

Positive crossratios, barycenters, trees and applications to maximal representations

Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti

Abstract. We study metric properties of maximal framed representations of fundamental groups of surfaces in symplectic groups over real closed fields, interpreted as actions on Bruhat–Tits buildings endowed with adapted Finsler norms. We prove that the translation length can be computed as intersection with a geodesic current, give sufficient conditions guaranteeing that such a current is a multicurve, and, if the current is a measured lamination, construct an isometric embedding of the associated tree in the building. These results are obtained as application of more general results of independent interest on positive crossratios and actions with compatible barycenters.

1. Introduction

Maximal framed representations in real closed fields

Let $\Sigma := \Gamma \backslash \mathcal{H}^2$ be the quotient of the Poincaré upper half-plane \mathcal{H}^2 by a torsion-free lattice $\Gamma < \mathrm{PSL}(2, \mathbf{R})$, and let G be a simple real algebraic group. The aim of higher Teichmüller theory is to single out and study special components or specific semi-algebraic subsets of the representation variety $\mathrm{Hom}(\Gamma, G)$ that consist of injective homomorphisms with discrete image; such components thus generalize Teichmüller space, for which $G = \mathrm{PSL}(2, \mathbf{R})$. Prominent examples are Hitchin components for real split groups G (for example, $G = \mathrm{PSL}(n, \mathbf{R})$ or $\mathrm{PSp}(2n, \mathbf{R})$) [18, 24, 28], maximal representations for Hermitian groups G (for example, $G = \mathrm{PU}(p, q)$ or $\mathrm{PSp}(2n, \mathbf{R})$) [12] and Θ -positive representations for $G = \mathrm{PO}(p, q)$ [22], see the end of the introduction for more context.

There have been many issues regarding representation varieties and the related character varieties that people have been interested in. This is the second of a series of papers devoted to the study of compactifications of the character varieties (see [10] for an overview of the results). Echoing Thurston’s compactification of Teichmüller space, the third named author defined the so-called *Weyl chamber length compactification* of a general character variety. While in many cases this can also be characterized in terms of geodesic currents [9], from the topological point of view it has in general weak properties, as, for example, it does not even distinguish among connected components. On the other hand,

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the (closed) real spectrum compactification that we defined in [10, 11] has many interesting properties. For example, it is a compact metrizable $\text{Out}(\Gamma)$ -space, it contains the character variety as an open dense subset and it enjoys a fixed point property *à la Brouwer*. Most importantly the real spectrum compactification reflects all the homological properties of the character variety and it surjects onto the Weyl chamber length compactification. Despite being defined in terms of prime ideals, its (closed) boundary points come naturally realized as representations into non-Archimedean real closed fields \mathbf{F} that are “small”, that is with finite transcendence degree over the field of real algebraic numbers, in particular, admit a canonical order compatible valuation. Interesting enough, going to the boundary along an ultrafilter ω gives a representation over the Robinson field $\mathbf{R}_{\omega, \sigma}$, where σ is a well-chosen sequence of scalars (see Section 8.1 and [37]). The Robinson field however is not small in the above sense, and the two statements can be reconciled by passing to equivalence classes appropriately defined.

While the above holds true for any simple real algebraic group, more information can be gathered by looking at higher Teichmüller components. In this context, Hitchin components and maximal components present different difficulties and features. For instance, that the notion of positivity for $\text{SL}(n, \mathbf{F})$ -representations leads to positive crossratios and hence currents is far from elementary (see [17]). A feature differentiating the Siegel space, the symmetric space associated to $\text{PSp}(2n, \mathbf{R})$, and the symmetric space associated to $\text{SL}(n, \mathbf{R})$ is that the former has a family of half-dimensional totally geodesic submanifolds whose intersection pattern has many features in common with that of geodesics in the hyperbolic plane. This leads to well-behaved barycenters playing a crucial role in our paper.

In this paper, we study the boundary points of the real spectrum compactification of the subset of maximal representations $\rho: \Gamma \rightarrow \text{PSp}(2n, \mathbf{R})$. To do so, we must exploit the geometry of the non-standard Siegel upper half-space over \mathbf{F} , which admits as canonical quotient a generalized building on which Γ acts. To this action we associate a geodesic current by means of a crossratio that in general is far from being continuous. If the field generated by the coefficients of the representation has discrete valuation, we show that the corresponding geodesic current is a multicurve. We also study the case in which the geodesic current is a measured lamination, and we show that in this case the dual tree of the measured lamination embeds isometrically and equivariantly in the generalized building.

Given a representation $\rho: \Gamma \rightarrow \text{PSp}(2n, \mathbf{F})$, where \mathbf{F} is a general real closed field, having a maximal Toledo invariant as defined in [10, Definition 18], admitting a maximal framing defined on a Γ -invariant non-empty subset of $\partial\mathcal{H}^2$, or admitting a maximal framing defined on the set of fixed points of hyperbolic elements are all equivalent conditions (see [10, Theorem 20] for a precise statement). This paper solely relies upon the third definition, which we now recall. If \mathbf{F}^{2n} is endowed with the standard symplectic form, let $\mathcal{L}(\mathbf{F}^{2n})$ be the space of Lagrangians in \mathbf{F}^{2n} . The Maslov cocycle (see Section 6.4 and [32]) classifies the orbits of $\text{PSp}(2n, \mathbf{F})$ on triples of pairwise transverse Lagrangians. Such a triple is *maximal* if the cocycle takes its maximal value n . Let $\partial\mathcal{H}^2$ be the bound-

ary of the hyperbolic plane, which we endow with the cyclic ordering on triples of points induced by the orientation of \mathcal{H}^2 .

Definition 1.1. Let $H_\Gamma \subset \partial\mathcal{H}^2$ be the set of fixed points of hyperbolic elements of Γ . A representation $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{F})$ is *maximal framed* if there is a ρ -equivariant map $\varphi: H_\Gamma \rightarrow \mathcal{L}(\mathbf{F}^{2n})$ sending positively oriented triples in H_Γ to maximal triples of Lagrangians.

We assume that the real closed field \mathbf{F} admits an order compatible \mathbf{R} -valued valuation v with value group $\Lambda = v(\mathbf{F}^\times) < \mathbf{R}$. Then the group $\mathrm{PSp}(2n, \mathbf{F})$ acts by isometries on a Λ -metric space¹ $\mathcal{B}_\mathbf{F}^n$ obtained as quotient of the Siegel upper half-space $\mathcal{S}_\mathbf{F}^n$ associated to $\mathrm{PSp}(2n, \mathbf{F})$ (see Section 6.2 and [10, §3.4]). If $\mathbf{F} = \mathbf{R}$, $\mathcal{B}_\mathbf{R}^n$ coincides with $\mathcal{S}_\mathbf{R}^n$, while if \mathbf{F} is non-Archimedean, then this metric space $\mathcal{B}_\mathbf{F}^n$ sits naturally inside the Bruhat–Tits building associated to $\mathrm{PSp}(2n, \mathbf{F})$ as a dense subset.² The latter relationship will play no role in this paper, but will be discussed in detail in [11] (see also [10, §2.1] and [27]). The translation length of an element $g \in \mathrm{PSp}(2n, \mathbf{F})$ acting on $\mathcal{B}_\mathbf{F}^n$ induces then the *length function*

$$L(g) := -2 \sum_{i=1}^n v(|\lambda_i|), \tag{1.1}$$

where $\lambda_1, \dots, \lambda_n, \lambda_n^{-1}, \dots, \lambda_1^{-1} \in \mathbf{F}(\sqrt{-1})$ are the eigenvalues of a representative $\bar{g} \in \mathrm{Sp}(2n, \mathbf{F})$ of $g \in \mathrm{PSp}(2n, \mathbf{F})$ counted with multiplicity and ordered in such a way that $|\lambda_1| \geq \dots \geq |\lambda_n| \geq 1$. Here we denote by $|\cdot|: \mathbf{F}(\sqrt{-1}) \rightarrow \mathbf{F}_+$ the absolute value, that is, the square root of the norm function on the quadratic extension $\mathbf{F}(\sqrt{-1})$ of \mathbf{F} .

In our first result, we construct a geodesic current on Σ encoding the length function $\gamma \mapsto L(\rho(\gamma))$ of a maximal framed representation. Recall that a *geodesic current* is a Γ -invariant positive Radon measure on the space $(\partial\mathcal{H}^2)^{(2)}$ of pairs of distinct points in $\partial\mathcal{H}^2$. The *Bonahon intersection* $i(\mu, \lambda)$ of two geodesic currents μ and λ extends the topological intersection number of homotopy classes of curves on Σ : in fact, a (non-oriented) closed geodesic $c \subset \Sigma$ gives rise to the current $\delta_c := \frac{1}{2} \sum_{(a,b) \in c} \delta_{(a,b)}$, where we sum over the set of oriented geodesics $(a, b) \in (\partial\mathcal{H}^2)^{(2)}$ lifting c . For such currents, $i(\delta_c, \delta_{c'})$ is the topological intersection number of $[c]$ and $[c']$.

Theorem 1.2. *Let \mathbf{F} be a real closed field with an order compatible valuation v , and let $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{F})$ be maximal framed. Then there is a geodesic current μ_ρ such that, for any closed geodesic $c \subset \Sigma$ and for every $\gamma \in \Gamma$ representing c ,*

$$i(\mu_\rho, \delta_c) = L(\rho(\gamma)).$$

The current μ_ρ is non-zero if and only if there exists a hyperbolic element $\gamma \in \Gamma$ with $v(\mathrm{tr}(\rho(\gamma))) < 0$.

¹In [10] denoted by $\mathcal{B}_{\mathrm{PSp}(2n, \mathbf{F})}$.

²Note, however, that the metric that we consider here is only bi-Lipschitz to the restriction of the CAT(0) metric on the Bruhat–Tits building. See Section 6.2 for the Finsler metric relevant to our purposes.

If $n = 1$, $\mathbf{F} = \mathbf{R}$ and $\rho: \Gamma \rightarrow \mathrm{PSL}(2, \mathbf{R})$ is the lattice embedding, then μ_ρ is the Liouville current [6], that is, the unique $\mathrm{PSL}(2, \mathbf{R})$ -invariant geodesic current. If Σ is compact, the current μ_ρ is unique [36]; if additionally $\mathbf{F} = \mathbf{R}$, Theorem 1.2 was proven by Martone–Zhang [34, Theorem 1.1]. Notice that in the case of $\mathrm{SL}(2, \mathbb{F})$, μ_ρ is always a measured lamination (see Section 8.2).

For the next result, we will need the notion of *systole* of a maximal framed representation ρ (following [9])

$$\mathrm{Syst}(\rho) := \inf\{L(\rho(\gamma)) : \gamma \in \Gamma, \gamma \text{ hyperbolic}\}.$$

The systole of any real maximal framed representation $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{R})$ is positive (see Section 7.3). On the other hand, for non-Archimedean real closed fields \mathbf{F} , many different possibilities can happen: if Σ is compact, all maximal framed representations $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{F})$ have vanishing systole (since μ_ρ is a measured lamination), while, in higher rank, there are many examples of non-Archimedean maximal framed representations with positive systole (see [9, Corollary 1.11]). For these representations, we have the following.

Corollary 1.3. *Assume that Σ is compact, and that \mathbf{F} is non-Archimedean. Let $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{F})$ be a maximal framed representation. If $\mathrm{Syst}(\rho) > 0$, then for every $x \in \mathcal{B}_{\mathbf{F}}^n$, the orbit map $\Gamma \rightarrow \mathcal{B}_{\mathbf{F}}^n$, $x \mapsto \rho(\gamma)x$ is a quasi-isometric embedding.*

We now give a robust criterion guaranteeing that the current μ_ρ is atomic. We say that a geodesic current is a *multicurve* if it is a finite sum of Γ -orbits of Dirac masses on (lifts of) non-necessarily simple closed geodesics and geodesics with endpoints in cusps.

Theorem 1.4. *Let $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ be maximal framed, and let $\mathbf{Q}(\rho) < \mathbf{F}$ be the field generated over \mathbf{Q} by matrix coefficients of ρ . If the restriction of $v: \mathbf{F}^\times \rightarrow \mathbf{R}$ to $\mathbf{Q}(\rho)$ is discrete, then, up to rescaling, the associated current μ_ρ is a multicurve.*

Using Strubel coordinates, we construct examples of representations to which Theorem 1.4 applies (see Section 8.3). Moreover, we will prove³ that a maximal framed representation $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ is always conjugate to a representation $\rho': \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F}_1)$ for a finite extension \mathbf{F}_1 of the field $\mathbf{Q}(\mathrm{tr}(\rho))$ generated by the traces of the representation ρ . As a result, Theorem 1.4 applies as soon as the field $\mathbf{Q}(\mathrm{tr}(\rho))$ generated by the traces of the representation ρ has discrete valuation; using this, we will show that multicurves are dense in both the real spectrum and Weyl chamber length compactifications of character varieties of maximal representations.

We turn now to the case in which the current μ_ρ in Theorem 1.2 is a measured lamination. In this case, we denote by $\mathcal{T}(\mu_\rho)$ the associated \mathbf{R} -tree, and by $\mathcal{V}(\mu_\rho)$ its vertex set (see Section 2.3 for the definition). We have then the following assertion.

³M. Burger, A. Iozzi, A. Parreau, and M. B. Pozzetti, Real spectrum of maximal character varieties. 2024, in preparation.

Theorem 1.5. *Let \mathbf{F} be a non-Archimedean real closed field, and let $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ be a maximal framed representation. If the associated current μ_ρ is a measured lamination, then there is a Γ -equivariant isometric embedding*

$$\mathcal{V}(\mu_\rho) \hookrightarrow \mathcal{B}_{\mathbf{F}}^n.$$

We will see that if $\mathbf{Q}(\rho)$ has discrete valuation, then $\mathcal{V}(\mu_\rho)$ is the vertex set of a simplicial tree (see Section 7.3 for a general statement).

Currents associated to positive crossratios

The proof of Theorem 1.2 relies on an abstract framework that is applicable to more general situations and that we shortly describe here. Let $X \subset \partial\mathcal{H}^2$ be a Γ -invariant non-empty subset, such as, for example, the set H_Γ of fixed points of hyperbolic elements in Γ , and let $X^{[4]}$ denote the set of positively ordered quadruples in X . A *positive crossratio* is a Γ -invariant function

$$[\cdot, \cdot, \cdot, \cdot]: X^{[4]} \rightarrow [0, \infty)$$

that is flip-invariant

$$[x_1, x_2, x_3, x_4] = [x_3, x_4, x_1, x_2]$$

and satisfies the property

$$[x_1, x_2, x_4, x_5] = [x_1, x_2, x_3, x_5] + [x_1, x_3, x_4, x_5]$$

whenever defined. While the flip invariance of the crossratio guarantees that the associated geodesic current descends to the space of unoriented geodesics, the interplay with crossratios makes it natural to work, as we do, with oriented geodesics. This is also convenient since it allows us to name the endpoints of the geodesics.

Our definition is considerably more general than others existing in the literature (see Remark 3.2 for a comparison). First, we only require our crossratio to be defined on a dense subset X of the boundary of the hyperbolic plane: this is important for some of the applications. For example, if the field \mathbf{F} is countable, $\mathcal{L}(\mathbf{F}^n)$ is also countable, and thus a maximal framing associated to a representation $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$, as well as the induced crossratio, can only be defined on a countable set. Furthermore, some representations, as for example, those defined via Fock–Goncharov or shear coordinates, only have a framing defined on a countable set not including the fixed points of hyperbolic elements. Second, we do not require any continuity on our crossratio. In many interesting examples, the crossratios arising from representations over non-Archimedean real closed fields are integer valued, and thus cannot be continuous. Dropping the continuity assumption on the crossratio also allows us to encompass the theory of crossratios arising from actions on trees (see Example 3.5).

If $\gamma \in \Gamma$ is hyperbolic and $\{\gamma_-, \gamma_+\} \subset X$, the *period*⁴ $\text{per}(\gamma)$ of γ with respect to $[\cdot, \cdot, \cdot, \cdot]$ is defined by

$$\text{per}(\gamma) := [\gamma_-, x, \gamma x, \gamma_+],$$

where $x \in X$ is any point such that $(\gamma_-, x, \gamma x, \gamma_+) \in X^{[4]}$. We show that the following holds.

Theorem 1.6. *Let $X \subset \partial\mathcal{H}^2$ be a Γ -invariant non-empty subset and $[\cdot, \cdot, \cdot, \cdot]$ a positive crossratio on X . Then there is a geodesic current μ on Σ such that for all hyperbolic $\gamma \in \Gamma$,*

$$\text{per}(\gamma) = i(\mu, \delta_\gamma).$$

The geodesic current μ depends continuously on the crossratio $[\cdot, \cdot, \cdot, \cdot]$.

The theorem has been previously shown by Martone–Zhang under the hypothesis that $X = \partial\mathcal{H}^2$ and the crossratio is continuous [34]. In general, the crossratio is not uniquely determined by its periods (see Example 3.4). For the last statement in Theorem 1.6, we consider the space $\mathcal{CR}^+(X)$ of positive crossratios as a closed convex cone in the topological vector space of crossratios on X with the topology of pointwise convergence. This last property will be used in the proof of the continuity of the map which to a point in the real spectrum compactification of maximal representations associates a geodesic current [11]; see also [10, Theorem 36].

The proof of Theorem 1.6 bypasses the possible discontinuities of the crossratio $[\cdot, \cdot, \cdot, \cdot]$ by forcing inner and outer regularity of the current μ and using its σ -additivity. As an application of the explicit construction we obtain the following assertion.

Corollary 1.7. *If the crossratio $[\cdot, \cdot, \cdot, \cdot]$ is integral valued, then the current μ is a multicurve.*

To deduce Theorems 1.2 and 1.4 from Theorem 1.6 and Corollary 1.7, we use the maximal framing to construct a positive crossratio $[\cdot, \cdot, \cdot, \cdot]_\rho$ on H_Γ whose periods satisfy the equality $\text{per}(\gamma) = L(\rho(\gamma))$. Then Theorem 1.6 provides a geodesic current with the required properties.

Maximal representations are not the only class of representations whose length function is given by the periods of a positive crossratio: this is the case for all *positively ratioed representations* [34] (a class that also includes Hitchin representations [29]), representations satisfying property H_k [2] and Θ -positive representations [3]. Corollary 1.7 can be used to study asymptotic properties of these representations as well.

Our approach using σ -additivity of geodesic currents has interesting applications even for representations in $\text{PSp}(2n, \mathbf{R})$, for which we cannot always assume that the crossratio is continuous. The simplest instance is for $\text{PSL}(2, \mathbf{R})$ if ρ sends an element representing a cusp of Γ to a hyperbolic element.

⁴See Section 3.3 for a more general definition of the period without the restriction that $\{\gamma_-, \gamma_+\} \subset X$.

Corollary 1.8. *Let $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{R})$ be a maximal representation, and let $K \subset \Sigma = \Gamma \backslash \mathcal{H}^2$ be a compact subset. Then there are constants $0 < c_1 \leq c_2$ such that for every $\gamma \in \Gamma$ representing a closed geodesic c contained in K ,*

$$c_1 \ell(c) \leq L(\rho(\gamma)) \leq c_2 \ell(c).$$

In particular, this holds uniformly for all γ representing simple closed geodesics.

