusin's theorems for functions taking values in separable metric spaces (see Theorem 7.5.1 in [21] and Appendix D in [20]). In this setting, we deal with maps taking value in $C(Y,Z)$, the space of continuous functions between the compact spaces Y and Z, which is separable.

Therefore, there exists a compact $K\subset X$ such that: (1) the maps $x\in K\mapsto f_n(x,\cdot)\in C(Y,Z)$ are continuous; (2) the restrictions $x\in K\mapsto f_n(x,\cdot)$ converge to $x\in K\mapsto f(x,\cdot)$, uniformly in the space $C(K,C(Y,Z))$; and (3) $\mathfrak{m}(X\setminus K)\leq\varepsilon$. Regarding point (2), this implies that the restrictions $f_n|_{K\times Y}\to f|_{K\times Y}$ converge uniformly in $K\times Y$.

We test the convergence of ν_n against a function $\varphi \in C_b^0(X \times Y \times Z)$:

$$\begin{aligned}
\Big| \int_{X\times Y\times Z} \varphi \, d\nu_n - \int_{X\times Y\times Z} \varphi \, d\nu \Big| \\
&\le \|\varphi\|_{C^0} \left(\nu_n((X \setminus K) \times Y \times Z) + \nu((X \setminus K) \times Y \times Z) \right) \\
&\quad + \Big| \int_{K\times Y\times Z} \varphi \, d\nu_n - \int_{K\times Y\times Z} \varphi \, d\nu \Big| \\
&= \|\varphi\|_{C^0} \left(\mathfrak{m}(X \setminus K) + \mathfrak{m}(X \setminus K) \right) + \Big| \int_{K\times Y\times Z} \varphi \, d\nu_n - \int_{K\times Y\times Z} \varphi \, d\nu \Big| \\
&\le 2\varepsilon \|\varphi\|_{C^0} + \Big| \int_{K\times Y\times Z} \varphi \, d\nu_n - \int_{K\times Y\times Z} \varphi \, d\nu \Big|.
\end{aligned}$$

Regarding the second term, we compute the integral

$$\int_{K\times Y\times Z} \varphi \, d\nu_n = \int_{K\times Y} \varphi(x, y, f_n(x, y)) \, \mu_n(dx\, dy).$$

Using compactness, one easily checks that $\varphi(x, y, f_n(x, y))$ converges to $\varphi(x, y, f(x, y))$ uniformly in $K \times Y$. For this reason, together with the fact that $\mu_n \rightharpoonup \mu$ weakly, we take the limit in the equation above, concluding the proof:

$$\lim_{n\to\infty} \int_{K\times Y} \varphi(x,y,f_n(x,y)) \mu_n(dx\,dy) = \int_{K\times Y} \varphi(x,y,f(x,y)) \mu(dx\,dy) = \int_{K\times Y\times Z} \varphi \, d\nu.$$

■

Corollary 7.8. *Consider the function $G \colon \overline{E} \times K \to \mathcal{P}(F) \times S$ defined as*

$$G(x, \gamma) = \Big(\gamma_\#(N t^{N-1} \mathcal{L}^1 \llcorner_{[0,1]}), \max \Big\{ \frac{N}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \Big\} \delta_{e_1(\gamma)} \Big),$$

and let $\sigma := (\mathrm{Id} \times G)_\# \tau$. Then it holds that $\sigma_R \rightharpoonup \sigma$ in the weak topology of measures.

Proof. We need only to check the hypotheses of the previous lemma. Due to Remark 2.2, the irreversiblity of the distance is not harmful. The set $\overline{E}$ is compact, hence Polish. The set K is compact, and so is $\mathcal{P}(F) \times S$ (with respect to the distance given by (7.17)). The maps G_R are measurable and continuous in the first and second variables, respectively. Finally, we need to see that for $\mathfrak{m}\llcorner_E$-a.e. x, the limit $G_R(x, \gamma) \to G(x, \gamma)$ holds uniformly in γ. Fix x and γ and pick $\psi \in C_b(F)$ a test function, and compute

$$\begin{aligned}
\Big| \int_F \psi(y) \, \gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]})(dy) - \int_F \psi(y) \, \gamma_\#(N t^{N-1} \mathcal{L}^1 \llcorner_{[0,1]})(dy) \Big| \\
&= \Big| \int_0^1 \psi(\gamma_t) (\tilde{h}_E^{x,R} - N t^{N-1}) \, dt \Big| \le \|\psi\|_{C(F)} \, \|\tilde{h}_E^{x,R} - N t^{N-1}\|_{L^\infty}.
\end{aligned}$$

The right-hand side of the inequality is independent of γ (but depends only on x and ψ), and converges to 0 by Theorem 6.11 (see in particular (6.16)). Therefore, the first component of $G_R(x, \gamma)$ converges (in the weak topology of $\mathcal{P}(F)$), uniformly with respect to γ (compare with (7.17)). For the other component, the proof is analogous, so we omit it. ■

We conclude this section with a proposition reporting all the relevant properties of the limit measure σ.

Proposition 7.9. *The measure σ satisfies the following disintegration formulae. For all $\psi \in L^1(E; \mathfrak{m}\llcorner_E)$,*

$$\int_E \psi(y)\, \mathfrak{m}(dy) = \int_Z \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp), \tag{7.20}$$

and for all $\psi \in L^1(\overline{E}; \mathsf{P}(E; \cdot))$,

$$\int_{\overline{E}} \psi(y)\, \mathsf{P}(E; dy) = \frac{N}{\rho} \int_Z \psi(e_1(\gamma))\psi\, \sigma(dx\, d\gamma\, d\mu\, dp). \tag{7.21}$$

Furthermore, for σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)), \quad \forall\, 0 \le t \le s \le 1, \tag{7.22}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{7.23}$$

$$x \in \gamma, \tag{7.24}$$

$$\mu = \gamma_\#(N t^{N-1} \mathcal{L}^1 \llcorner_{[0,1]}), \tag{7.25}$$

$$p = \tfrac{N}{\rho}\, \delta_{e_1(\gamma)}. \tag{7.26}$$

Proof. Equations (7.22)–(7.24) have been already proven (see equations (7.12)–(7.14)). Equation (7.25) follows from the definition of G. Similarly, the defintion of G and equation (7.23) imply (7.26):

$$p = \min\left\{\frac{N}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho}\right\} \delta_{e_1(\gamma)} = \frac{N}{\rho}\, \delta_{e_1(\gamma)}.$$

We prove now equation (7.20). Given a function $\psi \in C_b^0(F) = C_b^0(e_{(0,1)}(K))$, we define $L_\psi \colon \mathcal{P}(F) \to \mathbb{R}$ as $L_\psi(\mu) = \int_F \psi\, d\mu$. This last function is bounded and continuous with respect to the weak topology of $\mathcal{P}(F)$, thus we can compute the limit using (7.18) and (7.25):

$$\begin{aligned}
\int_E \psi\, d\mathfrak{m} &= \lim_{R\to\infty} \int_Z \int_F \psi(y)\, \mu(dy)\, \sigma_R(dx\, d\gamma\, d\mu\, dp) \\
&= \lim_{R\to\infty} \int_Z L_\psi(\mu)\, \sigma_R(dx\, d\gamma\, d\mu\, dp) = \int_Z \int_F \psi(y)\, \mu(dy)\, \sigma(dx\, d\gamma\, d\mu\, dp) \\
&= \int_Z \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp).
\end{aligned}$$

Using standard approximation arguments, we see that the equation above holds true also for any $\psi \in L^1(E; \mathfrak{m}\llcorner_E)$.

Regarding (7.21), one can analogously deduce that

$$\int_{\overline{E}} \psi(y)\, \mathsf{P}(E; dy) \ge \frac{N}{\rho} \int_Z \psi(e_1(\gamma))\, \sigma(dx\, d\gamma\, d\mu\, dp), \quad \forall \psi \in L^1(\overline{E}; \mathsf{P}(E; \cdot)),\ \psi \ge 0.$$

If we test the inequality above with $\psi = 1$, the inequality is saturated, thus the two measures have the same mass, so the inequality improves to an equality. ∎

7.4. Back to the classical localization notation

We are now in position to re-obtain a "classical" disintegration formula for the measure $\mathfrak{m}$, as well as for the relative perimeter of E.

We recall the definition of some of the objects that were introduced in Section 2.4. For instance, let $\Gamma_\infty := \{(x,y) : \varphi_\infty(x) - \varphi_\infty(y) = \mathsf{d}(x,y)\}$ and let $\mathcal{T}_\infty$ be the transport set, i.e., the family of points passing through only one non-degenerate transport curve. Let $\mathcal{A}_\infty$ the set of branching points (i.e., points where two of more non-degenerate transport curves pass). The sets of forward and backward branching points are defined as

$$\mathcal{A}_\infty^+ := \{x \in \mathcal{A}_\infty : \exists y \neq x \text{ such that } (x,y) \in \Gamma_\infty\}, \tag{7.27}$$

$$\mathcal{A}_\infty^- := \{x \in \mathcal{A}_\infty : \exists y \neq x \text{ such that } (y,x) \in \Gamma_\infty\}. \tag{7.28}$$

We recall that $\mathcal{A}_\infty = \mathcal{A}_\infty^+ \cup \mathcal{A}_\infty^-$ and that $\mathcal{A}_\infty$ is negligible. Let Q_∞ be the quotient set, and let $\mathfrak{Q}_\infty : \mathcal{T}_\infty \to Q_\infty$ be the quotient map; denote by $X_{\alpha,\infty} := \mathfrak{Q}^{-1}(\alpha)$ the disintegration rays, and let $g_\infty : \mathrm{Dom}(g_\infty) \subset Q_\infty \times [0,\infty) \to X$ be the standard parametrization of the rays.

