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Coregular submanifolds and Poisson submersions

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Abstract. In this paper, we analyze *submersions with Poisson fibres*. These are submersions whose total space carries a Poisson structure, on which the ambient Poisson structure pulls back, as a Dirac structure, to Poisson structures on each individual fibre. Our "Poisson–Dirac viewpoint" is prompted by natural examples of Poisson submersions with Poisson fibres – in toric geometry and in Poisson–Lie groups – whose analysis was not possible using the existing tools in the Poisson literature.

The first part of the paper studies the Poisson–Dirac perspective of inducing Poisson structures on submanifolds. This is a rich landscape, in which subtle behaviours abound, as illustrated by a surprising "jumping phenomenon" concerning the complex relation between the induced and the ambient symplectic foliations, which we discovered here. These pathologies, however, are absent from the well-behaved and abundant class of *coregular* submanifolds, with which we are mostly concerned here.

The second part of the paper studies Poisson submersions with Poisson fibres – the natural Poisson generalization of flat symplectic bundles. These Poisson submersions have coregular Poisson–Dirac fibres, and behave functorially with respect to such submanifolds. We discuss the subtle collective behavior of the Poisson fibres of such Poisson fibrations, and explain their relation to pencils of Poisson structures.

The third and final part applies the theory developed to Poisson submersions with Poisson fibres which arise in Lie theory. We also show that such submersions are a convenient setting for the associated bundle construction, and we illustrate this by producing new Poisson structures with a finite number of symplectic leaves.

Some of the points in the paper being fairly new, we illustrate the many fine issues that appear with an abundance of (counter-)examples.

1. Introduction

The condition that a submanifold of a Poisson manifold "inherit" a Poisson structure is somewhat subtle, since bivectors cannot be pulled back as such. However, we understand since the work of Ted Courant on Dirac structures [10] that there is a *canonical candidate* to an "induced Poisson structure". Indeed, a Poisson structure π_M on M may be regarded as a Dirac structure via its graph:

$$\mathrm{Gr}(\pi_M) = \{\pi_M^\sharp(\xi) + \xi \mid \xi \in T^*M\} \subset TM \oplus T^*M =: \mathbb{T}M.$$

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It turns out that every submanifold X inherits a canonical (pullback) Lagrangian family $i^!\,\mathrm{Gr}(\pi_M) \subset \mathbb{T}X$ (see Section 2 for more details), and *at most* one bivector $\pi_X \in \mathfrak{X}^2(X)$ can exist on X for which

$$\mathrm{Gr}(\pi_X) = i^!\,\mathrm{Gr}(\pi_M).$$

When that is the case, π_X is automatically Poisson, and we say that (X, π_X) is a *Poisson–Dirac submanifold* of (M, π_M). This notion was first introduced in [11], Section 8.

This is the most general recipe to "induce" Poisson structures on submanifolds, but it is rather difficult to check, in that the existence and smoothness of π_X are *assumed* as opposed to *deduced* from some more straightforwardly verifiable condition, and deciding whether π_X is smooth or not may be technically challenging in practice.

In the literature concerning submanifolds of a Poisson manifold, it was (mistakenly) thought that the induced symplectic foliation on a Poisson–Dirac submanifold would behave well with respect to the ambient symplectic foliation – in the sense that the leaves of the Poisson–Dirac submanifold would arise as (clean) intersections of the submanifold with the ambient leaves. For example, Poisson–Dirac submanifolds and their clean (or split) counterparts are confused in:

- the foundational paper [11] (cf. Definition 4 and Corollary 10 in [11]);
- in [39] (cf. Definitions 5 and 6 in [39]);
- in [3] (see Lemma A.1 in [3]);
- in [2] (see Theorem 2.1 in [2]).

It turns out, however, that the symplectic foliation of the Poisson–Dirac submanifold can be *wildly* different from that of the ambient manifold. In fact, the first contribution in this note is the discovery of a

> **Jumping phenomenon:** A leaf of a Poisson–Dirac submanifold need not lie inside a leaf of the ambient manifold;

see Examples 2.7 and 2.8.

One example of a condition on a submanifold X of a Poisson manifold (M, π_M), which turns out to be easy to check, and ensures that X has the structure of clean Poisson–Dirac submanifold, is to require that X be *split*; that is, that there exists a splitting $TM|_X = TX \oplus E$ with $\pi_M|_X = \pi_X$ modulo $\Gamma(\wedge^2 E)$. This condition turns out to be equivalent to demanding that Hamiltonian flows of (X, π_X) be the restriction of Hamiltonian flows of (M, π_M) – otherwise said, split manifolds are those Poisson–Dirac submanifolds in which the jumping phenomenon alluded to above is ruled out by fiat.

Coregular Poisson–Dirac submanifolds, first introduced by Courant (Theorem 3.2.1 in [10]), are a particular case of split submanifolds. A submanifold X is a coregular Poisson–Dirac submanifold if TX and the image $\pi_M^\sharp(N^*X)$ of the conormal bundle of X meet trivially, and their direct sum $TX \oplus \pi_M^\sharp(N^*X)$ is a vector subbundle of $TM|_X$. Said otherwise, these are split Poisson–Dirac submanifolds, in which the projection to the normal bundle of X

$$\mathrm{Q}_X : N^*X \longrightarrow NX, \quad \mathrm{Q}_X(\xi) = \pi_M^\sharp(\xi) + TX$$

has constant rank. Coregular submanifolds comprise, among others:

- *Poisson submanifolds*: submanifolds X to which every Hamiltonian vector field of (M, π_M) is tangent, that is,

$$\pi_M^\sharp(T^*M|_X) \subset TX.$$

- *Poisson transversals*: submanifolds X which meet each leaf of (M, π_M) transversally and symplectically, that is,

$$TM|_X = TX \oplus \pi_M^\sharp(N^*X).$$

- Any *point* in a Poisson manifold.

We devote Section 2 to a detailed description of this whole hierarchy of regularity conditions, with a heavy emphasis on (counter-)examples. This complements (and in some cases, corrects) the endeavours in [11,15,28,38], and to some extent those in [8,39] (which discuss distinguished submanifolds of Poisson manifolds which do not necessarily inherit Poisson structures).

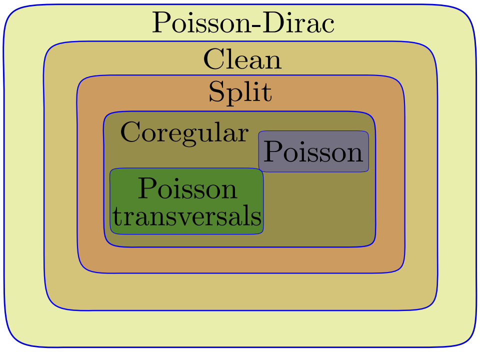

Figure 1. Hierarchy of induced Poisson structures.

The main object of interest in the present paper are submersions

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M)$$

between Poisson manifolds. The natural "compatibility" condition one can impose is:

(a) that p be a Poisson map:

$$\{f \circ p, g \circ p\}_\Sigma = \{f, g\}_M \circ p, \quad f, g \in C^\infty(M).$$

This is the correct Poisson-theoretic notion to ensure that p intertwines Hamiltonian flows of f and $f \circ p$. Note, however, that this does not imply that p intertwines the induced symplectic foliations on the total space and base – in the sense that the preimage of leaves is saturated. For example, symplectic realizations exist for any Poisson manifold [23,37], and those only intertwine symplectic foliations in this sense if (M, π_M) is symplectic.

In light of the discussion on submanifolds, another natural "compatibility" condition one can impose is:

(b) that fibres of p have an induced Poisson structure.

Definition. A submersion $p\colon (\Sigma, \pi_\Sigma) \to M$ from a Poisson manifold (Σ, π_Σ) has *Poisson fibres* if each of its fibres is a Poisson–Dirac submanifold.

Submersions with Poisson fibres encompass previously studied classes (such as vertical, coupling [4, 35, 36] and almost-coupling Poisson structures [4, 34]). All these cases assume a good collective behaviour of the Poisson–Dirac fibres – local triviality or, at the very least, the existence of a compatible Ehresmann connection. Our Poisson–Dirac standpoint makes no such assumptions.

Of great importance to what follows is that *Poisson submersions* with Poisson fibres

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M)$$

have in fact coregular fibres, and behave functorially in the following sense.

Theorem. *A Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ has Poisson fibres exactly when each vertical tangent space inherits a Poisson structure. In that case, its fibres are coregular.*

Moreover, for a coregular Poisson–Dirac submanifold $Y \subset M$,

(a) $X := p^{-1}(Y) \subset \Sigma$ *is a coregular Poisson–Dirac submanifold*;

(b) $p\colon (X, \pi_X) \to (Y, \pi_Y)$ *is a Poisson submersion with Poisson fibres.*

Such good functorial behavior of Poisson submersion with Poisson fibres, as opposed to general Poisson submersions, has implications regarding the symplectic foliations of the total space and base.

Theorem. *A Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ pulls symplectic leaves of M back to Poisson submanifolds of Σ, over which p restricts to coupling Poisson submersions.*

These results convey a clear picture of a Poisson submersion with Poisson fibres

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M);$$

it maps leaves of π_Σ into leaves of π_M, in which case the ensuing restrictions between symplectic leaves

$$p : \mathrm{S}_\Sigma(x) \longrightarrow \mathrm{S}_M(p(x))$$

are *flat symplectic bundles*. So, very much like a Poisson structure on a manifold makes precise the idea of a assembling symplectic manifolds, a Poisson submersion with Poisson fibres makes precise the idea of assembling flat symplectic bundles.

In fact, the good behavior with respect to the symplectic foliations (or even with respect to their distributions) *characterizes* Poisson submersion with Poisson fibres among Poisson submersions.

Theorem. *The following statements for a Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ are equivalent.*

(a) *It is a Poisson submersion with Poisson fibres.*

(b) *It maps symplectic leaves of (Σ, π_Σ) into symplectic leaves of (M, π_M).*

(c) *Its differential maps the characteristic distribution of (Σ, π_Σ) into the characteristic distribution of (M, π_M).*

Poisson submersions with Poisson fibres have their own version of the jumping phenomenon. A Poisson submersion with Poisson fibres over a symplectic manifold (that is, a coupling) induces, under a mild completeness assumption, *diffeomorphic* Poisson structures on its fibres. However:

> **Jumping phenomenon:** If the base manifold of Poisson submersions with Poisson fibres is not symplectic, the Poisson diffeomorphism type of fibres may very well *change* as we change symplectic leaves in the base. Remarkably, they need not even assemble into a vertical Poisson structure.

In fact, for a Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$, we have the following.

Theorem. *The Poisson fibres assemble into a vertical Poisson structure $\pi_V \in \Gamma(\wedge^2 V)$ exactly when a Poisson structure $\pi_H \in \mathfrak{X}^2(\Sigma)$ exists, whose symplectic foliation arises from Hamiltonian flows of functions on M. When that is the case, π_V and π_H commute and split π_Σ:*

$$\pi_\Sigma = \pi_V + \pi_H, \quad [\pi_H, \pi_V] = 0.$$

We refer to such objects as *orthogonal pencils*, to highlight the somewhat surprising fact that the Poisson structure on the total space arises from a pencil of Poisson structures.

Yet again, the landscape is fraught with subtleties (e.g., orthogonal pencils need not be almost-coupling), and we devote the bulk of the Section 3 to lay out the foundational aspects of the theory of Poisson submersions with Poisson fibres, striving to offer as many examples as a clear picture requires. Just as for submanifolds, our analysis yields a whole hierarchy of Poisson submersions.

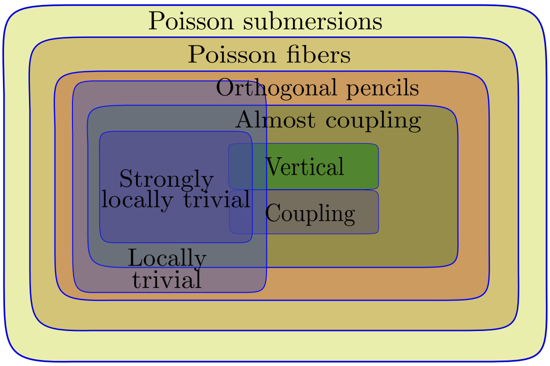

Figure 2. Hierarchy of Poisson submersions.

The remainder of the paper is devoted to applications and examples of Poisson submersions with Poisson fibres coming from Lie theory. The pointwise-to-global feature of coregular manifolds manifests itself in the following key result.

Lemma. *Let a Poisson–Lie group (G, π_G) have a Poisson action on a Poisson manifold (M, π_M). Then an orbit X of $G \curvearrowright M$ is a coregular Poisson–Dirac submanifold if and only if $T_x X \subset (T_x M, \pi_{M,x})$ has an induced Poisson structure for some $x \in X$.*

Applied to the action of a complex vector space A on a complex manifold M by holomorphic transformations, there corresponds to each *positive* bivector $\pi_A \in \wedge^2 A$ (that is, one whose leaves are Kähler manifolds) an induced A-invariant Poisson structure π_M on M, such that every complex subspace of A acts on (M, π_M) by Poisson diffeomorphisms, and has coregular Poisson–Dirac orbits (Lemma 4.5). One instance is the GIT presentation of a toric manifold M_Δ, a certain principal bundle $p\colon \Sigma_\Delta \to M_\Delta$ with structure group a complex torus, and total space Σ_Δ an open set in $\mathbb{C}^d$, constructed out of a Delzant polytope Δ.

Proposition. *Every positive bivector $\pi \in \wedge^2\mathbb{C}^d$ turns the GIT presentation $p\colon \Sigma_\Delta \to M_\Delta$ of the toric variety M_Δ into a Poisson submersion with Poisson fibres.*

Following [6], we call those Poisson structures which arise from nondegenerate, positive bivectors (and which therefore have finitely many leaves) *toric Poisson manifolds*.

The second class of examples concerns the "standard" (or Lu–Weinstein) Poisson–Lie group structure π_G on a compact connected semisimple Lie group G associated with a choice of maximal torus T (and root ordering). This structure (discovered in [27]) descends to a "standard" Poisson structure π_M on the manifold of full flags $M = G/T$, with finitely many leaves, the Bruhat cells.

Proposition. *The quotient map $p\colon (G, \pi_G) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres, inducing the trivial Poisson structure on fibres.*

We conclude the paper with a discussion of the associated bundle construction in the context of Poisson submersions. Namely, a right principal G-bundle $p\colon P \to M$ equipped with a G-invariant Poisson structure π_P determines a Poisson submersion

$$p : (P, \pi_P) \longrightarrow (M, \pi_M).$$

If G acts on the left of (X, π_X) by Poisson diffeomorphisms, the associated bundle $\Sigma := P \times_G X$ has an induced Poisson structure π_Σ, for which

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M)$$

is again a Poisson submersion. We will see that good behavior of the principal Poisson submersion $p\colon (P, \pi_P) \to (M, \pi_M)$ is inherited by the associated bundle $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ (Lemma 4.11), but also that conditions can be imposed on π_P and the action $G \curvearrowright (X, \pi_X)$ to ensure that the associated bundle behaves well even if the principal bundle itself does not (Lemma 4.12).

In the study of such "Poisson associated bundles", the need arises to impose some condition akin to local triviality. When the base of the submersion is symplectic, local triviality has a transparent, mandated meaning (see [14]), but its meaning for general Poisson submersions with Poisson fibres is less clear. For our purposes, it suffices to consider three notions, corresponding to the following local models of π_Σ: the product of Poisson structures on the base and on the fibre (*strongly locally trivial*), its gauge-transform by a closed two-form (*locally trivial*), and having the same singular foliation as those (*locally trivial foliation*). The delicate issue will be that this hierarchy of local triviality notions will not always pass from principal to associated bundles. However, we have the following.

Proposition. *The associated bundle has locally trivial foliation, provided the orbits of $G \curvearrowright (X, \pi_X)$ lie inside symplectic leaves, and*

(a) *either the Poisson structures on the fibres of $p\colon P \to M$ are all trivial,*

(b) *or the orbits of $G \curvearrowright X$ are isotropic submanifolds of the symplectic leaves of (X, π_X).*

In fact, in that case the leaf spaces of (Σ, π_Σ) and $(M, \pi_M) \times (X, \pi_X)$ are homeomorphic. This leads to the last application of the paper: the construction of Poisson submersions with Poisson fibres with finitely many leaves, using as models for base/fibres toric Poisson manifolds or manifolds of full flags.

Conventions. By a *singular foliation* $\mathcal{S}$ on a smooth manifold M we mean a partition of M into initial, immersed submanifolds, in the sense of Stefan–Sussmann [32, 33]. We note that all singular foliations in this paper will be the orbit partition of a Lie algebroid [1].

A *foliation* $\mathscr{F}$, tout court, is a singular foliation by equidimensional leaves (that is, an honest foliation in the usual sense). *Submanifold* is to be understood as an embedded submanifold – but in Appendix A we consider the case of embedded submanifolds whose connected components may have different dimensions.

For a Poisson manifold (M, π_M), we denote by $\mathrm{H}_f := \pi_M^\sharp(\mathrm{d}f)$ the Hamiltonian vector field of a (possibly time-dependent) function f on M, and by $\phi^{t,s}_{\mathrm{H}_f}$ its local flow.

The symplectic leaf through $x \in M$ (the orbit through Hamiltonian local flows) is denoted by $\mathrm{S}_M(x)$, and $\mathcal{S}_M$ stands for the ensuing singular foliation of M. In all cases but one, the singular foliations considered in the paper will be of this kind (see, e.g., Section 4.1 in [12]).

For references on the basics of Poisson geometry, with a viewpoint similar to the one espoused here, we also recommend [29], [25] and [28]. For those on Dirac structures, see e.g. [5] and [20], and for Lie group structure theory, see [24].

Terminology. The terminology concerning submanifolds of Poisson manifolds is quite inconsistent, and we have adopted our own. One reason for this inconsistency was the hitherto unknown jumping phenomenon we discuss in the paper. For instance,

- "Poisson–Dirac submanifolds" in our sense are those in Proposition 6 of [11], and in Definition 6 of [39].
- What we call "clean Poisson–Dirac submanifolds" are *also* called "Poisson–Dirac submanifolds" in the same works, see Definition 4 in [11] and Definition 5 in [39] – the jumping phenomenon was missing from the picture, so both classes were mistakenly conflated.
- What we call "split Poisson–Dirac submanifolds" were called "Poisson–Dirac submanifolds with a Dirac splitting" in Proposition 7 of [11], but merely called "Poisson–Dirac submanifolds" in [25] and in [28].
- What we call "coregular submanifold" appears without name in Theorem 3.2.1 of [10], as "Poisson–Dirac submanifolds of constant rank" in [11], p. 119, and as "pre-Poisson Poisson–Dirac submanifold" in [8]. Our terminology, however, agrees with that of [12] and that of [19] in the case where the induced Dirac structure is in fact Poisson. Our choice of names seeks to accurately reflect the properties which characterize each of the classes in the hierarchy of Poisson–Dirac submanifolds.

2. Submanifolds

Recall that the *generalized tangent bundle* $\mathbb{T}M := TM \oplus T^*M$ carries the canonical symmetric pairing

$$\langle \cdot, \cdot \rangle : \mathbb{T}M \times_M \mathbb{T}M, \quad \langle u + \xi, v + \eta \rangle := \iota_v \xi + \iota_u \eta, \tag{2.1}$$

and the Dorfman bracket on the space of sections:

$$[\cdot, \cdot] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \longrightarrow \Gamma(\mathbb{T}M), \quad [u + \xi, v + \eta] := [u, v] + \mathscr{L}_u \eta - \iota_v \mathrm{d}\xi. \tag{2.2}$$

A subspace $L_M \subset \mathbb{T}M$ is a *Lagrangian family* if it meets each fibre $\mathbb{T}_x M$ in a Lagrangian vector subspace. No continuity is assumed for the map $x \mapsto L_{M,x}$; see next example.

Example 2.1. The subspace

$$L \subset \mathbb{T}\mathbb{R}, \quad L_t = \begin{cases} T_t\mathbb{R}, & \text{if } t \neq 0, \\ T_0^*\mathbb{R}, & \text{if } t = 0, \end{cases}$$

is a discontinuous Lagrangian family on the real line.

