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Mordell–Weil groups and automorphism groups of elliptic $K3$ surfaces

Ichiro Shimada

Abstract. We present a method to calculate the action of the Mordell–Weil group of an elliptic $K3$ surface on the numerical Néron–Severi lattice of the $K3$ surface. As an application, we compute a finite generating set of the automorphism group of a $K3$ surface birational to the double plane branched along a 6-cuspidal sextic curve of torus type.

1. Introduction

We work over an algebraically closed field k.

Let X be a projective $K3$ $K3$ surface. We denote by S_X the *numerical Néron–Severi lattice* of X, that is, the group of numerical equivalence classes of divisors of X with the intersection pairing

$$\langle \quad \rangle : S_X \times S_X \to \mathbb{Z}.$$

Let $\mathrm{O}(S_X)$ denote the group of isometries of the lattice S_X. We investigate the automorphism group $\mathrm{Aut}(X)$ of X by means of the action

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X)$$

of $\mathrm{Aut}(X)$ on the lattice S_X.

Let $\phi\colon X \to \mathbb{P}^1$ be an elliptic fibration with a distinguished section $\zeta\colon \mathbb{P}^1 \to X$. In this case, we say that (ϕ, ζ) is a *Jacobian fibration*. We denote by $\mathrm{MW}(X, \phi, \zeta)$ the Mordell–Weil group of sections of ϕ with ζ being the zero element. An element $\sigma \in \mathrm{MW}(X, \phi, \zeta)$ acts on the generic fiber of ϕ by translation. Since X is minimal, this birational automorphism of X is an automorphism of X, and hence we have an embedding of $\mathrm{MW}(X, \phi, \zeta)$ into $\mathrm{Aut}(X)$. In this paper, we investigate the composite homomorphism

$$\mathrm{MW}(X, \phi, \zeta) \;\to\; \mathrm{Aut}(X) \;\to\; \mathrm{O}(S_X). \tag{1.1}$$

This homomorphism has been used in many situations in the study of automorphisms of $K3$ surfaces (see, for example, [26]). The purpose of this paper is to present a general algorithm to calculate (1.1) explicitly and to give applications.

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Borcherds' method ([5,6]) is a method to calculate a finite generating set of the image of $\mathrm{Aut}(X) \to \mathrm{O}(S_X)$ by means of a certain decomposition of the nef-and-big cone of X into a union of polyhedral cones. The first application of this method to the study of the automorphism group of a $K3$ surface was given by Kondō [16]. See also [31]. Since this method is based on lattice-theoretic computation, the geometric meaning of elements in the generating set obtained by this method is not clear in general. The homomorphism (1.1) helps us to express the generating set geometrically. See Remark 5.16.

As an application, we calculate the automorphism group of the complex $K3$ surface $X_{f,g}$ obtained as the minimal resolution of the double cover $\overline{X}_{f,g}$ of $\mathbb{P}^2$ defined by

$$w^2 = f(x,y,z)^2 + g(x,y,z)^3, \tag{1.2}$$

where f and g are very general homogeneous polynomials on $\mathbb{P}^2$ of degree 3 and 2, respectively. Here, being *very general* means that there exist at most countably many analytic subsets of $H^0(\mathbb{P}^2, \mathcal{O}(3)) \times H^0(\mathbb{P}^2, \mathcal{O}(2))$ with codimension ≥ 1 such that the pair (f, g) does not belong to any of them. We prove the following.

Theorem 1.1. *The automorphism group* $\mathrm{Aut}(X_{f,g})$ *of* $X_{f,g}$ *is generated by* 463 *involutions associated with double coverings* $X_{f,g} \to \mathbb{P}^2$ *and* 360 *elements of infinite order in Mordell–Weil groups of Jacobian fibrations of* $X_{f,g}$.

Here, by a *double covering*, we mean a generically finite morphism of degree 2.

Theorem 1.2. *The automorphism group* $\mathrm{Aut}(X_{f,g})$ *acts on the set of smooth rational curves on* $X_{f,g}$ *transitively.*

The branch curve of the finite double cover $\overline{X}_{f,g} \to \mathbb{P}^2$ is defined by the equation $f^2 + g^3 = 0$. This plane curve is called a 6*-cuspidal plane sextic of torus type*, and was studied intensively from various points of view. See, for example, [23]. In fact, Zariski [38] observed that there exists a 6-cuspidal plane sextic of *non*-torus type, and the seminal notion of *Zariski pairs* emerged from this observation. See [1] and [2]. In [9] and [29], this classical example of Zariski pairs was studied in relation to the theory of $K3$ surfaces. It would be an interesting problem to calculate the automorphism group of the $K3$ surface obtained from the 6-cuspidal plane sextic of non-torus type.

The generating set in Theorem 1.1 is constructed in such a way that we can clearly see the geometric meaning of each element. See Section 6 for more precise descriptions of these automorphisms. Remark that this generating set is not minimal at all.

In fact, we give divisors of $X_{f,g}$ whose classes generate S_X. Hence we can calculate, in principle, the equations of the double coverings and the Jacobian fibrations by the method given in [30]. The actual computation of the equations, however, would be very hard.

Theorem 1.1 is proved in the following three steps.

(a) We find many automorphisms of $X_{f,g}$ geometrically by the methods explained in Section 3 (especially Section 3.7) and Section 4.

(b) We find a finite generating set of $\mathrm{Aut}(X_{f,g})$ by Borcherds' method, which will be explained in Section 5.

(c) We then show that the group generated by the automorphisms obtained in Step (a) contains the generating set obtained in Step (b).

See [18], [19] and [34] for general finiteness results of the automorphism group of a $K3$ surface and its action on the nef-and-big cone.

This paper is organized as follows. After fixing some notions and notation about lattices in Section 2, we summarize in Section 3 various computational tools that are useful in the study of the geometry of $K3$ surfaces. These tools are based on an algorithm given in [30] to calculate $\mathrm{Sep}(v_1, v_2)$ of *separating* (-2)*-vectors* in a hyperbolic lattice. In Section 4, we present an algorithm to calculate the homomorphism (1.1). In Section 5, we review Borcherds' method. We employ a graph-theoretic formulation of Borcherds' method given in Section 4.1 of [7]. Sections 3–5 are intended to be summaries of computational methods in the study of $K3$ surfaces for future reference. In Section 6, we calculate $\mathrm{Aut}(X_{f,g})$ by means of all these algorithms, and prove Theorems 1.1 and 1.2. We used `GAP` [11] for the actual computation. Detailed computation data about $\mathrm{Aut}(X_{f,g})$ can be found in the author's webpage [28].

2. Notation and terminologies

By a *lattice*, we mean a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle \quad \rangle : L \times L \to \mathbb{Z},$$

which we call the *intersection form* (or the *intersection pairing*) of L. The group of isometries of a lattice L is denoted by $\mathrm{O}(L)$, which we let act on L from the *right*.

Let L be a lattice. Then the *dual lattice* $L^\vee$ of L is defined to be

$$\{x \in L \otimes \mathbb{Q} \mid \langle x, v \rangle \in \mathbb{Z} \text{ for all } v \in L\}.$$

The finite abelian group $A(L) := L^\vee / L$ is called the *discriminant group* of L. We say that L is *unimodular* if $L = L^\vee$.

A lattice L is said to be *even* if $\langle v, v \rangle \in 2\mathbb{Z}$ holds for all $v \in L$. A *root* of an even lattice L is a vector $r \in L$ such that $\langle r, r \rangle$ is either 2 or -2. A (-2)*-vector* of L is a root $r \in L$ such that $\langle r, r \rangle = -2$. Suppose that L is even and negative-definite. Then the set

$$\mathrm{Roots}(L) := \{r \in L \mid \langle r, r \rangle = -2\}$$

is finite. An even negative-definite lattice L is called a *root lattice* if L is generated by $\mathrm{Roots}(L)$. A root lattice has a basis consisting of roots whose dual graph is a Dynkin diagram of type ADE. See, for example, Section 1 in [10] for the definition of dual graphs, Dynkin diagrams, and their ADE-types.

A lattice L of rank $n > 1$ is said to be *hyperbolic* if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(1, n-1)$. Let L be an even hyperbolic lattice. A *positive cone* of L is one of the two connected components of the space

$$\{x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0\}.$$

Let $\mathcal{P}$ be a positive cone of L. We put

$$\mathrm{O}(L, \mathcal{P}) := \{g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P}\}.$$

We have $\mathrm{O}(L) = \mathrm{O}(L, \mathcal{P}) \times \{\pm 1\}$. For $v \in L \otimes \mathbb{R}$, we put

$$v^{\perp} := \{x \in L \otimes \mathbb{R} \mid \langle x, v \rangle = 0\}.$$

When $v \in \mathcal{P} \cap L$, the intersection $v^{\perp} \cap L$ is an even negative-definite sublattice of L, and hence we can effectively calculate the finite set

$$\mathrm{Roots}(v^{\perp} \cap L) = \{r \in L \mid \langle r, v \rangle = 0,\ \langle r, r \rangle = -2\}$$

of (-2)-vectors in L perpendicular to v.

For $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, we put

$$(v)^{\perp} := v^{\perp} \cap \mathcal{P} = \{x \in \mathcal{P} \mid \langle x, v \rangle = 0\},$$

which is a real hyperplane of $\mathcal{P}$. Let $v_1, v_2 \in L \otimes \mathbb{Q}$ be rational vectors in $\mathcal{P}$. Then we can calculate the finite set

$$\mathrm{Sep}(v_1, v_2) := \{r \in L \mid\ \langle r, v_1 \rangle > 0,\ \langle r, v_2 \rangle < 0,\ \langle r, r \rangle = -2\}$$

of (-2)-vectors *separating* v_1 *and* v_2. See [30] for the algorithm. As will be explained in Section 3, this algorithm is very useful in the study of $K3$ surfaces.

Definition 2.1. By a *chamber*, we mean a closed subset D of $\mathcal{P}$ such that

- D contains a non-empty open subset of $\mathcal{P}$, and
- D is defined by linear inequalities $\langle x, v_i \rangle \geq 0$ ($i \in I$), where v_i ($i \in I$) are vectors of $L \otimes \mathbb{R}$ with $\langle v_i, v_i \rangle < 0$ such that the family $\{(v_i)^{\perp} \mid i \in I\}$ of hyperplanes is locally finite in $\mathcal{P}$.

Definition 2.2. Let D be a chamber. A *wall* of D is a closed subset of D of the form $D \cap (v)^{\perp}$ such that the hyperplane $(v)^{\perp}$ is disjoint from the interior of D and such that $D \cap (v)^{\perp}$ contains a non-empty open subset of $(v)^{\perp}$. We say that a vector $v \in L \otimes \mathbb{R}$ *defines* a wall w of D if $w = D \cap (v)^{\perp}$ and $\langle x, v \rangle > 0$ for an interior point x of D (and hence $\langle x, v \rangle \geq 0$ for all $x \in D$). A defining vector of a wall of a chamber is unique up to positive multiplicative constant.

Definition 2.3. Let $\mathcal{F} := \{(v_\alpha)^{\perp} \mid \alpha \in F\}$ be a locally finite family of hyperplanes in $\mathcal{P}$. Then the closure in $\mathcal{P}$ of each connected component of

$$\mathcal{P} \setminus \bigcup_{\alpha \in F} (v_\alpha)^{\perp}$$

is a chamber. Let $\mathcal{C}_{\mathcal{F}}$ be the set of these chambers. In this situation, we say that $\mathcal{P}$ is *tessellated by the chambers in* $\mathcal{C}_{\mathcal{F}}$. If a subset N of $\mathcal{P}$ is the union of chambers in a subset of $\mathcal{C}_{\mathcal{F}}$, we say that N is *tessellated by chambers in* $\mathcal{C}_{\mathcal{F}}$.

Let w be a wall of a chamber $D \in \mathcal{C}_{\mathcal{F}}$. Then there exists a unique chamber $D' \in \mathcal{C}_{\mathcal{F}}$ such that $D \neq D'$ and $w \subset D'$. This chamber D' is called the chamber *adjacent to* D *across the wall* w.

A (-2)-vector $r \in L$ defines a reflection

$$s_r : x \mapsto x + \langle x, r \rangle r$$

into the mirror $(r)^{\perp}$. We have $s_r \in \mathrm{O}(L, \mathcal{P})$. Let $W(L)$ denote the subgroup of $\mathrm{O}(L, \mathcal{P})$ generated by all the reflections s_r with respect to (-2)-vectors r. We call $W(L)$ the *Weyl group* of L. Note that the family of hyperplanes $(r)^{\perp}$ defined by (-2)-vectors r is locally finite in $\mathcal{P}$.

Definition 2.4. A *standard fundamental domain of* $W(L)$ is the closure of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp},$$

where r runs through the set of (-2)-vectors.

Let D be a standard fundamental domain of $W(L)$. We put

$$\mathrm{O}(L, D) := \{g \in \mathrm{O}(L) \mid D^g = D\}.$$

Then we have $\mathrm{O}(L, \mathcal{P}) = W(L) \rtimes \mathrm{O}(L, D)$. The action of $\mathrm{O}(L, \mathcal{P})$ on $\mathcal{P}$ preserves the tessellation of $\mathcal{P}$ by the standard fundamental domains of $W(L)$.

3. The numerical Néron–Severi lattice of a $K3$ surface

Let X be a $K3$ surface, and let S_X be the lattice of numerical equivalence classes of divisors of X, which we call the *numerical Néron–Severi lattice* of X. For a divisor D of X, we denote by $[D] \in S_X$ the class of D. Suppose that S_X is of rank $n > 1$. Then S_X is an even hyperbolic lattice. Let $\mathcal{P}_X$ be the positive cone of S_X containing an ample class of X, and let $\overline{\mathcal{P}}_X$ be the closure of $\mathcal{P}_X$ in $S_X \otimes \mathbb{R}$. We put

$$\begin{aligned} N_X &:= \{x \in \mathcal{P}_X \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } X\}, \\ N_X^{\circ} &:= \text{the interior of } N_X, \\ \overline{N}_X &:= \text{the closure of } N_X \text{ in } \overline{\mathcal{P}}_X. \end{aligned}$$

The cone N_X is called the *nef-and-big cone* of X. If C is a smooth rational curve on X, then its class $[C]$ is a (-2)-vector of S_X. We put

$$\mathrm{Rats}(X) := \{[C] \in S_X \mid C \text{ is a smooth rational curve on } X\}.$$

We have the following.

Theorem 3.1. *The nef-and-big cone N_X is a standard fundamental domain of the Weyl group $W(S_X)$ of S_X. A (-2)-vector $r \in S_X$ belongs to* $\mathrm{Rats}(X)$ *if and only if r defines a wall of the chamber N_X.*

Suppose that we have an ample class $\boldsymbol{a} \in N_X^{\circ} \cap S_X$. Then Vinberg's algorithm [36] enables us to enumerate, for a given positive integer m, all the walls $N_X \cap (r)^{\perp}$ of N_X defined by $r \in \mathrm{Rats}(X)$ with $\langle r, \boldsymbol{a} \rangle \leq m$. (See (3.2) below.) Our algorithm [30] of calculating the set $\mathrm{Sep}(v_1, v_2)$ of separating (-2)-vectors provides us with an alternative method to investigate the nef-and-big cone N_X. Below are some examples.