We introduce the function $t_\alpha : \overline{X_{\alpha,\infty}} \to [0,\infty)$ defined as

$$t_\alpha(x) := (g_\infty(\alpha,\cdot))^{-1} = \mathsf{d}(g_\infty(\mathfrak{Q}_\infty(x),0),x);$$

the function t_α measures how much a point is translates from the starting point of the ray $X_{\alpha,\infty}$.

The following proposition guarantees that the geodesic on which the measure σ is supported lays on the transport set $\mathcal{T}_\infty$.

Proposition 7.10. *For σ-a.e. $(x,\gamma,\mu,p) \in Z$, it holds that $e_t(\gamma) \in \mathcal{T}_\infty$, for all $t \in (0,1)$.*

Proof. Clearly, for σ-a.e. $(x,\gamma,\mu,p) \in Z$, γ is non-degenerate, hence $e_t(\gamma) \notin \mathcal{D}$, where $\mathcal{D}$ is the set where no non-degenerate transport curve pass. Therefore we need only to check that $e_t(\gamma) \notin \mathcal{A}^\infty$. We will prove only that $e_t(\gamma) \neq \mathcal{A}_\infty^+$, for the case $e_t(\gamma) \neq \mathcal{A}_\infty^-$ is analogous. Fix $\varepsilon > 0$ and let

$$P := \{(x,\gamma,\mu,p) \in Z : e_\varepsilon(\gamma) \in \mathcal{A}_\infty^+ \text{ and conditions (7.20)–(7.26) hold}\}.$$

Notice that by definition of $\mathcal{A}_\infty^+$, if $(x,\gamma,\mu,p) \in P$, then $\gamma_t \in \mathcal{A}_\infty^+$, for all $t \in [0,\varepsilon]$, thus we can compute

$$\begin{aligned} 0 = \mathfrak{m}(\mathcal{A}_\infty^+) &= \int_Z \int_0^1 \mathbf{1}_{\mathcal{A}_\infty^+}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \\ &\geq \int_P \int_0^\varepsilon \mathbf{1}_{\mathcal{A}_\infty^+}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \geq \varepsilon^N \sigma(P), \end{aligned}$$

thus P is negligible. Fix now $(x,\gamma,\mu,p) \notin P$. By definition of $\mathcal{A}_\infty^+$ and P, we have that $\gamma_t \notin \mathcal{A}_\infty^+$, for all $t \in [\varepsilon, 1]$. By arbitrariness of ε, we deduce that for σ-a.e $(x,\gamma,\mu,p) \in Z$, it holds that $e_t(\gamma) \notin \mathcal{A}_\infty^+$, for all $t \in (0,1]$. ■

Corollary 7.11. *It holds that $E \subset \mathcal{T}_\infty$, and for σ-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $e_t(\gamma) \in \overline{X_{\mathfrak{Q}(x),\infty}}$ and*

$$e_t(\gamma) = g_\infty(\mathfrak{Q}(x), t_{\mathfrak{Q}(x)}(e_0(\gamma)) + \rho t). \tag{7.29}$$

Define

$$\hat{\mathfrak{q}} := \frac{1}{\mathfrak{m}(E)}\,(\mathfrak{Q}_\infty)_\#(\mathfrak{m}\llcorner_E) \ll (\mathfrak{Q}_\infty)_\#\mathfrak{m}\llcorner_{\mathcal{T}_\infty},$$

and let $\tilde{\mathfrak{q}}$ be a probability measure such that $(\mathfrak{Q}_\infty)_\#\mathfrak{m}\llcorner_{\mathcal{T}_\infty} \ll \tilde{\mathfrak{q}}$. The disintegration theorem gives the following two formulae:

$$\mathfrak{m}\llcorner_E = \int_{Q_\infty} \widehat{\mathfrak{m}}_{\alpha,\infty}\,\hat{\mathfrak{q}}(d\alpha), \quad \text{and} \quad \mathfrak{m}\llcorner_{\mathcal{T}_\infty} = \int_{Q_\infty} \widetilde{\mathfrak{m}}_{\alpha,\infty}\,\tilde{\mathfrak{q}}(d\alpha), \tag{7.30}$$

where the measures $\widehat{\mathfrak{m}}_{\alpha,\infty}$ and $\widetilde{\mathfrak{m}}_{\alpha,\infty}$ are supported on $X_{\alpha,\infty}$. By comparing the two expressions above, it turns out that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\,\widehat{\mathfrak{m}}_{\alpha,\infty} = \mathbf{1}_E\,\widetilde{\mathfrak{m}}_{\alpha,\infty}$. The localization Theorem 2.9 (see also Remark 2.10) ensures that the transport rays $(X_{\alpha,\infty}, F, \widetilde{\mathfrak{m}}_{\alpha,\infty})$ satisfy the oriented $\mathsf{CD}(0, N)$ condition. On the contrary, we cannot deduce the same condition for the other disintegration, because the reference measure is restricted to the set E, and not the transport set. Consider the densities $\hat{h}_\alpha$ and $\tilde{h}_\alpha$ given by

$$\widehat{\mathfrak{m}}_{\alpha,\infty} = (g_\infty(\alpha,\cdot))_\#(\hat{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}) \quad \text{and} \quad \widetilde{\mathfrak{m}}_{\alpha,\infty} = (g_\infty(\alpha,\cdot))_\#(\tilde{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}).$$

Clearly, it holds that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\,\hat{h}_\alpha(t) = \mathbf{1}_E(g(\alpha,t))\tilde{h}_\alpha(t)$, thus we can derive a somehow weaker concavity condition for the function $\hat{h}_\alpha^{1/(N-1)}$: for all $x_0, x_1 \in (0, |X_{\alpha,\infty}|)$ and for all $t \in [0,1]$, it holds that

$$\hat{h}_\alpha((1-t)x_0 + tx_1)^{1/(N-1)} \geq (1-t)\,\hat{h}_\alpha(x_0)^{1/(N-1)} + t\,\hat{h}_\alpha(x_1)^{1/(N-1)},$$

if $\hat{h}_\alpha((1-t)x_0 + tx_1) > 0$.

A natural consequence is the following "Bishop–Gromov inequality":

$$\text{the map } r \mapsto \frac{\hat{h}_\alpha(r)}{r^{N-1}} \text{ is decreasing on the set } \{r \in (0, |X_{\alpha,\infty}|) : \hat{h}_\alpha(r) > 0\}. \tag{7.31}$$

Define the full-measure set $\hat{Z} \subset Z$ as

$$\hat{Z} := \{(x,\gamma,\mu,p) \in Z : x \in E \cap \mathcal{T}_\infty, \text{ and the properties given by equations (7.20)–(7.21) and (7.29) hold}\}.$$

We partition $\hat{Z}$ as follows:

$$\hat{Z}_\alpha := \{(x,\gamma,\mu,p) \in \hat{Z} : \mathfrak{Q}_\infty(x) = \alpha\},$$

and we disintegrate the measure σ according to the partition $(\hat{Z}_\alpha)_{\alpha\in Q_\infty}$:

$$\sigma = \int_{Q_\infty} \sigma_\alpha\,\mathfrak{q}(d\alpha), \tag{7.32}$$

where the probability measures σ_α are supported on $\hat{Z}_\alpha$. Moreover, let $\nu_\alpha \in \mathcal{P}([0,\infty))$ be the measure given by

$$\nu_\alpha := \frac{1}{\mathfrak{m}(E)}\,(t_\alpha \circ e_0 \circ \pi_K)_\#(\sigma_\alpha)$$

(we recall that $t_\alpha = (g_\infty(\alpha,\cdot))^{-1}$ and $\pi_K(x,\gamma,\mu,p) = \gamma$).

The following proposition shows that the density $\hat{h}_\alpha$ can be seen as a convolution of the model density and the measure ν_α.