We say that a Lagrangian family is *smooth* if L_M is a (smooth) subbundle. For example, given a two-form $\omega \in \Omega^2(M)$, a subbundle $E \subset TM$ or a bivector $\pi \in \mathfrak{X}^2(M)$, the Lagrangian families

$$\mathrm{Gr}(\omega) = \{u + \iota_u \omega \,|\, u \in TM\}, \quad \mathrm{Gr}(E) = E \oplus E^\circ \quad \text{and} \quad \mathrm{Gr}(\pi) = \{\pi^\sharp(\xi) + \xi \,|\, \xi \in T^*M\}$$

are all smooth, and we refer to them as the *graphs* of ω, E and π, respectively.

A *Dirac structure* is a smooth Lagrangian family whose space of sections $\Gamma(L)$ is involutive with respect to the Dorfman bracket. Graphs corresponding to two-forms are Dirac structures if and only if the two-form is closed; similarly, the graph of a subbundle E is a Dirac structure if and only if E is the tangent bundle to a foliation, and the graph of a bivector is a Dirac structure if and only if the bivector is Poisson – equivalently, L is the graph of a bivector exactly when L meets the tangent bundle trivially:

$$L \cap TM = 0.$$

If M is equipped with a Dirac structure L_M, and $f : N \to M$ is any smooth map, there is an induced *pullback* Lagrangian family

$$f^!(L_M) := \{u + f^*(\xi) \in \mathbb{T}N \mid f_*(u) + \xi \in L_M\}.$$

Such a Lagrangian family may fail to be smooth:

Example 2.2. If L_M is the Poisson structure $\pi_M = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ on $M = \mathbb{R}^2$, and if $i : \mathbb{R} \to M$ is $i(t) = (t, 0)$, then $i^!(L_M)$ is the discontinuous Lagrangian family discussed in Example 2.1.

However, if the pullback Lagrangian family $f^!(L_N)$ happens to be smooth, then it is automatically a Dirac structure (Proposition 5.6 in [5]), which we refer to as a *pullback* Dirac structure.

In light of the discussion above, there is a canonical candidate to an *induced* Dirac structure on a submanifold X of a Dirac manifold (M, L_M):

> A Dirac structure L_M on M *induces* a Dirac structure L_X on a submanifold $i\colon X \to M$ if the Lagrangian family $i^!(L_M)$ is smooth, in which case we call $i^!(L_M)$ the *induced Dirac structure* on X.

Example 2.3. If $\mathscr{F}$ is a foliation on M, then $\mathrm{Gr}(\mathscr{F}) := T\mathscr{F} \oplus N^*\mathscr{F}$ is a Dirac structure on M. In fact, a Dirac structure L_M is of the form $\mathrm{Gr}(\mathscr{F})$ some foliation $\mathscr{F}$ on M exactly when $\mathrm{pr}_T(L_M) = \mathrm{pr}_{T^*}(L_M)^\circ$ as subsets of TM – from which it follows by lower semicontinuity of rank $\mathrm{pr}_T(L_M)$, and upper semicontinuity of rank $\mathrm{pr}_{T^*}(L_M)^\circ$, that $\mathrm{pr}_T(L_M)$ is an (involutive) subbundle of TM. Moreover, if $i\colon X \to M$ is a submanifold, then

$$i^!\,\mathrm{Gr}(\mathscr{F})_x = i_*^{-1}(T_x\mathscr{F}) \oplus i^*(N_x^*\mathscr{F}) = i^*(N_x^*\mathscr{F})^\circ \oplus i^*(N_x^*\mathscr{F})$$

for $x \in X$ shows that, if $i^!\,\mathrm{Gr}(\mathscr{F})$ is a Dirac structure on X, it must correspond to a foliation $\mathscr{F}_X$ on X. Note that $\mathscr{F}_X$ is nothing but the intersection of the ambient foliation $\mathscr{F}$ with X.

2.1. Poisson–Dirac submanifolds

Of special interest in this note is the case in which a Poisson structure induces, in the sense above, a Poisson structure on a submanifold:

Definition 2.4. A *Poisson–Dirac* submanifold $i\colon X \to M$ of a Poisson manifold (M, π_M) is one in which $i^!\,\mathrm{Gr}(\pi_M)$ is smooth (and hence Dirac), and $i^!\,\mathrm{Gr}(\pi_M)$ meets TX trivially.

That is, a Poisson–Dirac submanifold is one which inherits a Dirac structure which is Poisson.

Example 2.5. If A is a vector space, any constant bivector $\pi_A \in \wedge^2 A$ is Poisson. A linear subspace $B \subset A$ inherits a (constant) Dirac structure $i^!\mathrm{Gr}(\pi_A) \subset B \oplus B^*$, and this is Poisson exactly when

$$\pi_A^\sharp(B^\circ) \cap B = 0. \tag{2.3}$$

In that case, for each $\xi \in B^*$, the element $\pi_B^\sharp(\xi)$ is given by $\pi_A^\sharp(\widetilde{\xi})$, where $\widetilde{\xi} \in A^*$ is any element such that $\xi = \widetilde{\xi}|_B$ and $\pi_A^\sharp(\widetilde{\xi}) \in B$.

Lemma 2.6. *If (X, π_X) is a Poisson–Dirac submanifold of (M, π_M), and*

$$\mathrm{S}_X(x) \subset X \quad \textit{and} \quad \mathrm{S}_M(x) \subset M,$$

denote, respectively, the symplectic leaves of (X, π_X) and of (M, π_M) passing through $x \in X$, then

(a) $T_x\mathrm{S}_X(x) = T_xX \cap T_x\mathrm{S}_M(x)$;

(b) $\mathrm{S}_X(x)$ *contains every Hamiltonian curve of (M, π_M) which starts at x and stays inside X.*

Proof. The tangent space $T_x \mathrm{S}_X(x)$ is the image of $\pi_X^\sharp : T_x^* X \to T_x X$, which, by definition of π_X, consists of those $\pi_M^\sharp(\xi)$ which are tangent to X at x, and this proves (a). This in turn implies that if $f \in C^\infty(I \times M)$ is a function whose Hamiltonian flow $\phi_{\mathrm{H}_f}^{t,0}$ is such that $\phi_{\mathrm{H}_f}^{t,0}(x) \in X$, then $f_X := (\mathrm{id}, i)^*(f) \in C^\infty(I \times X)$ satisfies $\phi_{\mathrm{H}_{f_X}}^{t,0}(x) = \phi_{\mathrm{H}_f}^{t,0}(x)$, which proves (b). ■

There is an important subtlety concerning Poisson–Dirac submanifolds. Namely: if X inherits a Poisson structure π_X from (M, π_M), there exist two induced partitions of X, one by the leaves $\mathrm{S}_X(x)$ of π_X, and the other by the subsets $X \cap \mathrm{S}_M(x)$. In general, however, the latter partition can be wildly misbehaved, as the next example illustrates.

Example 2.7. *An example of a Poisson–Dirac submanifold in which the partition $X \cap \mathrm{S}_M$ is not by smooth manifolds.* Let i be the embedding

$$i : \mathbb{R}^2 \longrightarrow \Big(\mathbb{R}^4, \pi = \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial x_2} + x_3 \tfrac{\partial}{\partial x_3} \wedge \tfrac{\partial}{\partial x_4}\Big), \quad i(x, y) = (x, y, f(x, y)^2, f(x, y)^2),$$

where $f \in C^\infty(\mathbb{R}^2)$ is any smooth function. We claim that $i^! \mathrm{Gr}(\pi) = \mathrm{Gr}(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. Indeed, let $Z \subset \mathbb{R}^4$ denote the subset where $x_3 = 0$, and let $F := f^{-1}(0) \subset \mathbb{R}^2$ its preimage under i. Observe that, in the complement of Z, the given Poisson structure is symplectic, corresponding to the closed two-form

$$\omega = \mathrm{d}x_2 \wedge \mathrm{d}x_1 + \mathrm{d}x_4 \wedge \frac{\mathrm{d}x_3}{x_3} \in \Omega^2(\mathbb{R}^4 \setminus Z).$$

However, the pullback (as a Dirac structure) of the graph of a closed two-form coincides with the graph of the pullback form:

$$i^! \mathrm{Gr}(\omega) = \mathrm{Gr}(i^*(\omega)) = \mathrm{Gr}(\mathrm{d}x_2 \wedge \mathrm{d}x_1)|_{\mathbb{R}^2 \setminus F} = \mathrm{Gr}(\nu)|_{\mathbb{R}^2 \setminus F},$$

where $\nu = \partial x_1 \wedge \partial_{x_2} \in \mathfrak{X}^2(\mathbb{R}^2)$.

On the other hand, for $(x, y) \in F$, we have that

$$\mathrm{Gr}(\pi)_{i(x,y)} = \mathrm{Gr}(\nu)_{(x,y)} \times T^*_{(0,0)}\mathbb{R}^2,$$

whereas

$$\begin{aligned} i_{*,(x,y)} &: T_{(x,y)}\mathbb{R}^2 \longrightarrow T_{(x,y)}\mathbb{R}^2 \times T_{(0,0)}\mathbb{R}^2, \quad i_{*,(x,y)}(u) = (u, 0), \\ i^*_{(x,y)} &: T^*_{(x,y)}\mathbb{R}^2 \times T^*_{(0,0)}\mathbb{R}^2 \longrightarrow T^*_{(x,y)}\mathbb{R}^2, \quad i^*_{(x,y)}(\xi, \eta) = \xi, \end{aligned}$$

reduce to the canonical injection and projection in the first factor. Therefore,

$$\begin{aligned} i^! \mathrm{Gr}(\pi)_{(x,y)} &= \{u + i^*(\xi, \eta) \mid i_*(u) + (\xi, \eta) \in \mathrm{Gr}(\pi)_{i(x,y)}\} \\ &= \{u + \xi \mid (u, 0) + (\xi, \eta) \in \mathrm{Gr}(\pi)_{i(x,y)}\} = \{u + \xi \mid (u, 0) = \pi_{i(x,y)}(\xi, \eta)\} \\ &= \{u + \xi \mid u = \nu^\sharp_{i(x,y)}(\xi)\} = \mathrm{Gr}(\nu)_{(x,y)}. \end{aligned}$$

There are three important takeaways from this example:

(a) Any closed subset $F \subset \mathbb{R}^2$ arises as the preimage of a suitable smooth function f, see Section 2 in Chapter 2 of [21]. Therefore, the intersection of the embedding with a symplectic leaf can be very ill-behaved.

(b) Illustrating our previous point, if we take for f the smooth function

$$f(x,y) = \begin{cases} e^{-1/x^2}\big(y - \sin(\tfrac{1}{x})\big), & x \neq 0, \\ 0 & x = 0, \end{cases}$$

then $F \subset \mathbb{R}^2$ is a typical example of a connected topological space which fails to be path-connected. (So, in particular, it is *not* a manifold!)

Nevertheless, that X meets symplectic foliation in a rather pathological way does *not* prevent the existence of a Poisson–Dirac structure on X.

(c) Hamiltonian flows of the Poisson–Dirac structure need *not* lie in a single leaf of the ambient manifold. Indeed, any non-zero function with non-trivial zero locus is in fact a counterexample to the widely believed claim that the symplectic foliation of a Poisson–Dirac submanifold arises as the intersection of the ambient symplectic foliation with the submanifold. We call this a *jumping phenomenon* to highlight the counterintuitive but important fact that the Hamiltonian flow

$$t \mapsto \phi^{t,0}_{\mathrm{H}_{f_X}}(x)$$

of a function $f_X \in C^\infty(X)$ may "jump" between different leaves of the ambient manifold.

In the concrete example (b) above, something more striking occurs: because $X = i(\mathbb{R}^2)$ is *symplectic*, any two points are connected by a Hamiltonian path; however, because the set F (which i maps into a singular leaf) is connected but *not* path connected, there exist $p_0, p_1 \in F$ which cannot possibly be endpoints of a Hamiltonian path in the ambient manifold $\mathbb{R}^4$ which lies in X.

Because a general Poisson–Dirac submanifold may intersect the ambient symplectic foliation in a poorly behaved fashion, the following language is in order: given an arbitrary subset Y of a smooth manifold M, we shall say that two points y_0, y_1 in Y lie in the same *smooth path-connected component* if there exists a smooth curve $c\colon I \to M$ such that, for some $\varepsilon > 0$,

$$c|_{[0,\varepsilon]} = y_0, \quad c(I) \subset Y \quad \text{and} \quad c|_{[1-\varepsilon,1]} = y_1.$$

This is an equivalence relation on Y, and we denote by $\langle\langle Y \rangle\rangle_y$ the equivalence class of $y \in Y$. In this language, Lemma 2.6 asserts that the partition of X given by

$$\mathrm{S}'_X(x) := \langle\langle X \cap \mathrm{S}_M(x) \rangle\rangle_x \tag{2.4}$$

refines the partition S_X of X given by the symplectic leaves of the induced Poisson structure, because any smooth path inside an ambient symplectic leaf is in fact a Hamiltonian path. Even when the partition S'_X consists exclusively of smooth manifolds, it may still be a strict refinement of S_X:

Example 2.8. The embedding

$$i : \mathbb{R}^2 \longrightarrow \left(\mathbb{R}^4, \pi = \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial x_2} + x_3 \tfrac{\partial}{\partial x_3} \wedge \tfrac{\partial}{\partial x_4}\right), \quad i(x,y) = (x, y, x^3, 0),$$

meets the leaves in smooth submanifolds:

$$X \cap \mathrm{S}_M(ae_3 + be_4) = \begin{cases} \mathbb{R}_\pm \times \mathbb{R}, & \text{for } \pm a > 0, \\ \varnothing, & \text{for } a = 0 \neq b, \\ \{0\} \times \mathbb{R}, & \text{for } a = b = 0. \end{cases}$$

In this example, the induced Lagrangian family $i^!\mathrm{Gr}(\pi_M)$ is the (symplectic) Poisson structure $\mathrm{Gr}(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. This can be argued in exactly the same fashion as in Example 2.7: on the open set $i^{-1}(\mathbb{R}^4 \setminus Z)$, the family $i^!\mathrm{Gr}(\pi)$ is merely the graph of the pullback of the form which corresponds to π on $\mathbb{R}^4 \setminus Z$. Along points of $i^{-1}(Z)$, we again have that i_* and i^* are respectively the inclusion and the projection in the first factor, and so the computation along points of Z is *ipsis litteris* that of Example 2.7.

Therefore, the intersection of the Poisson–Dirac submanifold X with the symplectic foliation is a strict refinement of the partition of X by its own symplectic foliation – which consists of a single leaf.

2.2. Clean Poisson–Dirac submanifolds

In light of the discussion above, a first "regularity" condition to be imposed on Poisson–Dirac submanifolds is to require that there be no surprises when it comes to the induced singular foliation.

Definition 2.9. A Poisson–Dirac submanifold X of a Poisson manifold (M, π_M) is *clean* if the partition S_X by leaves of its induced Poisson structure π_X coincides with the partition S'_X induced by the partition S_M of M by leaves of π_M.

The adjective "clean" is justified by the following.

Lemma 2.10. *For a Poisson–Dirac submanifold X of (M, π_M), the following conditions are equivalent*:

(i) *X meets the leaves of (M, π_M) cleanly, that is, $X \cap \mathrm{S}_M(x)$ is an embedded submanifold of $\mathrm{S}_M(x)$ and $T(X \cap \mathrm{S}_M(x)) = TX \cap T\mathrm{S}_M(x)$;*

(ii) *X is a clean Poisson–Dirac submanifold.*

Proof. Recall that S'_X refines S_X, that is, each $\mathrm{S}_X(x)$ is a disjoint union

$$\mathrm{S}_X(x) = \coprod_{y \in \Upsilon(x)} \mathrm{S}'_X(y), \qquad \Upsilon(x) \subset \mathrm{S}_X(x).$$

(i) *implies* (ii). If X meets $\mathrm{S}_M(x)$ cleanly for every $x \in X$, then $\mathrm{S}'_X(x)$ (the smooth path connected component of $X \cap \mathrm{S}_M(x)$ through x) is by Lemma A.6 an initial submanifold of $\mathrm{S}_X(x)$, with

$$T_x\mathrm{S}'_X(x) = T_xX \cap T_x\mathrm{S}_M(x) = T_x\mathrm{S}_X(x).$$

Therefore, $\mathrm{S}'_X(x)$ is an open submanifold of $\mathrm{S}_X(x)$. Since the latter is connected and partitioned by S'_X, we deduce that

$$\mathrm{S}'_X(x) = \mathrm{S}_X(x)$$

for every $x \in X$. Therefore X is a clean Poisson–Dirac submanifold.

(ii) *implies* (i). By hypothesis, the smooth path connected component $\mathrm{S}'_X(x)$ of $X \cap \mathrm{S}_M(x)$ through x is the symplectic leaf $\mathrm{S}_X(x)$ of (X, π_X) through x. This implies that $X \cap \mathrm{S}_M(x)$ is a disjoint union of initial submanifolds $\mathrm{S}_X(x)$, for which according to (a) in Lemma 2.6 $T_y\mathrm{S}_X(x) = T_yX \cap T_y\mathrm{S}_M(x)$ for all $y \in \mathrm{S}_X(x)$. By Proposition A.7, this implies that X and $\mathrm{S}_M(x)$ meet cleanly. ∎

Remark 2.11. The cleanness condition for Poisson–Dirac submanifolds is reminiscent of the transversality condition for Poisson transversals. In the case of the latter, we demand that a submanifold X of a Poisson manifold (M, π_M) meet the leaves of M transversally,

$$T_x M = T_x X + T_x \mathrm{S}_M(x), \quad x \in X,$$

and that the intersections $X \cap \mathrm{S}_M(x)$ be symplectic submanifolds. It then *follows* that the connected components of $X \cap \mathrm{S}_M(x)$ are the symplectic leaves of a smooth Poisson structure on X.

The cleanness condition can be rightfully regarded as a relaxation of the transversality condition above. However, in contrast to Poisson transversals, a submanifold which meets the leaves of a Poisson structure cleanly and symplectically need *not* be Poisson–Dirac, as the next two examples show.

Example 2.12. *An example* (*see Example* 3 *in* [11]) *in which a submanifold which meets leaves cleanly and symplectically need not inherit an induced singular foliation.* On $\mathbb{C}^3$, equipped with complex coordinates (z_1, z_2, z_3), we consider the Poisson structure of constant rank corresponding to the foliation given by $\mathrm{d}z_2 = 0$ and $\mathrm{d}z_3 - z_2\,\mathrm{d}z_1 = 0$, and the pullback of the standard symplectic form $\frac{i}{2}\sum_1^3 \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ to leaves. Then the locus X of $z_3 = 0$ meets leaves cleanly and symplectically:

$$\mathrm{S}'_X(z_1, z_2, 0) = \begin{cases} \mathbb{C} \times \{(0,0)\} & \text{if } z_2 = 0, \\ \{(z_1, z_2, 0)\} & \text{if } z_2 \neq 0. \end{cases}$$

This shows that the partition S'_X cannot even arise from a singular foliation (for it is not lower-semicontinuous).

Example 2.13. *An example in which the singular foliation induced on a submanifold which meets leaves cleanly and symplectically need not come from a Poisson structure.* Let $X \subset \mathbb{R}^2$ be the open unit disk, and let $M := X \times X \subset \mathbb{R}^4$ be endowed with the Poisson structure

$$\pi_M = (x_1^2 + x_2^2 + x_1)\,\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (x_3^2 + x_4^2 - x_3)\,\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}.$$

This is the product of two dimensional Poisson structures which vanish in the circles of radius $1/2$ and center $(-1/2, 0)$ and $(1/2, 0)$, respectively.

