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Estimates for some bilinear wave operators

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Abstract. We consider some bilinear Fourier multiplier operators and give a bilinear version of Seeger, Sogge, and Stein's result for Fourier integral operators. Our results improve, for the case of Fourier multiplier operators, Rodríguez-López, Rule, and Staubach's result for bilinear Fourier integral operators. The sharpness of the results is also considered.

1. Introduction

The solution to the wave equation $\partial_t^2 u = \Delta u$ with the initial data $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ is given by

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|)\, \widehat{f}(\xi)\, d\xi + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \frac{\sin(t|\xi|)}{|\xi|}\, \widehat{g}(\xi)\, d\xi,$$

where $\widehat{f}$ denotes the Fourier transform of f (for the definition of Fourier transform, see Notation 1.6 below). Several basic properties of the mapping $(f,g) \mapsto u(t,\cdot)$ are derived from the estimate of the operator

$$Tf(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, e^{i|\xi|}\, (1+|\xi|^2)^{m/2}\, \widehat{f}(\xi)\, d\xi. \tag{1.1}$$

The purpose of this paper is to consider bilinear versions of this operator.

We begin with the definition of linear Fourier multiplier operators. For $\theta \in L^\infty(\mathbb{R}^n)$, the operator $\theta(D)$ is defined by

$$\theta(D) f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \theta(\xi)\, \widehat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. If X and Y are function spaces on $\mathbb{R}^n$ equipped with quasi-norms or seminorms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and if there exists a constant A such that

$$\|\theta(D) f\|_Y \le A \|f\|_X \quad \text{for all } f \in X \cap \mathcal{S},$$

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then we say that θ is a *Fourier multiplier* for $X \to Y$ and write $\theta \in \mathcal{M}(X \to Y)$. (Sometimes we write $\theta(\xi) \in \mathcal{M}(X \to Y)$ to mean $\theta(\cdot) \in \mathcal{M}(X \to Y)$.) The minimum of A that satisfies the above inequality is denoted by $\|\theta\|_{\mathcal{M}(X \to Y)}$.

Throughout this paper, H^p, $0 < p \le \infty$, denotes the Hardy space, and BMO denotes the space of bounded mean oscillation. We shall use the convention that $H^p = L^p$ if $1 < p \le \infty$. For H^p and BMO, see, e.g., Chapters III and IV in [20].

We recall classical results about the operator (1.1) and its generalizations. We use the following notation.

Definition 1.1. We write $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ to denote the set of all functions on $\mathbb{R}^n$ that are real-valued, homogeneous of degree 1, and C^∞ away from the origin.

The following theorem is due to Seeger, Sogge, and Stein [19].

Theorem A (Seeger–Sogge–Stein [19]). *Let* $1 \le p \le \infty$ *and* $m = -(n-1)|1/p - 1/2|$. *Assume* $\phi \in \mathcal{P}(\mathbb{R}^n)$. *Then*

$$e^{i\phi(\xi)}(1+|\xi|^2)^{m/2} \in \begin{cases} \mathcal{M}(H^p \to H^p) & \text{when } 1 \le p < \infty, \\ \mathcal{M}(\mathrm{BMO} \to \mathrm{BMO}) & \text{when } p = \infty. \end{cases}$$

In fact, this theorem is not given in [19] in exactly the same form as above; the result given in [19] is restricted to local estimates. However, Theorem A can be proved by a slight modification of the argument of [19]. Or one can appeal to the general results given by Ruzhansky and Sugimoto, see Theorems 1.2 and 2.2 in [18].

It is known that the number $-(n-1)|1/p - 1/2|$ given in Theorem A is optimal. In fact, for the typical case $\phi(\xi) = |\xi|$, the following theorem holds.

Theorem B. *If* $1 \le p \le \infty$, *and if*

$$e^{i|\xi|}(1+|\xi|^2)^{m/2} \in \begin{cases} \mathcal{M}(H^p \to H^p) & \text{when } 1 \le p < \infty, \\ \mathcal{M}(\mathrm{BMO} \to \mathrm{BMO}) & \text{when } p = \infty, \end{cases}$$

then $m \le -(n-1)|1/p - 1/2|$.

For a proof of Theorem B, see Theorem 1 in [10] or Section 6.13 in Chapter IX of [20].

The purpose of the present paper is to consider bilinear versions of Theorems A and B.

We recall the definition of bilinear Fourier multiplier operators. For a bounded measurable function $\sigma = \sigma(\xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$, the bilinear operator T_σ is defined by

$$T_\sigma(f, g)(x) = \frac{1}{(2\pi)^{2n}} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix\cdot(\xi+\eta)}\, \sigma(\xi, \eta)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, d\xi\, d\eta, \quad x \in \mathbb{R}^n,$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$. If X, Y and Z are function spaces on $\mathbb{R}^n$ equipped with quasi-norms or seminorms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively, and if there exists a constant A such that

$$\|T_\sigma(f, g)\|_Z \le A\|f\|_X \|g\|_Y \quad \text{for all } f \in X \cap \mathcal{S} \text{ and all } g \in Y \cap \mathcal{S},$$

then we say that σ is a *bilinear Fourier multiplier* for $X \times Y$ to Z and we write $\sigma \in \mathcal{M}(X \times Y \to Z)$. (Sometimes we write $\theta(\xi, \eta) \in \mathcal{M}(X \times Y \to Z)$ to mean $\theta(\cdot, \cdot) \in \mathcal{M}(X \times Y \to Z)$.) The smallest constant A that satisfies the above inequality is denoted by $\|\sigma\|_{\mathcal{M}(X \times Y \to Z)}$.

We shall consider the bilinear Fourier multiplier of the form

$$e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\sigma(\xi,\eta) \quad \text{with } \phi_1,\phi_2 \in \mathscr{P}(\mathbb{R}^n) \text{ and } \sigma \in S^m_{1,0}(\mathbb{R}^{2n}),$$

where the class $S^m_{1,0}(\mathbb{R}^{2n})$ is defined as follows.

Definition 1.2. For $m \in \mathbb{R}$, the class $S^m_{1,0}(\mathbb{R}^{2n})$ is defined to be the set of all C^∞ functions $\sigma = \sigma(\xi,\eta)$ on $\mathbb{R}^{2n}$ that satisfy the estimate

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_\alpha (1+|\xi|+|\eta|)^{m-|\alpha|-|\beta|}$$

for all multi-indices α and β.

In the theory of bilinear Fourier multipliers, a classical method is known that allows us to write a multiplier $\sigma \in S^m_{1,0}(\mathbb{R}^{2n})$ as a sum of multipliers of the product form $\theta_1(\xi)\,\theta_2(\eta)$. Using this method, we can deduce the following result from Theorem A.

Theorem 1.3. *Let $n \ge 2$, and let $1 \le p,q \le \infty$ be such that $1/p + 1/q = 1/r$. Assume $\phi_1,\phi_2 \in \mathscr{P}(\mathbb{R}^n)$ and $\sigma \in S^m_{1,0}(\mathbb{R}^{2n})$ with $m = -(n-1)(|1/p-1/2| + |1/q-1/2|)$. Then $e^{i(\phi_1(\xi)+\phi_2(\eta))}\sigma(\xi,\eta) \in \mathcal{M}(H^p \times H^q \to L^r)$, where L^r should be replaced by* BMO *when $r=\infty$.*

In fact, Rodríguez-López, Rule, and Staubach [17] considered more general operators, bilinear Fourier integral operators, and proved a theorem that almost covers Theorem 1.3. The statement of the theorem of [17] is, however, restricted to local estimate. We shall give a full proof of Theorem 1.3 in Section 3.

The main purpose of the present paper is to show that the number

$$m = -(n-1)\left(\left|\frac{1}{p}-\frac{1}{2}\right| + \left|\frac{1}{q}-\frac{1}{2}\right|\right)$$

in Theorem 1.3 can be improved, and to show that the improved m is optimal at least for certain (p,q).

The following is the first main theorem of this paper.

Theorem 1.4. *Let $n \ge 2$, and let $1 \le p,q \le \infty$ be such that $1/p + 1/q = 1/r$. Assume $\phi_1,\phi_2 \in \mathscr{P}(\mathbb{R}^n)$ and $\sigma \in S^m_{1,0}(\mathbb{R}^{2n})$ with $m = m_1(p,q)$, where*

$$m_1(p,q) = \begin{cases} -(n-1)\left(\left|\frac{1}{p}-\frac{1}{2}\right| + \left|\frac{1}{q}-\frac{1}{2}\right|\right) & \text{if } 1 \le p,q \le 2 \text{ or if } 2 \le p,q \le \infty, \\ -\left(\frac{1}{p}-\frac{1}{2}\right) - (n-1)\left(\frac{1}{2}-\frac{1}{q}\right) & \text{if } 1 \le p \le 2 \le q \le \infty \text{ and } \frac{1}{p}+\frac{1}{q} \le 1, \\ -(n-1)\left(\frac{1}{p}-\frac{1}{2}\right) - \left(\frac{1}{2}-\frac{1}{q}\right) & \text{if } 1 \le p \le 2 \le q \le \infty \text{ and } \frac{1}{p}+\frac{1}{q} \ge 1, \\ -(n-1)\left(\frac{1}{2}-\frac{1}{p}\right) - \left(\frac{1}{q}-\frac{1}{2}\right) & \text{if } 1 \le q \le 2 \le p \le \infty \text{ and } \frac{1}{p}+\frac{1}{q} \le 1, \\ -\left(\frac{1}{2}-\frac{1}{p}\right) - (n-1)\left(\frac{1}{q}-\frac{1}{2}\right) & \text{if } 1 \le q \le 2 \le p \le \infty \text{ and } \frac{1}{p}+\frac{1}{q} \ge 1. \end{cases}$$

Then $e^{i(\phi_1(\xi)+\phi_2(\eta))}\sigma(\xi,\eta) \in \mathcal{M}(H^p \times H^q \to L^r)$, where L^r should be replaced by BMO *when $r=\infty$.*

Compare the claims of Theorems 1.3 and 1.4. They are the same in the regions $1 \le p,q \le 2$ and $2 \le p,q \le \infty$, but are different outside of these regions. In the typical case $(p,q) = (1,\infty)$, Theorem 1.3 asserts that the multiplier $e^{i(\phi_1(\xi)+\phi_2(\eta))}\sigma(\xi,\eta)$ belongs to $\mathcal{M}(H^1 \times L^\infty \to L^1)$ if $\sigma \in S^{-(n-1)}_{1,0}(\mathbb{R}^{2n})$, whereas Theorem 1.4 asserts that the same holds if $\sigma \in S^{-n/2}_{1,0}(\mathbb{R}^{2n})$. The latter is stronger if $n \geq 3$. To be precise, observe that

$$m_1(p,q) > -(n-1)(|1/p - 1/2| + |1/q - 1/2|)$$

if $n \geq 3$ and $1 \leq p < 2 < q \leq \infty$ or $1 \leq q < 2 < p \leq \infty$. Thus Theorem 1.4 is an improvement of Theorem 1.3 for these n, p and q.

In order to show that the number $m_1(p,q)$ is in fact optimal for some (p,q), we consider the special case $\phi_1(\xi) = \phi_2(\xi) = |\xi|$. We write

$$X_r = \begin{cases} L^r & \text{if } 0 < r < \infty, \\ \mathrm{BMO} & \text{if } r = \infty. \end{cases} \tag{1.2}$$

For $p, q \in [1,\infty]$ given, we set $1/r = 1/p + 1/q$ and we consider a necessary condition on $m \in \mathbb{R}$ that allows the assertion

$$e^{i(|\xi|+|\eta|)}\sigma(\xi,\eta) \in \mathcal{M}(H^p \times H^q \to X_r) \quad \text{for all } \sigma \in S^m_{1,0}(\mathbb{R}^{2n}). \tag{1.3}$$

The following is the second main theorem of this paper.

Theorem 1.5. *Let $n \geq 2$.*

(1) *Let $1 \leq p,q \leq 2$ or $2 \leq p,q \leq \infty$. Then $m \in \mathbb{R}$ satisfies (1.3) only if $m \leq -(n-1) \times (|1/p - 1/2| + |1/q - 1/2|)$.*
(2) *Let $1 \leq p \leq 2 \leq q \leq \infty$ or $1 \leq q \leq 2 \leq p \leq \infty$, and assume $1/p + 1/q = 1$. Then $m \in \mathbb{R}$ satisfies (1.3) only if $m \leq -n|1/p - 1/2|$.*

This theorem implies that the number $m_1(p,q)$ of Theorem 1.4 is optimal for p and q in the range given in (1) and (2) of Theorem 1.5. The present authors do not know whether $m_1(p,q)$ is optimal for other p and q.

The contents of the rest of the paper are as follows. In Section 2, we collect some propositions concerning flag paraproduct, which we will use in the proof of Theorem 1.3. In order not to interrupt the stream of argument, we shall postpone rather long proofs of those propositions to Section 6. In Sections 3, 4 and 5, we prove Theorems 1.3, 1.4 and 1.5, respectively. The last section, Section 6, is devoted to the proofs of the propositions stated in Section 2.

We end this section by introducing some notations used throughout this paper.

Notation 1.6. We define the Fourier transform and the inverse Fourier transform on $\mathbb{R}^d$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\, dx \quad \text{and} \quad (g)^\vee(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x} g(\xi)\, d\xi.$$

Sometimes we use rude expressions $(f(x))^\wedge$ or $(g(\xi))^\vee$ to denote $(f(\cdot))^\wedge$ or $(g(\cdot))^\vee$, respectively.

We shall repeatedly use dyadic partitions of unity, which are defined as follows. Take a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and $\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1$ for $\xi \neq 0$. We define functions ζ and φ by $\zeta(\xi) = \sum_{j=1}^{\infty} \psi(2^{-j}\xi)$ and $\varphi(\xi) = 1 - \zeta(\xi)$.

We have

$$\zeta(\xi) = 0 \quad \text{if } |\xi| \le 1, \quad \zeta(\xi) = 1 \quad \text{if } |\xi| \ge 2,$$
$$\varphi(\xi) = 1 \quad \text{if } |\xi| \le 1, \quad \varphi(\xi) = 0 \quad \text{if } |\xi| \ge 2,$$
$$\sum\nolimits_{j=-\infty}^{k} \psi(2^{-j}\xi) = \varphi(2^{-k}\xi) \quad \text{for } \xi \ne 0, k \in \mathbb{Z}.$$

Notice, however, that we will also use the letters ψ, ζ and φ in a meaning different from the above.

For a smooth function θ on $\mathbb{R}^d$ and for a nonnegative integer N, we write

$$\|\theta\|_{C^N} = \max_{|\alpha|\le N} \sup_{\xi} \left|\partial_\xi^\alpha \theta(\xi)\right|.$$

The letter n denotes the dimension of the Euclidean space that we consider. Unless further restrictions are explicitly made, n is an arbitrary positive integer.

2. Some results from bilinear flag paraproducts

In this section, we give some results for the bilinear Fourier multipliers of the form

$$a_0(\xi,\eta)a_1(\xi)a_2(\eta).$$

This kind of multipliers, with a_0, a_1 and a_2 being 0-th order multipliers (i.e., the ones that generalize homogeneous functions of degree 0), are considered by Muscalu [14, 15] and Muscalu–Schlag [16], Chapter 8, where their mapping properties between L^p spaces are given. In this section, we consider the case where a_0, a_1 and a_2 are non-zero order multipliers, and give estimates including H^p and BMO. The results of this section will be used to prove Theorem 1.3.

Definition 2.1. For $m \in \mathbb{R}$ and $d \in \mathbb{N}$, the class $\dot{S}^m_{1,0}(\mathbb{R}^d)$ is defined to be the set of all C^∞ functions θ on $\mathbb{R}^d \setminus \{0\}$ such that, for all multi-indices α,

$$|\partial_\xi^\alpha \theta(\xi)| \le C_\alpha |\xi|^{m-|\alpha|}.$$

First, we recall a classical result about the bilinear Fourier multipliers in the class $\dot{S}^0_{1,0}(\mathbb{R}^{2n})$. The following result was established in the works of Coifman–Meyer [2,3,9], Kenig–Stein [8], Grafakos–Torres [7], and Grafakos–Kalton [5].

Proposition 2.2. *If $\sigma \in \dot{S}^0_{1,0}(\mathbb{R}^{2n})$, then $\sigma \in \mathcal{M}(H^p \times H^q \to L^r)$ for $0 < p, q \le \infty$ and $1/p + 1/q = 1/r > 0$, and also $\sigma \in \mathcal{M}(L^\infty \times L^\infty \to \mathrm{BMO})$.*

The proofs of the following two propositions will be given in Section 6.

Proposition 2.3. *Let $m_1, m_2 \le 0$, $m = m_1 + m_2$, $a_0 \in \dot{S}^m_{1,0}(\mathbb{R}^{2n})$, $a_1 \in \dot{S}^{-m_1}_{1,0}(\mathbb{R}^n)$ and $a_2 \in \dot{S}^{-m_2}_{1,0}(\mathbb{R}^n)$. Let $\sigma(\xi,\eta) = a_0(\xi,\eta)a_1(\xi)a_2(\eta)$. Then the following hold.*

(1) *$\sigma \in \mathcal{M}(H^p \times H^q \to L^r)$ for $0 < p, q < \infty$ and $1/p + 1/q = 1/r$.*

(2) *If $m_2 < 0$, then $\sigma \in \mathcal{M}(H^p \times \mathrm{BMO} \to L^p)$ for $0 < p < \infty$.*

(3) *If $m_1 < 0$, then $\sigma \in \mathcal{M}(\mathrm{BMO} \times H^q \to L^q)$ for $0 < q < \infty$.*

(4) *If $m_1, m_2 < 0$, then $\sigma \in \mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO})$.*

Proposition 2.4. *Let $m_1 \le 0$, $a_0 \in \dot{S}^{m_1}_{1,0}(\mathbb{R}^{2n})$, $a_1 \in \dot{S}^{-m_1}_{1,0}(\mathbb{R}^n)$, and define $\tau(\xi,\eta) = a_0(\xi,\eta)a_1(\xi)$. Then the following hold.*

(1) *$\tau \in \mathcal{M}(H^p \times L^\infty \to L^p)$ for $0 < p < \infty$.*

(2) *If $m_1 < 0$, then $\tau \in \mathcal{M}(\mathrm{BMO} \times L^\infty \to \mathrm{BMO})$.*

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we use the following lemma.

