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Singular Yamabe-type problems with an asymptotically flat metric

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Abstract. In this paper, we study the asymptotic symmetry and local behavior of positive solutions at infinity to the equation

$$-L_g u = |x|^\tau u^{\frac{n+2+2\tau}{n-2}}$$

outside a bounded set in $\mathbb{R}^n$, where $n \geq 3$, $-2 < \tau < 0$, and L_g is the conformal Laplacian with asymptotically flat Riemannian metric g. We prove that the solution, at ∞, either converges to a fundamental solution of the Laplace operator on the Euclidean space, or is asymptotically close to a Fowler-type solution defined on $\mathbb{R}^n \setminus \{0\}$.

1. Introduction

In this paper, we shall first discuss the asymptotic behavior at infinity of solutions of the equation

$$-L_g u = |x|^\tau u^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1, \tag{1.1}$$

with parameter $\tau \in (-2, 0)$, where B_1 is the unit ball with center 0 in $\mathbb{R}^n$, $n \geq 3$, and where g is a smooth Riemannian metric defined on $\mathbb{R}^n \setminus \overline{B}_1$ that satisfies the asymptotically flat condition

$$\sum_{i,j=1}^{n} |\nabla^m (g_{ij}(x) - \delta_{ij})| \leq C_0 |x|^{-a-m} \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1. \tag{1.2}$$

Here C_0 is a positive constant, $m = 0, 1, 2$, and $a \geq (n-2)/2$. The lower bound of a is the minimal flatness order required to define ADM mass in general relativity, see Bartnik [2] and Denisov–Solove [9]. The operator

$$L_g u = \Delta_g u - \frac{n-2}{4(n-1)} R_g u$$

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is the conformal Laplacian, where Δ_g is the Laplace–Beltrami operator and R_g is the scalar curvature associated with the metric g. More specifically, if g and $g' = u^{4/(n-2)} g$ are any two such metrics, with corresponding scalar curvature functions R_g and $R_{g'}$, respectively, then for any smooth function ϕ,

$$L_g(u\phi) = u^{\frac{n+2}{n-2}} L_{g'}\phi.$$

Notice that the problem with a singularity at infinity can be transformed into one with an isolated singularity on the punctured unit ball by a Kelvin transform. In more detail, for any $x \in \mathbb{R}^n \setminus B_1$, it follows that $z = x/|x|^2 \in B_1 \setminus \{0\}$, and

$$\begin{aligned} u(x)^{4/(n-2)} g_{ij}\, dx^i\, dx^j &= \Big(\frac{1}{|z|^{n-2}} u\Big(\frac{z}{|z|^2}\Big)\Big)^{4/(n-2)} |z|^4 g_{ij}\Big(\frac{z}{|z|^2}\Big) d\Big(\frac{z^i}{|z|^2}\Big) d\Big(\frac{z^j}{|z|^2}\Big) \\ &=: v(z)^{4/(n-2)} \hat{g}_{kl}\, dz^k\, dz^l, \end{aligned}$$

where $v(z) = \frac{1}{|z|^{n-2}} u(z/|z|^2)$ is the Kelvin transform, and

$$\hat{g}_{kl}(z)\, dz^k\, dz^l = \frac{1}{|z|^4} \sum_{i,j=1}^{n} g_{ij}\Big(\frac{z}{|z|^2}\Big) \big(\delta_{ik}|z|^2 - 2z^i z^k\big) \big(\delta_{jl}|z|^2 - 2z^j z^l\big)\, dz^k\, dz^l.$$

Therefore, by the conformal invariance of L_g, we have

$$-L_{\hat{g}} v = |z|^\tau v^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_1 \setminus \{0\}, \tag{1.3}$$

with $L_{\hat{g}} v = \Delta_{\hat{g}} v - c(n) R_{\hat{g}} v$, and

$$\sum_{i,j=1}^{n} |\nabla^m(\hat{g}_{ij}(z) - \delta_{ij})| \le \hat{C}\, |z|^{a-m} \quad \text{in } B_1 \setminus \{0\}.$$

Studying the behavior of the solution u of (1.1) as $x \to \infty$ is equivalent to identifying the asymptotic profile of the solution v of (1.3) as $z \to 0$. If the solution v of (1.3) can be extended as a continuous function near the origin 0, we say that 0 is a removable singularity. If 0 is a removable singularity of v, we also shall say that u has a removable singularity at infinity. To avoid using too many variables, from now on we will rename the z variable in $B_1 \setminus \{0\}$ as x, and $v(z)$ as $u(x)$. Therefore, we shall study the positive solutions $u(x)$ of

$$-L_{\hat{g}} u = |x|^\tau u^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_1 \setminus \{0\}, \tag{1.4}$$

where for $m = 0, 1, 2$, the metric $\hat{g}$ satisfies

$$\sum_{i,j=1}^{n} |\nabla^m(\hat{g}_{ij}(x) - \delta_{ij})| \le \hat{C}\, |x|^{a-m} \quad \text{in } B_1 \setminus \{0\}. \tag{1.5}$$

By Han–Xiong–Zhang [14], we also have the following bound for the scalar curvature:

$$|R_{\hat{g}}| \le C |x|^{a-2} \quad \text{in } B_1 \setminus \{0\}. \tag{1.6}$$

The equation (1.4) is closely related to the well-known Yamabe problem. The resolution of the Yamabe problem is an outstanding achievement: it was the first time that the existence problem of a nonlinear partial differential equation with critical exponent was completely solved affirmatively; see Yamabe [38], Trudinger [35], Aubin [1] and Schoen [30]. A great deal of work has been done on the equation (1.4).

Let us first look at the case $\tau = 0$. The existence of positive solutions for (1.4) is related to the study of local solutions of the singular Yamabe problem, which has been considered by Schoen [31], Mazzeo–Smale [28], Mazzeo–Pollack–Uhlenbeck [27], and Mazzeo–Pacard [26]. If $\hat{g}$ is flat and 0 is a non-removable singularity, using a rather complicated version of the Alexandrov reflection, Caffarelli–Gidas–Spruck [3] proved that the positive solutions satisfy

$$u(x) = u_0(|x|)\,(1 + o(1)) \quad \text{as } x \to 0,$$

where u_0 is a singular positive radial solution satisfying

$$-\Delta u_0 = u_0^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

We refer to these radial singular solutions on $\mathbb{R}^n \setminus \{0\}$ as Fowler solutions. After that, Korevaar–Mazzeo–Pacard–Schoen [16] presented a much simpler and more geometric derivation of this fact, and improved the remainder term $o(1)$: for some $\alpha > 0$,

$$u(x) = u_0(|x|)\,(1 + O(|x|^\alpha)) \quad \text{as } x \to 0.$$

See Han–Li–Li [12] for a higher order expansion of u. In a series of works [5,6,22,33,34], the local singular positive solutions to the prescribed scalar curvature equation

$$-\Delta u = K(x)\,u^{(n+2)/(n-2)} \quad \text{in } B_1 \setminus \{0\}$$

have been studied for a positive function $K(x)$ with appropriate flatness near $x = 0$. For more related papers about the isolated singularities problem for the Yamabe equation, see [4, 7, 8, 15, 18–21, 32, 36] and the references therein.

If $\hat{g}$ is not flat and 0 is a non-removable singularity, Marques [25] established the same asymptotic behavior for $3 \le n \le 5$. Xiong–Zhang [37] showed that this still holds for $n = 6$. Recently, Han–Xiong–Zhang [14] obtained that for $n \le 24$, the solution converges to a Fowler solution defined on $\mathbb{R}^n \setminus \{0\}$. The same conclusion also holds when $n > 24$, see again [14], under the additional condition that the solution grows no faster than the fundamental solution of the flat metric Laplacian at the singularity.

Now consider the case $\tau \ne 0$. If $\hat{g}$ is flat, Caffarelli–Gidas–Spruck [3] pointed out that equation (1.4) has no positive solution in any punctured ball for $|\tau| \ge 2$. Hence, we always assume $|\tau| < 2$. Li [17] proved that for $-2 < \tau < 0$, the asymptotic behavior of the positive solutions is

$$u(x) = \bar{u}(|x|)\,(1 + o(1)) \quad \text{as } x \to 0,$$

where $\bar{u}(r) = \fint_{\partial B_r} u\, d\mathbb{S}_r$ is the average of u on ∂B_r, and B_r is the ball in $\mathbb{R}^n$ with radius r and center 0.

It is a natural question to ask whether the theorems of Caffarelli–Gidas–Spruck [3] and Korevaar–Mazzeo–Pacard–Schoen [16] on a punctured ball still hold when $\tau \neq 0$ and the background metric $\hat{g}$ is not flat. In this paper, we shall give some answers if $\hat{g}$ satisfies (1.5). Clearly, it suffices to consider

$$a = \frac{n-2}{2}\cdot$$

The main result of this article is the following.

Theorem 1.1. *Let $-2 < \tau < 0$ and $n \geq 3$, and suppose that $u \in C^2(B_1 \setminus \{0\})$ is a positive solution of* (1.4) *with $\hat{g}$ satisfying* (1.5). *Then*

(1) *either 0 is a removable singularity, and there exists a positive constant A_0 such that*

$$u(x) = A_0 + O(|x|) \quad \text{as } x \to 0, \tag{1.7}$$

(2) *or there exist two positive constants C_1 and C_2, depending on n, τ and $\hat{C}$, such that*

$$C_1 |x|^{-(n-2)/2} \leq u(x) \leq C_2 |x|^{-(n-2)/2} \quad \text{as } x \to 0. \tag{1.8}$$

Furthermore, there exists $\alpha \in (0,1)$ such that

$$u(x) = u_\tau(|x|)\,(1 + O(|x|^\alpha)) \quad \text{as } x \to 0, \tag{1.9}$$

with $u_\tau(|x|)$ a Fowler-type solution, that is, a C^2 positive radial solution of

$$-\Delta u_\tau = |x|^\tau\, u_\tau^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Theorem 1.1 has two aspects of importance. On the one hand, the estimate (1.8) is optimal even for flat metrics. In a sense, Theorem 1.1 extends corresponding earlier results; for instance,

(i) when the metric $\hat{g}$ is flat and $\tau = 0$, Caffarelli–Gidas–Spruck [3] and Korevaar–Mazzeo–Pacard–Schoen [16] investigated Theorem 1.1 for $n \geq 3$;

(ii) when the metric $\hat{g}$ is flat and $-2 < \tau < 0$, Li [17] established Theorem 1.1 for $n \geq 3$;

(iii) when the metric $\hat{g}$ is not flat and $\tau = 0$, Theorem 1.1 was obtained by Marques [25] for $3 \leq n \leq 5$, Xiong–Zhang [37] for $n = 6$ and Han–Xiong–Zhang [14] for $n \geq 3$.

On the other hand, for $n > 24$, in Han–Xiong–Zhang [14] it is required in addition that the solution grows no faster than the fundamental solution of the flat metric Laplacian at the singularity. However, in the process of blow up analysis, we get that the limit equation for $-2 < \tau < 0$ is

$$-\Delta u = u^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n,$$

which is a subcritical type equation, so by the Liouville theorem, our results still hold when $n > 24$ without any additional condition, in contrast to [14].

As pointed out earlier in the article, the exterior formulation (1.1) is equivalent to problem (1.4) in the punctured unit ball, so by performing a Kelvin transform, we can establish the asymptotic behavior of positive solutions to the equation (1.1) with g satisfying (1.2).

Theorem 1.2. *Let* $-2 < \tau < 0$ *and* $n \geq 3$, *and suppose that* $u \in C^2(\mathbb{R}^n \setminus \overline{B}_1)$ *is a positive solution of* (1.1) *with* g *satisfying* (1.2). *Then*

(1) *either* ∞ *is removable, and there exists a positive constant* A_0 *such that*

$$u(x) = A_0 |x|^{2-n} + O(|x|^{1-n}) \quad \text{as } x \to \infty,$$

(2) *or there exist two positive constants* C_1 *and* C_2, *depending on* n, τ *and* C_0, *such that*

$$C_1 |x|^{-(n-2)/2} \leq u(x) \leq C_2 |x|^{-(n-2)/2} \quad \text{as } x \to \infty,$$

and there exists $\alpha \in (0, 1)$ *such that*

$$u(x) = u_\tau(|x|)(1 + O(|x|^{-\alpha})) \quad \text{as } x \to \infty.$$

These results extend the work of Han–Xiong–Zhang [14] from $\tau = 0$ to $-2 < \tau < 0$. This is also consistent with earlier results of Gidas–Spruck [10], who studied the isolated singularity located at infinity when g is flat.

The proof of Theorem 1.1 is divided into several parts. In the second section, we shall prove the upper bound in (1.8) near a singularity. The lower bound of (1.8) and the removability of the singularity will be obtained in Section 3. In Section 4, some important propositions and improved estimates will be established. In Section 5, we shall prove the asymptotic radial symmetry. Finally, Theorem 1.2 will be deduced from Theorem 1.1 in Section 6.

2. The upper bound near the singularity

This section is devoted to proving the upper bound in (1.8) near a singularity. Indeed, the condition (1.5) is not necessary in this part. Via the blow up technique, we obtain the following.