3.1. Finding an ample class

It is well known that a class $v \in S_X$ is ample if and only if $v \in N_X^\circ$. Let $\overline{X}$ be a normal surface birational to X, and let $h \in S_X$ be the pull-back of an ample class of $\overline{X}$ by the minimal resolution $X \to \overline{X}$. Then we have $h \in N_X$. It is known [3] that $\overline{X}$ has only rational double points as its singularities, and hence the exceptional locus of the desingularization $X \to \overline{X}$ is a union of smooth rational curves whose dual graph is a Dynkin diagram of type ADE. Let $r_1, \dots, r_\mu$ be the classes of smooth rational curves contracted by $X \to \overline{X}$. Then, *locally around* h, the chamber N_X is defined by $\langle x, r_i \rangle \geq 0$ for $i = 1, \dots, \mu$. Therefore a vector $v \in \mathcal{P}_X \cap S_X$ is ample if and only if

$$\mathrm{Sep}(h, v) = \emptyset, \quad \mathrm{Roots}(v^\perp \cap S_X) = \emptyset \quad \text{and} \quad \langle v, r_i \rangle > 0 \text{ for } i = 1, \dots, \mu.$$

If $a' \in S_X$ satisfies $\langle a', r_i \rangle > 0$ for $i = 1, \dots, \mu$, then

$$\boldsymbol{a} := mh + a'$$

is ample for sufficiently large integers m.

3.2. Nefness and ampleness

Suppose that we have an ample class $\boldsymbol{a} \in S_X$. We can characterize N_X as the unique standard fundamental domain of $W(S_X)$ containing $\boldsymbol{a}$. Let $v \in S_X$ be a vector with $\langle v, v \rangle > 0$. Then we have

$$v \in \mathcal{P}_X \iff \langle \boldsymbol{a}, v \rangle > 0.$$

When $v \in \mathcal{P}_X$ is the case, we have

$$v \in N_X \iff \mathrm{Sep}(\boldsymbol{a}, v) = \emptyset.$$

When $v \in N_X$ is the case, we have

$$v \in N_X^\circ \iff \mathrm{Roots}(v^\perp \cap S_X) = \emptyset.$$

3.3. The group $\mathrm{O}(S_X, N_X)$

Recall that $\mathrm{O}(S_X, N_X)$ is the subgroup of $\mathrm{O}(S_X, \mathcal{P}_X)$ consisting of all isometries g such that $N_X^g = N_X$. Suppose again that we have an ample class $\boldsymbol{a} \in S_X$. Let g be an element of $\mathrm{O}(S_X)$. Then we have

$$g \in \mathrm{O}(S_X, \mathcal{P}_X) \iff \langle \boldsymbol{a}, \boldsymbol{a}^g \rangle > 0.$$

When $g \in \mathrm{O}(S_X, \mathcal{P}_X)$ is the case, we have

$$g \in \mathrm{O}(S_X, N_X) \iff \mathrm{Sep}(\boldsymbol{a}, \boldsymbol{a}^g) = \emptyset, \tag{3.1}$$

because, for $g \in \mathrm{O}(S_X, \mathcal{P}_X)$, the chamber N_X^g is also a standard fundamental domain of $W(S_X)$.

3.4. The set $\mathrm{Rats}(X)$

Again we assume that we have an ample class $\boldsymbol{a} \in S_X$. Let $r \in S_X$ be a (-2)-vector such that $\langle \boldsymbol{a}, r\rangle > 0$. Then there exists an effective divisor D of X such that $r = [D]$. We have $r \in \mathrm{Rats}(X)$ if and only if D is irreducible.

Since D contains a smooth rational curve C such that $\langle [C], [D]\rangle < 0$ as an irreducible component, we have the following criterion, which is a geometric interpretation of Vinberg's algorithm [36] applied to (-2)-vectors:

$$r \in \mathrm{Rats}(X) \iff \langle r, r'\rangle \geq 0 \text{ for all } r' \in \mathrm{Rats}(X) \text{ with } \langle r', \boldsymbol{a}\rangle < \langle r, \boldsymbol{a}\rangle \tag{3.2}$$

Thanks to the algorithm to calculate $\mathrm{Sep}(v_1, v_2)$, we obtain another criterion.

Proposition 3.2. *Let $r \in S_X$ be a (-2)-vector with $\langle \boldsymbol{a}, r\rangle > 0$. We put*

$$a'_r := \boldsymbol{a} + \frac{\langle \boldsymbol{a}, r\rangle}{2}\, r.$$

Then $r \in \mathrm{Rats}(X)$ if and only if

$$\mathrm{Roots}(a'^{\perp}_r \cap S_X) = \{r, -r\} \quad \text{and} \quad \mathrm{Sep}(a'_r, \boldsymbol{a}) = \emptyset. \tag{3.3}$$

Proof. Since $\langle a'_r, r\rangle = 0$ and $\langle a'_r, a'_r\rangle > 0$, we have $a'_r \in (r)^{\perp} \subset \mathcal{P}_X$, and hence the set $\mathrm{Sep}(a'_r, \boldsymbol{a})$ makes sense. In fact, the point $a'_r \in (r)^{\perp}$ is the image of $\boldsymbol{a}$ by the orthogonal projection to the hyperplane $(r)^{\perp}$ in $\mathcal{P}$. In particular, we have $\{r, -r\} \subset \mathrm{Roots}(a'^{\perp}_r \cap S_X)$. Then Proposition 3.2 follows from Proposition 2.2 in [37]. We present a proof for the convenience of readers.

If (3.3) holds, then $a'_r \in N_X$ and a small neighborhood of a'_r in $(r)^{\perp}$ is contained in N_X. In particular, r is a defining (-2)-vector of a wall of N_X and hence $r \in \mathrm{Rats}(X)$. Conversely, suppose that $r \in \mathrm{Rats}(X)$. Then for any $r' \in \mathrm{Rats}(X)$ with $r' \neq r$, we have $\langle r, r'\rangle \geq 0$ and $\langle \boldsymbol{a}, r'\rangle > 0$, and hence

$$\langle a'_r, r'\rangle = \langle \boldsymbol{a}, r'\rangle + \frac{\langle \boldsymbol{a}, r\rangle\langle r, r'\rangle}{2} > 0.$$

Therefore (3.3) holds. ∎

3.5. Nefness of a vector of norm 0

Suppose again that we have $\boldsymbol{a} \in N_X^{\circ} \cap S_X$.

Proposition 3.3. *Let f be a non-zero vector in $\overline{\mathcal{P}}_X \cap S_X$ with $\langle f, f\rangle = 0$. Then $f \in \overline{N}_X$ if and only if $\mathrm{Sep}(a'_f, \boldsymbol{a}) = \emptyset$, where $a'_f := \boldsymbol{a} + \langle \boldsymbol{a}, f\rangle f$.*

Proof. First note that, since $f \in \overline{\mathcal{P}}_X \setminus \{0\}$, we have $\langle \boldsymbol{a}, f\rangle > 0$, $a'_f \in \mathcal{P}_X$, and hence $\mathrm{Sep}(a'_f, \boldsymbol{a})$ makes sense.

Suppose that $f \in \overline{N}_X$. Since $\boldsymbol{a} \in N_X^{\circ}$, we have $a'_f \in N_X^{\circ}$ and hence $\mathrm{Sep}(a'_f, \boldsymbol{a}) = \emptyset$. Suppose that $f \notin \overline{N}_X$. Then there exists a smooth rational curve C such that $\langle f, [C]\rangle < 0$. We put $r := [C]$. Then we have $\langle f, r\rangle \leq -1$. Since $\langle f, f\rangle = 0$ and $\langle f, \boldsymbol{a}\rangle > 0$, there exists an effective divisor F on X such that $f = [F]$. Then C is an irreducible component of F such that $C \neq F$, and hence $\langle \boldsymbol{a}, r\rangle < \langle \boldsymbol{a}, f\rangle$. The intersection point of $(r)^{\perp}$ and the open line segment

$$(\boldsymbol{a}, f) := \{p(t) = \boldsymbol{a} + tf \mid t \in \mathbb{R}_{>0}\} \subset \mathcal{P}_X$$

is equal to $p(t_0)$, where

$$t_0 := -\frac{\langle \boldsymbol{a}, r\rangle}{\langle f, r\rangle} \leq \langle \boldsymbol{a}, r\rangle < \langle \boldsymbol{a}, f\rangle.$$

Since $a'_f = p(\langle \boldsymbol{a}, f\rangle)$, the intersection point $p(t_0)$ is located on the open line segment $(\boldsymbol{a}, a'_f) \subset (\boldsymbol{a}, f)$. Therefore r is a (-2)-vector separating a'_f and $\boldsymbol{a}$. ■

3.6. Singularities of a normal surface birational to X

Suppose again that we have $\boldsymbol{a} \in N_X^{\circ} \cap S_X$. Let h be a vector in $N_X \cap S_X$, and let $\mathcal{L}$ be a line bundle whose class is h. Then, for some large positive integer m, the complete linear system $|\mathcal{L}^{\otimes m}|$ gives a birational morphism $X \to \overline{X}$ to a normal surface $\overline{X}$. See Saint-Donat [25]. The surface $\overline{X}$ is smooth if and only if $h \in N_X^{\circ}$. Suppose that $h \notin N_X^{\circ}$. Then the singularities of $\overline{X}$ consist of rational double points (see Artin [3]), and the set of classes of smooth rational curves contracted by the birational morphism $X \to \overline{X}$ is equal to

$$\{r \in \mathrm{Rats}(X) \mid \langle r, h\rangle = 0\} \;=\; \mathrm{Rats}(X) \cap \mathrm{Roots}(h^{\perp} \cap S_X).$$

3.7. Finding automorphisms from nef vectors of norm 2

Let $\boldsymbol{a} \in S_X$ be an ample class of X. Let h be a vector in $N_X \cap S_X$ with $\langle h, h\rangle = 2$. By a *double covering*, we mean a generically finite morphism of degree 2. By abuse of notation, we write $|h|$ for the complete linear system of a line bundle whose class is h. Then either one of the following holds (see Saint-Donat [25] or Nikulin [22]):

- (h1) The complete linear system $|h|$ is base-point free and defines a double covering $\pi(h)\colon X \to \mathbb{P}^2$, or
- (h2) $|h|$ has a fixed component Z, which is a smooth rational curve, and every member of $|h|$ is of the form $Z + E_1 + E_2$, where E_1 and E_2 are members of a pencil $|E|$ of elliptic curves such that $\langle [E], [Z]\rangle = 1$.

These two cases can be distinguished by the following criterion. We put

$$\mathcal{E} := \{e \in S_X \mid \langle e, e\rangle = 0, \langle e, h\rangle = 1\}.$$

Since the quadratic part of the intersection form $\langle\quad\rangle$ restricted to the affine hyperplane of $S_X \otimes \mathbb{R}$ defined by $\langle x, h\rangle = 1$ is negative-definite, the set $\mathcal{E}$ is finite and can be calculated effectively.

Case ($\mathcal{E}$1). If $\mathcal{E} = \emptyset$, then $|h|$ is base-point free. In this case, we say that h is a *polarization of degree* 2, and denote by $i(h) \in \mathrm{Aut}(X)$ the involution associated with the double covering $\pi(h)\colon X \to \mathbb{P}^2$ given by $|h|$. Let

$$X \;\to\; \overline{X} \;\to\; \mathbb{P}^2$$

be the Stein factorization of $\pi(h)$, and let $B(h) \subset \mathbb{P}^2$ be the branch curve of the finite double covering $\overline{X} \to \mathbb{P}^2$. We can calculate the set

$$\mathrm{Rats}(X) \cap \mathrm{Roots}(h^{\perp} \cap S_X)$$

of classes of smooth rational curves contracted by $\pi(h)$. Hence we obtain the ADE-type of $\mathrm{Sing}(B(h))$, and the invariant part

$$\{v \in S_X \otimes \mathbb{Q} \mid v^{i(h)} = v\}$$

of the action of $i(h)$ on $S_X \otimes \mathbb{Q}$. Indeed, applying to $\overline{X}$ the theory of *canonical resolutions* of rational double points due to Horikawa [12], we have a successive blowing up $Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ such that $X \to \mathbb{P}^2$ factors through a finite double covering $X \to Y$, and the invariant part is equal to the pull-back of the space $S_Y \otimes \mathbb{Q}$ of the numerical equivalence classes of curves on the rational surface Y. See [32] for detail. From this subspace, we can calculate the action of the involution $i(h)$ on S_X, because $i(h)$ acts on the orthogonal complement of the invariant subspace as the scalar multiplication by -1.

Remark 3.4. The equality $i(h) = i(h')$ of involutions does not imply $h = h'$ in general. See Remark 6.11, for example. The set of polarizations h of degree 2 that induce the same involution $i(h)$ is in one-to-one correspondence with the set of blowing-downs of Y to $\mathbb{P}^2$.

Case ($\mathcal{E}2$). Suppose that $\mathcal{E} \neq \emptyset$. Then we have a unique element $f \in \mathcal{E}$ such that

$$f \in \overline{N}_X \quad \text{and} \quad z := h - 2f \in \mathrm{Rats}(X).$$

We can find this f by the methods in Sections 3.5 and 3.4. Then f is the class of a fiber of a Jacobian fibration $\phi\colon X \to \mathbb{P}^1$, with z being the class of the zero section $\zeta\colon \mathbb{P}^1 \to X$. From these vectors f and z, we can calculate the Mordell–Weil group $\mathrm{MW}(X, \phi, \zeta)$ and its action on S_X by the algorithm explained in Section 4.

4. The action of a Mordell–Weil group on S_X

In this section, we assume that the characteristic of the base field k is $\neq 2, 3$ for simplicity. Let X be a $K3$ surface, and let $\boldsymbol{a} \in S_X$ be an ample class.

Let $\phi\colon X \to \mathbb{P}^1$ be a fibration whose general fiber is a curve of genus 1. Suppose that ϕ has a distinguished section $\zeta\colon \mathbb{P}^1 \to X$, that is, the pair (ϕ, ζ) is a *Jacobian fibration*. We denote by $\eta = \operatorname{Spec} k(\mathbb{P}^1)$ the generic point of the base curve $\mathbb{P}^1$. Then the generic fiber $E_\eta := \phi^{-1}(\eta)$ of ϕ is an elliptic curve defined over $k(\mathbb{P}^1)$ with the zero element being the $k(\mathbb{P}^1)$-rational point corresponding to ζ, and the set

$$\mathrm{MW}_\phi := \mathrm{MW}(X, \phi, \zeta)$$

of sections of ϕ has a structure of the abelian group with $\zeta = 0$. This group MW_ϕ is called the *Mordell–Weil group*. The group MW_ϕ acts on E_η via the translation $x \mapsto x +_E \sigma$ on E_η, where $\sigma \in \mathrm{MW}_\phi$ is a section and $+_E$ denotes the addition in the elliptic curve E_η. Since X is minimal, this automorphism of E_η gives an automorphism of X. Hence MW_ϕ embeds in $\mathrm{Aut}(X)$, and acts on the lattice S_X:

$$\mathrm{MW}_\phi \to \mathrm{Aut}(X) \to \mathrm{O}(S_X, \mathcal{P}_X). \tag{4.1}$$

Let $f \in S_X$ be the class of a fiber of ϕ, and let $z = [\zeta] \in S_X$ be the class of the image of ζ. Since the Jacobian fibration (ϕ, ζ) is uniquely determined by the classes f and z, we sometimes write $\mathrm{MW}(X, f, z)$ for $\mathrm{MW}(X, \phi, \zeta)$. The purpose of this section is to show that we can calculate the homomorphism (4.1) from the classes f, z and an ample class $\boldsymbol{a}$.