Lemma 3.1. *If $\phi \in \mathcal{P}(\mathbb{R}^n)$ and if $\theta \in C_0^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp}\theta \subset \{|\xi| \le 2\}$, then*

$$\left\|(e^{i\phi(\xi)}\theta(\xi))^\vee\right\|_{L^1} \le c\|\theta\|_{C^{n+1}},$$

where $c = c(n,\phi)$.

Proof. Write

$$e^{i\phi(\xi)}\,\theta(\xi) = \theta(\xi) + \sum_{j=-\infty}^{1} (e^{i\phi(\xi)} - 1)\,\theta(\xi)\,\psi(2^{-j}\xi),$$

where ψ is the function given in Notation 1.6. The inverse Fourier transform of $\theta(\xi)$ satisfies $|(\theta)^\vee(x)| \lesssim \|\theta\|_{C^{n+1}}(1+|x|)^{-n-1}$ and hence $\|(\theta)^\vee\|_{L^1} \lesssim \|\theta\|_{C^{n+1}}$. The function $(e^{i\phi(\xi)}-1)\,\theta(\xi)\,\psi(2^{-j}\xi)$ has support included in $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$, and satisfies the estimate

$$\left|\partial_\xi^\alpha\big((e^{i\phi(\xi)}-1)\,\theta(\xi)\,\psi(2^{-j}\xi)\big)\right| \lesssim \|\theta\|_{C^{n+1}}\,(2^j)^{1-|\alpha|}, \qquad |\alpha| \le n+1.$$

From this we obtain

$$\left|((e^{i\phi(\xi)}-1)\,\theta(\xi)\,\psi(2^{-j}\xi))^\vee(x)\right| \lesssim \|\theta\|_{C^{n+1}}\,2^{j(n+1)}\,(1+2^j|x|)^{-n-1},$$

and hence

$$\|((e^{i\phi(\xi)}-1)\,\phi(\xi)\,\theta(2^{-j}\xi))^\vee\|_{L^1} \lesssim \|\theta\|_{C^{n+1}}\,2^j.$$

Taking sum over $j \le 1$, we obtain $\|(e^{i\phi(\xi)}-1)\phi(\xi))^\vee\|_{L^1} \lesssim \|\theta\|_{C^{n+1}}$. ■

Proof of Theorem 1.3. We write $m_1 = -(n-1)|1/p - 1/2|$, $m_2 = -(n-1)|1/q - 1/2|$, and $1/p + 1/q = 1/r$. We also use the notation (1.2).

Using the functions ζ and φ of Notation 1.6, we decompose τ as

$$\tau(\xi,\eta) = \tau_1(\xi,\eta) + \tau_2(\xi,\eta) + \tau_3(\xi,\eta) + \tau_4(\xi,\eta),$$

where

$$\begin{aligned}
\tau_1(\xi,\eta) &= e^{i\phi_1(\xi)}\,\varphi(\xi)\,e^{i\phi_2(\eta)}\,\varphi(\eta)\,\sigma(\xi,\eta),\\
\tau_2(\xi,\eta) &= e^{i\phi_1(\xi)}\,\zeta(\xi)\,e^{i\phi_2(\eta)}\,\varphi(\eta)\,\sigma(\xi,\eta),\\
\tau_3(\xi,\eta) &= e^{i\phi_1(\xi)}\,\varphi(\xi)\,e^{i\phi_2(\eta)}\,\zeta(\eta)\,\sigma(\xi,\eta),\\
\tau_4(\xi,\eta) &= e^{i\phi_1(\xi)}\,\zeta(\xi)\,e^{i\phi_2(\eta)}\,\zeta(\eta)\,\sigma(\xi,\eta).
\end{aligned}$$

We shall prove that $\tau_i \in \mathcal{M}(H^p \times H^q \to X_r)$ for $i = 1,2,3,4$.

Firstly, the multiplier τ_1 is easy to handle. By Lemma 3.1, the inverse Fourier transform of $e^{i\phi_1(\xi)}\varphi(\xi)$ is in $L^1(\mathbb{R}^n)$, and hence $e^{i\phi_1(\xi)}\varphi(\xi) \in \mathcal{M}(H^p \to H^p)$, $1 \le p \le \infty$. Similarly, $e^{i\phi_2(\eta)}\varphi(\eta) \in \mathcal{M}(H^q \to H^q)$, $1 \le q \le \infty$. Also $\sigma \in \mathcal{M}(H^p \times H^q \to X_r)$, by Proposition 2.2. Combining these facts, we have that $\tau_1 \in \mathcal{M}(H^p \times H^q \to X_r)$.

Next, consider τ_2. We write this as

$$\tau_2(\xi,\eta) = \sigma(\xi,\eta)\,\tilde{\zeta}(\xi)\,|\xi|^{-m_1} \cdot e^{i\phi_1(\xi)}\,\zeta(\xi)\,|\xi|^{m_1} \cdot e^{i\phi_2(\eta)}\,\varphi(\eta),$$

where $\tilde{\zeta}$ is a C^∞ function on $\mathbb{R}^n$ such that $\tilde{\zeta}(\xi) = 1$ for $|\xi| \ge 1$ and $\tilde{\zeta}(\xi) = 0$ for $|\xi| \le 2^{-1}$. As we have seen above, $e^{i\phi_2(\eta)}\varphi(\eta) \in \mathcal{M}(H^q \to H^q)$ for $1 \le q \le \infty$. Theorem A implies

$$e^{i\phi_1(\xi)}\,\zeta(\xi)\,|\xi|^{m_1} \in \begin{cases} \mathcal{M}(H^p \to H^p) & \text{if } 1 \le p < \infty, \\ \mathcal{M}(\mathrm{BMO} \to \mathrm{BMO}) & \text{if } p = \infty. \end{cases}$$

Notice that $\sigma \in S^m_{1,0}(\mathbb{R}^{2n}) \subset \dot{S}^{m_1}_{1,0}(\mathbb{R}^{2n})$ and that $\tilde{\zeta}(\xi)|\xi|^{-m_1} \in \dot{S}^{-m_1}_{1,0}(\mathbb{R}^n)$. Hence Propositions 2.3 and 2.4 give

$$\sigma(\xi,\eta)\,\tilde{\zeta}(\xi)\,|\xi|^{-m_1} \in \begin{cases} \mathcal{M}(H^p \times H^q \to L^r) & \text{if } 1 \le p,q < \infty, \\ \mathcal{M}(H^p \times L^\infty \to L^p) & \text{if } 1 \le p < \infty \text{ and } q = \infty, \\ \mathcal{M}(\mathrm{BMO} \times H^q \to L^q) & \text{if } p = \infty \text{ and } 1 \le q < \infty, \\ \mathcal{M}(\mathrm{BMO} \times L^\infty \to \mathrm{BMO}) & \text{if } p = q = \infty \end{cases} \tag{3.1}$$

(notice that $m_1 < 0$ if $n \ge 2$ and $p = \infty$). Combining these results, we see that τ_2 belongs to the same multiplier class as in (3.1), which a fortiori implies $\tau_2 \in \mathcal{M}(H^p \times H^q \to X_r)$.

By symmetry, we also have $\tau_3 \in \mathcal{M}(H^p \times H^q \to X_r)$.

Finally, consider τ_4. We write this as

$$\tau_4(\xi,\eta) = \sigma(\xi,\eta)\,\tilde{\zeta}(\xi)\,|\xi|^{-m_1}\,\tilde{\zeta}(\eta)\,|\eta|^{-m_2} \cdot e^{i\phi_1(\xi)}\,\zeta(\xi)\,|\xi|^{m_1} \cdot e^{i\phi_2(\eta)}\,\zeta(\eta)\,|\eta|^{m_2},$$

where $\tilde{\zeta}$ is the same as above. Theorem A gives

$$e^{i\phi_1(\xi)}\,\zeta(\xi)\,|\xi|^{m_1} \in \begin{cases} \mathcal{M}(H^p \to H^p) & \text{if } 1 \le p < \infty, \\ \mathcal{M}(\mathrm{BMO} \to \mathrm{BMO}) & \text{if } p = \infty, \end{cases}$$

$$e^{i\phi_2(\eta)}\,\zeta(\eta)\,|\eta|^{m_2} \in \begin{cases} \mathcal{M}(H^q \to H^q) & \text{if } 1 \le q < \infty, \\ \mathcal{M}(\mathrm{BMO} \to \mathrm{BMO}) & \text{if } q = \infty. \end{cases}$$

Proposition 2.3 gives

$$\begin{aligned} &\sigma(\xi,\eta)\,\tilde{\zeta}(\xi)\,|\xi|^{-m_1}\,\tilde{\zeta}(\eta)\,|\eta|^{-m_2} \\ &\in \begin{cases} \mathcal{M}(H^p \times H^p \to L^r) & \text{if } 1 \le p,q < \infty, \\ \mathcal{M}(H^p \times \mathrm{BMO} \to L^p) & \text{if } 1 \le p < \infty \text{ and } q = \infty, \\ \mathcal{M}(\mathrm{BMO} \times H^q \to L^q) & \text{if } p = \infty \text{ and } 1 \le q < \infty, \\ \mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO}) & \text{if } p = q = \infty \end{cases} \end{aligned} \tag{3.2}$$

(notice that $m_1 < 0$ if $n \ge 2$ and $p = \infty$ and that $m_2 < 0$ if $n \ge 2$ and $q = \infty$). Now combining these results, we see that τ_4 belongs to the same multiplier class as in (3.2), which a fortiori implies $\tau_4 \in \mathcal{M}(H^p \times H^q \to X_r)$. This completes the proof of Theorem 1.3. ∎

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. For this, the key is to prove the assertion of Theorem 1.4 in the special case $p = 1$ and $q = \infty$, which we shall write here for the sake of reference.

Theorem 4.1. *If $n \geq 2$, $\phi_1, \phi_2 \in \mathcal{P}(\mathbb{R}^n)$, and $\sigma \in S^{-n/2}_{1,0}(\mathbb{R}^{2n})$, then $e^{i(\phi_1(\xi)+\phi_2(\eta))}\sigma(\xi,\eta) \in \mathcal{M}(H^1 \times L^\infty \to L^1)$.*

Theorem 1.4 can be deduced from this theorem and from Theorem 1.3. In fact, notice that, by obvious symmetry, we have $e^{i(\phi_1(\xi)+\phi_2(\eta))}\sigma(\xi,\eta) \in \mathcal{M}(L^\infty \times H^1 \to L^1)$ under the same assumptions on n and σ. Hence, if Theorem 4.1 is proved, then we can deduce the claims of Theorem 1.4 from the claims of Theorems 1.3 and 4.1 with the aid of complex interpolation. (For the interpolation argument, see, e.g., the proof of Theorem 2.2 in [1] or the proof of the 'if' part of Theorem 1.1 in [12].) Thus it is sufficient to prove Theorem 4.1.

To that end, we use the following lemmas.

Lemma 4.2. *Let $\phi_1, \phi_2 \in \mathcal{P}(\mathbb{R}^n)$ and let $\theta \in C_0^\infty(\mathbb{R}^{2n})$. Then $(e^{i(\phi_1(\xi)+\phi_2(\eta))}\theta(\xi,\eta))^\vee \in L^1(\mathbb{R}^{2n})$.*

Proof. Take a function $\widetilde{\theta} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\theta}(\xi)\widetilde{\theta}(\eta) = 1$ on $\operatorname{supp}\theta$. Then

$$e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\theta(\xi,\eta) = e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\widetilde{\theta}(\xi)\,\widetilde{\theta}(\eta)\,\theta(\xi,\eta).$$

Lemma 3.1 implies $(e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\widetilde{\theta}(\xi)\widetilde{\theta}(\eta))^\vee \in L^1(\mathbb{R}^{2n})$. Clearly $(\theta(\xi,\eta))^\vee \in L^1(\mathbb{R}^{2n})$. Hence the conclusion of the lemma follows. ∎

Lemma 4.3. *Let $n \geq 2$ and $\phi \in \mathcal{P}(\mathbb{R}^n)$, and set $R = \sup\{|\nabla\phi(\xi)| \mid |\xi| = 1\}$. Let ψ be a C^∞ function on $\mathbb{R}^n$ satisfying $\operatorname{supp}\psi \subset \{2^{-1} \leq |\xi| \leq 2\}$. Then the following hold.*

(1) *For each positive integer N, there exists a constant c_N, depending only on n, ϕ and N, such that*

$$\left|(e^{-i\phi(\xi)}\psi(2^{-j}\xi))^\vee(x)\right| \leq c_N \|\psi\|_{C^N} (2^j)^{n-N/2}|x|^{-N} \quad \textit{for } |x| > 2R \textit{ and } j \in \mathbb{N}.$$

(2) *There exists a constant c, depending only on n and ϕ, such that*

$$\left\|(e^{-i\phi(\xi)}\psi(2^{-j}\xi))^\vee(x)\right\|_{L^1} \leq c\|\psi\|_{C^{2n-1}}(2^j)^{(n-1)/2} \quad \textit{for all } j \in \mathbb{N}.$$

Proof. We write $f_j(x) = (e^{-i\phi(\xi)}\,\psi(2^{-j}\xi))^\vee(x)$.

To estimate $f_j(x)$, we follow the idea given by Seeger–Sogge–Stein [19]. Let $S^{n-1} = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$. For each $j \in \mathbb{N}$, take a sequence of points $\{\xi_j^\nu\}_\nu$ such that

$$\xi_j^\nu \in S^{n-1}, \quad \bigcup_\nu B(\xi_j^\nu, 2^{-j/2}) \cap S^{n-1} = S^{n-1},$$

$$\text{and} \quad \sum_\nu \mathbf{1}_{B(\xi_j^\nu, 2^{-j/2+1})}(\xi) \leq c \quad \text{for all } \xi \in S^{n-1},$$

where $B(x, r)$ denotes the ball with center x and radius r, and ν runs on an index set of cardinality $\approx (2^{j/2})^{n-1}$.

Take functions $\{\chi_j^\nu\}_\nu$ such that

$$\begin{aligned}
&\chi_j^\nu \ \text{ is homogeneous of degree 0 and } C^\infty \text{ on } \mathbb{R}^n \setminus \{0\},\\
&\{\xi \in S^{n-1} \mid \chi_j^\nu(\xi) \neq 0\} \subset \{\xi \in S^{n-1} \mid |\xi - \xi_j^\nu| < 2^{-j/2+1}\},\\
&|\partial_\xi^\alpha \chi_j^\nu(\xi)| \le c_\alpha (2^{j/2})^{|\alpha|} \ \text{ for all } \xi \in S^{n-1},\\
&\sum_\nu \chi_j^\nu(\xi) = 1 \ \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}.
\end{aligned}$$

Using this partition of unity, we decompose $f_j(x)$ as

$$f_j(x) = \sum_\nu f_j^\nu(x),$$

$$f_j^\nu(x) = \big(e^{-i\phi(\xi)}\,\psi(2^{-j}\xi)\,\chi_j^\nu(\xi)\big)^\vee(x) = \frac{1}{(2\pi)^n}\int e^{i(\xi\cdot x-\phi(\xi))}\,\psi(2^{-j}\xi)\,\chi_j^\nu(\xi)\,d\xi.$$

The key idea used below is that the oscillating factor $e^{-i\phi(\xi)}$ can be well approximated by $e^{-i\xi\cdot\nabla\phi(\xi_j^\nu)}$ on the support of $\psi(2^{-j}\xi)\,\chi_j^\nu(\xi)$. We write the phase function $\xi\cdot x-\phi(\xi)$ appearing in the last integral as

$$\begin{aligned}
\xi\cdot x-\phi(\xi) &= \xi\cdot(x-\nabla\phi(\xi)) = \xi\cdot(x-\nabla\phi(\xi_j^\nu)) + h_j^\nu(\xi),\\
h_j^\nu(\xi) &= \xi\cdot(\nabla\phi(\xi_j^\nu)-\nabla\phi(\xi)).
\end{aligned}$$

Then

$$f_j^\nu(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot(x-\nabla\phi(\xi_j^\nu))}\,\psi(2^{-j}\xi)\,\chi_j^\nu(\xi)\,e^{ih_j^\nu(\xi)}\,d\xi.$$