Proposition 7.12. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\widehat{h}_\alpha(r) = N\omega_N \,\mathsf{AVR}_X \int_{[0,\infty)} (r-t)^{N-1}\,\mathbf{1}_{(t,t+\rho)}(r)\,\nu_\alpha(dt), \qquad \forall r \in (0, |X_{\alpha,\infty}|).$$

Proof. Fix $\psi \in L^1(\mathfrak{m}\llcorner_E)$ and compute its integral using equations (7.20) and (7.32):

$$\begin{aligned}\int_E \psi(x)\,\mathfrak{m}(dx) &= \int_{\hat{Z}}\int_0^1 \psi(e_t(\gamma))Nt^{N-1}\,dt\,\sigma(dx\,d\gamma\,d\mu\,dp)\\ &= \int_{Q_\infty}\int_{\hat{Z}_\alpha}\int_0^1 \psi(e_t(\gamma))Nt^{N-1}\,dt\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,\mathfrak{q}(d\alpha).\end{aligned}$$

Fix now $\alpha \in Q_\infty$ and compute (recall (7.29) and the definition of $\widehat{Z}$)

$$\begin{aligned}&\int_{\hat{Z}_\alpha}\int_0^1 \psi(e_t(\gamma))Nt^{N-1}\,dt\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\\ &\quad= \int_{\hat{Z}_\alpha}\int_0^\rho \psi(e_{s/\rho}(\gamma))N\frac{s^{N-1}}{\rho^N}\,ds\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\\ &\quad= \int_{\hat{Z}_\alpha}\int_0^\rho \psi(g_\infty(\mathfrak{Q}(x), t(\alpha,\gamma_0)+s))N\frac{s^{N-1}}{\rho^N}\,ds\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\\ &\quad= \int_{\hat{Z}_\alpha}\int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha,r))N\frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N}\times\\ &\qquad\times \mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r)\,dr\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\\ &\quad= \int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha,r))\int_{\hat{Z}_\alpha} N\,\frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N}\\ &\qquad\times \mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r)\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,dr.\end{aligned}$$

Therefore, by the uniqueness of the disintegration, we can conclude

$$\begin{aligned}\hat{h}_\alpha(r) &= \int_{\hat{Z}_\alpha} N\frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N}\,\mathbf{1}_{(t(\alpha,\gamma_0),t(\alpha,\gamma_0)+\rho)}(r)\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\\ &= N\omega_N\,\mathsf{AVR}_X\int_{[0,\infty)}(r-t)^{N-1}\,\mathbf{1}_{(t,t+\rho)}(r)\,\nu_\alpha(dt). \qquad\blacksquare\end{aligned}$$

Using the fact that $\widehat{h}_\alpha$ is a convolution, we deduce that ν_α is indeed the Dirac delta.

Proposition 7.13. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that $\nu_\alpha = \delta_0$.*

Proof. Let $T := \inf\operatorname{supp}\nu_\alpha$. If we set $r \in (T, T+\rho)$, we can compute

$$\begin{aligned}\frac{\widehat{h}_{\alpha,\infty}(r)}{N\omega_N\,\mathsf{AVR}_X} &= \int_{[0,\infty)}(r-t)^{N-1}\,\mathbf{1}_{(t,t+\rho)}(r)\,\nu_\alpha(dt) = \int_{[T,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\ &\geq \int_{[T,r)}\Big(\frac{r-T}{2}\,\mathbf{1}_{[T,(r+T)/2]}(t)\Big)^{N-1}\,\nu_\alpha(dt) = \frac{(r-T)^{N-1}}{2^{N-1}}\,\nu_\alpha([T,\tfrac{r+T}{2}]).\end{aligned} \tag{7.33}$$

By definition of T, we have that $\nu_\alpha([T,(r+T)/2])>0$, hence $\widehat{h}_\alpha(r)>0$, for all $r\in(T,T+\rho)$. On the other hand,

$$\begin{aligned}\widehat{h}_{\alpha,\infty}(r) &= N\omega_N\,\mathsf{AVR}_X\int_{[T,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\ &\le N\omega_N\,\mathsf{AVR}_X(r-T)^{N-1}\,\nu_\alpha([T,r))\to 0.\quad \text{as } r\to T^+.\end{aligned}\tag{7.34}$$

We claim that $T=0$. Indeed, if $T>0$, then

$$\lim_{r\to T^+}\frac{\widehat{h}_\alpha(r)}{r^{N-1}}=0,$$

contradicting (7.31). We derive now the non-increasing function

$$(0,\rho)\ni r\mapsto \frac{\widehat{h}_\alpha(r)}{r^{N-1}}=\frac{N\omega_N\,\mathsf{AVR}_X}{r^{N-1}}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt),$$

obtaining

$$0\ge N\omega_N\,\mathsf{AVR}_X\Big(\frac{1-N}{r^N}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)+\frac{1}{r^{N-1}}\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\Big).$$

We compute the second term:

$$\begin{aligned}&\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\ &=\lim_{h\to 0}\int_{[r,r+h)}\frac{(r+h-t)^{N-1}}{h}\,\nu_\alpha(dt)+\lim_{h\to 0}\int_{[0,r)}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\,\nu_\alpha(dt)\\ &\ge 0+\int_{[0,r)}\lim_{h\to 0}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\,\nu_\alpha(dt)=(N-1)\int_{[0,r)}(r-t)^{N-2}\,\nu_\alpha(dt),\end{aligned}$$

yielding

$$\begin{aligned}0&\ge(1-N)\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)+r\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\ &\ge(N-1)\int_{[0,r)}(r(r-t)^{N-2}-(r-t)^{N-1})\,\nu_\alpha(dt)\\ &=(N-1)\int_{[0,r)}t(r-t)^{N-2}\,\nu_\alpha(dt).\end{aligned}$$

The inequality above gives $\nu_\alpha((0,r))=0$, for all $r\in(0,\rho)$, hence $\nu_\alpha(0,\rho)=0$. We deduce that

$$\widehat{h}_\alpha(r)=N\omega_N\,\mathsf{AVR}_X\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)=N\omega_N\,\mathsf{AVR}_X\,r^{N-1}\nu_\alpha(\{0\}),\qquad\forall r\in(0,\rho).$$

If $\nu_\alpha([\rho,\infty))=0$, then $\nu_\alpha=\delta_0$ (because ν_α has mass 1), completing the proof. Assume on the contrary that $\nu_\alpha([\rho,\infty))>0$, and let $S:=\inf\operatorname{supp}(\nu_\alpha\llcorner_{[\rho,\infty)})\ge\rho$. In this case, following the computations (7.33) and (7.34), with S in place of T, we deduce $\lim_{r\to S^+}\widehat{h}_\alpha(r)=0$, contradicting (7.31). ∎

Corollary 7.14. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, for σ_α-a.e. $(x, \gamma, \mu, p) \in Z_\alpha$, it holds that $e_t(\gamma) = g(\alpha, \rho t)$, $\forall t \in [0, 1]$.*

Proof. The fact that $\nu_\alpha = \delta_0$ implies $t_\alpha(\gamma_0) = 0$ for σ_α-a.e. $(x, \gamma\ \mu, p) \in \widehat{Z}_\alpha$, hence, recalling the disintegration formula (7.29) and the definition of $\widetilde{Z}$, we deduce that $e_t(\gamma) = g(\alpha, t_\alpha(e_0) + \rho t) = g(\alpha, \rho t)$. ■

The next corollary concludes the discussion of the limiting procedures of the disintegration.

Corollary 7.15. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r) = N\omega_N \,\mathsf{AVR}_X \mathbf{1}_{(0,\rho)}(r)\, r^{N-1}.$$

Moreover, the following disintegration formulae hold true:

$$\mathfrak{m}\llcorner_E = N\omega_N \,\mathsf{AVR}_X \int_{Q_\infty} (g_\infty(\alpha, \cdot))_\#(r^{N-1}\, \mathcal{L}^1\llcorner_{(0,\rho)})\, \hat{\mathfrak{q}}(d\alpha), \tag{7.35}$$

$$\mathsf{P}(E; \cdot) = \mathsf{P}(E) \int_{Q_\infty} \delta_{g_\infty(\alpha,\rho)}\, \hat{\mathfrak{q}}(d\alpha). \tag{7.36}$$

Proof. We need only to prove equation (7.36). Equation (7.21) and Corollary 7.14 yield

$$\begin{aligned}
\int_{\overline{E}} \psi(x)\, \mathsf{P}(E; dx) &= \frac{N}{\rho} \int_{\widehat{Z}} \psi(e_1(\gamma))\psi\, \sigma(dx\, d\gamma\, d\mu\, dp) \\
&= \frac{N}{\rho} \int_{Q_\infty} \int_{\widehat{Z}_\alpha} \psi(e_1(\gamma))\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp)\, \hat{\mathfrak{q}}(d\alpha) \\
&= \frac{N}{\rho} \int_{Q_\infty} \psi(g_\infty(\alpha, \rho)) \int_{\widehat{Z}_\alpha} \sigma_\alpha(dx\, d\gamma\, d\mu\, dp)\, \hat{\mathfrak{q}}(d\alpha),
\end{aligned}$$

for all $\psi \in L^1(\overline{E}; \mathsf{P}(E; \cdot))$. ■

8. E is a ball

The aim of this section is to prove that E coincides with a ball of radius ρ, and to extend the disintegration formula to the whole manifold. Before starting the proof, we give a topological technical lemma. This lemma is, in some sense, a weak formulation of the statement: let Ω be an open connected subset of a topological space X and let $E \subset X$ be any set; if $\Omega \cap E \neq \emptyset$ and $\Omega \setminus E \neq \emptyset$, then we have that $\partial E \cap \Omega \neq \emptyset$.

Lemma 8.1. *Let $(X, F, \mathfrak{m})$ be measured Finsler manifold (with possible infinite reversibility). Let $E \subset X$ be a Borel set and let $\Omega \subset X$ be an open connected set with finite measure. If $\mathfrak{m}(E \cap \Omega) > 0$ and $\mathfrak{m}(\Omega \setminus E) > 0$, then $\mathsf{P}(E; \Omega) > 0$.*

Proof. Assume first that the manifold is Riemannian. In this case, we can assume by contradiction that $\mathsf{P}(E; \Omega) = 0$, yielding that the BV function $\mathbf{1}_E$ is constant in Ω. But this contradicts the hypotheses $\mathfrak{m}(E \cap \Omega) > 0$ and $\mathfrak{m}(\Omega \setminus E) > 0$.