For Anosov representations, such a result follows directly from the more modern definitions [4, 26]. However, if Σ is not compact, a maximal representation is not necessarily Anosov since the images of parabolic elements can be unipotent (see for instance Section 8.3).

Actions with compatible barycenters

The proof of Theorem 1.5 is carried out in the framework of actions with compatible barycenters that we now define. Given an isometric Γ -action on a metric space (\mathcal{X}, d) , we say that a map

$$\beta: X^{(3)} \rightarrow \mathcal{X}$$

from the set $X^{(3)}$ of distinct triples in X to \mathcal{X} is a *barycenter compatible with the crossratio* $[\cdot, \cdot, \cdot, \cdot]$ if β is S_3 -invariant, Γ -equivariant and for every $(a, b, c, d) \in X^{[4]}$, we have

$$[a, b, c, d] = d(\beta(a, b, d), \beta(a, c, d)).$$

We show then that the following holds.

Theorem 1.9. *Let $X \subset \partial \mathcal{H}^2$ be a Γ -invariant non-empty subset and $[\cdot, \cdot, \cdot, \cdot]$ a positive crossratio on X . Assume that the geodesic current μ associated by Theorem 1.6 to the positive crossratio $[\cdot, \cdot, \cdot, \cdot]$ corresponds to a measured lamination. Then for every isometric Γ -action on a metric space \mathcal{X} admitting a barycenter compatible with the crossratio $[\cdot, \cdot, \cdot, \cdot]$, there is an isometric Γ -equivariant map*

$$\mathcal{V}(\mu) \rightarrow \mathcal{X}.$$

We will see that Theorem 1.9 always applies to a framed action of Γ on an \mathbf{R} -tree \mathcal{T} if the crossratio $[\cdot, \cdot, \cdot, \cdot]$ induced by the action is positive (Proposition 5.8). This crossratio is always positive in the case of the action on $\mathcal{B}_{\mathbf{F}}^1$ induced by a maximal framed representation in $\mathrm{SL}(2, \mathbf{F})$ (Theorem 8.1).

When $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ is a maximal framed representation, we will use the geometry of the Siegel space to define a barycenter map associating to every maximal triple (ℓ_1, ℓ_2, ℓ_3) of Lagrangians a point $B(\ell_1, \ell_2, \ell_3) \in \mathcal{B}_{\mathbf{F}}^n$. Given a representation $\rho: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ with maximal framing $\varphi: X \rightarrow \mathcal{L}(\mathbf{F}^{2n})$, we will show that the map

$$\beta(a, b, c) = B(\varphi(a), \varphi(b), \varphi(c))$$

defines a barycenter compatible with the crossratio $[\cdot, \cdot, \cdot, \cdot]_\rho$ previously defined. Thus Theorem 1.9 applies whenever μ_ρ corresponds to a measured lamination. Using [9, Corollary 1.9], we can find a collection of maximal subsurfaces $\Sigma' \subset \Sigma$ such that Theorem 1.9 holds for the restriction of ρ to Σ' .

Context of our work

When Σ is compact, Hitchin defined a component in the G -character variety of Γ when G is simple real split and proved that it is diffeomorphic to $\mathbf{R}^{\dim G(2g-2)}$ [24]; Labourie then introduced the concept of Anosov flow and used it to show that Hitchin representations are injective with discrete image. Guichard and Wienhard linked Hitchin representations to geometric structures on appropriate compact manifolds [21] which turn out to fiber over Σ [1]. Fock and Goncharov introduced a notion of positivity for G -representations of Γ that is stronger than our notion of maximal framing (Definition 1.1) and which for Σ compact recovers the Hitchin component [18].

Parallel to the split real case, the notion of maximal representations of Γ into a group of Hermitian type G is present since the 1980's in the work of Goldman [19] and Toledo [41]. Typically, for given G and Σ , there are many connected components of maximal representations, the question of their number and topology has been initiated in the work of Bradlow, García-Prada, Gothen [7] and are still topics of current research. That in general maximal representations are injective with discrete image was shown by Burger–Iozzi–Wienhard [12], and, when Σ is compact, the link to Anosov flows was established in [8].

More recently, Guichard–Wienhard generalized Fock–Goncharov positivity to Θ -positivity, a unifying framework encompassing Hitchin and maximal representations, as well as some representations in $\mathrm{PO}(p, q)$ and in groups G whose restricted root system is of type F_4 [22]. When Σ is compact, Θ -positive representations are Anosov [20], and if additionally the target group is $\mathrm{PO}(p, q)$, they form a union of connected components [3]. Guichard–Wienhard's notion of positivity depends on a subset Θ of the simple roots, which consists of a single root in the case of maximal representations, and is the whole set of simple roots in the case of Hitchin components.

When Σ is compact and G is classical, it was shown in [3, 34] that any crossratio associated to a fundamental weight of a root in Θ is positive; we expect that the same is true in general, and for any real closed field \mathbf{F} (see [17] for progress on Hitchin representations). As a result, we expect that Theorems 1.2 and 1.4 admit suitable generalizations to all representations in all higher Teichmüller spaces. These phenomena are, however, not limited to higher Teichmüller components: for example, *hyperconvex representations* are representations that, in general, do not exhaust components of the character variety, but have associated positive crossratios [2, 39]. We refer for more context and details to [38, 42].

Structure of the paper

In Section 2, we discuss preliminaries on geodesic currents and measured laminations. The new result is Proposition 2.1 that gives a useful 4-point characterization of measured laminations among geodesic currents that only involves a dense subset of $\partial\mathcal{H}^2$. In Section 3, we introduce positive crossratios and the associated periods. In Section 4, we construct the geodesic current associated to such a crossratio. Theorem 1.6 follows directly combining Propositions 4.3, 4.9 and 4.10, which are proven in this section. Corollary 1.7 follows from Proposition 4.12. In Section 5, we discuss barycenter maps, and prove Theorem 1.9. In Section 6, we review the geometry of the Siegel space over real closed fields from [13]; using this, we associate to a maximal framed action on the Λ -metric space $\mathcal{B}_n^{\mathbb{F}}$ a positive crossratio (Proposition 6.5), as well as a compatible barycenter map (Section 6.7). In Section 7, we prove the results on maximal framed representations: Theorems 1.2, 1.4, 1.5 and Corollaries 1.3 and 1.8. Section 8 collects interesting examples of maximal framed representations, illustrating various phenomena.

2. On geodesic currents and measured laminations

In this section, we recall the notions of geodesic currents and their Bonahon intersection (Section 2.1); then we establish a criterion for the support of a geodesic current to be a geodesic lamination (Section 2.2); we end by recalling the definition of the tree $\mathcal{T}(\mu)$ associated to a current μ of lamination type in terms of the straight pseudodistance (see [9, Section 4]) on \mathcal{H}^2 associated to a general current.

Let $\Sigma := \Gamma \backslash \mathcal{H}^2$ be a hyperbolic surface of finite area and denote by $\text{pr}: \mathcal{H}^2 \rightarrow \Sigma$ the covering map. The boundary $\partial\mathcal{H}^2$ of \mathcal{H}^2 is endowed with the natural cyclic order. For $(a, b) \in (\partial\mathcal{H}^2)^2$ with $a \neq b$, we will denote the associated *open interval* in $\partial\mathcal{H}^2$ by

$$I_{(a,b)} := \{x \in \partial\mathcal{H}^2 : (a, x, b) \text{ is positively oriented}\},$$

and the *left half-open interval* $I_{(a,b]}$, *right half-open interval* $I_{[a,b)}$ and *closed interval* $I_{[a,b]}$ accordingly, so for example,

$$I_{[a,b)} = \{a\} \cup I_{(a,b)}.$$

Given a subset $A \subset \partial\mathcal{H}^2$, we will denote

$$A^{[4]} := \{(x, y, z, t) \in A^4 : (x, y, z, t) \text{ is positively oriented}\}.$$

2.1. Geodesic currents

A *geodesic current* is a flip-invariant Γ -invariant positive Radon measure on the set of (oriented) geodesics in \mathcal{H}^2 , which we identify with

$$(\partial\mathcal{H}^2)^{(2)} := \{(x, y) \in (\partial\mathcal{H}^2)^2 : x \neq y\}.$$

Given a (non-oriented) geodesic $c \subset \Sigma$ that is either closed or joining two cusps, δ_c will be the geodesic current given by

$$\delta_c := \frac{1}{2} \sum_{(a,b)} \delta_{(a,b)},$$

where we sum over the set of oriented geodesics $(a, b) \in (\partial\mathcal{H}^2)^{(2)}$ lifting c .

Two geodesics $(a, b), (a', b') \in (\partial\mathcal{H}^2)^{(2)}$ are *transverse* if they intersect in a point. The group $\mathrm{PSL}(2, \mathbf{R})$, hence Γ , acts properly on the open subset $\mathcal{G} \subset (\partial\mathcal{H}^2)^{(2)} \times (\partial\mathcal{H}^2)^{(2)}$ of transverse pairs of geodesics. The *Bonahon intersection* $i(\mu, \nu)$ of two geodesic currents μ and ν is the (possibly infinite) $\mu \times \nu$ -measure of any Borel fundamental domain for Γ in \mathcal{G} . Note that, when ν has compact carrier,⁵ $i(\mu, \nu)$ is finite. In order to simplify notations, we set

$$i(\mu, c) := i(\mu, \delta_c)$$

for all closed geodesics $c \subset \Sigma$. We refer to [6] for more details.

2.2. Measured laminations and μ -short geodesics

We refer to [33, §8.3.4] for preliminaries on measured laminations. The equivalence between (1) and (2) in the next proposition is classical; we establish that the two conditions are also equivalent to (4), an additional 4-point characterization that uses only a dense subset of $\partial\mathcal{H}^2$.

Proposition 2.1. *Let μ be a geodesic current and X a dense subset of $\partial\mathcal{H}^2$. The following are equivalent:*

- (1) $\mathrm{supp}(\mu)$ is a lamination;
- (2) $i(\mu, \mu) = 0$;
- (3) $\mu(I_{(d,a)} \times I_{(b,c)}) \cdot \mu(I_{(a,b)} \times I_{(c,d)}) = 0$ for all (a, b, c, d) in $(\partial\mathcal{H}^2)^{[4]}$;
- (4) $\mu(I_{(d,a)} \times I_{(b,c)}) \cdot \mu(I_{(a,b)} \times I_{(c,d)}) = 0$ for all (a, b, c, d) in $X^{[4]}$.

Such a current will be called of lamination type.⁶

Proof. We first show (1) implies (2): if $i(\mu, \mu) > 0$, then there exist open subsets A, B of $(\partial\mathcal{H}^2)^{(2)}$ with $A \times B \subset \mathcal{G}$ and

$$(\mu \times \mu)(A \times B) = \mu(A)\mu(B) > 0,$$

hence $\mu(A), \mu(B) > 0$. Then there exist $g \in A \cap \mathrm{supp}(\mu)$ and $g' \in B \cap \mathrm{supp}(\mu)$. Since $A \times B \subset \mathcal{G}$, in particular g and g' intersect in a point, hence $\mathrm{supp}(\mu)$ is not a lamination.

⁵We recall that the *carrier* of a geodesic current μ is the closed subset $\mathrm{pr}(\bigcup_{g \in \mathrm{supp}(\mu)} g) \subset \Sigma$. The hypothesis that ν has compact carrier ensures that $i(\mu, \nu) < \infty$ and is needed in the proofs of the continuity of the Bonahon intersection.

⁶While in the introduction we identified with a slight abuse of notation measured laminations with currents with zero self intersection, we prefer to keep the objects distinct for the rest of the paper.

We now prove that (2) implies (3): since $i(\mu, \mu) = 0$, we have $(\mu \times \mu)(A \times B) = 0$ for all transverse Borel subsets A, B of $(\partial\mathcal{H}^2)^{(2)}$ (namely, every pair of geodesics $(a, b) \in A, (a', b') \in B$ intersects in one point). The claim follows as $A = I_{(d,a]} \times I_{(b,c]}$ and $B = I_{(a,b]} \times I_{(c,d]}$ are transverse.

It is clear that (3) implies (4).

Suppose now (4). Let g, g' be two geodesics in $\text{supp}(\mu)$. If they are transverse, then $g = (x, y), g' = (x', y')$ with (x, x', y, y') positively oriented. Then by density of X , there exists (a, b, c, d) in $X^{[4]}$ such that $(x, a, x', b, y, c, y', d)$ is positively oriented. Then $g \in I_{(d,a)} \times I_{(b,c)}$ and $g' \in I_{(a,b)} \times I_{(c,d)}$, and as g, g' are in the support of μ , we have $\mu(I_{(d,a)} \times I_{(b,c)}) > 0$ and $\mu(I_{(a,b)} \times I_{(c,d)}) > 0$, a contradiction. Hence $\text{supp}(\mu)$ is a lamination, proving (1). ■

An important concept in [9] was that of μ -short geodesic, namely a geodesic not intersecting in a point any geodesic in the support of μ ; observe that a geodesic (a, b) is μ -short if and only if

$$\mu(I_{(a,b)} \times I_{(b,a)}) = 0.$$

It follows from Proposition 2.1 that if the current μ is of lamination type, its support consists of μ -short geodesics.

2.3. The tree associated to a current of lamination type

We now recall the construction of the tree $\mathcal{T}(\mu)$ associated to a current μ of lamination type. We chose here a description adapted to our purposes, but this agrees with the standard construction described, for example, in [35] and [25, §11.12].

Given a geodesic current μ , we consider the straight pseudodistance on \mathcal{H}^2 [9, §4]

$$d_\mu(x, y) = \frac{1}{2} \{ \mu(\mathcal{G}_{[x,y]}^\uparrow) + \mu(\mathcal{G}_{[x,y]}^\downarrow) \},$$

where for a possibly empty geodesic segment $I \subset \mathcal{H}^2$, we define

$$\mathcal{G}_I^\uparrow = \{ (g_-, g_+) \in (\partial\mathcal{H}^2)^{(2)} : |g \cap I| = 1 \}$$

as the set of geodesics g that intersect transversely the geodesic segment I .

If μ is of lamination type, then the quotient metric space $X_\mu = \mathcal{H}^2 / \sim$, obtained by identifying points at d_μ -distance zero, is 0-hyperbolic in the sense of Gromov and can therefore be canonically embedded in a minimal \mathbf{R} -tree $\mathcal{T}(\mu)$. We will denote by $\mathcal{V}(\mu)$ the image in $\mathcal{T}(\mu)$ of the complementary regions \mathcal{R} of $\text{supp}(\mu)$. It corresponds to the set of branching points of $\mathcal{T}(\mu)$.

Since μ is Γ -invariant, the group Γ acts on $\mathcal{T}(\mu)$ and therefore on $\mathcal{V}(\mu)$ by isometries. A direct consequence of the definition of Bonahon intersection is that, for this action,

$$\ell_{\mathcal{T}(\mu)}(\gamma) = i(\mu, \delta_c)$$

for hyperbolic γ representing a closed geodesic c .

3. Positive crossratios

In this section, we introduce the notion of positive crossratio $[\cdot, \cdot, \cdot, \cdot]$, prove that its periods are well defined, and discuss examples.

3.1. Positive crossratios

Let $\Sigma = \Gamma \backslash \mathcal{H}^2$ be a finite area hyperbolic surface, and let $\Gamma < \text{PSL}(2, \mathbf{R})$ be its fundamental group realized as a torsion-free lattice in $\text{PSL}(2, \mathbf{R})$.

Definition 3.1. Let $X \subset \partial \mathcal{H}^2$ be a Γ -invariant non-empty subset. A *crossratio* on X is a real valued function $[\cdot, \cdot, \cdot, \cdot]$ defined on $X^{[4]}$ satisfying the following properties:

- (CR1) it is Γ -invariant;
- (CR2) $[x, y, z, t] = [z, t, x, y]$ for all $(x, y, z, t) \in X^{[4]}$;
- (CR3) $[x, y, z, t] + [x, z, w, t] = [x, y, w, t]$ whenever (x, y, z, w, t) is positively oriented.

The crossratio is in addition *positive* if

- (CR4) $[x, y, z, t] \geq 0$ for all $(x, y, z, t) \in X^{[4]}$.

Remark 3.2. There are many different non-equivalent notions of crossratio available in the literature, and there is no standard choice of the order of the arguments of the function $[\cdot, \cdot, \cdot, \cdot]$. More specifically,

- If $B(\cdot, \cdot, \cdot, \cdot)$ is a crossratio according to [34, Definition 2.4] (which agrees with [31, Definition 1.f]), then

$$[a, b, c, d] = B(a, d, c, b)$$

is a crossratio according to Definition 3.1. However, we do not require continuity, and our crossratio is defined only on a smaller set.

- If $\text{Cr}(\cdot, \cdot, \cdot, \cdot)$ is a crossratio according to [23, p. 1], then

$$[a, b, c, d] = \log \text{Cr}(b, c, d, a)$$

is a crossratio according to Definition 3.1. However, in [23, p. 1] the crossratio is Hölder continuous.

- If $\mathbb{B}(\cdot, \cdot, \cdot, \cdot)$ is a crossratio according to [29, p. 1], then

$$[a, b, c, d] = \log \mathbb{B}(a, c, d, b)$$

is a crossratio according to Definition 3.1. However, the definition in [29, p. 1] requires Hölder continuity and a much stronger positivity than what we impose, namely the strict inequality in (CR4).

As a direct consequence of (CR3), positive crossratios have the following monotonicity property, illustrated in Figure 1.

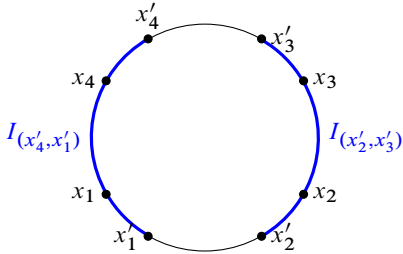


Figure 1. The configuration of points in Lemma 3.3.

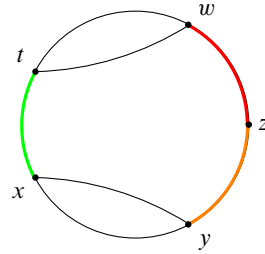


Figure 2. The additivity corresponding to CR3.

Lemma 3.3. For all (x_1, x_2, x_3, x_4) and (x'_1, x'_2, x'_3, x'_4) in $X^{[4]}$ such that

$$I_{(x_4, x_1)} \subset I_{(x'_4, x'_1)} \quad \text{and} \quad I_{(x_2, x_3)} \subset I_{(x'_2, x'_3)},$$

we have

$$[x_1, x_2, x_3, x_4] \leq [x'_1, x'_2, x'_3, x'_4]. \tag{3.1}$$

To gain some intuition on the properties (CR2) and (CR3), we recall that if $x, y, z, t \in \partial\mathcal{H}^2 = \mathbf{R} \cup \{\infty\}$ and

$$[x, y, z, t] = \ln \frac{(x - z)(y - t)}{(x - y)(z - t)}$$

is the logarithm of the usual crossratio, the Liouville measure \mathcal{L} has the property that

$$\mathcal{L}(I_{(t,x)} \times I_{(y,z)}) = [x, y, z, t]. \tag{3.2}$$

Thus (CR2) corresponds to the flip-invariance of \mathcal{L} ,

$$\mathcal{L}(I_{(t,x)} \times I_{(y,z)}) = \mathcal{L}(I_{(y,z)} \times I_{(t,x)})$$

and (CR3) to additivity (see Figure 2)

$$\mathcal{L}(I_{(t,x)} \times I_{(y,w)}) = \mathcal{L}(I_{(t,x)} \times I_{(y,z)}) + \mathcal{L}(I_{(t,x)} \times I_{(z,w)})$$

since

$$\mathcal{L}(I_{(t,x)} \times \{z\}) = 0.$$

3.2. Examples

There are two natural crossratios associated to a geodesic current.