The image of the origin under the diagonal embedding

$$i : X \longrightarrow M, \quad i(t,s) = (t,s,t,s),$$

is the leaf of π_M which consists of the origin alone, so the intersection of X with M is clean at that point. The image of a point (t,s) which satisfies $t^2 + s^2 \pm t = 0$ intersects transversely a two-dimensional ambient symplectic leaf. The complement of the union of the two circles $X_0 = X \setminus \{t^2 + s^2 \pm t = 0\}$ embeds in the four open symplectic leaves of π_M, and the induced Lagrangian family is

$$i^!\,\mathrm{Gr}(\pi_M)|_{X_0} = \mathrm{Gr}(\pi_{X_0}), \quad \pi_{X_0} = \frac{(t^2+s^2)^2 - t^2}{2t^2 + 2s^2}\,\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s}.$$

Observe that the above formula extends to the intersection of the image of X with the 2-dimensional leaves of M. Therefore X intersects the symplectic leaves of M cleanly in symplectic submanifolds, and these submanifolds fit into a foliation of X. However, the leafwise symplectic forms do not come from a smooth Poisson tensor on X, because

$$\lim_{t\to 0}\lim_{s\to 0}\mathrm{Gr}(\pi_{X_0}) = \langle \tfrac{\partial}{\partial s} - 2\,\mathrm{d}t, \tfrac{\partial}{\partial t} + 2\,\mathrm{d}s\rangle \neq \langle \mathrm{d}t, \mathrm{d}s\rangle = \lim_{s\to 0}\lim_{t\to 0}\mathrm{Gr}(\pi_{X_0})$$

shows that π_{X_0} does not even extend to a continuous bivector on X.

2.3. Split Poisson–Dirac submanifolds

Among all Poisson structures π_X on a Poisson–Dirac submanifold X of a Poisson manifold (M, π_M), the induced one is characterized by the property that Hamiltonian *vectors* of X are also Hamiltonian vectors for M. That is, for each $f \in C_c^\infty(X)$ and $x \in X$, there is an extension $\widetilde{f} \in C_c^\infty(M)$ of f such that

$$\mathrm{H}_f(x) = \mathrm{H}_{\widetilde{f}}(x).$$

This Poisson–Dirac submanifold is clean exactly when Hamiltonian *curves* in X are also Hamiltonian curves in M. That is, for each $f \in C_c^\infty(X \times I)$ and $x \in X$, there is an extension $\widetilde{f} \in C_c^\infty(M \times I)$ of f, such that

$$\phi^{t,0}_{\mathrm{H}_f}(x) = \phi^{t,0}_{\mathrm{H}_{\widetilde{f}}}(x).$$

This leads to the next step in our hierarchy of good behavior, in which we require that every Hamiltonian *flow* of X be the restriction of a Hamiltonian flow of M, as ensured by the following definition.

Definition 2.14. A Poisson–Dirac submanifold X of a Poisson manifold (M, π_M) is *split* if its Hamiltonian vector fields are the restriction of Hamiltonian vector fields of M.

In contrast to the clean condition, there is a more convenient formulation of Definition 2.14 involving a splitting condition: we say that an *orthogonal splitting* of a Poisson manifold (M, π_M) along a submanifold X is a splitting $TM|_X = TX \oplus E$ in which

$$\pi_M|_X = \pi_X + \pi_E, \quad \pi_X \in \Gamma(\wedge^2 TX) \quad \text{and} \quad \pi_E \in \Gamma(\wedge^2 E),$$

in which case it follows that $i^!\mathrm{Gr}(\pi_M) = \mathrm{Gr}(\pi_X)$, and so π_X is the Poisson structure induced on X by (M, π_M).[1]

Lemma 2.15. *The submanifolds along which a Poisson manifold has an orthogonal splitting are exactly the split Poisson–Dirac submanifolds.*

Proof. Let X be a submanifold of a Poisson manifold (M, π_M). If $E \subset TM|_X$ is an orthogonal splitting along X, then $E \times I \subset T(M \times I)|_{X\times I}$ is an orthogonal splitting

[1]These should not be confused with the Lie–Dirac submanifolds discussed in Section 8.3 of [11], and introduced in [38] under the name of “Dirac submanifolds".

for $(\pi_M, 0) \in \mathfrak{X}^2(M \times I)$ along $X \times I$. Let $f \in C_c^\infty(X \times I)$ be a smooth function, and consider $\alpha \in \Gamma(T^*(M \times I)|_{X\times I})$ defined by

$$\alpha|_{T(X\times I)} = \mathrm{d}f, \quad \alpha|_{E\times I} = 0.$$

Then $\alpha = \mathrm{d}\widetilde{f}|_{X\times I}$ for some function $\widetilde{f} \in C_c^\infty(M \times I)$, and the Hamiltonian flow of f is by construction the restriction to X of the Hamiltonian flow of $\widetilde{f}$.

Conversely, suppose X is a split Poisson–Dirac submanifold. Then an orthogonal splitting along X may be constructed as $E := \sigma(T^*X)^\circ$, where $\sigma: T^*X \to T^*M$ is any linear map satisfying

$$\sigma(\xi)|_X = \xi, \quad \pi_X^\sharp(\xi) = \pi_M^\sharp \sigma(\xi),$$

for all $\xi \in T^*X$. Because these conditions are convex, it suffices to show that a such linear map exists in an open neighborhood $U \subset X$ of a point $x \in X$. But if $x_1, \dots, x_n \in C_c^\infty(X)$ define a coordinate chart on U, and $\widetilde{x}_1, \dots, \widetilde{x}_n \in C_c^\infty(M)$ are the extensions granted by the split Poisson–Dirac condition, the linear map

$$T^*U \longrightarrow T^*M|_U, \quad \mathrm{d}x_i \mapsto \mathrm{d}\widetilde{x}_i,$$

does the job. ∎

Example 2.16. *An example* (*see Example* 6 *in* [11]) *of a clean Poisson–Dirac submanifold which is not split.* Consider the embedding $i: \mathbb{R}^2 \to (M, \pi_M)$, where $M = \mathbb{R}^4$ and

$$\pi_M = x_1^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}, \quad i(t,s) = (t^2, 0, t, s).$$

Then $i^!\mathrm{Gr}(\pi_M) = \mathrm{Gr}(t \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2})$ shows that i maps leaves into leaves, and so its image is a clean Poisson–Dirac submanifold. However, it is not split, since the Hamiltonian vector field of $s \in C^\infty(\mathbb{R}^2)$ is not i-related to any Hamiltonian vector field of M:

$$\mathrm{H}_s = -t \frac{\partial}{\partial t} \sim v = -2x_1 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_3} = -2x_1 \frac{\partial}{\partial x_1} + \mathrm{H}_{x_4}.$$

Here $v \in \mathfrak{X}(M)$ is not Hamiltonian, because no Hamiltonian vector field on (M, π_M) restricts to $x_1 \frac{\partial}{\partial x_1}$ on X.

Example 2.17. Poisson submanifolds X of a Poisson manifold (M, π_M) are exactly those split Poisson–Dirac submanifolds for which any splitting $TM|_X = TX \oplus E$ along X is orthogonal.

Example 2.18. Poisson transversals X in a Poisson manifold (M, π_M) have a unique orthogonal splitting $TM|_X = TX \oplus \pi_M^\sharp(N^*M)$ along X.

2.4. Coregular submanifolds

Among split Poisson–Dirac submanifolds, Poisson submanifolds and Poisson transversals are distinguished further by the property that they are either saturated by, or transverse to ambient symplectic leaves. That is, the symplectic directions transverse to the submanifold fit into a subbundle (the trivial one and a full normal bundle, respectively). This suggests that we consider those Poisson–Dirac submanifolds X of (M, π_M), for which the image of the map

$$\mathrm{Q}_X \colon N^*X \longrightarrow NX, \quad \mathrm{Q}_X(\xi) \colon= \pi_M^\sharp(\xi) + TX,$$

is a vector bundle.

Definition 2.19. A split Poisson–Dirac submanifold of a Poisson manifold (M, π_M) is a *coregular Poisson–Dirac* submanifold if $\mathrm{Q}_X \colon N^*X \to NX$ has locally constant rank.

Thus, Poisson submanifolds are those Poisson–Dirac submanifolds for which Q_X vanishes, while Poisson transversals are those for which Q_X is surjective.

Lemma 2.20. *For a submanifold X of a Poisson manifold (M, π_M), the following are equivalent*:

(i) *X is a coregular Poisson–Dirac submanifold*;

(ii) *$\pi_M^\sharp(N^*X) \subset TM|_X$ is a vector subbundle which meets TX trivially.*

Proof. If X is such that $\pi_M^\sharp(N^*X) \oplus TX$ is a vector subbundle of $TM|_X$, then any splitting $TM|_X = TX \oplus E$ in which $\pi_M^\sharp(N^*X) \subset E$ is an orthogonal splitting along X, and the rank of Q_X is that of the vector bundle $\pi_M^\sharp(N^*X)$. Hence (ii) implies (i). Conversely, a Poisson–Dirac submanifold X in a Poisson manifold (M, π_M) in particular has an induced Poisson structure, whence $\pi_M^\sharp(N^*X) \cap TX = 0$. Hence $\pi_M^\sharp(N^*X) \oplus TX$ is a vector subbundle of $TM|_X$ if $\mathrm{Q}_X \colon N^*X \to NX$ has constant rank. So (i) implies (ii). ∎

Example 2.21. Every point in a Poisson manifold is coregular.

Example 2.22. *An example of a split Poisson–Dirac submanifold which is not coregular.* Let $M = \mathfrak{so}(3)^*$ be dual to the Lie algebra of the group of oriented isometries of Euclidean 3-space, equipped with its canonical linear Poisson structure

$$\pi_M = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

whose leaves are concentric spheres and the origin. Consider the embedding $i \colon \mathbb{R} \to M$ given by $i(t) = (t, 0, 0)$. Then π_M has an orthogonal splitting along the image X of i, namely

$$E = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle, \quad \pi_M^\sharp(E^\circ) = 0.$$

Hence X is a split Poisson–Dirac submanifold. Note however that

$$\pi_M^\sharp(N^*X) = \begin{cases} E & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0, \end{cases}$$

whence X is not coregular.

The hierarchy of submanifolds illustrated in Figure 1 is therefore strict, since there exist:

- Poisson–Dirac submanifolds which are not clean (e.g., the jumping phenomenon of Example 2.8);
- clean Poisson–Dirac submanifolds which are not split (e.g., Example 2.16);
- split Poisson–Dirac manifolds which are not coregular (e.g., Example 2.22).

Nevertheless, for certain types of Poisson manifolds (M, π_M), certain classes in the hierarchy illustrated in Figure 1 may coincide. One such instance is the following.

Example 2.23. A submanifold X in a Poisson manifold (M, π_M) is called *coisotropic* if

$$\pi_M^\sharp(N^*X) \subset TX.$$

A coisotropic which is Poisson–Dirac is necessarily a Poisson submanifold.

Another useful example occurs when the ambient manifold has locally constant rank.

Proposition 2.24. *A Poisson–Dirac submanifold of a Poisson manifold of locally constant rank is necessarily coregular of locally constant rank.*

Proof. If (M, π_M) has locally constant rank, then

$$\mathrm{Gr}(\pi_M) = \mathcal{R}_{\omega_M} \mathrm{Gr}(\mathscr{F}) := \{u + \iota_u \omega_M + \xi \mid u + \xi \in \mathrm{Gr}(\mathscr{F})\},$$

where $T\mathscr{F} = \pi_M^\sharp(T^*M)$ and ω_M is a two-form on M for which the endomorphism $\pi_M^\sharp \omega_M^\sharp$ restricts to the identity on $T\mathscr{F}$. Then

$$i^! \mathrm{Gr}(\pi_M) = \mathcal{R}_{i^*(\omega_M)} i^! \mathrm{Gr}(\mathscr{F})$$

shows that $i^!\mathrm{Gr}(\pi_M)$ is Dirac exactly when $i^!\mathrm{Gr}(\mathscr{F})$ is Dirac, which according to Example 2.3, means that $i^!\mathrm{Gr}(\pi_M)$ must be a Poisson structure of locally constant rank if it is smooth. Moreover,

$$TX \cap T\mathscr{F}|_X \text{ smooth} \iff TX + T\mathscr{F}|_X \text{ smooth} \iff N^*X \cap N^*\mathscr{F} \text{ smooth.}$$

Because the leftmost space is $T\mathscr{F}_X$ and the rightmost is $\ker \mathrm{Q}_X$, X is coregular. ∎

3. Poisson submersions with Poisson fibres

We next turn to a "compatibility" condition for a surjective submersion between Poisson manifolds

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M). \tag{3.1}$$

There are two natural conditions to impose:

(a) that p be a Poisson map;

(b) that fibres of p be Poisson–Dirac submanifolds.

Condition (a) appears naturally in Poisson geometry.

Example 3.1. Let $G \rightrightarrows M$ be a Lie groupoid. A two-form $\omega_G \in \Omega^2(G)$ is *multiplicative* if

$$\mathrm{m}^*(\omega_G) = \mathrm{pr}_1^*(\omega_G) + \mathrm{pr}_2^*(\omega_G),$$

where m, s and t denote respectively the multiplication, source, and target maps.

Then, if ω_G is symplectic, a unique Poisson structure π_M exists on M, such that

$$\mathbf{s}\colon (G, \omega_G) \to (M, \pi_M) \quad \text{and} \quad \mathbf{t}\colon (G, \omega_G) \to (M, -\pi_M)$$

are Poisson maps.

Conversely, given a Poisson manifold (M, π_M), there exists a submersion as in (3.1), in which π_Σ is symplectic and p is Poisson – this is what is called a *symplectic realization* [9, 23]. A symplectic realization is said to be *complete* if, for every for a function $f \in C^\infty(M)$, the Hamiltonian vector field on Σ corresponding to $f \circ p$ is complete if that of f is. Complete symplectic realizations exist exactly when (M, π_M) arises from a symplectic groupoid (G, ω_G) as in Example 3.1, see Theorem 8 in [11].

Condition (b), on the other hand, appears naturally in the context of this paper if one is to interpret a submersion as a family $p^{-1}(x)$ of submanifolds of Σ, parametrized by M. The simplest condition to consider in this direction is the following.

Example 3.2 (Coupling Poisson structure). A submersion $p\colon (\Sigma, \pi_\Sigma) \to M$ from a Poisson manifold is *coupling* if its fibres are Poisson transversals.

For example, every Poisson manifold is coupling around an embedded symplectic leaf. More explicitly, if $(\mathrm{S}_M(x), \omega_{\mathrm{S}_M(x)})$ is a closed symplectic leaf of (M, π_M), and $p\colon M \supset U \to \mathrm{S}_M(x)$ is a tubular neighborhood of $\mathrm{S}_M(x)$, then (shrinking U if need be) $p\colon (U, \pi_M) \to \mathrm{S}_M(x)$ is coupling.

A simple example in which both conditions (a) and (b) are met is given by the following.

Example 3.3 (Vertical Poisson structures). Any vertical Poisson structure $\pi_\Sigma \in \Gamma(\wedge^2 V)$, $V = \ker p_*$, turns a surjective submersion $p\colon \Sigma \to M$ into a Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, 0)$ whose fibres are Poisson submanifolds.

The last two examples, with coregular Poisson–Dirac fibres, motivate the following definition.

Definition 3.4. A submersion $p\colon (\Sigma, \pi_\Sigma) \to M$ has *Poisson fibres* if its fibres are Poisson–Dirac submanifolds.

Example 3.5. Consider the submersion

$$p : \Sigma = \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = M, \quad p(x_1, x_2, x_3) = (x_1, x_2).$$

Then fibres of p are Poisson–Dirac submanifolds of Σ when equipped with the Poisson structure

$$\pi_\Sigma = x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \in \mathfrak{X}^2(\Sigma),$$

all fibres inheriting the zero Poisson structure. Note however that none of the fibres is a *coregular* Poisson–Dirac submanifold.

It turns out that the fibres of a *Poisson submersion* with Poisson fibres are automatically coregular. In fact,

Theorem 3.6. *A Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ in which each tangent space $T_x p^{-1}(px) \subset (T_x\Sigma, \pi_{\Sigma,x})$ inherits a Poisson structure (that is, $\pi_\Sigma^\sharp(V^\circ) \cap V = 0$) automatically has coregular Poisson–Dirac submanifolds for fibres. Moreover, for a coregular Poisson–Dirac submanifold $Y \subset M$,*

(a) *$X := p^{-1}(Y) \subset \Sigma$ is a coregular Poisson–Dirac submanifold*;

(b) *$p\colon (X, \pi_X) \to (Y, \pi_Y)$ is a Poisson submersion with Poisson fibres.*

Proof. For a coregular Poisson–Dirac submanifold Y of (M, π_M), we have that

$$TY \oplus \pi_M^\sharp(N^*Y) \subset TM|_Y$$

is a vector subbundle. Taking the preimage under p_*,

$$TX + p_*^{-1}(\pi_M^\sharp(N^*Y)) \subset T\Sigma|_X$$

is a vector subbundle. Now, because p is Poisson,

$$TX + p_*^{-1}(\pi_M^\sharp(N^*Y)) = TX + \pi_\Sigma^\sharp(N^*X),$$

and because the fibres of p are Poisson–Dirac,

$$TX \cap \pi_\Sigma^\sharp(N^*X) = V \cap \pi_\Sigma^\sharp(p^*(\ker Q_Y)) \subset V \cap \pi_\Sigma^\sharp(V^\circ) = 0.$$

By Lemma 2.20, it follows that X is a coregular Poisson–Dirac submanifold, and specializing to the case where Y is a point, we deduce that $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is a Poisson submersion with coregular Poisson–Dirac submanifolds as fibres. Moreover, the diagram of Poisson manifolds

$$\begin{tikzcd}
(X, \pi_X) \arrow[r, "j"] \arrow[d, "p|_X"'] & (\Sigma, \pi_\Sigma) \arrow[d, "p"] \\
(Y, \pi_Y) \arrow[r, "i"'] & (M, \pi_M)
\end{tikzcd}$$

satisfies the condition of Lemma 3 in [16], which implies that $p|_X\colon (X, \pi_X) \to (Y, \pi_Y)$ is a Poisson submersion. Explicitly: because i induces a Poisson structure, for any $u_Y + \xi_Y \in \mathrm{Gr}(\pi_Y)$ there is $\xi_M \in T^*M$ such that $\xi_Y = i^*(\xi_M)$ and $i_*(u_Y) + \xi_M \in \mathrm{Gr}(\pi_M)$. Because $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is Poisson, there exists $u_\Sigma \in T\Sigma$ such that $i_*(u_Y) = p_*(u_\Sigma)$ and $u_\Sigma + p^*(\xi_M) \in \mathrm{Gr}(\pi_\Sigma)$. Because $TX \simeq TY \times_{TM} T\Sigma$ and $pj = ip|_X$, it follows that $(u_Y, u_\Sigma) + (p|_X)^*(\xi_Y) \in \mathrm{Gr}(\pi_X)$, and this implies that $p|_X$ is a Poisson submersion. It has Poisson fibres because both

$$V_y \subset (T_y\Sigma, \pi_{\Sigma,y}) \quad \text{and} \quad T_xX \subset (T_x\Sigma, \pi_{\Sigma,x})$$

inherit Poisson structures $\pi_{V,y} \in \wedge^2 V_y$ and $\pi_{X,x} \in \wedge^2 T_xX$, for all $y \in \Sigma$ and $x \in X$, and therefore $V_x \subset (T_x\Sigma, \pi_{X,x})$ inherits the Poisson structure $\pi_{V,x}$. ■

In the spirit of the comment leading up to Proposition 2.24, one could say that, for the class of submanifolds which arise as fibres of Poisson submersions, the hierarchy of Figure 2 collapses. Other pertinent examples of this phenomenon can be found in Section 4.

Remark 3.7. A Poisson map $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is automatically transverse to any Poisson transversal $Y \subset M$, whose preimage $X := p^{-1}(Y) \subset \Sigma$ is again a Poisson transversal, and $p|_X\colon X \to Y$ becomes a Poisson map for the induced Poisson structures, see Lemma 1 in [16]. Theorem 3.6 may be regarded as a possible analogue to this previous statement in the context of coregular Poisson–Dirac submanifolds.