Theorem 2.1. *Let* $-2 < \tau < 0$, *and suppose that* $u \in C^2(B_1 \setminus \{0\})$ *is a positive solution of* (1.4). *Then there exists a positive constant* C, *depending on* n, τ *and* $\hat{C}$, *such that*

$$u(x) \leq C|x|^{-(n-2)/2} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{-n/2} \quad \text{as } x \to 0. \tag{2.1}$$

To obtain the theorem, using blow up analysis, we get the limit equation

$$-\Delta u = u^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n.$$

The classic Liouville theorem [10] tells us that $u = 0$ is the only nonnegative entire solution of

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n$$

with $1 < p < (n+2)/(n-2)$. If $\tau > 0$, then $\frac{n+2+2\tau}{n-2} > \frac{n+2}{n-2}$, which implies that the Liouville theorem becomes invalid. On the other hand, when $\hat{g}$ is not conformally flat, Han–Xiong–Zhang [14] have considered the case $\tau = 0$. Hence, we assume $-2 < \tau < 0$.

We recall now the doubling property (see Lemma 5.1 in [29]), which plays an important part in our proof. We denote by $B_R(x)$ the ball in $\mathbb{R}^n$ with radius R and center x. We write $B_R(0)$ as B_R for short. With C we denote a positive constant, which may differ from line to line.

Proposition 2.2. *Suppose that $\emptyset \neq D \subset \Sigma \subset \mathbb{R}^n$, that Σ is closed, and let $\Gamma = \Sigma \setminus D$. Let $M\colon D \to (0,\infty)$ be bounded on compact subsets of D. If for a fixed positive constant k, there exists $y \in D$ satisfying*

$$M(y)\,\mathrm{dist}(y,\Gamma) > 2k,$$

then there exists $x \in D$ such that

$$M(x) \geq M(y), \quad M(x)\,\mathrm{dist}(x,\Gamma) > 2k,$$

and for all $z \in D \cap B_{kM^{-1}(x)}(x)$,

$$M(z) \leq 2M(x).$$

The following result is proved with the help of the doubling property.

Lemma 2.3. *Let $1 < p < \frac{n+2}{n-2}$ and $0 < \alpha \leq 1$, and let $c(x) \in C^{2,\alpha}(\overline{B}_1)$ satisfy*

$$\|c\|_{C^{2,\alpha}(\overline{B}_1)} \leq C_1 \quad \text{and} \quad c(x) \geq C_2 \quad \text{in } \overline{B}_1, \tag{2.2}$$

for some positive constants C_1, C_2. Suppose that $u \in C^2(B_1)$ is a nonnegative solution of

$$-L_{\hat{g}} u = c(x) u^p \quad \text{in } B_1. \tag{2.3}$$

Then there exists a positive constant C, depending only on n, p, C_1, C_2 and $\hat{C}$, such that

$$|u(x)|^{(p-1)/2} + |\nabla u(x)|^{(p-1)/(p+1)} \leq C[\mathrm{dist}(x,\partial B_1)]^{-1} \quad \text{in } B_1.$$

Proof. Arguing by contradiction, for $k = 1,2,\ldots$, we assume that there exist nonnegative functions u_k satisfying (2.3) and points $y_k \in B_1$ such that

$$|u_k(y_k)|^{(p-1)/2} + |\nabla u_k(y_k)|^{(p-1)/(p+1)} > 2k\,[\mathrm{dist}(y_k,\partial B_1)]^{-1}. \tag{2.4}$$

Define

$$M_k(x) := |u_k(x)|^{(p-1)/2} + |\nabla u_k(x)|^{(p-1)/(p+1)}.$$

Via Proposition 2.2, for $D = B_1$ and $\Gamma = \partial B_1$, there exist $x_k \in B_1$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k[\mathrm{dist}(x_k,\partial B_1)]^{-1} \geq 2k, \tag{2.5}$$

and for any $z \in B_1$ and $|z - x_k| \leq kM_k^{-1}(x_k)$,

$$M_k(z) \leq 2M_k(x_k). \tag{2.6}$$

It follows from (2.5) that

$$\lambda_k := M_k^{-1}(x_k) \to 0 \quad \text{as } k \to \infty, \tag{2.7}$$

and

$$\mathrm{dist}(x_k,\partial B_1) > 2k\lambda_k, \quad \text{for } k = 1,2,\ldots \tag{2.8}$$

Consider

$$w_k(y) := \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y) \quad \text{in } B_k.$$

Combining this with (2.8) gives that, for any $y \in B_k$,

$$|x_k + \lambda_k y - x_k| \leq \lambda_k |y| \leq \lambda_k k < \frac{1}{2}\,\mathrm{dist}(x_k,\partial B_1),$$

that is,

$$x_k + \lambda_k y \in B_{\frac{1}{2}\mathrm{dist}(x_k,\partial B_1)}(x_k) \subset B_1.$$

Therefore, w_k is well defined in B_k, and

$$|w_k(y)|^{(p-1)/2} = \lambda_k |u_k(x_k + \lambda_k y)|^{(p-1)/2},$$
$$|\nabla w_k(y)|^{(p-1)/(p+1)} = \lambda_k |\nabla u_k(x_k + \lambda_k y)|^{(p-1)/(p+1)}.$$

From (2.6), we find that for all $y \in B_k$,

$$|u_k(x_k + \lambda_k y)|^{\frac{p-1}{2}} + |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{p+1}} \le 2\Big(|u_k(x_k)|^{\frac{p-1}{2}} + |\nabla u_k(x_k)|^{\frac{p-1}{p+1}}\Big),$$

that is,

$$|w_k(y)|^{(p-1)/2} + |\nabla w_k(y)|^{(p-1)/(p+1)} \le 2\lambda_k M_k(x_k) = 2. \tag{2.9}$$

Moreover, w_k satisfies

$$-L_{\hat{g}_k} w_k = c_k(y)\, w_k^p \quad \text{in } B_k, \tag{2.10}$$

where $\hat{g}_k(y) := \hat{g}(x_k + \lambda_k y)$ and $c_k(y) := c(x_k + \lambda_k y)$. Moreover, it follows that

$$|w_k(0)|^{(p-1)/2} + |\nabla w_k(0)|^{(p-1)/(p+1)} = 1.$$

Standard elliptic theory, see [11], then implies that, after passing to a subsequence, the sequence $\{w_k\}$ converges to some nonnegative function $w \in C^2_{\mathrm{loc}}(\mathbb{R}^n)$,

$$w_k \to w \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n).$$

On the other hand, we also obtain that $\{c_k\}$ is uniformly bounded in $\mathbb{R}^n$ by condition (2.2). For each $R > 0$ and for all $y, z \in B_R$, we have

$$|D^\beta c_k(y) - D^\beta c_k(z)| \le C_1 \lambda_k^{|\beta|} |\lambda_k(y-z)|^\alpha \le C_1 |y-z|^\alpha, \quad |\beta| = 0, 1, 2,$$

for k large enough. Therefore, by the Arzelà–Ascoli theorem, there exists a function $c \in C^2(\mathbb{R}^n)$ such that, after extracting a subsequence, $c_k \to c$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, by (2.7), we obtain

$$|c_k(y) - c_k(z)| \to 0 \quad \text{as } k \to \infty. \tag{2.11}$$

This implies that the function c is actually a constant A. By (2.2) again, $c_k \ge C_2 > 0$, and we conclude that A is a positive constant.

Therefore, we deduce that w satisfies

$$-\Delta w = A w^p \quad \text{in } \mathbb{R}^n \tag{2.12}$$

and

$$|w(0)|^{(p-1)/2} + |\nabla w(0)|^{(p-1)/(p+1)} = 1.$$

Since $p < (n+2)/(n-2)$, this contradicts the Liouville type result [10] that the only nonnegative entire solution of (2.12) is $w = 0$. This concludes the proof. ∎

Applying Lemma 2.3, now we can prove Theorem 2.1.

Proof of Theorem 2.1. For $x_0 \in B_{1/2}\setminus\{0\}$, we denote $R := \frac{1}{2}|x_0|$. Then for any $y \in B_1$, we have $\frac{1}{2}|x_0| < |x_0 + Ry| < \frac{3}{2}|x_0|$, and deduce that $x_0 + Ry \in B_1\setminus\{0\}$. Define

$$w(y) := R^{(n-2)/2} u(x_0 + Ry).$$

Therefore, we obtain that

$$-L_{\bar{g}} w = c(y)\, w^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_1,$$

where $\bar{g}(y) := \hat{g}(x_0 + Ry)$, and $c(y) := |y + x_0/R|^\tau$. Notice that

$$1 < \left| y + \frac{x_0}{R} \right| < 3 \quad \text{in } \overline{B}_1.$$

Moreover,

$$\|c\|_{C^3(\overline{B_1})} \le C \quad \text{and} \quad c(y) \ge 3^{-2} \quad \text{in } \overline{B}_1.$$

Applying Lemma 2.3, we obtain that

$$|w(0)|^{(2+\tau)/(n-2)} + |\nabla w(0)|^{(n+\tau)/(2+\tau)} \le C,$$

that is,

$$(R^{(n-2)/2} u(x_0))^{(2+\tau)/(n-2)} + (R^{(n-2)/2+1} |\nabla u(x_0)|)^{(n+\tau)/(2+\tau)} \le C.$$

Hence, we have

$$u(x_0) \le CR^{-(n-2)/2} \le C|x_0|^{-(n-2)/2} \quad \text{and} \quad |\nabla u(x_0)| \le CR^{-n/2} \le C|x_0|^{-n/2}.$$

Since $x_0 \in B_{1/2}\setminus\{0\}$ is arbitrary, Theorem 2.1 is proved. ∎

As a consequence of the upper bound, we get the following spherical Harnack inequality, which will be used later on.

Corollary 2.4. *Let* $-2 < \tau < 0$, *and suppose that* $u \in C^2(B_1\setminus\{0\})$ *is a positive solution of* (1.4) *with* $\hat{g}$ *satisfying* (1.5). *Then there exists a positive constant* C, *depending on* n, τ *and* $\hat{C}$, *such that*

$$\max_{r/2\le|x|\le 2r} u(x) \le C \min_{r/2\le|x|\le 2r} u(x), \tag{2.13}$$

and

$$|\nabla u(x)| + |x|\,|\nabla^2 u(x)| \le C|x|^{-1} u(x)$$

for every $0 < r < 1/4$, *where* C *is independent of* r.

Proof. For any $\bar{x} \in B_{1/4}\setminus\{0\}$, let $|\bar{x}| = r$ and consider

$$v_r(y) := r^{(n-2)/2} u(ry).$$

Then

$$-L_{\bar{g}} v_r = |y|^\tau\, v_r^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_{1/r} \setminus \{0\},$$

where $\bar{g}(y) := \hat{g}(ry)$. Thanks to the upper bound (2.1), we have

$$v_r \leq C \quad \text{in } B_4 \setminus B_{1/4}.$$

Now, applying standard elliptic theory and the Harnack inequality to v_r in the annulus $B_4 \setminus B_{1/4}$, we have that there exists a positive constant C, not depending on r, such that

$$\max_{1/4 \leq |y| \leq 4} v_r(y) \leq C \min_{1/4 \leq |y| \leq 4} v_r(y) \quad \text{and} \quad |\nabla v_r(y)| + |\nabla^2 v_r(y)| \leq C v_r(y),$$

After scaling back to u, the corollary follows immediately. ■

3. The lower bound and removability of the singularity

In this section, the lower bound of (1.8) and the removability of the singularity will be obtained. First of all, we define a Pohozaev-type integral for u as

$$P(r,u) = \int_{\partial B_r} \Big(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{1}{2} r |\nabla u|^2 + r \Big| \frac{\partial u}{\partial r} \Big|^2 + \frac{n-2}{2(n+\tau)} r^{\tau+1} u^{\frac{2(n+\tau)}{n-2}} \Big) d\mathbb{S}_r, \tag{3.1}$$

where $d\mathbb{S}_r$ is the standard area measure on ∂B_r. Inspired by the work of Caffarelli–Gidas–Spruck [3], we prove a removable singularity result as follows.

Theorem 3.1. *Let $-2 < \tau < 0$, and suppose that $u \in C^2(B_1 \setminus \{0\})$ is a positive solution of (1.4) with $\hat{g}$ satisfying (1.5). Then the limit*

$$\lim_{r \to 0} P(r,u) := P(u)$$

exists and

$$P(u) \leq 0.$$

Moreover, $P(u) = 0$ if and only if 0 is a removable singularity. If $P(u) < 0$, there exists a positive constant C, depending on n, τ and $\hat{C}$, such that

$$u(x) \geq C|x|^{-(n-2)/2} \quad \text{as } x \to 0. \tag{3.2}$$

To prove Theorem 3.1, we will suppose that $P(u) \geq 0$, and by analyzing the behavior of solutions of an ordinary differential inequality (see (3.10)) satisfied by the spherical average of the solution, we shall get that $P(u) \leq 0$, and $P(u) = 0$ if and only if 0 is a removable singularity. This leads to the lower bound estimate in (1.8) by Theorem 3.1. We point out that to overcome a non-trivial linear term in the differential inequality, we shall prove a refined estimate in Lemma 4.1 through the comparison principle.

Theorem 3.1 will follow from Propositions 3.2–3.4.

Proposition 3.2. *The limit*

$$\lim_{r \to 0} P(r,u) := P(u)$$

exists. The number $P(u)$ is called the Pohozaev invariant of the solution u.