Notice that the support of $\psi(2^{-j}\xi)\chi_j^\nu(\xi)$ is included in the set

$$E_{j,\nu} = \left\{\xi \;\middle|\; 2^{j-1}\le|\xi|\le 2^{j+1},\ \left|\frac{\xi}{|\xi|}-\xi_j^\nu\right|\le 2^{-j/2+1}\right\},$$

which has Lebesgue measure $|E_{j,\nu}| \approx (2^j)^{(n+1)/2}$. The functions appearing in the above integral satisfy the following estimates on $E_{j,\nu}$:

$$\begin{aligned}
|\partial_\xi^\alpha \psi(2^{-j}\xi)| &\le c_\alpha\,\|\psi\|_{C^{|\alpha|}}\,(2^j)^{-|\alpha|},\\
|\partial_\xi^\alpha \chi_j^\nu(\xi)| &\le c_\alpha\,(2^{j/2})^{-|\alpha|},\\
|(\xi_j^\nu\cdot\nabla_\xi)^k\,\chi_j^\nu(\xi)| &\le c_k\,(2^j)^{-k},\\
|\partial_\xi^\alpha e^{ih_j^\nu(\xi)}| &\le c_\alpha\,(2^{j/2})^{-|\alpha|},\\
|(\xi_j^\nu\cdot\nabla_\xi)^k e^{ih_j^\nu(\xi)}| &\le c_k\,(2^j)^{-k}.
\end{aligned}$$

By using these estimates and by integration by parts, we obtain the following two estimates:

$$|f_j^\nu(x)| \le c_N\,\|\psi\|_{C^N}\,(2^j)^{(n+1)/2}\big(1+2^{j/2}\,|x-\nabla\phi(\xi_j^\nu)|\big)^{-N}, \tag{4.1}$$

$$|f_j^\nu(x)| \le c_N\,\|\psi\|_{C^N}\,(2^j)^{(n+1)/2}\big(1+2^{j}\,|\xi_j^\nu\cdot(x-\nabla\phi(\xi_j^\nu))|\big)^{-N}. \tag{4.2}$$

Proof of (1). Suppose $|x| > 2R$. Then $|x - \nabla\phi(\xi_j^\nu)| \approx |x|$, and hence (4.1) gives

$$|f_j^\nu(x)| \lesssim \|\psi\|_{C^N} (2^j)^{(n+1)/2} (2^{j/2}|x|)^{-N}.$$

Taking sum over ν's of card $\approx (2^{j/2})^{n-1}$, we have

$$\begin{aligned}|f_j(x)| \le \sum_\nu |f_j^\nu(x)| &\lesssim \|\psi\|_{C^N} (2^j)^{(n+1)/2} (2^{j/2}|x|)^{-N} (2^{j/2})^{n-1} \\ &= \|\psi\|_{C^N} (2^j)^{n-N/2} |x|^{-N}.\end{aligned}$$

Proof of (2). Combining (4.2) and (4.1), we have

$$\begin{aligned}&|f_j^\nu(x)| \\ &\le c_N \|\psi\|_{C^N} (2^j)^{(n+1)/2} \big(1 + 2^j |\xi_j^\nu \cdot (x - \nabla\phi(\xi_j^\nu))|\big)^{-N/2} (1 + 2^{j/2}|x - \nabla\phi(\xi_j^\nu)|)^{-N/2} \\ &\le c_N \|\psi\|_{C^N} (2^j)^{(n+1)/2} (1 + 2^j |\xi_j^\nu \cdot (x - \nabla\phi(\xi_j^\nu))|)^{-N/2} (1 + 2^{j/2}|(x - \nabla\phi(\xi_j^\nu))'|)^{-N/2},\end{aligned}$$

where $(x - \nabla\phi(\xi_j^\nu))'$ denotes the orthogonal projection of $x - \nabla\phi(\xi_j^\nu)$ to the orthogonal complement of the line $\mathbb{R}\xi_j^\nu$. Taking $N = 2n - 1$ and integrating the above inequality, we have

$$\begin{aligned}\|f_j^\nu\|_{L^1} &\lesssim \|\psi\|_{C^{2n-1}} (2^j)^{(n+1)/2} \int_{\mathbb{R}^n} \big(1 + 2^j |\xi_j^\nu \cdot (x - \nabla\phi(\xi_j^\nu))|\big)^{-(2n-1)/2} \\ &\qquad \times \big(1 + 2^{j/2}|(x - \nabla\phi(\xi_j^\nu))'|\big)^{-(2n-1)/2} \, dx \\ &\approx \|\psi\|_{C^{2n-1}}.\end{aligned}$$

Taking sum over ν's of card $\approx (2^{j/2})^{n-1}$, we obtain the inequality as mentioned in (2). This completes the proof of Lemma 4.3. ∎

Lemma 4.4. *Let $n \ge 2$, $\phi \in \mathcal{P}(\mathbb{R}^n)$, and set $R = \sup\{|\nabla\phi(\xi)| \mid |\xi| = 1\}$. Let ζ be the function given in Notation 1.6, and let $\theta \in C_0^\infty(\mathbb{R}^n)$ satisfy* $\operatorname{supp}\theta \subset \{|\xi| \le 2\}$. *Then the following hold.*

(1) *For each positive integer $N > 2n$, there exists a constant c_N, depending only on n, ϕ and N, such that*

$$\big|(e^{i\phi(\xi)}\zeta(\xi)\theta(2^{-j}\xi))^\vee(x)\big| \le c_N \|\theta\|_{C^N} |x|^{-N} \quad \textit{for } |x| > 2R \textit{ and for all } j \in \mathbb{N}.$$

(2) *There exists a constant c, depending only on n and ϕ, such that*

$$\big\| \big(e^{i\phi(\xi)}\zeta(\xi)\theta(2^{-j}\xi)\big)^\vee \big\|_{L^1} \le c \|\theta\|_{C^{2n-1}} 2^{j(n-1)/2}.$$

Proof. From the definition of ζ and from the assumption on $\operatorname{supp}\theta$, we have

$$e^{i\phi(\xi)}\zeta(\xi)\theta(2^{-j}\xi) = \sum_{k=1}^{j+1} e^{i\phi(\xi)}\psi(2^{-k}\xi)\theta(2^{-j}\xi).$$

If $|x| > 2R$ and $1 \le k \le j+1$, then Lemma 4.3 (1) gives

$$\begin{aligned}\left|(e^{i\phi(\xi)}\psi(2^{-k}\xi)\theta(2^{-j}\xi))^{\vee}(x)\right| &\lesssim (2^k)^{n-N/2}\,|x|^{-N}\,\|\psi(\cdot)\,\theta(2^{k-j}\cdot)\|_{C^N}\\ &\lesssim (2^k)^{n-N/2}\,|x|^{-N}\,\|\theta\|_{C^N}.\end{aligned}$$

If $N > 2n$, then taking sum over k, we obtain the inequality mentioned in (1).

For $1 \le k \le j+1$, Lemma 4.3 (2) gives

$$\begin{aligned}\left\|(e^{i\phi(\xi)}\,\psi(2^{-k}\xi)\,\theta(2^{-j}\xi))^{\vee}(x)\right\|_{L^1} &\lesssim \|\psi(\cdot)\,\theta(2^{k-j}\cdot)\|_{C^{2n-1}}\,(2^k)^{(n-1)/2}\\ &\lesssim \|\theta\|_{C^{2n-1}}\,(2^k)^{(n-1)/2}.\end{aligned}$$

Taking sum over $k \le j+1$, we obtain the inequality mentioned in (2). Lemma 4.4 is proved. ∎

Proof of Theorem 4.1. We write $m = -n/2$ and assume $\sigma \in S^m_{1,0}(\mathbb{R}^{2n})$.

We use a dyadic partition of unity to decompose $\sigma(\xi,\eta)$. Let ψ, φ and ζ be the functions given in Notation 1.6. For $j \in \mathbb{N} \cup \{0\}$, we define ψ_j by

$$\psi_j(\xi) = \begin{cases}\varphi(\xi) & \text{if } j = 0,\\ \psi(2^{-j}\xi) & \text{if } j \ge 1.\end{cases}$$

Notice that $\sum_{j=0}^{\infty}\psi_j(\xi) = 1$ and $\sum_{j=0}^{k}\psi_j(\xi) = \varphi(2^{-k}\xi)$ for $k \in \mathbb{N} \cup \{0\}$.

We decompose σ as

$$\begin{aligned}\sigma(\xi,\eta) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sigma(\xi,\eta)\,\psi_j(\xi)\,\psi_k(\eta) = \sum_{j>k} + \sum_{j=k} + \sum_{j<k}\\ &= \sigma_{\mathrm{I}}(\xi,\eta) + \sigma_{\mathrm{II}}(\xi,\eta) + \sigma_{\mathrm{III}}(\xi,\eta),\end{aligned}$$

where $\sum_{j>k}$, $\sum_{j=k}$, and $\sum_{j<k}$ denote the sums of $\sigma(\xi,\eta)\,\psi_j(\xi)\,\psi_k(\eta)$ over $(j,k) \in (\mathbb{N} \cup \{0\})^2$ that satisfy the designated restrictions.

Consider the multiplier σ_{I}. This is written as

$$\sigma_{\mathrm{I}}(\xi,\eta) = \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\sigma(\xi,\eta)\,\psi_j(\xi)\,\psi_k(\eta) = \sum_{j=1}^{\infty}\sigma(\xi,\eta)\,\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta).$$

Take a function $\widetilde{\psi} \in C_0^{\infty}(\mathbb{R}^n)$ such that $\operatorname{supp}\widetilde{\psi} \subset \{3^{-1} \le |\xi| \le 3\}$ and $\widetilde{\psi}(\xi) = 1$ for $2^{-1} \le |\xi| \le 2$. Also take a function $\widetilde{\varphi} \in C_0^{\infty}(\mathbb{R}^n)$ such that $\operatorname{supp}\widetilde{\varphi} \subset \{|\xi| \le 3\}$ and $\widetilde{\varphi}(\xi) = 1$ for $|\xi| \le 2$. Then

$$\sigma_{\mathrm{I}}(\xi,\eta) = \sum_{j=1}^{\infty}\sigma(\xi,\eta)\,\widetilde{\psi}(2^{-j}\xi)\,\widetilde{\varphi}(2^{-j+1}\eta)\,\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta).$$

The function $\sigma(2^j\xi, 2^{j-1}\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\varphi}(\eta)$ is supported in $\{3^{-1} \le |\xi| \le 3\} \times \{|\eta| \le 3\}$, and satisfies the estimate

$$\left|\partial_\xi^\alpha\,\partial_\eta^\beta\{\sigma(2^j\xi, 2^{j-1}\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\varphi}(\eta)\}\right| \le C_{\alpha,\beta}\,2^{jm},$$

with $C_{\alpha,\beta}$ independent of $j \in \mathbb{N}$. Hence, by the Fourier series expansion, we can write

$$\sigma(2^j\xi, 2^{j-1}\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\varphi}(\eta) = \sum_{a,b\in\mathbb{Z}^n} c_{\mathrm{I},j}^{(a,b)}\, e^{ia\cdot\xi}\, e^{ib\cdot\eta} \quad \text{for } |\xi| < \pi,\ |\eta| < \pi,$$

with the coefficient satisfying

$$|c_{\mathrm{I},j}^{(a,b)}| \lesssim 2^{jm}\,(1+|a|)^{-L}\,(1+|b|)^{-L} \tag{4.3}$$

for any $L > 0$. Changing variables $\xi \to 2^{-j}\xi$ and $\eta \to 2^{-j+1}\eta$, and multiplying by the function $\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta)$, we obtain

$$\sigma(\xi,\eta)\,\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta) = \sum_{a,b\in\mathbb{Z}^n} c_{\mathrm{I},j}^{(a,b)}\, e^{ia\cdot 2^{-j}\xi}\, e^{ib\cdot 2^{-j+1}\eta}\,\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta).$$

Hence, σ_{I} is written as follows:

$$\begin{aligned}\sigma_{\mathrm{I}}(\xi,\eta) &= \sum_{a,b\in\mathbb{Z}^n}\sum_{j=1}^{\infty} c_{\mathrm{I},j}^{(a,b)}\, e^{ia\cdot 2^{-j}\xi}\, e^{ib\cdot 2^{-j+1}\eta}\,\psi(2^{-j}\xi)\,\varphi(2^{-j+1}\eta)\\ &= \sum_{a,b\in\mathbb{Z}^n}\sum_{j=1}^{\infty} c_{\mathrm{I},j}^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\varphi^{(b)}(2^{-j+1}\eta),\end{aligned}$$

where

$$\psi^{(\nu)}(\xi) = e^{i\nu\cdot\xi}\,\psi(\xi) \quad \text{and} \quad \varphi^{(\nu)}(\eta) = e^{i\nu\cdot\eta}\,\varphi(\eta) \quad \text{for } \nu \in \mathbb{Z}^n.$$

By a similar argument, σ_{II} and σ_{III} can be written as follows:

$$\sigma_{\mathrm{II}}(\xi,\eta) = \sigma(\xi,\eta)\psi_0(\xi)\psi_0(\eta) + \sum_{a,b\in\mathbb{Z}^n}\sum_{j=1}^{\infty} c_{\mathrm{II},j}^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\psi^{(b)}(2^{-j}\eta),$$

$$\sigma_{\mathrm{III}}(\xi,\eta) = \sum_{a,b\in\mathbb{Z}^n}\sum_{j=1}^{\infty} c_{\mathrm{III},j}^{(a,b)}\,\varphi^{(a)}(2^{-j+1}\xi)\,\psi^{(b)}(2^{-j}\eta),$$

where the coefficients $c_{\mathrm{II},j}^{(a,b)}$ and $c_{\mathrm{III},j}^{(a,b)}$ satisfy the same estimates as (4.3).

Hereafter, we shall consider a slightly general multiplier. We assume the multiplier $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(\xi,\eta) = \sum_{j=1}^{\infty} c_j\,\theta_1(2^{-j}\xi)\,\theta_2(2^{-j}\eta), \tag{4.4}$$

where $(c_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers satisfying

$$|c_j| \le 2^{jm} A, \quad j \in \mathbb{N}, \tag{4.5}$$

for some $A \in (0,\infty)$, and where θ_1 and θ_2 are functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\theta_1,\ \operatorname{supp}\theta_2 \subset \{|\xi| \le 2\}. \tag{4.6}$$

For such $\widetilde{\sigma}$, we shall prove the estimate

$$\big\| e^{i(\phi_1(\xi)+\phi_2(\eta))}\, \widetilde{\sigma}(\xi,\eta) \big\|_{\mathcal{M}(H^1\times L^\infty\to L^1)} \le cA\|\theta_1\|_{C^N}\|\theta_2\|_{C^N}, \tag{4.7}$$

with $c = c(n) \in (0,\infty)$ and $N = N(n) \in \mathbb{N}$.

If this is proved, then by applying it to $c_j = c^{(a,b)}_{\mathrm{I},j}$, $\theta_1 = \psi^{(a)}$ and $\theta_2 = \varphi^{(b)}(2\cdot)$, we obtain

$$\Big\| e^{i(\phi_1(\xi)+\phi_2(\eta))} \sum_{j=1}^{\infty} c^{(a,b)}_{\mathrm{I},j}\, \psi^{(a)}(2^{-j}\xi)\, \varphi^{(b)}(2^{-j+1}\eta) \Big\|_{\mathcal{M}(H^1\times L^\infty\to L^1)}$$
$$\lesssim (1+|a|)^{-L}(1+|b|)^{-L}\|\psi^{(a)}\|_{C^N}\|\varphi^{(b)}(2\cdot)\|_{C^N} \lesssim (1+|a|)^{-L+N}(1+|b|)^{-L+N},$$

and, thus, taking L sufficiently large and taking sum over $a, b \in \mathbb{Z}^n$, we obtain

$$e^{i(\phi_1(\xi)+\phi_2(\eta))}\, \sigma_{\mathrm{I}}(\xi,\eta) \in \mathcal{M}(H^1\times L^\infty\to L^1).$$

In the same way, we obtain

$$e^{i(\phi_1(\xi)+\phi_2(\eta))} \big(\sigma_{\mathrm{II}}(\xi,\eta) - \sigma(\xi,\eta)\,\psi_0(\xi)\,\psi_0(\eta)\big) \in \mathcal{M}(H^1\times L^\infty\to L^1)$$

and

$$e^{i(\phi_1(\xi)+\phi_2(\eta))}\, \sigma_{\mathrm{III}}(\xi,\eta) \in \mathcal{M}(H^1\times L^\infty\to L^1).$$

Since $e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\sigma(\xi,\eta)\,\psi_0(\xi)\,\psi_0(\eta)$ is also a multiplier for $H^1\times L^\infty\to L^1$ by virtue of Lemma 4.2, we will obtain the conclusion of the theorem.

Thus the proof is reduced to showing (4.7) for $\widetilde{\sigma}$ given by (4.4), (4.5), and (4.6).