We review the theory of elliptic $K3$ surfaces, and fix some notation. Since $\langle f, f \rangle = 0$, $\langle f, z \rangle = 1$ and $\langle z, z \rangle = -2$, the classes f and z generate a unimodular hyperbolic sublattice U_ϕ in S_X of rank 2. Let W_ϕ denote the orthogonal complement of U_ϕ in S_X. Since U_ϕ is unimodular, we have an orthogonal direct-sum decomposition

$$S_X = U_\phi \oplus W_\phi.$$

Since W_ϕ is negative-definite, we can calculate the set

$$\mathrm{Roots}(W_\phi) = \{r \in W_\phi \mid \langle r, r \rangle = -2\}.$$

Hence we can compute

$$\Theta_\phi := \mathrm{Roots}(W_\phi) \cap \mathrm{Rats}(X) \tag{4.2}$$

by Proposition 3.2. Let Σ_ϕ denote the sublattice of W_ϕ generated by $\mathrm{Roots}(W_\phi)$, and τ_ϕ the ADE-type of the root lattice Σ_ϕ. Here an ADE-*type* is a finite formal sum of the symbols A_ℓ, D_ℓ, and E_ℓ. See, for example, Section 1 in [10] for the definition of ADE-types of root lattices. Then we have the following proposition. The first part follows from the definition of $\mathrm{Rats}(X)$, and the second part follows from the classification of singular fibers of elliptic surfaces due to Kodaira and Néron. See Chapters 5 and 6 in [27].

Proposition 4.1. *The set Θ_ϕ defined by (4.2) is equal to the set of classes of smooth rational curves that are contracted to points by ϕ and are disjoint from the zero section ζ. The vectors in Θ_ϕ form a basis of the root lattice Σ_ϕ, and their dual graph is the Dynkin diagram of type τ_ϕ.*

Definition 4.2. The sublattice $U_\phi \oplus \Sigma_\phi$ of S_X is called the *trivial sublattice* of the Jacobian fibration (ϕ, ζ).

The following is of fundamental importance in the theory of Mordell–Weil groups of elliptic surfaces. This holds, not only for $K3$ surfaces, but for elliptic surfaces in general. See Chapter 6 in [27].

Theorem 4.3. *Let $[\ ] \colon \mathrm{MW}_\phi \to \mathrm{Rats}(X)$ denote the mapping that associates to each section $\sigma \in \mathrm{MW}_\phi$ the class $[\sigma] \in \mathrm{Rats}(X)$ of the image of σ. Then the composite*

$$\mathrm{MW}_\phi \xrightarrow{[\ ]} \mathrm{Rats}(X) \hookrightarrow S_X \twoheadrightarrow S_X/(U_\phi \oplus \Sigma_\phi) \tag{4.3}$$

is an isomorphism of abelian groups.

Remark 4.4. By the isomorphism (4.3), Shioda [33] (see also [27]) introduced a structure of the positive-definite lattice (with a $\mathbb{Q}$-*valued* intersection form) on the free $\mathbb{Z}$-module $\mathrm{MW}_\phi/(\text{torsion})$. This lattice is called the *Mordell–Weil lattice*. The norm of the Mordell–Weil lattice is very useful, for example, in finding good generators of MW_ϕ. See Section 6.6.

For a vector $v \in S_X$, we denote by $s(v) \in \mathrm{MW}_\phi$ the section that corresponds to the class $v \bmod (U_\phi \oplus \Sigma_\phi) \in S_X/(U_\phi \oplus \Sigma_\phi)$ by the isomorphism (4.3). First, we will explain a method to calculate $[s(v)] \in \mathrm{Rats}(X)$ for a given $v \in S_X$.

We review the Kodaira–Néron theory of singular fibers of an elliptic surface in more detail. See Chapters 5 and 6 in [27], [14, 15], [20], and Table in page 46 of [35]. Recall that Θ_ϕ is the set of classes of smooth rational curves in fibers of ϕ that is disjoint from the zero section ζ, and that the dual graph of Θ_ϕ is the Dynkin diagram of type τ_ϕ. Let

$$\Theta_\phi = \Theta_1 \sqcup \cdots \sqcup \Theta_n \tag{4.4}$$

be the decomposition according to the decomposition of the Dynkin diagram into connected components. Then two elements $r = [C]$ and $r' = [C']$ of Θ_ϕ, where C and C' are smooth rational curves on X, belong to the same Θ_ν if and only if ϕ maps C and C' to the same point. Hence the set $\{\Theta_1, \dots, \Theta_n\}$ is in one-to-one correspondence with the set

$$\{p \in \mathbb{P}^1 \mid \phi^{-1}(p) \text{ is reducible}\} = \{p_1, \dots, p_n\}$$

in such a way that $p_\nu \in \mathbb{P}^1$ is the point $\phi(C)$ for $[C] \in \Theta_\nu$. We put

$$\rho(\nu) := \mathrm{Card}(\Theta_\nu) \quad \text{and} \quad \tau_\nu := \text{the ADE-type of } \Theta_\nu.$$

In particular, each τ_ν is either A_ℓ, D_ℓ, or E_ℓ, and we have $\tau_\phi = \tau_1 + \cdots + \tau_n$. Recall that Σ_ϕ is the root lattice generated by Θ_ϕ. Let Σ_ν be the sublattice of Σ_ϕ generated by the elements of Θ_ν. We have an orthogonal direct-sum decomposition

$$\Sigma_\phi = \Sigma_1 \oplus \cdots \oplus \Sigma_n.$$

The fiber $\phi^{-1}(p_\nu)$ consists of $\rho(\nu) + 1$ smooth rational curves

$$C_{\nu,0},\ C_{\nu,1},\ \dots,\ C_{\nu,\rho(\nu)}$$

such that $\Theta_\nu = \{[C_{\nu,1}], \dots, [C_{\nu,\rho(\nu)}]\}$ and that $C_{\nu,0}$ intersects the zero section ζ. The dual graph of

$$\widetilde{\Theta}_\nu := \{[C_{\nu,0}]\} \cup \Theta_\nu$$

is the *affine* Dynkin diagram of type τ_ν. We number the smooth rational curves in $\widetilde{\Theta}_\nu$ as in Figure 4.1.

The divisor $\phi^*(p_\nu)$ is written as

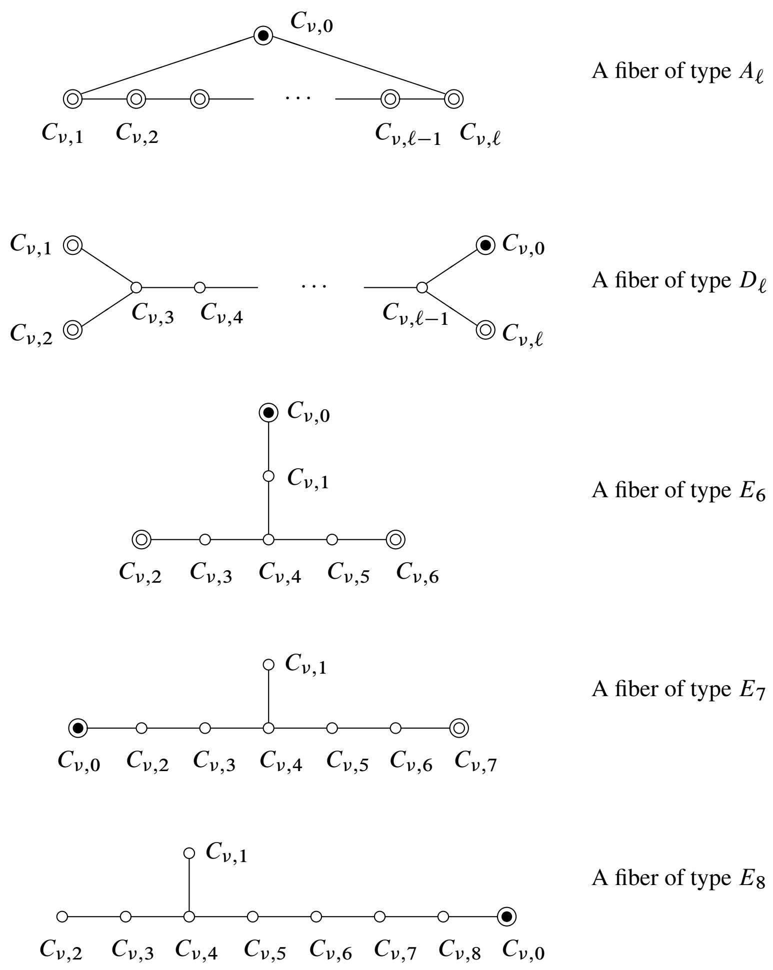
$$\phi^*(p_\nu) = \sum_{j=0}^{\rho(\nu)} m_{\nu,j} C_{\nu,j} \quad (m_{\nu,j} \in \mathbb{Z}_{>0}),$$

where the coefficients $m_{\nu,j}$ are given in Table 4.1. We put

$$J_\nu := \{j \mid m_{\nu,j} = 1\}.$$

We have $0 \in J_\nu$, and the class $[C_{\nu,0}]$ is calculated by

$$[C_{\nu,0}] = f - \sum_{j=1}^{\rho(\nu)} m_{\nu,j} [C_{\nu,j}]. \tag{4.5}$$



$C_{\nu,0}$ is indicated by ◉, and $C_{\nu,j}$ for $j \in J_\nu - \{0\}$ is indicated by ◎.

Figure 4.1. Reducible fibers.

(It is well known that $m_{\nu,j}$ with $j > 0$ are the coefficients of the highest root of the root system Θ_ν.) Let $\phi^*(p_\nu)^\sharp$ denote the smooth part of the divisor $\phi^*(p_\nu)$:

$$\phi^*(p_\nu)^\sharp = \bigcup_{j \in J_\nu} C^\circ_{\nu,j},$$

where $C^\circ_{\nu,j}$ is $C_{\nu,j}$ minus the intersection points of $C_{\nu,j}$ with other irreducible components of $\phi^{-1}(p_\nu)$. By Kodaira–Néron theory, we can equip $\phi^*(p_\nu)^\sharp$ with the structure of an abelian Lie group. See Section 5.6.1 in [27]. (When we work over $\mathbb{C}$, this group structure

τ_ν	$j =\ \ 0, 1, 2, \dots, \rho(\nu)$
A_ℓ	$1, 1, 1, \dots, 1, 1$
D_ℓ	$1, 1, 1, 2, \dots, 2, 1$
E_6	$1, 2, 1, 2, 3, 2, 1$
E_7	$1, 2, 2, 3, 4, 3, 2, 1$
E_8	$1, 3, 2, 4, 6, 5, 4, 3, 2$

Table 4.1. Coefficients $m_{\nu,j}$.

is obtained as the limit of the group structures of general fibers of ϕ.) Then the set J_ν, which is regarded as the set of connected components $C^\circ_{\nu,j}$ of $\phi^*(p_\nu)^\sharp$, also has a natural structure of an abelian group as a quotient group of $\phi^*(p_\nu)^\sharp$. The element $0 \in J_\nu$ is the zero element. See Table 4.2, which is copied from Table in page 46 of [35], for the precise description of the group structure of J_ν.

τ_ν	J_ν	Group structure
A_ℓ	$\{0, 1, \dots, \ell\}$	cyclic group $\mathbb{Z}/(\ell+1)\mathbb{Z}$: the sum of $a, b \in J_\nu$ is $c \in J_\nu$ such that $a + b \equiv c \bmod (\ell+1)$
D_ℓ (ℓ : even)	$\{0, 1, 2, \ell\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
D_ℓ (ℓ : odd)	$\{0, 1, 2, \ell\}$	$\mathbb{Z}/4\mathbb{Z}$ generated by $1 \in J_\nu$ with $\ell \in J_\nu$ being of order 2
E_6	$\{0, 2, 6\}$	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\{0, 7\}$	$\mathbb{Z}/2\mathbb{Z}$
E_8	$\{0\}$	trivial

Table 4.2. Group structure of J_ν (see Table in page 46 of [35]).

Let $\Sigma_\nu^\vee$ be the dual lattice of Σ_ν, and let $\gamma_{\nu,1}, \dots, \gamma_{\nu,\rho(\nu)}$ be the basis of $\Sigma_\nu^\vee$ dual to the basis $[C_{\nu,1}], \dots, [C_{\nu,\rho(\nu)}]$ of Σ_ν. We also put

$$\gamma_{\nu,0} := 0 \ \in\ \Sigma_\nu^\vee.$$

For $j = 0, 1, \dots, \rho(\nu)$, we denote by $\bar\gamma_{\nu,j}$ the element $\gamma_{\nu,j} \pmod{\Sigma_\nu}$ of the discriminant group $A(\Sigma_\nu) = \Sigma_\nu^\vee/\Sigma_\nu$ of Σ_ν. The following is the key observation for our method.

Lemma 4.5. *The map $j \mapsto \bar\gamma_{\nu,j}$ gives an isomorphism $J_\nu \cong \Sigma_\nu^\vee/\Sigma_\nu$ of abelian groups.*

Proof. We compare Table 4.2 calculated in the Kodaira–Néron theory with the discriminant groups $\Sigma_\nu^\vee/\Sigma_\nu$ of root lattices of type A_ℓ, D_ℓ, and E_ℓ. The order of $\Sigma_\nu^\vee/\Sigma_\nu$ is classically known, and coincides with $|J_\nu|$. We equip the vector space $\mathbb{R}^n$ with the standard basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ and with the negative-definite intersection form $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle := -\delta_{ij}$.

(1) *The case* $\tau_\nu = A_\ell$.

We embed Σ_ν into $\mathbb{R}^{\ell+1}$ by $[C_{\nu,j}] \mapsto \boldsymbol{e}_j - \boldsymbol{e}_{j+1}$ so that

$$\Sigma_\nu = \{(x_1, \dots, x_{\ell+1}) \in \mathbb{Z}^{\ell+1} \mid x_1 + \dots + x_{\ell+1} = 0\}.$$

Then we have

$$\gamma_{\nu,j} = \frac{1}{\ell+1}\Big(-\sum_{k=1}^{j} (\ell+1-j)\,\boldsymbol{e}_k + \sum_{k=j+1}^{\ell+1} j\,\boldsymbol{e}_k \Big) \in \Sigma_\nu \otimes \mathbb{Q}.$$

It is easy to check that $j\gamma_{\nu,1} - \gamma_{\nu,j} \in \Sigma_\nu$. Hence $j \mapsto \bar{\gamma}_{\nu,j}$ gives an isomorphism from $\mathbb{Z}/(\ell+1)\mathbb{Z}$ to $\Sigma_\nu^\vee/\Sigma_\nu$.

(2) *The case* $\tau_\nu = D_\ell$.