Proof. A Pohozaev-type identity, see [24], asserts that for any $0 < s \le r < 1$, we have

$$\begin{aligned} P(r,u) - P(s,u) &= -\int_{s\le|x|\le r} \Big(x\cdot\nabla u + \frac{n-2}{2}u\Big)(L_{\hat g}-\Delta)u\,dx \\ &\quad + \frac{n-2}{2(n+\tau)}\int_{s\le|x|\le r}(x\cdot\nabla(|x|^\tau))\,u^{\frac{2(n+\tau)}{n-2}}\,dx \\ &\quad + \Big(\frac{n(n-2)}{2(n+\tau)} - \frac{n-2}{2}\Big)\int_{s\le|x|\le r}|x|^\tau u^{\frac{2(n+\tau)}{n-2}}\,dx \\ &= -\int_{s\le|x|\le r}\Big(x\cdot\nabla u + \frac{n-2}{2}u\Big)(L_{\hat g}-\Delta)u\,dx \end{aligned}$$

By Theorem 2.1, Corollary 2.4 and the flatness condition (1.5) on $\hat g$, we have

$$\Big|\Big(x\cdot\nabla u + \frac{n-2}{2}u\Big)(L_{\hat g}-\Delta)u\Big| \le C|x|^{(n-2)/2-n},$$

for some positive constant C independent of x. This implies that for any $0 < s < r < 1$,

$$|P(r,u) - P(s,u)| \le r^{(n-2)/2}.$$

Hence the Pohozaev-type integral shows that the limit

$$\lim_{r\to 0} P(r,u) := P(u)$$

exists. ∎

Proposition 3.3. *Assume that* $P(u) \ge 0$. *Then*

$$\liminf_{x\to 0}|x|^{(n-2)/2}u(x) = 0.$$

Proof. Supposing the opposite, there exist two positive constants c_1 and c_2 such that for any $0 < |x| < 1/2$,

$$c_1 \le |x|^{(n-2)/2}u(x) \le c_2. \tag{3.3}$$

Let $\{r_k\}$ be any sequence of positive numbers such that $r_k \to 0$ as $k \to +\infty$. Define

$$f_k(x) := r_k^{(n-2)/2}u(r_kx).$$

Therefore, we have

$$-L_{\bar g}f_k = |x|^\tau f_k^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_{1/r_k}\setminus\{0\},$$

where $\bar g(x) := \hat g(r_kx)$. By (3.3), we obtain that

$$c_1 \le |x|^{(n-2)/2}f_k(x) \le c_2 \quad \text{in } B_{1/(2r_k)}\setminus\{0\}.$$

Thus, the sequence $\{f_k\}$ is locally uniformly bounded away from the origin. By standard elliptic estimates, there exists a subsequence of $\{f_k\}$ (still denoted by $\{f_k\}$) which converges as $k \to +\infty$ to a positive solution f in compact subsets of $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$-\Delta f = |x|^\tau f^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

and

$$c_1 |x|^{-(n-2)/2} \leq f(x) \leq c_2 |x|^{-(n-2)/2}.$$

Consider $P(r, f)$ defined as in (3.1). In this case, by the proof of Lemma 3.2, we can deduce that $P(r, f)$ is a constant independent of r. Next we shall show that it determines a unique negative constant,

$$P(f) < 0. \tag{3.4}$$

Indeed, we denote $|x| = r$, $t = -\log r$, $\theta = x/|x|$, and define

$$\psi(t, \theta) := |x|^{(n-2)/2} f(x).$$

Then

$$\psi_{tt} + \Delta_\theta \psi - \frac{(n-2)^2}{4} \psi + \psi^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } \mathbb{R} \times \mathbb{S}^{n-1}.$$

Let $\bar{f}(r) = \fint_{\partial B_r} f \, d\mathbb{S}_r$ be the average of f on ∂B_r, and let $\beta(t) := e^{-t\,(n-2)/2} \bar{f}(r)$. By [17], we have

$$\psi(t, \theta) = r^{(n-2)/2} \bar{f}(r)(1 + O(r)) = \beta(t)(1 + O(e^{-t})) \quad \text{as } t \to \infty.$$

Hence, for $r/2 \leq |x| \leq 2r$, we have

$$-\Delta(f - \bar{f}) = r^\tau (f^{\frac{n+2+2\tau}{n-2}} - \bar{f}^{\frac{n+2+2\tau}{n-2}}) = r^\tau \bar{f}^{\frac{n+2+2\tau}{n-2}} O(r) \quad \text{as } r \to 0.$$

Standard elliptic estimates [11] give that, as $r \to 0$,

$$|\nabla(f - \bar{f})| \leq C\Big(\frac{\sup|f - \bar{f}|}{r} + r \sup r^\tau \bar{f}^{\frac{n+2+2\tau}{n-2}} O(r)\Big) \leq C \sup\big(\bar{f} + r^2 r^\tau \bar{f}^{\frac{n+2+2\tau}{n-2}}\big).$$

Since f satisfies $\Delta f + |x|^\tau f^{\frac{4+2\tau}{n-2}} f = 0$, with $|x|^\tau f^{\frac{4+2\tau}{n-2}} \leq C|x|^{-2}$, and from [10], f also satisfies the Harnack inequality, and $\sup f$ is comparable to $f(x)$ for $r/2 \leq |x| \leq 2r$, so as $r \to 0$,

$$|\nabla(f - \bar{f})| \leq C\left(\bar{f} + r^{2+\tau} \bar{f}^{\frac{4+2\tau}{n-2}} \bar{f}\right) \leq C \bar{f}(r).$$

In particular,

$$\frac{\partial}{\partial r}(f - \bar{f}) \leq C \bar{f}(r) \quad \text{and} \quad |\nabla_\theta(f - \bar{f})| \leq C r \bar{f}(r).$$

Together with $\psi(t, \theta) - \beta(t) = r^{(n-2)/2}(f - \bar{f})$, we conclude that

$$\frac{\partial}{\partial t}(\psi(t, \theta) - \beta(t)) = \beta\, O(e^{-t}) \quad \text{as } t \to \infty, \tag{3.5}$$

$$|\nabla_\theta(\psi(t, \theta) - \beta(t))| = \beta\, O(e^{-t}) \quad \text{as } t \to \infty. \tag{3.6}$$

Since $P(r,f) = P(t,\psi)$, with

$$P(t,\psi) = \frac{|\mathbb{S}^{n-1}|}{2}\int_{\mathbb{S}^{n-1}}\Big[\psi_t^2(t,\theta) - |\nabla_\theta\psi(t,\theta)|^2 - \frac{(n-2)^2}{4}\,\psi^2(t,\theta) + \frac{n-2}{n+\tau}\,\psi^{\frac{2(n+\tau)}{n-2}}(t,\theta)\Big]\,d\mathbb{S}_1,$$

and using (3.5) and (3.6), we convert $P(t,f) - P(s,f)$ with $t \ge s$ into

$$H(t) - (\beta^2(t) + \beta_t^2(t))\,O(e^{-t}) = H(s) - (\beta^2(s) + \beta_t^2(s))\,O(e^{-s}),$$

where

$$H(t) = \beta_t^2(t) - \frac{(n-2)^2}{4}\,\beta^2(t) + \frac{n-2}{n+\tau}\,\beta^{\frac{2(n+\tau)}{n-2}}(t).$$

It is clear that $P(r,f)$ is a constant independent of r, so it determines a unique constant $H_\infty = \lim_{t\to\infty} H(t)$. Hence, to prove (3.4), we just need to check that $H_\infty < 0$. By the above argument, we can write

$$H(t) = H_\infty + (\beta^2(t) + \beta_t^2(t))\,O(e^{-t}),$$

that is,

$$\beta_t^2(t) = \frac{(n-2)^2}{4}\,\beta^2(t) - \frac{n-2}{n+\tau}\,\beta^{\frac{2(n+\tau)}{n-2}}(t) + H_\infty + (\beta^2(t) + \beta_t^2(t))\,O(e^{-t}), \tag{3.7}$$

which implies that the behavior of β is completely determined by the roots of the right-hand side of the above equality. Hence, for $\tau > -2$, we conclude that

$$0 \ge H_\infty \ge -\frac{2+\tau}{n+\tau}\Big(\frac{n-2}{2}\Big)^{\frac{2(n+\tau)}{2+\tau}}.$$

We show now that f has a removable singularity in the case $H_\infty = 0$. By (3.7), β cannot have a local minimum and must ultimately decrease monotonically to zero, which implies that

$$\lim_{t\to\infty}\beta(t) = \lim_{t\to\infty}\beta_t(t) = 0,$$

and that there exists $T > 0$ such that, for $t > T$,

$$\beta_t(t) < 0.$$

From (3.7), we obtain that for any $0 < \mu < (n-2)/2$, and as $t \to +\infty$,

$$\beta_t^2 - \Big(\frac{n-2}{2} - \mu\Big)^2\beta^2 = ((n-2)\mu - \mu^2)\,\beta^2 - \frac{n-2}{n+\tau}\,\beta^{\frac{2(n+\tau)}{n-2}}(t) + (\beta^2(t) + \beta_t^2(t))\,O(e^{-t}),$$

and we can choose T large enough such that for $t > T$,

$$\beta_t^2 \ge \Big(\frac{n-2}{2} - \mu\Big)^2\beta^2,$$

which implies

$$\frac{\beta_t^2}{\beta^2} \ge \Big(\frac{n-2}{2} - \mu\Big)^2, \quad \text{that is,} \quad \frac{-\beta_t}{\beta} \ge \frac{n-2}{2} - \mu.$$

Integrating the differential inequality, we have that, for $t > T$,

$$\beta(t) \le \beta(T) \exp\Big(\Big(\frac{n-2}{2} - \mu\Big)(T-t)\Big).$$

Hence, there exists $r_0(\mu) > 0$ small enough such that for $x \in B_{r_0}$,

$$f(x) \le C(\mu)|x|^{-\mu}$$

and since μ can be chosen small enough, this implies that $f \in L^p(B_{r_0})$ for arbitrary large p. Then elliptic theory tells us that the function f has to be smooth around the origin. Therefore, 0 is a removable singularity.

Since

$$c_1|x|^{-(n-2)/2} \le f(x) \le c_2|x|^{-(n-2)/2},$$

we can conclude that in this case

$$P(f) < 0,$$

which implies that

$$0 > P(f) = \lim_{k\to+\infty} P(r, f_k) = \lim_{k\to\infty} P(r_k r, u) = \lim_{r\to 0} P(r,u) \ge 0,$$

which is a contradiction. Thus, we obtain that $\liminf_{x\to 0}|x|^{(n-2)/2}u(x) = 0$. This finishes the proof of Proposition 3.3. ∎

Proposition 3.4. *Assume* $\liminf_{x\to 0}|x|^{(n-2)/2}u(x) = 0$. *Then*

$$\lim_{x\to 0}|x|^{(n-2)/2}u(x) = 0. \tag{3.8}$$

Proof. We prove (3.8) by contradiction. Since $\liminf_{x\to 0}|x|^{(n-2)/2}u(x) = 0$, if the conclusion of the proposition did not hold, we would have $\limsup_{x\to 0}|x|^{(n-2)/2}u(x) > 0$. Let $|x| = r$, $t = -\ln r$, and let $\bar{u}(r) = \fint_{\partial B_r} u\, d\mathbb{S}_r$ be the average of u on ∂B_r. Define

$$w(t) := e^{-t(n-2)/2}\,\bar{u}(r).$$

Observe that the upper bound in Theorem 2.1 implies that $w(t)$ is bounded. A direct computation gives that

$$\bar{u}_r = -e^{\frac{n}{2}t}\Big(\frac{n-2}{2}w + w_t\Big),$$

and this implies that $|w_t(t)|$ is bounded. Furthermore,

$$\bar{u}_{rr} = e^{\frac{n+2}{2}t}\Big(\frac{n(n-2)}{4}w + (n-1)w_t + w_{tt}\Big).$$

Therefore, we have

$$\bar{u}_{rr} + \frac{n-1}{r}\bar{u}_r = e^{\frac{n+2}{2}t}\Big(w_{tt} - \frac{(n-2)^2}{4}w\Big). \tag{3.9}$$

Choosing a fixed $s < r$ and applying the divergence theorem, we get

$$\Big(\int_{B_r\setminus B_s} \Delta u(x)\Big)_r = \Big(\int_{B_r} \Delta u(x) - \int_{B_s} \Delta u(x)\Big)_r = \Big(\int_{\partial B_r} u_r(x)\Big)_r$$
$$= \Big(\int_{\partial B_1} u_r(rx)\, r^{n-1}\Big)_r = \frac{n-1}{r}\int_{\partial B_1} u_r(rx)\, r^{n-1} + \int_{\partial B_1} u_{rr}(rx)\, r^{n-1}$$
$$= \frac{n-1}{r}\int_{\partial B_r} u_r(x) + r^{n-1}|\mathbb{S}^{n-1}|\Big(\fint_{\partial B_r} u_r(x)\Big)_r.$$

It follows that

$$\bar u_{rr} = \Big(\fint_{\partial B_r} u_r\Big)_r = -\frac{n-1}{r}\frac{r^{1-n}}{|\mathbb{S}^{n-1}|}\int_{\partial B_r} u_r + \frac{r^{1-n}}{|\mathbb{S}^{n-1}|}\Big(\int_{B_r\setminus B_s} \Delta u\Big)_r$$
$$= -\frac{n-1}{r}\bar u_r + \frac{r^{1-n}}{|\mathbb{S}^{n-1}|}\Big(\int_{B_r\setminus B_s} \Delta u - L_{\hat g} u - |x|^\tau u^{\frac{n+2+2\tau}{n-2}}\Big)_r$$
$$= -\frac{n-1}{r}\bar u_r + \frac{r^{1-n}}{|\mathbb{S}^{n-1}|}\Big(\int_{B_r\setminus B_s} (\Delta - \Delta_{\hat g})u + c(n)\, R_{\hat g} u - |x|^\tau u^{\frac{n+2+2\tau}{n-2}}\Big)_r.$$

Hence,

$$\bar u_{rr} + \frac{n-1}{r}\bar u_r = \fint_{\partial B_r} \Big((\Delta - \Delta_{\hat g})u + c(n)\, R_{\hat g} u - |x|^\tau u^{\frac{n+2+2\tau}{n-2}}\Big).$$