Let us point out in passing that – in contrast to Poisson transversals – Poisson maps need *not* meet coregular Poisson–Dirac submanifolds cleanly, and even when they do (as in the hypotheses of Theorem 3.6), their preimage need not be Poisson–Dirac, as the next two examples illustrate.

Example 3.8. *An example of a Poisson map which does not meet leaves cleanly.* Consider the surjective Poisson map

$$\begin{aligned} \psi : \big(\mathbb{R}^4, \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial x_2} + \tfrac{\partial}{\partial x_3} \wedge \tfrac{\partial}{\partial x_4}\big) &\longrightarrow \big(\mathbb{R}^2, x_1 \tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial x_2}\big), \\ \psi(x_1, x_2, x_3, x_4) &= (x_2, x_3 x_4 - x_1 x_2). \end{aligned}$$

Then ψ does not meet cleanly the leaf consisting of the origin alone, since

$$\psi^{-1}(0) = \{(x_1, 0, x_3, x_4) \mid x_3 x_4 = 0\}$$

is not a manifold.

Example 3.9. *An example of a Poisson submersion whose fibres are not Poisson–Dirac.* Let the cotangent bundle T^*M of a smooth manifold be equipped with its canonical symplectic form $\omega_{\mathrm{can}} \in \Omega^2(T^*M)$,

$$\omega_{\mathrm{can}} := -\mathrm{d}\lambda_{\mathrm{can}}, \quad \lambda_{\mathrm{can}} \in \Omega^1(T^*M), \quad \lambda_{\mathrm{can}}(v)_\xi := \langle \xi, p_*(v) \rangle, \quad v \in T_\xi T^*M.$$

If $\pi_{\mathrm{can}} \in \mathfrak{X}^2(T^*M)$ denotes the Poisson structure corresponding to ω_{can}, the canonical projection

$$p\colon (T^*M, \pi_{\mathrm{can}}) \longrightarrow (M, 0)$$

defines a Poisson submersion, none of whose fibres is Poisson–Dirac. Observe that this example shows that the hypothesis in Theorem 3.6 that tangent spaces inherit Poisson structures cannot be removed.

3.1. Couplings over leaves

As discussed in the previous section, Poisson submersions with Poisson fibres are modeled on both vertical Poisson structures and coupling Poisson submersions, in the same way as coregular Poisson–Dirac submanifolds were modeled on Poisson submanifolds and Poisson transversals.

In order to introduce our remaining *dramatis personæ*, we carry out a closer examination of the coupling condition to pave the way for the discussion to follow.

A Poisson bivector $\pi_\Sigma \in \mathfrak{X}^2(\Sigma)$ is *coupling* for a surjective submersion $p\colon \Sigma \to M$ when the fibres of p are Poisson transversals. This is equivalent to $\mathrm{Gr}(\pi_\Sigma)$ being transverse to the Dirac structure $\mathrm{Gr}(V)\colon = V \oplus V^\circ$ corresponding to the foliation by fibres of p. This is in turn equivalent (see [4, 35]) to the existence of

(a) an Ehresmann connection H,

(b) a bivector $\pi \in \Gamma(\wedge^2 V)$,

(c) a form $\omega \in \Gamma(\wedge^2 V^\circ)$,

such that

$$\mathrm{Gr}(\pi_\Sigma) = \mathcal{R}_\omega(H) \oplus \mathcal{R}_\pi(H^\circ)\colon = \{u + \pi^\sharp(\xi) + \omega^\sharp(u) + \xi \mid u + \xi \in H \oplus H^\circ\},$$

in which case

$$\begin{array}{llll} \text{(a)} & \pi \text{ is Poisson;} & \text{(b)} & \mathscr{L}_{\mathrm{h}(u)}\pi = 0; \\ \text{(c)} & \mathrm{curv}(u,v) = \pi^\sharp\, \mathrm{d}\omega(\mathrm{h}(v),\mathrm{h}(u)), & \text{(d)} & \mathrm{d}\omega(\mathrm{h}(u),\mathrm{h}(v),\mathrm{h}(w)) = 0; \end{array}$$

for all $u, v, w \in \mathfrak{X}(M)$. Here h denotes the horizontal lift of H and $\mathrm{curv}(u,v) := \mathrm{h}([u,v]) - [\mathrm{h}(u),\mathrm{h}(v)]$ denotes its curvature. Finally, if the submersion

$$p\colon (\Sigma, \pi_\Sigma) \longrightarrow (M, \pi_M)$$

has connected fibres, then it is a Poisson map for some Poisson structure π_M on M exactly when ω is closed, in which case π_M is symplectic, ω is the pullback of the closed form on M corresponding to π_M, and H is involutive (see Corollary 1 in [17]).

Example 3.10 (Symplectic base). For a Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$, we have that

$$\mathrm{Gr}(\pi_\Sigma) \cap \mathrm{Gr}(V) = \{\pi_\Sigma^\sharp(p^*\xi) + p^*\xi \mid \xi \in \ker \pi_M^\sharp\}.$$

Therefore a Poisson submersion is automatically coupling if its base (M, π_M) is symplectic.

Theorem 3.11. *A Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ pulls symplectic leaves of M back to Poisson submanifolds of Σ, and p restricts to coupling Poisson submersions*

$$p|_{\mathrm{S}_\Sigma(x)} : (\mathrm{S}_\Sigma(x), \omega_{\mathrm{S}_\Sigma(x)}) \longrightarrow (\mathrm{S}_M(px), \omega_{\mathrm{S}_M(px)})$$

for each $x \in \Sigma$.

Proof. The symplectic leaf $(\mathrm{S}_M(px), \omega_{\mathrm{S}_M(px)})$ of (M, π_M) through $px \in M$ is in particular a coisotropic submanifold; hence so its preimage $p^{-1}(\mathrm{S}_M(px)) \subset \Sigma$ under the Poisson map p. However, because $\mathrm{S}_M(px)$ is a coregular Poisson–Dirac submanifold, it follows from Theorem 3.6 that $p^{-1}\mathrm{S}_M(px)$ is a coregular Poisson–Dirac submanifold, and that

$$p|_{p^{-1}\mathrm{S}_M(px)} : (p^{-1}\mathrm{S}_M(px), \pi_{p^{-1}\mathrm{S}_M(px)}) \longrightarrow (\mathrm{S}_M(px), \omega_{\mathrm{S}_M(px)})$$

is a Poisson submersion with Poisson fibres.

This implies, by Example 2.23, that $p^{-1}\mathrm{S}_M(px)$ is a Poisson submanifold, and by Example 3.10 that $p\colon (p^{-1}\mathrm{S}_M(px), \pi_{p^{-1}\mathrm{S}_M(px)}) \to (S_M, \pi_{S_M})$ is a coupling Poisson submersion. Hence, if $\mathrm{S}_\Sigma(x)$ denotes the symplectic leaf of Σ through x, then

$$p|_{\mathrm{S}_\Sigma(x)} : (\mathrm{S}_\Sigma(x), \omega_{\mathrm{S}_\Sigma(x)}) \longrightarrow (\mathrm{S}_M(px), \omega_{\mathrm{S}_M(px)})$$

is a Poisson map (because the inclusion of a symplectic leaf is Poisson), and because $\mathrm{S}_M(px)$ is symplectic, $p|_{\mathrm{S}_\Sigma(x)}\colon \mathrm{S}_\Sigma(x) \to \mathrm{S}_M(px)$ must be a submersion, and again by Example 3.10, it is automatically coupling. ∎

Beware that such couplings over leaves need *not* be surjective, as illustrated by the example below.

Example 3.12. *An example of a Poisson submersion with Poisson fibres whose restriction over a leaf is not surjective.* Consider the surjective Poisson submersion

$$p : (\Sigma, \pi_\Sigma) = \left(\mathbb{R}^3, \tfrac{\partial}{\partial x} \wedge \tfrac{\partial}{\partial y} + (1+z^2)\tfrac{\partial}{\partial z} \wedge \tfrac{\partial}{\partial y}\right) \longrightarrow \left(\mathbb{R}^2, \tfrac{\partial}{\partial x} \wedge \tfrac{\partial}{\partial y}\right) = (M, \pi_M),$$
$$p(x, y, z) = (x, y).$$

Note that

$$\mathrm{S}_\Sigma(0) = \{(\arctan(z), y, z) \mid y, z \in \mathbb{R}\},$$

and therefore the restriction $p|_{\mathrm{S}_\Sigma(0)}\colon \mathrm{S}_\Sigma(0) \to \mathrm{S}_M(0) = M$ is not surjective.

3.2. Pencils

By the description of a coupling Poisson submersion

$$p\colon (\Sigma, \pi_\Sigma) \longrightarrow (M, \omega_M),$$

given above, the Poisson structure π_Σ on the total space splits as a sum

$$\pi_\Sigma = \pi_V + \pi_H,$$

where π_V is the vertical Poisson structure inducing the given Poisson structures on fibres, and π_H is a regular Poisson structure obtained by pulling ω_M back to a flat Ehresmann connection on $p\colon \Sigma \to M$. This trivially implies that π_V, π_H *commute.*

This phenomenon holds true in the setting proposed in [34] to generalize the coupling condition:

Example 3.13 (Almost-coupling). An *almost-coupling* Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is a Poisson submersion such that there exists an Ehresmann connection $H \subset T\Sigma$ for which $\pi_\Sigma(H^\circ, V^\circ) = 0$. A such connection splits π_Σ as a sum $\pi_\Sigma = \pi_V + \pi_H$, where π_V and π_H are the vertical and horizontal bivectors defined by

$$\pi_V^\sharp(\xi) = \begin{cases} 0, & \xi \in V^\circ, \\ \pi_\Sigma^\sharp(\xi), & \xi \in H^\circ, \end{cases} \quad \text{and} \quad \pi_H^\sharp(\xi) = \begin{cases} 0, & \xi \in H^\circ, \\ \pi_\Sigma^\sharp(\xi), & \xi \in V^\circ. \end{cases}$$

Moreover, because p is Poisson, it follows that the horizontal bivector π_H is of the form $\pi_H = \mathrm{h}(\pi_M)$, where $\mathrm{h}: \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(\Sigma)$ denotes the horizontal lift (of multivectors) associated with H. This implies that in the induced bigrading $\mathfrak{X}^{p,q}(\Sigma) = \Gamma(\wedge^p V \otimes \wedge^q H)$, we have

$$[\pi_V, \pi_V] \in \mathfrak{X}^{3,0}(\Sigma), \quad [\pi_V, \pi_H] \in \mathfrak{X}^{2,1}(\Sigma) \quad \text{and} \quad [\pi_H, \pi_H] \in \mathfrak{X}^{1,2}(\Sigma) \oplus \mathfrak{X}^{0,3}(\Sigma).$$

Hence $\pi_\Sigma = \pi_V + \pi_H$ Poisson implies that π_V and π_H are commuting Poisson structures:

$$[\pi_V, \pi_V] = 0, \quad [\pi_V, \pi_H] = 0 \quad \text{and} \quad [\pi_H, \pi_H] = 0.$$

This motivates our next definition.

Definition 3.14. An *orthogonal pencil* is a Poisson submersion $p: (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ in which π_Σ splits into commuting Poisson structures $\pi_\Sigma = \pi_V + \pi_H$, where $\pi_V \in \Gamma(\wedge^2 V)$ is a vertical bivector, and $\pi_H^\sharp(T^*\Sigma) \cap V = 0$.

Note that an orthogonal pencil $p: (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is automatically a Poisson submersion with Poisson fibres, and that the splitting $\pi_\Sigma = \pi_V + \pi_H$ is unique. Moreover, an almost-coupling submersion $p: (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is tantamount to an orthogonal pencil for which π_H is tangent to an Ehresmann connection for p.

Example 3.15. *An example of an orthogonal pencil which is not almost-coupling.* Consider on $\Sigma := \mathbb{C}^2$, with coordinates

$$z_0 = x_0 + iy_0 \in \mathbb{C} \quad \text{and} \quad z_1 = x_1 + iy_1 \in \mathbb{C},$$

and with Euler and rotational vector fields

$$\mathscr{E}_i = x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \quad \text{and} \quad \mathscr{V}_i = x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}.$$

Then there is a surjective Poisson submersion

$$p : (\Sigma, \pi_\Sigma) \to (M, \pi_M), \quad p(z_0, z_1) = z_0,$$

where $M := \mathbb{C}$ and

$$\pi_\Sigma = |z_0|^2 (\mathscr{E}_0 + \mathscr{E}_1) \wedge (\mathscr{V}_0 + \mathscr{V}_1), \quad \pi_M = |z_0|^2 \mathscr{E}_0 \wedge \mathscr{V}_0.$$

Note that

$$\pi_\Sigma^\sharp(T^*\Sigma) = \begin{cases} \langle \mathscr{E}_0 + \mathscr{E}_1, \mathscr{V}_0 + \mathscr{V}_1 \rangle & \text{if } z_0 \neq 0, \\ 0 & \text{if } z_0 = 0, \end{cases}$$

meets the vertical bundle of $p: \Sigma \to M$ trivially, and therefore $p: (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres in which fibres all have the trivial Poisson structure. This implies that this is in fact an orthogonal pencil. However, it is not almost-coupling: an Ehresmann connection for which it is almost-coupling would coincide with $H = \pi_\Sigma^\sharp(T^*\Sigma)$ on the locus $z_0 \neq 0$, but

$$\lim_{x_0 \to 0} \lim_{x_1 \to 0} \lim_{y_0 \to 0} \lim_{y_1 \to 0} H = \langle \tfrac{\partial}{\partial x_0}, \tfrac{\partial}{\partial y_0} \rangle \neq \langle \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial y_1} \rangle = \lim_{x_1 \to 0} \lim_{x_0 \to 0} \lim_{y_1 \to 0} \lim_{y_0 \to 0} H$$

shows that $H|_{z_0 \neq 0}$ cannot extend to a global Ehresmann connection.

While general Poisson submersions with Poisson fibres need not admit orthogonal splittings, they have in some sense a canonical candidate for the job.

Proposition 3.16. *The equivalence relation on the total space of a Poisson submersion with Poisson fibres* $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ *in which* $x_0, x_1 \in \Sigma$ *are equivalent if and only if there is* $f \in C^\infty_c(M \times I)$ *such that*

$$\phi^{1,0}_{\mathrm{H}_{f\circ p}}(x_0) = x_1$$

defines a singular foliation $\mathcal{S}_H$*, each of whose leaves carries a canonical symplectic form.*

Proof. First observe that, because p is a Poisson map,

$$\pi^\sharp_\Sigma \circ p^* : \Omega^1(M) \longrightarrow \mathfrak{X}(\Sigma)$$

defines a Lie algebra map, when $\Omega^1(M)$ is given the Koszul bracket

$$[\xi, \eta]^{\pi_M} := \mathscr{L}_{\pi^\sharp_M(\xi)}\eta - \iota_{\pi^\sharp_M(\eta)}\mathrm{d}\xi.$$

This implies that the pullback vector bundle $A := p^*(T^*M)$ over Σ carries a structure of Lie algebroid, with bracket and anchor determined by

$$[p^*(\xi), p^*(\eta)]_A = p^*[\xi, \eta]^{\pi_M} \quad \text{and} \quad \rho_A(p^*(\xi)) = \pi^\sharp_\Sigma \circ p^*(\xi).$$

In the usual manner, A determines on Σ a singular foliation $\mathcal{S}_H$, whose tangent space is $\pi^\sharp_\Sigma(V^\circ)$. Hence its leaves are the equivalence classes of the equivalence relation described in the statement, and they are submanifolds of the symplectic leaves of Σ: $\mathcal{S}_H(x) \subset \mathrm{S}_\Sigma(x)$. Therefore,

$$\omega_{\mathcal{S}_H(x)}(\mathrm{H}_{f\circ p}, \mathrm{H}_{g\circ p}) = \{f, g\} \circ p$$

is just the restriction of the symplectic form on $\mathrm{S}_\Sigma(x)$ to $\mathcal{S}_H(x)$. ∎

Remark 3.17. The following asymmetry is noteworthy: while the singular horizontal foliation $\mathcal{S}_H$ is defined for all Poisson submersions with Poisson fibres, the partition into leaves of the Poisson–Dirac structures on fibres need not in general define a singular foliation, as Example 3.19 below illustrates.

In contrast to almost-coupling Poisson submersions, for a general Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$, the singular foliation $\mathcal{S}_H$, with leaves canonically equipped with symplectic forms (described in Proposition 3.16), need not in general arise from a Poisson structure, just as, for a general Poisson submersion with Poisson fibres, the induced Poisson structures $\pi_{p^{-1}(x)}$ on the fibres $p^{-1}(x) \subset \Sigma$ need not vary smoothly with $x \in M$. That is, there need not be any vertical bivector $\pi_V \in \Gamma(\wedge^2 V)$ which restricts on $p^{-1}(x)$ to $\pi_{p^{-1}(x)}$. In fact, these conditions are simultaneously satisfied.

Theorem 3.18. *For a Poisson submersion with Poisson fibres* $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$*, the following assertions are equivalent*:

(i) *It admits an orthogonal pencil.*

(ii) *There is a Poisson structure* $\pi_H \in \mathfrak{X}^2(\Sigma)$ *whose symplectic leaves are those of* $\mathcal{S}_H$.

(iii) *The Poisson structures on fibres assemble into a vertical Poisson structure* $\pi_V \in \Gamma(\wedge^2 V)$.

(iv) *The linear family*

$$\mathrm{Gr}(\pi_\Sigma) \cap (V \oplus T^*\Sigma) + V^\circ \subset \mathbb{T}^*\Sigma$$

is a vector bundle.

Proof. If π_Σ splits into a pencil $\pi_\Sigma = \pi_V + \pi_H$, where π_V is a vertical bivector, and $\pi_H^\sharp(T^*\Sigma) \cap V = 0$, then

$$\pi_H^\sharp(\xi) = \pi_\Sigma^\sharp(\xi), \quad \xi \in V^\circ, \tag{3.2}$$

which implies that the tangent space at $x \in \Sigma$ to the leaf $\mathcal{S}_H(x)$ of the singular foliation $\mathcal{S}_H$ of Proposition 3.16 coincides with the tangent space $\pi_H^\sharp(T_x^*\Sigma)$ to the leaf of π_H through x, and the symplectic form on those spaces coincides as well, being the pullback of that on the ambient space. Therefore (i) implies (ii).

On the other hand, if a Poisson structure π_H on Σ exists whose singular symplectic foliation coincides with that of $\mathcal{S}_H$, then π_H is unique, and satisfies (3.2) – which is to say that $\pi_V := \pi_\Sigma - \pi_H \in \mathfrak{X}^2(\Sigma)$ is a vertical bivector, which induces on the fibres of p the same Poisson structure as π_Σ does. Therefore (ii) implies (iii).