We shall make a further reduction. As in the proof of Theorem 1.3, using the functions φ and ζ of Notation 1.6, we decompose the multiplier $e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\widetilde{\sigma}(\xi,\eta)$ into four parts:

$$e^{i(\phi_1(\xi)+\phi_2(\eta))}\,\widetilde{\sigma}(\xi,\eta) = \tau_1(\xi,\eta) + \tau_2(\xi,\eta) + \tau_3(\xi,\eta) + \tau_4(\xi,\eta),$$

where

$$\begin{aligned}
\tau_1(\xi,\eta) &= e^{i\phi_1(\xi)}\,\varphi(\xi)\,e^{i\phi_2(\eta)}\,\varphi(\eta)\,\widetilde{\sigma}(\xi,\eta),\\
\tau_2(\xi,\eta) &= e^{i\phi_1(\xi)}\,\zeta(\xi)\,e^{i\phi_2(\eta)}\,\varphi(\eta)\,\widetilde{\sigma}(\xi,\eta),\\
\tau_3(\xi,\eta) &= e^{i\phi_1(\xi)}\,\varphi(\xi)\,e^{i\phi_2(\eta)}\,\zeta(\eta)\,\widetilde{\sigma}(\xi,\eta),\\
\tau_4(\xi,\eta) &= e^{i\phi_1(\xi)}\,\zeta(\xi)\,e^{i\phi_2(\eta)}\,\zeta(\eta)\,\widetilde{\sigma}(\xi,\eta).
\end{aligned}$$

The multipliers τ_1, τ_2, and τ_3 are easy to handle. For τ_1, its inverse Fourier transform is given by

$$(\tau_1(\xi,\eta))^\vee(x,y) = \sum_{j=1}^{\infty} c_j \big(e^{i\phi_1(\xi)}\,\varphi(\xi)\,\theta_1(2^{-j}\xi)\big)^\vee(x) \big(e^{i\phi_2(\eta)}\,\varphi(\eta)\,\theta_2(2^{-j}\eta)\big)^\vee(y).$$

By Lemma 3.1, we have

$$\big\|(e^{i\phi_1(\xi)}\,\varphi(\xi)\,\theta_1(2^{-j}\xi))^\vee\big\|_{L^1} \lesssim \|\theta_1\|_{C^{n+1}} \tag{4.8}$$

and we get a similar estimate with θ_2 in place of θ_1. Thus

$$\begin{aligned}
&\|(\tau_1)^{\vee}\|_{L^1(\mathbb{R}^{2n})} \\
&\quad \le \sum_{j=1}^{\infty} 2^{jm} A \big\| (e^{i\phi_1(\xi)} \varphi(2^{-j}\xi)\, \theta_1(2^{-j}\xi))^{\vee} \big\|_{L^1(\mathbb{R}^n)} \big\| (e^{i\phi_2(\eta)} \varphi(\eta)\, \theta_2(2^{-j}\eta))^{\vee} \big\|_{L^1(\mathbb{R}^n)} \\
&\quad \lesssim \sum_{j=1}^{\infty} 2^{jm} A \|\theta_1\|_{C^{n+1}} \|\theta_2\|_{C^{n+1}} \approx A \|\theta_1\|_{C^{n+1}} \|\theta_2\|_{C^{n+1}},
\end{aligned}$$

which implies

$$\|\tau_1\|_{\mathcal{M}(H^1\times L^\infty\to L^1)} \lesssim A \|\theta_1\|_{C^{n+1}} \|\theta_2\|_{C^{n+1}}.$$

For τ_2, we use the estimate

$$\big\| (e^{i\phi_1(\xi)} \zeta(\xi)\, \theta_1(2^{-j}\xi))^{\vee} \big\|_{L^1(\mathbb{R}^n)} \lesssim 2^{j(n-1)/2} \|\theta_1\|_{C^{2n-1}},$$

which is given in Lemma 4.4 (2). Using this together with (4.8), we obtain

$$\begin{aligned}
\|\tau_2\|_{\mathcal{M}(H^1\times L^\infty\to L^1)} &\le \|(\tau_2)^{\vee}\|_{L^1(\mathbb{R}^{2n})} \\
&= \Big\| \sum_{j=1}^{\infty} c_j \big(e^{i\phi_1(\xi)} \zeta(\xi)\, \theta_1(2^{-j}\xi)\big)^{\vee}(x) \big(e^{i\phi_2(\eta)} \varphi(\eta)\, \theta_2(2^{-j}\eta)\big)^{\vee}(y) \Big\|_{L^1_{x,y}(\mathbb{R}^{2n})} \\
&\lesssim \sum_{j=1}^{\infty} 2^{jm} A\, 2^{j(n-1)/2} \|\theta_1\|_{C^{2n-1}} \|\theta_2\|_{C^{n+1}} \approx A \|\theta_1\|_{C^{2n-1}} \|\theta_2\|_{C^{n+1}},
\end{aligned}$$

where the last $\approx$ holds because $m < -(n-1)/2$. Similarly, we have

$$\|\tau_3\|_{\mathcal{M}(H^1\times L^\infty\to L^1)} \le \|(\tau_3)^{\vee}\|_{L^1(\mathbb{R}^{2n})} \lesssim A \|\theta_1\|_{C^{n+1}} \|\theta_2\|_{C^{2n-1}}.$$

Thus the rest of the proof is the estimate for τ_4. Our purpose is to prove the estimate

$$\|T_{\tau_4}(f,g)\|_{L^1} \lesssim A \|\theta_1\|_{C^N} \|\theta_2\|_{C^N} \|f\|_{H^1} \|g\|_{L^\infty}.$$

To prove this, by virtue of the atomic decomposition of H^1, it is sufficient to prove the uniform estimate of $\|T_{\tau_4}(f,g)\|_{L^1}$ for H^1-atoms f. By translation, we may assume that the H^1-atoms are supported on balls centered at the origin. Thus we assume

$$\operatorname{supp} f \subset \{|x| \le r\}, \quad \|f\|_{L^\infty} \le r^{-n} \quad \text{and} \quad \int f(x)\, dx = 0,$$

and we shall prove

$$\|T_{\tau_4}(f,g)\|_{L^1} \lesssim A \|\theta_1\|_{C^N} \|\theta_2\|_{C^N} \|g\|_{L^\infty}.$$

Recall that the bilinear operator T_{τ_4} is given by

$$T_{\tau_4}(f,g)(x) = \sum_{j=1}^{\infty} c_j (e^{i\phi_1(D)} \zeta(D)\, \theta_1(2^{-j}D) f)(x)\, (e^{i\phi_2(D)} \zeta(D)\, \theta_2(2^{-j}D) g)(x).$$

We set $R = 1 + \max_{i=1,2} \sup\{|\nabla\phi_i(\xi)| \mid |\xi| = 1\}$.

Firstly, consider the case $r > R$. In this case, we estimate the L^1 norm as

$$\begin{aligned}\|T_{\tau_4}(f,g)\|_{L^1} &\le \sum_{j=1}^{\infty} 2^{jm} A \big\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f\big\|_{L^1}\big\|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)g\big\|_{L^\infty}\\ &=: (\star).\end{aligned}$$

For the L^∞-norm involving g, we use Lemma 4.4(2) to obtain

$$\begin{aligned}\big\|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)g\big\|_{L^\infty} &\le \big\|\big(e^{i\phi_2(\eta)}\zeta(\eta)\theta_2(2^{-j}\eta)\big)^\vee\big\|_{L^1}\|g\|_{L^\infty}\\ &\lesssim 2^{j(n-1)/2}\|\theta_2\|_{C^{2n-1}}\|g\|_{L^\infty}.\end{aligned} \tag{4.9}$$

For the L^1 norm of $e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)$ on $|x| \le 3r$, we use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}&\big\|(e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f)(x)\big\|_{L^1(|x|\le 3r)}\\ &\quad\lesssim r^{n/2}\big\|(e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f)(x)\big\|_{L^2(|x|\le 3r)} \lesssim r^{n/2}\|\theta_1\|_{C^0}\|f\|_{L^2} \lesssim \|\theta_1\|_{C^0}.\end{aligned}$$

If $|x| > 3r$ and $|y| \le r$, then $|x-y| > 2r > 2R$. Hence, for $|x| > 3r$, using Lemma 4.4(1), we see that

$$\begin{aligned}\big|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\big| &= \Big|\int \big(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi)\big)^\vee(x-y)\,f(y)\,dy\Big|\\ &\lesssim \int_{|y|\le r}\|\theta_1\|_{C^N}|x-y|^{-N}|f(y)|\,dy \lesssim \|\theta_1\|_{C^N}|x|^{-N},\end{aligned}$$

which implies

$$\big\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\big\|_{L^1(|x|>3r)} \lesssim \|\theta_1\|_{C^N}\int_{|x|>3r}|x|^{-N}\,dx \lesssim \|\theta_1\|_{C^N}.$$

Combining the above estimates, we have

$$\big\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\big\|_{L^1} \lesssim \|\theta_1\|_{C^N}. \tag{4.10}$$

Now from (4.9) and (4.10), we obtain

$$(\star) \lesssim \sum_{j=1}^{\infty} 2^{jm}A\|\theta_1\|_{C^N}2^{j(n-1)/2}\|\theta_2\|_{C^N}\|g\|_{L^\infty} \approx A\|\theta_1\|_{C^N}\|\theta_2\|_{C^N}\|g\|_{L^\infty},$$

where the last $\approx$ holds because $m < -(n-1)/2$.

Secondly, we assume $r \le R$ and estimate the L^1 norm of $T_{\tau_4}(f,g)(x)$ on $|x| > 3R$. We estimate this as

$$\begin{aligned}T_{\tau_4}(f,g)(x)\|_{L^1(|x|>3R)} &\le \sum_{j=1}^{\infty} 2^{jm}A\big\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\big\|_{L^1(|x|>3R)}\\ &\qquad\times\big\|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)g(x)\big\|_{L^\infty(|x|>3R)}\\ &=: (\star\star).\end{aligned}$$

For the L^∞ norm involving g, we have (4.9). If $|x| > 3R$ and $|y| \le r \le R$, then $|x-y| > 2R$. Hence, for $|x| > 3R$, Lemma 4.4 (1) yields

$$\begin{aligned}\left|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\right| &= \left|\int \left(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi)\right)^\vee(x-y)\,f(y)\,dy\right| \\ &\lesssim \int_{|y|\le r} \|\theta_1\|_{C^N}\,|x-y|^{-N}\,|f(y)|\,dy \lesssim \|\theta_1\|_{C^N}\,|x|^{-N}.\end{aligned}$$

This implies

$$\left\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\right\|_{L^1(|x|>3R)} \lesssim \|\theta_1\|_{C^N}.$$

Thus we obtain

$$(\star\star) \lesssim \sum_{j=1}^{\infty} 2^{jm}A\|\theta_1\|_{C^N}\,2^{j(n-1)/2}\,\|\theta_2\|_{C^N}\,\|g\|_{L^\infty} \approx A\|\theta_1\|_{C^N}\|\theta_2\|_{C^N}\|g\|_{L^\infty},$$

where we used $m < -(n-1)/2$ again.

Thirdly, we assume $r \le R$ and estimate the L^1 norm of $T_{\tau_4}(f,g)(x)$ on $|x| \le 3R$. We set $B = \{x \in \mathbb{R}^n \mid |x| \le 5R\}$ and decompose g as $g = g\mathbf{1}_B + g\mathbf{1}_{B^c}$.

For the $L^1(|x| \le 3R)$ norm of $T_{\tau_4}(f, g\mathbf{1}_{B^c})(x)$, we have

$$\begin{aligned}\|T_{\tau_4}(f,g\mathbf{1}_{B^c})(x)\|_{L^1(|x|\le 3R)} &\le \sum_{j=1}^{\infty} 2^{jm}A\left\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\right\|_{L^1(|x|\le 3R)} \\ &\quad\times \left\|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)(g\mathbf{1}_{B^c})(x)\right\|_{L^\infty(|x|\le 3R)} \\ &=: (\star\star\star).\end{aligned}$$

Using Lemma 4.4 (2), we have

$$\begin{aligned}\left\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x)\right\|_{L^1(|x|\le 3R)} &\le \left\|(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee\right\|_{L^1}\|f\|_{L^1} \\ &\lesssim 2^{j(n-1)/2}\,\|\theta_1\|_{C^N}.\end{aligned}$$

If $|x| \le 3R$ and $|y| > 5R$, then $|x-y| > 2R$. Hence, for $|x| \le 3R$, we use Lemma 4.4 (1) to have

$$\begin{aligned}\left|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)(g\mathbf{1}_{B^c})(x)\right| &= \left|\int_{|y|>5R} (e^{i\phi_2(\eta)}\zeta(\eta)\theta_2(2^{-j}\eta))^\vee(x-y)\,g(y)\,dy\right| \\ &\lesssim \int_{|y|>5R} \|\theta_2\|_{C^N}\,|x-y|^{-N}\,\|g\|_{L^\infty}\,dy \approx \|\theta_2\|_{C^N}\|g\|_{L^\infty}.\end{aligned}$$

Thus

$$(\star\star\star) \lesssim \sum_{j=1}^{\infty} 2^{jm}A\,2^{j(n-1)/2}\,\|\theta_1\|_{C^N}\|\theta_2\|_{C^N}\|g\|_{L^\infty} \approx A\|\theta_1\|_{C^N}\|\theta_2\|_{C^N}\|g\|_{L^\infty},$$

where we used $m < -(n-1)/2$ again.

Finally, we estimate the L^1 norm of $T_{\tau_4}(f, g\mathbf{1}_B)(x)$ on $|x| \le 3R$. For this, we use the Cauchy–Schwarz inequality to have

$$\begin{aligned}
&\|T_{\tau_4}(f, g\mathbf{1}_B)(x)\|_{L^1(|x|\le 3R)} \\
&\le \sum_{j=1}^{\infty} 2^{jm} A \big\| e^{i\phi_1(D)} \zeta(D)\theta_1(2^{-j}D) f \big\|_{L^2} \big\| e^{i\phi_2(D)} \zeta(D)\theta_2(2^{-j}D)(g\mathbf{1}_B) \big\|_{L^2} \\
&=: (\star\star\star\star).
\end{aligned}$$

For the L^2 norm involving $g\mathbf{1}_B$, we have

$$\|e^{i\phi_2(D)}\zeta(D)\theta_2(2^{-j}D)(g\mathbf{1}_B)\|_{L^2} \lesssim \|\theta_2\|_{C^0}\|g\mathbf{1}_B\|_{L^2} \lesssim \|\theta_2\|_{C^0}\|g\|_{L^\infty}. \tag{4.11}$$

We estimate the L^2 norm of $e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f$ in two ways. Firstly, we have

$$\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f\|_{L^2} \lesssim \|\theta_1\|_{C^0}\|f\|_{L^2} \lesssim r^{-n/2}\|\theta_1\|_{C^0}. \tag{4.12}$$

On the other hand, using the moment condition of f, we can write

$$\begin{aligned}
&e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f(x) \\
&\quad = \int \big\{(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee(x-y) - (e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee(x)\big\} f(y)\,dy \\
&\quad = \iint_{\substack{0<t<1\\|y|\le r}} \nabla(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee(x-ty)\cdot(-y)\; f(y)\,dt\,dy.
\end{aligned}$$

Hence

$$\begin{aligned}
\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f\|_{L^2} &\lesssim \big\|\nabla(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee\big\|_{L^2}\int_{|y|\le r}|y|\,|f(y)|\,dy \\
&\lesssim r\,\big\|\nabla(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee\big\|_{L^2}.
\end{aligned}$$

By Plancherel's theorem,

$$\big\|\nabla(e^{i\phi_1(\xi)}\zeta(\xi)\theta_1(2^{-j}\xi))^\vee\big\|_{L^2} \approx \|\xi\,\zeta(\xi)\theta_1(2^{-j}\xi)\|_{L^2} \lesssim 2^{j(n/2+1)}\|\theta_1\|_{C^0}.$$

Hence

$$\big\|e^{i\phi_1(D)}\zeta(D)\theta_1(2^{-j}D)f\big\|_{L^2} \lesssim 2^{j(n/2+1)}\,r\,\|\theta_1\|_{C^0}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we obtain

$$\begin{aligned}
(\star\star\star\star) &\lesssim \sum_{j=1}^{\infty} 2^{jm}A\,\min\{r^{-n/2},\,2^{j(n/2+1)}\,r\}\,\|\theta_1\|_{C^0}\|\theta_2\|_{C^0}\|g\|_{L^\infty} \\
&= A\|\theta_1\|_{C^0}\|\theta_2\|_{C^0}\|g\|_{L^\infty}\sum_{j=1}^{\infty}\min\{(2^jr)^{-n/2},2^jr\} \approx A\|\theta_1\|_{C^0}\|\theta_2\|_{C^0}\|g\|_{L^\infty},
\end{aligned}$$

where we used $m = -n/2$. This completes the proof of Theorem 4.1. ∎

5. Necessary conditions on *m*

In this section, we shall prove Theorem 1.5.

In fact, we shall prove a stronger theorem by considering a multiplier of a special form. Take a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\theta \subset \{3^{-1} \le |\xi| \le 3\}$ and $\theta(\xi) = 1$ for $2^{-1} \le |\xi| \le 2$. We consider the multiplier

$$\sigma_j(\xi, \eta) = 2^{jm}\, \theta(2^{-j}\xi)\, \theta(2^{-j}\eta), \quad j \in \mathbb{N}.$$

This multiplier satisfies the inequalities

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma_j(\xi, \eta)| \le C_{\alpha,\beta}\, (1 + |\xi| + |\eta|)^{m-|\alpha|-|\beta|},$$

with $C_{\alpha,\beta}$ independent of $j \in \mathbb{N}$. Thus if the assertion (1.3) holds then, by the closed graph theorem, it follows that there exists a constant $A = A(n, m, p, q, \theta)$ such that

$$\big\| 2^{jm}\, e^{i(|\xi|+|\eta|)}\, \theta(2^{-j}\xi)\, \theta(2^{-j}\eta) \big\|_{\mathcal{M}(H^p \times H^q \to X_r)} \le A \quad \text{for all } j \in \mathbb{N}. \tag{5.1}$$

We shall prove that the conditions given in Theorem 1.5 are already necessary for (5.1). In fact, we prove the following theorem, which asserts that the claims of Theorem 1.5 hold if we replace the condition (1.3) by the condition (5.1).

Theorem 5.1. *Let $n \ge 2$.*

(1) *Let $0 < p, q \le 2$ or $2 \le p, q \le \infty$. Then $m \in \mathbb{R}$ satisfies (5.1) only if $m \le -(n-1) \times (|1/p - 1/2| + |1/q - 1/2|)$.*

(2) *Let $1 \le p \le 2 \le q \le \infty$ or $1 \le q \le 2 \le p \le \infty$ and assume $1/p + 1/q = 1$. Then $m \in \mathbb{R}$ satisfies (5.1) only if $m \le -n|1/p - 1/2|$.*

To prove this theorem, we use the following lemma.