Example 3.4. If μ is a current, it is easily checked that

$$[a, b, c, d]_{\mu}^{+} := \mu(I_{(d,a)} \times I_{(b,c)})$$

defines a positive crossratio on $\partial\mathcal{H}^2$. Similarly,

$$[a, b, c, d]_{\mu}^{-} := \mu(I_{[d,a]} \times I_{[b,c]})$$

defines a positive crossratio on $\partial\mathcal{H}^2$. Note that these two crossratios may be different (for example, this is the case if $\mu = \delta_c$ for some closed geodesic c), but have the same periods.

Framed actions on trees give other fundamental examples of crossratios.

Example 3.5. If \mathcal{T} is a real tree, we denote by $[\cdot, \cdot, \cdot, \cdot]_{\mathcal{T}}$ the usual crossratio on the boundary $\partial_{\infty}\mathcal{T}$ of the tree \mathcal{T} : for every pairwise distinct $(a, b, c, d) \in \partial_{\infty}\mathcal{T}^4$, $[a, b, c, d]_{\mathcal{T}}$ is the signed distance, on the oriented geodesic from a to d , from the orthogonal projection $\beta_{\mathcal{T}}(a, b, d)$ of b to the orthogonal projection $\beta_{\mathcal{T}}(a, c, d)$ of c . Note that

$$|[a, b, c, d]_{\mathcal{T}}| = d(\beta_{\mathcal{T}}(a, b, d), \beta_{\mathcal{T}}(a, c, d)), \tag{3.3}$$

where d denotes the distance in \mathcal{T} .

A *framed action* of Γ on \mathcal{T} is an action by isometries $\rho: \Gamma \rightarrow \text{Isom}(\mathcal{T})$ admitting an injective equivariant map (a *framing*) $\varphi: X \rightarrow \partial_{\infty}\mathcal{T}$, where X is some Γ -invariant non-empty subset of $\partial\mathcal{H}^2$. Then the crossratio $[\cdot, \cdot, \cdot, \cdot]_{\mathcal{T}}$ on $\partial_{\infty}\mathcal{T}$ induces a crossratio $[\cdot, \cdot, \cdot, \cdot]_{\varphi}$ on X defined by

$$[x_1, x_2, x_3, x_4]_{\varphi} := [\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)]_{\mathcal{T}}$$

for every $(x_1, x_2, x_3, x_4) \in X^{[4]}$.

Example 3.6. An example of such situation is given by the Γ -action on the \mathbf{R} -tree $\mathcal{T}(\mu)$ associated to a current of lamination type. Let $X \subset \partial\mathcal{H}^2$ be the set of fixed points of hyperbolic elements whose axis are transverse to the geodesic lamination $\text{supp}(\mu)$. Then for every such $\gamma \in \Gamma$ with $\{\gamma_-, \gamma_+\} \subset X$, the element γ acts on $\mathcal{T}(\mu)$ with strictly positive translation length

$$\ell_{\mathcal{T}(\mu)}(\gamma) = i(\mu, \delta_c)$$

(see Section 2.3) and has thus an attractive fixed point $\varphi(\gamma_+)$ and a repulsive one $\varphi(\gamma_-)$ in $\partial_{\infty}\mathcal{T}(\mu)$. Then $\varphi: X \rightarrow \partial_{\infty}\mathcal{T}(\mu)$ is a framing and it follows from the definition of the distance on $\mathcal{T}(\mu)$ that

$$\mu(I_{[x_4, x_1]} \times I_{[x_2, x_3]}) = [\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)]_{\mathcal{T}}.$$

It follows from the discussion recalled in Section 2.2 that the crossratio is positive.

Example 3.5 inspires the following definition.

Definition 3.7. We say that a crossratio is *ultrametric* if it satisfies

$$\text{(CRU)} \quad [a, b, c, d] \times [b, c, d, a] = 0 \text{ for all } (a, b, c, d) \text{ in } X^{[4]}.$$

The following is clear.

Proposition 3.8. *The crossratio induced by a framed action on an \mathbf{R} -tree is ultrametric.*

The following is a corollary of Proposition 2.1.

Proposition 3.9. *The crossratio*

$$[a, b, c, d] := \mu(I_{(d,a)} \times I_{(b,c)})$$

associated to a lamination type current μ is ultrametric.

3.3. The periods of the crossratio

Let now $[\cdot, \cdot, \cdot, \cdot]$ be a positive crossratio defined on $X \subset \partial\mathcal{H}^2$, and let $\gamma \in \Gamma$ be hyperbolic such that $\{\gamma_-, \gamma_+\} \subset X$. The additivity property (CR3) of the crossratio implies that the value $[\gamma_-, x, \gamma x, \gamma_+]$ is independent of $x \in I_{(\gamma_-, \gamma_+)} \cap X$. This justifies the following.

Definition 3.10. If $[\cdot, \cdot, \cdot, \cdot]$ is a positive crossratio on X and $\gamma \in \Gamma$ is hyperbolic such that $\{\gamma_-, \gamma_+\} \subset X$, the *period* of γ is defined by

$$\text{per}(\gamma) := [\gamma_-, x, \gamma x, \gamma_+]$$

for one (any) $x \in I_{(\gamma_-, \gamma_+)} \cap X$.

The purpose of this section is to extend the definition of the period of a crossratio defined on a Γ -invariant set $X \subset \partial\mathcal{H}^2$ to hyperbolic elements $\gamma \in \Gamma$ whose endpoints do not necessarily belong to the set X .

This is achieved by the following.

Proposition 3.11. *Let $[\cdot, \cdot, \cdot, \cdot]$ be a positive crossratio on X , and let $\gamma \in \Gamma$ be a hyperbolic element. Choose monotone sequences $(x_n), (x'_n) \subset X$ with limit γ_- and $(y_n), (y'_n) \subset X$ with limit γ_+ . Assume furthermore that $(x'_n, \gamma_-, x_n, y_n, \gamma_+, y'_n)$ is positively oriented. Then, for all $x \in X$,*

$$\lim_{n \rightarrow \infty} [x_n, x, \gamma x, y_n] = \lim_{n \rightarrow \infty} [x'_n, x, \gamma x, y'_n].$$

Proof. Up to passing to a subsequence, we can and will assume that $(x_0, x, \gamma x, y_0)$ is positive (see Figure 3). Since by (CR2) and (CR3), we have

$$[x_n, x, \gamma x, y_n] = [x_n, x, \gamma x, x'_n] + [x'_n, x, \gamma x, y'_n] + [y'_n, x, \gamma x, y_n],$$

it is enough to show that

$$\lim_{n \rightarrow \infty} [x_n, x, \gamma x, x'_n] = 0 \tag{3.4}$$

and the analogous statement for $[y'_n, x, \gamma x, y_n]$.

Since the crossratio is positive (CR4), the limit in equation (3.4) is monotone. Since both x_n and $\gamma^{-n}x_0$ tend to γ_- , it is in turn enough to show that

$$\lim_{n \rightarrow \infty} [\gamma^{-n}x_0, x, \gamma x, \gamma^{-n}x'_0] = 0.$$

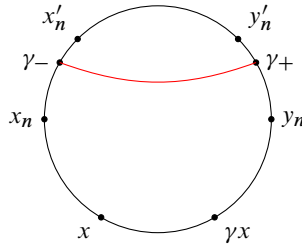


Figure 3. The configuration in Proposition 3.11.

This follows since, for every N ,

$$\begin{aligned} \infty > [x_0, x, y'_0, x'_0] &\geq [x_0, x, \gamma^N x, x'_0] \\ &\geq \sum_{j=0}^{N-1} [x_0, \gamma^j x, \gamma^{j+1} x, x'_0] \\ &= \sum_{j=0}^{N-1} [\gamma^{-j} x_0, x, \gamma x, \gamma^{-j} x'_0]. \end{aligned}$$

Here in the last equality, we used the fact that the crossratio is Γ -invariant. The claim for $[y'_n, x, \gamma x, y_n]$ follows analogously. ■

Thanks to Proposition 3.11, we can extend Definition 3.10 to the following.

Definition 3.12. If $[\cdot, \cdot, \cdot, \cdot]$ is a positive crossratio on X and $\gamma \in \Gamma$ is hyperbolic, the *period* of γ is

$$\text{per}(\gamma) := \lim_{\substack{s, t \in X \\ s \rightarrow \gamma_- \\ t \rightarrow \gamma_+}} [s, x, \gamma x, t]$$

for one (any) $x \in I_{(\gamma_-, \gamma_+)} \cap X$.

4. The geodesic current associated to a positive crossratio

In this section, we prove Theorem 1.6 and Corollary 1.7. The proof of Theorem 1.6 is carried out in three steps: in Section 4.1, we use a crossratio to construct a geodesic current $\mu_{[\cdot, \cdot, \cdot, \cdot]}$; in Section 4.2, we relate the periods of the crossratio $[\cdot, \cdot, \cdot, \cdot]$ and the intersection of curves with $\mu_{[\cdot, \cdot, \cdot, \cdot]}$; in Section 4.3, we conclude the proof of Theorem 1.6 by showing that $\mu_{[\cdot, \cdot, \cdot, \cdot]}$ depends continuously on the crossratio $[\cdot, \cdot, \cdot, \cdot]$. The fact that an integer valued crossratio leads to a multicurve (Corollary 1.7) is shown in Section 4.4. We conclude the section discussing in Section 4.5 how crossratios and geodesic currents can be restricted to subsurfaces; this is for future reference and will be used in the study of the real spectrum compactification of maximal representations.

4.1. Construction of the current

The aim of this section is to show that a positive crossratio always leads to a geodesic current. This is done in Proposition 4.3. The strategy of the proof is first to associate to the crossratio a finitely additive set function r defined on the family of proper rectangles with vertices in $X^{[4]}$ (Proposition 4.1), and then to build a canonical Radon measure μ out of r .

Fix a Γ -invariant non-empty subset $X \subset \partial\mathcal{H}^2$, and a positive crossratio

$$[\cdot, \cdot, \cdot, \cdot]: X^{[4]} \rightarrow [0, \infty).$$

A rectangle in $(\partial\mathcal{H}^2)^{(2)}$ is the product $R = I \times J$ of two disjoint intervals $I, J \subset \partial\mathcal{H}^2$. It is called *proper* if its closure in $(\partial\mathcal{H}^2)^{(2)}$ is compact, that is, $\bar{I} \cap \bar{J} = \emptyset$, and I and J have non-empty interior. The *vertices* of R are then the unique positively oriented 4-tuple (a, b, c, d) in $(\partial\mathcal{H}^2)^{[4]}$ such that d, a are the endpoints of I , and b, c are the endpoints of J , equivalently,

$$I_{(d,a)} \times I_{(b,c)} \subset R \subset I_{[d,a]} \times I_{[b,c]}.$$

For $\mathcal{A} \subset \partial\mathcal{H}^2$, we denote by $\mathcal{R}(\mathcal{A})$ the family of all proper rectangles with vertices in $\mathcal{A}^{[4]}$.

If R is a proper rectangle with vertices (a, b, c, d) in $X^{[4]}$, we define

$$r(R) = [a, b, c, d],$$

the *crossratio* of the rectangle R . It follows directly from the additivity property (CR3) of the crossratio that this defines a finitely additive positive function on the family $\mathcal{R}(X)$ of all proper rectangles with vertices in $X^{[4]}$.

Proposition 4.1. *The function $r: \mathcal{R}(X) \rightarrow \mathbf{R}$ satisfies the following:*

- (1) *If a rectangle $R \in \mathcal{R}(X)$ is the union $R = R_1 \sqcup R_2$ of two rectangles with disjoint interior in $\mathcal{R}(X)$, then*

$$r(R) = r(R_1) + r(R_2).$$

- (2) *For all R, R' in $\mathcal{R}(X)$, if $R \subset R'$, then $r(R) \leq r(R')$.*

Remark 4.2. The function $r: \mathcal{R}(X) \rightarrow \mathbf{R}$ may not be σ -additive, even restricting to the family of left half-open rectangles $I_{(d,a]} \times I_{[b,c]}$ with $(a, b, c, d) \in X^{[4]}$. For example, setting

$$[a, b, c, d] := \delta_e(I_{[d,a)} \times I_{[b,c)})$$

for a closed curve e corresponding to some hyperbolic $\gamma \in \Gamma$, we get a positive crossratio $[\cdot, \cdot, \cdot, \cdot]$ on $X^{[4]} = (\partial\mathcal{H}^2)^{[4]}$ whose associated function r is not σ -additive on $\mathcal{R}(X)$. Take a, c such that $(a, \gamma_+, c, \gamma_-) \in X^{[4]}$ and $I_{[\gamma_-,a)} \times I_{[\gamma_+,c)}$ contains no other point of the orbit of (γ_-, γ_+) . Let $d_n \downarrow \gamma_-$ in $I_{(\gamma_-,a)}$ and $b_n \downarrow \gamma_+$ in $I_{(\gamma_+,c)}$. Let $R_n = I_{(d_n,a]} \times I_{[b_n,c]}$. Then $R = I_{(\gamma_-,a]} \times I_{[\gamma_+,c]}$ is the increasing union of the R_n , and $r(R_n) = [a, b_n, c, d_n] = 0$ for all n whereas $r(R) = [a, \gamma_+, c, \gamma_-] = 1$, contradicting σ -additivity. The problem is due to the fact that this crossratio is not continuous at $(a, \gamma_+, c, \gamma_-)$.

We now construct the measure μ . Recall that for a rectangle $R \in \mathcal{R}(X)$ with vertices (a, b, c, d) , we set $r(R) = [a, b, c, d]$. Furthermore, we denote by $\overset{\circ}{R}$ (resp. \bar{R}) the open (resp. closed) rectangle with the same vertices as R .

Proposition 4.3. *There exists a unique positive Radon measure μ on $(\partial\mathcal{H}^2)^{(2)}$ satisfying one of the following equivalent conditions:*

- (1) $\mu(\overset{\circ}{R}) \leq r(R) \leq \mu(\bar{R})$ for any (proper) rectangle $R \in \mathcal{R}(X)$.
- (2) For any (proper) rectangles $R, R' \in \mathcal{R}(X)$ with $\bar{R}' \subset \overset{\circ}{R}$, we have $\mu(R') \leq r(R)$ and $r(R') \leq \mu(R)$.
- (3) For all (proper) open rectangles $R \in \mathcal{R}(\partial\mathcal{H}^2)$,

$$\mu(R) = \sup\{r(R') : R' \in \mathcal{R}(X) \text{ and } \bar{R}' \subset R\}.$$

- (4) For all (proper) closed rectangles $R \in \mathcal{R}(\partial\mathcal{H}^2)$,

$$\mu(R) = \inf\{r(R') : R' \in \mathcal{R}(X) \text{ and } R \subset \overset{\circ}{R}'\}.$$

Definition 4.4. We call the measure μ in Proposition 4.3 the *geodesic current associated to the positive crossratio* $[\cdot, \cdot, \cdot, \cdot] : X^{[4]} \rightarrow \mathbf{R}$, and denote it by $\mu_{[\cdot, \cdot, \cdot, \cdot]}$ if we want to emphasize the dependence on $[\cdot, \cdot, \cdot, \cdot]$.

Proposition 4.3 implies the following “outer and inner” continuity properties of the current.

Proposition 4.5. *Let (a, b, c, d) be a positively oriented quadruple in $(\partial\mathcal{H}^2)^{[4]}$. Let $(a_n, b_n, c_n, d_n)_{n \geq 1}$ be a sequence in $X^{[4]}$ converging to (a, b, c, d) . Then*

- (1) *If $d_n, a_n \in I_{(d,a)}$ and $b_n, c_n \in I_{(b,c)}$ for all $n \geq 1$, then $\mu(I_{(d,a)} \times I_{(b,c)}) = \lim_n [a_n, b_n, c_n, d_n]$.*
- (2) *If $a_n, b_n \in I_{(a,b)}$ and $c_n, d_n \in I_{(c,d)}$ for all $n \geq 1$, then $\mu(I_{[d,a]} \times I_{[b,c]}) = \lim_n [a_n, b_n, c_n, d_n]$.*

Proof. We prove the first assertion (the second is similar). We have by (3)

$$\mu(I_{(d,a)} \times I_{(b,c)}) = \sup\{[a', b', c', d'] : a', d' \in I_{(d,a)}, b', c' \in I_{(b,c)} \text{ and } (a', b', c', d') \in X^{[4]}\}. \tag{4.1}$$

Let $(a', b', c', d') \in X^{[4]}$ with $a', d' \in I_{(d,a)}$ and $b', c' \in I_{(b,c)}$. For n large enough, we have

$$I_{[d',a']} \subset I_{[d_n,a_n]} \quad \text{and} \quad I_{[b',c']} \subset I_{[b_n,c_n]},$$

and hence

$$[a', b', c', d'] \leq [a_n, b_n, c_n, d_n],$$

which by (4.1) implies (1). ■

Proof of Proposition 4.3. We begin by proving that the conditions are equivalent. Let μ be any positive Radon measure on $(\partial\mathcal{H}^2)^{(2)}$.

It is clear that (1) implies (2).

We now prove that (2) implies (3). Consider an open rectangle R in $\mathcal{R}(\partial\mathcal{H}^2)$. First observe that for every R' in $\mathcal{R}(X)$ such that $\overline{R'} \subset R$, we have by (2) that $r(R') \leq \mu(R)$, hence

$$\sup\{r(R') : R' \in \mathcal{R}(X) \text{ and } \overline{R'} \subset R\} \leq \mu(R).$$

By density of X in $\partial\mathcal{H}^2$, we can now take an increasing sequence of rectangles R_n in $\mathcal{R}(X)$ with union R such that $\overline{R_n} \subset \overset{\circ}{R}_{n+1}$. We have by (2) that

$$\mu(R_{n-1}) \leq r(R_n) \leq \mu(R_{n+1}),$$

in particular, by σ -additivity of μ we have $\mu(R) = \lim_n \mu(R_n) = \lim_n r(R_n)$.

We now prove that (3) implies (4). Consider a proper closed rectangle R in $\partial\mathcal{H}^2$. For every R'' in $\mathcal{R}(X)$ such that $R \subset \overset{\circ}{R}''$, there is an open R' in $\mathcal{R}(X)$ such that $R \subset R'$ and $\overline{R'} \subset \overset{\circ}{R}''$. Then we have $\mu(R) \leq \mu(R') \leq r(R'')$ by (3). Hence

$$\mu(R) \leq \inf\{r(R'') : R'' \in \mathcal{R}(X) \text{ and } R \subset \overset{\circ}{R}''\}.$$

Let now R_n be a decreasing sequence of open rectangles in $\mathcal{R}(X)$ with intersection R , such that $\overline{R_{n+1}} \subset R_n$. Then $\mu(R) = \lim_n \mu(R_n)$, and by (3) we have $r(R_n) \leq \mu(R_{n-1})$ and

$$\mu(R_n) = \sup\{r(R') : R' \in \mathcal{R}(X) \text{ and } \overline{R'} \subset R_n\} \leq r(R_n).$$

hence $\mu(R) = \lim_n r(R_n)$.