Next observe that if a vertical bivector $\pi_V \in \Gamma(\wedge^2 V)$ induces on the fibres of p the same Poisson structure as π_Σ, then

$$\mathrm{Gr}(\pi_V) = \mathrm{Gr}(\pi_\Sigma) \cap (V \oplus T^*\Sigma) + V^\circ,$$

and therefore (iii) implies (iv). Finally, note that the pullback of the Lagrangian family

$$L_\Sigma := \mathrm{Gr}(\pi_\Sigma) \cap (V \oplus T^*\Sigma) + V^\circ$$

under the inclusion $i\colon X \to \Sigma$ of any fibre X of p coincides with that of $\mathrm{Gr}(\pi_\Sigma)$:

$$i^!(L_\Sigma) = i^!\mathrm{Gr}(\pi_\Sigma).$$

This implies that $L_\Sigma = \mathrm{Gr}(\pi_V)$ for a vertical Poisson bivector $\pi_V \in \Gamma(\wedge^2 V)$. Because on an open, dense subset $U \subset \Sigma$ on which the rank of $\pi_H^\sharp := \pi_\Sigma^\sharp - \pi_V^\sharp$ is locally constant, there is an Ehresmann connection $H \subset TU$ for p to which π_H is tangent. Hence $p\colon (U, \pi_\Sigma) \to (pU, \pi_M)$ is an almost-coupling Poisson submersion, and H induces the splitting $\pi_\Sigma|_U = \pi_V|U + \pi_H|_U$. By Example 3.13, $\pi_V|_U$ and $\pi_H|_U$ are commuting Poisson structures, and because U is dense in Σ, this implies that $\pi_\Sigma = \pi_V + \pi_H$ is an orthogonal pencil. ■

Example 3.19. *An example of a Poisson submersion with Poisson fibres which is not an orthogonal pencil.* The quotient map of the action of scalar multiplication on $\Sigma = \mathbb{C}^2 \setminus \{0\}$,

$$\mathbb{C}^\times \times \Sigma \longrightarrow \Sigma, \quad w \cdot (z_0, z_1) := (wz_0, wz_1),$$

gives rise to a submersion

$$p : \Sigma \longrightarrow M = \mathbb{C}P^1, \quad p(z_0, z_1) := [z_0 : z_1],$$

whose vertical bundle is spanned by $\mathscr{E}_0 + \mathscr{E}_1$ and $\mathscr{V}_0 + \mathscr{V}_1$ (using the notation of Example 3.15). The Poisson bivector

$$\pi_\Sigma = \tfrac{1}{4}(\mathscr{E}_0 - \mathscr{E}_1) \wedge (\mathscr{V}_0 - \mathscr{V}_1)$$

is $\mathbb{C}^\times$-invariant, and thus gives rise to a Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$, where $\pi_M = \mathscr{E} \wedge \mathscr{V}$, and where $\mathscr{E}, \mathscr{V} \in \mathfrak{X}(\mathbb{C}P^1)$ are vector fields which restrict in a standard affine chart restrict to the Euler and rotational vector fields (or their opposites) of $\mathbb{C}$. Note that the image of π_Σ is spanned by $\mathscr{E}_0 - \mathscr{E}_1$ and $\mathscr{V}_0 - \mathscr{V}_1$; hence the induced Poisson structure on the fibre $\Sigma_{[z_0:z_1]}$ over $[z_0 : z_1] \in M$ is trivial if $z_0 z_1 \neq 0$, and it is symplectic otherwise. Hence the Poisson structures on fibres do not even vary continuously, and by Theorem 3.18, no orthogonal pencil exists, nor is $\mathcal{S}_H$ the partition into leaves of a Poisson structure on Σ.

4. Examples from Lie theory

4.1. Poisson–Lie groups

A splitting of a Lie algebra $\mathfrak{d}$ into Lie subalgebras $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ is called a *Manin triple* if $\mathfrak{d}$ is equipped with an invariant, symmetric bilinear pairing $\langle\cdot,\cdot\rangle\colon S^2\mathfrak{d} \to \mathbb{R}$ for which the subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ are Lagrangian.

A *G-invariant Manin triple* $(\mathfrak{d}, \langle\cdot,\cdot\rangle, \mathfrak{g}, \mathfrak{h})$ is a Manin triple equipped with a choice of Lie group G with Lie algebra $\mathfrak{g}$, and an extension $\mathrm{Ad}\colon G \curvearrowright \mathfrak{d}$ of the adjoint action of G on $\mathfrak{g}$ which integrates the Lie bracket action $\mathrm{ad}\colon \mathfrak{g} \curvearrowright \mathfrak{d}$ and such that

$$\mathrm{Ad}_g[v, w] = [\mathrm{Ad}_g(v), \mathrm{Ad}_g(w)], \quad \langle v, w\rangle = \langle \mathrm{Ad}_g(v), \mathrm{Ad}_g(w)\rangle$$

for all $g \in G$ and $v, w \in \mathfrak{d}$. In that case, the quotient representation $\mathrm{Ad}\colon G \curvearrowright \mathfrak{d}/\mathfrak{g} = \mathfrak{g}^*$ is the coadjoint action.

To a G-invariant Manin triple there corresponds a *Poisson–Lie group* structure π_G on G (see [13]; see also Section 5 in [29], whose perspective we espouse here) – that is, a Poisson bivector $\pi_G \in \mathfrak{X}^2(G)$, for which multiplication

$$\mathrm{m} : (G, \pi_G) \times (G, \pi_G) \longrightarrow (G, \pi_G)$$

is a Poisson map. Explicitly, this means that, for all $g_1, g_2 \in G$,

$$\pi_{G, g_1 g_2} = l_{g_1 *}\pi_{G, g_2} + r_{g_2 *}\pi_{G, g_1},$$

where $l_{g_1}(g) = g_1 g$ and $r_{g_2}(g) = g g_2$ stand for left- and right-multiplication. Indeed, the G-invariant Manin triple $(\mathfrak{d}, \langle\cdot,\cdot\rangle, \mathfrak{g}, \mathfrak{h})$ defines an infinitesimal *dressing action* $\varrho\colon \mathfrak{d} \to \Gamma(TG)$, uniquely determined by the condition that

$$\mathrm{Ad}_g(\iota_{\varrho(v)}\theta^L_g) = \mathrm{pr}_{\mathfrak{g}}\mathrm{Ad}_g(v), \quad (g, v) \in G \times \mathfrak{d}, \tag{4.1}$$

where $\theta^L \in \Omega^1(G; \mathfrak{g})$ denotes the left-invariant Maurer–Cartan form of G. The infinitesimal dressing action extends in fact to a linear map

$$\varepsilon : \mathfrak{d} \longrightarrow \Gamma(\mathbb{T}G), \quad \varepsilon(v) = \varrho(v) + \langle\theta^L, v\rangle, \tag{4.2}$$

which satisfies

$$\langle v, w\rangle = \langle \varepsilon(v), \varepsilon(w)\rangle, \quad [v, w] = [\varepsilon(v), \varepsilon(w)],$$

and for the ensuing isomorphism $\varepsilon: G \times \mathfrak{d} \to \mathbb{T}G$, we have

$$TG = \varepsilon(G \times \mathfrak{g}), \quad \mathrm{Gr}(\pi_G) = \varepsilon(G \times \mathfrak{h}).$$

Note that, by definition of π_G, Poisson submanifolds of (G, π_G) are unions of orbits of the infinitesimal action $\rho: \mathfrak{h} \to \mathfrak{X}(G)$; that is, $\mathfrak{h}$-invariant submanifolds of G.

4.2. Coregular Poisson–Dirac submanifolds from orbits

Let (G, π_G) be a Poisson–Lie group. An action $\alpha: G \times M \to M$ of G on a Poisson manifold (M, π_M) is *Poisson* if

$$\alpha : (G, \pi_G) \times (M, \pi_M) \longrightarrow (M, \pi_M)$$

is a Poisson map.

Remark 4.1. A Poisson action of (G, π_G) on (M, π_M) need not act by Poisson diffeomorphisms of (M, π_M), unless G is equipped with the trivial Poisson structure $\pi_G = 0$.

Lemma 4.2 (Orbits). *For an orbit X of a Poisson action of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π), the following assertions are equivalent*:

(i) *it is a coregular Poisson–Dirac submanifold*;

(ii) *it is Poisson–Dirac*;

(iii) $T_x X \subset (T_x M, \pi_{M,x})$ *is Poisson–Dirac for some $x \in X$.*

Proof. We need only show that condition (iii) implies condition (i). Let $g \in G$ and let $\xi \in N^*_{gx} X$. Because $\alpha^{-1}(X) = G \times X$, we have that

$$\alpha^*(\xi) = (\alpha^*_x(\xi), \alpha^*_g(\xi)) = (0, \alpha^*_g(\xi)) \in N^*_{(g,x)}(G \times X) \subset T^*_g G \times T^*_x M.$$

Because α is a Poisson map,

$$\pi^\sharp_{M,gx}(\xi) = \alpha_*(\pi_G, \pi_M)^\sharp_{(g,x)} \alpha^*(\xi) = \alpha_*(0, \pi^\sharp_{M,x} \alpha^*_g(\xi)) = \alpha_{g*} \pi^\sharp_{M,x} \alpha^*_g(\xi) = 0,$$

In particular, this implies that

$$\pi^\sharp_{M,gx}(N^*_{gx} X) = \alpha_{g*} \pi^\sharp_{M,x}(N^*_x X),$$
$$\pi^\sharp_{M,gx}(N^*_{gx} X) \cap T_{gx} X = \alpha_{g*}\big(\pi^\sharp_{M,x}(N^*_x X) \cap T_x X\big),$$

where in the second equality we used condition (iii). Hence X is a coregular Poisson–Dirac submanifold. ∎

Example 4.3. If a vector space A acts on a smooth manifold M, with action $\alpha: A \times M \to M$, and induced infinitesimal action

$$\rho : A \longrightarrow \mathfrak{X}(M), \quad \rho(v)_x := \tfrac{d}{dt} \alpha(e^{-t} v, x)|_{t=0},$$

the induced map $\wedge^2\rho\colon \wedge^2\mathfrak{g} \to \mathfrak{X}^2(M)$ maps into A-invariant Poisson bivectors on M, and

$$\alpha : (A, 0) \times (M, \pi_M) \longrightarrow (M, \pi_M), \quad \pi_M := \wedge^2\rho(\pi_A),$$

is a Poisson action for any $\pi_A \in \wedge^2 A$. Because the tangent space at $x \in M$ to the orbit $A \cdot x \subset M$ contains by construction $\pi_M^\sharp(T_x^* M)$, they are all Poisson submanifolds.

Note in the setting of Example 4.3 that, for any subspace $B \subset A$, the restricted action

$$\alpha : (B, 0) \times (M, \pi_M) \longrightarrow (M, \pi_M)$$

is Poisson, but it need not be the case that its orbits $B \cdot x$ are coregular Poisson–Dirac submanifolds. For example, if $A = \mathbb{C}$ acts by translations on $M = \mathbb{C}$ with its standard symplectic structure, orbits of the subgroup $B = \mathbb{R}$ are not Poisson–Dirac.

Let us borrow from symplectic geometry a useful setting in which orbits are automatically coregular Poisson–Dirac submanifolds.

Definition 4.4. Let a vector space A be equipped with a complex structure $J \in \mathrm{End}(A)$. A bivector $\pi_A \in \wedge^2 A$ is *positive* if J makes the symplectic leaves of π_A into Kähler manifolds.

Explicitly, π_A is positive if $S := \pi_A^\sharp(A^*)$ is a complex subspace, $J\colon (S, \omega_S) \to (S, \omega_S)$ is a symplectic automorphism, and

$$g_S : S \times S \longrightarrow \mathbb{R}, \quad g_S(u, v) := \omega_S(u, Jv),$$

is symmetric and positive-definite. Note that if π_A is positive and $B \subset A$ is J-invariant, then for $\xi \in B^\circ$ such that $\pi_A^\sharp(\xi) \in B$,

$$\begin{aligned} 0 &= \langle \xi, J\pi_A^\sharp(\xi)\rangle = \pi_A(\xi, J^*\xi) = -\omega_S(\pi_A^\sharp(\xi), \pi_A^\sharp(J^*\xi)) = \omega_S(\pi_A^\sharp(\xi), J\pi_A^\sharp(\xi)) \\ &= g_S(\pi_A^\sharp(\xi), \pi_A^\sharp(\xi)) \end{aligned}$$

implies that the pertinent set $\pi_A^\sharp(B^\circ) \cap B$ in (2.3) is trivial, and so B is a coregular Poisson–Dirac submanifold in (A, π_A).

Lemma 4.5. *Let A be a complex vector space equipped with a positive bivector $\pi_A \in \wedge^2 A$. An action of A on a complex manifold M by holomorphic transformations induces an A-invariant Poisson structure π_M on M, with the property that, for all complex subspace $B \subset A$, the induced action $(B, 0) \curvearrowright (M, \pi_M)$ is Poisson and has coregular Poisson–Dirac submanifolds as orbits.*

Proof. By Lemma 4.2, $B \cdot x \subset M$ is a coregular Poisson–Dirac submanifold if and only if

$$T_x(B \cdot x) \subset (T_x M, \pi_{M,x})$$

has an induced Poisson structure, and this happens exactly when

$$T_x(B \cdot x) \subset (T_x(A \cdot x), \pi_{A\cdot x, x})$$

has an induced Poisson structure, where $\pi_{A\cdot x}$ is the Poisson structure on the Poisson submanifold $A\cdot x\subset(M,\pi_M)$ (as in Example 4.3). Because $T_x(B\cdot x)\subset T_x(A\cdot x)$ is a complex subspace, it suffices to check that $\pi_{A\cdot x,x}$ is positive. And that is the case because the complex-linear infinitesimal action at x,

$$\rho_x:(A,\pi_A)\longrightarrow(T_x(A\cdot x),\pi_{A\cdot x,x})$$

induces an identification

$$\rho_x:(A/A_x,\pi_{A/A_x})\xrightarrow{\sim}(T_x(A\cdot x),\quad A_x:=\ker(\rho_x),$$

where π_{A/A_x} is in turn identified with the restriction of π_A to $\wedge^2A_x^\circ$ –and is therefore positive. ∎

Example 4.6. On $A=\mathbb{C}$, the bivector $\pi_A=\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$ is positive. Acting on $M=\mathbb{C}$ by translations, there ensues $\pi_M=\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$, while the action $w\cdot z=e^wz$ produces $\pi_M=(x^2+y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$.

4.3. Poisson submersions with Poisson fibres from toric varieties

The previous results can be used to produce Poisson submersions with Poisson fibres associated with the quotient presentation of a (smooth) projective toric variety. By a *toric variety* we mean a variety with a Zariski open subset identified to an algebraic torus whose action on itself by multiplication extends to an action on the variety. Toric varieties are given by the combinatorial data encoded in a fan. Projective toric varieties can also be described via suitable polytopes, a viewpoint which is useful to highlight the symplectic geometry nature of toric varieties.

Example 4.7. We recall Delzant's Hamiltonian quotient construction of symplectic toric manifolds and the necessary modifications to induce the projective toric variety structure via a GIT (for Geometric Invariant Theory) quotient. A symplectic manifold (M^{2n},ω_M) is *toric* if it comes equipped with an effective Hamiltonian action

$$T\curvearrowright(M,\omega_M)\xrightarrow{\mu_M}\mathfrak{t}^*,$$

where T is an n-dimensional (compact) torus $T\simeq\mathbb{T}^n$ and $\mathfrak{t}$ denotes its Lie algebra. A *polytope* $\Delta\subset\mathfrak{t}^*$ is a compact subset of the form

$$\Delta=\bigcap_{i=1}^{d}\{\xi\in\mathfrak{t}^*\mid\langle\xi,u_i\rangle\geqslant c_i\},\tag{4.3}$$

where $c_i\in\mathbb{R}$ and $u_1,\dots,u_d\in\mathfrak{t}$, which are thought of as normal to the faces of Δ. Such a polytope is called *Delzant* if $u_1,\dots,u_d$ can be rescaled to lie in the lattice $\Lambda\subset\mathfrak{t}$ which is the kernel of the exponential map $\exp:\mathfrak{t}\to T$, and at each vertex of Δ the vectors normal to faces of Δ through the vertex form a $\mathbb{Z}$-basis of Λ.

To a Delzant polytope $\Delta\subset\mathfrak{t}^*$ as in (4.3) one associates the exact sequence of tori:

$$1\longrightarrow N\xrightarrow{i}\mathbb{T}^d\xrightarrow{p}T\longrightarrow 1$$

where p is uniquely determined by the condition that $p_*(e_i) = u_i$, where $e_1, \ldots, e_d$ stands for the standard basis of $\mathbb{R}^d = \mathrm{Lie}(\mathbb{T}^d)$. If $\omega_{\mathrm{std}} \in \Omega^2(\mathbb{C}^d)$ denotes the standard symplectic structure $\omega_{\mathrm{std}} = \frac{i}{2}\sum_{i=1}^d \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, the standard action of $\mathbb{T}^d$ on $\mathbb{C}^d$ by multiplication gives rise to a Hamiltonian action

$$\mathbb{T}^d \curvearrowright (\mathbb{C}^d, \omega_{\mathrm{std}}) \xrightarrow{\mu} \mathrm{Lie}(\mathbb{T}^d)^*, \quad \mu(z) = \sum_{i=1}^d \Big(\frac{|z_i|^2}{2} + c_i\Big) e_i.$$

Then $\mu_N := i^*\mu \colon \mathbb{C}^d \to \mathfrak{n}^*$ is a moment map for the action of the subtorus $N \curvearrowright (\mathbb{C}^d, \omega_{\mathrm{std}})$. The map μ_N is proper, zero is a regular value, and the action of N on $\mu_N^{-1}(0)$ is free. Consequently, $M_\Delta := \mu_N^{-1}(0)/N$ is a compact smooth manifold endowed with a residual action of $T = \mathbb{T}^d/N$. This action is Hamiltonian for the symplectic form on M coming from Hamiltonian reduction of ω_{std} and the image of the moment map is Δ.

To obtain the algebro-geometric quotient construction $\mu_N^{-1}(0)$ is enlarged to an open dense subset $\Sigma_\Delta \subset \mathbb{C}^d$ which can be described in several equivalent ways: it is the saturation of $\mu_N^{-1}(0)$ by the action of the complexification $N_{\mathbb{C}}$ of N, it is the subset of $\mathbb{C}^d$ where $N_{\mathbb{C}}$ acts freely and with closed orbits, and it is the collection of orbits the standard action $(\mathbb{C}^\times)^d \curvearrowright \mathbb{C}^d$ which intersect $\mu_N^{-1}(0)$. More explicitly, the orbits of $(\mathbb{C}^\times)^d \curvearrowright \mathbb{C}^d$ are parametrized by subsets I of $\{1, 2, \ldots, d\}$:

$$\mathbb{C}^d_I := \{z \in \mathbb{C}^d \mid z_i = 0 \;\Leftrightarrow\; i \in I\}.$$

Each subset $F(\Delta)_I \subset \Delta$,

$$F(\Delta)_I := \{\xi \in \Delta \mid \langle \xi, u_i\rangle = c_i \;\Leftrightarrow\; i \in I\},$$

is a face of Δ if nonempty, and

$$\Sigma_\Delta = \bigcup_{F(\Delta)_I \neq \varnothing} \mathbb{C}^d_I$$

These equivalent descriptions give a canonical identification of the compact quotient M_Δ with the GIT quotient $\Sigma_\Delta/N_{\mathbb{C}}$. The outcome is a complex (projective) structure on M_Δ together with a complex action of $(\mathbb{C}^\times)^d/N_{\mathbb{C}}$ with an open dense orbit.

Proposition 4.8. *Let $p\colon \Sigma_\Delta \to M_\Delta$ be the GIT quotient construction of the toric variety M_Δ. Then every positive bivector $\pi \in \wedge^2 \mathbb{C}^d$ induces Poisson structures π_Σ on Σ_Δ and π_M on M_Δ, for which the quotient map $p\colon (\Sigma_\Delta, \pi_\Sigma) \to (M_\Delta, \pi_M)$ is a Poisson submersion with Poisson fibres.*

Proof. The complex vector space $A := \mathbb{C}^d$ acts on $\mathbb{C}^d$ by the holomorphic transformation $(w_1, \ldots, w_d) \cdot (z_1, \ldots, z_d) := (e^{w_1}z_1, \ldots, e^{w_d}z_d)$. By Lemma 4.5, there is an induced A-invariant Poisson structure Π on $\mathbb{C}^d$, with the property that $(\mathfrak{n}_{\mathbb{C}}, 0) \curvearrowright (\mathbb{C}^d, \Pi)$ is a Poisson action with coregular Poisson–Dirac submanifolds as orbits, where $\mathfrak{n}_{\mathbb{C}} = \mathrm{Lie}(N_{\mathbb{C}})$. Because $\Sigma_\Delta \subset \mathbb{C}^d$ is a union of A-orbits, it is a Poisson submanifold, with induced Poisson structure $\pi_\Sigma := \Pi|_{\Sigma_\Delta}$. Because π_Σ is $N_{\mathbb{C}}$-invariant, the quotient map $p\colon \Sigma_\Delta \to M_\Delta$ pushes π_Σ to a Poisson structure π_M on M_Δ (cf. [6]), whose fibres are coregular Poisson–Dirac submanifolds of $(\Sigma_\Delta, \pi_\Sigma)$. ∎

When the positive bivector $\pi \in \wedge^2\mathbb{C}^d$ is nondegenerate, the leaves of (M_Δ, π_M) are the orbits of the complex torus action, and so are finite in number. We refer to such manifolds as *toric Poisson manifolds* (cf. [6]). If complex conjugation is an anti-Poisson automorphism of $(\mathbb{C}^d, \pi)$, we say that π is *totally real*.