Using (3.9), it follows that

$$w_{tt} - \frac{(n-2)^2}{4} w = r^{(n+2)/2}\fint_{\partial B_r} \Big((\Delta - \Delta_{\hat g})u + c(n)\, R_{\hat g} u - |x|^\tau u^{\frac{n+2+2\tau}{n-2}}\Big).$$

Applying the spherical Harnack inequality obtained in Lemma 2.4, we have that

$$c_2\, r^\tau\, \bar u^{\frac{n+2+2\tau}{n-2}} \le \fint_{\partial B_r} |x|^\tau u^{\frac{n+2+2\tau}{n-2}} \le c_3\, r^\tau\, \bar u^{\frac{n+2+2\tau}{n-2}},$$

and together with (1.6) and (1.5), it follows that

$$\Big|\fint_{\partial B_r} (\Delta - \Delta_{\hat g})u + c(n)\, R_{\hat g} u\Big| \le c_1 |x|^{(n-2)/2-2}\, \bar u,$$

where c_1, c_2 and c_3 are positive constants. With these estimates, we obtain

$$-c_3\, w^{\frac{n+2+2\tau}{n-2}} - c_1 e^{-\frac{n-2}{2}t} w \le w_{tt} - \frac{(n-2)^2}{4} w \le -c_2\, w^{\frac{n+2+2\tau}{n-2}} + c_1 e^{-\frac{n-2}{2}t} w. \tag{3.10}$$

On the other hand, as $\liminf_{x\to 0} |x|^{(n-2)/2} u(x) = 0$ and $\limsup_{x\to 0} |x|^{(n-2)/2} u(x) > 0$, we deduce that

$$0 = \liminf_{t\to\infty} w(t) < \limsup_{t\to\infty} w(t) < \infty.$$

Making use of (3.10), we can choose $\varepsilon_0 > 0$ sufficiently small so that we are able to construct sequences

$$\bar{t}_i < t_i < t_i^*, \quad \text{with} \quad \lim_{i\to\infty} \bar{t}_i = \infty,$$

such that

$$w(\bar{t}_i) = w(t_i^*) = \varepsilon_0, \tag{3.11}$$

and t_i is the unique minimum point of w in $(\bar{t}_i, t_i^*)$,

$$w_t(t_i) = 0, \quad \text{with} \quad \lim_{i\to+\infty} w(t_i) = 0.$$

Hence, w is decreasing in $(\bar{t}_i, t_i)$ and increasing in (t_i, t_i^*).

By (3.10), for $\bar{t}_i \le t \le t_i$, we have the estimates

$$\frac{2}{n-2}\ln\frac{w(t)}{w(t_i)} - c \le t_i - t \le \Big(\frac{2}{n-2} + ce^{-\frac{n-2}{2}t_i}\Big)\ln\frac{w(t)}{w(t_i)} + c, \tag{3.12}$$

and for $t_i \le t \le t_i^*$, we have the estimates

$$\frac{2}{n-2}\ln\frac{w(t)}{w(t_i)} - c \le t - t_i \le \Big(\frac{2}{n-2} + ce^{-\frac{n-2}{2}t_i}\Big)\ln\frac{w(t)}{w(t_i)} + c. \tag{3.13}$$

We shall give a proof in the next section, which can be obtained by Lemma 4.1.

On the other hand, since there exists a diffeomorphism between the half cylinder and the punctured ball, it will be more convenient work in cylindrical coordinates. More explicitly, the diffeomorphism

$$\Phi : (\mathbb{R}\times\mathbb{S}^{n-1}, g_{\mathrm{cyl}} = dt^2 + d\theta^2) \to (\mathbb{R}^n\setminus\{0\}, \delta)$$

is given by $\Phi(t,\theta) = e^{-t}\theta$, with inverse $\Phi^{-1}(x) = (-\log|x|, x|x|^{-1})$. One also verifies that

$$\Phi^*\delta = e^{-2t} g_{\mathrm{cyl}}.$$

We denote $|x| = r$, $t = -\log r$, $\theta = x/|x|$, and define

$$v(t,\theta) := |x|^{(n-2)/2}u(x) \quad \text{and} \quad \widetilde{g} := e^{2t}\Phi^*\hat{g} = (e^{\frac{n-2}{2}t})^{\frac{4}{n-2}}\Phi^*\hat{g}.$$

Then

$$-L_{\widetilde{g}}v = v^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}\times\mathbb{S}^{n-1},$$

where

$$L_{\widetilde{g}}v = \Delta_{\widetilde{g}}v - \frac{n-2}{4(n-1)}(R_{\widetilde{g}} - e^{-2t}R_{\Phi^*\hat{g}})$$

is the conformal Laplacian. It is also useful to recall that in cylindrical coordinates, we have

$$R_{\widetilde{g}} - e^{-2t}R_{\Phi^*\hat{g}} = (n-2)(n-1) + 2(n-1)e^{-t}\frac{\partial_r\sqrt{|\hat{g}|}}{|\hat{g}|}\circ\Phi,$$

and

$$\widetilde{g} = dt^2 + d\theta^2 + O(e^{-2t}).$$

Thus, $P(r,u)$ can be written as

$$P(t,v)=\frac{|\mathbb{S}^{n-1}|}{2}\int_{\mathbb{S}^{n-1}}\Big[v_t^2(t,\theta)-|\nabla_\theta v(t,\theta)|^2-\frac{(n-2)^2}{4}v^2(t,\theta)+\frac{n-2}{n+\tau}v^{\frac{2(n+\tau)}{n-2}}(t,\theta)\Big]\,d\mathbb{S}_1.$$

We denote

$$\bar{r}_i=e^{-\bar{t}_i},\quad r_i=e^{-t_i}\quad\text{and}\quad r_i^*=e^{-t_i^*}.$$

Observe that $\bar{r}_i>r_i>r_i^*$. Using the Harnack inequality and the gradient estimates on $u(x)$ of Corollary 2.4 we see that, in terms of $v(t,\theta)$, we have

$$|\nabla v(t,\theta)|=O(1)\,w(t)\quad\text{as } t\to\infty,$$

uniformly for $\theta\in\mathbb{S}^{n-1}$, so it follows that as $i\to\infty$,

$$\int_{\mathbb{S}^{n-1}}\big[v_t^2(t_i,\theta)-|\nabla_\theta v(t_i,\theta)|^2\big]\,d\theta\to 0,$$

and using that $\lim_{i\to\infty}w(t_i)=0$ in Proposition 3.2, we have

$$P(u)=\lim_{i\to+\infty}P(r_i,u)=0. \tag{3.14}$$

Next we claim that, for $|x|=r_i$ and as $i\to\infty$,

$$u(x)=\bar{u}(r_i)(1+o(1)), \tag{3.15}$$

and

$$|\nabla u(x)|=-\bar{u}_r(r_i)(1+o(1)). \tag{3.16}$$

Indeed, let

$$h_i(y):=r_i^{(n-2)/2}u(r_iy).$$

By the choice of r_i, $\bar{h}_i(1)=w(t_i)\to 0$ as $i\to\infty$, using the Harnack inequality of Corollary 2.4, and the fact that h_i converges to 0 uniformly in subsets of $\mathbb{R}^n\setminus\{0\}$. Taking

$$\eta_i(y)=\frac{h_i(y)}{h_i(e_1)},$$

where $e_1=(1,0,\dots,0)\in\mathbb{R}^n$, we have

$$-L_{\bar{g}}\eta_i=(r_i^{(n-2)/2}u(r_ie_1))^{\frac{4+2\tau}{n-2}}|y|^\tau\eta_i^{\frac{n+2+2\tau}{n-2}}\quad\text{in } B_{1/r_i}\setminus\{0\},$$

where $\bar{g}(y):=\hat{g}(r_iy)$. Note that by the Harnack inequality in Corollary 2.4, η_i is locally uniformly bounded in $\mathbb{R}^n\setminus\{0\}$. Hence, elliptic estimates tell us that there exists a subsequence $\{\eta_i\}$ which converges to a nonnegative function η in $C^2_{\text{loc}}(\mathbb{R}^n\setminus\{0\})$ satisfying

$$-\Delta\eta=0\quad\text{in }\mathbb{R}^n\setminus\{0\},$$

and $\eta(e_1) = 1$. By the Bocher theorem, we have

$$\eta = a_1|x|^{2-n} + b_1,$$

and $a_1 + b_1 = 1$. On the other hand, by the fact that

$$\begin{aligned} r^{(n-2)/2}\,\bar{\eta}_i(r) = r^{(n-2)/2} \fint_{\partial B_r} \eta_i(r) &= \frac{1}{u(r_i e_1)}\, r^{(n-2)/2}\,\bar{u}(r r_i) \\ &= \frac{1}{u(r_i e_1)}\, r_i^{-(n-2)/2}\, w(-\ln(r r_i)), \end{aligned}$$

and since t_i is the unique minimum point of w in $(\bar{t}_i, t_i^*)$, which implies that

$$\frac{d}{dr}(r^{(n-2)/2}\,\bar{\eta}_i(r))\Big|_{r=1} = 0,$$

we conclude that $\frac{d}{dr}(r^{(n-2)/2}\,\eta(r))|_{r=1} = 0$, and it follows that $a_1 = b_1 = 1/2$. Hence, we conclude that

$$\eta = \frac{1}{2}\,|y|^{2-n} + \frac{1}{2}.$$

Note that if $|y| = 1$, then $\eta = 1$. This implies that for any $|y| = 1$,

$$u(r_i y) = u(r_1 e)\,(1 + o(1)) \quad \text{as } i \to \infty,$$

which after averaging gives

$$\bar{u}(r_i) = u(r_i e)\,(1 + o(1)).$$

On the other hand, for any $|y| = 1$, we also have

$$u(r_i y) = u(r_1 e)\,(1 + o(1)) \quad \text{as } i \to \infty.$$

Hence, we conclude that for $|x| = r_i$,

$$\bar{u}(r_i) = \bar{u}(r_1 e)\,(1 + o(1)) \quad \text{as } i \to \infty,$$

which implies that for $|x| = r_i$,

$$u(x) = \bar{u}(r_i)(1 + o(1)) \quad \text{as } i \to \infty.$$

This establishes (3.15), and we obtain (3.16) analogously.

Making use of (3.15) and (3.16), we have as $i \to \infty$,

$$|v(t_i,\theta) - w(t_i)| = o(1)\,w(t_i),$$

and

$$|\nabla_\theta v(t_i,\theta)| = o(1)\,w(t_i), \quad |v_t(t_i,\theta)| = o(1)\,w(t_i)$$

uniformly for $\theta \in \mathbb{S}^{n-1}$. Then we conclude that, as $i \to \infty$,

$$P(t_i, v) = |\mathbb{S}^{n-1}|\Big[-\frac{1}{2}\,\frac{(n-2)^2}{4}\,w^2(t_i)(1 + o(1)) + \frac{n-2}{n+\tau}\,w^{\frac{2(n+\tau)}{n-2}}(t_i)\,(1 + o(1))\Big].$$

Hence, using that $P(t_i, v) = P(r_i, u)$, we have that, for i sufficiently large,

(3.17) $$w^2(t_i) \le C_n |P(r_i, u)|.$$

From the Pohozaev-type identity in Lemma 3.2, it follows that

$$|P(r_i, u)| \le \int_{r_i \le |x| \le r_i^*} \Big| \Big(x \cdot \nabla u + \frac{n-2}{2} u \Big) (L_{\hat g} - \Delta) u \Big| \, dx + \int_{|x| \le r_i^*} \Big| \Big(x \cdot \nabla u + \frac{n-2}{2} u \Big) (L_{\hat g} - \Delta) u \Big| \, dx.$$

On the other hand, Corollary 2.4, (1.5) and (2.1) give that

$$\Big| \Big(x \cdot \nabla u + \frac{n-2}{2} u \Big) (L_{\hat g} - \Delta) u \Big| \le C |x|^{(n-2)/2-n}.$$

Then,

$$I_1 := \int_{|x| \le r_i^*} \Big| \Big(x \cdot \nabla u + \frac{n-2}{2} u \Big) (L_{\hat g} - \Delta) u \Big| \, dx \le C e^{-\frac{n-2}{2} t_i^*}.$$

By the first inequality in (3.13), we have for $t_i \le t \le t_i^*$,

$$w(t) \le C w(t_i) \, e^{-\frac{n-2}{2} t_i} \, e^{\frac{n-2}{2} t},$$

which implies that for $r_i^* \le |x| \le r_i$,

(3.18) $$u(x) \le C w(t_i) \, e^{-\frac{n-2}{2} t_i} \, |x|^{2-n}.$$

By Corollary 2.4 and (1.5), we also have

$$\Big| \Big(x \cdot \nabla u + \frac{n-2}{2} u \Big) (L_{\hat g} - \Delta) u \Big| \le C u^2(x) |x|^{(n-2)/2-2}.$$

Hence, using (3.18), we have

$$\begin{aligned} I_2 &:= \int_{r_i \le |x| \le r_i^*} \Big| \Big(x \cdot \nabla u + \frac{n-2}{2} \, u \Big) \, (L_{\hat g} - \Delta) u \Big| \, dx \\ &\le C w^2(t_i) \, e^{-(n-2) t_i} \int_{|x| \le r_i^*} |x|^{2+(n-2)/2-2n} \, dx \le C w^2(t_i) \, e^{-(n-2) t_i} \, (r_i^*)^{(n+2)/2-n}. \end{aligned}$$