We now drop the Riemannianity hypothesis. As we stressed out in Remark 2.2, there exists a Riemannian metric g such that its dual metric g^{-1} in T^*X satisfies $\sqrt{g^{-1}(\omega,\omega)} \leq F^*(\omega)$, for all $\omega \in T^*X$. By definition of perimeter, there exists a sequence $u_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_n \to \mathbf{1}_E$ in L^1_{loc} and $\int_\Omega |\partial u_n|\, d\mathfrak{m} \to \mathsf{P}_{(X,F,\mathfrak{m})}(E;\Omega)$. Since $g^{-1}(du_n, du_n) \leq F^*(-du_n) = |\partial u_n|$ a.e. in Ω, we conclude that $\mathsf{P}_{(X,g,\mathfrak{m})}(E;\Omega) \leq \mathsf{P}_{(X,F,\mathfrak{m})}(E;\Omega)$. ∎

Proposition 8.2. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\varphi_\infty(g_\infty(\alpha,0)) \leq \operatorname*{ess\,sup}_E \varphi_\infty, \quad \textit{and} \quad \varphi_\infty(g_\infty(\alpha,\rho)) \geq \operatorname*{ess\,inf}_E \varphi_\infty.$$

Proof. We prove only the first inequality; the second has the same proof. In order to ease the notation, define $M := \operatorname{ess\,sup}_E \varphi_\infty$. Let $H := \{\alpha \in Q_\infty : \varphi_\infty(g_\infty(\alpha,0)) \geq M + 2\varepsilon\}$. Consider the following measure on E:

$$\mathfrak{n}(T) := N\omega_N \,\mathsf{AVR}_X \int_H \int_0^\varepsilon \mathbf{1}_T(g_\infty(\alpha,r))\, r^{N-1}\, dr\, \hat{\mathfrak{q}}(d\alpha), \quad \forall T \subset E \text{ Borel}.$$

Clearly, $\mathfrak{n} \ll \mathfrak{m}$ (compare with (7.35)), thus $\varphi_\infty(x) \leq M$, for $\mathfrak{n}$-a.e. $x \in E$. If we compute the integral

$$\begin{aligned}
0 &\geq \int_E (\varphi_\infty(x) - M)\, \mathfrak{n}(dx) = N\omega_N \,\mathsf{AVR}_X \int_H \int_0^\varepsilon (\varphi_\infty(g_\infty(\alpha,t)) - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\
&= N\omega_N \,\mathsf{AVR}_X \int_H \int_0^\varepsilon (\varphi_\infty(g_\infty(\alpha,0)) - t - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\
&\geq N\omega_N \,\mathsf{AVR}_X \int_H \int_0^\varepsilon \varepsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) = \varepsilon^N \hat{\mathfrak{q}}(H),
\end{aligned}$$

we can deduce that $\hat{\mathfrak{q}}(H) = 0$ and, by arbitrariness of ε, we conclude. ∎

Theorem 8.3. *There exists a (unique) point $o \in X$ such that, up to a negligible set, we have $E = B^+(o,\rho)$, where $\rho = (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{1/N}$. Moreover, it holds that*

$$\varphi_\infty(o) = \operatorname*{ess\,sup}_E \varphi_\infty = \max_{B^+(o,\rho)} \varphi_\infty. \tag{8.1}$$

Proof. Define $\tilde{E} := \operatorname{supp} \mathbf{1}_E$. Recall that by definition of support, $\tilde{E} = \bigcap_C C$, where the intersection is taken among all closed sets C such that $\mathfrak{m}(E \setminus C) = 0$; and in particular $\mathfrak{m}(E \setminus \tilde{E}) = 0$. Let $o \in \arg\max_{\tilde{E}} \varphi_\infty$. By definition of $\tilde{E}$, we have that $\max_{\tilde{E}} \varphi_\infty = \operatorname{ess\,sup}_E \varphi_\infty$, deducing the first equality of (8.1). The other equality in (8.1) will follow from the fact $E = B^+(o,\rho)$ (up to a negligible set).

It is sufficient to prove only that $B^+(o,\rho) \subset E$, for the other inclusion is automatic Indeed, the Bishop–Gromov inequality, together with the definition of asymptotic volume ratio, yields

$$\mathfrak{m}(E) \geq \mathfrak{m}(B^+(o,\rho)) \geq \omega_N \mathsf{AVR}_X \rho^N = \mathfrak{m}(E),$$

and the equality of measures improves to an equality of sets.

Fix now $\varepsilon > 0$ and define $A = B^+(o, \rho - \varepsilon)$. If $\mathfrak{m}(A \setminus E) = 0$, then we deduce that $B^+(o, \rho - \varepsilon) \subset E$ and, by arbitrariness of ε, we can conclude.

Suppose on the contrary that $\mathfrak{m}(A \setminus E) > 0$. Clearly, A is connected and $\mathfrak{m}(A \cap E) > 0$ (otherwise $o \notin \widetilde{E}$), so we can apply Lemma 8.1 obtaining $\mathsf{P}(E; A) > 0$. Define $H = \{\alpha \in Q_\infty : g_\infty(\alpha, \rho) \in A\}$. The set H is non-negligible because (recall (7.36))

$$0 < \frac{\mathsf{P}(E; A)}{\mathsf{P}(E)} = \int_{Q_\infty} \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \int_H \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \hat{\mathfrak{q}}(H).$$

By Lipschitz-continuity of φ_∞, we deduce

$$\varphi_\infty(x) \geq \varphi_\infty(o) - \rho + \varepsilon \geq M - \rho + \varepsilon, \quad \forall x \in A = B^+(o, \rho - \varepsilon),$$

hence

$$\varphi_\infty(g_\infty(\alpha, \rho)) \geq M - \rho + \varepsilon, \quad \forall \alpha \in H.$$

Continuing the chain of inequalities, we arrive at

$$\varphi_\infty(g_\infty(\alpha, 0)) = \varphi_\infty(g_\infty(\alpha, \rho)) + \rho \geq M + \varepsilon, \quad \forall \alpha \in H.$$

The line above, together with the fact that $\hat{\mathfrak{q}}(H) > 0$, contradicts Proposition 8.2. ■

8.1. $\varphi_\infty(x)$ coincides with $-\mathsf{d}(o, x)$

The present section is devoted to proving that $\varphi_\infty(x) = -\mathsf{d}(o, x) + \varphi_\infty(o)$.

Proposition 8.4. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\mathsf{d}(o, g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho]. \tag{8.2}$$

Proof. By the 1-lipschitzianity of φ_∞ and the fact that $E = B^+(o, \rho)$ (up to a negligible set), we deduce that $\varphi_\infty(x) \geq \varphi_\infty(o) - \rho$, for $\mathfrak{m}$-a.e. $x \in E$. Henceforth, Proposition 8.2 and equation (8.1) yield

$$\varphi_\infty(g_\infty(\alpha, 0)) \leq \varphi_\infty(o) \quad \text{and} \quad \varphi_\infty(g_\infty(\alpha, \rho)) \geq \varphi_\infty(o) - \rho.$$

Since $\frac{d}{dt}\varphi_\infty(g_\infty(\alpha, t)) = -1$, $t \in (o, \rho)$, the inequalities above are saturated, i.e., it holds that

$$\varphi_\infty(g_\infty(\alpha, t)) = \varphi_\infty(o) - t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty.$$

Using again the 1-lipschitzianity of φ_∞, we arrive at

$$\mathsf{d}(o, g_\infty(\alpha, t)) \geq \varphi_\infty(o) - \varphi_\infty(g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty. \tag{8.3}$$

Now fix $\varepsilon > 0$ and let $C = \{\alpha \in Q_\infty : \mathsf{d}(o, g_\infty(\alpha, 0)) > (1 + \Lambda_F)\varepsilon\}$, where Λ_F is the reversibility constant. Define the function $f(t) := \inf\{\mathsf{d}(o, g_\infty(\alpha, t)) : \alpha \in C\}$. Clearly, f is Λ_F-Lipschitz and satisfies $f(0) \geq (1 + \Lambda_F)\varepsilon$, hence $f(t) \geq (1 + \Lambda_F)\varepsilon - \Lambda_F t$, yielding (cf. (8.3))

$$f(t) \geq \max\{((1 + \Lambda_F)\varepsilon - \Lambda_F t), t\} \geq \varepsilon.$$

The inequality above implies that $g_\infty(\alpha, t) \notin B^+(o, \varepsilon)$ for all $t \in [0, 1]$, for all $\alpha \in C$. We compute $\mathfrak{m}(B^+(0, \varepsilon))$ using the disintegration formula (7.35):

$$\begin{aligned}\frac{\mathfrak{m}(B^+(o,\varepsilon))}{N\omega_N\,\mathsf{AVR}_X} &= \int_{Q_\infty}\int_0^\rho \mathbf{1}_{B^+(o,\varepsilon)}(g_\infty(\alpha,t))\,t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B^+(o,\varepsilon)}(g_\infty(\alpha,t))\,t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha).\end{aligned}$$