4.4. Poisson submersions with Poisson fibres from varieties of full flags

A closed subgroup K of a Poisson–Lie group (G, π_G) is a *Poisson–Lie subgroup* if it is a Poisson submanifold of (G, π_G); otherwise said, if π_G is tangent to K, $\pi_G|_K = \pi_K \in \mathfrak{X}^2(K)$, in which case (K, π_K) becomes a Poisson–Lie group in its own right.

In the following proposition, we look at different ways in which a closed subgroup interacts with the ambient Poisson–Lie group structure (cf. Section 4 in [27] and Proposition 2 in [31]).

Proposition 4.9. *Let a Poisson–Lie group (G, π_G) correspond to the G-invariant Manin triple $(\mathfrak{d}, \langle\cdot,\cdot\rangle, \mathfrak{g}, \mathfrak{h})$, and for a connected, closed subgroup $K \subset G$ with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$, denote by $p: G \to M := G/K$ the quotient map under the action*

$$K \curvearrowright G, \quad \alpha(k, g) = gk^{-1}.$$

(a) *π_G is K-invariant $\iff [\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{h}$, in which case π_G vanishes along K.*
(b) *K is a Poisson submanifold $\iff \mathfrak{k}^\circ \subset \mathfrak{h}$ is an ideal $\iff [\mathfrak{k}, \mathfrak{k}^\circ] \subset \mathfrak{h}$.*
(c) *A Poisson structure π_M on M exists, for which the quotient map $p: (G, \pi_G) \to (M, \pi_M)$ is a Poisson submersion $\iff \mathfrak{k}^\circ \subset \mathfrak{h}$ is a subalgebra.*
(d) *If K is a Poisson submanifold, then fibres of $p: G \to M$ are Poisson–Dirac submanifolds if and only if*
$$\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{k} \oplus \mathfrak{k}^\circ) \subset \mathfrak{k}^\circ.$$
In that case, the Poisson structures on fibres are all trivial exactly when
$$\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{k} \oplus \mathfrak{h}) \subset \mathfrak{h}.$$

Proof. First note that the infinitesimal action $\mathfrak{k} \to \mathfrak{X}(G)$ is given by the map ϱ of (4.1), since
$$\frac{d}{dt} g \exp(-tv)^{-1} = \frac{d}{dt} l_g(\exp(tv)) = (l_g)_*(v) = v_g^L = \varrho(v)_g.$$
If we let $V \subset TG$ stand for the vertical bundle of $p: G \to M$, then

$$V = \{\varrho(v)_g \mid (g, v) \in G \times \mathfrak{k}\}, \quad V^\circ = \{\langle \theta_g^L, w\rangle \mid w \in \mathfrak{k}^\circ\}$$

Indeed, V is spanned by left-translates of $\mathfrak{k}$, and

$$\begin{aligned} V_g^\circ &= \{\langle \theta_g^L, w\rangle \mid \iota_{\varrho(v)_g}\langle \theta_g^L, w\rangle = \langle \iota_{\varrho(v)_g}\theta_g^L, w\rangle = \langle \mathrm{Ad}_{g^{-1}} \mathrm{pr}_{\mathfrak{g}} \mathrm{Ad}_g(v), w\rangle = 0,\ v \in \mathfrak{k}\} \\ &= \{\langle \theta_g^L, w\rangle \mid \langle v, w\rangle = 0,\ v \in \mathfrak{k}\} = \{\langle \theta_g^L, w\rangle \mid w \in \mathfrak{k}^\perp\} \\ &= \{\langle \theta_g^L, w\rangle \mid w \in \mathfrak{k}^\circ\}, \end{aligned}$$

where we used the fact that $\mathfrak{k}^{\perp} = \mathfrak{g} \oplus \mathfrak{k}^{\circ}$, and that $\langle \theta^L, \mathfrak{g}\rangle = 0$. Because the map $\varepsilon\colon \mathfrak{d} \to \Gamma(\mathbb{T}G)$ of (4.2) is an algebra map, we have

$$\varepsilon[v, w] = [\varepsilon(v), \varepsilon(w)] = [\varrho(v), \varrho(w) + \langle \theta^L, w\rangle] = [\varrho(v), \pi_G^{\sharp}\langle \theta^L, w\rangle + \langle \theta^L, w\rangle]$$

for all $v \in \mathfrak{k}$ and $w \in \mathfrak{h}$, and therefore,

$$[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{h} \quad \Longleftrightarrow \quad [\varrho(\mathfrak{k}), \Gamma(\mathrm{Gr}\,\pi_G)] \subset \Gamma(\mathrm{Gr}\,\pi_G),$$

which is the same as saying that the infinitesimal action $\varrho\colon \mathfrak{k} \to \mathfrak{X}(G)$ is by Poisson automorphisms of (G, π_G). Because K is assumed to be connected, this is in turn equivalent to demanding that $\pi_G \in \mathfrak{X}^2(G)$ be K-invariant. By multiplicativity of π_G,

$$\pi_{G,gk^{-1}} = (l_g)_*(\pi_{G,k^{-1}}) + (r_{k^{-1}})_*(\pi_{G,k}) = (r_{k^{-1}})_*(\pi_{G,k}), \quad (k,g) \in K \times G.$$

This implies that π_G vanishes along K, and hence that $(K, 0) \curvearrowright (G, \pi_G)$ is a Poisson action, whose orbits are the fibres of $p\colon G \to M$. This proves (a). Observe next that K is a Poisson submanifold if and only if

$$\pi_G^{\sharp}(N^*K) = \{\varrho(w)_k \mid (k, w) \in K \times \mathfrak{k}^{\circ}\}$$

is trivial, which is tantamount to saying that $\mathrm{Ad}_K(\mathfrak{k}^{\circ}) \subset \mathfrak{h}$, or, equivalently, that $[\mathfrak{k}, \mathfrak{k}^{\circ}] \subset \mathfrak{h}$ – which by invariance of $\langle \cdot, \cdot \rangle$ is yet equivalent to $[\mathfrak{k}^{\circ}, \mathfrak{h}] \subset \mathfrak{k}^{\circ}$. This proves (b). Moreover, because

$$\varepsilon(G \times \mathfrak{k}^{\circ}) = \{\rho(w)_g + \langle \theta_g^L, w\rangle \mid (g, w) \in G \times \mathfrak{k}^{\circ}\} = \mathcal{R}_{\pi_G}(V^{\circ}),$$

it follows that

$$\varepsilon(G \times \mathrm{Gr}(\mathfrak{k})) = \mathcal{R}_{\pi_G}\mathrm{Gr}(V).$$

Again by invariance of $\langle \cdot, \cdot \rangle$, we have that

$$\mathrm{Gr}(\mathfrak{k}) \ \text{ is a subalgebra} \quad \Longleftrightarrow \quad \mathfrak{k}^{\circ} \ \text{ is a subalgebra} \quad \Longleftrightarrow \quad [\mathfrak{k}, \mathfrak{k}^{\circ}] \subset \mathrm{Gr}(\mathfrak{k}),$$

and therefore $\mathfrak{k}^{\circ}$ is a subalgebra exactly when $\mathcal{R}_{\pi_G}\mathrm{Gr}(V)$ is a Dirac structure on G. Because $\mathcal{R}_{\pi_G}\mathrm{Gr}(V)$ contains the vertical bundle, it is a basic Dirac structure according to Proposition 1 in [17], that is, it is the pullback of a Dirac structure L_M on M,

$$\mathcal{R}_{\pi_G}\mathrm{Gr}(V) = p^{!}(L_M),$$

in which case

$$L_M = p_! \mathcal{R}_{\pi_G}\mathrm{Gr}(V) = p_! \mathrm{Gr}(\pi_G).$$

Because L_M arises from pushing forward a Dirac structure, it must itself be the graph of a Poisson structure π_M on M, for which $p\colon (G, \pi_G) \to (M, \pi_M)$ is a Poisson map. This proves (c).

Observe next that, if (K, π_K) is a Poisson–Lie subgroup of (G, π_G), then

$$\alpha : (K, -\pi_K) \times (G, \pi_G) \longrightarrow (G, \pi_G)$$

is a Poisson action, whose orbits are the fibres of $p\colon G \to M$. Invoking Lemma 4.2, we deduce that the fibres of p have an induced Poisson structure exactly when the intersection

$$\pi_G^{\sharp}(V^{\circ}) \cap V = \big\{\varrho(v)_g \mid \exists\, w \in \mathfrak{k}^{\circ},\ \varrho(v)_g = \varrho(w)_g\big\} = \big\{\varrho(v)_g \mid v \in \mathrm{pr}_{\mathfrak{g}} \ker(\varrho_g|_{\mathfrak{k}\oplus\mathfrak{k}^{\circ}})\big\}$$

is trivial. This is equivalent to the condition that

$$(g, v, w) \in G \times \mathfrak{k} \times \mathfrak{k}^\circ, \quad \mathrm{Ad}_g(v + w) \in \mathfrak{h} \quad \Longrightarrow \quad v = 0,$$

that is, that $\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{k} \oplus \mathfrak{k}^\circ) \subset \mathfrak{k}^\circ$. Arguing in exactly the same manner, we conclude that

$$\pi_G^\sharp(T^*G) \cap V = \{\varrho(v)_g \mid \exists\, w \in \mathfrak{h},\ \varrho(v)_g = \varrho(w)_g\}$$

is trivial (that is, that the induced structures on fibres are trivial) exactly when $\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{k} \oplus \mathfrak{h}) \subset \mathfrak{h}$. ■

On a compact connected semisimple Lie group a choice of maximal torus and root order gives rise to a "standard" Poisson–Lie group structure which descends to the *manifolds of full flags* [27].

Proposition 4.10. *Let a connected compact Lie group G be equipped with its "standard" Poisson structure π_G, and let (M, π_M) be the manifold of full flags. Then $p\colon (G, \pi_G) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres, and the Poisson structures on fibres are all trivial.*

Proof. Let us fix a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t}$. Then $\mathfrak{t}_\mathbb{C}$ is a Cartan subalgebra for the complex semisimple Lie algebra $\mathfrak{d} := \mathfrak{g}_\mathbb{C}$; hence there is a splitting of vector spaces

$$\mathfrak{d} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{d}_\alpha,$$

where Δ is the set of all roots $\alpha \in \mathfrak{t}_\mathbb{C}^* \setminus \{0\}$.

The Killing form $\mathrm{B}_\mathfrak{d} \in S^2\mathfrak{d}^*$ of $\mathfrak{d}$ is an invariant, nondegenerate, symmetric bilinear pairing. It is nondegenerate on $\mathfrak{t}_\mathbb{C} \times \mathfrak{t}_\mathbb{C}$ and on $\mathfrak{d}_\alpha \times \mathfrak{d}_{-\alpha}$, and it vanishes on $\mathfrak{d}_\alpha \times \mathfrak{d}_\beta$ if $\alpha + \beta \neq 0$. Moreover, there is a splitting of real vector spaces

$$\mathfrak{t}_\mathbb{C} = \mathfrak{t} \oplus \mathfrak{a}, \quad \mathfrak{a} := \{v \in \mathfrak{t}_\mathbb{C} \mid \alpha \in \Delta \ \Rightarrow\ \alpha(v) \in \mathbb{R}\},$$

and $\mathrm{B}_\mathfrak{d}|_{\mathfrak{t}\times\mathfrak{t}}$ is negative-definite and $\mathrm{B}_\mathfrak{d}|_{\mathfrak{a}\times\mathfrak{a}}$ is positive-definite. Let us now fix a set of positive roots $\Delta^+ \subset \Delta$, and write $\Delta = \Delta^+ \coprod \Delta^-$, where $\Delta^- := -\Delta^+$. Define

$$\mathfrak{n}_\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{d}_\alpha, \quad \mathfrak{b}_\pm := \mathfrak{t}_\mathbb{C} \oplus \mathfrak{b}_\pm.$$

Then $\mathfrak{n}_\pm$ are nilpotent Lie algebras, and $\mathfrak{b}_\pm$ are solvable Lie algebras; in fact, $[\mathfrak{b}_\pm, \mathfrak{b}_\pm] = \mathfrak{n}_\pm$. Moreover, the splittings $\mathfrak{b}_\pm = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}_\pm$ can be described in terms of the spectrum of the adjoint map $\mathrm{ad}(v)\colon \mathfrak{d} \to \mathfrak{d}$, in the sense that, for $v \in \mathfrak{b}_\pm$,

(S1) $v \in \mathfrak{t} \oplus \mathfrak{n}_\pm \iff \mathrm{Spec}(\mathrm{ad}(v)) \subset i\mathbb{R}$;

(S2) $v \in \mathfrak{a} \oplus \mathfrak{n}_\pm \iff \mathrm{Spec}(\mathrm{ad}(v)) \subset \mathbb{R}$;

(S3) $v \in \mathfrak{n}_\pm \iff \mathrm{Spec}(\mathrm{ad}(v)) = \{0\}$.

Define further

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{n}_+ \quad \text{and} \quad \langle v, w \rangle := \Im \mathrm{B}_\mathfrak{d}(v, w).$$

Then $(\mathfrak{d}, \langle\cdot,\cdot\rangle, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, and because G is connected, there are no choices in the representation $\mathrm{Ad} : G \curvearrowright \mathfrak{d}$, so we may regard $(\mathfrak{d}, \langle\cdot,\cdot\rangle, \mathfrak{g}, \mathfrak{h})$ as G-invariant Manin triple. The Poisson structure $\pi_G \in \mathfrak{X}^2(G)$ that corresponds to it is the "standard" or *Lu–Weinstein* Poisson structure on G. By item (a) in Proposition 4.9, the fact that

$$[\mathfrak{t}, \mathfrak{h}] = \mathfrak{n}_+ \subset \mathfrak{h}$$

implies that $p\colon G \to G/T$ pushes π_G to a Poisson structure π_M on $M = G/T$. On the other hand, suppose $(g, x, y, z, w) \in G \times \mathfrak{a} \times \mathfrak{n}_\pm \times \mathfrak{t} \times \mathfrak{n}_\pm$ is such that

$$\mathrm{Ad}_g(x + y) = z + w.$$

By (S1), $\mathrm{Spec}(\mathrm{ad}(z + w)) \subset i\mathbb{R}$, and by (S2),

$$\mathrm{Spec}(\mathrm{ad}\mathrm{Ad}_g(x + y)) = \mathrm{Spec}(\mathrm{ad}(x + y)) \subset \mathbb{R}.$$

Thus (S3) implies that $x = 0$ and $z = 0$. That is,

$$\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{t} \oplus \mathfrak{n}_+) \subset \mathfrak{n}_+,$$

which implies by item (d) in Proposition 4.9 that the fibres of $p\colon G \to M$ are Poisson–Dirac submanifolds. In fact, by the same argument one deduces that

$$\mathrm{Ad}_G(\mathfrak{h}) \cap (\mathfrak{t} \oplus \mathfrak{h}) \subset \mathfrak{h},$$

which is to say that all Poisson structures on fibres are trivial. ■

4.5. Poisson structures on associated bundles

Recall that if $p\colon P \to M$ is a right principal G-bundle, and G acts on the left on a manifold X, we refer to the quotient of the free left action

$$G \curvearrowright P \times X, \quad g \cdot (y, x) := (yg^{-1}, gx),$$

as the *associated bundle*

$$p : \Sigma := P \times_G X \longrightarrow M, \quad p[y, x] = p(y).$$

The first observation is that, when P is equipped with a G-invariant Poisson structure, the associated bundle Σ has an induced Poisson structure – and certain good properties of the principal bundle are inherited by its associated bundle:

Lemma 4.11. *Let a right principal G-bundle $p\colon P \to M$ be endowed with a G-invariant Poisson structure π_P, and let G act on the left of a Poisson manifold (X, π_X) by Poisson diffeomorphisms. Then there exist Poisson structures π_M on M and π_Σ on $\Sigma := P \times_G X$, for which*

$$\begin{tikzcd} (P, \pi_P) \times (X, \pi_X) \arrow[r, "q"] \arrow[d, "\mathrm{pr}_1"'] & (\Sigma, \pi_\Sigma) \arrow[d, "p"] \\ (P, \pi_P) \arrow[r, "p"'] & (M, \pi_M) \end{tikzcd}$$

are all Poisson submersions. If $p\colon (P, \pi_P) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres, or an orthogonal pencil, then so is $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$.

Proof. Because $\pi_P \in \mathfrak{X}^2(P)$ is G-invariant, there exists a unique Poisson structure $\pi_M \in \mathfrak{X}^2(M)$ such that $p\colon (P, \pi_P) \to (M, \pi_M)$ is a Poisson submersion. Likewise, the Poisson structure (π_P, π_X) on $P \times X$ is invariant under the diagonal action

$$G \curvearrowright P \times X, \quad g(y, x) = (yg^{-1}, gx),$$

and a unique Poisson structure π_Σ exists on Σ, for which $q : (P, \pi_P) \times (X, \pi_X) \to (\Sigma, \pi_\Sigma)$ is a Poisson submersion. In fact, $(\pi_P, 0)$ and $(0, \pi_X)$ are both G-invariant, and if we denote by $\pi_H, \pi_V \in \mathfrak{X}^2(\Sigma)$ the respective Poisson bivectors which correspond to them, we have a splitting into commuting Poisson structures

$$\pi_\Sigma = \pi_H + \pi_V,$$

in which π_V is tangent to the fibres of $p\colon \Sigma \to M$.

Suppose on the one hand that $p\colon (P, \pi_P) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres. Then

$$\pi_\Sigma^\sharp(V_\Sigma^\circ) = q_*(\pi_P^\sharp, \pi_X^\sharp)\, q^*(V_\Sigma^\circ) = q_*(\pi_P^\sharp, \pi_X^\sharp)(V_P^\circ \times X) = q_*(\pi_P^\sharp(V_P^\circ) \times X)$$

is a vector bundle, which meets V_Σ trivially because $q_*(\pi_P^\sharp(\xi), 0) \in V_\Sigma$ implies that $\pi_P^\sharp(\xi) \in V_P$. Hence $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ would be a Poisson submersion with Poisson fibres as well. Suppose on the other hand that $p\colon (P, \pi_P) \to (M, \pi_M)$ is an orthogonal pencil. By Theorem 3.18, this is equivalent to the vertical Poisson structures on fibres assembling into a smooth, vertical Poisson structure $\tilde{\pi}_V \in \Gamma(\wedge^2 V_\Sigma)$ – in which case $\pi_V := q_*\tilde{\pi}_V + \pi_X \in \Gamma(\wedge^2 V_\Sigma)$ is a smooth, vertical Poisson structure on Σ, which induces on fibres the same Poisson structure as π_Σ. Hence $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is an orthogonal pencil. ∎

Even when the principal bundle itself fails to be sufficiently well-behaved, one may sometimes impose conditions on the Poisson structure π_P and the action $G \curvearrowright (X, \pi_X)$ to require better behavior of its associated bundle, as in the following result.