We embed Σ_ν into $\mathbb{R}^\ell$ by

$$[C_{\nu,1}] \mapsto -\boldsymbol{e}_1 - \boldsymbol{e}_2, \quad [C_{\nu,2}] \mapsto \boldsymbol{e}_1 - \boldsymbol{e}_2, \quad [C_{\nu,j}] \mapsto \boldsymbol{e}_{j-1} - \boldsymbol{e}_j \quad (j = 3, \dots, \ell),$$

so that we have

$$\Sigma_\nu = \{(x_1, \dots, x_\ell) \in \mathbb{Z}^\ell \mid x_1 + \dots + x_\ell \in 2\mathbb{Z}\}.$$

The vectors $\gamma_{\nu,j} \in \Sigma_\nu \otimes \mathbb{Q}$ are given by

$$\gamma_{\nu,1} = \frac{1}{2}\sum_{k=1}^{\ell} \boldsymbol{e}_k, \quad \gamma_{\nu,2} = -\frac{1}{2}\boldsymbol{e}_1 + \frac{1}{2}\sum_{k=2}^{\ell} \boldsymbol{e}_k, \quad \gamma_{\nu,j} = \sum_{k=j}^{\ell} \boldsymbol{e}_k \quad (j = 3, \dots, \ell).$$

It is easy to see that $\bar{\gamma}_{\nu,0} = 0$, $\bar{\gamma}_{\nu,1}$, $\bar{\gamma}_{\nu,2}$, $\bar{\gamma}_{\nu,\ell}$ form the group isomorphic, via $\bar{\gamma}_{\nu,j} \mapsto j$, to the group $J_\nu = \{0, 1, 2, \ell\}$ described in Table 4.2. Note that, for $j = 3, \dots, \ell - 1$, the element $\bar{\gamma}_{\nu,j}$ is either equal to $\bar{\gamma}_{\nu,0} = 0$ or equal to $\bar{\gamma}_{\nu,\ell}$.

(3) *The case* $\tau_\nu = E_6$.

Using the basis $[C_{\nu,1}], \dots, [C_{\nu,6}]$ of Σ_ν, we can write

$$\gamma_{\nu,2} = -\frac{1}{3}(3, 4, 5, 6, 4, 2) \quad \text{and} \quad \gamma_{\nu,6} = -\frac{1}{3}(3, 2, 4, 6, 5, 4) \equiv 2\gamma_{\nu,2} \pmod{\Sigma_\nu}.$$

Hence we have $\Sigma_\nu^\vee/\Sigma_\nu = \{\bar{\gamma}_{\nu,0}, \bar{\gamma}_{\nu,2}, \bar{\gamma}_{\nu,6}\} \cong \mathbb{Z}/3\mathbb{Z}$. Note that we have $\bar{\gamma}_{\nu,5} = \bar{\gamma}_{\nu,2}$, $\bar{\gamma}_{\nu,3} = \bar{\gamma}_{\nu,6}$, $\bar{\gamma}_{\nu,1} = \bar{\gamma}_{\nu,4} = \bar{\gamma}_{\nu,0}$ in $\Sigma_\nu^\vee/\Sigma_\nu$.

(4) *The case* $\tau_\nu = E_7$.

Using the basis $[C_{\nu,1}], \dots, [C_{\nu,7}]$ of Σ_ν, we can write

$$\gamma_{\nu,7} = -\frac{1}{2}(3, 2, 4, 6, 5, 4, 3).$$

Hence we have $\Sigma_\nu^\vee/\Sigma_\nu = \{\bar{\gamma}_{\nu,0}, \bar{\gamma}_{\nu,7}\} \cong \mathbb{Z}/2\mathbb{Z}$. Note that we have $\bar{\gamma}_{\nu,1} = \bar{\gamma}_{\nu,5} = \bar{\gamma}_{\nu,7}$ and $\bar{\gamma}_{\nu,2} = \bar{\gamma}_{\nu,3} = \bar{\gamma}_{\nu,4} = \bar{\gamma}_{\nu,6} = \bar{\gamma}_{\nu,0}$ in $\Sigma_\nu^\vee/\Sigma_\nu$.

(5) *The case* $\tau_\nu = E_8$. Trivial. ■

A section $\sigma \in \mathrm{MW}_\phi$ intersects $\phi^{-1}(p_\nu)$ at a single point $\mathrm{sp}_\nu(\sigma)$, and the intersection is transverse. Hence the intersection point $\mathrm{sp}_\nu(\sigma)$ is a smooth point of the fiber, that is, we have $\mathrm{sp}_\nu(\sigma) \in \phi^*(p_\nu)^\sharp$. Thus we have the *specialization map*

$$\mathrm{sp}_\nu : \mathrm{MW}_\phi \to \phi^*(p_\nu)^\sharp.$$

By the definition of the group structure on $\phi^*(p_\nu)^\sharp$, the map sp_ν is a group homomorphism. (See Section 5.6.1 in [27].) The inclusion $\Sigma_\nu \hookrightarrow S_X$ gives rise to the restriction homomorphism $S_X \to \Sigma_\nu^\vee$, which we write as

$$v \mapsto v|_\nu.$$

For $\sigma \in \mathrm{MW}_\phi$, we have

$$[\sigma]|_\nu = \gamma_{\nu, j[\sigma]},$$

where $j[\sigma] \in J_\nu$ is the index of the connected component of $\phi^*(p_\nu)^\sharp$ intersecting σ, or equivalently, containing the point $\mathrm{sp}_\nu(\sigma)$. The kernel of the composite of $S_X \to \Sigma_\nu^\vee$ and $\Sigma_\nu^\vee \to \Sigma_\nu^\vee/\Sigma_\nu$ contains the trivial sublattice $U_\phi \oplus \Sigma_\phi$. Hence, by Theorem 4.3, the natural mapping

$$\mathrm{MW}_\phi \xrightarrow{[\ ]} S_X \xrightarrow{|_\nu} \Sigma_\nu^\vee \twoheadrightarrow \Sigma_\nu^\vee/\Sigma_\nu \tag{4.6}$$

is a group homomorphism. By definition, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{MW}_\phi & \xrightarrow{(4.6)} & \Sigma_\nu^\vee/\Sigma_\nu \\ {\scriptstyle \mathrm{sp}_\nu} \downarrow & & \downarrow \wr \quad \text{by Lemma 4.5} \\ \phi^*(p_\nu)^\sharp & \twoheadrightarrow & J_\nu, \end{array} \tag{4.7}$$

where the lower horizontal arrow is the natural quotient homomorphism.

Suppose that a vector $v \in S_X$ is given. Then the class $[s(v)] \in S_X$ of the section $s(v) \in \mathrm{MW}_\phi$ corresponding to v mod $(U_\phi \oplus \Sigma_\phi)$ by (4.3) satisfies the following:

(i) $\langle [s(v)], [s(v)] \rangle = -2$ and $\langle [s(v)], f \rangle = 1$. Hence, by the orthogonal direct-sum decomposition $S_X = U_\phi \oplus W_\phi$, we have $[s(v)] = tf + z + w$, where $w \in W_\phi$ and $t = -\langle w, w \rangle/2$.

(ii) $[s(v)] \equiv v \bmod U_\phi \oplus \Sigma_\phi$. In particular, for each $\nu = 1, \dots, n$, we have

$$([s(v)] - v)|_\nu \in \Sigma_\nu.$$

(iii) For each $\nu = 1, \dots, n$, there exists a unique index $j(v) \in J_\nu$ such that $[s(v)]|_\nu = \gamma_{\nu, j(v)}$. This $j(v)$ is the index j of the connected component $C^\circ_{\nu,j}$ that contains the intersection point $\mathrm{sp}_\nu(s(v))$ of $s(v)$ and $\phi^{-1}(p_\nu)$, and hence $j(v)$ is the image of v by $S_X \to J_\nu$ in the diagrams (4.6) and (4.7).

Therefore the following calculations compute the class $[s(v)]$.

Step 1. Let $v' \in W_\phi$ be the image of v by the projection to W_ϕ under the orthogonal direct-sum decomposition $S_X = U_\phi \oplus W_\phi$.

Step 2. For each $\nu = 1, \dots, n$, calculate the element $\delta_\nu(v') := v'|_\nu \bmod \Sigma_\nu$ of the discriminant group $\Sigma_\nu^\vee/\Sigma_\nu$, and find the index $j(v) \in J_\nu$ such that $\delta_\nu(v')$ is equal to $\bar{\gamma}_{\nu, j(v)}$.

Then the element $v'|_\nu - \gamma_{\nu,j(v)}$ of $\Sigma_\nu^\vee$ belongs to Σ_ν. We calculate the integers $\alpha_{\nu,k}$ such that

$$v'|_\nu - \gamma_{\nu,j(v)} = \sum_{k=1}^{\rho(\nu)} \alpha_{\nu,k}\,[C_{\nu,k}].$$

Step 3. We put

$$v'' := v' - \sum_{\nu=1}^{n}\sum_{k=1}^{\rho(\nu)} \alpha_{\nu,k}\,[C_{\nu,k}].$$

Then we have

$$[s(v)] = tf + z + v'',$$

where $t := -\langle v'', v''\rangle/2$.

Next, we explain how to calculate, for a given vector $v \in S_X$, the isometry

$$g(s(v)) \in \mathrm{O}(S_X, \mathcal{P}_X)$$

induced by the translation $x \mapsto x +_E s(v)$ on E_η by the section $s(v) \in \mathrm{MW}_\phi$, where $+_E$ is the addition on the elliptic curve E_η over $k(\mathbb{P}^1)$. Let m be the Mordell–Weil rank of ϕ:

$$m := \dim(\mathrm{MW}_\phi \otimes \mathbb{Q}) = \operatorname{rank} S_X - 2 - \sum_{\nu=1}^{n} \rho(\nu),$$

where the second equality follows from Theorem 4.3. We choose vectors $u_1, \dots, u_m \in S_X$ such that their images by

$$S_X \to (S_X/(U_\phi \oplus \Sigma_\phi)) \otimes \mathbb{Q}$$

form a basis of $\mathrm{MW}_\phi \otimes \mathbb{Q}$. Then $S_X \otimes \mathbb{Q}$ is spanned by

(4.8) $f,\ z = [s(0)],\ [s(u_1)], \dots, [s(u_m)],$ and the vectors $[C_{\nu,1}], \dots, [C_{\nu,\rho(\nu)}]$ in Θ_ν for $\nu = 1, \dots, n$.

Therefore, to calculate $g(s(v))$, it is enough to calculate the images of vectors in (4.8) by $g(s(v))$. It is obvious that

$$\begin{aligned} f^{g(s(v))} &= f, \\ z^{g(s(v))} &= [s(v)], \\ [s(u_\mu)]^{g(s(v))} &= [s(u_\mu + v)] \quad \text{for} \quad \mu = 1, \dots, m. \end{aligned}$$

Hence it remains only to calculate the image by $g(s(v))$ of the classes in Θ_ν. Note that $g(s(v))$ induces a permutation on the set $\widetilde{\Theta}_\nu = \{[C_{\nu,0}]\} \cup \Theta_\nu$ that preserves the subset J_ν of classes of reduced irreducible components. By the method described in Step 2 above, we calculate the index $j(v) \in J_\nu$, which is the image of $s(v) \in \mathrm{MW}_\phi$ by the composite of $\mathrm{sp}_\nu\colon \mathrm{MW}_\phi \to \phi^*(p_\nu)^\sharp$ and $\phi^*(p_\nu)^\sharp \to J_\nu$. The translation of $\phi^*(p_\nu)^\sharp$ by $\mathrm{sp}_\nu(s(v))$ induces the translation of J_ν by $j(v)$. Checking each Dynkin diagram of type A_ℓ, D_ℓ and E_ℓ, we see that this permutation of J_ν extends *uniquely* to a permutation of $\widetilde{\Theta}_\nu$ that preserves the dual graph. See Table 4.3, in which we abbreviate $\widetilde{\Theta}_\nu = \{[C_{\nu,0}], \dots, [C_{\nu,\rho(\nu)}]\}$ as $\{0, 1, \dots, \rho(\nu)\}$. Hence the image of each element of $\widetilde{\Theta}_\nu$ by $g(s(v))$ is computed. Using (4.5), we can calculate the action of $g(s(v))$ on the classes of Θ_ν.

τ_ν	J_ν	$j(v)$	Permutation of $\widetilde{\Theta}_\nu$
A_ℓ	$\mathbb{Z}/(\ell+1)\mathbb{Z}$	a	$i \mapsto (i+a) \bmod (\ell+1)$
D_ℓ	$(\mathbb{Z}/2\mathbb{Z})^2$	0	id
(ℓ : even)		1	$0 \leftrightarrow 1,\quad 2 \leftrightarrow \ell,\quad k \leftrightarrow \ell+2-k \quad (2<k<\ell)$
		2	$0 \leftrightarrow 2,\quad 1 \leftrightarrow \ell,\quad k \leftrightarrow \ell+2-k \quad (2<k<\ell)$
		ℓ	$0 \leftrightarrow \ell,\quad 1 \leftrightarrow 2,\quad k \leftrightarrow k \quad (2<k<\ell)$
D_ℓ	$\mathbb{Z}/4\mathbb{Z}$	0	id
(ℓ : odd)		1	$0 \mapsto 1 \mapsto \ell \mapsto 2 \mapsto 0,\quad k \leftrightarrow \ell+2-k \quad (2<k<\ell)$
		2	$0 \mapsto 2 \mapsto \ell \mapsto 1 \mapsto 0,\quad k \leftrightarrow \ell+2-k \quad (2<k<\ell)$
		ℓ	$0 \leftrightarrow \ell,\quad 1 \leftrightarrow 2,\quad k \leftrightarrow k \quad (2<k<\ell)$
E_6	$\mathbb{Z}/3\mathbb{Z}$	0	id
		2	$0 \mapsto 2 \mapsto 6 \mapsto 0,\quad 1 \mapsto 3 \mapsto 5 \mapsto 1,\quad 4 \mapsto 4$
		6	$0 \mapsto 6 \mapsto 2 \mapsto 0,\quad 1 \mapsto 5 \mapsto 3 \mapsto 1,\quad 4 \mapsto 4$
E_7	$\mathbb{Z}/2\mathbb{Z}$	0	id
		7	$0 \leftrightarrow 7,\quad 1 \leftrightarrow 1,\quad 4 \leftrightarrow 4,\quad 2 \leftrightarrow 6,\quad 3 \leftrightarrow 5$
E_8	0	0	id

Table 4.3. Permutations of $\widetilde{\Theta}_\nu$.

5. Borcherds' method

5.1. An algorithm on a graph

We recall an algorithm introduced in [7]. Let (V, E) be a simple non-oriented connected graph, where V is the set of vertices and E is the set of edges, which is a set of non-ordered pairs of distinct elements of V:

$$E \subset \binom{V}{2}.$$

We say that $v, v' \in V$ are *adjacent* if $\{v, v'\} \in E$. The set V may be infinite. The assumption that (V, E) be connected is important. Suppose that a group G acts on (V, E) from the right. For vertices $v, v' \in V$, we put

$$T_G(v, v') := \{g \in G \mid v^g = v'\},$$

and define the G*-equivalence relation* $\sim$ on V by

$$v \sim v' \iff T_G(v, v') \neq \emptyset.$$

Thus we have two relations on V, the adjacency relation and the G-equivalence relation. Suppose that V_0 is a non-empty subset of V with the following properties.

(a) If $v, v' \in V_0$ are distinct, then v and v' are not G-equivalent.

(b) If a vertex $v \in V$ is adjacent to a vertex in V_0, then v is G-equivalent to a vertex in V_0.

We put

$$\widetilde{V}_0 := \{v \in V \mid v \text{ is adjacent to a vertex in } V_0\}.$$

Then, for each $v \in \widetilde{V}_0$, there exists a vertex $u_0(v) \in V_0$ such that $T_G(v, u_0(v)) \neq \emptyset$. Note that $u_0(v) \in V_0$ is unique by assumption (a). We choose an element $h(v)$ from $T_G(v, u_0(v))$ for each $v \in \widetilde{V}_0$, and put

$$\mathcal{H} := \{h(v) \mid v \in \widetilde{V}_0\}. \tag{5.1}$$

Proposition 5.1 (Proposition 4.1 of [7]). *The subset $V_0 \subset V$ is a complete set of representatives of the orbit decomposition of V by G, and the group G is generated by the union of $\mathcal{H}$ and the stabilizer subgroup* $\mathrm{Stab}_G(v_0) = T_G(v_0, v_0)$ *of a vertex $v_0 \in V_0$.*

In Section 4.1 of [7], we presented an algorithm to obtain V_0 and $\mathcal{H}$ under the assumption that (V, E) and G have certain *local effectiveness properties*.