On the other hand, combining (3.11), (3.12) and (3.13), we get that

$$t_i^* - t_i \le \Big(\frac{2}{n-2} + C e^{-\frac{n-2}{2} t_i} \Big) \ln \frac{\varepsilon_0}{w(t_i)} + C,$$

and

$$\frac{2}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} \le t_i - \bar t_i + C.$$

Thus,

$$t_i^* - t_i \le \frac{n-2}{2} \Big(\frac{2}{n-2} + C e^{-\frac{n-2}{2} t_i} \Big) (t_i - \bar t_i + C) + C.$$

Moreover,

$$e^{-\frac{n-2}{2}t_i}(t_i - \bar{t}_i) \le e^{-\frac{n-2}{2}t_i} t_i \le 1,$$

with bound independent of i, so we conclude that

$$t_i^* - t_i \le t_i - \bar{t}_i + C_1 \tag{3.19}$$

for some constant C_1 independent of i. Then, via (3.19), we can more precisely estimate I_2, as follows:

$$\begin{aligned} I_2 &\le C w^2(t_i)\, e^{-(n-2)t_i} \, (r_i^*)^{2+(n-2)/2-n} = C w^2(t_i)\, e^{-(n-2)t_i + \frac{n-2}{2} t_i^*} \\ &= C w^2(t_i)\, e^{\frac{n-2}{2}(t_i^* - 2t_i)} \le C w^2(t_i)\, e^{-\frac{n-2}{2}\bar{t}_i}. \end{aligned}$$

Thus,

$$|P(r_i, u)| \le I_1 + I_2 \le C w^2(t_i)\, e^{-\frac{n-2}{2}\bar{t}_i} + C e^{-\frac{n-2}{2}\bar{t}_i^*}.$$

Then, choosing i large enough, from (3.17) we obtain

$$w^2(t_i) \le C e^{-\frac{n-2}{2}\bar{t}_i^*},$$

and it follows that

$$\ln \frac{1}{w(t_i)} \ge \frac{n-2}{4} t_i^* - \frac{1}{2} \ln C. \tag{3.20}$$

From the first inequality of (3.12) and the first inequality of (3.13), we have

$$t_i - \bar{t}_i \ge \frac{2}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} - C \quad \text{and} \quad t_i^* - t_i \ge \frac{2}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} - C.$$

Summing them up, we have

$$t_i^* - \bar{t}_i \ge \frac{4}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} - C = \frac{4}{n-2} \ln \frac{1}{w(t_i)} + \frac{4}{n-2} \ln \varepsilon_0 - C.$$

Combining with (3.20), we have

$$t_i^* - \bar{t}_i \ge t_i^* - C,$$

which implies $\bar{t}_i \le C$. This contradicts $\bar{t}_i \to \infty$. Notice that the positive constants C above may differ from line to line. Hence, we finish the proof of Proposition 3.4. ∎

With the help of the above propositions, we prove now Theorem 3.1.

Proof of Theorem 3.1. Applying Proposition 3.2, we have the existence of $P(u)$. On the one hand, if 0 is a removable singularity of the solution u, it is obvious that $P(u) = 0$. To prove the theorem, we suppose that $P(u) \ge 0$. With the help of Proposition 3.3 and Proposition 3.4, we get that $\lim_{x \to 0} |x|^{(n-2)/2} u(x) = 0$, and in the following we shall show that 0 is an removable singularity of the solution u and $P(u) = 0$. Hence, we conclude that $P(u) \le 0$.

We start by showing that 0 is a removable singularity. Suppose that $P(u) \geq 0$. Using Proposition 3.4, we have

$$\lim_{x \to 0} |x|^{(n-2)/2} u(x) = 0,$$

which implies that

$$\lim_{t \to \infty} w(t) = 0. \tag{3.21}$$

Via the first inequality in (3.10), we conclude that there exist $T_1 > 0$ sufficiently large and $\varepsilon_0 > 0$ sufficiently small such that, for $t > T_1$,

$$w \leq \varepsilon_0 \quad \text{and} \quad w_{tt} \geq 0,$$

that is, w is convex. As a result, we also have for $t > T_1$,

$$w_t(t) < 0. \tag{3.22}$$

For any $0 < \delta < n-2$, by (3.10) and (3.21), T_1 can be chosen large enough so that for $t > T_1$,

$$w_{tt} - \Big(\frac{n-2}{2} - \delta\Big)^2 w \geq ((n-2)\delta - \delta^2 - c_3 w^{\frac{4+2\tau}{n-2}} - c_1 e^{-at}) w \geq 0. \tag{3.23}$$

Combining (3.22) and (3.23), we have

$$\frac{d}{dt} w_t^2 - \Big(\frac{n-2}{2} - \delta\Big)^2 w^2 \leq 0,$$

which implies that $w_t^2 - ((n-2)/2 - \delta)^2 w^2$ is non-increasing. Since $\lim_{t \to \infty} w(t) = 0$, we conclude that

$$w_t^2 - \Big(\frac{n-2}{2} - \delta\Big)^2 w^2 \geq 0.$$

Integrating the differential inequality, we have for $t > T_1$,

$$w(t) \leq w(T_1) \exp\Big(\Big(\frac{n-2}{2} - \delta\Big)(T_1 - t)\Big).$$

Going back to the original u and applying the Harnack inequality obtained in Lemma 2.4, we have that there exists $r_0(\delta)$ such that if $|x| < r_0(\delta)$,

$$u(x) \leq c(\delta) |x|^{-\delta}$$

Since δ is arbitrarily small, we have that $u \in L^p(B_{r_0})$ for arbitrarily large p. Elliptic theory then tells us that the function u has to be smooth around the origin. Therefore, 0 is a removable singularity.

Thus, we conclude that $P(u) \leq 0$ and the equality holds if and only if 0 is a removable singularity of the solution u. If $P(u) < 0$, Proposition 3.3 gives that

$$\liminf_{x \to 0} |x|^{(n-2)/2} u(x) > 0.$$

This finishes the proof. ∎

Proof of the first part in Theorem 1.1. Combining Theorem 2.1 and Theorem 3.1, we conclude that if $u \in C^2(B_1 \setminus \{0\})$ is a positive solution of (1.4) with $\hat{g}$ satisfying (1.5), then either 0 is a removable singularity or there exist two positive constants C_1 and C_2, depending on n, τ and $\hat{C}$, such that

$$C_1 |x|^{-(n-2)/2} \le u(x) \le C_2 |x|^{-(n-2)/2} \quad \text{as } x \to 0.$$

The proof of (1.7) will be given in Section 6. ■

4. Proof of the estimates (3.12) and (3.13)

For completeness, we give detailed proofs of the estimates (3.12) and (3.13), which were used in the previous section. Indeed, the left-hand side inequalities in (3.12) and (3.13) can be obtained by using Lemma 6.2 in [14]. With the following lemma, the right-hand side inequalities in (3.12) and (3.13) will follow directly from (4.1) and (4.2).

Lemma 4.1. *Suppose that a, b, c, t_1 and t_2 are positive numbers, and w is a $C^2([t_1, t_2])$ positive function satisfying*

$$w_{tt}(t) - \Big(\frac{(n-2)^2}{4} - be^{-at}\Big) w(t) + \frac{(n+\tau)(n-2)c}{(2+\tau)^2}\, w(t)^{\frac{n+2+2\tau}{n-2}} \ge 0$$

and

$$w(t) \le \varepsilon_0$$

for ε_0 small enough, which will be fixed later. Then there exist positive constants t^*, C_1, C_2, C_3 and C_4, depending only on a, b, c, t_1 and t_2, such that for $t^* \le t_1 \le t_2$, we have $w(t) \le \varepsilon_0$ on $[t_1, t_2]$,

(i) *If $w_t(t_2) \le 0$, then for $t_1 \le t \le t_2$, there holds*

$$t_2 - t \le \Big(\frac{2}{n-2} + C_1 e^{-at_1}\Big) \ln \frac{w(t)}{w(t_2)} + C_2. \tag{4.1}$$

(ii) *If $w_t(t_1) \ge 0$, then for $t_1 \le t \le t_2$, there holds*

$$t_1 - t \le \Big(\frac{2}{n-2} + C_3 e^{-at_1}\Big) \ln \frac{w(t)}{w(t_2)} + C_4. \tag{4.2}$$

Proof. We just prove (4.1); the inequality (4.2) follows with a similar argument. To that end, we will use the following Claim, to be proved later.

Claim. *There exists a $C^2([t_1, t_2])$ positive function ζ that satisfies*

$$\zeta_{tt}(t) - \Big(\frac{(n-2)^2}{4} - be^{-at}\Big) \zeta(t) + \frac{(n+\tau)(n-2)c}{(2+\tau)^2}\, \zeta(t)^{\frac{n+2+2\tau}{n-2}} \le 0, \tag{4.3}$$

with $\zeta(t_2) = w(t_2)$. Moreover, for $t_1 \le t \le t_2$, we have that $\zeta_t(t) \ge 0$ and

$$t_2 - t \le \Big(\frac{2}{n-2} + C_1 e^{-at_1}\Big) \ln \frac{\zeta(t)}{\zeta(t_2)} + C_2. \tag{4.4}$$

With the help of this Claim, we can give a proof of (4.1). Indeed, let

$$z(t) := w(t) - \zeta(t).$$

It follows that

$$z_{tt}(t) - \Big(\frac{(n-2)^2}{4} - be^{-at}\Big) z(t) + \frac{(n+\tau)(n+2+2\tau)c}{(2+\tau)^2} \zeta(t)^{\frac{4+2\tau}{n-2}} z(t) \geq 0,$$

with

$$z(t_2) = 0, \quad z_t(t_2) \leq 0,$$

and

$$(n+2+2\tau)\,\zeta(t)^{\frac{4+2\tau}{n-2}} = \begin{cases} (n-2)\,\frac{w(t)^{\frac{n+2+2\tau}{n-2}} - \zeta(t)^{\frac{n+2+2\tau}{n-2}}}{w(t)-\zeta(t)}, & \text{if } w(t) \neq \zeta(t), \\ (n+2+2\tau)\, w(t)^{\frac{4+2\tau}{n-2}}, & \text{if } w(t) = \zeta(t). \end{cases}$$

Note that $\zeta(t), w(t) \leq \varepsilon_0$ for $t_1 \leq t \leq t_2$, so by taking ε_0 sufficiently small, and t^* sufficiently large, we then have

$$\frac{(n-2)^2}{4} - be^{-at} - \frac{(n+\tau)(n+2+2\tau)c}{(2+\tau)^2} \zeta(t)^{\frac{4+2\tau}{n-2}} \geq 0.$$

It follows from Lemma 6.1 in [14] that for $t_1 \leq t \leq t_2$, we have

$$w(t) - \zeta(t) \geq 0. \tag{4.5}$$

Together with (4.4) and $w(t_2) = \zeta(t_2)$, we have, for $t_1 \leq t \leq t_2$,

$$t_2 - t \leq \Big(\frac{2}{n-2} + Ce^{-at_1}\Big) \ln \frac{\zeta(t)}{w(t_2)} + C.$$

From (4.5), we conclude that for $t_1 \leq t \leq t_2$,

$$t_2 - t \leq \Big(\frac{2}{n-2} + Ce^{-at_1}\Big) \ln \frac{w(t)}{w(t_2)} + C,$$

that is (4.1). With a similar argument, (4.2) follows. Hence, the upper bounds in (3.12) and (3.13) follow. ■

We prove now the Claim.

Proof of the Claim. Consider

$$\zeta(t) := B \cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})),$$

where $\alpha > 0$ will be fixed later, $cB^{\frac{(4+2\tau)}{n-2}} = 1$, and $\bar{t} \geq t_2$ is such that $\zeta(t_2) = w(t_2)$, and $\cosh^{-2}(\alpha(t-\bar{t})) \leq \varepsilon_0$ for $\varepsilon_0 > 0$ small. A direct calculation yields

$$\zeta_t(t) = \frac{B\alpha(2-n)}{2+\tau} \cosh^{\frac{-\tau-n}{2+\tau}}(\alpha(t-\bar{t})) \sinh(\alpha(t-\bar{t})), \tag{4.6}$$

and

$$\zeta_{tt}(t) = \frac{B(2-n)^2}{(2+\tau)^2}\,\alpha^2 \cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})) + \frac{B(2-n)(n+\tau)}{(2+\tau)^2}\,\alpha^2 \cosh^{\frac{-2-n-2\tau}{2+\tau}}(\alpha(t-\bar{t})).$$

Hence, we have

$$\begin{aligned}
\zeta_{tt}(t) - &\Big(\frac{(n-2)^2}{4} - be^{-at}\Big)\zeta(t) + \frac{(n+\tau)(n-2)c}{(2+\tau)^2}\,\zeta^{\frac{n+2+2\tau}{n-2}}(t) \\
&= \frac{(n-2)^2 B}{4}\Big[\frac{-4(n+\tau)}{(2+\tau)^2(n-2)}\,\frac{\alpha^2-1}{\cosh^2(\alpha(t-\bar{t}))}\Big]\cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})) \\
&\quad + \frac{(n-2)^2 B}{4}\Big[\frac{4}{(2+\tau)^2}\,\alpha^2 - 1 + \frac{4}{(n-2)^2}\,be^{-at}\Big]\cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})) \\
&\le \frac{(n-2)^2 B}{4}\Big[\frac{4(n+\tau)}{(2+\tau)^2(n-2)}\,(1-\alpha^2)\varepsilon_0\Big]\cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})) \\
&\quad + \frac{(n-2)^2 B}{4}\Big[\frac{4}{(2+\tau)^2}\,\alpha^2 - 1 + \frac{4}{(n-2)^2}\,be^{-at}\Big]\cosh^{\frac{2-n}{2+\tau}}(\alpha(t-\bar{t})).
\end{aligned}$$

By a direct calculation, we get that

$$\begin{aligned}
\frac{4(n+\tau)}{(2+\tau)^2(n-2)}\,(1-\alpha^2)\varepsilon_0 &+ \frac{4}{(2+\tau)^2}\,\alpha^2 - 1 + \frac{4}{(n-2)^2}\,be^{-at} \\
&\le \frac{12}{(2+\tau)^2}\,(1-\alpha^2)\varepsilon_0 + \frac{4}{(2+\tau)^2}\,\alpha^2 - 1 + \frac{4}{(n-2)^2}\,be^{-at}.
\end{aligned}$$

Let $\alpha = \sqrt{be^{-at}}$. Then, for t^* sufficiently large and $\varepsilon_0 > 0$ sufficiently small, it follows that

$$\frac{12}{(2+\tau)^2}\,(1-\alpha^2)\varepsilon_0 + \frac{4}{(2+\tau)^2}\,\alpha^2 - 1 + \frac{4}{(n-2)^2}\,be^{-at} \le 0$$

holds for $t^* \le t_1 \le t \le t_2$. Hence,

$$\zeta_{tt}(t) - \Big(\frac{(n-2)^2}{4} - be^{-at}\Big)\zeta(t) + \frac{(n+\tau)(n-2)c}{(n+\tau)^2}\,\zeta(t)^{\frac{n+2+2\tau}{n-2}} \le 0, \tag{4.7}$$

that is (4.3).