Lemma 4.12. *Let a right principal G-bundle $p\colon P \to M$ be endowed with a G-invariant Poisson structure π_P, and let G act on the left of a Poisson manifold (X, π_X) by Poisson diffeomorphisms. Let $W_G \subset V_P \times TX$ denote the vertical bundle to $q\colon P \times X \to \Sigma$, and V_P the vertical bundle to $p\colon P \to M$. If $\pi_P^\sharp\colon W_G^\circ \to TP \times X$ meets $V_P \times X$ in a vector bundle, then $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is an orthogonal pencil.*

Proof. Observe that by the characterization of Theorem 3.18, $p\colon (\Sigma, \pi_H) \to (M, \pi_M)$ is an orthogonal pencil if and only if $\mathrm{Gr}(\pi_H) \cap (V_\Sigma \oplus T^*\Sigma) + V_\Sigma^\circ$ is a vector bundle. Because q is a Poisson submersion, this is the case exactly when

$$q^! \mathrm{Gr}(\pi_H) \cap \big((V_P \times TX) \oplus (T^*P \times T^*X)\big) + V_P^\circ \times X$$

is a vector bundle, and since $q^! \mathrm{Gr}(\pi_H) = \mathcal{R}_{\pi_H} \mathrm{Gr}(W_Q)$, that is implied by the hypothesis that

$$(\pi_P^\sharp, 0)^{-1}(V_P \times X) \subset W_Q^\circ$$

is a vector bundle. ∎

Let us (tentatively) say that a Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is *strongly locally trivial* if every point $x \in M$ lies in an open set $U \subset M$, over which there exists a diffeomorphism $\phi\colon U \times X \xrightarrow{\sim} p^{-1}(U)$ such that

$$\mathrm{Gr}(\pi_M)|_U \times \mathrm{Gr}(\pi_X) = \phi^! \,\mathrm{Gr}(\pi_\Sigma).$$

Any two fibres of a strongly locally trivial Poisson submersion are Poisson diffeomorphic. It is also worthwhile to consider the case of Poisson submersions in which nearby fibres are *gauge-equivalent*: we call a Poisson submersion with Poisson fibres *locally trivial* if

$$\mathcal{R}_\omega(\mathrm{Gr}(\pi_M)|_U \times \mathrm{Gr}(\pi_X)) = \phi^! \,\mathrm{Gr}(\pi_\Sigma) \tag{4.4}$$

for a closed two-form $\omega \in \Omega^2(U \times X)$ for which

$$\mathrm{id} + (0, \pi_X)^\sharp \omega : U \times TX \longrightarrow U \times TX$$

is an isomorphism.

Note that the latter condition is tantamount to requiring that $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ has Poisson fibres.

Remark 4.13. Strongly locally trivial Poisson submersions are locally trivial. In favorable circumstances, one may employ the Moser trick, see Lemma 4 in [15], to attempt to promote a local trivialization into a strong one. This strategy (or any strategy, for that matter) may fail, however: below we present an example which is locally trivial, but not strongly locally trivial for completeness reasons:

Example 4.14. *An example of a locally trivial Poisson submersion with Poisson fibres which is not strongly locally trivial.* Let $M = \mathbb{R}_+$ have coordinate t, and let

$$X \subset \mathbb{R}^2, \quad X = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}.$$

Consider on M the trivial Poisson structure $\pi_M = 0$, and on X the standard Poisson structure $\pi_X = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. Consider the closed two-form

$$\omega := -\mathrm{d}(t\alpha), \quad \alpha := \tfrac{1}{2}(x_1 \,\mathrm{d}x_2 - x_2 \,\mathrm{d}x_1),$$

on $\Sigma := M \times X$, equipped with the Poisson structure

$$\mathrm{Gr}(\pi_\Sigma) := \mathcal{R}_\omega(\mathrm{Gr}(\pi_M) \times \mathrm{Gr}(\pi_X)) = \mathrm{Gr}\big(\tfrac{1}{1+t}\, \pi_X\big).$$

Then the canonical projection

$$p : (\Sigma, \pi_\Sigma) \longrightarrow (M, 0)$$

is a Poisson submersion with Poisson fibres, which is locally trivial by construction.

However, no strong local trivialization can exist. For example, suppose a strong local trivialization would exist around, say $t = 1$. Then for some open set $1 \in U \subset M$, there would exist a smooth embedding φ,

$$\varphi : U \times X \xrightarrow{\sim} \Sigma,$$

such that $p\varphi = p$, and

$$\varphi : (U, \pi_U) \times (X, \pi_X) \longrightarrow \big(\Sigma, \tfrac{1}{1+t}\,\pi_X\big)$$

is a Poisson map. A such map would induce, for all $t \in U$, Poisson diffeomorphisms

$$\varphi_t : (X, \pi_X) \xrightarrow{\sim} \big(X, \tfrac{1}{1+t}\,\pi_X\big),$$

which is impossible since the symplectic areas change.

Example 4.15. A Poisson submersion $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is *complete* if the Hamiltonian vector field of a function $f \circ p \in C^\infty(\Sigma)$ is complete if that of $f \in C^\infty(M)$ is complete. It follows from Theorem 3.11 that a complete Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ restricts over each leaf $\mathrm{S}_M(x)$ of (M, π_M) to a strongly locally trivial Poisson submersion

$$p : (p^{-1}\mathrm{S}_M(x), \pi_{p^{-1}\mathrm{S}_M(x)}) \to (\mathrm{S}_M(x), \pi_{\mathrm{S}_M(x)}).$$

Indeed, recall that a Poisson submersion is coupling exactly when its horizontal foliation is an Ehreshmann connection. In such a situation there is a mandatory strategy to attempt to produce a strong local trivialization around any given fibre: to integrate the flows of Hamiltonian vector fields of (appropriate) functions pulled back from the base. The success of the construction for every fibre is equivalent to the completeness of the (coupling) Poisson submersion.

Lemma 4.16. *A locally trivial Poisson submersion is an orthogonal pencil. A strongly locally trivial Poisson submersion is almost-coupling.*

Proof. Let $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ be a locally trivial Poisson submersion. As we already observed, p has Poisson fibres, so in order to show that it is an orthogonal pencil, it suffices by Theorem 3.18 to check that the Poisson–Dirac structures on fibres vary smoothly. This is a local matter in M, which we need only check over a trivialization

$$\Phi := \phi_* \mathcal{R}_\omega : (U, \pi_M) \times (X, \pi_X) \xrightarrow{\sim} (p^{-1}(U), \pi_\Sigma).$$

But

$$\mathcal{R}_\omega(\mathrm{Gr}(\pi_M)|_U \times \mathrm{Gr}(\pi_X)) = \phi^! \mathrm{Gr}(\pi_\Sigma), \quad \mathrm{id} + (0, \pi_X)^\sharp \omega : U \times TX \xrightarrow{\sim} U \times TX,$$

exhibits the fibres of p as gauge-transformations of π_X by the restriction of the smooth form ω to fibres, which is clearly smooth. Thus a locally trivial Poisson submersion is an orthogonal pencil.

When the submersion is strongly locally trivial (so that ω may be chosen to vanish identically), then Ehresmann connections on $\mathrm{pr}\colon U \times X \to U$ which make it into an almost-coupling submersion form a non-empty convex set, and so a global Ehresmann connection can be built out of a partition of unity subordinated to a trivializing cover, which turns $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ into an almost-coupling submersion. ∎

Let π_P be an invariant Poisson structure on the total space of the right principal G-bundle $p\colon P \to M$. Then p pushes π_P to a Poisson structure π_M on M, i.e., $p\colon (P, \pi_P) \to (M, \pi_M)$ is a Poisson submersion. Such a "Poisson principal bundle" $p\colon (P, \pi_P) \to (M, \pi_M)$ is *equivariantly* locally trivial if around each $x \in M$ there is an open set $U \subset M$, over which there exists a principal bundle trivialization $\phi\colon U \times X \xrightarrow{\sim} p^{-1}(U)$ and a G-invariant closed two-form $\omega \in \Omega^2(U \times G)^G$ for which

$$\mathcal{R}_\omega(\mathrm{Gr}(\pi_M)|_U \times \mathrm{Gr}(\pi_G)) = \phi^! \mathrm{Gr}(\pi_P)$$

and

$$\mathrm{id} + (0, \pi_G)^\sharp \omega : U \times TG \xrightarrow{\sim} U \times TG$$

is an isomorphism.

Remark 4.17. A symplectic principal bundle with symplectic fibres is always equivariantly locally trivial: the symplectic orthogonal to fibres defines a *complete* principal (symplectic) connection, see Proposition 6.6 in [7].

There are Poisson principal bundles which are locally trivial but not equivariantly so, as the example below illustrates.

Example 4.18. Consider the trivial principal G-bundle $p\colon (G \times \mathbb{R}, \pi_P) \to \mathbb{R}$ endowed with a G-invariant vertical Poisson structure. That π_P be strongly locally trivial is equivalent to $\pi_{p^{-1}(x)}$ having nearby Poisson diffeomorphic fibres (by means of a smooth 1-parameter family of Poisson diffeomorphism). Equivariant strong local triviality amounts to realizing such diffeomorphisms by Lie group automorphisms.

Let $G := \mathrm{Aff}(\mathbb{R}) \times \mathrm{Aff}(\mathbb{R})$, where $\mathrm{Aff}(\mathbb{R})$ is the group of affine motions of the line. Consider the the 1-parameter family of linear 2-forms on the Lie algebra of G

$$e_1^* \wedge e_2^* + x e_1^* \wedge e_3^* + e_3^* \wedge e_4^* \in \wedge^2 \mathfrak{g}^*, \quad x \in \mathbb{R},$$

where

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{aff}(\mathbb{R})$$

and $e_3 = e_1$, $e_4 = e_2$ belong to a second copy of $\mathrm{aff}(\mathbb{R})$. Let π_P be the Poisson structure corresponding to the 1-parameter family of right-invariant symplectic forms which integrates the previous family of linear 2-forms. For different positive values of x, the right-invariant symplectic forms induced on G cannot be related by a Lie group automorphism, see Proposition 2.4 in [30]. However, the Moser trick in logarithmic coordinates produces a symplectic trivialization.

Lemma 4.19. *Bundles associated to strongly, equivariantly locally trivial principal bundles are strongly locally trivial.*

Proof. Let π_P be an invariant Poisson structure on the total space of the right principal G-bundle $p\colon P \to M$ which makes it into a Poisson submersion with Poisson fibres. If π_P is equivariantly strongly locally trivial, around each point $x \in M$ there is a local bundle trivialization $\phi\colon U \times G \to P$ and a G-invariant Poisson structure π_G on G, such that

$(\pi_M|_U, \pi_G) = \phi^*\pi_P$. Then $(\phi, \mathrm{id}_X)\colon U \times G \times X \to P \times X$ induces a local trivialization $\overline{\phi}\colon U \times X \to \Sigma$ of the associated bundle $\Sigma = P \times_G X$, making

$$\begin{tikzcd} U \times G \times X \arrow[r, "(\phi, \mathrm{id}_X)"] \arrow[d, "(\mathrm{id}_M, \alpha)"'] & P \times X \arrow[d, "q"] \\ U \times X \arrow[r, "\overline{\phi}"'] & \Sigma \end{tikzcd} \tag{4.5}$$

commute, where $\alpha\colon G \times X \to X$ is the action map. Now, π_G being invariant under the right action of G on itself means that $\pi_{G,g} = (r_g)_*(\pi_{G,e})$ for all $g \in G$. Observe that

$$\alpha_*(v^L_g, u) = -\rho(v) + (\alpha_g)_*(u), \quad v \in \mathfrak{g},\ u \in TX,$$

where $v^L \in \mathfrak{X}(G)$ denotes the left-invariant vector field corresponding to v, and $\rho(v) \in \mathfrak{X}(X)$ denotes the infinitesimal action of v. This implies that

$$\alpha : (G, \pi_G) \times (X, \pi_X) \longrightarrow (X, \widetilde{\pi}_X), \quad \widetilde{\pi}_X := \wedge^2\rho(\pi_{G,e}) + \pi_X, \tag{4.6}$$

is a Poisson map, and therefore

$$\overline{\phi} : (U, \pi_M) \times (X, \widetilde{\pi}_X) \xrightarrow{\sim} (\Sigma, \pi_\Sigma)$$

is a Poisson diffeomorphism, whence π_Σ is strongly locally trivial. ■

In yet another tentative flavor of local triviality of a Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$, one could require that it have a *locally trivial foliation*; namely, that around each point in M there exist a trivialization $\phi\colon U \times X \xrightarrow{\sim} p^{-1}(U)$ with the property that

$$\mathrm{Gr}(\pi_M)|_U \times \mathrm{Gr}(\pi_X) \quad \text{and} \quad \phi^!\,\mathrm{Gr}(\pi_\Sigma) \quad \text{induce the same singular foliation on } U \times X.$$

Clearly, locally trivial Poisson submersions have locally trivial foliation.

Example 4.20. *An example of a Poisson submersion with Poisson fibres which has locally trivial foliation, and yet is not locally trivial.* Consider the submersion

$$p : \Sigma = \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{C}P^1 = M$$

of Example 3.19, but now equipped with the Poisson structure

$$\pi_\Sigma = \tfrac{1}{2}(\mathscr{E}_0 \wedge \mathscr{V}_0 + \mathscr{E}_1 \wedge \mathscr{V}_1).$$

This bivector is invariant under the $\mathbb{C}^\times$-action, and hence p pushes π_Σ to a Poisson structure π_M on M. Consider the diagram of open embeddings

$$\begin{tikzcd} \mathbb{C}\times\mathbb{C}^\times \arrow[r, "\phi_0"] \arrow[d, "\mathrm{pr}_1"'] & \Sigma \arrow[d, "p"] & \mathbb{C}\times\mathbb{C}^\times \arrow[l, "\phi_1"'] \arrow[d, "\mathrm{pr}_1"] \\ \mathbb{C} \arrow[r] & M & \mathbb{C}, \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} (z_0, z_1) \arrow[r] \arrow[d] & (z_1, z_0 z_1) \arrow[d] & (z_0 z_1, z_1) \arrow[d] & (z_0, z_1) \arrow[l] \arrow[d] \\ z_0 \arrow[r] & {[1 : z_0]}, & {[z_0 : 1]} & z_0. \arrow[l] \end{tikzcd}$$

Then both ϕ_i are Poisson embeddings for the Poisson structure

$$\phi_i : (\mathbb{C} \times \mathbb{C}^\times, \Pi) \longrightarrow (\Sigma, \pi_\Sigma), \quad \Pi = (\mathscr{E}_0 - \tfrac{1}{2}\mathscr{E}_1) \wedge (\mathscr{V}_0 - \tfrac{1}{2}\mathscr{V}_1) + \tfrac{1}{4}(\mathscr{E}_1 \wedge \mathscr{V}_1).$$

From this, we read off that $\mathrm{pr}_1\colon (\mathbb{C} \times \mathbb{C}^\times, \Pi) \to (\mathbb{C}, \mathscr{E}_0 \wedge \mathscr{V}_0)$ is a Poisson submersion with Poisson fibres $(\mathbb{C}^\times, \frac{1}{4}\mathscr{E}_1 \wedge \mathscr{V}_1)$, and that

$$\mathrm{Gr}(\Pi) \quad \text{and} \quad \mathrm{Gr}(\mathscr{E}_0 \wedge \mathscr{V}_0) \times \mathrm{Gr}(\tfrac{1}{4}\mathscr{E}_1 \wedge \mathscr{V}_1) \tag{4.7}$$

have the same singular foliation. From this, it follows that $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is a Poisson submersion with Poisson fibres with locally trivial foliation. Note that both (M, π_M) and (Σ, π_Σ) have exactly three symplectic leaves. We claim that this Poisson submersion with Poisson fibres is not locally trivial around the fibres of p over the singular points of (M, π_M) – that is, around the two singular leaves of (Σ, π_Σ). Indeed, suppose by contradiction that there exist

- an open neighborhood $U \subset \mathbb{C}$ of 0,
- a closed two-form $\omega \in \Omega^2(U \times \mathbb{C}^\times)$,
- a diffeomorphism $\varphi : U \times \mathbb{C}^\times \xrightarrow{\sim} U \times \mathbb{C}^\times$ of the form

$$\varphi(x_0, y_0, x_1, y_1) = (x_0, y_0, u_1, v_1), \quad (x_0, y_0) \in U, \ (x_1, y_1) \in \mathbb{C}^\times,$$

with the property that

$$\mathcal{R}_\omega \mathrm{Gr}(\mathscr{E}_0 \wedge \mathscr{V}_0 + \tfrac{1}{4}\mathscr{E}_1 \wedge \mathscr{V}_1) = \varphi^! \mathrm{Gr}(\Pi). \tag{4.8}$$

Because

$$\mathrm{Gr}(\Pi)|_{U\setminus\{0\}\times\mathbb{C}^\times} = \mathrm{Gr}(\sigma_1) \quad \text{and} \quad \mathrm{Gr}(\mathscr{E}_0 \wedge \mathscr{V}_0 + \tfrac{1}{4}\mathscr{E}_1 \wedge \mathscr{V}_1)|_{U\setminus\{0\}\times\mathbb{C}^\times} = \mathrm{Gr}(\sigma_2)$$

for symplectic forms $\sigma_1, \sigma_2 \in \Omega^2(U \setminus \{0\})$, condition (4.8) then reads

$$\omega = \varphi^*\sigma_1 - \sigma_2.$$

Note that

$$\Big\langle \omega, \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial y_0} \Big\rangle = \frac{2}{(x_0^2 + y_0^2)(u_1^2 + v_1^2)}(f), \tag{4.9}$$

where

$$\begin{aligned} f =& - x_0\big(v_1\big(\tfrac{\partial v_1}{\partial x_0} + \tfrac{\partial u_1}{\partial y_0}\big) + u_1\big(\tfrac{\partial v_1}{\partial y_0} + \tfrac{\partial u_1}{\partial x_0}\big)\big) + y_0\big(u_1\big(\tfrac{\partial v_1}{\partial x_0} + \tfrac{\partial u_1}{\partial y_0}\big) - v_1\big(\tfrac{\partial v_1}{\partial y_0} + \tfrac{\partial u_1}{\partial x_0}\big)\big) \\ &- 2(x_0^2 + y_0^2)\big(\tfrac{\partial u_1}{\partial x_0}\tfrac{\partial v_1}{\partial y_0} - \tfrac{\partial u_1}{\partial y_0}\tfrac{\partial v_1}{\partial x_0}\big) - \tfrac{1}{2}(u_1^2 + v_1^2). \end{aligned}$$

We thus see that

$$\lim_{(x_0,y_0,x_1,y_1)\to(0,0,1,0)} f = -\frac{1}{2}$$

implies that

$$\lim_{(x_0,y_0,x_1,y_1)\to(0,0,1,0)} \langle \omega, \tfrac{\partial}{\partial x_0} \wedge \tfrac{\partial}{\partial y_0} \rangle = \lim_{(x_0,y_0,x_1,y_1)\to(0,0,1,0)} \tfrac{2}{(x_0^2+y_0^2)(u_1^2+v_1^2)} f = -\infty.$$

This shows that (4.9) cannot extend to a smooth function on the whole $U \times \mathbb{C}^\times$, and therefore no such ω can exist (on the whole $U \times \mathbb{C}^\times$). Therefore, the Poisson submersion with Poisson fibres $p\colon (\Sigma, \pi_\Sigma) \to (M, \pi_M)$ is not locally trivial, in spite of having a locally trivial foliation.

Example 4.21. *An example of a bundle associated to locally trivial principal bundle whose foliation is not locally trivial.* The translation action

$$G = \mathbb{R}^2 \curvearrowright P = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_3 > 0\}, \quad (a,b)\cdot(y_1,y_2,y_3) = (a+y_1, b+y_2, y_3),$$

turns

$$p : P \longrightarrow M = \mathbb{R}_+, \quad p(y_1, y_2, y_3) = y_3,$$

into a principal G-bundle. The Poisson structure

$$\pi_P = -y_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$$

on P is G-invariant, and $p\colon (P, \pi_P) \to (M, 0)$ is a locally trivial Poisson submersion with Poisson fibres. Let G act on the Poisson manifold

$$X = \mathbb{R}^2, \quad \pi_X = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2},$$

by $(a,b)\cdot(x_1,x_2) = (a+x_1, b+x_2)$. The submanifold

$$\Sigma \subset P \times X, \quad \Sigma = \{(0,0)\} \times \mathbb{R}_+ \times \mathbb{R}^2,$$

is a full slice to the action of G, and is therefore identified with the quotient $\Sigma = P \times_G X$. The associated Poisson structure is

$$\pi_\Sigma = (1 - y_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2},$$

which does not have locally trivial foliation around $y_3 = 1$.

Next, we observe the following.