5.2. Period condition

In this subsection, we assume that the base field k is the complex number field $\mathbb{C}$, and introduce *period condition* on elements of $\mathrm{O}(S_X)$. The period condition is, however, also defined when X is a supersingular $K3$ surface in positive characteristic. See, for example, [17].

Let L be an even lattice, and let $A(L) = L^\vee/L$ be the discriminant group of L. We define a quadratic form

$$q(L) : A(L) \to \mathbb{Q}/2\mathbb{Z}$$

by $q(x \bmod L) := \langle x, x\rangle \bmod 2\mathbb{Z}$. This finite quadratic form is called the *discriminant form* of L, which was introduced by Nikulin [21]. Let M be a primitive sublattice of L, and N the orthogonal complement of M in L. Then we have natural embeddings

$$M \oplus N \subset L \subset L^\vee \subset M^\vee \oplus N^\vee.$$

Suppose that L is unimodular, that is, $L^\vee = L$. Then the submodule

$$L/(M \oplus N) \subset A(M) \times A(N)$$

is a graph of an isomorphism $A(M) \cong A(N)$, which induces an isomorphism

$$\iota_L\colon q(M) \cong -q(N).$$

Nikulin [21] proved the following.

Proposition 5.2. *Suppose that L is unimodular. Let G_N be a subgroup of* $\mathrm{O}(N)$, *and let $q(G_N) \subset$* $\mathrm{Aut}(q(N))$ *denote the image of $G_N \subset$* $\mathrm{O}(N)$ *by the natural homomorphism* $\mathrm{O}(N) \to \mathrm{Aut}(q(N))$. *Then an isometry g_M of M extends to an isometry g_L of L such that its restriction $g_L|N$ to N is an element of G_N if and only if the action of g_M on $q(M)$ belongs to $q(G_N)$ via the isomorphism* $\mathrm{Aut}(q(M)) \cong \mathrm{Aut}(q(N))$ *induced by* $\iota_L\colon q(M) \cong -q(N)$.

We apply this result to the primitive embedding of S_X into the even unimodular lattice $H^2(X,\mathbb{Z})$ of rank 22 defined by the cup product. Let T_X denote the orthogonal complement of S_X in $H^2(X,\mathbb{Z})$, which we call the *transcendental lattice* of X. Then $H^2(X,\mathbb{Z})$ induces an isomorphism

$$\iota_H : q(S_X) \cong -q(T_X).$$

Note that T_X is the minimal primitive submodule of $H^2(X,\mathbb{Z})$ such that $T_X \otimes \mathbb{C}$ contains the period $H^{2,0}(X) = \mathbb{C}\omega_X \subset H^2(X,\mathbb{C})$ of X, where ω_X is a non-zero holomorphic 2-form on X.

Definition 5.3. We put

$$\mathrm{O}(T_X,\omega_X) := \{g_T \in \mathrm{O}(T_X) \mid g_T \otimes \mathbb{C} \text{ preserves } H^{2,0}(X)\}.$$

Then we say that $g_S \in \mathrm{O}(S_X)$ satisfies the *period condition* if the action of g_S on $q(S_X)$ is equal to the action on $q(T_X)$ of some of $g_T \in \mathrm{O}(T_X,\omega_X)$ via the isomorphism of the finite quadratic forms $\iota_H : q(S_X) \cong -q(T_X)$ induced by $H^2(X,\mathbb{Z})$.

By Proposition 5.2, we see that an isometry $g_S \in \mathrm{O}(S_X)$ extends to an isometry of $H^2(X,\mathbb{Z})$ preserving the period $H^{2,0}(X)$ if and only if g_S satisfies the period condition. By Torelli theorem [24] (see also Chapter VIII of [4]), we obtain the following.

Theorem 5.4. *We put*

$$G := \mathrm{Im}(\mathrm{Aut}(X) \to \mathrm{O}(S_X,\mathcal{P}_X)).$$

Then $g \in \mathrm{O}(S_X,\mathcal{P}_X)$ belongs to G if and only if g preserves N_X and satisfies the period condition.

Example 5.5. Suppose that $\operatorname{rank} T_X \geq 3$ and assume that ω_X is very general in the period domain $\mathcal{Q}$ in $\mathbb{P}_*(T_X \otimes \mathbb{C})$. (See Chapter VIII of [4] for the definition of the period domain.) Then we have

$$\mathrm{O}(T_X,\omega_X) = \{\pm 1\}, \tag{5.2}$$

and hence $g_S \in \mathrm{O}(S_X)$ satisfies the period condition if and only if the action of g_S on the discriminant group $A(S_X)$ is 1 or -1.

We give a proof of (5.2). The period domain $\mathcal{Q}$ is an open subset (in the classical topology) of a smooth quadratic hypersurface in $\mathbb{P}_*(T_X \otimes \mathbb{C})$, and hence we have

$$\dim \mathcal{Q} = \operatorname{rank} T_X - 2 > 0.$$

For $\gamma \in \mathrm{O}(T_X)$, let $V_{\gamma,\lambda} \subset T_X \otimes \mathbb{C}$ denote the eigenspace of γ with eigenvalue $\lambda \in \mathbb{C}$. If $\gamma \notin \{\pm 1\}$, then $\dim V_{\gamma,\lambda} < \operatorname{rank} T_X$ and hence $\mathbb{P}_*(V_{\gamma,\lambda}) \cap \mathcal{Q}$ is a proper analytic subspace of $\mathcal{Q}$ for any λ. Since a countable union of proper analytic subspaces of a positive-dimensional connected complex manifold cannot cover the total space, we have (5.2) for ω_X very general in $\mathcal{Q}$.

Suppose moreover that $-1 \in \mathrm{O}(T_X,\omega_X)$ acts on $A(T_X)$ non-trivially (that is, the abelian group $A(T_X) \cong A(S_X)$ is not 2-elementary). By Proposition 5.2, there exists no isometry g_H of the overlattice $H^2(X,\mathbb{Z})$ of $S_X \oplus T_X$ such that $g_H|S_X = 1$ and $g_H|T_X = -1$. Since $\mathrm{Aut}(X)$ acts on $H^2(X,\mathbb{Z})$ faithfully, the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(S_X,\mathcal{P}_X)$ is injective.

Remark 5.6. For supersingular $K3$ surfaces, we have to prove (5.2) in a different method, because the period domain is a subvariety of codimension > 1 in a Grassmannian variety. See [17].

5.3. Tessellation by L_{26}/S_X-chambers

Let L_{26} denote an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. We choose a positive cone $\mathcal{P}_{26}$ of L_{26}. A standard fundamental domain of $W(L_{26})$ was determined by Conway [8] by means of Vinberg's algorithm [36].

Definition 5.7. A vector $\mathbf{w} \in L_{26}$ is called a *Weyl vector* if $\mathbf{w}$ is a non-zero primitive vector of L_{26} contained in $\partial\overline{\mathcal{P}}_{26}$ (in particular, we have $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ and hence $\mathbb{Z}\mathbf{w} \subset (\mathbb{Z}\mathbf{w})^{\perp}$) such that $(\mathbb{Z}\mathbf{w})^{\perp}/\mathbb{Z}\mathbf{w}$ is isomorphic to the negative-definite Leech lattice.

Definition 5.8. Let $\mathbf{w}$ be a Weyl vector. A (-2)-vector $r \in L_{26}$ is said to be a *Leech root* with respect to $\mathbf{w}$ if $\langle \mathbf{w}, r \rangle = 1$. We then put

$$\mathbf{C}(\mathbf{w}) := \{x \in \mathcal{P}_{26} \mid \langle x, r \rangle \geq 0 \text{ for all Leech roots } r \text{ with respect to } \mathbf{w}\}.$$

Theorem 5.9 (Conway [8]). (1) *The mapping* $\mathbf{w} \mapsto \mathbf{C}(\mathbf{w})$ *gives a bijection from the set of Weyl vectors to the set of standard fundamental domains of* $W(L_{26})$.

(2) *Let* $\mathbf{w}$ *be a Weyl vector. Then the mapping* $r \mapsto \mathbf{C}(\mathbf{w}) \cap (r)^{\perp}$ *gives a bijection from the set of Leech roots with respect to* $\mathbf{w}$ *to the set of walls of the chamber* $\mathbf{C}(\mathbf{w})$.

Definition 5.10. We call a standard fundamental domain of $W(L_{26})$ a *Conway chamber*. Hence $\mathcal{P}_{26}$ is tessellated by the Conway chambers.

Suppose that we have a primitive embedding

$$\iota : S_X \hookrightarrow L_{26}.$$

Replacing ι by $-\iota$ if necessary, we assume that ι maps $\mathcal{P}_X$ into $\mathcal{P}_{26}$, and regard $\mathcal{P}_X$ as a subspace of $\mathcal{P}_{26}$:

$$\mathcal{P}_X = \iota^{-1}(\mathcal{P}_{26}) = (S_X \otimes \mathbb{R}) \cap \mathcal{P}_{26}.$$

Definition 5.11. An L_{26}/S_X*-chamber* is a chamber D of $\mathcal{P}_X$ that is obtained as the intersection $\mathcal{P}_X \cap \mathbf{C}(\mathbf{w})$ of $\mathcal{P}_X$ with a Conway chamber $\mathbf{C}(\mathbf{w})$.

The tessellation of $\mathcal{P}_{26}$ by the Conway chambers induces a tessellation of $\mathcal{P}_X$ by the L_{26}/S_X-chambers. By definition, the nef-and-big cone N_X, which is a standard fundamental domain of $W(S_X)$, is tessellated by L_{26}/S_X-chambers. In other words, the tessellation of $\mathcal{P}_X$ by the L_{26}/S_X-chambers is a refinement of the tessellation by the standard fundamental domains of $W(S_X)$.

Definition 5.12. We define a graph (V, E) by the following: the set V of vertices is the set of L_{26}/S_X-chambers contained in N_X, and the set E of edges is the set of pairs of adjacent L_{26}/S_X-chambers.

Let G be the image of the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(S_X, \mathcal{P}_X)$. Suppose that

(5.3) the period condition for $g \in \mathrm{O}(S_X)$ is that the action of g on the discriminant group $A(S_X)$ be 1 or -1.

See Example 5.5 for a case where this assumption is satisfied. Then, by Proposition 5.2, every element $g \in G$ extends to an isometry of L_{26}. In particular, the action of G preserves the tessellation of $\mathcal{P}_X$ by the L_{26}/S_X-chambers. Since the action of G preserves N_X, we obtain the following.

Proposition 5.13. *If* (5.3) *holds, then G acts on the graph* (V, E).

Definition 5.14. Let $D = \mathcal{P}_X \cap \mathbf{C}(\mathbf{w})$ be an L_{26}/S_X-chamber. For each wall w of D, there exists a unique defining vector v of w in the dual lattice $S_X^\vee$ that is primitive in $S_X^\vee$. (See Definition 2.2.) We call this vector $v \in S_X^\vee$ the *primitive defining vector* of the wall w.

Note that a Conway chamber has infinitely many walls. For the graph (V, E) to have local effectiveness properties in [7], it needs that each L_{26}/S_X-chamber has only a finite number of walls. We consider the following assumption:

(5.4) The orthogonal complement of S_X in L_{26} cannot be embedded in the negative-definite Leech lattice.

This holds, for example, if the orthogonal complement contains at least one (-2)-vector.

Proposition 5.15 ([31]). *Suppose that* (5.4) *holds. Then each L_{26}/S_X-chamber has only a finite number of walls. If $D = \mathcal{P}_X \cap \mathbf{C}(\mathbf{w})$ is an L_{26}/S_X-chamber obtained by the Conway chamber $\mathbf{C}(\mathbf{w})$ associated with a Weyl vector $\mathbf{w}$, then we can calculate the primitive defining vectors of walls of D from $\mathbf{w}$. Moreover, for each wall w of D, we can calculate a Weyl vector $\mathbf{w}'$ such that $D' = \mathcal{P}_X \cap \mathbf{C}(\mathbf{w}')$ is the L_{26}/S_X-chamber adjacent to D across the wall w.*

Thus, under assumptions (5.3) and (5.4), the local effectiveness properties in [7] hold for (V, E) and G, and we can apply the algorithm in Section 4.1 of [7] to (V, E) and G.

Remark 5.16. The amount of the computation of this method is estimated by $|V_0| = |V/G|$, that is, the number of the orbits of the action of $\mathrm{Aut}(X)$ on the set of L_{26}/S_X-chambers contained in N_X.

In practice, it seems that Borcherds' method carried out without using computer (for example, [16]) can only deal with the case where $|V_0| = 1$. Some cases with $|V_0| > 1$ were treated in [31], where V_0 is of size about $10^3 \sim 10^4$. However, the geometric description of the generators of $\mathrm{Aut}(X)$ was not given for these cases. We also have observed some cases where $|V_0|$ is too large for Borcherds' method to terminate in a reasonable time (for example, [13]).

In the case of the present article (see Section 6), we have $|V_0| = 7$. Since this is not so large, we have managed to obtain geometric generators.

Remark 5.17. It has been *empirically* observed that $|V_0|$ is small when the orthogonal complement of $\iota\colon S_X \hookrightarrow L_{26}$ contains a root lattice as a sublattice of finite index.

6. Computation of $\mathrm{Aut}(X_{f,g})$

In this section, we prove Theorems 1.1 and 1.2. For simplicity, we write X for the $K3$ surface $X_{f,g}$. Recall that the polynomials f and g in the defining equation (1.2) of $\overline{X}_{f,g}$ are assumed to be very general. We use this assumption throughout this section.

6.1. The lattice S_X

First, we describe the lattice S_X and the nef-and-big cone N_X. Let $H \subset X$ denote the pull-back of a line of $\mathbb{P}^2$, and let us put

$$\boldsymbol{h} := [H] \in S_X.$$

The singular locus of the branch curve $B(\boldsymbol{h}) = \{f^2 + g^3 = 0\} \subset \mathbb{P}^2$ of the finite double covering $\overline{X}_{f,g} \to \mathbb{P}^2$ consists of six ordinary cusps $\bar{p}_1, \dots, \bar{p}_6$, which are located at the locus defined by $f = g = 0$. Hence the singularities of $\overline{X}_{f,g}$ consist of six rational double points $p_1, \dots, p_6$ of type A_2, where p_i is located over $\bar{p}_i$. Let $E_i^{(+)}$ and $E_i^{(-)}$ denote the exceptional curves that are contracted to the point $p_i \in \operatorname{Sing}(\overline{X}_{f,g})$ by the desingularization $X \to \overline{X}_{f,g}$. We put

$$\boldsymbol{e}_i^{(+)} := [E_i^{(+)}] \in S_X \quad \text{and} \quad \boldsymbol{e}_i^{(-)} := [E_i^{(-)}] \in S_X.$$

Let $\overline{\Gamma} \subset \mathbb{P}^2$ be the conic defined by $g = 0$. Then $\overline{\Gamma}$ passes through the six cusps $\bar{p}_1, \dots, \bar{p}_6$ of $B(\boldsymbol{h})$. Hence the strict transform of $\overline{\Gamma}$ in X is a disjoint union of two smooth rational curves $\Gamma^{(+)}$ and $\Gamma^{(-)}$. We put

$$\boldsymbol{\gamma}^{(+)} := [\Gamma^{(+)}] \in S_X \quad \text{and} \quad \boldsymbol{\gamma}^{(-)} := [\Gamma^{(-)}] \in S_X.$$

For each $i \in \{1, \dots, 6\}$, the curve $\Gamma^{(+)}$ intersects one of $E_i^{(+)}$ or $E_i^{(-)}$ and is disjoint from the other. Interchanging $E_i^{(+)}$ and $E_i^{(-)}$ if necessary, we can assume that

$$\langle \boldsymbol{\gamma}^{(+)}, \boldsymbol{e}_i^{(+)} \rangle = 1 \quad \text{and} \quad \langle \boldsymbol{\gamma}^{(+)}, \boldsymbol{e}_i^{(-)} \rangle = 0$$

hold for $i = 1, \dots, 6$. Then we have the following (see also [29]).