If $\mathbf{1}_{B^+(o,\varepsilon)}(g_\infty(\alpha, t)) = 1$, then inequality (8.3) yields $t \le \varepsilon$, so we continue the computation:

$$\begin{aligned}\frac{\mathfrak{m}(B^+(o,\varepsilon))}{N\omega_N\,\mathsf{AVR}_X} &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B^+(o,\varepsilon)}(g_\infty(\alpha,t))\,t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &= \int_{Q_\infty\setminus C}\int_0^\varepsilon \mathbf{1}_{B^+(o,\varepsilon)}(g_\infty(\alpha,t))\,t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha)\\ &\le \int_{Q_\infty\setminus C}\int_0^\varepsilon t^{N-1}\,dt\,\hat{\mathfrak{q}}(d\alpha) = (\hat{\mathfrak{q}}(Q_\infty) - \hat{\mathfrak{q}}(C))\,\frac{\varepsilon^N}{N}.\end{aligned}$$

On the other hand, the Bishop–Gromov inequality yields

$$\mathfrak{m}(B^+(o,\varepsilon)) \ge \frac{\varepsilon^N}{\rho^N}\,\mathfrak{m}(B^+(o,\rho)) = \frac{\varepsilon^N}{\rho^N}\,\mathfrak{m}(E) = \varepsilon^N\omega_N\,\mathsf{AVR}_X.$$

The comparison of the two previous inequality gives $\hat{\mathfrak{q}}(C) = 0$. By arbitrariness of ε, we deduce that $g_\infty(\alpha, 0) = o$ for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$.

Finally, using again (8.3), we conclude

$$t \le \mathsf{d}(o, g_\infty(\alpha,t)) \le \mathsf{d}(o, g_\infty(\alpha,0)) + \mathsf{d}(g_\infty(\alpha,0), g_\infty(\alpha,t)) = t,$$

for all $t \in [0, \rho]$, for $\hat{\mathfrak{q}}$-a.e $\alpha \in Q_\infty$. ∎

Corollary 8.5. *It holds that, for all $x \in B^+(o, \rho)$, $\varphi_\infty(x) = \varphi_\infty(o) - \mathsf{d}(o, x)$.*

Proof. If $x \in E \cap \mathcal{T}_\infty$, then $x = g(\alpha, t)$, for some t, with $\alpha = \mathfrak{Q}_\infty(x)$. By the previous proposition, we may assume that $g_\infty(\alpha, 0) = o$, hence we have that

$$\varphi_\infty(x) - \varphi_\infty(o) = \varphi_\infty(g_\infty(\alpha,t)) - \varphi_\infty(g_\infty(\alpha,0)) = -\mathsf{d}(g_\infty(\alpha,0), g_\infty(\alpha,t)) = -\mathsf{d}(o,x).$$

Since $\mathcal{T}_\infty \cap E$ has full measure in $B^+(o, \rho)$, we conclude. ∎

8.2. Localization of the whole space

We can now extend the localization given in Section 7.4 to the whole space X. Since we do not know the behaviour of φ_∞ outside $B^+(o, \rho)$, we take as reference the 1-Lipschitz function $-\mathsf{d}(o, \cdot)$, which coincides with φ_∞ on $B^+(o, \rho)$: we disintegrate using $-\mathsf{d}(o, \cdot)$ and we see that this disintegration coincides with the one given Section 7.4 in the set E. From this fact, and the geometric properties of the space, we will conclude.

We recall some of the concepts introduced in Subsection 2.4, applied to the 1-Lipschitz function $-\mathsf{d}(o,\cdot)$. The set $\mathcal{D}$ where no non-degenerate transport curve pass is empty, for we can connect o to any point with a minimal geodesic. The set of branching points, $\mathcal{A}$, contains only o and possibly elements of the boundary; this follows from the uniqueness of the geodesics. For this reason, the transport set $\mathcal{T}$ coincides with $X\setminus\{o\}$. Let $Q\subset\mathcal{T}$ be a measurable section, and let $\mathfrak{Q}\colon\mathcal{T}\to Q$ be the quotient map; let $X_\alpha:=\mathfrak{Q}^{-1}(\alpha)$ be the disintegration rays, and let $g\colon\mathrm{Dom}(g)\subset Q\times\mathbb{R}\to X$ be the standard parametrization. The map $t\mapsto g(\alpha,t)$ is the unitary speed parametrization of the geodesic connecting o to α, and then maximally extended. Define $\mathfrak{q}:=\frac{1}{\mathfrak{m}(E)}\mathfrak{Q}_\#(\mathfrak{m}\llcorner_E)$. Using the $\mathsf{CD}(0,N)$ condition, one immediately sees that $\mathfrak{Q}_\#(\mathfrak{m})\ll\mathfrak{q}$.

We are in position to use Theorem 2.9 (compare with Remark 2.10), hence there exists a unique disintegration for the measure $\mathfrak{m}$:

$$\mathfrak{m}=\int_Q\mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha), \tag{8.4}$$

where the measures $\mathfrak{m}_\alpha$ are supported on X_α and the transport rays $(X_\alpha,F,\mathfrak{m}_\alpha)$ satisfy the oriented $\mathsf{CD}(0,N)$ condition. We denote by $h_\alpha\colon(0,|X_\alpha|)\to\mathbb{R}$ the density function satisfying $\mathfrak{m}_\alpha=(g(\alpha,\cdot))_\#(h_\alpha\mathcal{L}^1\llcorner_{(0,|X_\alpha|)})$.

The next two propositions bind together the disintegration obtained in Section 7.4 (in particular Corollary 7.15) with the disintegration given by (8.4).

Proposition 8.6. *There exists a (unique) measurable map $L\colon\mathrm{Dom}(L)\subset Q_\infty\to Q$ such that $\mathcal{D}(L)$ has full $\hat{\mathfrak{q}}$-measure in Q_∞ and it holds*

$$L(\mathfrak{Q}_\infty(x))=\mathfrak{Q}(x),\quad\forall x\in B^+(o,\rho)\cap\mathcal{T}_\infty\cap\mathcal{T}\quad\textit{and}\quad\mathfrak{q}=L_\#\hat{\mathfrak{q}}.$$

Proof. Since $\varphi_\infty=\varphi_\infty(o)-\mathsf{d}(o,\cdot)$ on $B^+(o,\rho)$, we have that the partitions $(X_{\alpha,\infty})_{\alpha\in Q_\infty}$ and $(X_\alpha)_{\alpha\in Q}$ agree on the set $B^+(o,\rho)\cap\mathcal{T}_\infty\cap\mathcal{T}$. Consider the set

$$H:=\{(x,\alpha,\beta)\in(B^+(o,\rho)\cap\mathcal{T}_\infty\cap\mathcal{T})\times Q_\infty\times Q:\mathfrak{Q}_\infty(x)=\alpha\text{ and }\mathfrak{Q}(x)=\beta\},$$

and let $G:=\pi_{Q_\infty\times Q}(H)$ be the projection of H on the second and third variables. For what we have said, G is the graph of a map $L\colon\mathrm{Dom}(L)\subset Q_\infty\to Q$. The other properties easily follow. ∎

Proposition 8.7. *For $\mathfrak{q}$-a.e. $\alpha\in Q$, it holds that $|X_\alpha|\geq\rho$ and*

$$h_\alpha(r)=N\omega_N\,\mathsf{AVR}_X\,r^{N-1},\quad\forall r\in[0,\rho].$$

Proof. Using equation (8.2), we deduce that for $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, it holds that

$$g_\infty(\alpha,t)=g(L(\alpha),t),\quad\forall t\in(0,\min\{\rho,|X_\alpha|\}).$$

Since in the disintegration (7.35), all rays have length ρ, we deduce that $|X_\alpha|\geq\rho$. Moreover, we obtain $\hat{\mathfrak{m}}_{\alpha,\infty}=(\mathfrak{m}_{L(\alpha)})\llcorner_E$, concluding. ∎

Theorem 8.8. *For $\mathfrak{q}$-a.e. $\alpha\in Q$, it holds that $|X_\alpha|=\infty$ and*

$$h_\alpha(r)=N\omega_N\,\mathsf{AVR}_X\,r^{N-1},\quad\forall r>0.$$

Proof. Fix $\varepsilon > 0$ and let

$$C := \Big\{\alpha \in Q : \lim_{R\to\infty} \int_0^R h_\alpha/R^N < \omega_N \mathsf{AVR}_X(1-\varepsilon)\Big\},$$

with the convention that the limit above is 0 if $|X_\alpha| < \infty$. The limit always exists, and it is not larger than $\omega_N \,\mathsf{AVR}_X$ by the Bishop–Gromov inequality applied to each transport ray. We compute AVR_X using the disintegration:

$$\begin{aligned}
\omega_N \,\mathsf{AVR}_X &= \lim_{R\to\infty} \frac{\mathfrak{m}(B^+(o,R))}{R^N} = \lim_{R\to\infty} \int_Q \int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha) \\
&= \int_Q \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha) \\
&= \int_C \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha) + \int_{Q\setminus C} \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\,dt\,\mathfrak{q}(d\alpha) \\
&\le \int_C \omega_N \,\mathsf{AVR}_X(1-\varepsilon)\,\mathfrak{q}(d\alpha) + \int_{Q\setminus C} \omega_N \,\mathsf{AVR}_X\,\mathfrak{q}(d\alpha) = \omega_N \,\mathsf{AVR}_X(1-\varepsilon\mathfrak{q}(C)),
\end{aligned}$$

thus $\mathfrak{q}(C) = 0$. By arbitrariness of ε, we deduce that $\lim_{R\to\infty} \int_0^R h_\alpha/R^N = \omega_N \mathsf{AVR}_X$, hence $h_\alpha(t) = N\omega_N \,\mathsf{AVR}_X t^{N-1}$, for $\mathfrak{q}$-a.e. $\alpha \in \tilde{Q}$. ∎

The proof of Theorem 1.4 is therefore concluded. As described in the introduction, Theorem 1.5 is an immediate consequence.