We finally check that (4) implies (1). Consider any rectangle R in $\mathcal{R}(X)$. As $r(R) \leq r(R')$ for all R' containing R , taking infimum on R' containing \overline{R} in their interior, we get by (4) that $r(R) \leq \mu(\overline{R})$. Now write the open rectangle $\overset{\circ}{R}$ as an increasing union $\overset{\circ}{R} = \cup \uparrow R_n$ of closed rectangles R_n in $\mathcal{R}(X)$ with $R_n \subset \overset{\circ}{R}_{n+1}$. Then by (4), we have that $\mu(R_n) \leq r(R_{n+1}) \leq r(R)$. As $\mu(R_n) \rightarrow \mu(\overset{\circ}{R})$ by σ -additivity, we deduce $\mu(\overset{\circ}{R}) \leq r(R)$.

We now prove the existence of μ satisfying (2). The strategy of the construction of μ is to use the finitely additive function r to define the integral of compactly supported continuous functions. This leads by the Riesz representation theorem to a Radon measure μ .

A *simple function* is a linear combination $g = \sum_{i=1}^n \alpha_i \chi_{R_i}$ of characteristic functions of rectangles R_i in $\mathcal{R}(X)$. Define $E(g)$ by

$$E(g) := \sum_{i=1}^n \alpha_i r(R_i).$$

The additivity property of r on $\mathcal{R}(X)$ (Proposition 4.1) shows that $E(g)$ is independent of the representation of g as linear combination of characteristic functions of proper rectangles in $\mathcal{R}(X)$. It implies that if g_1 and g_2 are simple functions, then

$$E(g_1 + g_2) = E(g_1) + E(g_2).$$

This property also shows that if g_1, g_2 are simple and $g_1 \leq g_2$, then $E(g_1) \leq E(g_2)$.

If now $f \geq 0$ is a continuous function on $(\partial\mathcal{H}^2)^{(2)}$ with compact support, define

$$I_+(f) := \sup\{E(g) : 0 \leq g \leq f, g \text{ is simple}\},$$

$$I_-(f) := \inf\{E(g) : f \leq g, g \text{ is simple}\}.$$

Then by uniform continuity of f and density of X , we have $I_-(f) = I_+(f) =: I(f)$. The additivity of I on positive continuous functions with compact support then follows from the fact that I_- is super-additive, and I_+ is subadditive. Then I extends to all continuous functions with compact support as a positive linear functional on the space of continuous functions with compact support, hence corresponds to a Radon measure μ .

We now prove that μ satisfies (2). Let R, R' be rectangles with $\overline{R'} \subset \overset{\circ}{R}$. As there is a continuous function f with compact support such that $\chi_{R'} \leq f \leq \chi_R$, we have $\mu(R') \leq I(f) \leq E(\chi_R) = r(R)$ whenever $R \in \mathcal{R}(X)$, and $r(R') = E(\chi_{R'}) \leq I(f) \leq \mu(R)$ whenever $R' \in \mathcal{R}(X)$.

Uniqueness comes from (3), as the class of proper open rectangles in $(\partial\mathcal{H}^2)^{(2)}$ is stable under finite intersection and generates the Borel σ -algebra. ■

Remark 4.6. It follows from Proposition 4.3 (4) that also for all pencils $P = \{a\} \times I_{[b,c]}$,

$$\mu(P) = \inf\{r(R') : R' \in \mathcal{R}(X) \text{ and } P \subset \overset{\circ}{R'}\}.$$

Proposition 4.7. *If a positive crossratio is ultrametric, then the associated current μ is of lamination type.*

Proof. By Proposition 2.1, it is enough to prove that, for all $(a, b, c, d) \in X^{[4]}$,

$$\mu(I_{(d,a)} \times I_{(b,c)}) \cdot \mu(I_{(a,b)} \times I_{(c,d)}) = 0.$$

Let (a, b, c, d) in $X^{[4]}$. As the crossratio is ultrametric, we have either $[a, b, c, d] = 0$ or $[b, c, d, a] = 0$. As $\mu(I_{(d,a)} \times I_{(b,c)}) \leq [a, b, c, d]$ and $\mu(I_{(a,b)} \times I_{(c,d)}) \leq [b, c, d, a]$ (see Proposition 4.3 (1)), this implies that $\mu(I_{(d,a)} \times I_{(b,c)}) = 0$ or $\mu(I_{(a,b)} \times I_{(c,d)}) = 0$. ■

4.2. Periods and intersections

We now turn to the problem of identifying the periods of a positive crossratio with the intersections of the corresponding current. Let then $[\cdot, \cdot, \cdot, \cdot]$ be a positive crossratio defined on X , and let $\gamma \in \Gamma$ be hyperbolic. Recall from Section 3.3 the definition of the period of γ with respect to the crossratio $[\cdot, \cdot, \cdot, \cdot]$.

In the following proposition, we will use the well-known fact that if γ is a hyperbolic element representing a closed geodesic $c \subset \Sigma$, μ is a geodesic current, and $x \in \partial\mathcal{H}^2$ is any fixed point distinct from γ_+ and γ_- , then the intersection $i(\mu, \delta_c)$ can be computed as

$$i(\mu, \delta_c) = \mu(I_{(\gamma_+, \gamma_-)} \times I_{(x, \gamma x)}).$$

Indeed, $I_{(\gamma_+, \gamma_-)} \times I_{(x, \gamma x)}$ is a Borel fundamental domain for the $\langle \gamma \rangle$ -action on $I_{(\gamma_+, \gamma_-)} \times I_{(\gamma_-, \gamma_+)}$ because (γ_-, γ_+) is the disjoint union of the $\langle \gamma \rangle$ -translates of $(x, \gamma x)$.

We will also need that geodesic currents do not weight pencils based at fixed points of hyperbolic elements and not containing their axis. To be more precise, let H_Γ denote the subset of $\partial\mathcal{H}^2$ consisting of the fixed points of hyperbolic elements in Γ . For every $a \in H_\Gamma$, we choose γ such that $a = \gamma_-$ and denote by \bar{a} the point γ_+ .

Lemma 4.8 ([33, Proposition 8.2.8]). *Let μ be a geodesic current, $a \in H_\Gamma$ and $I \subset \partial\mathcal{H}^2$ be a closed interval with $\{a, \bar{a}\} \cap I = \emptyset$. Then $\mu(\{a\} \times I) = 0$.*

Proposition 4.9. *Let μ be the geodesic current associated to a positive crossratio $[\cdot, \cdot, \cdot, \cdot]$, and let c be a closed geodesic represented by a hyperbolic element $\gamma \in \Gamma$. Then*

$$\text{per}(\gamma) = i(\mu, \delta_c).$$

Proof. As in Proposition 3.11, we choose monotone sequences $(x_n), (x'_n) \subset X$ with limit γ_- and $(y_n), (y'_n) \subset X$ with limit γ_+ , and assume that $(x'_n, \gamma_-, x_n, y_n, \gamma_+, y'_n)$ is positively oriented (see also Figure 3). We have that

$$\text{per}(\gamma) = \lim_{n \rightarrow \infty} [x_n, x, \gamma x, y_n] \tag{4.2}$$

and also

$$\text{per}(\gamma) = \lim_{n \rightarrow \infty} [x'_n, x, \gamma x, y'_n], \tag{4.3}$$

where $x \in I_{(\gamma_-, \gamma_+)} \cap X$ is arbitrary.

For any $x \in I_{(\gamma_-, \gamma_+)} \cap X$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} i(\mu, \delta_c) &= \mu(I_{(\gamma_+, \gamma_-)} \times I_{(x, \gamma x)}) \\ &= \mu(I_{(\gamma_+, \gamma_-)} \times I_{(x, \gamma x)}) + \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}) \\ &\leq [x_n, x, \gamma x, y_n] + \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}), \end{aligned}$$

where the inequality follows from Proposition 4.3 (1). By (4.2), this implies that

$$i(\mu, \delta_c) \leq \text{per}(\gamma) + \mu(I_{(\gamma_-, \gamma_+)} \times \{x\}).$$

Next we have

$$i(\mu, \delta_c) = \mu(I_{(\gamma_+, \gamma_-)} \times I_{(x, \gamma x)}) = \mu(I_{(\gamma_+, \gamma_-)} \times I_{[x, \gamma x]}) - \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}).$$

Using Lemma 4.8, this equals

$$\mu(I_{[\gamma_+, \gamma_-]} \times I_{[x, \gamma x]}) - \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}),$$

which, again by Proposition 4.3 (1), implies

$$\begin{aligned} i(\mu, \delta_c) &= \mu(I_{[\gamma_+, \gamma_-]} \times I_{[x, \gamma x]}) - \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}) \\ &\geq [x'_n, x, \gamma x, y'_n] - \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}). \end{aligned}$$

Then it follows from (4.3) that

$$i(\mu, \delta_c) \geq \text{per}(\gamma) - \mu(I_{(\gamma_-, \gamma_+)} \times \{x\})$$

and thus

$$|i(\mu, \delta_c) - \text{per}(\gamma)| \leq \mu(I_{(\gamma_+, \gamma_-)} \times \{x\})$$

for any $x \in I_{(\gamma_-, \gamma_+)} \cap X$.

Now fix a closed interval $I_{[a,b]} \subset I_{(\gamma_-, \gamma_+)}$ with non-empty interior. Then

$$\sum_{x \in X \cap [a,b]} \mu(I_{(\gamma_+, \gamma_-)} \times \{x\}) \leq \mu(I_{(\gamma_+, \gamma_-)} \times I_{[a,b]}) < \infty$$

and since $X \cap [a, b]$ is infinite, this implies the existence of a sequence $(x_n)_{n \geq 1}$ in $X \cap [a, b]$ with

$$\lim_{n \rightarrow \infty} \mu(I_{(\gamma_+, \gamma_-)} \times \{x_n\}) = 0,$$

which implies that $i(\mu, \delta_c) = \text{per}(\gamma)$. ■

4.3. The current depends continuously on the crossratio

The vector space $\mathcal{C}\mathcal{R}(X)$ of crossratios on X is a topological vector space for the topology of pointwise convergence, and the space $\mathcal{C}\mathcal{R}^+(X)$ of positive crossratios is a closed convex cone in it. We observe moreover that the map

$$\mathcal{C}\mathcal{R}^+(X) \rightarrow \mathcal{C}(\Sigma), \quad [\cdot, \cdot, \cdot, \cdot] \mapsto \mu_{[\cdot, \cdot, \cdot, \cdot]}$$

from positive crossratios to the space $\mathcal{C}(\Sigma)$ of geodesic currents is surjective. In fact, if μ is a geodesic current, one verifies using the regularity of μ that $\mu_{[\cdot, \cdot, \cdot, \cdot]}^+ = \mu$ for the crossratio $[\cdot, \cdot, \cdot, \cdot]_\mu^+$ of Example 3.4.

Recall from Section 4.2 that H_Γ denotes the subset of $\partial\mathcal{H}^2$ consisting of the fixed points of hyperbolic elements in Γ , and for $a \in H_\Gamma$, we denote by \bar{a} the other fixed point of any element fixing a .

We call a Γ -invariant subset $S_\Gamma \subset H_\Gamma$ *symmetric* if $\bar{\xi} \in S_\Gamma$ whenever $\xi \in S_\Gamma$.

Proposition 4.10. *Let $S_\Gamma \subset H_\Gamma$ be a Γ -invariant symmetric subset such that $\{(\xi, \bar{\xi}) : \xi \in S_\Gamma\}$ is dense in $(\partial\mathcal{H}^2)^2$. Then the map*

$$\mathcal{C}\mathcal{R}^+(S_\Gamma) \rightarrow \mathcal{C}(\Sigma), \quad [\cdot, \cdot, \cdot, \cdot] \mapsto \mu_{[\cdot, \cdot, \cdot, \cdot]}$$

is continuous.

Observe that Proposition 4.10 applies to the subset $S_\Gamma = H_\Gamma$ in particular. In the proof of Proposition 4.10, we will focus on a special subset of $S_\Gamma^{[4]}$: we say that a quadruple $(a, b, c, d) \in S_\Gamma^{[4]}$ is *good* if $\{\bar{a}, \bar{d}\} \cap I_{[b,c]} = \emptyset$ and $\{\bar{b}, \bar{c}\} \cap I_{[d,a]} = \emptyset$.

Lemma 4.11. *Let $[\cdot, \cdot, \cdot, \cdot] \in \mathcal{C}\mathcal{R}^+(S_\Gamma)$, and let μ be the associated current. For every good quadruple $(a, b, c, d) \in S_\Gamma^{[4]}$, we have*

$$\mu(I_{(d,a)} \times I_{(b,c)}) = \mu(I_{[d,a]} \times I_{[b,c]}) = [a, b, c, d].$$

Proof. This follows immediately from the inequalities in Proposition 4.3 (1) and Lemma 4.8 applied to $\{d\} \times I_{[b,c]}$, $\{a\} \times I_{[b,c]}$, $I_{[d,a]} \times \{b\}$ and $I_{[d,a]} \times \{c\}$. ■

Proof of Proposition 4.10. Let $[\cdot, \cdot, \cdot, \cdot]_n$ and $[\cdot, \cdot, \cdot, \cdot]$ be positive crossratios on S_Γ and μ_n, μ the associated geodesic currents. Assume that

$$\lim_n [\cdot, \cdot, \cdot, \cdot]_n = [\cdot, \cdot, \cdot, \cdot].$$

We have to show that for every positive continuous function f on $(\partial\mathcal{H}^2)^{(2)}$ with compact support, $\lim_{n \rightarrow \infty} \mu_n(f) = \mu(f)$. Using a finite partition of unity, we may assume that the support $\text{supp}(f)$ is contained in some rectangle $I_{[d,a]} \times I_{[b,c]}$ with (a, b, c, d) positive.

Fix an interval $I_{[a_0, b_0]} \subset I_{(a,b)}$ and observe that for every open set $J \subset I_{(b_0, a_0)}$, the set

$$S_J := \{\xi \in J \cap S_\Gamma : \bar{\xi} \in I_{[a_0, b_0]}\}$$

is dense in J . Choose then $a' \in S_{(a, a_0)}, b' \in S_{(b_0, b)}, \{c', d'\} \subset S_{(c, d)}$ with (a', b', c', d') positive and observe that the quadruple (a', b', c', d') is good (see Figure 4).

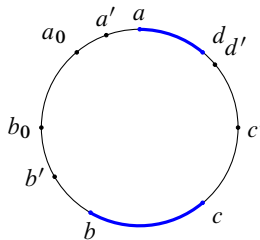


Figure 4. The configuration of points in the proof of Proposition 4.10.

Fix some distance inducing the topology on $(\partial\mathcal{H}^2)^{(2)}$, fix $\varepsilon > 0$, and let $\delta > 0$ be such that if $R \subset (\partial\mathcal{H}^2)^{(2)}$ is any closed rectangle of diameter $\text{diam}(R) < \delta$, then

$$\max_R f - \min_R f < \varepsilon.$$

Fix now a cover of $\text{supp}(f)$ by closed rectangles $R_i = I_{[x_i, y_i]} \times I_{[z_i, w_i]}$, for $i = 1, 2, \dots, N$, such that

- (1) $\text{diam}(R_i) < \delta$;
- (2) the interiors of the rectangles are pairwise disjoint;
- (3) $R_i \subset I_{(d', a')} \times I_{(b', c')}$;
- (4) $\{x_i, y_i\} \subset S_{(d', a')}$ and $\{z_i, w_i\} \subset S_{(b', c')}$.

Observe that for every $1 \leq i \leq N$, the quadruple (x_i, z_i, w_i, y_i) is good. As a result, we have (see Lemma 4.11)

- (5) $\mu(R_i) = [x_i, z_i, w_i, y_i], \mu_n(R_i) = [x_i, z_i, w_i, y_i]_n;$
- (6) $\mu(R_i \cap R_j) = 0, \mu_n(R_i \cap R_j) = 0$ for all $i \neq j$.

It follows then that

$$\left| \int f d\mu_n - \sum_{i=1}^N (\min_{R_i} f) \mu_n(R_i) \right| \leq \sum_{i=1}^N \varepsilon \mu_n(R_i) \leq \varepsilon \mu_n(I_{[d', a']} \times I_{[b', c']}) = \varepsilon [a', b', c', d']_n$$

and similarly,

$$\left| \int f d\mu - \sum_{i=1}^N (\min_{R_i} f) \mu(R_i) \right| \leq \varepsilon [a', b', c', d'].$$

From these inequalities, the assumption that $\lim_n [\cdot, \cdot, \cdot, \cdot]_n = [\cdot, \cdot, \cdot, \cdot]$ and (5), we deduce that

$$\overline{\lim}_{n \rightarrow \infty} \left| \int f d\mu - \int f d\mu_n \right| \leq 2\varepsilon [a', b', c', d'],$$

and hence

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu. \quad \blacksquare$$

4.4. Integral crossratios

The goal of the section is to prove the following assertion.

Proposition 4.12. *If the positive crossratio*

$$[\cdot, \cdot, \cdot, \cdot]: X^{[4]} \rightarrow [0, \infty)$$

takes values in the non-negative integers $\mathbf{N} = \{0, 1, 2, \dots\}$, then the associated current corresponds to an integral geodesic multicurve.

Proof. We first show that μ is purely atomic. It follows from Proposition 4.3 (3) and (4) that $\mu(R) \in \mathbf{N}$ for all proper open and proper closed rectangles. Let $g \in \text{supp}(\mu) \subset (\partial \mathcal{H}^2)^{(2)}$, and let R_n be a decreasing sequence of open rectangles with $\cap R_n = \{g\}$. Since $\mu(R_n) \in \mathbf{N}$, we have either

- (1) $\mu(R_n) \geq 1$ for all $n \geq 1$ and $\mu(\{g\}) \geq 1$ or
- (2) there exists n_0 such that $\mu(R_n) = 0$ for all $n \geq n_0$.

The second case cannot happen since g is in the support of μ . As a result, μ is purely atomic with \mathbf{N} -valued atoms.

We now show that all geodesics in the support of μ are either closed or connect two cusps. Let $(a, b) \in (\partial \mathcal{H}^2)^{(2)}$ be such an atom. Then $\Gamma \cdot (a, b)$ meets every compact subset

of $(\partial\mathcal{H}^2)^{(2)}$ in only finitely many points. As a result, if $g \subset \mathcal{H}^2$ is the geodesic connecting (a, b) , $\text{pr}(g) \subset \Sigma$ is a closed subset, where, as always, $\text{pr}: \mathcal{H}^2 \rightarrow \Sigma$ denotes the universal covering map. Thus either g corresponds to a periodic geodesic or $\text{pr}(g)$ is a geodesic connecting two cusps.