Proposition 6.1 (Degtyarev [9]). *The $\mathbb{Q}$-vector space $S_X \otimes \mathbb{Q}$ is of dimension 13, and is generated by the classes*

$$\boldsymbol{h}, \boldsymbol{e}_1^{(+)}, \boldsymbol{e}_1^{(-)}, \dots, \boldsymbol{e}_6^{(+)}, \boldsymbol{e}_6^{(-)}. \tag{6.1}$$

The sublattice $S_{X,0}$ of S_X generated by the classes in (6.1) is of index 3 in S_X. The lattice S_X is generated by $S_{X,0}$ and the class $\boldsymbol{\gamma}^{(+)}$.

By Proposition 6.1, a vector v of $S_X \otimes \mathbb{Q}$ is uniquely determined by the list of intersection numbers

$$\langle v, \boldsymbol{h} \rangle, \ \langle v, \boldsymbol{e}_1^{(+)} \rangle, \ \langle v, \boldsymbol{e}_1^{(-)} \rangle, \dots, \langle v, \boldsymbol{e}_6^{(+)} \rangle, \ \langle v, \boldsymbol{e}_6^{(-)} \rangle.$$

Moreover, an isometry g of S_X is specified by the images of the classes in (6.1) by g. For example, the involution $i(\boldsymbol{h})$ associated with the double covering $\pi(\boldsymbol{h})\colon X \to \mathbb{P}^2$ defined by $|\boldsymbol{h}|$ is given by

$$\boldsymbol{h}^{i(\boldsymbol{h})} = \boldsymbol{h}, \quad (\boldsymbol{e}_i^{(+)})^{i(\boldsymbol{h})} = \boldsymbol{e}_i^{(-)}, \quad (\boldsymbol{e}_i^{(-)})^{i(\boldsymbol{h})} = \boldsymbol{e}_i^{(+)} \quad (i = 1, \dots, 6).$$

The vector $\boldsymbol{a} \in S_X \otimes \mathbb{Q}$ defined by

$$\langle \boldsymbol{a}, \boldsymbol{h} \rangle = 8, \quad \langle \boldsymbol{a}, \boldsymbol{e}_i^{(+)} \rangle = 1, \quad \langle \boldsymbol{a}, \boldsymbol{e}_i^{(-)} \rangle = 1 \quad (i = 1, \dots, 6) \tag{6.2}$$

is a vector of $\mathcal{P}_X \cap S_X$, and satisfies

$$\langle \boldsymbol{a}, \boldsymbol{a} \rangle = 20, \quad \text{Roots}(\ \boldsymbol{a}^{\perp} \cap S_X\ ) = \emptyset, \quad \text{Sep}(\boldsymbol{h}, \boldsymbol{a}) = \emptyset.$$

Hence $\boldsymbol{a}$ is ample (see Section 3.1). By this ample class $\boldsymbol{a}$, we can specify the nef-and-big cone N_X in $\mathcal{P}_X$.

Next, we investigate the period condition of X. We consider the moduli space $\mathcal{M}$ of lattice-polarized $K3$ surfaces (X', η'), where X' is a $K3$ surface and η' is an isometry $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ that induces an embedding $S_X \hookrightarrow S_{X'}$. Then $\mathcal{M}$ is covered by the period domain $\mathcal{Q} \subset \mathbb{P}_*(T_X \otimes \mathbb{C})$. If (X', η') is very general in $\mathcal{M}$, then we have $S_X = S_{X'}$. Looking at the lattice $S_X = S_{X'}$, we obtain the following.

Proposition 6.2 (Degtyarev [9]). *If (X', η') is very general in $\mathcal{M}$, then there exist homogeneous polynomials f' and g' of degree* 3 *and* 2*, respectively, such that X' is birational to the double plane defined by $w^2 = f'^2 + g'^3$.*

Remark 6.3. The following naive dimension count may help in understanding Proposition 6.2: the dimension of the parameter space of pairs (f', g') of homogeneous polynomials of degree 3 and 2 modulo linear transformation is equal to

$$\dim H^0(\mathbb{P}^2, \mathcal{O}(3)) + \dim H^0(\mathbb{P}^2, \mathcal{O}(2)) - \dim \mathrm{GL}(3, \mathbb{C}) = 7 = \operatorname{rank} T_X - 2 = \dim \mathcal{Q}.$$

See also [29] for the proof of Proposition 6.2.

Since f and g are very general, we see that X is very general in $\mathcal{M}$, and hence we can assume that ω_X is very general in the period domain $\mathcal{Q}$. Therefore, by (5.2) in Example 5.5, we have

$$\mathrm{O}(T_X, \omega_X) = \{\pm 1\}. \tag{6.3}$$

The discriminant group $A(S_X)$ of S_X is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4$. Hence, using Example 5.5, we obtain the following.

Proposition 6.4. *The natural representation of* $\mathrm{Aut}(X)$ *on S_X is faithful.*

We will consider $\mathrm{Aut}(X)$ as a subgroup of $\mathrm{O}(S_X, \mathcal{P}_X)$ from now on. By Theorem 5.4 and (3.1), we have the following.

Proposition 6.5. *An element $g \in \mathrm{O}(S_X, \mathcal{P}_X)$ belongs to* $\mathrm{Aut}(X)$ *if and only if g acts on $A(S_X)$ as* 1 *or* -1*, and* $\mathrm{Sep}(\boldsymbol{a}, \boldsymbol{a}^g) = \emptyset$ *holds.*

We introduce an auxiliary group M, which makes the descriptions of N_X and $\mathrm{Aut}(X)$ much easier. Let M be the subgroup of $\mathrm{O}(S_X, \mathcal{P}_X)$ consisting of elements g satisfying $\boldsymbol{h}^g = \boldsymbol{h}$ and

$$\{\boldsymbol{e}_1^{(+)}, \boldsymbol{e}_1^{(-)}, \dots, \boldsymbol{e}_6^{(+)}, \boldsymbol{e}_6^{(-)}\}^g = \{\boldsymbol{e}_1^{(+)}, \boldsymbol{e}_1^{(-)}, \dots, \boldsymbol{e}_6^{(+)}, \boldsymbol{e}_6^{(-)}\}.$$

Then M is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_6$, generated by the involution $i(\boldsymbol{h})$ and permutations $\sigma \in S_6$ given by

$$\boldsymbol{h}^\sigma = \boldsymbol{h}, \quad \boldsymbol{e}_i^{(+)\sigma} = \boldsymbol{e}_{i^\sigma}^{(+)}, \quad \boldsymbol{e}_i^{(-)\sigma} = \boldsymbol{e}_{i^\sigma}^{(-)}.$$

For each $g \in M$, we have $\boldsymbol{a} = \boldsymbol{a}^g$, and hence $M \subset \mathrm{O}(S_X, N_X)$. The discriminant form $q(S_X)$ of S_X is isomorphic to

$$\left(\left[\frac{1}{2}\right], \mathbb{Z}/2\mathbb{Z}\right) \oplus \left(\left[\frac{4}{3}\right], \mathbb{Z}/3\mathbb{Z}\right)^{\oplus 3} \oplus \left(\left[\frac{2}{3}\right], \mathbb{Z}/3\mathbb{Z}\right).$$

Here $([\alpha], \mathbb{Z}/m\mathbb{Z})$ denotes a cyclic group $A = \langle \gamma \rangle$ of order m generated by γ equipped with the quadratic form $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ such that $q(\gamma) = \alpha$. The natural homomorphism $\mathrm{O}(S_X) \to \mathrm{Aut}(q(S_X))$ maps M to $\mathrm{Aut}(q(S_X))$ isomorphically. Note that $\iota(\boldsymbol{h})$ acts on $A(S_X)$ as -1. Hence we have

$$M \cap \mathrm{Aut}(X) = \{1, i(\boldsymbol{h})\}.$$

Remark 6.6. By means of the methods in Section 3.4, we can make the list of classes of smooth rational curves C on X with $\langle [C], \boldsymbol{h} \rangle = m$ for each non-negative integer m. The size $\nu(m)$ of this list is as follows: when m is odd, we have $\nu(m) = 0$, whereas for m even, we have

m	0	2	4	6	8	10	12	14	...
$\nu(m)$	12	17	0	492	720	492	8292	8730	...

For i, j with $1 \le i \le 6$, $1 \le j \le 6$, and $i \ne j$, let $\ell_{ij} \subset \mathbb{P}^2$ denote the line passing through the singular points $\bar{p}_i$ and $\bar{p}_j$ of the branch curve $B(\boldsymbol{h})$, and let $\tilde{\ell}_{ij} \subset X$ be the strict transform of ℓ_{ij}. The $\nu(2) = 17$ smooth rational curves on X of degree 2 with respect to $\boldsymbol{h}$ are the lifts $\Gamma^{(\pm)}$ of the conic $\overline{\Gamma} \subset \mathbb{P}^2$ and the curves $\tilde{\ell}_{ij}$.

6.2. Automorphisms of X

By the method in Section 3.7, we find many automorphisms of X from nef vectors of norm 2. Among them, we have the following automorphisms:

- type (a): the involution $i(\boldsymbol{h})$,
- type (b): 90 involutions $i(h_{IJ})$ associated with polarizations h_{IJ} of degree 2 such that $\langle h_{IJ}, \boldsymbol{h} \rangle = 6$ and that $\mathrm{Sing}(B(h_{IJ}))$ is of type $A_3 + A_5$,
- type (c): 12 involutions $i(h_\alpha^\pm)$ associated with polarizations $h_\alpha^\pm$ of degree 2 such that $\langle h_\alpha^\pm, \boldsymbol{h} \rangle = 4$ and that $\mathrm{Sing}(B(h_\alpha^\pm))$ is of type $A_2 + 5A_1$,
- type (d): 360 involutions $i(h_{\pm J})$ associated with polarizations $h_{\pm J}$ of degree 2 such that $\langle h_{\pm J}, \boldsymbol{h} \rangle = 14$, and that $\mathrm{Sing}(B(h_{\pm J}))$ is of type $D_4 + A_5$, and
- type (e): 360 translations associated with sections $\boldsymbol{e}_j^{(\pm)}$ of infinite order of 120 Jacobian fibrations $\phi\colon X \to \mathbb{P}^1$ defined by $(f_\phi, z_\phi) = (f_{\pm I}, \boldsymbol{e}_i^{(\pm)})$ with $\langle f_{\pm I}, \boldsymbol{h} \rangle = 4$ such that MW_ϕ is torsion-free of rank 4 and that the reducible fibers of ϕ are of type $D_4 + A_3$.

See subsections below for more precise descriptions of these automorphisms. We will show, by Borcherds' method, that these automorphisms generate $\mathrm{Aut}(X)$.

6.3. Primitive embedding $S_X \hookrightarrow L_{26}$

To apply Borcherds' method, we embed S_X into L_{26} primitively. Let R_0 be a negative-definite root lattice of type $A_1 + 6A_2$ with a basis

$$\alpha,\ \beta_1^{(+)}, \beta_1^{(-)},\ \ldots,\ \beta_6^{(+)}, \beta_6^{(-)} \tag{6.4}$$

consisting of roots that form the dual graph as in Figure 6.1.

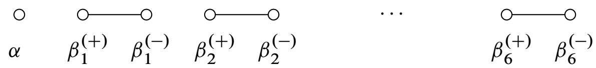


Figure 6.1. Basis of R_0.

Let

$$\alpha^\vee,\ \beta_1^{(+)\vee}, \beta_1^{(-)\vee},\ \ldots,\ \beta_6^{(+)\vee}, \beta_6^{(-)\vee}$$

be the basis of the dual lattice $R_0^\vee$ that is dual to the basis (6.4). Then

$$R := R_0 + \mathbb{Z}(\beta_1^{(+)\vee} + \cdots + \beta_6^{(+)\vee}) \subset R_0^\vee$$

is an even lattice whose discriminant form is isomorphic to $-q(S_X)$. Recall that the natural homomorphism $\mathrm{O}(S_X) \to \mathrm{Aut}(q(S_X))$ maps M to $\mathrm{Aut}(q(S_X))$ isomorphically, and hence is surjective. Therefore, by Nikulin [21], there exists a *unique* (up to the action of $\mathrm{O}(S_X)$) even unimodular overlattice of $S_X \oplus R$ in which S_X and R are both primitive. Taking this unimodular overlattice as L_{26}, we find a primitive embedding

$$\iota : S_X \hookrightarrow L_{26}.$$

We consider the tessellation of $N_X \subset \mathcal{P}_X$ by the L_{26}/S_X-chambers associated with this primitive embedding. Let (V, E) be the graph of L_{26}/S_X-chambers contained in N_X (see Definition 5.12). By (6.3) and Propositions 5.13 and 5.15, we see that the group $G = \mathrm{Aut}(X) \subset \mathrm{O}(S_X, \mathcal{P}_X)$ acts on the graph (V, E), and we can apply the algorithm in Section 4.1 of [7].

Remark 6.7. Primitive embeddings of S_X into L_{26} are not unique. In fact, the genus of negative-definite even lattices containing the isomorphism class of R consists of 26 isomorphism classes.

The image $\iota(\boldsymbol{a}) \in \mathcal{P}_{26} \cap L_{26}$ of the ample class $\boldsymbol{a} \in S_X$ defined by (6.2) satisfies

$$\mathrm{Roots}(([\iota(\boldsymbol{a})] \hookrightarrow L_{26})^\perp) = \mathrm{Roots}((\iota\colon S_X \hookrightarrow L_{26})^\perp) \cong \mathrm{Roots}(R), \tag{6.5}$$

where $[\iota(\boldsymbol{a})]$ is the sublattice of L_{26} generated by $\iota(\boldsymbol{a})$. Hence $\boldsymbol{a}$ is an interior point of an L_{26}/S_X-chamber, which we denote by D_0. Moreover, we have

$$\mathrm{Sep}_{26}(\iota(\boldsymbol{a}), \iota(\boldsymbol{h})) = \emptyset,$$

where we denote by Sep_{26} the set of separating (-2)-vectors in L_{26}. Hence the class $\boldsymbol{h}$ is a point of D_0. We choose a vector $\tilde{\boldsymbol{a}} \in \mathcal{P}_L \cap L_{26}$ that satisfies

$$\mathrm{Roots}(([\tilde{\boldsymbol{a}}] \hookrightarrow L_{26})^\perp) = \emptyset \quad \text{and} \quad \mathrm{Sep}_{26}(\iota(\boldsymbol{a}), \tilde{\boldsymbol{a}}) = \emptyset.$$

Then $\tilde{\boldsymbol{a}}$ is an interior point of a Conway chamber $\mathbf{C}_0$ such that $\iota^{-1}(\mathbf{C}_0) = D_0$. We can calculate a subset of the set of roots $\tilde{r}$ of L_{26} such that $\mathbf{C}_0 \cap (\tilde{r})^\perp$ is a wall of $\mathbf{C}_0$, either by Vinberg's algorithm [36], or by calculating $\mathrm{Sep}_{26}(\tilde{\boldsymbol{a}}, \boldsymbol{v})$, where $\boldsymbol{v} \in \mathscr{P}_{26} \cap L_{26}$ are randomly chosen vectors. If this subset is large enough, these roots $\tilde{r}$ span $L_{26} \otimes \mathbb{Q}$ and hence the Weyl vector $\mathbf{w}_0$ of the Conway chamber $\mathbf{C}_0$ is calculated by solving the equations $\langle \mathbf{w}_0, \tilde{r} \rangle = 1$.