A. The relative perimeter as a Borel measure

This appendix is devoted to proving that the relative perimeter can be extended uniquely to a Borel measure. Notice that in the result that follow, it is not needed the fact that $\Lambda_F < \infty$, the forward-completeness, and local forward convexity. We follow the line traced in [34].

We recall the definition of relative perimeter: fixed a Borel set $E \subset \Omega$ of a measured Finsler manifold $(X, F, \mathfrak{m})$, and fixed $\Omega \subset X$, we define the perimeter of E relative to Ω as

$$\mathsf{P}(E;\Omega) := \inf\Big\{\liminf_{n\to\infty} \int_\Omega |\partial u_n|\,d\mathfrak{m} : u_n \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \text{ and } u_n \to \mathbf{1}_E \text{ in } L^1_{\mathrm{loc}}(\Omega)\Big\}.$$

The infimum is clearly realized by a certain sequence u_n. Using a truncation argument, we may assume that u_n takes values in $[0, 1]$; moreover, by passing to subsequences, we may also assume that u_n converges also in the $\mathfrak{m}$-a.e. sense. If, in addition, Ω has finite measure, we may also assume (by the dominated convergence theorem) that $u_n \to \mathbf{1}_E$ in $L^1(\Omega)$. These assumptions will always be assumed tacitly, when dealing with a sequence realizing the minimum in the definition of perimeter.

The slope satisfies calculus rules, in the $\mathfrak{m}$-a.e. sense:

$$|\partial(f+g)| \le |\partial f| + |\partial g|, \quad |\partial(-f)| \le \Lambda_F |\partial f|, \tag{A.1}$$

$$|\partial(fg)| \le f|\partial g| + g|\partial f|, \quad \text{if } f, g \ge 0, \tag{A.2}$$

$$|\partial(fg)| \le \Lambda_F(|f||\partial g| + |g||\partial f|). \tag{A.3}$$

The proof is straightforward, once we know that $|\partial f|(x) = F^*(-df(x))$, for $\mathfrak{m}$-a.e. $x \in X$.

The next lemma permits us to join two Lipschitz functions defined on overlapping domains.

Lemma A.1. *Let $(X, F, \mathfrak{m})$ be a measured Finsler manifold. Let $N, M \subset X$ be two open sets such that $\partial M \cap \partial N = \emptyset$ and $\Lambda_{F,M\cap N} < \infty$. Then there exist an open set H such that $\overline{H} \subset N \cap M$ and a constant $c = c(M, N)$ such that the following happen. For all $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$, $v \in \mathrm{Lip}_{\mathrm{loc}}(N)$, for all $\varepsilon > 0$, there exists a function $w \in \mathrm{Lip}_{\mathrm{loc}}(M \cup N)$ such that*

$$w = u \text{ in } M \setminus N, \quad w = v \text{ in } N \setminus M, \quad \min\{u, v\} \leq w \leq \max\{u, v\} \text{ in } M \cap N,$$

and it holds that

$$\int_{M\cup N} |\partial w| \, d\mathfrak{m} \leq \int_M |\partial u| \, d\mathfrak{m} + \int_N |\partial v| \, d\mathfrak{m} + c \int_H |v - w| \, d\mathfrak{m} + \varepsilon. \tag{A.4}$$

Proof. The hypothesis $\partial M \cap \partial N = \emptyset$ yields $\overline{M \setminus N} \cap \overline{N \setminus M} = \emptyset$. Define

$$d := \inf\{\mathsf{d}(x, y), x \in M \setminus N, y \in N \setminus M\},$$

and consider the function $\varphi\colon M \cup N \to \mathbb{R}$ defined as

$$\varphi(x) := \max\left\{1 - \frac{3}{d} \sup_{y\in B^+(M\setminus N,d/3)} \mathsf{d}(y, x), 0\right\}.$$

The function φ is $(3/d)$-Lipschitz and attains the values 1 and 0 in a neighborhood of $M \setminus N$ and $N \setminus M$, respectively. Define $H = \varphi^{-1}((0, 1))$. Clearly, it holds $\overline{H} \subset M \cap N$. Fix now $\varepsilon > 0$ and find $k \in \mathbb{N}$ such that

$$\int_H (|\partial u| + |\partial v|) \, d\mathfrak{m} \leq \Lambda_{F,M\cap N}^{-2} \varepsilon k.$$

Define H_i and ψ_i $(i = 1, \ldots, k)$ as

$$H_i = \varphi^{-1}\Big(\Big(\frac{i-1}{k}, \frac{i}{k}\Big)\Big) \quad \text{and} \quad \psi_i = \min\Big\{3\Big(k\varphi - i + \frac{2}{3}\Big)^+, 1\Big\}.$$

Clearly, ψ_i is $(9k/d)$-Lipschitz and it is locally constant outside H_i. Define $w_i = \psi_i u + (1 - \psi_i)v$. We compute the slope of w_i in H_i using the calculus rules for the slope:

$$\begin{aligned}
|\partial w_i| &= |\partial(v + \psi_i(u - v))| \leq |\partial v| + |\partial(\psi_i(u - v))| \\
&\leq |\partial v| + \Lambda_{F,M\cap N}|\partial\psi_i||u - v| + \Lambda_{F,M\cap N}|\partial(u - v)|\psi_i \\
&\leq |\partial v| + \frac{9k}{d}\Lambda_{F,M\cap N}|u - v| + \Lambda_{F,M\cap N}^2(|\partial u| + |\partial v|).
\end{aligned}$$

Outside H_i, the slope of w_i is either $|\partial u$ or ∂v. Integrating over $M \cup N$, we obtain

$$\begin{aligned}
\int_{M\cup N} |\partial w_i| \, d\mathfrak{m} \leq \int_M |\partial u| \, d\mathfrak{m} + \int_N |\partial v| \, d\mathfrak{m} + \frac{9k\Lambda_{F,M\cap N}}{d} \int_{H_i} |u - v| \, d\mathfrak{m} \\
+ \Lambda_{F,M\cap N}^2 \int_{H_i} (|\partial u| + |\partial v|) \, d\mathfrak{m}.
\end{aligned}$$

Summing over i and dividing by k, we deduce that

$$\frac{1}{k}\sum_{i=1}^{k}\int_{M\cup N}|\partial w_i|\,d\mathfrak{m} \le \int_M |\partial u|\,d\mathfrak{m} + \int_N |\partial v|\,d\mathfrak{m} + \frac{9\Lambda_{F,M\cap N}}{d}\int_H |u-v|\,d\mathfrak{m} + \varepsilon,$$

hence there exists an index i_0 such that $w = w_{i_0}$ satisfies (A.4), with $c = 9\Lambda_{F,M\cap N}/d$. ∎

Theorem A.2. *Let $(X, F, \mathfrak{m})$ be a measured Finsler manifold, and let $E \subset X$ be a Borel set. Then the following hold.*

(1) (*Monotonicity*) $\mathsf{P}(E; A) \le \mathsf{P}(E; B)$, *if* $A \subset B$,

(2) (*Superadditivity*) $\mathsf{P}(E; A \cup B) \ge \mathsf{P}(E; A) + \mathsf{P}(E; B)$, *if* $A \cap B = \emptyset$,

(3) (*Inner regularity*) $\mathsf{P}(E; A) = \sup\{\mathsf{P}(E; B) : B \subset A$ *has compact closure in* $A\}$,

(4) (*Subadditivity*) $\mathsf{P}(E; A \cup B) \le \mathsf{P}(E; A) + \mathsf{P}(E; B)$,

for all open sets A and B.

Moreover, if for any Borel set A we define $\mathsf{P}(E; A) := \inf\{\mathsf{P}(E; B) : B \supset A$ *is open*$\}$, *then the map* $A \mapsto \mathsf{P}(E; A)$ *is a Borel measure.*

Proof. The monotonicity and superadditivity are immediate consequences of the definition of perimeter. Let us consider the inner regularity. Fix an open set A such that $\sup\{\mathsf{P}(E; B) : B \subset A\} < \infty$ (otherwise, the proof is trivial). Find a sequence $(A_j)_j$ of open sets with compact closure such that $\overline{A_j} \subset A_{j+1}$, and $\bigcup_j A_j = A$, and define

$$C_j = A_{2j} \setminus \overline{A_{2j-3}}.$$

Since $C_{2j} \cap C_{2k} = \emptyset$, if $j \neq k$, by superadditivity, we have that $\sum_j \mathsf{P}(E; C_{2j}) < \infty$, and analogously $\sum_j \mathsf{P}(E; C_{2j+1}) < \infty$. Fix $\varepsilon > 0$; there exists J such that

$$\sum_{j=J}^{\infty} \mathsf{P}(E; C_j) \le \varepsilon 2^{-4}.$$

Let $A := C_{J+2}$, $B' := A_{J+1}$, $F_h := C_{J+h-1}$, and $G_h := \bigcup_{i=1}^{h} F_i$; all these sets have compact closure, thus the irreversibility constant is finite on these sets.