Lemma 5.2. *Let ψ be a C^∞ function on $\mathbb{R}$ such that*

$$\operatorname{supp}\psi \subset \{t \in \mathbb{R} \mid 2^{-1} \le t \le 2\}, \quad \psi(t) \ge 0, \quad \psi(t) \not\equiv 0,$$

and set

$$h_j(x) = (e^{-i|\xi|}\psi(2^{-j}|\xi|))^\vee(x),$$

which is the inverse Fourier transform of the radial function $e^{-i|\xi|}\psi(2^{-j}|\xi|)$ on $\mathbb{R}^n$. Then the following hold.

(1) *For each $L > 0$, there exists a constant c_L, depending only on n, ψ, and L, such that*

$$|h_j(x)| \le c_L\, 2^{j(n+1)/2} \left(1 + 2^j |1 - |x||\right)^{-L}$$

for all $j \in \mathbb{N}$ and all $x \in \mathbb{R}^n$.

(2) *There exist $\delta, c_0 \in (0, \infty)$ and $j_0 \in \mathbb{N}$, depending only on n and ψ, such that*

$$\left| e^{i((n-2)\pi/4 + \pi/4)}\, 2^{-j(n+1)/2}\, h_j(x) - c_0 \right| \le \frac{c_0}{10}$$

if $1 - \delta 2^{-j} < |x| < 1 + \delta 2^{-j}$ and $j > j_0$.

(3) *For each* $0 < p \le \infty$,

$$\|h_j\|_{H^p} \approx \|h_j\|_{L^p} \approx 2^{j((n+1)/2-1/p)}, \quad j \in \mathbb{N},$$

where the implicit constants in $\approx$ *depend only on* n, ψ, *and* p.

Proof. The assertion (3) follows from (1) and (2). In fact, the inequality $\|h_j\|_{H^p} \approx \|h_j\|_{L^p}$ holds because the support of the inverse Fourier transform of h_j is included in the annulus $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$. The estimate of $\|h_j\|_{L^p} \lesssim 2^{j((n+1)/2-1/p)}$ follows from (1) and the converse estimate $\|h_j\|_{L^p} \gtrsim 2^{j((n+1)/2-1/p)}$ follows from (2). Thus we only need to prove (1) and (2).

Since $h_j(x)$ is the inverse Fourier transform of a radial function, it is written in terms of Bessel function as

$$h_j(x) = (2\pi)^{-n/2} \int_0^\infty J_{(n-2)/2}(|x|t)\,(|x|t)^{-(n-2)/2}\,\psi(2^{-j}t)\,e^{-it}\,t^{n-1}\,dt,$$

where $J_{(n-2)/2}$ is the Bessel function (see, e.g., Theorem 3.3 on p. 155 of [21]; this formula holds for $n = 1$ as well, since $(2\pi)^{-1/2} J_{-1/2}(s)\,s^{1/2} = \pi^{-1}\cos s$).

Proof of (1). Firstly, we estimate of $h_j(x)$ for $2^j|x| \le 1$. For this, we use the power series expansion

$$(2\pi)^{-n/2} J_{(n-2)/2}(s)\,s^{-(n-2)/2} = \sum_{m=0}^{\infty} a_m\,s^m,$$

whose radius of convergence is ∞. Integrating term by term, we have

$$\begin{aligned} h_j(x) &= \int_0^\infty \sum_{m=0}^{\infty} a_m (|x|t)^m\,\psi(2^{-j}t)\,e^{-it}\,t^{n-1}\,dt = \sum_{m=0}^{\infty} a_m\,|x|^m (\psi(2^{-j}t)\,t^{m+n-1})^\wedge(1) \\ &= \sum_{m=0}^{\infty} a_m\,|x|^m\,(2^j)^{m+n}\,(\psi(t)\,t^{m+n-1})^\wedge(2^j). \end{aligned}$$

The function arising in the last expression satisfies

$$\operatorname{supp}(\psi(t)\,t^{m+n-1}) \subset \{t \in \mathbb{R} \mid 2^{-1} \le t \le 2\},$$
$$\Big|\Big(\frac{d}{dt}\Big)^{\ell} (\psi(t)\,t^{m+n-1})\Big| \le c_\ell\,(1+m)^\ell\,2^m.$$

Hence, for any $L' \in \mathbb{N}$, we have

$$|(\psi(t)\,t^{m+n-1})^\wedge(2^j)| \le c_{L'}\,(1+m)^{L'}\,2^m\,(2^j)^{-L'}.$$

Thus, for $2^j|x| \le 1$, we have

$$\begin{aligned} |h_j(x)| &\le \sum_{m=0}^{\infty} |a_m|\,|x|^m\,(2^j)^{m+n} c_{L'}\,(1+m)^{L'}\,2^m\,(2^j)^{-L'} \\ &\le c_{L'}(2^j)^{n-L'} \sum_{m=0}^{\infty} |a_m|\,(1+m)^{L'}\,2^m = \widetilde{c}_{L'}\,(2^j)^{n-L'}. \end{aligned}$$

Since L' can be taken arbitrarily large, the above implies the desired estimate of $h_j(x)$ for $2^j|x| \le 1$.

Next, we estimate $h_j(x)$ for $2^j|x| > 1$. For this, we use the asymptotic expansion of the Bessel function, which reads as

$$\begin{aligned}(2\pi)^{-n/2}\, &J_{(n-2)/2}(s)\, s^{-(n-2)/2} \\ &= b^+ e^{is} s^{1/2-n/2} + b^- e^{-is} s^{1/2-n/2} + e^{is} R^+(s) + e^{-is} R^-(s),\end{aligned}$$

where

$$b^{\pm} = (2\pi)^{-(n+1)/2}\, e^{\mp i((n-2)\pi/4+\pi/4)}$$

and the remainder terms $R^{\pm}(s)$ satisfy

$$\left(\frac{d}{ds}\right)^{\ell} R^{\pm}(s) = O(s^{1/2-n/2-1-\ell}) \quad \text{as } s \to \infty, \text{ for } \ell = 0, 1, 2, \dots \tag{5.2}$$

Corresponding to the asymptotic expansion formula given above, we write

$$\begin{aligned} h_j(x) &= b^+ \int_0^\infty e^{i|x|t} (|x|t)^{1/2-n/2}\, \psi(2^{-j}t)\, e^{-it}\, t^{n-1}\, dt \\ &\quad + b^- \int_0^\infty e^{-i|x|t} (|x|t)^{1/2-n/2}\, \psi(2^{-j}t)\, e^{-it}\, t^{n-1}\, dt \\ &\quad + \int_0^\infty e^{i|x|t} R^+(|x|t)\, \psi(2^{-j}t)\, e^{-it}\, t^{n-1}\, dt \\ &\quad + \int_0^\infty e^{-i|x|t} R^-(|x|t)\, \psi(2^{-j}t)\, e^{-it}\, t^{n-1}\, dt \\ &= I_j^+(x) + I_j^-(x) + K_j^+(x) + K_j^-(x). \end{aligned}$$

We shall estimate each of $I_j^+(x)$, $I_j^-(x)$, $K_j^+(x)$, and $K_j^-(x)$ for $2^j|x| > 1$.

(a) *Estimate of* $I_j^+(x)$ *for* $2^j|x| > 1$. The term $I_j^+(x)$ is written as

$$\begin{aligned} I_j^+(x) &= b^+ \left\{(|x|t)^{1/2-n/2}\, \psi(2^{-j}t)\, t^{n-1}\right\}^{\wedge}(1-|x|) \\ &= b^+ |x|^{1/2-n/2} (2^j)^{1/2+n/2} (\psi(t)\, t^{-1/2+n/2})^{\wedge}(2^j(1-|x|)). \end{aligned} \tag{5.3}$$

Since $(\psi(t)\, t^{-1/2+n/2})^{\wedge}$ is a rapidly decreasing function, we have

$$|I_j^+(x)| \lesssim |x|^{1/2-n/2} (2^j)^{1/2+n/2} \left(1 + 2^j |1-|x||\right)^{-L'}$$

for any $L' > 0$. Hence,

$$\begin{aligned} 2^{-j} < |x| \le 2^{-1} &\Rightarrow |I_j^+(x)| \lesssim (2^{-j})^{1/2-n/2} (2^j)^{1/2+n/2} (2^j)^{-L'} = (2^j)^{n-L'}, \\ |x| > 2^{-1} &\Rightarrow |I_j^+(x)| \lesssim (2^j)^{1/2+n/2} \left(1 + 2^j |1-|x||\right)^{-L'}. \end{aligned}$$

For any given $L > 0$, the above estimates with a sufficiently large L' imply that

$$|I_j^+(x)| \lesssim (2^j)^{1/2+n/2} \left(1 + 2^j\, |1-|x||\right)^{-L}, \quad 2^j|x| > 1. \tag{5.4}$$

(b) *Estimate of* $I_j^-(x)$ *for* $2^j|x| > 1$. The function $I_j^-(x)$ is written as

$$I_j^-(x) = b^- \left\{(|x|t)^{1/2-n/2}\, \psi(2^{-j}t)\, t^{n-1}\right\}^\wedge(1+|x|).$$

Hence, by the same reason as in the case of $I_j^+(x)$,

$$|I_j^-(x)| \lesssim |x|^{1/2-n/2}\,(2^j)^{1/2+n/2}\,(2^j|1+|x||)^{-L'}$$

for any $L' > 0$. Restricting to the region $2^j|x| > 1$, we have

$$|I_j^-(x)| \lesssim (2^j)^{n-L'}\,(1+|x|)^{-L'}, \quad 2^j|x| > 1. \tag{5.5}$$

(c) *Estimate of* $K_j^+(x)$ *for* $2^j|x| > 1$. The integral $K_j^+(x)$ is written as

$$K_j^+(x) = \left\{R^+(|x|t)\,\psi(2^{-j}t)\,t^{n-1}\right\}^\wedge(1-|x|).$$

The function $R^+(|x|t)\,\psi(2^{-j}t)\,t^{n-1}$ is supported on $\{2^{j-1} \le t \le 2^{j+1}\}$. If $2^j|x| > 1$, then (5.2) implies

$$\left|\partial_t^\ell\left\{R^+(|x|t)\,\psi(2^{-j}t)\,t^{n-1}\right\}\right| \lesssim |x|^{1/2-n/2-1}\,(2^j)^{1/2+n/2-2-\ell}, \quad \ell = 0, 1, 2, \dots,$$

which, via Fourier transform, yields

$$\left|\left\{R^+(|x|t)\,\psi(2^{-j}t)\,t^{n-1}\right\}^\wedge(1-|x|)\right| \lesssim |x|^{1/2-n/2-1}(2^j)^{1/2+n/2-1}\left(1+2^j|1-|x||\right)^{-L'}$$

for any $L' > 0$. Hence

$$\begin{aligned} 2^{-j} < |x| \le 2^{-1} \;&\Rightarrow\; |K_j^+(x)| \lesssim (2^{-j})^{1/2-n/2-1}\,(2^j)^{1/2+n/2-1}\,(2^j)^{-L'} = (2^j)^{n-L'}, \\ |x| > 2^{-1} \;&\Rightarrow\; |K_j^+(x)| \lesssim (2^j)^{1/2+n/2-1}\left(1+2^j|1-|x||\right)^{-L'}. \end{aligned}$$

For any $L > 0$, the above estimates with L' sufficiently large imply

$$\left|K_j^+(x)\right| \lesssim (2^j)^{1/2+n/2-1}\left(1+2^j|1-|x||\right)^{-L}, \quad 2^j|x| > 1. \tag{5.6}$$

(d) *Estimate of* $K_j^-(x)$ *for* $2^j|x| > 1$. The integral $K_j^-(x)$ is written as

$$K_j^-(x) = \left\{R^-(|x|t)\,\psi(2^{-j}t)\,t^{n-1}\right\}^\wedge(1+|x|).$$

If $2^j|x| > 1$, then by the same reasoning as above, we obtain

$$\left|\left\{R^-(|x|t)\,\psi(2^{-j}t)\,t^{n-1}\right\}^\wedge(1+|x|)\right| \lesssim |x|^{1/2-n/2-1}(2^j)^{1/2+n/2-1}\,(2^j(1+|x|))^{-L'}$$

for any $L' > 0$. Hence

$$|K_j^-(x)| \lesssim (2^j)^{n-L'}\,(1+|x|)^{-L'}, \quad 2^j|x| > 1. \tag{5.7}$$

Now from (5.4), (5.5), (5.6), and (5.7), we obtain the estimate of $h_j(x)$ for $2^j|x| > 1$ as claimed in the lemma. Thus the claim (1) is proved.

Proof of (2). The equality (5.3) and the equality

$$b^{+} = (2\pi)^{-(n+1)/2} e^{-i((n-2)\pi/4+\pi/4)}$$

give

$$\text{(5.8)} \quad \begin{aligned} &e^{i((n-2)\pi/4+\pi/4)} (2^j)^{-n/2-1/2} I_j^{+}(x) \\ &\qquad = (2\pi)^{-(n+1)/2} |x|^{1/2-n/2} (\psi(t)\, t^{-1/2+n/2})^{\wedge}(2^j(1-|x|)). \end{aligned}$$

We set

$$c_0 = (2\pi)^{-(n+1)/2} (\psi(t)\, t^{-1/2+n/2})^{\wedge}(0).$$

This is a positive number, since ψ is nonnegative and not identically equal to 0. Then, from (5.8) and from continuity of the functions, it follows that there exists a number $\delta > 0$ such that

$$1 - 2^{-j}\delta < |x| < 1 + 2^{-j}\delta \quad \Longrightarrow \quad \left| e^{i((n-2)\pi/4+\pi/4)} (2^j)^{-n/2-1/2} I_j^{+}(x) - c_0 \right| \le \frac{c_0}{20}.$$

On the other hand, the estimates of (5.5), (5.6), and (5.7) imply that there exists a constant $c_1 = c_1(n, \psi)$ such that

$$1 - 2^{-j} < |x| < 1 + 2^{-j} \quad \Longrightarrow \quad |I_j^{-}(x)| + |K_j^{+}(x)| + |K_j^{-}(x)| \le c_1\, 2^{j(n/2+1/2-1)}.$$

Hence, the estimate claimed in (2) of the lemma holds if we take j_0 large enough so that $c_1 2^{-j_0} \le c_0/20$. This completes the proof of Lemma 5.2. ■

Proof of Theorem 5.1. We define the operator S_j by

$$S_j h = \left(e^{i|\xi|} \theta(2^{-j}\xi)\, \widehat{h}(\xi) \right)^{\vee}.$$

We divide the proof into three cases.

Case 1: $0 < p, q \le 2$.

Assume (5.1) holds, or equivalently, that

$$\text{(5.9)} \qquad 2^{jm} \|S_j f \cdot S_j g\|_{X_r} \le A \|f\|_{H^p} \|g\|_{H^q} \quad \text{for all } j \in \mathbb{N}.$$

Take ψ as in Lemma 5.2 and set

$$f_j(x) = (\psi(2^{-j}|\xi|))^{\vee}(x) \quad \text{for } j \in \mathbb{N}.$$

We shall test (5.9) to $f = g = f_j$.

Since the support of the Fourier transform of the function f_j is included in the annulus $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$, and since $f_j(x) = 2^{jn}(\psi(|\cdot|))^{\vee}(2^j x)$, it follows that

$$\|f_j\|_{H^p} \approx \|f_j\|_{L^p} \approx 2^{j(n-n/p)},$$

and a similar estimate holds for $\|f_j\|_{H^q}$. On the other hand, by the choice of the functions θ and ψ, we have

$$S_j f_j = (e^{i|\xi|} \psi(2^{-j}|\xi|))^{\vee}.$$

Hence, by Lemma 5.2, there exist $\delta \in (0, \infty)$ and $j_0 \in \mathbb{N}$ such that

$$|S_j f_j(x)| \gtrsim 2^{j(n+1)/2} \mathbf{1}_{\{2^j |1-|x||<\delta\}} \quad \text{for } j > j_0.$$

Thus

$$\|(S_j f_j)^2\|_{L^r} \gtrsim (2^{j(n+1)/2})^2 \left(\int \mathbf{1}_{\{2^j |1-|x||<\delta\}} \, dx \right)^{1/r} \approx 2^{j(n+1-1/r)} \quad \text{for } j > j_0.$$

Hence, if (5.9) holds, then testing it to $f = g = f_j$ we have

$$2^{jm} \cdot 2^{j(n+1-1/r)} \lesssim 2^{j(n-n/p)} \cdot 2^{j(n-n/q)} \quad \text{for } j > j_0,$$

which is possible only when $m \leq -(n-1)(1/p - 1/2 + 1/q - 1/2)$.

Case 2: $2 \leq p, q \leq \infty$.