Next, we shall prove that ζ satisfies (4.4). Indeed, by (4.6) and for $t_1 \le t \le t_2 \le \bar{t}$,

$$\zeta_t(t) \ge 0. \tag{4.8}$$

Combining this with (4.7), we get that

$$\zeta_{tt}(t) - \Big(\frac{(n-2)^2}{4} - be^{-at_2}\Big)\zeta(t) + \frac{(n+\tau)(n-2)c}{(n+\tau)^2}\,\zeta(t)^{\frac{n+2+2\tau}{n-2}} \le 0.$$

Using (4.8), it is easy to see that

$$\zeta_{tt}(t)\,\zeta_t(t) - \Big(\frac{(n-2)^2}{4} - be^{-at_2}\Big)\zeta(t)\zeta_t(t) + \frac{(n+\tau)(n-2)c}{(2+\tau)^2}\,\zeta(t)^{\frac{n+2+2\tau}{n-2}}\,\zeta_t(t) \le 0,$$

which implies that

$$\frac{d}{dt}\Big[\zeta_t^2(t)-\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)zeta^2(t)+\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}\Big]\le 0.$$

Hence, we conclude that the function

$$\zeta_t^2(t)-\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\zeta^2(t)+\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}$$

is nonincreasing for $t_1\le t\le t_2$. Then

$$\begin{aligned}\zeta_t^2(t)-\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\zeta^2(t)+\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}&\\ \ge \zeta_t^2(t_2)-\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\zeta^2(t_2)+\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}&\\ \ge -\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\zeta^2(t_2)+\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}.&\end{aligned}$$

By (4.8), we conclude that

$$-\frac{1}{\zeta_t(t)}\le\frac{1}{\sqrt{g(\zeta(t))-g(\zeta(t_2))}},$$

with

$$g(\zeta(t))=\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\zeta(t)^2-\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta(t)^{\frac{2(n+\tau)}{n-2}}.$$

Integrating the inequality above,

$$t_2-t\le\int_{\zeta(t_2)}^{\zeta(t)}\frac{d\zeta}{\sqrt{g(\zeta(t))-g(\zeta(t_2))}}. \tag{4.9}$$

By scaling, $\eta=\zeta(t)/\zeta(t_2)$,

$$\int_{\zeta(t_2)}^{\zeta(t)}\frac{d\zeta}{\sqrt{g(\zeta(t))-g(\zeta(t_2))}}=\int_1^{\zeta(t)/\zeta(t_2)}\frac{\sqrt{\eta^2-1}}{\sqrt{\bar g(\eta)-\bar g(1)}}\,\frac{d\eta}{\sqrt{\eta^2-1}},$$

with

$$\bar g(\eta)=\Big(\frac{(n-2)^2}{4}-be^{-at_2}\Big)\eta^2-\frac{(n-2)^2c}{(2+\tau)^2}\,\zeta^{\frac{4+2\tau}{n-2}}(t_2)\,\eta^{\frac{2(n+\tau)}{n-2}}(t).$$

Since

$$1\le\eta\le\frac{\zeta(t)}{\zeta(t_2)}\le\frac{1}{\zeta(t_2)}\,B\varepsilon_0^{\frac{(n-2)}{2(2+\tau)}},$$

we have

$$\frac{\zeta^{\frac{4+2\tau}{n-2}}(t_2)\,(\eta^{\frac{2(n+\tau)}{n-2}}(t)-1)}{\eta^2(t)-1}\le C_1\,\zeta(t_2)^{\frac{4+2\tau}{n-2}}\,\eta^{\frac{4+2\tau}{n-2}}(t)\le C\varepsilon_0.$$

Then, for t^* sufficiently large and $\varepsilon_0 > 0$ sufficiently small, we have

$$\frac{\sqrt{\eta^2 - 1}}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} \leq \frac{2}{n-2} + C_1 e^{-at_2} + C\varepsilon_0 \leq \frac{2}{n-2} + C_1 e^{-at_1} + C\varepsilon_0.$$

Finally, since

$$\int_1^{\zeta(t)/\zeta(t_2)} \frac{d\eta}{\sqrt{\eta^2 - 1}} \leq C + \ln \frac{\zeta(t)}{\zeta(t_2)},$$

we have

$$\int_{\zeta(t_2)}^{\zeta(t)} \frac{d\zeta}{\sqrt{g(\zeta(t)) - g(\zeta(t_2))}} \leq \Big(\frac{2}{n-2} + Ce^{-at_1}\Big) \ln \frac{\zeta(t)}{\zeta(t_2)} + C.$$

Together with (4.9), we obtain (4.4). ∎

5. The asymptotic radial symmetry

The asymptotically radial symmetric expression (1.9) will be proved in this section. In detail, using the result in Theorem 1.1 that there exist two positive constants C_1 and C_2, depending on n, τ and $\hat{C}$, such that

$$C_1 |x|^{-(n-2)/2} \leq u(x) \leq C_2 |x|^{-(n-2)/2}, \quad |\nabla u(x)| \leq C|x|^{-n/2} \quad \text{as } x \to 0, \tag{5.1}$$

we are able to show that

$$-|x|^{-\tau} u^{-\frac{n+2+2\tau}{n-2}} \Delta u = 1 + O(|x|^a) \quad \text{as } x \to 0.$$

Then, motivated by the ideas of Han–Li–Teixeira [13] and Taliaferro–Zhang [34], and using a remarkable change of variables due to Mazzeo–Pollack–Uhlenbeck [27], we can get a differential inequality whose zero order term has a negative coefficient. Hence the maximum principle can be used. Then we are able to show that the solution of (1.4) is asymptotically radially symmetric.

Theorem 5.1. *Let $-2 < \tau < 0$, and suppose that $u \in C^2(B_1 \setminus \{0\})$ is a positive solution of (1.4) with $\hat{g}$ satisfying (1.5), and that 0 is a non-removable singularity. Then there exists $\alpha \in (0,1)$ such that*

$$u(x) = u_\tau(|x|)(1 + O(|x|^\alpha)) \quad \text{as } x \to 0, \tag{5.2}$$

where $u_\tau(|x|)$ is a Fowler-type solution.

First of all, we denote $|x| = r$, $t = -\log r$, $\theta = x/|x|$, and define

$$v(t,\theta) := |x|^{(n-2)/2} u(x) \quad \text{and} \quad K(t,\theta) := -|x|^{-\tau} u^{-\frac{n+2+2\tau}{n-2}} \Delta u.$$

Then

$$v_{tt} + \Delta_\theta v - \frac{(n-2)^2}{4} v + K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } (0,\infty) \times \mathbb{S}^{n-1}, \tag{5.3}$$

where Δ_θ is the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$. This, with (5.1), gives that

$$0 < \liminf_{t\to\infty} v(t,\theta) \le \limsup_{t\to\infty} v(t,\theta) < \infty. \tag{5.4}$$

When $x \to 0$, using (5.1) and (1.5), we get that

$$\begin{aligned} K(t,\theta) &= -|x|^{-\tau} u^{-\frac{n+2+2\tau}{n-2}} L_{\hat g} u - |x|^{-\tau} u^{-\frac{n+2+2\tau}{n-2}} (\Delta - L_{\hat g}) u \\ &= 1 + O(|x|^{(n+2)/2} \cdot |x|^{(n-2)/2-2-(n-2)/2}) = 1 + O(|x|^{(n-2)/2}). \end{aligned}$$

Thus, we have for $t \to \infty$,

$$\max_{\theta\in\mathbb{S}^{n-1}} |K(t,\theta) - 1| = O(e^{-\frac{n-2}{2}t}). \tag{5.5}$$

Multiplying (5.3) by $v_t(t,\theta)$ and then integrating over $\mathbb{S}^{n-1}$, we get for $t > 0$,

$$\frac{d}{dt} Q(t,v) = -\int_{\mathbb{S}^{n-1}} (K(t,\theta) - 1)\, v^{\frac{n+2+2\tau}{n-2}}\, v_t,$$

where

$$Q(t,v) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \Big[v_t^2(t,\theta) - |\nabla_\theta v(t,\theta)|^2 - \frac{(n-2)^2}{4} v^2(t,\theta) + \frac{n-2}{n+\tau} v^{\frac{2(n+\tau)}{n-2}}(t,\theta) \Big].$$

It follows from (5.3), (5.4) and (5.5) that v and its first order partial derivatives are bounded for t large. Thus, we conclude that

$$Q(v) = \lim_{t\to\infty} Q(t,v)$$

exists, and that for $t \to \infty$,

$$Q(t,v) = Q(v) + O(e^{-\frac{n-2}{2}t}). \tag{5.6}$$

We prove now some lemmas which will be used in the proof of Theorem 5.1.

Lemma 5.2.

$$Q(v) < 0.$$

Proof. Let $v_j(t,\theta) := v(t+t_j,\theta)$, with $t_j \to \infty$. Then (5.3) gives that

$$(v_j)_{tt} + \Delta_\theta v_j - \frac{(n-2)^2}{4} v_j + K(t+t_j,\theta)\, v_j^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } (-t_j,\infty)\times\mathbb{S}^{n-1}. \tag{5.7}$$

Therefore, it follows from (5.3), (5.4) and (5.5) that some subsequence of v_j, which we denote again by v_j, converges to v_τ in $C^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^{n-1})$, where v_τ is bounded between two positive constants, and satisfies

$$(v_\tau)_{tt} + \Delta_\theta v_\tau - \frac{(n-2)^2}{4} v_\tau + v_\tau^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } \mathbb{R}\times\mathbb{S}^{n-1}. \tag{5.8}$$

Thus by the results of [3] and [23], $v_\tau(t,\theta) = v_\tau(t)$ independent of θ, and so we have that, as $t \to \infty$,

$$Q(t, v_j) \to \frac{|\mathbb{S}^{n-1}|}{2} \Big[(v_\tau')^2(t) - \frac{(n-2)^2}{4} v_\tau^2(t) + \frac{n-2}{n+\tau} v_\tau^{\frac{2(n+\tau)}{n-2}}(t) \Big].$$

On the other hand, as $t \to \infty$,

$$Q(t, v_j) = Q(t + t_j, v) \to Q(v).$$

Hence, the uniqueness of the limit gives that

$$Q(v) = \frac{|\mathbb{S}^{n-1}|}{2} \Big[(v_\tau')^2(t) - \frac{(n-2)^2}{4} v_\tau^2(t) + \frac{n-2}{n+\tau} v_\tau^{\frac{2(n+\tau)}{n-2}}(t) \Big].$$

Arguing as for (3.4), we deduce that $Q(v) < 0$. ∎

Lemma 5.3. *As $t \to \infty$,*

$$\max_{\theta \in \mathbb{S}^{n-1}} |v(t,\theta) - \bar v(t)| \to 0, \tag{5.9}$$

$$\max_{\theta \in \mathbb{S}^{n-1}} |v_t(t,\theta) - \bar v'(t)| \to 0, \tag{5.10}$$

$$\max_{\theta \in \mathbb{S}^{n-1}} |\nabla_\theta v(t,\theta)| \to 0, \tag{5.11}$$

where $\bar v(t) = \fint_{\partial B_r} v(t,\theta)\, d\mathbb{S}_r$ is the average of $v(t,\theta)$ on ∂B_r.