By definition of perimeter, there exists a sequence $\psi_{m,h} \in \mathrm{Lip}_{\mathrm{loc}}(F_h)$ such that $\psi_{m,h} \to \mathbf{1}_E$ in $L^1(F_h)$ and

$$\int_{F_h} |\partial \psi_{m,h}|\,d\mathfrak{m} \le \mathsf{P}(E; F_h) + 2^{-2-m-h}.$$

Notice that G_h has compact closure, hence $\Lambda_{F,G_n\cap F_{h+1}} < \infty$, thus we are in position to use Lemma A.1 applied to the sets G_h and F_{h+1}. Said lemma gives a set $H_h \subset G_h \cap F_{h+1}$ and a constant c_h, that will be used soon. Clearly, up to passing to subsequences, we can assume that

$$c_h \int_{H_h} |\psi_{m,h+1} - \psi_{m,h}|\,d\mathfrak{m} \le \varepsilon 2^{-10-h}.$$

We define inductively on h a sequence of functions $u_{m,h} \colon G_h \to \mathbb{R}$ as follows.

For the initial step, take $u_{m,1} = \psi_{m,1}$.

For the inductive step, apply Lemma A.1 to the functions $u_{m,h}$ and $\psi_{m,h+1}$ obtaining a function $u_{m,h+1}$ such that

$$\int_{G_{h+1}} |\partial u_{m,h+1}|\, d\mathfrak{m} \le \int_{G_h} |\partial u_{m,h}|\, d\mathfrak{m} + \int_{F_{h+1}} |\partial \psi_{m,h+1}|\, d\mathfrak{m} \\ + c_h \int_{H_h} |u_{m,h} - \psi_{m,h+1}|\, d\mathfrak{m} + \varepsilon 2^{-10-h}.$$

Since $u_{m,h+1} = \psi_{m,h+1}$ on $F_{h+1} \setminus G_h$ and $u_{m,h+1} = u_{m,h}$ on $G_h \setminus F_{h+1}$, we can deduce by induction that

$$\int_{G_{h+1}} |\partial u_{m,h+1}|\, d\mathfrak{m} \le \sum_{i=1}^{h+1} \int_{F_i} |\partial \psi_{m,i}|\, d\mathfrak{m} + \sum_{i=1}^{h} \Big(c_i \int_{H_i} |\psi_{m,i} - \psi_{m,i+1}|\, d\mathfrak{m} + \varepsilon 2^{-10-i} \Big) \\ \le \sum_{i=1}^{h+1} \int_{F_i} |\partial \psi_{m,i}|\, d\mathfrak{m} + \varepsilon 2^{-8} \le \sum_{i=1}^{h+1} \mathsf{P}(E; F_i) + 2^{-m} + \varepsilon 2^{-8}.$$

We define $u_m(x) = u_{m,h}(x)$ whenever $x \in G_{h-1}$ (the definition is well posed), and we integrate its slope:

$$\int_{A \setminus \overline{B'}} |\partial u_m|\, d\mathfrak{m} \le \lim_{h \to \infty} \int_{G_h} |\partial u_{m,h-1}|\, d\mathfrak{m} \le \sum_{h=1}^{\infty} \mathsf{P}(E; F_h) + 2^{-m} + \varepsilon 2^{-8} \\ = \sum_{h=1}^{\infty} \mathsf{P}(E; C_{J+h-1}) + 2^{-m} + \varepsilon 2^{-8} \le \varepsilon 2^{-3} + 2^{-m}.$$

The sequence u_m converges to $\mathbf{1}_E$ in $L^1(G_h)$ for all h, hence it converges in $L^1_{\mathrm{loc}}(A \setminus \overline{B'})$.

We take now $v_m \in \mathrm{Lip}_{\mathrm{loc}}(B)$ converging to $\mathbf{1}_E$ in $L^1(B)$ such that

$$\mathsf{P}(E; B) \le \int_B |\partial v_m|\, d\mathfrak{m} + 2^{-m}.$$

We are in position to use Lemma A.1 again with the sets $A \setminus \overline{B'}$ and B and find an open set H and a constant c such that, for all m, there exists a function $w_m \colon A \to \mathbb{R}$ such that

$$\int_A |\partial w_m|\, d\mathfrak{m} \le \int_{A \setminus \overline{B'}} |\partial u_m|\, d\mathfrak{m} + \int_B |\partial v_m|\, d\mathfrak{m} + \int_H |u_m - v_m|\, d\mathfrak{m} + \varepsilon 2^{-3} \\ \le 2^{-m} + \varepsilon 2^{-3} + 2^{-m} + \int_H |u_m - \mathbf{1}_E|\, d\mathfrak{m} + \int_H |\mathbf{1}_E - v_m|\, d\mathfrak{m} + \varepsilon 2^{-3} \\ \le 2^{1-m} + \varepsilon + \int_{G_3} |u_m - \mathbf{1}_E|\, d\mathfrak{m} + \int_B |\mathbf{1}_E - v_m|\, d\mathfrak{m}.$$

By taking the limit as $m \to \infty$, we deduce that $\mathsf{P}(E; A) \le \mathsf{P}(E; B) + \varepsilon$, concluding the proof of the inner regularity.

We prove now the subadditivity. Fix A and B two open sets, and let A' and B' be compactly included in A and B, respectively. We will prove that $\mathsf{P}(E; A' \cup B') \le$

$\mathsf{P}(E;A') + \mathsf{P}(E;B')$. From this fact and the inner regularity, the subadditivity will follow. Consider $u_n \in \mathrm{Lip}_{\mathrm{loc}}(A')$ and $v_n \in \mathrm{Lip}_{\mathrm{loc}}(B')$ converging in L^1 to $\mathbf{1}_E$ such that

$$\int_{A'} |\partial u_n| \, d\mathfrak{m} \le \mathsf{P}(E;A') + \frac{1}{n} \quad \text{and} \quad \int_{B'} |\partial v_n| \, d\mathfrak{m} \le \mathsf{P}(E;B') + \frac{1}{n}\cdot$$

Apply Lemma A.1 to the sets A' and B', and find $H \subset A' \cap B'$ and $c > 0$ such that, for all $n > 0$, there exists a function w_n satisfying

$$\int_{A'\cup B'} |\partial_n w_n| \, d\mathfrak{m} \le \int_{A'} |\partial_n u_n| \, d\mathfrak{m} + \int_{B'} |\partial_n v_n| \, d\mathfrak{m} + c \int_H |u_n - v_n| \, d\mathfrak{m} + \frac{1}{n}\cdot$$

We conclude by taking the limit as $n \to \infty$.

The fact that the relative perimeter can be extended to a Borel measure is a consequence of a well-known theorem of De Giorgi and Letta [18], that states that the conditions we have just proven are sufficient to obtain such a measure. ■

B. Relaxation of the Minkowski content

In this appendix, we give a proof of the fact that the perimeter can be seen as the relaxation on the Minkowski content. The proof follows the lines of [3], with some extra attention to the irreversibility of the space. In the case $X = \mathbb{R}^d$, this was already proven in [15], with a different technique.

Proposition B.1. *Let $(X, F, \mathfrak{m})$ be a measured Finsler manifold and let $E \subset X$ be a Borel set. Then it holds that*

$$\mathfrak{m}^+(E) \ge \mathsf{P}(E).$$

Proof. We just consider the case $\mathfrak{m}^+(E) < \infty$ (the other is trivial). This implies that $\mathfrak{m}(\overline{E} \setminus E) = 0$, hence, without loss of generality, we may assume that E is closed. Consider the ε^{-1}-Lipschitz function

$$f_\varepsilon(x) := \max\Big\{1 - \frac{1}{\varepsilon} \sup_{y \in B^+(E,\varepsilon^2)} \mathsf{d}(y,x), 0\Big\}.$$

Clearly, $f_\varepsilon \to \mathbf{1}_E$ in $L^1(\mathfrak{m})$. In $B^+(E,\varepsilon^2)$ it is equal to 1, hence $|\partial f_\varepsilon|(x) = 0$, for all $x \in E$. Conversely, in $X \setminus B^+(E, \varepsilon + \varepsilon^2)$ it attains its minimum, hence $|\partial f_\varepsilon|(x) = 0$ for all $x \in X \setminus B^+(E, \varepsilon + \varepsilon^2)$. We compute the integral

$$\begin{aligned}\int_X |\partial f_\varepsilon|(x) \, \mathfrak{m}(dx) &= \int_{B^+(E,\varepsilon+\varepsilon^2)\setminus E} |\partial f_\varepsilon|(x) \, \mathfrak{m}(dx) \le \int_{B^+(E,\varepsilon+\varepsilon^2)\setminus E} \frac{1}{\varepsilon} \, \mathfrak{m}(dx) \\ &= \frac{\mathfrak{m}(B^+(E,\varepsilon+\varepsilon^2)\setminus E)}{\varepsilon} = (1+\varepsilon) \frac{\mathfrak{m}(B^+(E,\varepsilon+\varepsilon^2)) - \mathfrak{m}(E)}{\varepsilon + \varepsilon^2}\cdot\end{aligned}$$

By taking the inferior limit as $\varepsilon \to 0$, we conclude. ■

The previous proposition guarantees that the lower semicontinuous envelope of the Minkowski content is not smaller than the perimeter. The reverse is a bit more difficult and, at a certain point, we will require forward-completeness.