Remark 6.8. The ADE-type of the roots in (6.5) is $A_1 + 6A_2$. Hence the hyperplanes perpendicular to these roots decompose $R \otimes \mathbb{R}$ into 2×6^6 regions. Therefore there exist exactly 2×6^6 Conway chambers $\mathbf{C}$ such that $\iota^{-1}(\mathbf{C}) = D_0$.

Thus we prepared all the data necessary to start the algorithm of Section 4.1 in [7] to calculate a complete set V_0 of the representatives of V/G and a finite generating set of $G = \mathrm{Aut}(X)$. We executed this algorithm. The computation terminated and yielded the following.

Proposition 6.9. *The set V_0 consists of the following seven L_{26}/S_X-chambers*:

$$D_0,\ D_1^{(1)},\ D_1^{(2)},\ D_1^{(3)},\ D_1^{(4)},\ D_1^{(5)},\ D_1^{(6)}.$$

We will describe each of these L_{26}/S_X-chambers in V_0, and during the description, we present automorphisms in the set $\mathscr{H}$ defined by (5.1).

We use the following convention. Let D be an L_{26}/S_X-chamber, and let $\mathbf{C}$ be a Conway chamber such that $\iota^{-1}(\mathbf{C}) = D$. Let $\mathbf{w}$ be the Weyl vector of $\mathbf{C}$. For a wall w of D, let $v \in S_X^\vee$ be the primitive defining vector of w (see Definition 5.14), and we put

$$n(w) := \langle v, v \rangle, \quad a(w) := \langle \mathbf{w}, \iota(v) \rangle, \quad h(w) := \langle \boldsymbol{h}, v \rangle.$$

These rational numbers are useful in classifying walls.

6.4. The L_{26}/S_X-chamber D_0

The initial L_{26}/S_X-chamber D_0 contains the ample class $\boldsymbol{a}$ in its interior. The stabilizer subgroup of D_0 in G is $\{1, i(\boldsymbol{h})\}$. The group M leaves D_0 invariant. The chamber D_0 has 110 walls, and the action of M decomposes the walls of D_0 into four orbits o_1, o_2, o_3 and o_4 of sizes 2, 12, 6 and 90, respectively. The data of these orbits are given in Table 6.1.

	size	n	a	h	
o_1	2	-2	1	2	$\boldsymbol{\gamma}^{(\pm)}$
o_2	12	-2	1	0	$\boldsymbol{e}_i^{(\pm)}$
o_3	6	$-3/2$	$3/2$	1	isom with $D_1^{(\alpha)}$
o_4	90	$-2/3$	3	2	isom with D_0

Table 6.1. Walls of D_0.

The orbit o_1 of size 2 consists of $(\boldsymbol{\gamma}^{(\pm)})^\perp \cap D_0$. The orbit o_2 of size 12 consists of $(\boldsymbol{e}_i^{(\pm)})^\perp \cap D_0$. Hence the L_{26}/S_X-chamber adjacent to D_0 across a wall in o_1 or o_2 is not contained in N_X.

The orbit o_3 of size 6 consists of the walls $(v_\alpha)^\perp \cap D_0$ whose primitive defining vectors v_α are given by

$$\langle v_\alpha, \boldsymbol{h}\rangle = 1, \quad \langle v_\alpha, \boldsymbol{e}_i^{(+)}\rangle = \langle v_\alpha, \boldsymbol{e}_i^{(-)}\rangle = \begin{cases} 1 & \text{if } i = \alpha, \\ 0 & \text{if } i \neq \alpha. \end{cases}$$

Let $D_1^{(\alpha)}$ be the L_{26}/S_X-chamber adjacent to D_0 across the wall $(v_\alpha)^\perp \cap D_0$. Then $D_1^{(\alpha)}$ is contained in N_X, but is not G-equivalent to D_0, and any two of $D_1^{(1)}, \dots, D_1^{(6)}$ are not G-equivalent to each other. Hence these chambers $D_1^{(\alpha)}$ ($\alpha = 1, \dots, 6$) are added to V_0 as new representatives of V/G.

The walls w_{IJ} in the orbit o_4 of size 90 are indexed by ordered pairs (I, J), where I and J are subsets of $\{1, \dots, 6\}$ satisfying $|I| = |J| = 2$ and $I \cap J = \emptyset$. The primitive defining vector $v_{IJ} \in S_X^\vee$ of $w_{IJ} \in o_4$ is given by

$$\begin{aligned} &\langle v_{IJ}, \boldsymbol{h}\rangle = 2, \\ &\langle v_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle v_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 0, \qquad \text{if } i \notin I \cup J, \\ &\langle v_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 1, \quad \langle v_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 0, \qquad \text{if } i \in I, \\ &\langle v_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle v_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 1, \qquad \text{if } i \in J. \end{aligned}$$

The L_{26}/S_X-chamber D_{IJ} adjacent to D_0 across the wall w_{IJ} is G-equivalent to D_0. An automorphism $g_{IJ} \in G$ that maps D_0 to D_{IJ} isomorphically is given as follows. Let h_{IJ} be a vector of $S_X \otimes \mathbb{Q}$ defined by

$$\begin{aligned} &\langle h_{IJ}, \boldsymbol{h}\rangle = 6, \\ &\langle h_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle h_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 0, \qquad \text{if } i \notin I \cup J, \\ &\langle h_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 1, \quad \langle h_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 1, \qquad \text{if } i \in I, \\ &\langle h_{IJ}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle h_{IJ}, \boldsymbol{e}_i^{(-)}\rangle = 3, \qquad \text{if } i \in J. \end{aligned} \tag{6.6}$$

Then $h_{IJ} \in S_X$ and $\langle h_{IJ}, h_{IJ}\rangle = 2$. We confirm $\mathrm{Sep}(h_{IJ}, \boldsymbol{a}) = \emptyset$, and hence $h_{IJ} \in N_X$. The complete linear system $|h_{IJ}|$ is proved to be fixed-component free by the criterion in Section 3.7. The involution $i(h_{IJ})$ associated with the double covering $\pi(h_{IJ})\colon X \to \mathbb{P}^2$ given by $|h_{IJ}|$ maps D_0 to D_{IJ} isomorphically. Therefore

$$i(h_{IJ})^{-1} = i(h_{IJ}) \in T_G(D_{IJ}, u_0(D_{IJ}))$$

in the notation of Section 5.1, where $u_0(D_{IJ}) = D_0$. These involutions $i(h_{IJ})$ are the involutions of type (b) in Section 6.2.

Remark 6.10. Suppose that $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$, and

$$\{1, \dots, 6\} - (I \cup J) = \{k_1, k_2\}.$$

Then the smooth rational curves on X contracted by the double covering $\pi(h_{IJ})\colon X \to \mathbb{P}^2$ are as in Figure 6.2, where $\tilde{\ell}_{j_1 j_2}$ is the curve given in Remark 6.6. In particular, the singular locus of the branch curve $B(h_{IJ})$ is of type $A_3 + A_5$.

Remark 6.11. We have $v_{IJ}{}^{i(\boldsymbol{h})} = v_{JI}$, $h_{IJ}{}^{i(\boldsymbol{h})} \neq h_{JI}$, and can confirm that the involution $i(h_{IJ}{}^{i(\boldsymbol{h})}) = i(\boldsymbol{h}) i(h_{IJ}) i(\boldsymbol{h})$ is equal to $i(h_{JI})$.

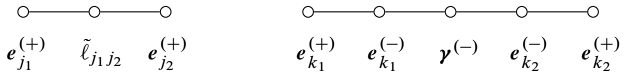


Figure 6.2. Exceptional curves of $\pi(h_{IJ})$.

6.5. The L_{26}/S_X-chamber $D_1^{(\alpha)}$

The stabilizer subgroup of $D_1^{(\alpha)}$ in G is $\{1, i(\boldsymbol{h})\}$. On the other hand, the group M acts on the set $\{D_1^{(1)}, \dots, D_1^{(6)}\}$ transitively. Let M_α be the stabilizer subgroup of $D_1^{(\alpha)}$ in M. Then M_α is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_5$. The chamber $D_1^{(\alpha)}$ has 110 walls, and the action of M_α decomposes the walls of $D_1^{(\alpha)}$ into seven orbits $o'_1, \dots, o'_7$. The data of these orbits are given in Table 6.2.

	size	n	a	h	
o'_1	1	$-3/2$	$3/2$	-1	back to D_0
o'_2	2	-2	1	2	$\boldsymbol{\gamma}^{(\pm)}$
o'_3	5	-2	1	2	$\tilde{\ell}_{\alpha\beta}$ $(\beta \neq \alpha)$
o'_4	10	-2	1	0	$\boldsymbol{e}_\beta^{(\pm)}$ $(\beta \neq \alpha)$
o'_5	2	$-3/2$	$3/2$	1	isom with $D_1^{(\alpha)}$
o'_6	30	$-1/6$	$7/2$	3	isom with $D_1^{(\beta)}$ $(\beta \neq \alpha)$
o'_7	60	$-2/3$	3	2	isom with $D_1^{(\beta)}$ $(\beta \neq \alpha)$

Table 6.2. Walls of $D_1^{(\alpha)}$.

The orbit o'_1 consists of a single wall, and the adjacent L_{26}/S_X-chamber across this wall is D_0, which means that this wall is a wall in the orbit o_3 of walls of D_0 viewed from the opposite side.

The orbit o'_2 of size 2 consists of $(\boldsymbol{\gamma}^{(\pm)})^\perp \cap D_1^{(\alpha)}$, the orbit o'_3 of size 5 consists of $(\tilde{\ell}_{\alpha\beta})^\perp \cap D_1^{(\alpha)}$ with $\beta \neq \alpha$, and the orbit o'_4 of size 10 consists of $(\boldsymbol{e}_\beta^{(\pm)})^\perp \cap D_1^{(\alpha)}$ with $\beta \neq \alpha$. The adjacent L_{26}/S_X-chambers across these walls are therefore not contained in N_X.

The orbit o'_5 is of size 2. One of the walls in o'_5 is defined by a vector $v_\alpha^+ \in S_X^\vee$ satisfying

$$\begin{aligned} &\langle v_\alpha^+, \boldsymbol{h} \rangle = 1, \\ &\langle v_\alpha^+, \boldsymbol{e}_\alpha^{(+)} \rangle = 2, \quad \langle v_\alpha^+, \boldsymbol{e}_\alpha^{(-)} \rangle = -1, \\ &\langle v_\alpha^+, \boldsymbol{e}_\beta^{(+)} \rangle = 0, \quad \langle v_\alpha^+, \boldsymbol{e}_\beta^{(-)} \rangle = 0 \qquad (\beta \neq \alpha), \end{aligned}$$

and the other wall in o'_5 is defined by the vector

$$v_\alpha^- := (v_\alpha^+)^{i(\boldsymbol{h})}.$$

The adjacent L_{26}/S_X-chamber D_α^+ across the wall $(v_\alpha^+)^\perp \cap D_1^{(\alpha)}$ is G-equivalent to $D_1^{(\alpha)}$. Indeed, the following automorphism $i(h_\alpha^+) \in G$ maps $D_1^{(\alpha)}$ to D_α^+ isomorphically. Let h_α^+ be the vector defined by

$$\begin{aligned}
&\langle h_\alpha^+, \boldsymbol{h}\rangle = 4,\\
&\langle h_\alpha^+, \boldsymbol{e}_\alpha^{(+)}\rangle = 2, \quad \langle h_\alpha^+, \boldsymbol{e}_\alpha^{(-)}\rangle = 0,\\
&\langle h_\alpha^+, \boldsymbol{e}_\beta^{(+)}\rangle = 0, \quad \langle h_\alpha^+, \boldsymbol{e}_\beta^{(-)}\rangle = 1 \quad (\beta \neq \alpha).
\end{aligned}$$

Then we have $h_\alpha^+ \in S_X$ and $\langle h_\alpha^+, h_\alpha^+\rangle = 2$. We confirm $\mathrm{Sep}(h_\alpha^+, \boldsymbol{a}) = \emptyset$ and hence $h_\alpha^+ \in N_X$. The complete linear system $|h_\alpha^+|$ is proved to be fixed-component free by the criterion in Section 3.7. Then we can confirm by direct computation that the involution $i(h_\alpha^+)$ associated with the double covering $\pi(h_\alpha^+)\colon X \to \mathbb{P}^2$ given by $|h_\alpha^+|$ induces $D_1^{(\alpha)} \cong D_\alpha^+$. It is obvious that the automorphism $i(h_\alpha^-) := i(\boldsymbol{h})i(h_\alpha^+)i(\boldsymbol{h})$ maps $D_1^{(\alpha)}$ to the adjacent L_{26}/S_X-chamber D_α^- across the wall $(v_\alpha^-)^\perp \cap D_1^{(\alpha)}$. Therefore we have

$$i(h_\alpha^\pm) = i(h_\alpha^\pm)^{-1} \in T_G(D_\alpha^\pm, u_0(D_\alpha^\pm))$$

in the notation of Section 5.1. These involutions $i(h_\alpha^\pm)$ are the involutions of type (c) in Section 6.2.

Remark 6.12. The branch curve $B(h_\alpha^+)$ of the double covering $\pi(h_\alpha^+)$ has the singularities of type $A_2 + 5A_1$. The exceptional curves over the singular point of type A_2 are $\boldsymbol{\gamma}^{(-)}$ and $\boldsymbol{e}_\alpha^{(-)}$, whereas the exceptional curves over the singular points of type A_1 are $\boldsymbol{e}_\beta^{(+)}$ for $\beta \neq \alpha$. In particular, the involution $i(h_\alpha^+)$ interchanges $\boldsymbol{\gamma}^{(-)}$ and $\boldsymbol{e}_\alpha^{(-)}$.

The description of the orbit o_6' is rather complicated, and hence is postponed to the next subsection.