Proof. Suppose that (5.9) does not hold. Then there exist $\varepsilon > 0$ and $t_j \to \infty$ such that the function $v_j(t,\theta) = v(t + t_j, \theta)$ satisfies

$$\max_{\theta \in \mathbb{S}^{n-1}} v_j(t,\theta) - \min_{\theta \in \mathbb{S}^{n-1}} v_j(t,\theta) \ge \varepsilon, \tag{5.12}$$

and

$$(v_j)_{tt} + \Delta_\theta v_j - \frac{(n-2)^2}{4} v_j + K(t + t_j, \theta)\, v_j^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } (-t_j, \infty) \times \mathbb{S}^{n-1}.$$

From (5.3), (5.4) and (5.5), it follows that some subsequence of v_j converges to v_τ in $C^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^{n-1})$, where v_τ satisfies (5.8), independently of θ, and is bounded between positive constants. This is a contradiction with (5.12). Hence, we obtain (5.9). The proofs of (5.10) and (5.11) are similar. ∎

Lemma 5.4. *As $t \to \infty$, we have*

$$\bar v_{tt} - \frac{(n-2)^2}{4} \bar v + \bar v^{\frac{n+2+2\tau}{n-2}} = o(1), \tag{5.13}$$

and

$$\frac{|\mathbb{S}^{n-1}|}{2} \Big[(\bar v')^2(t) - \frac{(n-2)^2}{4} \bar v^2(t) + \frac{n-2}{n+\tau} \bar v^{\frac{2(n+\tau)}{n-2}}(t) \Big] = Q(v) + o(1). \tag{5.14}$$

Proof. Averaging (5.3), we have

$$\bar{v}_{tt} + \overline{\Delta_\theta v} - \frac{(n-2)^2}{4}\,\bar{v} + \overline{K(t,\theta)\,v^{\frac{n+2+2\tau}{n-2}}} = 0 \quad \text{in } (0,\infty)\times\mathbb{S}^{n-1}. \tag{5.15}$$

Using the divergence theorem, we have as $t\to\infty$,

$$\begin{aligned}
&\bar{v}_{tt} - \frac{(n-2)^2}{4}\,\bar{v} + \bar{v}^{\frac{n+2+2\tau}{n-2}}\\
&= \bar{v}_{tt} - \frac{(n-2)^2}{4}\,\bar{v} + \bar{v}^{\frac{n+2+2\tau}{n-2}} - \overline{\left(v_{tt} + \Delta_\theta v - \frac{(n-2)^2}{4}\,v + K(t,\theta)\,v^{\frac{n+2+2\tau}{n-2}}\right)}\\
&= \bar{v}^{\frac{n+2+2\tau}{n-2}} - \overline{K(t,\theta)\,v^{\frac{n+2+2\tau}{n-2}}} = \bar{v}^{\frac{n+2+2\tau}{n-2}} - \overline{v^{\frac{n+2+2\tau}{n-2}}} - \overline{(K(t,\theta)-1)\,v^{\frac{n+2+2\tau}{n-2}}} = o(1),
\end{aligned}$$

where (5.5) and (5.9) were used in the last step; this is (5.13).

Since

$$\begin{aligned}
Q(t,\bar{v}) &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \left[(\bar{v}')^2(t) - \frac{(n-2)^2}{4}\,\bar{v}^2(t) + \frac{n-2}{n+\tau}\,\bar{v}^{\frac{2(n+\tau)}{n-2}}(t)\right] d\mathbb{S}_1\\
&= \frac{|\mathbb{S}^{n-1}|}{2}\left[(\bar{v}')^2(t) - \frac{(n-2)^2}{4}\,\bar{v}^2(t) + \frac{n-2}{n+\tau}\,\bar{v}^{\frac{2(n+\tau)}{n-2}}(t)\right],
\end{aligned}$$

together with (5.6) and Lemma 5.3, we have as $t\to\infty$,

$$Q(t,\bar{v}) = (Q(t,\bar{v}) - Q(t,v)) + Q(t,v) = o(1) + Q(v).$$

This finishes the proof of this lemma. ■

Proof of Theorem 5.1. It suffices to prove that there exists a C^2 positive solution $v_\tau(t)$ satisfying

$$v_\tau'' - \frac{(n-2)^2}{4}\,v_\tau + v_\tau^{\frac{n+2+2\tau}{n-2}} = 0 \quad \text{in } \mathbb{R}, \tag{5.16}$$

such that, as $t\to+\infty$ and for some $\alpha\in(0,1)$,

$$\max_{\theta\in\mathbb{S}^{n-1}} |v(t,\theta) - v_\tau(t)| = O(e^{-\alpha t}). \tag{5.17}$$

Arguing as in the proof of Lemma 5.2, we deduce the existence of v_τ satisfying (5.16).

Next, we shall show that $t\to\infty$,

$$v(t,\theta) = \bar{v}(t) + O(e^{-\alpha t}), \tag{5.18}$$

and

$$\bar{v}(t) = v_\tau(t) + O(e^{-\alpha t}), \tag{5.19}$$

which imply that (5.17) follows.

For this purpose, subtracting (5.15) from (5.3), we find that $V = v - \bar{v}$ satisfies

$$V_{tt} + \Delta_\theta V - \frac{(n-2)^2}{4} V + \widehat{V} = 0 \quad \text{in } (0,\infty) \times \mathbb{S}^{n-1}, \tag{5.20}$$

where

$$\widehat{V} = K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}} - \overline{K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}}}.$$

Multiplying (5.20) by V and integrating over $\mathbb{S}^{n-1}$, we get for $t \in (0,+\infty)$,

$$\int_{\mathbb{S}^{n-1}} \Big(V V_{tt} + V \Delta_\theta V - \frac{(n-2)^2}{4} V^2 + V \widehat{V} \Big) = 0. \tag{5.21}$$

Define

$$\omega(t) := \Big(\int_{\mathbb{S}^{n-1}} V^2(t,\theta) \Big)^{1/2}$$

Then $\omega(t)$ is nonnegative, continuous for $t > 0$, and it is C^2 on those intervals where $\omega(t)$ is positive. If $V = v - \bar{v} \equiv 0$, then (5.18) follows. If not, then $\omega(t)$ is positive. This means that

$$|\omega(t)\omega'(t)| = \Big| \int_{\mathbb{S}^{n-1}} V V_t \Big| \le \Big(\int_{\mathbb{S}^{n-1}} V_t^2 \Big)^{1/2} \omega(t),$$

which implies that

$$|\omega'(t)| \le \Big(\int_{\mathbb{S}^{n-1}} V_t^2 \Big)^{1/2},$$

and

$$\omega(t)\omega''(t) + \omega'(t)^2 = \int_{\mathbb{S}^{n-1}} V_t^2 + \int_{\mathbb{S}^{n-1}} V V_{tt} \ge \omega'(t)^2 + \int_{\mathbb{S}^{n-1}} V V_{tt}.$$

That is,

$$\omega(t)\omega''(t) \ge \int_{\mathbb{S}^{n-1}} V V_{tt}. \tag{5.22}$$

Recall that the eigenvalues of $-\Delta_\theta$ on $\mathbb{S}^{n-1}$ can be arranged as an increasing sequence $\{\lambda_k\}$:

$$\lambda_0 = 0, \quad \lambda_1 = \cdots = \lambda_n = n-1, \quad \lambda_{n+1} = 2n, \ldots$$

with $\lambda_k \to \infty$ as $k \to \infty$. Using the fact that the smallest nonzero eigenvalue is $n-1$, we conclude that for $t > 0$,

$$-(n-1) \int_{\mathbb{S}^{n-1}} V^2 \ge \int_{\mathbb{S}^{n-1}} V\, \Delta_\theta V. \tag{5.23}$$

Since $\int_{\mathbb{S}^{n-1}} V d\mathbb{S}_1 = 0$, we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} V \widehat{V} &= \int_{\mathbb{S}^{n-1}} V K v^{\frac{n+2+2\tau}{n-2}} - \overline{K v^{\frac{n+2+2\tau}{n-2}}} \int_{\mathbb{S}^{n-1}} V \\
&= \int_{\mathbb{S}^{n-1}} V K v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \int_{\mathbb{S}^{n-1}} V = \int_{\mathbb{S}^{n-1}} V \big(K v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \big) \\
&= \int_{\mathbb{S}^{n-1}} V \big(v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \big) + \int_{\mathbb{S}^{n-1}} V v^{\frac{n+2+2\tau}{n-2}} (K-1).
\end{aligned}$$

On the other hand, the mean value theorem gives that

$$\int_{\mathbb{S}^{n-1}} V\big(v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}}\big) = \frac{n+2+2\tau}{n-2}\,\xi^{\frac{4+2\tau}{n-2}} \int_{\mathbb{S}^{n-1}} V(v-\bar{v}),$$

where $\xi = \xi(t,\theta)$ is between v and $\bar{v}$, and as $t \to \infty$, $\xi = \bar{v}(1+o(1))$. It follows that

$$\int_{\mathbb{S}^{n-1}} V\big(v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}}\big) = \frac{n+2+2\tau}{n-2}\,\xi^{\frac{4+2\tau}{n-2}}(1+o(1))\,\omega^2.$$

Using the Hölder inequality, we have as $t \to \infty$,

$$\int_{\mathbb{S}^{n-1}} V v^{\frac{n+2+2\tau}{n-2}}(K-1) \le \omega(t)\Big(\int_{\mathbb{S}^{n-1}} (\bar{v}^{\frac{n+2+2\tau}{n-2}}(K-1))^2\Big)^{1/2} = \omega(t)\,O(e^{-at}),$$

where (5.4) and (5.5) are used in the last inequality. Thus we have as $t \to \infty$,

$$\int_{\mathbb{S}^{n-1}} V\hat{V} \le \frac{n+2+2\tau}{n-2}\,\xi^{\frac{4+2\tau}{n-2}}(1+o(1))\,\omega^2 + \omega(t)\,O(e^{-at}). \tag{5.24}$$

Applying (5.22), (5.23) and (5.24) to (5.21), we conclude that as $t \to \infty$,

$$\omega\omega'' - \frac{(n-2)^2}{4}\omega^2 - (n-1)\omega^2 + \frac{n+2+2\tau}{n-2}\,\xi^{\frac{4+2\tau}{n-2}}(1+o(1))\,\omega^2 + \omega(t)O(e^{-at}) \ge 0.$$

Hence, for t large and $\omega(t) > 0$, there exists a positive constant C_1 such that as $t \to \infty$,

$$\omega'' - \frac{n^2}{4}\omega + \frac{n+2+2\tau}{n-2}\,\bar{v}^{\frac{4+2\tau}{n-2}}(1+o(1))\,\omega \ge -C_1 e^{-at}. \tag{5.25}$$

Let

$$h(t) := \bar{v}^{\frac{-n-\tau}{n-2}}(t)\,\omega(t).$$

From (5.4) and (5.9), we obtain that

$$\lim_{t\to\infty} h(t) = 0,$$

and with the help of (5.13) and (5.14), applied to (5.25), we obtain that as $t \to \infty$,

$$Lh := h'' + a_1(t)h' - a_2(t)h > -C_1 e^{-at}, \tag{5.26}$$

where

$$a_1(t) = \frac{2(n+\tau)}{n-2}\,\frac{\bar{v}_t}{\bar{v}},$$

and as $t \to \infty$,

$$a_2(t) = \frac{(2+\tau)(n+\tau)}{(n-2)^2}\,\frac{1}{\bar{v}^2}\Big(\frac{-2}{|\mathbb{S}^{n-1}|}\,Q(v) + o(1)\Big).$$

By Lemma 5.2 and Lemma 5.3, we can choose positive constants a_0, b_0 and t_0 such that for $t \ge t_0$, we have $-b_0 < a_1 < b_0$ and $a_2(t) > a_0$. Choose $\alpha \in (0,1)$ such that $-\alpha^2 - \alpha b_0 + a_0 > 0$. Let t_0 large enough such that for $t \ge t_0$,

$$-\alpha^2 - \alpha b_0 + a_0 > C_1 e^{(\alpha-a)t}.$$

Define

$$\widehat{h}(t) := Me^{-\alpha t}.$$

There exists some positive constant $M > 1$ such that

$$\widehat{h}(t_0) > h(t_0).$$

A direct calculation shows that, as $t \geq t_0$,

$$\begin{aligned} L\widehat{h} &= Me^{-\alpha t}(\alpha^2 - \alpha a_1 - a_2) < Me^{-\alpha t}(\alpha^2 + \alpha b_0 - a_0) \\ &< -Me^{-\alpha t} C_1 e^{(\alpha - a)t} < -C_1 e^{-at}. \end{aligned} \tag{5.27}$$

Hence, together with (5.26) and (5.27), we have, for $t \geq t_0$,

$$L(h - \widehat{h}) > 0,$$

and

$$(h - \widehat{h})(t_0) < 0 \quad \text{and} \quad \lim_{t \to +\infty} (h - \widehat{h})(t) = 0.$$

Notice that the zero order term for the differential operator L has a negative coefficient $-a_2$. Hence, by the maximum principle, we conclude that for $t \geq t_0$,

$$(h - \widehat{h})(t) < 0.$$

It follows that $h(t) = O(e^{-\alpha t})$ as $t \to \infty$. Together with (5.4), this implies that $\omega(t) = O(e^{-\alpha t})$ as $t \to \infty$. Hence, we conclude that, as $t \to \infty$,

$$\|V\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} = O(e^{-\alpha t}).$$

Since

$$v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} = \frac{n+2+2\tau}{n-2}\, \xi^{\frac{4+2\tau}{n-2}} (v - \bar{v}),$$

where $\xi = \xi(t, \theta)$ is between v and $\bar{v}$, and bounded, it follows that for $t \to +\infty$,

$$\left\| v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \leq C \|V\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})},$$

where C is a positive constant. Hence, by Jensens's inequality we have, for $t \to \infty$,

$$\begin{aligned} \|\widehat{V}\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} &= \left\| K(t,\theta) v^{\frac{n+2+2\tau}{n-2}} - \overline{K(t,\theta) v^{\frac{n+2+2\tau}{n-2}}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\leq \left\| K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\quad + \left\| \overline{K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\leq 2 \left\| K(t,\theta)\, v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\leq 2 \left\| (K(t,\theta) - 1)\, v^{\frac{n+2+2\tau}{n-2}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\quad + 2 \left\| v^{\frac{n+2+2\tau}{n-2}} - \bar{v}^{\frac{n+2+2\tau}{n-2}} \right\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \\ &\leq C \left(e^{-\frac{n-2}{2} t} + \|V\|_{L^2((t-1,t+1)\times \mathbb{S}^{n-1})} \right). \end{aligned}$$

After a finite number of iterations of standard elliptic theory applied to (5.20), we conclude that, as $t \to \infty$,

$$\|v - \bar{v}\|_{C^1((t-1,t+1)\times\mathbb{S}^{n-1})} = \|V\|_{C^1((t-1,t+1)\times\mathbb{S}^{n-1})} = O(e^{-\alpha t}). \tag{5.28}$$

This establishes (5.18).