We consider the "semigroup" $(T_t)_{t\geq 0}$ given by the formula

$$T_t f(x) := \sup_{y\in B^-(x,t)} f(y), \quad T_0 f = f.$$

Note that the ball in the supremum is backward. The semigroup T_t enjoys the following immediate property.

Lemma B.2. *It holds that $T_{t+s}f \geq T_t(T_s f)$ and, if f is locally Lipschitz,*

$$\limsup_{t\to 0^+} \frac{T_t f - f}{t} \leq |\partial f|, \quad \mathfrak{m}\text{-a.e. in } X.$$

Proof. For the first part, fix $x \in X$, and $\varepsilon > 0$. By definition, there exists y such that $\mathsf{d}(y,x) < t$ and $(T_t(T_s f))(x) \leq (T_s f)(y) + \varepsilon$. Similarly, there exists z such that $\mathsf{d}(z,y) < s$ and $(T_s f)(y) \leq f(z) + \varepsilon$. By the triangular inequality, we have that $\mathsf{d}(z,x) < t+s$, thus

$$(T_{t+s}f)(x) \geq f(z) \geq (T_s f)(y) - \varepsilon \geq (T_t(T_s f))(x) - 2\varepsilon.$$

By arbitrariness of ε, we conclude the first part.

Regarding the second part, fix $x \in X$. By a direct computation, we deduce

$$\begin{aligned}\limsup_{t\to 0^+} \frac{(T_t f)(x) - f(x)}{t} &= \inf_{r>0} \sup_{t\in(0,r)} \frac{\sup_{y\in B^-(x,t)} f(y) - f(x)}{t} \\ &= \inf_{r>0} \sup_{t\in(0,r)} \sup_{y\in B^-(x,t)} \frac{(f(y)-f(x))^+}{t} \leq \inf_{r>0} \sup_{t\in(0,r)} \sup_{y\in B^-(x,t)} \frac{(f(y)-f(x))^+}{\mathsf{d}(y,x)} \\ &= \limsup_{y\to x} \frac{(f(y)-f(x))^+}{\mathsf{d}(y,x)}.\end{aligned}$$

If x is a point where f is differentiable, then the last term of the inequality above is equal to $F^*(-df) = |\partial f|(x)$, concluding the proof. ∎

We prove now a sort of coarea formula.

Lemma B.3. *Consider $(X, F, \mathfrak{m})$ a measured Finsler manifold. If $f\colon X \to [0,\mathbb{R})$ is a Lipschitz function with compact support, it holds that*

$$\int_0^\infty \mathfrak{m}^+(\{f \geq t\})\, dt \leq \int_X |\partial f|(x)\, \mathfrak{m}(dx).$$

Proof. In the first place, we notice that $\int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)\, dt = f(x)$. Fix $t \geq 0$ and $h > 0$. If $x \in B^+(\{f \geq t\}, h)$, then $(T_h f)(x) \geq t$, or in other words, $\mathbf{1}_{B^+(\{f\geq t\},h)} \leq \mathbf{1}_{\{(T_h f)\geq t\}}$. By integrating over t, we obtain

$$\int_0^\infty \mathbf{1}_{B^+(\{f\geq t\},h)}(x)\, dt \leq \int_0^\infty \mathbf{1}_{\{(T_h f)\geq t\}}(x)\, dt \leq (T_h f)(x).$$

By subtracting the first equation to the inequality above, integrating over x and using Fubini's theorem, we obtain

$$\int_0^\infty \frac{\mathfrak{m}(B^+(\{f \geq t\}, h)) - \mathfrak{m}(\{f \geq t\})}{h}\, dt \leq \int_X \frac{(T_h f)(x) - f(x)}{h}\, \mathfrak{m}(dx).$$

The set $\{f \geq 0\}$ is compact, hence for $h > 0$ sufficiently small, $B^+(\{f \geq 0\}, h)$ is compact. Moreover, $(T_h f - f)/h$ is smaller than the Lipschitz constant of f, hence the integrand in the right-hand side is dominated by an L^1 function. We take the inferior and superior limit in the left-hand side and right-hand side, respectively, of the inequality above; Fatou's lemma brings us to the conclusion. ∎

We now prove that we can, without loss of generality, assume that the functions of a sequence attaining the minimum in the definition of the perimeter have compact support.

Proposition B.4. *Let $(X, F, \mathfrak{m})$ be a forward-complete measured Finsler manifold, and let $E \subset X$ be a Borel set with finite measure. Then there exists a sequence of Lipschitz functions with compact support, $(w_n)_n$, such that $w_n \to \mathbf{1}_E$ in L^1 and $\mathsf{P}(E) = \lim_{n\to\infty} \int_X |\partial w_n| \, d\mathfrak{m}$.*

Proof. Fix $E \subset X$ with finite measure, such that $\mathsf{P}(E) < \infty$ (otherwise, the proof is trivial). Let $A_n := B^+(o, n)$ for some o, fixed once and for all. Up to taking subsequences, we can assume that $\mathfrak{m}(E \setminus A_n) \leq 2^{-n}$. Let φ_n be the 3-Lipschitz function given by

$$\varphi_n(x) := \Big(1 - 3 \inf_{y \in B^+(A_n, \frac{1}{3})} \mathsf{d}(y, x)\Big)^+.$$

This function takes value 1 and 0 in a neighborhood of $\overline{A_n}$ and $X \setminus A_{n+1}$, respectively. By definition of perimeter, there exists a sequence $u_n \colon A_n \to [0, 1]$ of locally Lipschitz function such that

$$\mathsf{P}(A_n) \geq \int_{A_n} |\partial u_n| \, d\mathfrak{m} - 2^{-n} \quad \text{and} \quad \|u_n - \mathbf{1}_E\|_{L^1(A_n)} \leq 2^{-n}.$$

Define the function $w_n := \varphi_n u_{n+1}$. This function is Lipschitz with compact support. We compute its distance to $\mathbf{1}_E$:

$$\int_X |w_n - \mathbf{1}_E| \, d\mathfrak{m} \leq \int_{A_n} |u_{n+1} - \mathbf{1}_E| \, d\mathfrak{m} + 2\,\mathfrak{m}(E \cap A_{n+1} \setminus A_n) + \mathfrak{m}(E \setminus A_{n+1}) \leq 2^{3-n},$$

thus $w_n \to \mathbf{1}_E$ in $L^1(X)$. Using the fact $|\partial w_n| \leq \varphi_n |\partial u_{n+1}| + |\partial \varphi_n| u_{n+1}$, we deduce

$$\begin{aligned}
\int_X |\partial w_n| \, d\mathfrak{m} &\leq \int_{A_n} |\partial u_{n+1}| \, d\mathfrak{m} + \int_{A_{n+1} \setminus A_n} \varphi_n |\partial u_{n+1}| \, d\mathfrak{m} + \int_{A_{n+1} \setminus A_n} u_{n+1} \, d\mathfrak{m} \\
&\leq \int_{A_{n+1}} |\partial u_{n+1}| \, d\mathfrak{m} + 2^{-1-n} \leq \mathsf{P}(E; A_n) + 2^{-n} \leq \mathsf{P}(E) + 2^{-n}.
\end{aligned}$$ ∎

Theorem B.5. *Consider $(X, F, \mathfrak{m})$ a forward-complete measured Finsler manifold. Let $E \subset X$ be a Borel set with finite measure. Then there exists $(E_n)_n$, a sequence of compact sets, such that $\mathfrak{m}(E_n \triangle E) \to 0$ and*

$$\mathsf{P}(E) \geq \limsup_{n\to\infty} \mathfrak{m}^+(E_n).$$

Proof. Proposition B.4 guarantees the existence of a sequence $(f_n)_n$ of Lipschitz functions with compact support such that $f_n \to \mathbf{1}_E$ in $L^1(\mathfrak{m})$ and

$$\mathsf{P}(E) = \lim_{n\to\infty} \int_X |\partial f_n|(x) \, \mathfrak{m}(dx).$$

Clearly, we may assume that $0 \le f_n \le 1$. Fix $\varepsilon \in (0, 1/2)$. By Lemma B.3, there exists $t_n^\varepsilon \in (\varepsilon, 1-\varepsilon)$ such that

$$\mathfrak{m}^+(\{f_n \ge t_n^\varepsilon\}) \le \frac{1}{1-2\varepsilon} \int_X |\partial f_n|(x)\, \mathfrak{m}(dx).$$

Define $E_n^\varepsilon := \{f_n \ge t_n^\varepsilon\}$. Since $\mathfrak{m}(E_n^\varepsilon \triangle E) \to 0$, by taking an appropriate choice of $\varepsilon = \varepsilon_n$, we conclude. ■

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