There is a compact subset $K \subset \Sigma$ such that every biinfinite geodesic, as well as every closed geodesic, meets K . Thus if \mathcal{A} denotes the set of atoms of μ , there is a compact subset $C \subset (\partial\mathcal{H}^2)^{(2)}$ such that for all $a \in \mathcal{A}$, $\Gamma \cdot a \cap C \neq \emptyset$. This implies that

$$\mu = \sum_{c \in F} n_c \delta_c,$$

where F is a finite set of geodesics either periodic or connecting to cusps, δ_c is the geodesic current corresponding to c , and $n_c \in \mathbf{N}^*$. ■

4.5. Restriction to a subsurface

We conclude the section discussing how the construction of the geodesic current associated to a positive crossratio behaves with respect to restriction to subsurfaces. This will be useful in the study of maximal representations.

Let $\Sigma' \subset \Sigma$ be a subsurface with geodesic boundary. Let $\mathcal{G}(\Sigma') \subset (\partial\mathcal{H}^2)^{(2)}$ be the set of geodesics whose projection lies in the interior $\overset{\circ}{\Sigma}'$ of Σ' . If μ is a current on Σ , we define $\mu|_{\Sigma'} \in \mathcal{C}(\Sigma)$ by

$$\mu|_{\Sigma'} := \chi_{\mathcal{G}(\Sigma')} \mu,$$

where $\chi_{\mathcal{G}(\Sigma')}$ is the characteristic function of $\mathcal{G}(\Sigma')$.

We write $i(\mu, \partial\Sigma') = 0$ when $i(\mu, c) = 0$ for every boundary component c of Σ' . This is the case precisely when no geodesic in the support of μ intersects $\partial\Sigma'$; thus in that case, we have

$$i(\mu, c) = i(\mu|_{\Sigma'}, c)$$

for every closed geodesic c contained in Σ' .

We now choose a finite area hyperbolization $\Sigma_0 = \Gamma_0 \backslash \mathcal{H}^2$ of $\overset{\circ}{\Sigma}'$ and a corresponding identification $h: \Gamma_0 \rightarrow \pi_1(\Sigma') < \Gamma$. We denote by $\phi: \partial\mathcal{H}^2 \rightarrow \partial\mathcal{H}^2$ an injective, monotone h -equivariant map. This is a quasi-conjugacy that opens all the cusps corresponding to geodesic boundary components of Σ' . The preceding discussion leads to the following result.

Proposition 4.13. *Let μ be a current on Σ . Then*

$$\mu_0(A) := \mu((\phi \times \phi)(A))$$

defines a current $\mu_0 = \phi^ \mu$ on Σ_0 , that we will call the current induced by μ on Σ_0 . We have the following properties:*

- (1) $\phi^* \mu = \phi^* \mu|_{\Sigma'}$.
- (2) If $\mu|_{\Sigma'}$ has compact carrier included in $\overset{\circ}{\Sigma}'$, then μ_0 has compact carrier in Σ_0 .

(3) Let μ, ν be currents on Σ . Assume that $\mu|_{\Sigma'}$ and $\nu|_{\Sigma'}$ have compact carrier included in $\overset{\circ}{\Sigma}'$. Then

$$i(\mu|_{\Sigma'}, \nu|_{\Sigma'}) = i(\mu_0, \nu_0).$$

(4) We have

$$i(\mu_0, \gamma) = i(\mu|_{\Sigma'}, h(\gamma))$$

for all hyperbolic $\gamma \in \Gamma_0$. In particular, if $i(\mu, \partial\Sigma') = 0$, then

$$i(\mu_0, \gamma) = i(\mu, h(\gamma))$$

for all hyperbolic $\gamma \in \Gamma_0$.

(5) Assume that $i(\mu, \partial\Sigma') = 0$. If μ is the current associated to a positive crossratio $b \in \mathcal{C}\mathcal{R}(H_\Gamma)$, then μ_0 is the current associated to the positive crossratio $b_0 = \phi^*b$ in $\mathcal{C}\mathcal{R}(H_{\Gamma_0})$.

5. Equivariant tree embeddings

In this section, we discuss barycenter maps compatible with positive crossratios and prove Theorem 1.9. In Section 5.2, we discuss a first class of actions to which Theorem 1.9 applies: framed actions on trees.

5.1. Actions with compatible crossratio and barycenter

Let $\rho: \Gamma \rightarrow \text{Isom}(\mathcal{X})$ be an action by isometries on a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and $X \subset \partial\mathcal{H}^2$ be a non-empty Γ -invariant subset.

Definition 5.1. (1) A \mathcal{X} -valued barycenter map on X (or just a barycenter map if the context is clear) is a map

$$\beta: X^{(3)} \rightarrow \mathcal{X}$$

defined on the set $X^{(3)}$ of triples of distinct points in X that verifies

- (a) β is S_3 -invariant;
- (b) β is Γ -equivariant.

(2) A barycenter map is compatible with a positive crossratio $[\cdot, \cdot, \cdot, \cdot]$ defined on X if, whenever $(a, b, c, d) \in X^{[4]}$,

$$[a, b, c, d] = d_{\mathcal{X}}(\beta(a, b, d), \beta(a, c, d)).$$

Condition (2) is inspired by the construction of a crossratio induced by a framed action on a tree, as in Example 3.5. Indeed, we have the following.

Example 5.2. Let $\rho: \Gamma \rightarrow \text{Isom}(\mathcal{T})$ be a framed action of Γ on a real tree \mathcal{T} with framing $\varphi: X \rightarrow \partial_\infty\mathcal{T}$. Then the usual barycenter $B: \partial_\infty\mathcal{T}^{(3)} \rightarrow \mathcal{T}$ induces a barycenter map on X compatible with the positive crossratio $[\cdot, \cdot, \cdot, \cdot]_\varphi$.

The goal of the section is to prove the following assertion.

Theorem 5.3. *Let $\rho: \Gamma \rightarrow \text{Isom}(X)$ be an isometric action. Assume that there are a positive crossratio on X and a compatible barycenter map, that the geodesic current μ associated to the positive crossratio is of lamination type, and let $\mathcal{V}(\mu)$ be the set of vertices of the \mathbf{R} -tree $\mathcal{T}(\mu)$ associated to μ . Then there is an equivariant isometric embedding*

$$\mathcal{V}(\mu) \hookrightarrow X.$$

In order to define the embedding, we will show that each complementary region of the geodesic lamination $\tilde{\mathcal{L}} := \text{supp}(\mu)$ leads to a well-defined barycenter. In this section, we will work with complementary regions \mathcal{R} of the lamination $\tilde{\mathcal{L}}$, namely connected components of the complement of $\tilde{\mathcal{L}}$ in \mathcal{H}^2 . Given such a complementary region \mathcal{R} , we will show in Proposition 5.5 that the map that to three points $a, b, c \in (\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)) \cap X$ associates their barycenter is constant if the points are in different connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$. Here $\mathcal{R}(\infty)$ is the intersection of $\partial\mathcal{H}^2$ with the closure of \mathcal{R} in $\overline{\mathcal{H}^2}$. In Lemma 5.7, we show that the obtained map is isometric.

As a first step in the proof of Proposition 5.5, we show that the crossratio of 4-tuples separated by the lamination vanishes.

Lemma 5.4. *Let \mathcal{R} be a complementary region of $\tilde{\mathcal{L}}$, and let (a, b, c, d) be positively oriented such that $\{a, b, c, d\} \subset \partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$ and $\{a, b\}$, as well as $\{c, d\}$, are in different connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$. Then*

$$\mu(I_{[d,a]} \times I_{[b,c]}) = 0.$$

In particular, if in addition $(a, b, c, d) \in X^{[4]}$,

$$[a, b, c, d] = 0.$$

Proof. Since $\text{supp}(\mu) = \tilde{\mathcal{L}}$, we have that $\mu((\partial\mathcal{H}^2)^{(2)} \setminus \tilde{\mathcal{L}}) = 0$. Thus it suffices to show that under the hypotheses of the lemma,

$$I_{[d,a]} \times I_{[b,c]} \subset (\partial\mathcal{H}^2)^{(2)} \setminus \tilde{\mathcal{L}}.$$

Assume there is a geodesic $g \in \tilde{\mathcal{L}}$ connecting $I_{[d,a]}$ to $I_{[b,c]}$. Then \mathcal{R} must be contained in one of the half-planes determined by g , and hence either $\{a, b\}$ or $\{c, d\}$ are contained in the same connected component of $(\partial\mathcal{H}^2)^{(2)} \setminus \mathcal{R}(\infty)$, contradicting the hypothesis. The second statement follows directly from Proposition 4.3 (1). ■

With the use of Lemma 5.4, we proceed to define the barycenter of a complementary region of $\tilde{\mathcal{L}}$.

Proposition 5.5. *Let \mathcal{R} be a complementary region of $\tilde{\mathcal{L}}$. Then $\beta(a, b, c)$ is independent of the choice of $\{a, b, c\} \subset X$, provided a, b, c lie in three distinct connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$.*

Proof. We split the proof in three easy steps.

Step (1). Given three connected components I_1, I_2, I_3 of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$, $\beta(a, b, c)$ is independent of the choices $a \in I_1 \cap X, b \in I_2 \cap X$ and $c \in I_3 \cap X$.

Indeed, given $\{a, a'\} \subset I_1 \cap X$, we may assume, modulo exchanging b and c , and also a and a' , that $(c, a, a', b) \in X^{[4]}$. By Lemma 5.4, we have then that

$$d_X(\beta(c, a, b), \beta(c, a', b)) = [c, a, a', b] = 0.$$

Thus $\beta(a, b, c) = \beta(a', b, c)$.

Given now $b' \in I_2 \cap X$ and $c' \in I_3 \cap X$, we conclude, using the S_3 -invariance of β , that

$$\beta(a', b', c') = \beta(a', b', c) = \beta(a', b, c) = \beta(a, b, c).$$

Step (2). Let I_1, I_2, I_3, I_4 be distinct components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$, $\{a, b, c, c'\} \subset X$ with $a \in I_1, b \in I_2, c \in I_3$ and $c' \in I_4$. Then $\beta(a, b, c) = \beta(a, b, c')$.

We distinguish two cases.

(2a) If c, c' are in the same connected component of $\partial\mathcal{H}^2 \setminus \{a, b\}$, then possibly upon permuting a, b and c, c' , we may assume that $(b, c, c', a) \in X^{[4]}$. By Lemma 5.4, this implies that

$$d_X(\beta(b, c, a), \beta(b, c', a)) = [b, c, c', a] = 0.$$

Thus $\beta(b, c, a) = \beta(b, c', a)$.

(2b) If instead c, c' are in distinct connected components of $\partial\mathcal{H}^2 \setminus \{a, b\}$, we may assume, possibly permuting a, b and c, c' , that (a, c', b, c) is positively oriented. Using (2a) in the second and fourth equalities, we obtain

$$\beta(a, b, c') = \beta(a, c', b) = \beta(a, c', c) = \beta(c, a, c') = \beta(c, a, b) = \beta(a, b, c),$$

which shows the assertion.

Step (3). We finish now the proof of the proposition. Let $\{a, b, c, a', b', c'\} \subset X$ with $a \in I_1, b \in I_2, c \in I_3, a' \in I'_1, b' \in I'_2$ and $c' \in I'_3$, where I_1, I_2, I_3 and I'_1, I'_2, I'_3 are distinct connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$. We can assume, up to reordering the indices that $I_j \neq I_k$ for $j \neq k$. Then it follows from (1) and (2) that

$$\beta(a, b, c) = \beta(a', b, c) = \beta(a', b', c) = \beta(a', b', c'). \quad \blacksquare$$

Definition 5.6. The *barycenter* $\beta(\mathcal{R})$ of a complementary region \mathcal{R} of $\tilde{\mathcal{L}}$ is the point $\beta(a, b, c)$ for any choice $\{a, b, c\} \subset X$ of points lying in distinct components of $\partial\mathcal{H}^2 \setminus \mathcal{R}(\infty)$.

Taking into account the discussion in Section 2.3, the following lemma concludes the proof of Theorem 5.3.

Lemma 5.7. For the distance d_μ on the set $\mathcal{V}(\mu)$ of complementary regions of $\tilde{\mathcal{L}}$, we have

$$d_X(\beta(\mathcal{R}_1), \beta(\mathcal{R}_2)) = d_\mu(\mathcal{R}_1, \mathcal{R}_2)$$

for all $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{V}(\mu)$.

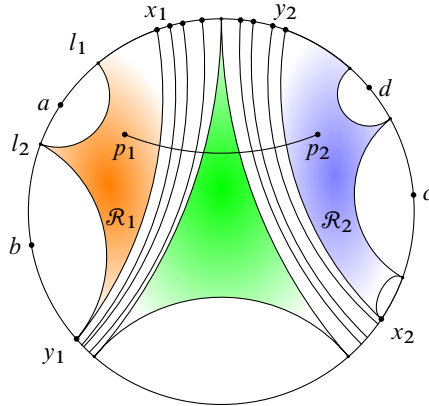


Figure 5. The proof of Lemma 5.7.

Proof. Let (x_1, y_1) be the endpoints of the geodesic in $\partial\mathcal{R}_1$ separating \mathcal{R}_1 from \mathcal{R}_2 and (x_2, y_2) the endpoints of the geodesic in $\partial\mathcal{R}_2$ separating \mathcal{R}_2 from \mathcal{R}_1 , ordered so that $(x_1, y_1, x_2, y_2) \in X^{[4]}$. Choose $a, b \in I_{(x_1, y_1)}$ in different connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}_1(\infty)$ and $c, d \in I_{(x_2, y_2)}$ in different connected components of $\partial\mathcal{H}^2 \setminus \mathcal{R}_2(\infty)$ in such a way that $(a, b, c, d) \in X^{[4]}$ (see Figure 5).

Then

$$d_{\mathcal{X}}(\beta(\mathcal{R}_1), \beta(\mathcal{R}_2)) = d_{\mathcal{X}}(\beta(a, b, d), \beta(a, c, d)) = [a, b, c, d].$$

Since the geodesics bounding \mathcal{R}_1 and \mathcal{R}_2 are all μ -short, we have

$$\mu(\{a\} \times I_{[b, c]}) \leq \mu(I_{(l_1, l_2)} \times I_{(l_2, l_1)}) = 0,$$

where $\{l_1, l_2\}$ is the geodesic in $\partial\mathcal{R}_1$ separating a from \mathcal{R}_1 , and similarly

$$\mu(\{d\} \times I_{[b, c]}) = 0.$$

As a result, from Proposition 4.3 (1) we have the equality

$$\mu(I_{[d, a]} \times I_{[b, c]}) = [a, b, c, d] = \mu(I_{(d, a)} \times I_{(b, c)}).$$

If now $p_i \in \mathcal{R}_i$, the set of leaves in $\tilde{\mathcal{L}}$ that intersect the segment (p_1, p_2) is exactly the set of leaves in $\tilde{\mathcal{L}}$ that separate \mathcal{R}_1 from \mathcal{R}_2 . This is also the same as the set of leaves in $\tilde{\mathcal{L}}$ that connect $I_{[d, a]}$ to $I_{[b, c]}$. The assertion then follows from the above considerations, recalling that $d_{\mu}(\mathcal{R}_1, \mathcal{R}_2)$ is the measure of this set,

$$\begin{aligned} d_{\mathcal{X}}(\beta(\mathcal{R}_1), \beta(\mathcal{R}_2)) &= [a, b, c, d] = \mu(I_{[d, a]} \times I_{[b, c]}) \\ &= d_{\mu}(\mathcal{R}_1, \mathcal{R}_2). \end{aligned}$$

■

5.2. Framed actions on trees

Theorem 5.3 applies to framed actions on trees.

Proposition 5.8. *Let $\rho: \Gamma \rightarrow \text{Isom}(\mathcal{T})$ be an action by isometries on a real tree \mathcal{T} with a framing $\varphi: X \rightarrow \partial_\infty \mathcal{T}$. Suppose that the associated crossratio $[\cdot, \cdot, \cdot, \cdot]_\varphi$ is positive, and denote by μ_ρ the associated geodesic current. Then μ_ρ corresponds to a measured lamination, and there is a Γ -equivariant isometric embedding*

$$\mathcal{T}(\mu_\rho) \hookrightarrow \mathcal{T}.$$

In particular, for all hyperbolic $\gamma \in \Gamma$,

$$\ell_{\mathcal{T}}(\rho(\gamma)) = i(\mu_\rho, \gamma).$$

Proof. By Proposition 3.8, the crossratio $[\cdot, \cdot, \cdot, \cdot]_\varphi$ is ultrametric, hence by Proposition 4.7, the current μ_ρ is of lamination type.

The barycenter $\beta(x, y, z)$ of $(x, y, z) \in X^{(3)}$ is the point $\beta_{\mathcal{T}}(x, y, z)$ in the tree \mathcal{T} which is barycenter of $(\varphi(x), \varphi(y), \varphi(z))$. Then β is by construction an equivariant barycenter map compatible with the crossratio (by (3.3) in Example 3.5 and S_3 -invariance of $\beta_{\mathcal{T}}$), hence induces an equivariant isometry

$$\Psi: \mathcal{V}(\mu_\rho) \hookrightarrow \mathcal{T}$$

of the vertices of the dual tree $\mathcal{T}(\mu_\rho)$ by Theorem 5.3. Since \mathcal{T} is uniquely geodesic, we can extend Ψ to $\mathcal{T}(\mu_\rho)$. Then for all $\gamma \in \Gamma$, we have $\ell_{\mathcal{T}}(\rho(\gamma)) = \ell_{\Psi(\mathcal{T}(\mu_\rho))}(\gamma)$ as $\Psi(\mathcal{T}(\mu_\rho))$ is a convex subset of \mathcal{T} , and $\ell_{\Psi(\mathcal{T}(\mu_\rho))}(\gamma) = \ell_{\mathcal{T}(\mu_\rho)}(\gamma)$ as Ψ is isometric. Now for hyperbolic γ representing a closed geodesic c , we have $\ell_{\mathcal{T}(\mu_\rho)}(\gamma) = i(\mu_\rho, \delta_c)$, hence $\ell_{\mathcal{T}}(\rho(\gamma)) = i(\mu_\rho, \gamma)$. ■

6. The geometry of the Siegel spaces over real closed fields

The goal of this section is to recall facts about the geometry of Siegel spaces over real closed fields needed to show that maximal framed actions give rise to a positive crossratio and admit a compatible barycenter (Lemma 6.6 and Proposition 6.7).

6.1. Real closed fields

Recall that an ordered field is a field \mathbf{F} endowed with a total order relation \leq satisfying

- (1) if $x \leq y$, then $x + z \leq y + z$ for all $z \in \mathbf{F}$;
- (2) if $0 \leq x$ and $0 \leq y$, then $0 \leq xy$.

The fields \mathbf{Q} and \mathbf{R} with their usual order are examples of ordered fields; while some fields admit no ordering, like \mathbf{C} , others admit many, like $\mathbf{R}(X)$.