Assume (5.9) holds. Using the function ψ of Lemma 5.2, we set

$$\widetilde{f}_j = (e^{-i|\xi|} \psi(2^{-j}|\xi|))^{\vee} \quad \text{for } j \in \mathbb{N}.$$

Then Lemma 5.2 gives the estimate

$$\|\widetilde{f}_j\|_{H^p} \approx \|\widetilde{f}_j\|_{L^p} \approx 2^{j((n+1)/2-1/p)},$$

and a similar estimate holds for $\|\widetilde{f}_j\|_{H^q}$. On the other hand,

$$S_j \widetilde{f}_j(x) = (\psi(2^{-j}|\xi|))^{\vee}(x) = 2^{jn} (\psi(|\cdot|))^{\vee}(2^j x),$$

and hence

$$\left\|(S_j \widetilde{f}_j)^2\right\|_{X_r} = \left\|2^{2jn} (\psi(|\cdot|))^{\vee}(2^j x)^2\right\|_{X_r} \approx 2^{j(2n-n/r)}.$$

Hence, if (5.9) holds, then by testing it to $f = g = \widetilde{f}_j$ we have

$$2^{jm} \cdot 2^{j(2n-n/r)} \lesssim 2^{j((n+1)/2-1/p)} \cdot 2^{j((n+1)/2-1/q)},$$

which is possible only when $m \leq -(n-1)(1/2 - 1/p + 1/2 - 1/q)$.

Case 3: $1 \leq p \leq 2 \leq q \leq \infty$ *or* $1 \leq q \leq 2 \leq p \leq \infty$ and $1/p + 1/q = 1$.

By the symmetry of the situation, it suffices to consider the case $1 \leq p \leq 2 \leq q \leq \infty$. Thus we assume that $1 \leq p \leq 2 \leq q \leq \infty$ and $1/p + 1/q = 1/r = 1$. We assume (5.1) holds, or equivalently, that

$$2^{jm} \|S_j f \cdot S_j g\|_{L^1} \leq A \|f\|_{H^p} \|g\|_{L^q} \quad \text{for all } j \in \mathbb{N}, \tag{5.10}$$

and prove that this is possible only when $m \leq -n/p + n/2$.

We use the same function f_j that was used in the proof of Case 1:

$$f_j(x) = (\psi(2^{-j}|\xi|))^{\vee}(x) \quad \text{for } j \in \mathbb{N},$$

where ψ is the function given in Lemma 5.2.

As we have seen in Case 1,

$$\|f_j\|_{H^p} \approx 2^{j(n-n/p)}. \tag{5.11}$$

On the other hand,

$$S_j f_j(x) = \left(e^{i|\xi|}\,\psi(2^{-j}|\xi|)\right)^{\vee}(x) = \overline{(e^{-i|\xi|}\,\psi(2^{-j}|\xi|))^{\vee}(-x)}$$

and, hence, Lemma 5.2 (2) gives that, for $1-\delta 2^{-j} < |x| < 1+\delta 2^{-j}$ and $j > j_0$,

$$\left|e^{-i((n-2)\pi/4+\pi/4)}\,2^{-j(n+1)/2}\,S_j f_j(x) - c_0\right| \le \frac{c_0}{10}. \tag{5.12}$$

For a sequence of complex numbers $\alpha = (\alpha_\ell)_{\ell\in\mathbb{Z}^n}$, we define $g_{j,\alpha}$ by

$$g_{j,\alpha}(x) = \sum_{\ell\in\mathbb{Z}^n} \alpha_\ell\, f_j(x-\delta' 2^{-j}\ell),$$

where δ' is a sufficiently small positive number; for the succeeding argument, the choice $\delta'=\delta/(2\sqrt{n})$ will suffice.

We shall prove

$$\|g_{j,\alpha}\|_{L^q} \lesssim 2^{j(n-n/q)}\,\|\alpha\|_{\ell^q}. \tag{5.13}$$

In fact, since $f_j(x) = 2^{jn}(\psi(|\cdot|))^{\vee}(2^j x)$ and since $(\psi(|\cdot|))^{\vee}$ is a Schwartz function, we have $|f_j(x)| \lesssim 2^{jn}\,(1+2^j|x|)^{-L}$ for any $L>0$. Thus, if $2\le q<\infty$, then Hölder's inequality yields

$$\begin{aligned}
|g_{j,\alpha}(x)| &\lesssim \sum_{\ell\in\mathbb{Z}^n} |\alpha_\ell|\,2^{jn}\left(1+2^j|x-\delta' 2^{-j}\ell|\right)^{-L}\\
&\le \Big(\sum_{\ell\in\mathbb{Z}^n} |\alpha_\ell|^q\,2^{jnq}\left(1+2^j|x-\delta' 2^{-j}\ell|\right)^{-L}\Big)^{1/q}\Big(\sum_{\ell\in\mathbb{Z}^n}\left(1+2^j|x-\delta' 2^{-j}\ell|\right)^{-L}\Big)^{1-1/q}\\
&\approx \Big(\sum_{\ell\in\mathbb{Z}^n} |\alpha_\ell|^q\,2^{jnq}\left(1+2^j|x-\delta' 2^{-j}\ell|\right)^{-L}\Big)^{1/q},
\end{aligned}$$

and hence,

$$\|g_{j,\alpha}\|_{L^q} \lesssim \Big(\int \sum_{\ell\in\mathbb{Z}^n} |\alpha_\ell|^q\,2^{jnq}\,(1+2^j|x-\delta' 2^{-j}\ell|)^{-L}\,dx\Big)^{1/q} \approx \|\alpha\|_{\ell^q}\,2^{j(n-n/q)}.$$

An obvious modification gives (5.13) for $q=\infty$ as well.

Since the operator S_j is linear and commutes with translation, we have

$$S_j g_{j,\alpha} = \sum_{\ell\in\mathbb{Z}^n} \alpha_\ell (S_j f_j)(x-\delta' 2^{-j}\ell).$$

Now we test (5.10) to $f=f_j$ and $g=g_{j,\alpha}$. Then by (5.11) and (5.13) we have

$$2^{jm}\Big\|S_j f_j(x)\sum_{\ell\in\mathbb{Z}^n}\alpha_\ell\,S_j f_j(x-\delta' 2^{-j}\ell)\Big\|_{L^1_x} \lesssim 2^{jn}\,\|\alpha\|_{\ell^q}$$

(recall that $1/p + 1/q = 1$). We take the dual form of this inequality, which reads as

$$2^{jm}\Big\| \int S_j f_j(x)\, S_j f_j(x - \delta' 2^{-j}\ell)\, \varphi(x)\, dx \Big\|_{\ell^{q'}_{\ell}} \lesssim 2^{jn}\, \|\varphi\|_{L^\infty}. \tag{5.14}$$

We define the cube Q_ν in $\mathbb{R}^n$ by

$$Q_\nu = \delta' 2^{-j}(\nu + (0,1]^n), \quad \nu \in \mathbb{Z}^n.$$

Then each Q_ν is a cube with side length $\delta' 2^{-j}$ and all of them constitute a partition of $\mathbb{R}^n$. Let $(\varepsilon_\nu)_{\nu\in\mathbb{Z}^n}$ be any sequence of ± 1, and apply (5.14) to $\varphi(x) = \sum_{\nu\in\mathbb{Z}^n} \varepsilon_\nu \mathbf{1}_{Q_\nu}(x)$. Then we obtain

$$2^{jm}\Big(\sum_{\ell\in\mathbb{Z}^n} \Big| \sum_{\nu\in\mathbb{Z}^n} \varepsilon_\nu \int_{Q_\nu} S_j f_j(x)\, S_j f_j(x - \delta' 2^{-j}\ell)\, dx \Big|^{q'} \Big)^{1/q'} \lesssim 2^{jn}.$$

Notice that this inequality holds uniformly for all choices of $\varepsilon_\nu = \pm 1$. We take the q'-th power of the above inequality, take average over all choices of $\varepsilon_\nu = \pm 1$, and use Kintchine's inequality; this yields

$$\sum_{\ell\in\mathbb{Z}^n} \Big(\sum_{\nu\in\mathbb{Z}^n} \Big| \int_{Q_\nu} S_j f_j(x)\, S_j f_j(x - \delta' 2^{-j}\ell)\, dx \Big|^2 \Big)^{q'/2} \lesssim 2^{j(n-m)q'}. \tag{5.15}$$

We shall estimate the left-hand side of (5.15) from below. For $v \in \mathbb{R}^n$, we define

$$\Sigma(v) = \{x \in \mathbb{R}^n \mid |x| = |x - v| = 1\}.$$

If $0 < |v| < 2$, then $\Sigma(v)$ is a $n-2$ dimensional sphere of radius $\sqrt{1 - 4^{-1}|v|^2}$. Thus, in particular, if $0 < |v| < 1$ and $\eta > 0$ is sufficiently small, then the n-dimensional Lebesgue measure of the η-neighborhood of $\Sigma(v)$ satisfies

$$|\text{the } \eta\text{-neighborhood of } \Sigma(v)| \approx \eta^2. \tag{5.16}$$

Suppose $\ell \in \mathbb{Z}^n$ satisfies

$$0 < |\delta' 2^{-j}\ell| < 1 \tag{5.17}$$

and consider $\nu \in \mathbb{Z}^n$ that satisfies

$$\operatorname{dist}\big(Q_\nu, \Sigma(\delta' 2^{-j}\ell)\big) < \frac{\delta 2^{-j}}{2}. \tag{5.18}$$

Then, for each $x \in Q_\nu$, there exists an $x' \in \Sigma(\delta' 2^{-j}\ell)$ such that

$$|x - x'| < \operatorname{diam} Q_\nu + \frac{\delta 2^{-j}}{2} = \delta 2^{-j},$$

and, since this x' satisfies $|x'| = |x' - \delta' 2^{-j}\ell| = 1$, we have

$$1 - \delta 2^{-j} < |x| < 1 + \delta 2^{-j} \quad \text{and} \quad 1 - \delta 2^{-j} < |x - \delta' 2^{-j}\ell| < 1 + \delta 2^{-j}.$$

Hence, by (5.12), we see that

$$\left|e^{-i((n-2)\pi/4+\pi/4)}\, 2^{-j(n+1)/2}\, S_j f_j(x) - c_0\right| \le \frac{c_0}{10},$$
$$\left|e^{-i((n-2)\pi/4+\pi/4)}\, 2^{-j(n+1)/2}\, S_j f_j\big(x-\delta' 2^{-j}\big) - c_0\right| \le \frac{c_0}{10},$$

for all $x \in Q_\nu$ and all $j > j_0$, which implies that

$$\left|\int_{Q_\nu} S_j f_j(x) S_j f_j(x-\delta' 2^{-j}\ell)\,dx\right| \approx 2^{j(n+1)/2}\, 2^{j(n+1)2}\, 2^{-jn} = 2^j \quad \text{for } j > j_0. \tag{5.19}$$

All the cubes Q_ν that satisfy (5.18) certainly cover the $(\frac{1}{2}\delta 2^{-j})$-neighborhood of the set $\Sigma(\delta' 2^{-j}\ell)$. Conversely, since $\operatorname{diam} Q_\nu = 2^{-1}\delta 2^{-j}$, all Q_ν that satisfy (5.18) are included in the $(\delta 2^{-j})$-neighborhood of $\Sigma(\delta' 2^{-j}\ell)$. Hence, by (5.16), we see that

$$\operatorname{card}\big\{\nu \in \mathbb{Z}^n \mid \nu \text{ satisfies } (5.18)\big\} \approx \frac{2^{-2j}}{2^{-jn}} = 2^{j(n-2)} \tag{5.20}$$

for each ℓ satisfying (5.17). Also we have obviously

$$\operatorname{card}\{\ell \in \mathbb{Z}^n \mid \ell \ \text{ satisfies } \ (5.17)\} \approx 2^{jn}. \tag{5.21}$$

From (5.19), (5.20), and (5.21), we see that the left-hand side of (5.15) is

$$\begin{aligned} &\ge \sum_{\ell:(5.17)} \Big(\sum_{\nu:(5.18)} \Big| \int_{Q_\nu} S_j f_j(x) S_j f_j(x-\delta' 2^{-j}\ell)\,dx \Big|^2 \Big)^{q'/2} \\ &\approx \sum_{\ell:(5.17)} \big((2^j)^2 \cdot 2^{j(n-2)}\big)^{q'/2} \approx 2^{j(nq'/2+n)} \quad \text{for all } j > j_0. \end{aligned}$$

Thus (5.15) implies $2^{j(nq'/2+n)} \lesssim 2^{j(n-m)q'}$ for $j > j_0$, which is possible only when $m \le -n/2 + n/q = n/2 - n/p$. This completes the proof of Theorem 5.1. ■

6. Proofs of Propositions 2.3 and 2.4

6.1. Proof of Proposition 2.3

In order to prove Proposition 2.3, we use the following lemmas. The first two lemmas are given in [13].

Lemma 6.1 (Lemma 2.5 in [13]). *Let* $0 < p, q \le \infty$ *and* $1/p + 1/q = 1/r > 0$. *Assume that* ψ *and* ϕ *are functions on* $\mathbb{R}^n$ *such that* $\operatorname{supp}\psi \subset \{a^{-1} \le |\xi| \le a\}$ *and*

$$|\partial_x^\alpha (\psi)^\vee(x)| \le A(1+|x|)^{-L} \quad \textit{for } |\alpha| = 0, 1,$$
$$|\partial_x^\beta (\phi)^\vee(x)| \le B(1+|x|)^{-L} \quad \textit{for } |\beta| \le L',$$

where $a, A, B \in (0,\infty)$, *and* L, L' *are sufficiently large integers determined by* p, q, *and* n. *Then*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \big|\psi(2^{-j}D)f \cdot \varphi(2^{-j}D)g\big|^2 \Big)^{1/2} \Big\|_{L^r} \le cAB\|f\|_{H^p}\|g\|_{H^q},$$

where $c = c(n, p, q, a)$ is a positive constant. Moreover, if $p = \infty$, then $\|f\|_{H^p}$ can be replaced by $\|f\|_{\mathrm{BMO}}$.

Lemma 6.2 (Lemma 2.7 in [13]). *Let $0 < p, q \le \infty$ and $1/p + 1/q = 1/r > 0$. Assume that ψ_1 and ψ_2 are functions on $\mathbb{R}^n$ such that $\operatorname{supp}\psi_1$, $\operatorname{supp}\psi_2 \subset \{a^{-1} \le |\xi| \le a\}$ and*

$$|\partial_x^\alpha (\psi_1)^\vee(x)| \le A(1+|x|)^{-L} \quad \textit{for } |\alpha| \le L',$$
$$|\partial_x^\beta (\psi_2)^\vee(x)| \le B(1+|x|)^{-L} \quad \textit{for } |\beta| \le L',$$

where $a, A, B \in (0,\infty)$ and L, L' are sufficiently large integers determined by p, q, and n. Then

$$\Big\| \sum_{j\in\mathbb{Z}} \big|\psi_1(2^{-j}D)f \cdot \psi_2(2^{-j}D)g\big| \Big\|_{L^r} \le cAB\|f\|_{H^p}\|g\|_{H^q},$$

where $c = c(n, p, q, a)$ is a positive constant. Moreover, if $p = \infty$ (respectively, $q = \infty$), then $\|f\|_{H^p}$ (respectively, $\|g\|_{H^q}$) can be replaced by $\|f\|_{\mathrm{BMO}}$ (respectively, $\|g\|_{\mathrm{BMO}}$).

Lemma 6.3. *Let $m_2 < 0$, and suppose the multiplier τ is given by*

$$\tau(\xi,\eta) = \sum_{j-k\ge 3} c_{j,k}\, \psi_1(2^{-j}\xi)\, \psi_2(2^{-k}\eta),$$

where $(c_{j,k})$ a sequence of complex numbers satisfying $|c_{j,k}| \le 2^{(j-k)m_2}$, and ψ_1 and ψ_2 are functions in $C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi_1$, $\operatorname{supp}\psi_2 \subset \{2^{-1} \le |\xi| \le 2\}$. Then τ belongs to the following multiplier classes:

$$\mathcal{M}(H^p \times H^q \to L^r), \quad 0 < p, q < \infty, \ \ 1/p + 1/q = 1/r,$$
$$\mathcal{M}(H^p \times \mathrm{BMO} \to L^p), \quad 0 < p < \infty,$$
$$\mathcal{M}(\mathrm{BMO} \times H^q \to L^q), \quad 0 < q < \infty,$$
$$\mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO}).$$

Moreover, in each case, the multiplier norm of τ is bounded by $c\|\psi_1\|_{C^N}\|\psi_2\|_{C^N}$, with $c = c(n, m_2, p, q)$ and $N = N(n, p, q)$.

Proof. We divide the proof into several cases.

Case 1. $H^p \times H^q \to L^r$, $0 < p, q < \infty$, $1/p + 1/q = 1/r$.

From the assumption $|c_{j,k}| \le 2^{(j-k)m_2}$ with $m_2 < 0$, we can use Schur's lemma (see, e.g., Appendix A in [4]) as follows:

$$\begin{aligned} |T_\tau(f,g)(x)| &= \Big| \sum_{j-k\ge 3} c_{j,k}\, \psi_1(2^{-j}D)f(x)\, \psi_2(2^{-k}D)g(x) \Big| \\ &\le \sum_{j-k\ge 3} 2^{(j-k)m_2}\, |\psi_1(2^{-j}D)f(x)|\, |\psi_2(2^{-k}D)g(x)| \\ &\lesssim \|\psi_1(2^{-j}D)f(x)\|_{\ell^2_j}\, \|\psi_2(2^{-k}D)g(x)\|_{\ell^2_k}. \end{aligned}$$

The above inequality, together with Hölder's inequality and the Littlewood–Paley inequalities, gives

$$\begin{aligned}\|T_\tau(f,g)\|_{L^r} &\lesssim \big\| \|\psi_1(2^{-j}D)f(x)\|_{\ell^2_j} \big\|_{L^p_x} \big\| \|\psi_2(2^{-k}D)g(x)\|_{\ell^2_k} \big\|_{L^q_x} \\ &\lesssim \|\psi_1\|_{C^N} \|f\|_{H^p} \|\psi_2\|_{C^N} \|g\|_{H^q},\end{aligned}$$

which is the desired estimate.