We describe the orbit o_7' of size 60. Suppose that $\beta \in \{1, \dots, 6\}$ and $F = \{i_1, i_2\} \subset \{1, \dots, 6\}$ satisfy $i_1 \neq i_2$, $\beta \neq \alpha$ and $\{\alpha, \beta\} \cap \{i_1, i_2\} = \emptyset$. Let $v_{\beta F}^{(+)} \in S_X^\vee$ be the vector defined by

$$\begin{aligned}
&\langle v_{\beta F}^{(+)}, \boldsymbol{h}\rangle = 2,\\
&\langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(+)}\rangle = 1, \quad \langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(-)}\rangle = 0 \quad \text{if } i \in \{\alpha, \beta\},\\
&\langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(-)}\rangle = 1 \quad \text{if } i \in F,\\
&\langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle v_{\beta F}^{(+)}, \boldsymbol{e}_i^{(-)}\rangle = 0 \quad \text{otherwise.}
\end{aligned}$$

We then put

$$v_{\beta F}^{(-)} := \left(v_{\beta F}^{(+)}\right)^{i(\boldsymbol{h})}.$$

The orbit o_7' consists of walls $(v_{\beta F}^{(+)})^\perp \cap D_1^{(\alpha)}$ and $(v_{\beta F}^{(-)})^\perp \cap D_1^{(\alpha)}$. The adjacent L_{26}/S_X-chamber $D_{\beta F}^{(\pm)}$ across the wall $(v_{\beta F}^{(\pm)})^\perp \cap D_1^{(\alpha)}$ is G-equivalent to $D_1^{(\beta)}$. We put $A := \{\alpha, \beta\}$, and consider the polarization h_{AF} of degree 2 defined by (6.6) with $I = A$ and $J = F$. The involution $i(h_{AF})$, which is an involution of type (b) in Section 6.2, maps

$D_1^{(\beta)}$ to $D_{\beta F}^{(+)}$ isomorphically, whereas the involution $i(h_{FA})$ maps $D_1^{(\beta)}$ to $D_{\beta F}^{(-)}$ isomorphically. (See Remark 6.11.) Therefore we have

$$i(h_{AF}) = i(h_{AF})^{-1} \in T_G(D_{\beta F}^{(+)}, u_0(D_{\beta F}^{(+)})),$$
$$i(h_{FA}) = i(h_{FA})^{-1} \in T_G(D_{\beta F}^{(-)}, u_0(D_{\beta F}^{(-)})),$$

in the notation of Section 5.1.

6.6. The orbit o_6'

In the following, for a sign $\sigma \in \{+, -\}$, let $\bar{\sigma}$ denote the opposite sign: $\{\sigma, \bar{\sigma}\} = \{+, -\}$. First, we define automorphisms $g'_{\sigma Ij}$ and $g''_{\sigma J}$.

Let $\mathcal{I}$ be the set of ordered triples

$$I = (\{i_1\}, \{i_2, i_3, i_4\}, \{i_5, i_6\})$$

such that $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$. We have $|\mathcal{I}| = 60$. For a pair of $\sigma \in \{+, -\}$ and $I \in \mathcal{I}$, we have the configuration of smooth rational curves as in Figure 6.3.

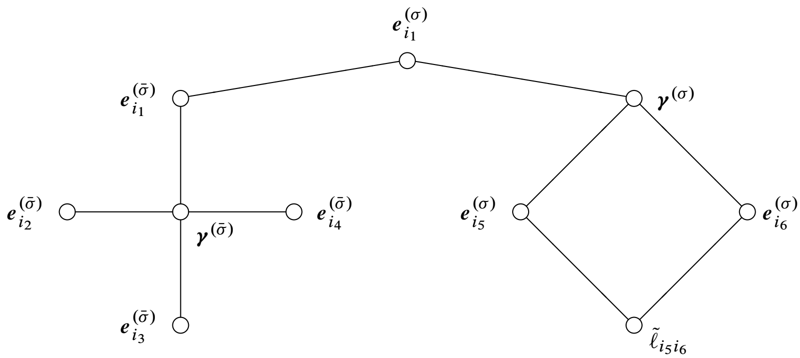


Figure 6.3. Configuration for a Jacobian fibration.

Then

$$f_\phi := f_{\sigma I} := e_{i_1}^{(\bar{\sigma})} + e_{i_2}^{(\bar{\sigma})} + e_{i_3}^{(\bar{\sigma})} + e_{i_4}^{(\bar{\sigma})} + 2\gamma^{(\bar{\sigma})} = \gamma^{(\sigma)} + e_{i_5}^{(\sigma)} + e_{i_6}^{(\sigma)} + \tilde{\ell}_{i_5 i_6}$$

is the class of a fiber of an elliptic fibration $\phi\colon X \to \mathbb{P}^1$ with

$$z_\phi := z_{\sigma I} := e_{i_1}^{(\sigma)}$$

being the class of a section.

Thus we obtain a Jacobian fibration ϕ with the zero section z_ϕ, and its Mordell–Weil group

$$\mathrm{MW}_\phi := \mathrm{MW}(X, f_\phi, z_\phi) \subset G = \mathrm{Aut}(X).$$

Calculating the set $\Theta_\phi = \mathrm{Roots}(W_\phi) \cap \mathrm{Rats}(X)$, we see that the ADE-type of the reducible fibers of $\phi\colon X \to \mathbb{P}^1$ is $D_4 + A_3$. Hence the rank of MW_ϕ is 4. Since the trivial sublattice of ϕ, which is of rank 9 generated by the classes of the ten curves in Figure 6.3, is primitive in S_X, we see that MW_ϕ is torsion free. A Gram matrix of the Mordell–Weil lattice MW_ϕ (see Remark 4.4) is

$$\frac{3}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{bmatrix}.$$

The numbers $n(s)$ of elements with small Mordell–Weil norms s in MW_ϕ are given as follows:

s	$9/4$	3	$21/4$	6	$\dots$
$n(s)$	12	14	16	30	$\dots$

Among these, we have the following sections of ϕ:

- The six smooth rational curves $\tilde{\ell}_{j_1 j_2}$, where $j_1 \in \{i_2, i_3, i_4\}$ and $j_2 \in \{i_5, i_6\}$, satisfy $\langle \tilde{\ell}_{j_1 j_2}, f\rangle = 1$, and hence they are sections of ϕ. Their Mordell–Weil norms are $9/4$.
- The three smooth rational curves $\boldsymbol{e}_j^{(\sigma)}$, where $j \in \{i_2, i_3, i_4\}$, also satisfy $\langle \boldsymbol{e}_j^{(\sigma)}, f\rangle = 1$, and hence they are sections of ϕ. Their Mordell–Weil norms are equal to 3.

These $6 + 3$ sections $\tilde{\ell}_{j_1 j_2}$ and $\boldsymbol{e}_j^{(\sigma)}$ generate MW_ϕ.

Definition 6.13. For $j \in \{i_2, i_3, i_4\}$, we denote by $g'_{\sigma I j}$ the automorphism of X obtained as the translation by the section $\boldsymbol{e}_j^{(\sigma)} \in \mathrm{MW}_\phi$. This is the automorphism of type (e) in Section 6.2.

Let $\mathcal{J}$ be the set of ordered 4-tuples

$$J = (\{i_1\}, \{i_2, i_3\}, \{i_4, i_5\}, \{i_6\})$$

such that $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$. We have $|\mathcal{J}| = 180$. For a pair of $\sigma \in \{+, -\}$ and $J \in \mathcal{J}$, let $h_{\sigma J}$ be the vector of $S_X \otimes \mathbb{Q}$ defined by

$$\begin{aligned}
&\langle h_{\sigma J}, \boldsymbol{h}\rangle = 14, \\
&\langle h_{\sigma J}, \boldsymbol{e}_{i_1}^{(\sigma)}\rangle = 1 \quad \text{and} \quad \langle h_{\sigma J}, \boldsymbol{e}_{i_1}^{(\bar\sigma)}\rangle = 0, \\
&\langle h_{\sigma J}, \boldsymbol{e}_{i}^{(\sigma)}\rangle = 4 \quad \text{and} \quad \langle h_{\sigma J}, \boldsymbol{e}_{i}^{(\bar\sigma)}\rangle = 0 \quad \text{for } i = i_2 \text{ and } i = i_3, \\
&\langle h_{\sigma J}, \boldsymbol{e}_{i}^{(\sigma)}\rangle = 0 \quad \text{and} \quad \langle h_{\sigma J}, \boldsymbol{e}_{i}^{(\bar\sigma)}\rangle = 5 \quad \text{for } i = i_4 \text{ and } i = i_5, \\
&\langle h_{\sigma J}, \boldsymbol{e}_{i_6}^{(\sigma)}\rangle = 5 \quad \text{and} \quad \langle h_{\sigma J}, \boldsymbol{e}_{i_6}^{(\bar\sigma)}\rangle = 4.
\end{aligned}$$

Then $h_{\sigma J} \in S_X$ and $\langle h_{\sigma J}, h_{\sigma J}\rangle = 2$. We confirm $\mathrm{Sep}(h_{\sigma J}, \boldsymbol{a}) = \emptyset$, and hence $h_{\sigma J} \in N_X$. The complete linear system $|h_{\sigma J}|$ is proved to be fixed-component free by the criterion in Section 3.7.

Definition 6.14. We denote by $g''_{\sigma J}$ the involution $i(h_{\sigma J})$. This is the involution of type (d) in Section 6.2.

Remark 6.15. The smooth rational curves on X contracted by the double covering $\pi(h_{\sigma J})$ associated with $|h_{\sigma J}|$ are as in Figure 6.4. In particular, $\mathrm{Sing}(B(h_{\sigma J}))$ is of type $D_4 + A_5$.

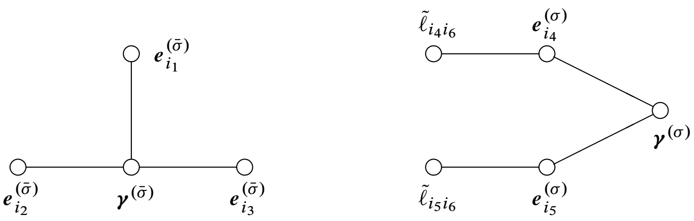


Figure 6.4. Exceptional curves of $\pi(h_{\sigma J})$.

We now describe the orbit o_6' of walls of $D_1^{(\alpha)}$. The size of o_6' is 30. Suppose that $\beta \in \{1,\dots,6\}$ and $F = \{i_1, i_2\} \subset \{1,\dots,6\}$ satisfy $i_1 \neq i_2$, $\beta \neq \alpha$ and $\{\alpha,\beta\} \cap \{i_1,i_2\} = \emptyset$. Let $u := u_{\beta F} \in S_X^\vee$ be the vector defined by

$$\begin{aligned}
&\langle u, \boldsymbol{h}\rangle = 3,\\
&\langle u, \boldsymbol{e}_\alpha^{(+)}\rangle = 1, \quad \langle u, \boldsymbol{e}_\alpha^{(-)}\rangle = 1,\\
&\langle u, \boldsymbol{e}_\beta^{(+)}\rangle = 0, \quad \langle u, \boldsymbol{e}_\beta^{(-)}\rangle = 0,\\
&\langle u, \boldsymbol{e}_i^{(+)}\rangle = 0, \quad \langle u, \boldsymbol{e}_i^{(-)}\rangle = 1 \quad \text{if } i \in F,\\
&\langle u, \boldsymbol{e}_i^{(+)}\rangle = 1, \quad \langle u, \boldsymbol{e}_i^{(-)}\rangle = 0 \quad \text{if } i \notin \{\alpha,\beta\} \cup F.
\end{aligned}$$

The orbit o_6' consists of walls $(u_{\beta F})^\perp \cap D_1^{(\alpha)}$. The L_{26}/S_X-chamber $D_{\alpha\beta F}$ adjacent to $D_1^{(\alpha)}$ across the wall $(u_{\beta F})^\perp \cap D_1^{(\alpha)}$ is G-equivalent to $D_1^{(\beta)}$. An automorphism $g_{\alpha\beta F} \in G$ that maps $D_1^{(\beta)}$ to $D_{\alpha\beta F}$ isomorphically is given as follows. We put

$$K := \{1,\dots,6\} \setminus (\{\alpha,\beta\} \ \cup \ F).$$

Then we have

$$g_{\alpha\beta F} = g'_{+I\beta} \cdot g''_{+J} = g'_{-I'\beta} \cdot g''_{-J'}, \tag{6.7}$$

where

$$\begin{aligned}
I &= (\{\alpha\}, K \cup \{\beta\}, F) \in \mathcal{I}, \quad & J &= (\{\beta\}, K, F, \{\alpha\}) \in \mathcal{J},\\
I' &= (\{\alpha\}, F \cup \{\beta\}, K) \in \mathcal{I}, \quad & J' &= (\{\beta\}, F, K, \{\alpha\}) \in \mathcal{J}.
\end{aligned}$$

Therefore we have

$$g_{\alpha\beta F}^{-1} \in T_G(D_{\alpha\beta F}, u_0(D_{\alpha\beta F}))$$

in the notation of Section 5.1.

Remark 6.16. The equality (6.7) was found by trying small combinations of the automorphisms of type (a)–(e).

6.7. Proof of Theorem 1.1

Any two distinct elements of V_0 are not G-equivalent. Any L_{26}/S_X-chamber that is contained in N_X and is adjacent to an element of V_0 is G-equivalent to an element of V_0. Hence, by Proposition 5.1, the set V_0 is a complete set of representatives of V/G.

As the set $\mathcal{H}$ defined by (5.1), we can take the set consisting of the identity element 1, all involutions of type (b) and (c), and the automorphisms $g_{\alpha\beta F}^{-1}$, where $g_{\alpha\beta F}$ is given by (6.7) and is a product of automorphisms of type (d) and (e). The stabilizer subgroup $\mathrm{Stab}_G(D_0)$ of the initial element $D_0 \in V_0$ is $\{1, i(\boldsymbol{h})\}$. Hence, by Proposition 5.1, the group $G = \mathrm{Aut}(X)$ is generated by the automorphisms of type (a)–(e). ■

Remark 6.17. This generating set is very redundant.

6.8. Proof of Theorem 1.2

We prove that $G = \mathrm{Aut}(X)$ acts on $\mathrm{Rats}(X)$ transitively. Let r be an arbitrary element of $\mathrm{Rats}(X)$. Since r defines a wall of N_X, there exists an L_{26}/S_X-chamber D contained in N_X such that r defines a wall of D. We have an automorphism $g \in G$ such that $D^g \in V_0$. By the description of walls of the representative L_{26}/S_X-chambers in V_0, we see that r^g is one of the $12 + 2 + 15$ smooth rational curves $\boldsymbol{e}_\alpha^{(\pm)}$, $\boldsymbol{\gamma}^{(\pm)}$, and $\tilde{\ell}_{ij}$. The action of $i(\boldsymbol{h})$ gives $\boldsymbol{e}_\alpha^{(+)} \leftrightarrow \boldsymbol{e}_\alpha^{(-)}$ and $\boldsymbol{\gamma}^{(+)} \leftrightarrow \boldsymbol{\gamma}^{(-)}$. By Remark 6.10, the involution $i(h_{IJ})$ of type (b) interchanges $\boldsymbol{e}_{j_1}^{(+)}$ and $\boldsymbol{e}_{j_2}^{(+)}$ (see Figure 6.2). By Remark 6.12, the involution $i(h_\alpha^+)$ of type (c) interchanges $\boldsymbol{\gamma}^{(-)}$ and $\boldsymbol{e}_\alpha^{(-)}$. As was shown in Section 6.6, the elliptic fibration $X \to \mathbb{P}^1$ given by $f_{\sigma I}$ has sections $\boldsymbol{e}_j^{(\sigma)}$ and $\tilde{\ell}_{j_1 j_2}$, and hence they belong to the same G-orbit. Therefore these $12 + 2 + 15$ smooth rational curves are in the same G-orbit. ■

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Ichiro Shimada
Mathematics Program, Graduate School of Advanced Science and Engineering,
Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 Japan;
ichiro-shimada@hiroshima-u.ac.jp