On the other hand, since

$$\begin{aligned} Q(t,\bar{v}) &= \frac{|\mathbb{S}^{n-1}|}{2}\left[(\bar{v}'(t))^2 - \frac{(n-2)^2}{4}\,\bar{v}^2(t) + \frac{n-2}{n+\tau}\,\bar{v}^{\frac{2(n+\tau)}{n-2}}(t)\right] \\ &= Q(t,\bar{v}) - Q(t,v) + Q(t,v), \end{aligned}$$

using (5.28), (5.11) and the argument in Lemma 5.2, we conclude that, as $t \to \infty$,

$$Q(t,\bar{v}) = O(e^{-\alpha t}) + Q(v) = O(e^{-\alpha t}) + Q(v_\tau),$$

where $v_\tau(t)$ is some solution of (5.16), and

$$Q(v_\tau) = \frac{|\mathbb{S}^{n-1}|}{2}\left[(v_\tau'(t))^2 - \frac{(n-2)^2}{4}\,v_\tau^2(t) + \frac{n-2}{n+\tau}\,v_\tau^{\frac{2(n+\tau)}{n-2}}(t)\right].$$

Hence, as $t \to \infty$,

$$(\bar{v}'(t))^2 = \frac{(n-2)^2}{4}\,(\bar{v}(t))^2 - \frac{n-2}{n+\tau}\,\bar{v}^{\frac{2(n+\tau)}{n-2}}(t) + \frac{2}{|\mathbb{S}^{n-1}|}\,Q(v_\tau) + O(e^{-\alpha t}),$$

and we obtain that the behavior of $\bar{v}$ is completely determined by the roots of the right-hand side of the above equality. From Lemma 5.2, we know that $Q(v_\tau) < 0$, so we conclude that as $t \to \infty$,

$$\bar{v}(t) = v_0(t) + O(e^{-\alpha t}). \tag{5.29}$$

Therefore, combining (5.28) with (5.29), we conclude that as $t \to \infty$,

$$\max_{\theta\in\mathbb{S}^{n-1}} |v(t,\theta) - v_\tau(t)| = O(e^{-\alpha t}),$$

that is (5.17). Hence, we complete the proof of Theorem 5.1. ∎

Proof of the second part in Theorem 1.1. Using Theorem 5.1, we conclude that if 0 is a non-removable singularity, the asymptotic radial symmetry can be deduced directly. ∎

6. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $u \in C^2(\mathbb{R}^n \setminus \overline{B}_1)$ be a positive solution of (1.1) with g satisfying (1.2). Using the Kelvin transform $v(z) = \frac{1}{|z|^{n-2}} u\,(z/|z|^2)$, we have

$$-L_{\hat{g}} v = |z|^\tau\, v^{\frac{n+2+2\tau}{n-2}} \quad \text{in } B_1 \setminus \{0\},$$

with $L_{\hat{g}} v = \Delta_{\hat{g}} v - c(n) R_{\hat{g}} v$, and for $m = 0, 1, 2$, we have

$$\sum_{i,j=1}^{n} |\nabla^m(\hat{g}_{ij}(z) - \delta_{ij})| \le \hat{C}|z|^{a-m} \quad \text{in } B_1 \setminus \{0\}.$$

If 0 is a removable singularity for v, the limit

$$A_0 := \lim_{z \to 0} v(z) = v(0) > 0$$

exists, and rescaling back to u, we conclude that ∞ is a removable singularity, and

$$u(x) \to A_0 |x|^{2-n} \quad \text{as } x \to \infty.$$

Moreover, Theorem 1.2 gives that for v, there exists a positive constant C, depending on n, τ and $\hat{C}$, such that

$$v(z) \le C|z|^{-(n-2)/2} \quad \text{and} \quad |\nabla v(z)| \le C|z|^{-n/2} \quad \text{as } z \to 0,$$

which implies

$$u(x) = A_0 |x|^{2-n} + O(|x|^{1-n}) \quad \text{as } x \to \infty.$$

Rescaling back to v, we also have

$$v(z) = A_0 + O(|z|) \quad \text{as } z \to 0,$$

that is (1.7).

If 0 is a non-removable singularity for v, Theorem 1.1 gives that there exist two positive constants C_1 and C_2 depending on n, τ and $\hat{C}$ such that

$$C_1 |z|^{-(n-2)/2} \le v(z) \le C_2 |z|^{-(n-2)/2} \quad \text{as } z \to 0.$$

Furthermore, there exists $\alpha \in (0,1)$ such that

$$v(z) = v_\tau(|z|)(1 + O(|z|^\alpha)) \quad \text{as } z \to 0,$$

with $v_\tau(|z|)$ is a Fowler-type solution. Rescaling back to u, we conclude that ∞ is a non-removable singularity for u, and we have

$$C_1 |x|^{-(n-2)/2} \le u(x) \le C_2 |x|^{-(n-2)/2} \quad \text{as } x \to \infty,$$

and

$$u(x) = u_\tau(|x|)(1 + O(|x|^{-\alpha})) \quad \text{as } x \to \infty,$$

with $u_\tau(|x|)$ is a Fowler-type solution, which is a C^2 positive radial solution of

$$-\Delta u_\tau = |x|^\tau u_\tau^{\frac{n+2+2\tau}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \qquad \blacksquare$$

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References

[1] Aubin, T.: Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)* **55** (1976), no. 3, 269–296. Zbl 0336.53033 MR 0431287

[2] Bartnik, R.: The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.* **39** (1986), no. 5, 661–693. Zbl 0598.53045 MR 0849427

[3] Caffarelli, L. A., Gidas, B. and Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42** (1989), no. 3, 271–297. Zbl 0702.35085 MR 0982351

[4] Caju, R., do Ó, J. M. and Santos, A. S.: Qualitative properties of positive singular solutions to nonlinear elliptic systems with critical exponent. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 6, 1575–1601. Zbl 1425.35039 MR 4002167

[5] Chen, C.-C. and Lin, C.-S.: Estimates of the conformal scalar curvature equation via the method of moving planes. *Comm. Pure Appl. Math.* **50** (1997), no. 10, 971–1017. Zbl 0958.35013 MR 1466584

[6] Chen, C.-C. and Lin, C.-S.: On the asymptotic symmetry of singular solutions of the scalar curvature equations. *Math. Ann.* **313** (1999), no. 2, 229–245. Zbl 0927.35034 MR 1679784

[7] Chruściel, P. T., Pacard, F. and Pollack, D.: Singular Yamabe metrics and initial data with exactly Kottler–Schwarzschild–de Sitter ends. II. Generic metrics. *Math. Res. Lett.* **16** (2009), no. 1, 157–164. Zbl 1170.83004 MR 2480569

[8] Chruściel, P. T. and Pollack, D.: Singular Yamabe metrics and initial data with exactly Kottler–Schwarzschild–de Sitter ends. *Ann. Henri Poincaré* **9** (2008), no. 4, 639–654. Zbl 1144.83004 MR 2413198

[9] Denisov, V. I. and Solov'ev, V. O.: Energy defined in general relativity on the basis of the traditional Hamiltonian approach has no physical meaning. *Teoret. Mat. Fiz.* **56** (1983), no. 2, 301–314. Zbl 0541.53024 MR 0718105

[10] Gidas, B. and Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1981), no. 4, 525–598. Zbl 0465.35003 MR 0615628

[11] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order*. Grundlehren Math. Wiss. 224, Springer, Berlin-New York, 1977. Zbl 0361.35003 MR 0473443

[12] Han, Q., Li, X. and Li, Y.: Asymptotic expansions of solutions of the Yamabe equation and the σ_k-Yamabe equation near isolated singular points. *Comm. Pure Appl. Math.* **74** (2021), no. 9, 1915–1970. Zbl 1482.35070 MR 4287691

[13] Han, Z.-C., Li, Y. and Teixeira, E. V.: Asymptotic behavior of solutions to the σ_k-Yamabe equation near isolated singularities. *Invent. Math.* **182** (2010), no. 3, 635–684. Zbl 1211.53064 MR 2737708

[14] Han, Z.-C., Xiong, J. and Zhang, L.: Asymptotic behavior of solutions to the Yamabe equation with an asymptotically flat metric. *J. Funct. Anal.* **285** (2023), no. 4, article no. 109982, 55 pp. Zbl 1523.58015 MR 4584986

[15] Jin, T. and Xiong, J.: Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **38** (2021), no. 4, 1167–1216. Zbl 1465.35396 MR 4266239

[16] Korevaar, N., Mazzeo, R., Pacard, F. and Schoen, R.: Refined asymptotics for constant scalar curvature metrics with isolated singularities. *Invent. Math.* **135** (1999), no. 2, 233–272. Zbl 0958.53032 MR 1666838

[17] Li, C.: Local asymptotic symmetry of singular solutions to nonlinear elliptic equations. *Invent. Math.* **123** (1996), no. 2, 221–231. Zbl 0849.35009 MR 1374197

[18] Li, Y.: Conformally invariant fully nonlinear elliptic equations and isolated singularities. *J. Funct. Anal.* **233** (2006), no. 2, 380–425. Zbl 1293.35112 MR 2214582

[19] Li, Y. and Nguyen, L.: A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound. *J. Funct. Anal.* **266** (2014), no. 6, 3741–3771. Zbl 1288.53030 MR 3165241

[20] Li, Y. and Nguyen, L.: A fully nonlinear version of the Yamabe problem on locally conformally flat manifolds with umbilic boundary. *Adv. Math.* **251** (2014), 87–110. Zbl 1291.53049 MR 3130336

[21] Li, Y., Wei, J. and Xu, H.: Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ on lattices in $\mathbb{R}^n$. *J. Reine Angew. Math.* **743** (2018), 163–211. Zbl 1410.35047 MR 3859272

[22] Lin, C.-S.: Estimates of the scalar curvature equation via the method of moving planes. III. *Comm. Pure Appl. Math.* **53** (2000), no. 5, 611–646. Zbl 1035.53052 MR 1737506

[23] Lions, P.-L.: Isolated singularities in semilinear problems. *J. Differential Equations* **38** (1980), no. 3, 441–450. Zbl 0458.35033 MR 0605060

[24] Marques, F. C.: A priori estimates for the Yamabe problem in the non-locally conformally flat case. *J. Differential Geom.* **71** (2005), no. 2, 315–346. Zbl 1101.53019 MR 2197144

[25] Marques, F. C.: Isolated singularities of solutions to the Yamabe equation. *Calc. Var. Partial Differential Equations* **32** (2008), no. 3, 349–371. Zbl 1143.35323 MR 2393072

[26] Mazzeo, R. and Pacard, F.: Constant scalar curvature metrics with isolated singularities. *Duke Math. J.* **99** (1999), no. 3, 353–418. Zbl 0945.53024 MR 1712628

[27] Mazzeo, R., Pollack, D. and Uhlenbeck, K.: Moduli spaces of singular Yamabe metrics. *J. Amer. Math. Soc.* **9** (1996), no. 2, 303–344. Zbl 0849.58012 MR 1356375

[28] Mazzeo, R. and Smale, N.: Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere. *J. Differential Geom.* **34** (1991), no. 3, 581–621. Zbl 0759.53029 MR 1139641

[29] Poláčik, P., Quittner, P. and Souplet, P.: Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.* **139** (2007), no. 3, 555–579. Zbl 1146.35038 MR 2350853

[30] Schoen, R. M.: Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.* **20** (1984), no. 2, 479–495. Zbl 0576.53028 MR 0788292

[31] Schoen, R. M.: The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation. *Comm. Pure Appl. Math.* **41** (1988), no. 3, 317–392. Zbl 0674.35027 MR 0929283

[32] Silva Santos, A.: A construction of constant scalar curvature manifolds with Delaunay-type ends. *Ann. Henri Poincaré* **10** (2010), no. 8, 1487–1535. Zbl 1214.53034 MR 2639545

[33] Taliaferro, S. D. and Zhang, L.: Arbitrarily large solutions of the conformal scalar curvature problem at an isolated singularity. *Proc. Amer. Math. Soc.* **131** (2003), no. 9, 2895–2902. Zbl 1125.35349 MR 1974347

[34] Taliaferro, S. D. and Zhang, L.: Asymptotic symmetries for conformal scalar curvature equations with singularity. *Calc. Var. Partial Differential Equations* **26** (2006), no. 4, 401–428. Zbl 1175.35055 MR 2235880

[35] Trudinger, N. S.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **22** (1968), 265–274. Zbl 0159.23801 MR 0240748

[36] Xiong, J.: The critical semilinear elliptic equation with isolated boundary singularities. *J. Differential Equations* **263** (2017), no. 3, 1907–1930. Zbl 1368.35127 MR 3634700

[37] Xiong, J. and Zhang, L.: Isolated singularities of solutions to the Yamabe equation in dimension 6. *Int. Math. Res. Not. IMRN* (2022), no. 12, 9571–9597. Zbl 1491.35243 MR 4436213

[38] Yamabe, H.: On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.* **12** (1960), 21–37. Zbl 0096.37201 MR 0125546

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