Example 6.1. The orders on $\mathbf{R}(X)$ admit the following description for $\varepsilon \in \{-, +\}$:

- (1) $>_{\varepsilon\infty}$: if $f \in \mathbf{R}(X)$, $f >_{\varepsilon\infty} 0$ if $f(t) > 0$ for $t \rightarrow \varepsilon\infty$.
- (2) $>_{a\varepsilon}$ for $a \in \mathbf{R}$: we say that $f >_{a+} 0$ if there exists $\eta = \eta(f) \in \mathbf{R}$, $\eta > 0$, with $f(t) > 0$ on the interval $(a, a + \eta)$, and $f >_{a-} 0$ if $f(t) > 0$ on the interval $(a - \eta, a)$.

A basic fact is that every ordered field \mathbf{F} admits a real closure $\overline{\mathbf{F}}^r$, that is, a maximal algebraic extension of \mathbf{F} to which the order extends. Such a real closure is then unique up to a unique \mathbf{F} -isomorphism. An ordered field \mathbf{F} is then *real closed* if the ordering does not extend to any proper algebraic extension. Two useful characterizations are the following:

- (1) the field \mathbf{F} is ordered and $\mathbf{F}(t)$ is algebraically closed, with $t = \sqrt{-1}$;
- (2) the field \mathbf{F} is ordered, every positive element is a square and any odd degree polynomial has a root.

An important consequence of (2), which is implicit in most of the geometric properties of the Siegel space we use, is that any symmetric matrix with coefficient in a real closed field is orthogonally similar to a diagonal one.

An **R-valued valuation** is a map

$$v: \mathbf{F} \rightarrow \mathbf{R} \cup \{\infty\},$$

where $v: (\mathbf{F}^\times, \cdot) \rightarrow (\mathbf{R}, +)$ is a group homomorphism, $v(0) = \infty$, and if we define the *norm* of $x \in \mathbf{F}$ by $\|x\|_v := e^{-v(x)}$ if $x \neq 0$ and $\|0\|_v := 0$, then

$$\|x + y\|_v \leq \|x\|_v + \|y\|_v.$$

An important valuation to keep in mind is $-\ln(|\cdot|): \mathbf{R} \rightarrow \mathbf{R} \cup \{\infty\}$. Indeed, valuations offer a replacement for the logarithm in more general real closed fields. The valuation is *order compatible* if $|x| \leq |y|$ implies that $v(x) \geq v(y)$, and it is *non-Archimedean* if $v(x + y) \geq \min\{v(x), v(y)\}$, that is if $\|x + y\|_v \leq \max\{\|x\|_v, \|y\|_v\}$.

Example 6.2. The following are examples of ordered fields:

- (1) The field \mathbf{R} of real numbers and the field $\overline{\mathbf{Q}}^r$ of real algebraic numbers; both are real closed.
- (2) Let ω be a non-principal ultrafilter on \mathbf{N} . The quotient \mathbf{R}_ω of the ring $\mathbf{R}^{\mathbf{N}}$ by the equivalence relation $(x_n) \sim (y_n)$ if $\omega(\{n : x_n = y_n\}) = 1$, ordered in such a way that positive elements are the classes of the sequences (x_n) such that $\omega(\{n : x_n > 0\}) = 1$, is a real closed field called the field of the *hyperreals*. It does not admit any order compatible \mathbf{R} -valued valuation.
- (3) Let $\sigma \in \mathbf{R}_\omega$ be a positive infinitesimal, that is, σ can be represented by a sequence $(\sigma_n)_{n \geq 0}$ with $\lim \sigma_n = 0$ and $\sigma_n > 0$. Then

$$\mathcal{O}_\sigma := \{x \in \mathbf{R}_\omega : |x| < \sigma^{-k} \text{ for some } k \in \mathbf{Z}\}$$

is a valuation ring with maximal ideal

$$\mathcal{I}_\sigma = \{x \in \mathbf{R}_\omega : |x| < \sigma^k \text{ for all } k \in \mathbf{Z}\}.$$

The quotient $\mathbf{R}_{\omega,\sigma} := \mathcal{O}_\sigma / \mathcal{I}_\sigma$ is a real closed field, called the *Robinson field*. It admits an order compatible valuation

$$v(x) = \lim_\omega \frac{\ln |x_n|}{\ln \sigma_n},$$

where $(x_n)_{n \geq 0}$ represents x , that leads to a non-Archimedean norm

$$\|x\|_v := e^{-v(x)}.$$

(4) Let G be a totally ordered Abelian group, and let

$$\mathcal{H}(G) := \left\{ f = \sum_{\sigma \in G} a_\sigma X^\sigma : a_\sigma \in \mathbf{R} \text{ and } \text{supp}(f) \subset G \text{ is well ordered} \right\}$$

be the set of formal power series with exponents in G and coefficients in \mathbf{R} , where $\text{supp}(f) := \{\sigma \in G : a_\sigma \neq 0\}$. This is an \mathbf{R} -vector space, and the restriction on supports allows one to define a ring structure that extends the ordinary multiplication on the group ring $\mathbf{R}[G]$ of G . In fact, $\mathcal{H}(G)$ turns out to be a field, called the *Hahn field with exponents* G . It is ordered by setting $f > 0$ if $a_{\sigma_0} > 0$, where $\sigma_0 = \min(\text{supp}(f))$ and it admits a G -valued compatible valuation $v(f) = \sigma_0$. Moreover, $\mathcal{H}(G)$ is real closed if G is divisible. For all of the above statements, see [14, Theorem 2.15].

(5) (Compare with Section 8.3) If $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, then the ring morphism

$$\mathbf{R}[x, y] \rightarrow \mathcal{H}(\mathbf{R}), \quad P \mapsto P(x, x^\alpha)$$

extends to $\mathbf{R}(x, y)$. In this way, we obtain an order on $\mathbf{R}(x, y)$ for which $P \in \mathbf{R}(x, y)$ is positive if and only if for some $\varepsilon > 0$, $P(t, t^\alpha) > 0$ for all $t \in (0, \varepsilon)$.

6.2. The Siegel upper half-space and the space $\mathcal{B}_\mathbf{F}^n$

Let \mathbf{F} be a real closed field, and ι be a square root of -1 . Endow $V = \mathbf{F}^{2n}$ with the standard symplectic form

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle := {}^t x_1 y_2 - {}^t y_1 x_2,$$

where $x_i, y_i \in \mathbf{F}^n$. The vector space $\text{Sym}_n(\mathbf{F})$ of symmetric matrices admits a partial order defined by setting

$$X \ll Y \quad \text{if } Y - X \text{ is positive definite.}$$

The *Siegel upper half-space* is the semi-algebraic set

$$\mathcal{S}_\mathbf{F}^n := \{Z = X + \iota Y : X, Y \in \text{Sym}_n(\mathbf{F}) \text{ and } Y \gg 0\}$$

on which

$$\mathrm{Sp}(2n, \mathbf{F}) := \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tAD - {}^tCB = \mathrm{Id}, {}^tAC = {}^tCA, {}^tBD = {}^tDB \right\}$$

acts by fractional linear transformations

$$g_*Z := (AZ + B)(CZ + D)^{-1}, \tag{6.1}$$

transitively. Of course, this action descends to an action of $\mathrm{P}\mathrm{Sp}(2n, \mathbf{F})$. The stabilizer of ${}^t\mathrm{Id}_n \in \mathcal{S}_{\mathbf{F}}^n$ in $\mathrm{Sp}(2n, \mathbf{F})$ is

$$K = \mathrm{Sp}(2n, \mathbf{F}) \cap \mathrm{O}(2n) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : {}^tAA + {}^tBB = \mathrm{Id}_n, {}^tAB \text{ is symmetric} \right\}.$$

If $\mathbf{F} = \mathbf{R}$, then $\mathcal{S}_{\mathbf{R}}^n$ is the symmetric space associated to $\mathrm{P}\mathrm{Sp}(2n, \mathbf{R})$. If, instead, the real closed field \mathbf{F} is endowed with an order compatible non-Archimedean valuation v , then $\mathrm{P}\mathrm{Sp}(2n, \mathbf{F})$ acts by isometries on a $v(\mathbf{F})$ -metric space $\mathcal{B}_{\mathbf{F}}^n$: a metric quotient of $\mathcal{S}_{\mathbf{F}}^n$ whose construction we now recall. See [10, 11] for generalizations of this construction.

On $\mathcal{S}_{\mathbf{F}}^n$, we define a multiplicative \mathbf{F} -valued distance function as follows. Since \mathbf{F} is real closed, any pair (Z_1, Z_2) with $Z_i \in \mathcal{S}_{\mathbf{F}}^n$, for $i = 1, 2$, is $\mathrm{P}\mathrm{Sp}(2n, \mathbf{F})$ -congruent to a unique pair $({}^t\mathrm{Id}_n, {}^tD)$, where $D = \mathrm{diag}(d_1, \dots, d_n)$, $d_1 \geq \dots \geq d_n \geq 1$ in \mathbf{F} . We then set

$$D(Z_1, Z_2) := \prod_{i=1}^n d_i.$$

Proposition 6.3. *The function D is a $\mathrm{P}\mathrm{Sp}(2n, \mathbf{F})$ -invariant multiplicative distance function on $\mathcal{S}_{\mathbf{F}}^n$, namely, for all $Z_1, Z_2, Z_3 \in \mathcal{S}_{\mathbf{F}}^n$,*

- (MD1) $D(Z_1, Z_2) \in \mathbf{F}_{\geq 1}$, with equality of and only if $Z_1 = Z_2$;
- (MD2) $D(Z_1, Z_2) = D(Z_2, Z_1)$;
- (MD3) $D(Z_1, Z_2) \leq D(Z_1, Z_3)D(Z_3, Z_2)$.

Proof. (MD1) and (MD2) are clear. We consider the standard action of $\mathrm{Sp}(2n, \mathbf{F})$ on $W = \bigwedge^n(\mathbf{F}^{2n})$, endowed with the standard scalar product, which is K -invariant as $K \subset \mathrm{O}(2n)$. For $a = \mathrm{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1})$ with $a_1 \geq \dots \geq a_n \geq 1$ in \mathbf{F} , we easily see that

$$\max_{w \in W, w \neq 0} \frac{\|aw\|}{\|w\|} = \prod_{i=1}^n a_i$$

(the biggest eigenvalue of a in W). For $g \in \mathrm{Sp}(2n, \mathbf{F})$, the operator norm of g on W is given by

$$\|g\| := \max_{w \in W, w \neq 0} \frac{\|gw\|}{\|w\|}.$$

Since $g = kak'$ for some k, k' in K and a as before (by the Cartan decomposition), and

$$D(\iota \text{Id}_n, g_* \iota \text{Id}_n) = D(\iota \text{Id}_n, a_* \iota \text{Id}_n) = 2\|a\| = 2\|g\|,$$

(MD3) follows from submultiplicativity of the operator norm and transitivity of the action of $\text{Sp}(2n, \mathbf{F})$. ■

On $\mathcal{S}_{\mathbf{F}}^n$, we define an *associated pseudodistance* d^1 as follows:

$$d^1(Z_1, Z_2) := -v(D(Z_1, Z_2)) = -\sum_{i=1}^n v(d_i). \tag{6.2}$$

The triangle inequality for d^1 comes from (MD3). We denote by $\mathcal{B}_{\mathbf{F}}^n$ the metric quotient of $\mathcal{S}_{\mathbf{F}}^n$ with respect to the pseudodistance d^1 . We will not need it, but note in passing that $\mathcal{B}_{\mathbf{F}}^n$ can be identified with the quotient $\text{PSp}(2n, \mathbf{F}) / \text{PSp}(2n, \mathbf{U})$, where $\mathbf{U} := \{x \in \mathbf{F} : \|x\|_v \leq 1\}$.

6.3. Embedding in \mathbf{K} -Lagrangians

In the classical case, the Borel embedding of $\mathcal{S}_{\mathbf{R}}^n$ into the complex Grassmannian provides a way to endow $\mathcal{S}_{\mathbf{R}}^n$ with structures defined on the Grassmannian, such as, for example, a crossratio. We recall from [13] the analogous picture in the case of a general real closed field.

Let $\mathbf{K} = \mathbf{F}(\iota)$ be the algebraic closure of \mathbf{F} , and let us also denote by $\langle \cdot, \cdot \rangle$ the \mathbf{K} -linear extension of the standard symplectic form to \mathbf{K}^{2n} and by $\sigma: \mathbf{K}^{2n} \rightarrow \mathbf{K}^{2n}$ the complex conjugation. Given matrices $Z_1, Z_2 \in M_n(\mathbf{K})$, we will denote by $\langle \left(\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix} \right) \rangle$ the subspace of \mathbf{K}^{2n} generated by the column vectors. We denote by $\mathcal{L}(\mathbf{K}^{2n})$ the submanifold of $\text{Gr}_n(\mathbf{K}^{2n})$ consisting of subspaces that are isotropic for the form $\langle \cdot, \cdot \rangle$. The map

$$\text{Sym}_n(\mathbf{K}) \rightarrow \mathcal{L}(\mathbf{K}^{2n}), \quad Z \mapsto \left\langle \begin{pmatrix} Z \\ \text{Id}_n \end{pmatrix} \right\rangle$$

gives a bijection between $\text{Sym}_n(\mathbf{K})$ and the subset of $\mathcal{L}(\mathbf{K}^{2n})$ of all Lagrangians transverse to

$$\ell_{\infty} := \left\langle \begin{pmatrix} \text{Id}_n \\ 0 \end{pmatrix} \right\rangle.$$

This map intertwines the action of $\text{PSp}(2n, \mathbf{K})$ on $\text{Sym}_n(\mathbf{K})$ by fractional linear transformations (6.1) and the standard action on $\mathcal{L}(\mathbf{K}^{2n})$. This bijection maps $\mathcal{S}_{\mathbf{F}}^n$ to the projective model

$$\mathcal{D}_{\mathbf{F}} := \{L \in \mathcal{L}(\mathbf{K}^{2n}) : -\iota \langle \cdot, \sigma(\cdot) \rangle|_{L \times L} \gg 0\}$$

and sends $\text{Sym}_n(\mathbf{F})$ to

$$\{\ell \otimes \mathbf{K} : \ell \in \mathcal{L}(\mathbf{F}^{2n}), \ell \text{ is transverse to } \ell_{\infty}\}.$$

6.4. Maximal triples and intervals

We associate to a triple (ℓ_1, ℓ_2, ℓ_3) of pairwise transverse Lagrangians in $\mathcal{L}(\mathbf{F}^{2n})$ the quadratic form $Q_{(\ell_1, \ell_2, \ell_3)}$ on ℓ_1 defined by

$$Q_{(\ell_1, \ell_2, \ell_3)}(v) := \langle v, v' \rangle,$$

where $v' \in \ell_3$ is the unique vector such that $v + v' \in \ell_2$. If $\ell = \langle \langle \begin{smallmatrix} X \\ \text{Id}_n \end{smallmatrix} \rangle \rangle$ and $\ell' = \langle \langle \begin{smallmatrix} X' \\ \text{Id}_n \end{smallmatrix} \rangle \rangle$ are pairwise transverse, then in the coordinates

$$\mathbf{F}^n \rightarrow \ell, \quad w \mapsto \begin{pmatrix} X \\ \text{Id}_n \end{pmatrix} w, \tag{6.3}$$

the quadratic form $Q_{(\ell, \ell', \ell_\infty)}$ is represented by $X' - X$.

Two triples of pairwise transverse Lagrangians (ℓ_1, ℓ_2, ℓ_3) and (m_1, m_2, m_3) are $\text{PSp}(2n, \mathbf{F})$ -congruent if and only if the associated quadratic spaces $(\ell_1, Q_{(\ell_1, \ell_2, \ell_3)})$ and $(m_1, Q_{(m_1, m_2, m_3)})$ are isomorphic or, equivalently, if and only if $Q_{(\ell_1, \ell_2, \ell_3)}$ and $Q_{(m_1, m_2, m_3)}$ have the same signature [13, Proposition 2.5]. The value of the *Maslov cocycle* on (ℓ_1, ℓ_2, ℓ_3) is the signature of $Q_{(\ell_1, \ell_2, \ell_3)}$

$$\tau(\ell_1, \ell_2, \ell_3) := \text{sign } Q_{(\ell_1, \ell_2, \ell_3)}.$$

The triple (ℓ_1, ℓ_2, ℓ_3) is *maximal* if $\tau(\ell_1, \ell_2, \ell_3) = n$, the maximal value the Maslov cocycle can take. Similarly, we say that a triple (ℓ_1, ℓ_2, ℓ_3) is *minimal* if $\tau(\ell_1, \ell_2, \ell_3) = -n$; it is easy to verify that (ℓ_1, ℓ_2, ℓ_3) is maximal if and only if (ℓ_2, ℓ_1, ℓ_3) is minimal. The group $\text{PSp}(2n, \mathbf{F})$ acts transitively on pairs of transverse Lagrangians in $\mathcal{L}(\mathbf{F}^{2n})$ and on maximal triples.

Given $\ell, \ell' \in \mathcal{L}(\mathbf{F}^{2n})$, we define the *interval*

$$I_{(\ell, \ell')} := \{m \in \mathcal{L}(\mathbf{F}^{2n}) : (\ell, m, \ell') \text{ is maximal}\}.$$

Lemma 6.4 (Compare [13, Lemma 2.10]). *Let $\ell = \langle \langle \begin{smallmatrix} X \\ \text{Id}_n \end{smallmatrix} \rangle \rangle$ and $\ell' = \langle \langle \begin{smallmatrix} X' \\ \text{Id}_n \end{smallmatrix} \rangle \rangle$. If the triple $(\ell, \ell', \ell_\infty)$ is maximal, then*

$$I_{(\ell, \ell')} = \left\{ \left\langle \left\langle \begin{pmatrix} Y \\ \text{Id}_n \end{pmatrix} \right\rangle \right\rangle : X \ll Y \ll X' \right\}.$$

Proof. Let $m = \langle \langle \begin{smallmatrix} Y \\ \text{Id}_n \end{smallmatrix} \rangle \rangle$. Using the maximality of $(\ell, \ell', \ell_\infty)$, the cocycle relation applied to the quadruple $(\ell, m, \ell', \ell_\infty)$ reads

$$\tau(m, \ell', \ell_\infty) + \tau(\ell, m, \ell_\infty) = \tau(\ell, m, \ell') + n.$$

From this relation, it follows that (ℓ, m, ℓ') is maximal if and only if (m, ℓ', ℓ_∞) and (ℓ, m, ℓ_∞) are both maximal, that is, $Y \ll X'$ and $X \ll Y$. ■

For $X, X' \in \text{Sym}_n(\mathbf{F})$ with $X \ll X'$, we will also denote by $I_{(X, X')}$ the set

$$I_{(X, X')} := \{Y \in \text{Sym}_n(\mathbf{F}) : X \ll Y \ll X'\}$$

and by $I_{(X, \infty)}$ the set

$$I_{(X, \infty)} := \{Y \in \text{Sym}_n(\mathbf{F}) : X \ll Y\}.$$

6.5. Crossratios

In this subsection, we recall the endomorphism valued crossratio from [13, §4.1] on quadruples of Lagrangians. This, together with a maximal framing, will allow us in Section 7.1 to associate to any maximal framed representation ρ a positive crossratio as in Section 3.