Case 2. $H^p \times \mathrm{BMO} \to L^p$, $0 < p < \infty$.

Observe that, if $j - k \geq 3$, then the support of the Fourier transform of the function $\psi_1(2^{-j}D)f \cdot \psi_2(2^{-k}D)g$ is included in the annulus $\{2^{j-2} \leq |\zeta| \leq 2^{j+2}\}$. Hence, the Littlewood–Paley theory for H^p gives

$$\begin{aligned}\Big\| \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D)f \cdot \psi_2(2^{-k}D)g \Big\|_{L^p} &\lesssim \Big\| \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D)f \cdot \psi_2(2^{-k}D)g \Big\|_{H^p} \\ &\lesssim \Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k} \psi_1(2^{-j}D)f(x)\psi_2(2^{-k}D)g(x) \Big\|_{\ell^2_j} \Big\|_{L^p_x} =: (\star).\end{aligned}$$

Since

$$\|\psi_2(2^{-k}D)g\|_{L^\infty} \lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}}$$

(see, e.g., Section 4.3.3 in Chapter IV of [20]), and since

$$\sum_{k=-\infty}^{j-3} |c_{j,k}| \leq \sum_{k=-\infty}^{j-3} 2^{(j-k)m_2} \approx 1,$$

we obtain

$$(\star) \lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}} \big\| \|\psi_1(2^{-j}D)f(x)\|_{\ell^2_j} \big\|_{L^p_x} \lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}} \|\psi_1\|_{C^N} \|f\|_{H^p},$$

which is the desired estimate.

Case 3. $\mathrm{BMO} \times H^q \to L^q$, $1 < q < \infty$.

By the same reason as in Case 2, the Littlewood–Paley theory for L^q, $1 < q < \infty$, yields

$$\begin{aligned}&\Big\| \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D)f \cdot \psi_2(2^{-k}D)g \Big\|_{L^q} \\ &\qquad \lesssim \Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k}\, \psi_1(2^{-j}D)f(x)\, \psi_2(2^{-k}D)g(x) \Big\|_{\ell^2_j} \Big\|_{L^q_x} =: (\star\star).\end{aligned}$$

Take a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\theta(\eta) = 1$ for $|\eta| \leq 2$. Then, for $j - k \geq 3$, we have

$$\psi_2(2^{-k}D)g(x) = \theta(2^{-j}D)\, \psi_2(2^{-k}D)g(x) = \int 2^{jn} (\theta)^\vee (2^j(x-y))\, \psi_2(2^{-k}D)g(y)\, dy.$$

Combining this formula with the inequality

$$|\psi_2(2^{-k}D)g(y)| \lesssim \|\psi_2\|_{C^N} Mg(y),$$

where M is the Hardy–Littlewood maximal operator, and with the inequality

$$|(\theta)^\vee(z)| \lesssim (1+|z|^2)^{-L/2},$$

we have

$$|\psi_2(2^{-k}D)g(x)| \lesssim \|\psi_2\|_{C^N}\, S_j(Mg)(x),$$

where S_j is defined by

$$S_j h(x) = \int 2^{jn}\,(1+|2^j(x-y)|^2)^{-L/2}\,h(y)\,dy,$$

with $L>0$ sufficiently large. Hence

$$\begin{aligned}(\star\star) &\lesssim \Big\| \Big\| \sum_{k=-\infty}^{j-3} 2^{(j-k)m_2}\,|\psi_1(2^{-j}D)f(x)|\,\|\psi_2\|_{C^N} S_j(Mg)(x) \Big\|_{\ell^2_j} \Big\|_{L^q_x} \\ &\approx \|\psi_2\|_{C^N} \Big\| \big\| |\psi_1(2^{-j}D)f(x)|\, S_j(Mg)(x) \big\|_{\ell^2_j} \Big\|_{L^q_x} \\ &\lesssim \|\psi_2\|_{C^N}\|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\|Mg\|_{L^q} \approx \|\psi_2\|_{C^N}\|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\|g\|_{L^q},\end{aligned}$$

where the second $\lesssim$ follows from Lemma 6.1 and the last $\approx$ holds because $q>1$.

Case 4. $\mathrm{BMO}\times H^q \to L^q$, $0<q\le 1$.

By virtue of the atomic decomposition for H^q, it is sufficient to show the uniform estimate of $\|T_\tau(f,g)\|_{L^q}$ for all H^q-atoms g. By translation, it is sufficient to consider the H^q-atoms supported on balls centered at the origin. Thus we assume

$$\operatorname{supp} g \subset \{|x|\le r\}, \quad \|g\|_{L^\infty} \le r^{-n/q}, \quad \int g(x)x^\alpha\,dx = 0 \quad \text{for } |\alpha| \le [n/q-n],$$

and we shall prove

$$\|T_\tau(f,g)\|_{L^q} \lesssim \|\psi_1\|_{C^N}\|\psi_2\|_{C^N}\|f\|_{\mathrm{BMO}}.$$

By the same reason as in Case 2, the Littlewood–Paley theory for H^q reduces the proof to the estimate of

$$\Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k}\,\psi_1(2^{-j}D)f(x)\,\psi_2(2^{-k}D)g(x) \Big\|_{\ell^2_j} \Big\|_{L^q_x}.$$

We first estimate the L^q norm on $|x|\le 2r$. Using Hölder's inequality and using the result proved in Case 3 (with $q=2$), we have

$$\begin{aligned}&\Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k}\,\psi_1(2^{-j}D)f(x)\,\psi_2(2^{-k}D)g(x) \Big\|_{\ell^2_j} \Big\|_{L^q(|x|\le 2r)} \\ &\qquad\lesssim r^{n/q-n/2} \Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k}\,\psi_1(2^{-j}D)f(x)\,\psi_2(2^{-k}D)g(x) \Big\|_{\ell^2_j} \Big\|_{L^2(|x|\le 2r)} \\ &\qquad\lesssim r^{n/q-n/2}\,\|\psi_1\|_{C^N}\,\|\psi_2\|_{C^N}\,\|f\|_{\mathrm{BMO}}\|g\|_{L^2} \lesssim \|\psi_1\|_{C^N}\,\|\psi_2\|_{C^N}\,\|f\|_{\mathrm{BMO}}.\end{aligned}$$

Next, we estimate the L^q norm on $|x| > 2r$. Using the inequality

$$\|\psi_1(2^{-j}D)f(x)\|_{L^\infty} \lesssim \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}},$$

we have

$$\begin{aligned}
&\Big\| \Big\| \sum_{k=-\infty}^{j-3} c_{j,k}\,\psi_1(2^{-j}D)f(x)\,\psi_2(2^{-k}D)g(x)\Big\|_{\ell^2_j}\Big\|_{L^q(|x|>2r)} \\
&\lesssim \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\Big\|\Big\|\sum_{k=-\infty}^{j-3} 2^{(j-k)m_2}\,|\psi_2(2^{-k}D)g(x)|\Big\|_{\ell^2_j}\Big\|_{L^q(|x|>2r)} \\
&\le \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\Big\|\sum_{k=-\infty}^{\infty} \|2^{(j-k)m_2}\|_{\ell^2(j\ge k+3)}\,|\psi_2(2^{-k}D)g(x)|\Big\|_{L^q(|x|>2r)} \\
&\approx \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\Big\|\sum_{k=-\infty}^{\infty} |\psi_2(2^{-k}D)g(x)|\Big\|_{L^q(|x|>2r)} \\
&\le \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\,\Big\|\|\psi_2(2^{-k}D)g(x)\|_{L^q(|x|>2r)}\Big\|_{\ell^q_k}. \qquad (6.1)
\end{aligned}$$

To estimate the L^q-norm of the functions $\psi_2(2^{-k}D)g(x)$ on $|x| > 2r$, we write

$$\psi_2(2^{-k}D)g(x) = \int_{|y|\le r} 2^{kn}(\psi_2)^\vee(2^k(x-y))\,g(y)\,dy.$$

Then, using the size estimate of g and the moment condition on g, we have

$$|\psi_2(2^{-k}D)g(x)| \lesssim \|\psi_2\|_{C^N}\,2^{kn}\,(1+2^k|x|)^{-L}\,r^{-n/q+n}\min\left\{1,\ (2^k r)^{[n/q-n]+1}\right\}$$

for $|x| > 2r$ (see inequalities (2.7) and (2.8) in [13]). Hence

$$\begin{aligned}
&\|\psi_2(2^{-k}D)g(x)\|_{L^q(|x|>2r)} \\
&\quad\lesssim \|\psi_2\|_{C^N}\,r^{-n/q+n}\min\left\{1,\ (2^k r)^{[n/q-n]+1}\right\}\left\|2^{kn}(1+2^k|x|)^{-L}\right\|_{L^q(|x|>2r)} \\
&\quad\approx \|\psi_2\|_{C^N}\min\left\{(2^k r)^{-L+n},\ (2^k r)^{n-n/q+[n/q-n]+1}\right\}. \qquad (6.2)
\end{aligned}$$

From (6.1) and (6.2), we obtain

$$\begin{aligned}
&\Big\|\Big\|\sum_{k=-\infty}^{j-3} c_{j,k}\,\psi_1(2^{-j}D)f(x)\,\psi_2(2^{-k}D)g(x)\Big\|_{\ell^2_j}\Big\|_{L^q(|x|>2r)} \\
&\quad\lesssim \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\|\psi_2\|_{C^N}\left\|\min\left\{(2^k r)^{-L+n},\ (2^k r)^{n-n/q+[n/q-n]+1}\right\}\right\|_{\ell^q_k} \\
&\quad\lesssim \|\psi_1\|_{C^N}\|f\|_{\mathrm{BMO}}\|\psi_2\|_{C^N}.
\end{aligned}$$

Case 5. BMO × BMO → BMO.

By the duality between BMO and H^1, it is sufficient to show the following inequality:

$$\Big| \int \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D) f(x)\ \psi_2(2^{-k}D) g(x)\, h(x)\, dx \Big| \lesssim \|\psi_1\|_{C^N} \|f\|_{\mathrm{BMO}} \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}} \|h\|_{H^1}. \tag{6.3}$$

Notice that if $j-k \geq 3$, then the support of the Fourier transform of the function $\psi_1(2^{-j}D)f \cdot \psi_2(2^{-k}D)g$ is included in the annulus $\{2^{j-2} \leq |\zeta| \leq 2^{j+2}\}$. Thus, if we take a function $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\widetilde{\psi} \subset \{2^{-3} \leq |\zeta| \leq 2^3\}$ and $\widetilde{\psi}(\zeta) = 1$ on $2^{-2} \leq |\zeta| \leq 2^2$, then the integral in (6.3) can be written as

$$\begin{aligned} &\int \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D) f(x)\ \psi_2(2^{-k}D) g(x)\, h(x)\, dx \\ &\qquad = \int \sum_{j-k\geq 3} c_{j,k}\, \psi_1(2^{-j}D) f(x)\ \psi_2(2^{-k}D) g(x)\, \widetilde{\psi}(2^{-j}D) h(x)\, dx. \end{aligned}$$

Hence, using the estimate

$$\|\psi_2(2^{-k}D)g\|_{L^\infty} \lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}}$$

and the assumption $|c_{j,k}| \leq 2^{(j-k)m_2}$, $m_2 < 0$, we have that the left-hand side of (6.3) is

$$\begin{aligned} &\lesssim \int \sum_{j-k\geq 3} 2^{(j-k)m_2} |\psi_1(2^{-j}D) f(x)|\, |\psi_2(2^{-k}D) g(x)|\, |\widetilde{\psi}(2^{-j}D) h(x)|\, dx \\ &\lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}} \int \sum_{j=-\infty}^{\infty} |\psi_1(2^{-j}D) f(x)|\, |\widetilde{\psi}(2^{-j}D) h(x)|\, dx \\ &\lesssim \|\psi_2\|_{C^N} \|g\|_{\mathrm{BMO}} \|\psi_1\|_{C^N} \|f\|_{\mathrm{BMO}} \|h\|_{H^1}, \end{aligned}$$

where the last $\lesssim$ follows from Lemma 6.2. This completes the proof of Lemma 6.3. ■

Lemma 6.4. *Suppose the multiplier* τ *is defined by*

$$\tau(\xi,\eta) = \sum_{j=-\infty}^{\infty} c_j\, \psi_1(2^{-j}\xi)\, \phi(2^{-j+3}\eta)$$

with a sequence of complex numbers (c_j) *satisfying* $|c_j| \leq 1$ *and with* $\psi_1, \phi \in C_0^\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp}\psi_1 \subset \{2^{-1} \leq |\xi| \leq 2\}$ *and* $\operatorname{supp}\phi \subset \{|\eta| \leq 2\}$. *Then* τ *belongs to the following multiplier classes*:

$$\begin{aligned} &\mathcal{M}(H^p \times H^q \to L^r), \quad 0 < p,q < \infty,\ \ 1/p + 1/q = 1/r, \\ &\mathcal{M}(H^p \times L^\infty \to L^p), \quad 0 < p < \infty, \\ &\mathcal{M}(\mathrm{BMO} \times H^q \to L^q), \quad 0 < q < \infty, \\ &\mathcal{M}(\mathrm{BMO} \times L^\infty \to \mathrm{BMO}). \end{aligned}$$

Moreover, in each case, the multiplier norm of τ *is bounded by* $c\|\psi_1\|_{C^N}\|\phi\|_{C^N}$ *with* $c = c(n,p,q)$ *and* $N = N(n,p,q)$.

Proof. From the assumptions on the supports of ψ_1 and ϕ, it follows that the support of the Fourier transform of $\psi_1(2^{-j}D)f \cdot \phi(2^{-j+3}D)g$ is included in the annulus $\{2^{j-2} \le |\zeta| \le 2^{j+2}\}$. Hence, for $0 < r < \infty$, the Littlewood–Paley theory implies

$$\Big\| \sum_{j=-\infty}^{\infty} c_j\, \psi_1(2^{-j}D)f \cdot \phi(2^{-j+3}D)g \Big\|_{L^r} \lesssim \Big\| \sum_{j=-\infty}^{\infty} c_j\, \psi_1(2^{-j}D)f \cdot \phi(2^{-j+3}D)g \Big\|_{H^r}$$
$$\lesssim \big\| \| c_j\, \psi_1(2^{-j}D)f \cdot \phi(2^{-j+3}D)g \|_{\ell^2_j} \big\|_{L^r_x} =: (\star).$$

By Lemma 6.1, we have that

$$(\star) \lesssim \|\psi_1\|_{C^N} \|\phi\|_{C^N} \begin{cases} \|f\|_{H^p} \|g\|_{H^q}, & \text{if } 0 < p, q < \infty \text{ and } 1/p + 1/q = 1/r, \\ \|f\|_{H^p} \|g\|_{L^\infty}, & \text{if } 0 < p < \infty \text{ and } p = r, \\ \|f\|_{\mathrm{BMO}} \|g\|_{H^q}, & \text{if } 0 < q < \infty \text{ and } q = r. \end{cases}$$

These prove the claims for the first three multiplier classes.

We shall prove $\tau \in \mathcal{M}(\mathrm{BMO} \times L^\infty \to \mathrm{BMO})$. By the same argument as given in Case 5 of the proof of Lemma 6.3, it is sufficient to show the inequality

$$(6.4) \qquad \Big| \int \sum_{j=-\infty}^{\infty} c_j\, \psi_1(2^{-j}D)f(x)\, \phi(2^{-j+3}D)g(x)\, \widetilde{\psi}(2^{-j}D)h(x)\, dx \Big|$$
$$\lesssim \|\phi\|_{C^N} \|g\|_{L^\infty} \|\psi_1\|_{C^N} \|f\|_{\mathrm{BMO}} \|h\|_{H^1},$$

where $\widetilde{\psi}$ is the same function as given there. In the present case, using the assumption $|c_j| \le 1$ and the inequality

$$\|\phi(2^{-j+3}D)g\|_{L^\infty} \lesssim \|\phi\|_{C^N} \|g\|_{L^\infty},$$

we see that the left-hand side of (6.4) is

$$\lesssim \|\phi\|_{C^N} \|g\|_{L^\infty} \int \sum_{j=-\infty}^{\infty} |\psi_1(2^{-j}D)f(x)|\, |\widetilde{\psi}(2^{-j}D)h(x)|\, dx.$$

Now (6.4) follows from Lemma 6.2. This completes the proof of Lemma 6.4. ■

Proof of Proposition 2.3. We use several well-known methods developed in the theory of bilinear Fourier multiplier operators. We first decompose $\sigma(\xi, \eta)$ by using the usual dyadic partition of unity. Let ψ, ζ, and φ be the functions as given in Notation 1.6.

We decompose σ into three parts:

$$\sigma(\xi,\eta) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sigma(\xi,\eta)\, \psi(2^{-j}\xi)\, \psi(2^{-k}\eta) = \sum_{j-k\ge 3} + \sum_{|j-k|\le 2} + \sum_{j-k\le -3}$$
$$= \sigma_{\mathrm{I}}(\xi,\eta) + \sigma_{\mathrm{II}}(\xi,\eta) + \sigma_{\mathrm{III}}(\xi,\eta),$$

where $\sum_{j-k\ge 3}$, $\sum_{|j-k|\le 2}$, and $\sum_{j-k\le -3}$ denote the sums of $\sigma(\xi,\eta)\psi(2^{-j}\xi)\,\psi(2^{-k}\eta)$ over $j, k \in \mathbb{Z}$ that satisfy the designated restrictions. We shall consider each of σ_{I}, σ_{II}, and σ_{III}.