Proposition 4.22. *Let a right principal G-bundle $p\colon P \to M$ be endowed with a G-invariant Poisson structure π_P, which makes it into a Poisson submersion with Poisson fibres. If G acts by Poisson diffeomorphisms on a Poisson manifold (X, π_X), with orbits contained in the symplectic leaves of (X, π_X), and*

(a) *either the Poisson structures on the fibres of $p\colon P \to M$ are all trivial,*

(b) *or the orbits of $G \curvearrowright X$ are isotropic submanifolds of the symplectic leaves of (X, π_X),*

then the induced Poisson structure π_Σ on the associated bundle $\Sigma = P \times_G X \to M$ has locally trivial foliation, and its leaf space is homeomorphic to that of $\mathrm{Gr}(\pi_M) \times \mathrm{Gr}(\pi_X)$.

Observe that condition (b) is satisfied for example when G is abelian and the action of G on X admits an (infinitesimal) moment map.

Proof. First observe that, because orbits of $G \curvearrowright X$ are tangent to the symplectic leaves of (X, π_X), there is a canonical singular foliation $\mathcal{S}_\Sigma$ on Σ induced by the singular foliations $\mathcal{S}_M$ on M and $\mathcal{S}_X$ on X, determined by the Poisson structures π_M and π_X: if $g_{ij}\colon U_i \cap U_j \to G$ is a cocycle representing $P \to M$, its composition with the action homomorphism $\psi\colon G \to \mathrm{Diff}(X)$ is a cocycle representing $\Sigma \to M$, and the product singular foliations $(U_i, \mathcal{S}_M) \times (X, \mathcal{S}_X)$ descend under the identifications

$$\phi_{ij} : U_i \cap U_j \times X \xrightarrow{\simeq} U_i \cap U_j \times X, \quad \phi_{ij}(y,x) = (y, \psi(g_{ij}(y))x),$$

to a singular foliation $\mathcal{S}_\Sigma$ on Σ, which is locally trivial by its very construction, and which is independent of the choice of cocycle. Observe moreover that the hypothesis that the action of G is tangent to leaves of $\mathcal{S}_X$ implies that the cocycle (U_i, ϕ_{ij}) induces the identity map

$$U_i \cap U_j \times X/\mathcal{S}_X \xrightarrow{\simeq} U_i \cap U_j \times X/\mathcal{S}_X,$$

where $X/\mathcal{S}_X$ is the leaf space of $\mathcal{S}_X$ – that is, the topological space obtained from X by identifying points which lie in the same leaf, equipped with the quotient topology. Therefore

$$\Sigma/\mathcal{S}_\Sigma \simeq M/\mathcal{S}_M \times X/\mathcal{S}_X.$$

We claim that, under either of the hypotheses (a) or (b) above, the singular foliation on Σ corresponding to the Poisson structure π_Σ coincides with the locally trivial singular foliation constructed in the preceding paragraph. The key observation is that it suffices to check this claim for coupling submersions:

The preimage under $\Sigma \to M$ of a leaf of π_M is saturated for both $\mathcal{S}_\Sigma$ and π_Σ. For each $y \in P$, the preimage $P|_{\mathrm{S}_M(py)} = p^{-1}\mathrm{S}_M(py)$ of the symplectic leaf of (M, π_M) through py is a Poisson submanifold of (P, π_P), and

$$p : (P|_{\mathrm{S}_M(py)}, \pi_P|_{\mathrm{S}_M(py)}) \longrightarrow (\mathrm{S}_M(py), \omega_{\mathrm{S}_M(py)}) \tag{4.10}$$

is a coupling Poisson submersion. It is also a principal G-bundle equipped with a G-invariant Poisson structure, and its associated bundle

$$p : (\Sigma|_{\mathrm{S}_M(py)}, \pi_\Sigma|_{\mathrm{S}_M(py)}) \longrightarrow (\mathrm{S}_M(py), \omega_{\mathrm{S}_M(py)}), \quad \Sigma|_{\mathrm{S}_M(py)} = P|_{\mathrm{S}_M(py)} \times_G X, \tag{4.11}$$

is the preimage of $\mathrm{S}_M(py)$ under $p\colon \Sigma \to M$, which (again by Theorem 3.11) is a Poisson submanifold of (Σ, π_Σ).

As a consequence, in order the prove the proposition, it suffices to verify the following.

If π_M is symplectic, then π_Σ induces $\mathcal{S}_\Sigma$. For in that case $p\colon (P, \pi_P) \to (M, \pi_M)$ is a coupling Poisson submersion, which, being G-invariant, is strongly locally trivial (as in Example 4.15). By Remark (4.17), a local trivialization

$$\phi : (U, \pi_M) \times (G, \pi_G) \longrightarrow (P, \pi_P)$$

of (4.11) induces a local trivialization

$$\bar{\phi} : (U, \pi_M) \times (X, \tilde{\pi}_X) \longrightarrow (\Sigma, \pi_\Sigma),$$

where $\tilde{\pi}_X := \wedge^2\rho(\pi_{G,e}) + \pi_X$ in the notation of (4.6). The proof concludes with the observation that, under either assumption (a) or (b) in the statement, $\tilde{\pi}_X$ and π_X induce the same singular foliation $\mathcal{S}_X$ on X. For (a), this is straightforward, whereas for (b) one still needs to argue that $\rho(\mathfrak{g}) \subset \widetilde{\pi}_X^\sharp(T^*X)$. But for any $v \in \mathfrak{g}$ and any $x \in X$, since the orbit $G \cdot x$ is isotropic, we have

$$\rho(v)_x = \pi_X^\sharp(\xi_x), \quad \xi_x \in N_x^*(G \cdot x),$$

which gives $\widetilde{\pi}_X^\sharp(\xi_x) = \pi_X^\sharp(\xi_x)$, since $\wedge^2\rho(\pi_{G,e})$ vanishes on $N_x^*(G \cdot x)$. Therefore

$$(U_i, \pi_M) \times (X, \widetilde{\pi}_X) \quad \text{defines} \quad \phi^* \mathcal{S}_\Sigma = \mathcal{S}_M|_{U_i} \times \mathcal{S}_X.$$

Hence $\mathcal{S}_\Sigma$ is the singular foliation induced by π_Σ. ■

4.6. Poisson structures with finitely many leaves.

As a final application of our methods, we construct associated bundles in which the Poisson structure on the total space has a finite number of symplectic leaves. The building blocks of our construction are two classes of Poisson manifolds with finitely many leaves:

(i) *toric Poisson manifolds* (coming from a non-degenerate positive bivector as described in Proposition 4.8), whose symplectic leaves are orbits of the action of a complex torus;

(ii) *manifolds of full flags* (as described in Proposition 4.10), whose symplectic leaves are Bruhat cells [27].

Proposition 4.23. *Let (G, π_G) be a compact, connected semisimple Lie group with its "standard" Poisson structure corresponding to the maximal torus $T \subset G$. Let (X, π_X) be*

- *either a manifold of full flags G'/T', again with its "standard" Poisson structure,*
- *or a toric Poisson manifold $T'_{\mathbb{C}} \curvearrowright M_\Delta$.*

Then any group homomorphism $T \to T'$ determines on the associated bundle $\Sigma = G \times_T X$ a Poisson structure π_Σ with a finite number of symplectic leaves.

Proof. By Proposition 4.10, the Poisson submersion $(G, \pi_G) \to (G/T, \pi_{G/T})$ has Poisson fibres with the trivial Poisson structure. We argue that the orbits of the torus action on both manifolds of full flags and toric varieties lie inside symplectic leaves – and thus fall within the hypotheses of Proposition 4.22. This in particular implies that the leaf space of the induced Poisson structure π_Σ on the ensuing associated bundle Σ is homeomorphic to the product of the leaf space of (M, π_X) and that of (X, π_X).

For a manifold of full flags G'/T', the left action of T' on (G', π'_G) is by Poisson diffeomorphisms (since inversion on G is an anti-Poisson map), and since $(G'/T', \pi_{G'/T'})$ has finitely many leaves, the orbits of $T' \curvearrowright G'/T'$ lie inside symplectic leaves.

For a toric Poisson manifold $T' \curvearrowright M_\Delta$, the action of $T'_{\mathbb{C}}$ on M_Δ is by Poisson diffeomorphism, and its orbits are exactly the symplectic leaves of the Poisson structure. ■

Example 4.24. Consider the Poisson–Lie group $(\mathrm{SU}_2, \pi_{\mathrm{SU}_2})$ corresponding to SO_2 and the toric Poisson manifold $(\mathbb{C}P^1, \pi_{\mathbb{C}P^1})$ arising from $\mathbb{C}^2 \setminus \{0\}$. Then a degree $k \in \mathbb{Z}$ homomorphism $z \mapsto z^k$ determines Poisson submersions with Poisson fibres

$$\mathrm{SU}_2 \times_{\mathrm{SO}_2} (\mathrm{SU}_2/\mathrm{SO}_2) \to \mathrm{SU}_2/\mathrm{SO}_2, \quad \mathrm{SU}_2 \times_{\mathrm{SO}_2} \mathbb{C}P^1 \to \mathrm{SU}_2/\mathrm{SO}_2$$

which have locally trivial foliation (and have respectively four and six leaves).

Proposition 4.25. *Let* $p\colon (P_\Delta, \pi_P) \to (M_\Delta, \pi_M)$ *be the* ($N_{\mathbb{C}}$*-principal) GIT presentation of the Poisson toric manifold* M_Δ*, and let* (X, π_X) *be*

- *either a manifold of full flags* G'/T'*,*
- *or a toric Poisson manifold* $T'_{\mathbb{C}} \curvearrowright M'_\Delta$ *whose Poisson structure is induced by a totally real bivector.*

Then, a group homomorphism $N \to T'$ *determines on the associated bundle* $\Sigma = P_\Delta \times_N X$ *a Poisson structure* π_Σ *with a finite number of symplectic leaves.*

Proof. To apply Proposition 4.22, it suffices to show that the orbits of $T' \curvearrowright (X, \pi_X)$ are isotropic. (Note that the fibres of $p\colon (P_\Delta, \pi_P) \to (M_\Delta, \pi_M)$ are symplectic).

For manifolds of full flags this is a consequence of the (global symplectic) charts in [26] for the Bruhat cells: the symplectic form splits as a product of area forms and the torus action is induced from a Cartesian product of rotations in the plane.

For toric Poisson manifolds, this is a consequence of the fact that the positive, nondegenerate bivector of which it is a quotient is totally real, and therefore the fixed-point set of complex conjugation (that is, the Lie algebra of T') is Lagrangian. ∎

Example 4.26. Consider the Poisson–Lie group $(\mathrm{SU}_2, \pi_{\mathrm{SU}_2})$ corresponding to SO_2 and a totally real toric Poisson manifold $(\mathbb{C}P^1, \pi_{\mathbb{C}P^1})$ arising from $\mathbb{C}^2 \setminus \{0\}$. Then a degree $k \in \mathbb{Z}$ homomorphism $z \mapsto z^k$ determines Poisson submersions with Poisson fibres

$$(\mathbb{C}^2 \setminus \{0\}) \times_{\mathrm{SO}_2} (\mathrm{SU}_2/\mathrm{SO}_2) \to \mathbb{C}P^1, \quad (\mathbb{C}^2 \setminus \{0\}) \times_{\mathrm{SO}_2} \mathbb{C}P^1 \to \mathbb{C}P^1,$$

which have locally trivial foliation (and have respectively six and nine leaves). The diffeomorphism type of these associated bundles (and those in Example 4.24) is determined by the parity of k, and it would be interesting to understand in which cases the Poisson structures with the same number of symplectic leaves are Poisson diffeomorphic.

A. Appendix

Symplectic leaves of a Poisson manifold (M, π) are not in general embedded submanifolds. However, because of the Weinstein splitting, the following is true: for each $x \in M$, there exists a diffeomorphism

$$\varphi : U \xrightarrow{\sim} V \times W,$$

where U is an open, connected neighborhood U of x in M, V is an open, connected neighborhood of x in $\mathrm{S}_M(x)$, and $x \in W \subset M$ is a connected submanifold transverse to V, with the property that

$$\varphi|_V = \mathrm{id}_V, \quad \varphi|_W = \mathrm{id}_W,$$

and, for each leaf $\mathrm{S}_M(y)$ of π,

$$\varphi(U \cap \mathrm{S}_M(y)) = V \times \Lambda(y), \quad \Lambda(y) \subset W,$$

with $\Lambda(x)$ being at most countable. This suggests the notion of a *leaf-like submanifold* (see Appendix B in [18]), which includes as examples leaves of singular foliations or Lie algebroids.

Definition A.1. A subset S of a smooth manifold M is a *leaf-like submanifold* if around each point $x \in S$, there is an open neighborhood U of x in M, together with a diffeomorphism

$$\phi : U \xrightarrow{\sim} V \times W$$

from U into the product of connected manifolds V and W, such that

$$\phi(U \cap S) = V \times \Lambda$$

for a subset $\Lambda \subset W$ which is at most countable.

The submanifolds $S_w^U := \phi^{-1}(V \times \{w\}) \subset U \cap S$, as w ranges in the set Λ, are called the *plaques* of S over U. Each plaque is an embedded submanifold, and plaques partition $U \cap S$:

$$U \cap S = \coprod_{w \in \Lambda} S_w^U.$$

As explained in Appendix B of [18], leaf-like manifolds are *initial* submanifolds; that is, they are abstract manifolds equipped with an injective immersion $j : S \to M$, with the property that, if $f : N \to M$ is a smooth map whose image lies in S, then the unique set-theoretic map $\tilde{f} : N \to S$ through which f factors is smooth. This implies that the differentiable structure on S in uniquely determined by that of M.

Definition A.2. Let $X \subset M$ be an embedded submanifold, and let $S \subset M$ be a leaf-like submanifold. We say that X and S *intersect cleanly* if

(a) $X \cap S$ is an embedded submanifold of S,

(b) $T(X \cap S) = TX \cap TS$.

Remark A.3. Condition (a) in Definition A.2 should be clarified. Because manifolds are supposed to be second-countable, they can have at most countably many connected components. Hence, when we say $X \cap S$ is a submanifold, it is implied that $X \cap S$ is the disjoint union of *countably many* connected (second-countable) submanifolds. However, we do allow connected components to have *different dimensions* (as in Example A.5).

This definition recovers the notion of clean intersection of manifolds, see Appendix C in [22], when S is also embedded.

Remark A.4. In Definition A.2, the notions of embedded- and leaf-like submanifolds play asymmetric roles, in that we do not require that the intersection $X \cap S$ be a leaf-like submanifold of X. As the example below shows, that need not always be the case.

Example A.5. Let M be the 3-torus endowed with a Kronecker-type foliation, meaning that every one-dimensional leaf is dense. Select a leaf S and fix a foliated chart $\varphi : U \to \mathbb{R}^3$ so that the vertical axis corresponds to a plaque S_i belonging to S. Define $X \subset U$ to be the preimage by φ of the helicoid in $\mathbb{R}^3$ with axis the vertical axis:

$$\mathbb{R}^2 \to \mathbb{R}^3, \quad (t, s) \mapsto (t\cos(s), t\sin(s), s).$$

Then the intersection of X and S is clean and consists of S_i and a countable collection of points which accumulate in S_i. Therefore $X \cap S$ is not a leaf-like submanifold.

In contrast, under Definition A.2, a clean intersection of an embedded submanifold with a leaf-like submanifold is always an initial submanifold of the former.

Lemma A.6. *If an embedded submanifold $X \subset M$ intersects a leaf-like submanifold $S \subset M$ cleanly, then $X \cap S$ is an initial submanifold of X.*

Proof. By hypothesis, $X \cap S$ is an embedded submanifold of S. Denote by

$$i : X \longrightarrow M, \quad j : S \longrightarrow M, \quad i' : X \cap S \longrightarrow S \quad \text{and} \quad j' : X \cap S \longrightarrow X$$

the implied immersions. Because S is leaf-like, j is initial, and because X is embedded, i and i' are initial. Consider a smooth map $f\colon N \to X$, whose image lies inside $X \cap S$. Then the image of the composition $i \circ f\colon N \to M$ with the inclusion $i\colon X \to M$ lies inside the image of the inclusion $j\colon S \to M$, and the latter, being an initial submanifold, has a smooth lift $\widetilde{f}\colon N \to S$ such that

$$i \circ f = j \circ \widetilde{f}.$$

The image of $\widetilde{f}$ is contained in the embedded submanifold $X \cap S$, which is also initial, and therefore $\widetilde{f}$ has a smooth lift $\widehat{f}\colon N \to X \cap S$, with

$$\widetilde{f} = i' \circ \widehat{f}.$$

Hence the lift $F := j' \circ \widehat{f}$ of f is smooth:

$$\begin{tikzcd}
X \cap S \arrow[rr, "i'"] \arrow[dd, "j'"'] & & S \arrow[dd, "j"] \\
& N \arrow[ul, "\widehat{f}"'] \arrow[ur, "\widetilde{f}"] \arrow[dl, "f"] & \\
X \arrow[rr, "i"'] & & M,
\end{tikzcd}$$

and this shows that j' is initial as well. ■

Proposition A.7. *Let X be an embedded submanifold, and let S be a leaf-like submanifold of a smooth manifold M. Then the following assertions are equivalent*:

(i) *X and S intersect cleanly.*

(ii) *$X \cap S$ is a disjoint union of (a priori uncountably many) initial submanifolds Z, for which*

$$T_z Z = T_z X \cap T_z S \quad \text{for each } z \in Z.$$

(iii) *Every $z \in X \cap S$ is the center of a coordinate chart $\varphi\colon M \supset U \to \mathbb{R}^m$ which is adapted to X and to the plaque of S over U which passes through z.*

Proof. (i) *implies* (ii). By definition of clean intersection, if X and S meet cleanly, then

$$X \cap S = \coprod Z_i$$

is a disjoint union of at most countably many embedded, connected submanifolds $Z_i \subset S$, with

$$T_z Z_i = T_z X \cap T_z S, \quad z \in Z_i.$$

Hence (i) (trivially) implies (ii).

(ii) *implies* (iii). Assume now that

$$X \cap S = \coprod Z_i$$

is a disjoint union of *possibly uncountably many* connected, initial submanifolds $Z_i \subset M$, with

$$T_z Z_i = T_z X \cap T_z S, \quad z \in Z_i.$$

Because S is leaf-like, we can find an open neighborhood U of any $z \in Z_i$, for which

$$U \cap S = \coprod S_a,$$

is a disjoint union of at most countably many plaques. Write $U \cap Z_i$ as a countable disjoint union of its connected components:

$$U \cap Z_i = \coprod_{A(i)} W_\beta.$$

Then $U \cap X \cap S_a$ is a union of such connected components. Let S_0 denote the plaque of S over U which passes through z, and W_0 the connected component of $U \cap X \cap S_0$ through z. Then $T_z W_0 = T_z X \cap T_z S_0$ ensures that one can build as in Proposition C.3.1 of [22] a coordinate chart (U, φ) of M which is centered at z and is adapted to both X and S_0. So (ii) implies (iii).

(iii) *implies* (i). If around each $z \in X \cap S$ one can find a chart $\varphi\colon M \supset U \to \mathbb{R}^m$ which is centered at z, and

$$\varphi(U \cap X) = \varphi(U) \cap W_X, \quad \varphi(U \cap S_0) = \varphi(U) \cap W_S,$$

where S_0 denotes the plaque of S over U through z, and W_X and W_S denote vector subspaces of $\mathbb{R}^m$, then

$$\varphi(U \cap X \cap S_0) = \varphi(U) \cap W_X \cap W_S$$

shows that

$$\varphi|_{U \cap S_0} : U \cap S_0 \longrightarrow W_S$$

is a coordinate chart of S adapted to $U \cap X \cap S_0$. This shows that $X \cap S$ is an embedded submanifold of S, and moreover, that

$$z \in X \cap S \quad \Longrightarrow \quad T_z(X \cap S) = T_z X \cap T_z S.$$

Therefore, (iii) implies (i). ∎

Remark A.8. The following subtlety in the formulation of Proposition A.7 is worth mentioning: we raised in item (ii) the *a priori* possibility that $X \cap S$ be the disjoint union of *uncountably* many submanifolds – and therefore *not* itself a submanifold, see Remark A.3 – only to rule out that possibility by concluding that $X \cap S$ must in fact be an embedded submanifold of S.

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