Given a quadruple of Lagrangians $(\ell_1, \ell_2, \ell_3, \ell_4)$ in $\mathcal{L}(\mathbf{F}^{2n})$ with $\ell_1 \cap \ell_2 = \ell_3 \cap \ell_4 = \{0\}$, their *crossratio* is the endomorphism of ℓ_1 given by

$$R(\ell_1, \ell_2, \ell_3, \ell_4) = p_{\ell_1}^{\|\ell_2} \circ p_{\ell_4}^{\|\ell_3}|_{\ell_1},$$

where $p_{\ell_j}^{\|\ell_i}$ denotes the projection of \mathbf{F}^{2n} to ℓ_j parallel to the complementary space ℓ_i . One verifies that for all $g \in \text{Sp}(2n, \mathbf{F})$,

$$R(g\ell_1, g\ell_2, g\ell_3, g\ell_4) = gR(\ell_1, \ell_2, \ell_3, \ell_4)g^{-1},$$

and hence

$$\det R(\ell_1, \ell_2, \ell_3, \ell_4)$$

is $\text{PSp}(2n, \mathbf{F})$ -invariant. If, for $j \in \{1, 2, 3, 4\}$, $\ell_j \cap \ell_\infty = \{0\}$, then $\ell_j = \langle (\begin{smallmatrix} X_j \\ \text{Id}_n \end{smallmatrix}) \rangle$ for some $X_j \in \text{Sym}_n(\mathbf{F})$. By [13, Lemma 4.2], the matrix representing $R(\ell_1, \ell_2, \ell_3, \ell_4)$ in the basis of ℓ_1 given by (6.3) is

$$(X_1 - X_2)^{-1}(X_2 - X_4)(X_3 - X_4)^{-1}(X_1 - X_3). \tag{6.4}$$

We will denote such matrix by $R(\ell_1, \ell_2, \ell_3, \ell_4)$, with an abuse of notation.

Proposition 6.5 (Compare [13, Lemma 4.4]). *Let $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ be pairwise transverse Lagrangians. Then*

- (1) $R(\ell_1, \ell_2, \ell_4, \ell_5) = R(\ell_1, \ell_2, \ell_3, \ell_5)R(\ell_1, \ell_3, \ell_4, \ell_5)$.
- (2) $R(\ell_1, \ell_2, \ell_3, \ell_4)$ is conjugate to $R(\ell_3, \ell_4, \ell_1, \ell_2)$.
- (3) We have that $\det R(\ell_1, \ell_2, \ell_4, \ell_5) = \det R(\ell_1, \ell_2, \ell_3, \ell_5) \det R(\ell_1, \ell_3, \ell_4, \ell_5)$ and $\det R(\ell_1, \ell_2, \ell_3, \ell_4) = \det R(\ell_3, \ell_4, \ell_1, \ell_2)$.
- (4) If $(\ell_1, \ell_2, \ell_3, \ell_4)$ is a maximal quadruple, then all eigenvalues of $R(\ell_1, \ell_2, \ell_3, \ell_4)$ belong to \mathbf{F} and are strictly larger than one. In particular, $\det R(\ell_1, \ell_2, \ell_3, \ell_4) > 1$.

Proof. (1) The right-hand side of the identity equals

$$p_{\ell_1}^{\|\ell_2} p_{\ell_5}^{\|\ell_3} p_{\ell_1}^{\|\ell_3} p_{\ell_5}^{\|\ell_4}|_{\ell_1}.$$

Using that $\mathbf{F}^{2n} = \ell_1 + \ell_3$, one verifies that $p_{\ell_5}^{\|\ell_3} p_{\ell_1}^{\|\ell_3}|_{\ell_5} = \text{Id}_{\ell_5}$, which implies the assertion.

- (2) This was shown in [13, Lemma 4.4 (3)].
- (3) These are immediate consequences of (1) and (2).

(4) Since $\mathrm{Sp}(2n, \mathbf{F})$ acts transitively on maximal triples, we may assume that

$$\ell_1 = \left\langle \begin{pmatrix} 0 \\ \mathrm{Id}_n \end{pmatrix} \right\rangle, \quad \ell_2 = \left\langle \begin{pmatrix} \mathrm{Id}_n \\ \mathrm{Id}_n \end{pmatrix} \right\rangle, \quad \ell_3 = \left\langle \begin{pmatrix} Y \\ \mathrm{Id}_n \end{pmatrix} \right\rangle \quad \text{and} \quad \ell_4 = \ell_\infty.$$

By maximality, $\mathrm{Id}_n \ll Y$ (see Lemma 6.4). A computation gives $R(\ell_1, \ell_2, \ell_3, \ell_4) = Y$, which concludes the proof. ■

6.6. \mathbf{F} -tubes and orthogonal projections

If $\ell, \ell' \in \mathcal{L}(\mathbf{F}^{2n})$, the \mathbf{F} -tube determined by ℓ, ℓ' is the semi-algebraic subset of $S_{\mathbf{F}}^n$ given by the equation

$$\mathcal{Y}_{\ell, \ell'} := \left\{ Z \in S_{\mathbf{F}}^n : R\left(\ell, \left\langle \begin{pmatrix} Z \\ \mathrm{Id}_n \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \bar{Z} \\ \mathrm{Id}_n \end{pmatrix} \right\rangle, \ell'\right) = -\mathrm{Id}_n \right\},$$

where \bar{Z} is the complex conjugate of Z , see [13, §4.2].

If $\mathbf{F} = \mathbf{R}$ and $n = 1$, the \mathbf{R} -tube $\mathcal{Y}_{\ell, \ell'}$ is the geodesic between ℓ and ℓ' while, for $n \geq 1$, $\mathcal{Y}_{\ell, \ell'}$ is a symmetric subspace of $S_{\mathbf{R}}^n$. The \mathbf{R} -tube $\mathcal{Y}_{\ell, \ell'}$ is a Lagrangian submanifold and is isometric to the symmetric space associated to $\mathrm{GL}(n, \mathbf{R})$. In general, for all $g \in \mathrm{P}\mathrm{Sp}(2n, \mathbf{F})$,

$$\mathcal{Y}_{g\ell, g\ell'} = g(\mathcal{Y}_{\ell, \ell'}), \tag{6.5}$$

and if we denote

$$\ell_0 := \left\langle \begin{pmatrix} 0 \\ \mathrm{Id}_n \end{pmatrix} \right\rangle,$$

then

$$\mathcal{Y}_{\ell_0, \ell_\infty} := \{tY : Y \in \mathrm{Sym}_n(\mathbf{F}), Y \gg 0\}.$$

We will often write $\mathcal{Y}_{0, \infty}$ for $\mathcal{Y}_{\ell_0, \ell_\infty}$.

If $(\ell_1, \ell_2, \ell_3, \ell_4)$ is a maximal quadruple, then $\mathcal{Y}_{\ell_1, \ell_3}$ and $\mathcal{Y}_{\ell_2, \ell_4}$ meet exactly in one point. Such \mathbf{F} -tubes are called *orthogonal* if

$$R(\ell_1, \ell_2, \ell_3, \ell_4) = 2 \mathrm{Id}_n$$

(see [13, Proposition 4.7 and Definition 4.14]). If $\mathbf{F} = \mathbf{R}$, the tubes are orthogonal if and only if they are orthogonal as submanifolds of the Riemannian manifold $S_{\mathbf{R}}^n$. Given any point $p \in I_{(\ell_1, \ell_3)} \cup I_{(\ell_3, \ell_1)}$, there exists a unique \mathbf{F} -tube $\mathcal{Y}_{\ell_2, \ell_4}$ orthogonal to $\mathcal{Y}_{\ell_1, \ell_3}$ “with endpoint p ” in the following sense:

$$\begin{cases} \ell_2 := p & \text{if } p \in I_{(\ell_1, \ell_3)}, \\ \ell_4 := p & \text{if } p \in I_{(\ell_3, \ell_1)}. \end{cases}$$

In this way, we obtain a map

$$\mathrm{pr}_{\mathcal{Y}_{\ell_1, \ell_3}} : I_{(\ell_1, \ell_3)} \cup I_{(\ell_3, \ell_1)} \rightarrow \mathcal{Y}_{\ell_1, \ell_3}$$

called the *orthogonal projection*.

In the case of $\mathcal{Y}_{0,\infty}$, the map $\text{pr}_{\mathcal{Y}_{0,\infty}}$ is given by

$$\text{pr}_{\mathcal{Y}_{0,\infty}}(Y) = \begin{cases} \iota Y & \text{if } Y \in I_{(\ell_0, \ell_\infty)}, \\ -\iota Y & \text{if } Y \in I_{(\ell_\infty, \ell_0)}. \end{cases}$$

In view of (6.5), this implies in particular that the restrictions of $\text{pr}_{\mathcal{Y}_{\ell_1, \ell_3}}$ to $I_{(\ell_1, \ell_3)}$ and $I_{(\ell_3, \ell_1)}$ are both bijective.

Lemma 6.6. *Assume $(\ell, \ell_1, \ell_2, \ell')$ is maximal. Then for the pseudodistance d^1 on $\mathcal{S}_{\mathbf{F}}^n$ (see (6.2) in Section 6.2), we have*

$$d^1(\text{pr}_{\mathcal{Y}_{\ell, \ell'}}(\ell_1), \text{pr}_{\mathcal{Y}_{\ell, \ell'}}(\ell_2)) = -v(\det R(\ell, \ell_1, \ell_2, \ell')).$$

Proof. We may assume that $\ell = \ell_0$ and $\ell' = \ell_\infty$, and set $\ell_j := \langle (\frac{Y_j}{\text{Id}_n}) \rangle$ for $j = 1, 2$. Then we have $0 \ll Y_1 \ll Y_2$, and in particular the eigenvalues r_1, \dots, r_n of the symmetric matrix $Y_1^{-1/2} Y_2 Y_1^{-1/2}$ are all greater than 1. By invariance of the distance d^1 , we have

$$\begin{aligned} d^1(\iota Y_1, \iota Y_2) &= d^1(\iota \text{Id}_n, \iota Y_1^{-1/2} Y_2 Y_1^{-1/2}) = \sum_{j=1}^n -v(r_j) = -v\left(\prod_{j=1}^n r_j\right) \\ &= -v(\det(Y_1^{-1} Y_2)) = -v(\det R(\ell, \ell_1, \ell_2, \ell')), \end{aligned}$$

where the last equality uses (6.4). This proves the lemma. ■

6.7. Barycenters

Assume that \mathbf{F} is non-Archimedean. If the triple of Lagrangians (ℓ_1, ℓ_2, ℓ_3) is either maximal or minimal, then $\ell_2 \in I_{(\ell_1, \ell_3)} \cup I_{(\ell_3, \ell_1)}$. We define

$$b(\ell_1, \ell_2, \ell_3) := \text{pr}_{\mathcal{Y}_{\ell_1, \ell_3}}(\ell_2) \in \mathcal{S}_{\mathbf{F}}^n.$$

The *barycenter* of the triple (ℓ_1, ℓ_2, ℓ_3) is the point

$$B(\ell_1, \ell_2, \ell_3) = \pi(b(\ell_1, \ell_2, \ell_3)),$$

where $\pi: \mathcal{S}_{\mathbf{F}}^n \rightarrow \mathcal{B}_{\mathbf{F}}^n$ is the metric quotient introduced in Section 6.2. Our goal is to show the following (cf. [13, Lemma 7.5]).

Proposition 6.7. *The point $B(\ell_1, \ell_2, \ell_3)$ is invariant under permutation of the arguments.*

First we need to establish a formula for the (pseudo-)distance in $\mathcal{S}_{\mathbf{F}}^1$.

Lemma 6.8. *If $z_1 = x_1 + \iota y_1, z_2 = x_2 + \iota y_2 \in \mathcal{S}_{\mathbf{F}}^1$, then*

$$d(z_1, z_2) = \max\left\{-v\left(\frac{(x_1 - x_2)^2}{y_1 y_2}\right), -v\left(\frac{y_1}{y_2}\right), -v\left(\frac{y_2}{y_1}\right)\right\}.$$

Proof. Recall that for a general real closed field \mathbf{F} , if $z_1, z_2 \in \mathcal{S}_{\mathbf{F}}^1$,

$$d(z_1, z_2) = \ln \|T + \sqrt{T^2 - 1}\|_v,$$

where $T = 1 + \frac{|z_1 - z_2|^2}{2y_1 y_2}$. Since \mathbf{F} is non-Archimedean, then for $a, b \in \mathbf{F}$, $a \geq 0, b \geq 0$, we have

$$\|a + b\|_v = \|\max\{a, b\}\|_v.$$

Hence

$$\|T + \sqrt{T^2 - 1}\|_v = \|T\|_v \quad \text{and} \quad d(z_1, z_2) = \ln \|T\|_v.$$

Since

$$T = 1 + \frac{|z_1 - z_2|^2}{2y_1 y_2} = \frac{(x_1 - x_2)^2}{2y_1 y_2} + \frac{y_1}{2y_2} + \frac{y_2}{2y_1},$$

then

$$\|T\|_v = \max \left\{ \left\| \frac{(x_1 - x_2)^2}{y_1 y_2} \right\|_v, \left\| \frac{y_1}{y_2} \right\|_v, \left\| \frac{y_2}{y_1} \right\|_v \right\},$$

where we took into account that $\|n\|_v = 1$ for any $n \in \mathbf{Z} \setminus \{0\}$. ■

Proof of Proposition 6.7. As $\text{PSP}(2n, \mathbf{F})$ acts transitively on maximal and minimal triples, we may assume that $(\ell_1, \ell_2, \ell_3) := (\ell_0, \langle (\begin{smallmatrix} Y \\ \text{Id}_n \end{smallmatrix}) \rangle, \ell_\infty)$, where $Y = \pm \text{Id}_n$, depending on whether (ℓ_1, ℓ_2, ℓ_3) is maximal or minimal. A computation then gives

$$\text{pr}_{y_{0,\infty}}(\pm \text{Id}_n) = \iota \text{Id}_n, \quad \text{pr}_{y_{\ell_\infty, \ell_2}}(0) = (\pm 1 + \iota) \text{Id}_n, \quad \text{pr}_{y_{\ell_2, \ell_0}}(\infty) = \frac{\pm 1 + \iota}{2} \text{Id}_n.$$

Observe that if $Z = \text{diag}(z_1, \dots, z_n)$, $W = \text{diag}(w_1, \dots, w_n) \in \mathcal{S}_{\mathbf{F}}^n$, then

$$d^1(Z, W) = \sum_{i=1}^n d(z_i, w_i).$$

Thus in order to compute the distances between various projections, we just have to compute the following in $\mathcal{S}_{\mathbf{F}}^1$:

$$\begin{aligned} d(\iota, \iota \pm 1) &= \ln \max\{\|1\|_v, \|\iota\|_v, \|\iota \pm 1\|_v\} = 0, \\ d\left(\iota, \left(\frac{1 + \iota}{2}\right)\right) &= \ln \max\left\{\left\|\frac{1}{2}\right\|_v, \left\|\frac{1 + \iota}{2}\right\|_v, \|2\|_v\right\} = 0. \end{aligned}$$

This concludes the proof. ■

7. Applications to maximal framed representations

In this section, we prove Theorems 1.2 and 1.5, as well as Corollaries 1.3 and 1.8 from the introduction.

7.1. The geodesic current μ_ρ and Theorem 1.2

Let $\rho: \Gamma \rightarrow \mathrm{PSp}(2n, \mathbf{F})$ be a maximal framed representation with maximal framing $\varphi: H_\Gamma \rightarrow \mathcal{L}(\mathbf{F}^{2n})$. Define then for every $(x_1, x_2, x_3, x_4) \in H_\Gamma^{[4]}$,

$$[x_1, x_2, x_3, x_4]_\rho := -v(\det R(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))).$$

It follows from Proposition 6.5 that $[\cdot, \cdot, \cdot, \cdot]_\rho$ is a positive crossratio on H_Γ , and hence (Propositions 4.3 and 4.9) there is a geodesic current μ_ρ such that $\mathrm{per}(\gamma) = i(\mu_\rho, \delta_c)$ for every closed geodesic c represented by a hyperbolic element $\gamma \in \Gamma$.

Next, for the computation of $\mathrm{per}(\gamma)$, recall that $g \in \mathrm{Sp}(2n, \mathbf{F})$ is called *Shilov hyperbolic* if there is a g -invariant decomposition $\mathbf{F}^{2n} = \ell_+ \oplus \ell_-$ into Lagrangians such that all eigenvalues of $g|_{\ell_-}$ have absolute value strictly smaller than one (and thus all eigenvalues of $g|_{\ell_+}$ have absolute value strictly larger than one) [13, Definition 2.12]. An element $g \in \mathrm{PSp}(2n, \mathbf{F})$ is *Shilov hyperbolic* if any of its lifts to $\mathrm{Sp}(2n, \mathbf{F})$ is. Of course, the decomposition is uniquely determined by g and does not depend on the lift. We will need the following assertion.

Lemma 7.1 ([13, Lemma 7.9]). *Let $g \in \mathrm{Sp}(2n, \mathbf{F})$ be Shilov hyperbolic, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $g|_{\ell_+}$. Then for any $\ell \in I(\ell_-, \ell_+)$, we have*

$$\det R(\ell_-, \ell, g\ell, \ell_+) = \left(\prod_{i=1}^n \lambda_i \right)^2.$$

Proof. We may assume by transitivity of the action of $\mathrm{Sp}(2n, \mathbf{F})$ on pairs of transverse Lagrangians that $(\ell_-, \ell_+) = (\ell_0, \ell_\infty)$. Furthermore, $g = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, where A is the matrix of $g|_{\ell_+}$. Then ℓ corresponds to a matrix $X \in \mathrm{Sym}_n(\mathbf{F})$ with $X \gg 0$, and (6.4) gives

$$R(\ell_-, \ell, g\ell, \ell_+) = X^{-1}AXA^t.$$

This implies that

$$\det R(\ell_-, \ell, g\ell, \ell_+) = (\det A)^2,$$

hence the lemma. ■

Let $\gamma \in \Gamma$ be hyperbolic with attractive and repulsive fixed points γ_+ and γ_- . It follows from [13, Theorem 1.9] that $\rho(\gamma)$ is Shilov hyperbolic with corresponding decomposition $\mathbf{F}^{2n} = \varphi(\gamma_-) \oplus \varphi(\gamma_+)$. In particular, the maximal framing $\varphi: H_\Gamma \rightarrow \mathcal{L}(\mathbf{F}^{2n})$ (recall Definition 1.1) is uniquely determined by the representation ρ . Furthermore, if $\lambda_1(\gamma), \dots, \lambda_n(\gamma)$ are the eigenvalues of $\rho(\gamma)$ with $|\lambda_1(\gamma)| \geq \dots \geq |\lambda_n(\gamma)| \geq 1$, then Lemma 7.1 implies that given $x \in I(\gamma_-, \gamma_+)$,

$$\begin{aligned} \mathrm{per}(\gamma) &= -v(\det R(\varphi(\gamma_-), \varphi(x), \rho(\gamma)\varphi(x), \varphi(\gamma_+))) \\ &= 2 \sum_{i=1}^n -v(\lambda_i(\gamma)) = L(\rho(\gamma)) \end{aligned}$$

for the length function $L: \mathrm{PSp}(2n, \mathbf{F}) \rightarrow \mathbf{R}$ introduced in equation (1.1).