Case 1. For the multiplier σ_{II}, we shall prove the following:

$$\begin{aligned}
&\sigma_{\mathrm{II}} \in \mathcal{M}(H^p \times H^q \to L^r), \quad 0 < p, q < \infty, \ \ 1/p + 1/q = 1/r,\\
&\sigma_{\mathrm{II}} \in \mathcal{M}(H^p \times \mathrm{BMO} \to L^p), \quad 0 < p < \infty,\\
&\sigma_{\mathrm{II}} \in \mathcal{M}(\mathrm{BMO} \times H^q \to L^q), \quad 0 < q < \infty,\\
&\sigma_{\mathrm{II}} \in \mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO}).
\end{aligned}$$

To prove this, observe that $|\xi| \approx |\eta| \approx 2^j$ on the support of $\psi(2^{-j}\xi)\psi(2^{-k}\eta)$ with $|j-k| \le 2$. From this we see that $\sigma_{\mathrm{II}} \in \dot{S}^0_{1,0}(\mathbb{R}^{2n})$. Hence Proposition 2.2 implies that σ_{II} is a bilinear Fourier multiplier for the following spaces:

$$\begin{aligned}
&H^p \times H^q \to L^r, \quad 0 < p, q < \infty, \ \ 1/p + 1/q = 1/r,\\
&H^p \times L^\infty \to L^p, \quad 0 < p < \infty,\\
&L^\infty \times H^q \to L^q, \quad 0 < q < \infty,\\
&L^\infty \times L^\infty \to \mathrm{BMO}.
\end{aligned}$$

We shall prove that the space L^∞ in the above can be replaced by BMO.

We use the Fefferman–Stein decomposition of BMO, which asserts that every $g \in \mathrm{BMO} \cap L^2$ can be written as

$$g = g_0 + \sum_{\ell=1}^{n} R_\ell g_\ell, \quad \text{with} \quad \sum_{\ell=0}^{n} \|g_\ell\|_{L^\infty} \approx \|g\|_{\mathrm{BMO}},$$

where $R_\ell h = (-i|\xi|^{-1}\xi_\ell \widehat{h}(\xi))^\vee$ is the Riesz transform. (If $g \in \mathrm{BMO} \cap L^2$, then we can take $g_\ell \in L^\infty \cap L^2$, and the equality $g = g_0 + \sum_{\ell=1}^{n} R_\ell g_\ell$ holds without modulo constants; see [11].) Thus

$$T_{\sigma_{\mathrm{II}}}(f, g) = T_{\sigma_{\mathrm{II}}}(f, g_0) + \sum_{\ell=1}^{n} T_{\sigma_{\mathrm{II}}}(f, R_\ell g_\ell) = T_{\sigma_{\mathrm{II}}}(f, g_0) + \sum_{\ell=1}^{n} T_{\sigma_{\ell,\mathrm{II}}}(f, g_\ell),$$

where

$$\sigma_\ell(\xi, \eta) = \sigma(\xi, \eta)(-i|\eta|^{-1}\eta_\ell) = a_0(\xi, \eta)\, a_1(\xi)\, a_2(\eta)(-i|\eta|^{-1}\eta_\ell)$$

and $\sigma_{\ell,\mathrm{II}}$ is defined in the same way as $\sigma \mapsto \sigma_{\mathrm{II}}$. Since the multiplier $a_2(\eta)(-i|\eta|^{-1}\eta_\ell)$ belongs to $\dot{S}^{-m_2}_{1,0}(\mathbb{R}^n)$, we can apply the result $\sigma_{\mathrm{II}} \in \mathcal{M}(H^p \times L^\infty \to L^p)$ to $\sigma_{\ell,\mathrm{II}}$ to see that

$$\|T_{\sigma_{\mathrm{II}}}(f, g)\|_{L^p} \lesssim \|f\|_{H^p} \sum_{\ell=0}^{n} \|g_\ell\|_{L^\infty} \approx \|f\|_{H^p} \|g\|_{\mathrm{BMO}}.$$

Thus $\sigma_{\mathrm{II}} \in \mathcal{M}(H^p \times \mathrm{BMO} \to L^p)$. The claims $\sigma_{\mathrm{II}} \in \mathcal{M}(\mathrm{BMO} \times H^q \to L^q)$ and $\sigma_{\mathrm{II}} \in \mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO})$ are proved in the same way.

Case 2. For the multiplier σ_{I}, we shall prove the following:

$$\sigma_{\mathrm{I}} \in \mathcal{M}(H^p \times H^q \to L^r) \quad \text{if } 0 < p, q < \infty, \ \ 1/p + 1/q = 1/r, \tag{6.5}$$

$$\sigma_{\mathrm{I}} \in \mathcal{M}(H^p \times \mathrm{BMO} \to L^p) \quad \text{if } m_2 < 0 \ \text{ and } \ 0 < p < \infty, \tag{6.6}$$

$$\sigma_{\mathrm{I}} \in \mathcal{M}(\mathrm{BMO} \times H^q \to L^q) \quad \text{if } 0 < q < \infty, \tag{6.7}$$

$$\sigma_{\mathrm{I}} \in \mathcal{M}(\mathrm{BMO} \times \mathrm{BMO} \to \mathrm{BMO}) \quad \text{if } m_2 < 0. \tag{6.8}$$

Proof of (6.5) *in the case* $m_2 = 0$. We write $\sigma_{\rm I}(\xi,\eta) = b(\xi,\eta)\,a_2(\eta)$, with

$$
\begin{aligned}
b(\xi,\eta) &= \sum_{j-k\ge 3} a_0(\xi,\eta)\,a_1(\xi)\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta) \\
&= \sum_{j=-\infty}^{\infty} a_0(\xi,\eta)\,a_1(\xi)\,\psi(2^{-j}\xi)\,\varphi(2^{-j+3}\eta).
\end{aligned} \tag{6.9}
$$

Since $m_2 = 0$ and $m = m_1$ in the present case, we see that $b \in \dot{S}^0_{1,0}(\mathbb{R}^{2n})$. Thus, Proposition 2.2 implies that $b \in \mathcal{M}(H^p \times H^q \to L^r)$. Also since $a_2 \in \dot{S}^0_{1,0}(\mathbb{R}^n)$ in the present case, the classical multiplier theorem for linear operators implies $a_2 \in \mathcal{M}(H^q \to H^q)$. Hence $\sigma_{\rm I} \in \mathcal{M}(H^p \times H^q \to L^r)$.

Proof of (6.5) *in the case* $m_2 < 0$. Notice that $\sigma_{\rm I}$ is supported in $|\xi| \ge 2|\eta|$ and satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma_{\rm I}(\xi,\eta)| \le C_{\alpha,\beta}\Big(\frac{|\xi|}{|\eta|}\Big)^{m_2} |\xi|^{-|\alpha|}\,|\eta|^{-|\beta|}.$$

Since $m_2 < 0$, a result of Grafakos and Kalton (see Theorem 7.4 in [6]) implies that $\sigma_{\rm I} \in \mathcal{M}(H^p \times H^q \to L^r)$.

Another proof of (6.5) *in the case* $m_2 < 0$. Here we shall give a direct proof of (6.5) for the case $m_2 < 0$, which uses only a classical method.

Take a function $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\widetilde{\psi} \subset \{3^{-1} \le |\xi| \le 3\}$ and $\widetilde{\psi}(\xi) = 1$ for $2^{-1} \le |\xi| \le 2$. Then

$$\sigma_{\rm I}(\xi,\eta) = \sum_{j-k\ge 3} \sigma(\xi,\eta)\,\widetilde{\psi}(2^{-j}\xi)\,\widetilde{\psi}(2^{-k}\eta)\,\psi(2^{-j}\xi)\psi(2^{-k}\eta).$$

Consider the function

$$\sigma(2^j\xi, 2^k\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\psi}(\eta) = a_0(2^j\xi, 2^k\eta)\,a_1(2^j\xi)\,a_2(2^k\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\psi}(\eta)$$

with $j - k \ge 3$. This function is supported in $\{3^{-1} \le |\xi| \le 3\} \times \{3^{-1} \le |\eta| \le 3\}$ and satisfies the estimate

$$|\partial_\xi^\alpha \partial_\eta^\beta \{\sigma(2^j\xi, 2^k\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\psi}(\eta)\}| \le C_{\alpha,\beta}\, 2^{(j-k)m_2}$$

with $C_{\alpha,\beta}$ independent of $j,k \in \mathbb{Z}$. Hence using the Fourier series expansion, we can write

$$\sigma(2^j\xi, 2^k\eta)\,\widetilde{\psi}(\xi)\,\widetilde{\psi}(\eta) = \sum_{a,b\in\mathbb{Z}^n} c_{j,k}^{(a,b)}\, e^{ia\cdot\xi}\, e^{ib\cdot\eta} \quad \text{for } |\xi| < \pi, |\eta| < \pi,$$

with the coefficient satisfying

$$|c_{j,k}^{(a,b)}| \lesssim 2^{(j-k)m_2}\,(1+|a|)^{-L}\,(1+|b|)^{-L} \tag{6.10}$$

for any $L > 0$. Changing variables $\xi \to 2^{-j}\xi$ and $\eta \to 2^{-k}\eta$ and multiplying by the function $\psi(2^{-j}\xi)\,\psi(2^{-k}\eta)$, we obtain

$$\sigma(\xi,\eta)\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta) = \sum_{a,b\in\mathbb{Z}^n} c_{j,k}^{(a,b)}\, e^{ia\cdot 2^{-j}\xi}\, e^{ib\cdot 2^{-k}\eta}\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta).$$

Thus σ_{I} is written as

$$\sigma_{\mathrm{I}}(\xi,\eta)=\sum_{a,b\in\mathbb{Z}^n}\sum_{j-k\geq 3}c_{j,k}^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\psi^{(b)}(2^{-k}\eta), \tag{6.11}$$

with

$$\psi^{(a)}(\xi)=e^{ia\cdot\xi}\,\psi(\xi)\quad\text{and}\quad\psi^{(b)}(\eta)=e^{ib\cdot\eta}\,\psi(\eta). \tag{6.12}$$

Now, applying Lemma 6.3 to $\psi_1=\psi^{(a)}$ and $\psi_2=\psi^{(b)}$, we obtain

$$\Big\|\sum_{j-k\geq 3}c_{j,k}^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\psi^{(b)}(2^{-j}\eta)\Big\|_{\mathcal{M}(H^p\times H^q\to L^r)}$$
$$\lesssim(1+|a|)^{-L}(1+|b|)^{-L}\,\|\psi^{(a)}\|_{C^N}\,\|\psi^{(b)}\|_{C^N}\lesssim(1+|a|)^{-L+N}(1+|b|)^{-L+N}.$$

Taking L sufficiently large and taking sum over $a,b\in\mathbb{Z}^n$, we obtain (6.5).

Proof of (6.6). Using (6.11), (6.12), and (6.10), we can derive (6.6) from Lemma 6.3.

Proof of (6.7). If $m_2<0$, then by using (6.11), (6.12), and (6.10), we can derive (6.7) from Lemma 6.3.

Assume $m_2=0$. Then we write σ_{I} as $\sigma_{\mathrm{I}}(\xi,\eta)=b(\xi,\eta)a_2(\eta)$, with b given by (6.9). Since $a_2\in\dot{S}^0_{1,0}(\mathbb{R}^n)$ in the present case ($m_2=0$), the linear multiplier theorem implies $a_2\in\mathcal{M}(H^q\to H^q)$. Hence (6.7) will follow if we prove $b\in\mathcal{M}(\mathrm{BMO}\times H^q\to L^q)$. By the same argument given in the proof of (6.5), we can write b as

$$b(\xi,\eta)=\sum_{a,b\in\mathbb{Z}^n}\sum_{j=-\infty}^{\infty}c_j^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\varphi^{(b)}(2^{-j+3}\eta),$$

where

$$|c_j^{(a,b)}|\lesssim(1+|a|)^{-L}(1+|b|)^{-L}, \tag{6.13}$$

$$\psi^{(a)}(\xi)=e^{ia\cdot\xi}\,\psi(\xi),\quad\varphi^{(b)}(\eta)=e^{ib\cdot\eta}\,\varphi(\eta). \tag{6.14}$$

Now we apply Lemma 6.4 to $\psi_1=\psi^{(a)}$ and $\phi=\varphi^{(b)}$ to obtain

$$\Big\|\sum_{j-k\geq 3}c_{j,k}^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\varphi^{(b)}(2^{-j+3}\eta)\Big\|_{\mathcal{M}(\mathrm{BMO}\times H^q\to L^q)}$$
$$\lesssim(1+|a|)^{-L}(1+|b|)^{-L}\,\|\psi^{(a)}\|_{C^N}\,\|\psi^{(b)}\|_{C^N}\lesssim(1+|a|)^{-L+N}(1+|b|)^{-L+N}.$$

Taking L sufficiently large and taking sum over $a,b\in\mathbb{Z}^n$, we obtain that $b\in\mathcal{M}(\mathrm{BMO}\times H^q\to L^q)$.

Proof of (6.8). This is also derived from Lemma 6.3 by using (6.11), (6.12), and (6.10).

Case 3. For the multiplier σ_{III}, the following hold:

$$\sigma_{\mathrm{III}}\in\mathcal{M}(H^p\times H^q\to L^r)\quad\text{if }\ 0<p,q<\infty,\ \ 1/p+1/q=1/r,$$
$$\sigma_{\mathrm{III}}\in\mathcal{M}(H^p\times\mathrm{BMO}\to L^p)\quad\text{if }\ 0<p<\infty,$$
$$\sigma_{\mathrm{III}}\in\mathcal{M}(\mathrm{BMO}\times H^q\to L^q)\quad\text{if }\ m_1<0\ \text{ and }\ 0<q<\infty,$$
$$\sigma_{\mathrm{III}}\in\mathcal{M}(\mathrm{BMO}\times\mathrm{BMO}\to\mathrm{BMO})\quad\text{if }\ m_1<0.$$

In fact, these follow from the results for σ_{I} by the obvious symmetry.

This completes the proof of Proposition 2.3. ∎

6.2. Proof of Proposition 2.4

Let ψ and φ be the functions as given in Notation 1.6. In the same way as in the proof of Proposition 2.3, we decompose τ into three parts:

$$\tau(\xi,\eta)=\tau_{\mathrm{I}}(\xi,\eta)+\tau_{\mathrm{II}}(\xi,\eta)+\tau_{\mathrm{III}}(\xi,\eta),$$

where

$$\tau_{\mathrm{I}}(\xi,\eta)=\sum_{j-k\ge 3} a_0(\xi,\eta)\,a_1(\xi)\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta),$$
$$\tau_{\mathrm{II}}(\xi,\eta)=\sum_{|j-k|\le 2} a_0(\xi,\eta)\,a_1(\xi)\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta),$$
$$\tau_{\mathrm{III}}(\xi,\eta)=\sum_{j-k\le -3} a_0(\xi,\eta)\,a_1(\xi)\,\psi(2^{-j}\xi)\,\psi(2^{-k}\eta).$$

We shall prove that each of τ_{I}, τ_{II}, and τ_{III} belongs to the multiplier class as mentioned in the proposition.

(1) Let $0<p<\infty$. The multipliers τ_{II} and τ_{III} belong to $\mathcal{M}(H^p\times \mathrm{BMO}\to L^p)$. In fact, these are proved in Cases 1 and 3 of the proof of Proposition 2.3.

We shall prove that $\tau_{\mathrm{I}}\in\mathcal{M}(H^p\times L^\infty\to L^p)$. By the same argument as in the proof of Proposition 2.3 (see the proof of (6.7)), we can write τ_{I} as

$$\tau_{\mathrm{I}}(\xi,\eta)=\sum_{a,b\in\mathbb{Z}^n}\sum_{j=-\infty}^{\infty} c_j^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\varphi^{(b)}(2^{-j+3}\eta), \tag{6.15}$$

with $c_j^{(a,b)}$ satisfying (6.13) and $\psi^{(a)}$ and $\varphi^{(b)}$ defined by (6.14). Then Lemma 6.4 gives

$$\Big\|\sum_{j=-\infty}^{\infty} c_j^{(a,b)}\,\psi^{(a)}(2^{-j}\xi)\,\varphi^{(b)}(2^{-j+3}\eta)\Big\|_{\mathcal{M}(H^p\times L^\infty\to L^p)}$$
$$\lesssim (1+|a|)^{-L}(1+|b|)^{-L}\|\psi^{(a)}\|_{C^N}\|\varphi^{(b)}\|_{C^N}\lesssim (1+|a|)^{-L+N}(1+|b|)^{-L+N}.$$

Taking L sufficiently large and taking sum over $a,b\in\mathbb{Z}^n$, we obtain $\tau_{\mathrm{I}}\in\mathcal{M}(H^p\times L^\infty\to L^p)$. Thus the part (1) is proved.

(2) Here we assume $m_1<0$. By the results proved in Cases 1 and 3 in the proof of Proposition 2.3, the multipliers τ_{II} and τ_{III} belong to $\mathcal{M}(\mathrm{BMO}\times\mathrm{BMO}\to\mathrm{BMO})$. Recall that the multiplier τ_{I} is written as (6.15), with $c_j^{(a,b)}$ satisfying (6.13) and $\psi^{(a)}$ and $\varphi^{(b)}$ defined by (6.14). Hence we can prove that $\tau_{\mathrm{I}}\in\mathcal{M}(\mathrm{BMO}\times L^\infty\to\mathrm{BMO})$ by using Lemma 6.4. Thus the part (2) of Proposition 2.4 is proved. This completes the proof of Proposition 2.4.

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