If there is a hyperbolic element $\gamma \in \Gamma$ with $-v(\text{tr}(\rho(\gamma))) > 0$, then necessarily $\rho(\gamma)$ has an eigenvalue with the same property and thus $\text{per}(\gamma) > 0$. Vice-versa, if μ_ρ is non-zero, there exists a proper closed rectangle $R = I_{[a,b]} \times I_{[c,d]}$ with $\mu_\rho(R) > 0$. Since Γ is a lattice, we find a hyperbolic element $\gamma \in \Gamma$ with $\gamma_+ \in I_{(b,c)}$, $\gamma_- \in I_{(d,a)}$. For such γ , the intersection $i(\mu_\rho, \delta_\gamma) > 0$, and thus $\text{per}(\gamma) > 0$, which implies that $\|\lambda_1(\gamma) \dots \lambda_n(\gamma)\|_v > 1$. Observe that the coefficients of the characteristic polynomial of $\bigwedge^n \rho(\gamma)$ are symmetric functions in the eigenvalues of $\rho(\gamma)$ and hence polynomials with rational coefficients of $\text{tr}(\rho(\gamma^s))$, $s \in \mathbf{N}$; if now we assume by contradiction that $-v(\text{tr}(\rho(\gamma^s))) \leq 0$ for all $s \in \mathbf{N}$, that is $\text{tr}(\rho(\gamma^s))$ belong to the ring $\mathcal{O} := \{x \in \mathbf{F} : \|x\|_v \leq 1\}$, we conclude since $\mathbf{Q} \subset \mathcal{O}$ that the coefficients of the characteristic polynomial of $\bigwedge^n \rho(\gamma)$ belong to \mathcal{O} as well. But $\lambda_1(\gamma) \dots \lambda_n(\gamma) \in \mathbf{F}$ is a root of this monic polynomial, and since $\mathcal{O} \subset \mathbf{F}$ is a valuation ring, it is integrally closed in \mathbf{F} (see [16, Theorem 3.1.3(1)]). This implies $\|\lambda_1(\gamma) \dots \lambda_n(\gamma)\|_v \leq 1$, a contradiction, that concludes the proof of Theorem 1.2.

7.2. Displacing representations and Corollary 1.3

Assume that Σ is compact, and let $\rho: \Gamma \rightarrow \text{PSp}(2n, \mathbf{F})$ be a maximal framed representation. If

$$\text{Syst}(\rho) := \inf_{\gamma \neq e} L(\rho(\gamma)) > 0,$$

it follows from Theorem 1.2 that the associated geodesic current μ_ρ has $\text{Syst}(\mu_\rho) > 0$; hence there exist c_1, c_2 such that for every $\gamma \in \Gamma$,

$$c_1 \ell(\gamma) \leq L(\rho(\gamma)) \leq c_2 \ell(\gamma), \tag{7.1}$$

where $\ell(\gamma)$ is the hyperbolic length of γ [9, Theorem 1.3]. The Γ -action on $\mathcal{B}_{\mathbf{F}}^n$ induced by ρ is displacing by (7.1). It follows then from [15, Theorem 2.0.2 and Lemma 2.0.1] that for every $x \in \mathcal{B}_{\mathbf{F}}^n$, the map $\gamma \mapsto \rho(\gamma)x$ is a quasi-isometric embedding. This proves Corollary 1.3.

7.3. Maximal representations in $\text{PSp}(2n, \mathbf{R})$ and Corollary 1.8

If Σ is not necessarily compact, the inequalities in (7.1) do not necessarily hold. However, applying [9, Corollary 1.5 (2)] to the current μ_ρ , we deduce the following.

Corollary 7.2. *Let $\rho: \Gamma \rightarrow \text{PSp}(2n, \mathbf{F})$ be a representation admitting a maximal framing defined on H_Γ . Assume that $\text{Syst}_\Sigma(\rho) > 0$. Then for every compact subset $K \subset \Sigma$, there are constants $0 < c_1 \leq c_2$ such that*

$$c_1 \ell(c) \leq L(\rho(\gamma)) \leq c_2 \ell(c)$$

for every $\gamma \in \Gamma$ representing a closed geodesic c contained in K . In particular, there exist constants c_1, c_2 such that this holds for all γ representing simple closed geodesics.

Assume now that $\mathbf{F} = \mathbf{R}$, and let $\rho: \Gamma \rightarrow \text{PSp}(2n, \mathbf{R})$ be a maximal representation. Then there is a maximal framing φ defined on $\partial \mathcal{H}^2$ [12, Theorem 8] and hence, by Theorem 1.2,

a geodesic current μ_ρ with $i(\mu_\rho, \delta_c) = L(\rho(\gamma))$, for every closed geodesic c represented by a hyperbolic element $\gamma \in \Gamma$. The collar lemma [13, Theorem 1.9] then implies that if γ, η are intersecting hyperbolic elements,

$$(e^{(L(\rho(\gamma)))/n} - 1)(e^{(L(\rho(\eta)))/n} - 1) \geq 1.$$

This implies that, if γ is self-intersecting,

$$L(\rho(\gamma)) \geq n(\ln 2),$$

and that there are at most $3g - 3 + p$ conjugacy classes of hyperbolic elements γ with $L(\rho(\gamma)) \leq n(\ln 2)$. In particular, $\text{Syst}(\mu_\rho) > 0$. Corollary 1.8 then follows from Corollary 7.2.

7.4. Lamination type currents and Theorem 1.5

Let \mathbf{F} be non-Archimedean. Given a maximal framed representation, as above, and with the notation of Section 6.7, define the *barycenter* of $(x, y, z) \in H_\Gamma^{(3)}$

$$\beta_\rho(x, y, z) := B(\varphi(x), \varphi(y), \varphi(z)) \in \mathcal{B}_\mathbf{F}^n.$$

It follows from Proposition 6.7 that β is indeed a barycenter according to Definition 5.1 and from Lemma 6.6 that it is compatible with the crossratio $[\cdot, \cdot, \cdot, \cdot]_\rho$ defined in Section 7.1.

If now μ_ρ is of lamination type, we deduce from Theorem 5.3 that there is a well-defined equivariant isometric embedding

$$\mathcal{V}(\mu_\rho) \hookrightarrow \mathcal{B}_\mathbf{F}^n$$

from the set of vertices $\mathcal{V}(\mu_\rho)$ of the \mathbf{R} -tree $\mathcal{T}(\mu_\rho)$ associated to μ_ρ into the metric space $\mathcal{B}_\mathbf{F}^n$.

7.5. The value group of $[\cdot, \cdot, \cdot, \cdot]_\rho$ and Theorem 1.4

Let $\rho: \Gamma \rightarrow \text{Sp}(2n, \mathbf{F})$ be maximal framed with framing $\varphi: H_\Gamma \rightarrow \mathcal{L}(\mathbf{F}^{2n})$, and set as always

$$[x_1, x_2, x_3, x_4]_\rho := -v(\det R(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))).$$

Theorem 7.3. *Let $\Lambda := v(\mathbf{Q}(\rho))$ be the value group of the field $\mathbf{Q}(\rho)$ generated over \mathbf{Q} by the matrix coefficients of ρ . Then*

$$[x_1, x_2, x_3, x_4]_\rho \in \frac{1}{(8n)!} \Lambda$$

for all $(x_1, x_2, x_3, x_4) \in H_\Gamma^{[4]}$.

Proof. We might assume that \mathbf{F} is the real closure of $\mathbf{Q}(\rho)$. Let $(x_1, x_2, x_3, x_4) \in H_{\mathbf{F}}^{[4]}$, and let γ_i be hyperbolic with $(\gamma_i)_- = x_i$. If $c_{\gamma_1}, c_{\gamma_2}, c_{\gamma_3}, c_{\gamma_4}$ are the characteristic polynomials of $\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3), \rho(\gamma_4)$, then $c_{\gamma_i} \in \mathbf{Q}(\rho)[X]$. Since $\mathbf{F}[i]$ is algebraically closed, $\rho(\gamma_i)$ splits in $\mathbf{F}[i]$. If \mathbf{L} is the splitting field in $\mathbf{F}[i]$ of $c_{\gamma_1}c_{\gamma_2}c_{\gamma_3}c_{\gamma_4} \in \mathbf{Q}(\rho)[X]$, then

$$[\mathbf{L} : \mathbf{Q}(\rho)] \leq (8n)!$$

Observe that the field \mathbf{L} depends on $\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3), \rho(\gamma_4)$.

It is now easy to see that the Lagrangians $\varphi(x_i) \subset \mathbf{F}^{2n}$ are defined over $\mathbf{L} \cap \mathbf{F}$, as a result we can represent them by $\langle (\begin{smallmatrix} X_i \\ \text{Id}_n \end{smallmatrix}) \rangle$ with $X_i \in \text{Sym}_n(\mathbf{L} \cap \mathbf{F})$, which implies that $\det R(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)) \in (\mathbf{L} \cap \mathbf{F})^\times$. We conclude using [30, Chapter XII, §4, Proposition 12] which says that the index of Λ in $v((\mathbf{L} \cap \mathbf{F})^\times)$ is at most $(8n)!$. This concludes the proof. ■

In particular, if $\mathbf{Q}(\rho)$ has discrete valuation, we can assume, up to rescaling the valuation, that the crossratio $[x_1, x_2, x_3, x_4]_\rho$ is integer valued. Theorem 1.4 is therefore a direct application of Proposition 4.12.

8. Examples of maximal framed representations

In this section, we collect several interesting examples of maximal framed representations over non-Archimedean real closed fields.

8.1. Ultralimits of representations and asymptotic cones

Let $(\rho_k)_{k \geq 1}$ be a sequence of maximal representations into $\text{Sp}(2n, \mathbf{R})$ and ω a non-principal ultrafilter on \mathbf{N} . This gives rise to a representation $\rho_\omega: \Gamma \rightarrow \text{Sp}(2n, \mathbf{R}_\omega)$, where \mathbf{R}_ω is the field of hyperreals and $\rho_\omega(\Gamma) \subset \text{Sp}(2n, \mathcal{O}_\sigma)$, where the infinitesimal σ is defined below. Denote by $\mathbf{R}_{\omega, \sigma}$ the Robinson field. The representation $\rho_{\omega, \sigma}$ obtained by composing ρ_ω with the projection

$$\text{Sp}(2n, \mathbf{R}_\omega) \rightarrow \text{Sp}(2n, \mathbf{R}_{\omega, \sigma})$$

is a maximal framed representation of Γ into $\text{Sp}(2n, \mathbf{R}_{\omega, \sigma})$ [13, Corollary 10.4]; its framing is defined on $\partial \mathcal{H}^2$, and Theorem 1.2 applies.

This construction is closely related to asymptotic cones, as we now recall. Denoting by d the $\text{Sp}(2n, \mathbf{R})$ -invariant Riemannian distance on the Siegel n -space, we say that a sequence of scales $(\lambda_k)_{k \in \mathbf{N}} \in (\mathbf{R}_{>0})^\mathbf{N}$ is *adapted* (to the sequence $(\rho_k)_{k \in \mathbf{N}}$) if for one, and hence every, finite generating set $S \subset \Gamma$

$$\lim_{\omega} \frac{\max_{\gamma \in S} d(\rho_k(\gamma) \iota \text{Id}_n, \iota \text{Id}_n)}{\lambda_k} < +\infty.$$

We obtain then an action ${}^\omega \rho_\lambda: \Gamma \rightarrow \text{Isom}({}^\omega \mathcal{X}_\lambda)$, on the asymptotic cone ${}^\omega \mathcal{S}_\lambda$ of the sequence of pointed metric spaces given by $(\mathcal{S}_{\mathbf{R}}^n, \iota \text{Id}_n, \frac{d}{\lambda_k})$.

If we set $\sigma := (e^{-\lambda_k})_{k \geq 1} \in \mathbf{R}_\omega$, then the asymptotic cone ${}^\omega \mathcal{X}_\lambda$ can be identified with the metric space $\mathcal{B}_{\mathbf{R}_\omega, \sigma}^n$ and, under this identification, ${}^\omega \rho_\lambda$ corresponds to $\rho_{\omega, \sigma}$ (see, for example, [37]).

8.2. Maximal representations in $\mathrm{SL}(2, \mathbf{F})$

Let \mathbf{F} be a real closed field with an order compatible non-Archimedean valuation, and let $\mathcal{T}^{\mathbf{F}} \supset \mathcal{B}_{\mathbf{F}}^1$ be the \mathbf{R} -tree associated to $\mathrm{SL}_2(\mathbf{F})$. Then $\mathbb{P}^1(\mathbf{F})$ identifies with a subset of $\partial_\infty \mathcal{T}^{\mathbf{F}}$ and the restriction to $\mathbb{P}^1(\mathbf{F})$ of the crossratio of $\partial_\infty \mathcal{T}^{\mathbf{F}}$ is the standard crossratio $[\cdot, \cdot, \cdot, \cdot]_{\mathbf{F}}$ in $\mathbb{P}^1(\mathbf{F})$.

Therefore, any representation $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{F})$ with framing $\varphi: H_\Gamma \rightarrow \mathbb{P}^1(\mathbf{F})$, gives a framed action on $\mathcal{T}^{\mathbf{F}}$. Note that the associated crossratio

$$[x_1, x_2, x_3, x_4]_\varphi = [\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)]_{\mathbf{F}}$$

is positive if the framing φ is maximal.

Proposition 5.8 implies the following.

Theorem 8.1. *Let \mathbf{F} be a real closed field with an order compatible non-Archimedean valuation. Let $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{F})$ be a representation with a maximal framing $\varphi: H_\Gamma \rightarrow \mathbb{P}^1(\mathbf{F})$. Denote by μ_ρ the geodesic current associated to the positive crossratio induced by φ on X . Then μ_ρ corresponds to a measured lamination, and there is a Γ -equivariant isometric embedding*

$$\mathcal{V}(\mu_\rho) \hookrightarrow \mathcal{T}^{\mathbf{F}}.$$

In particular, for all hyperbolic $\gamma \in \Gamma$, $\ell(\rho(\gamma)) = i(\mu_\rho, \gamma)$.

8.3. Unipotent representations of the thrice punctured sphere

Let $\Gamma < \mathrm{PSL}(2, \mathbf{R})$ be the (unique up to conjugation) lattice such that $\Gamma \backslash \mathcal{H}^2$ is the thrice punctured sphere. Then Γ admits a presentation

$$\Gamma = \langle c_1, c_2, c_3 : c_3 c_2 c_1 \rangle,$$

where c_1, c_2, c_3 are parabolic elements representing the three inequivalent cusps of Γ . Already, in this elementary example we are able to illustrate interesting features. For every $\alpha \in \mathbf{R}$, we construct maximal framed representations $\rho_\alpha: \Gamma \rightarrow \mathrm{Sp}(4, \mathcal{H}(\mathbf{R}))$, where $\mathcal{H}(\mathbf{R})$ is the Hahn field with exponents \mathbf{R} (see Example 6.2(4)), that have the following properties:

- (1) for $\alpha \leq 1/2$, the corresponding length functions $\gamma \mapsto L(\rho_\alpha(\gamma))$ are not proportional, and hence the corresponding currents μ_{ρ_α} are distinct in the space of projectivized currents;
- (2) for $\alpha \in \mathbf{Q}$, the associated geodesic current μ_{ρ_α} is a multicurve.

To this end, we use the explicit coordinates obtained by Strubel on the set of $\mathrm{Sp}(2n, \mathbf{R})$ -conjugacy classes of maximal representations of Γ into $\mathrm{Sp}(2n, \mathbf{R})$. Namely, let

$$\bar{B} := \{A \in \mathrm{GL}(n, \mathbf{R}) : \mathrm{spec}(A) \subset \{z \in \mathbf{C} : |z| \leq 1\}\},$$

where $\mathrm{spec}(A)$ denotes the set of eigenvalues of A , and

$$R = \{(X_1, X_2, X_3) \in \bar{B}^3 : X_3 {}^tX_2^{-1}X_1 \text{ is symmetric positive definite}\}.$$

Then, given $X := (X_1, X_2, X_3) \in R$, the formulas

$$\begin{aligned} \rho_X(c_1) &= \begin{pmatrix} X_1 & 0 \\ X_1 + X_2^{-1}X_3 & {}^tX_1^{-1} \end{pmatrix}, \\ \rho_X(c_3) &= \begin{pmatrix} {}^tX_3^{-1} & -{}^tX_3^{-1} - X_1^{-1}X_2 \\ 0 & X_3 \end{pmatrix} \end{aligned}$$

determine a representation $\Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ that is maximal [40, Theorem 2]. Moreover, every maximal representation of Γ into $\mathrm{Sp}(2n, \mathbf{R})$ is conjugate to a ρ_X for $X \in R$ and $\mathrm{Sp}(2n, \mathbf{R})$ -conjugacy classes of maximal representations correspond to $\mathrm{O}(n)$ -conjugacy classes in R for the diagonal conjugation action of $\mathrm{O}(n)$.

If now \mathbf{F} is a real closed field, we set

$$\bar{B}_{\mathbf{F}} = \{A \in \mathrm{GL}(n, \mathbf{F}) : \mathrm{spec}(A) \subset \{z \in \mathbf{F}(\sqrt{-1}) : |z| \leq 1\}\}$$

and

$$R_{\mathbf{F}} := \{(X_1, X_2, X_3) \in \bar{B}_{\mathbf{F}}^3 : X_3 {}^tX_2^{-1}X_1 \text{ is symmetric positive definite}\}.$$

The following gives a source of maximal framed representations over any real closed field \mathbf{F} .

Proposition 8.2. *For every $X := (X_1, X_2, X_3) \in R_{\mathbf{F}}$, the formulas for $\rho_X(c_1)$ and $\rho_X(c_3)$ define a maximal framed representation $\rho_X: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$.*

The proof of Proposition 8.2 is beyond the scope of this paper. We just mention that it is an easy application of the Tarski–Seidenberg principle (see [5, Proposition 5.1.3]) and the fact that R parametrizes a semi-algebraic subset of $\mathrm{Hom}(\Gamma, \mathrm{Sp}(2n, \mathbf{R}))$; the principle implies that since $\rho_X: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{R})$ is maximal framed for any $X \in R$, for any real closed field \mathbf{F} containing \mathbf{R} and any $X \in R_{\mathbf{F}}$, the representation $\rho_X: \Gamma \rightarrow \mathrm{Sp}(2n, \mathbf{F})$ is maximal framed as well (see [11] for further applications of the Tarski–Seidenberg principle to representation varieties).

We are interested in the case in which $\rho_X(c_1)$, $\rho_X(c_2)$ and $\rho_X(c_3)$ are all unipotent, which is equivalent to $X_1, -X_2, X_3$ being unipotent. This is never the case if $n = 1$ as one can see from the above formulas, while already for $\mathrm{Sp}(4, \mathbf{R})$ there are interesting examples with unipotent boundary holonomy. We restrict here to the subset of R consisting of triple

(X_1, X_2, X_3) such that $X_1, -X_2, X_3$ are unipotent and $X_3^t X_2^{-1} X_1 = \text{Id}$. The quotient by $O(2)$ -conjugation of such triples can be parametrized by

$$\left\{ \begin{pmatrix} 1 & \frac{4}{x} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3+y & -x \\ \frac{(y-2)^2}{x} & 1-y \end{pmatrix}, \begin{pmatrix} 1+y & \frac{y^2}{x} \\ -x & 1-y \end{pmatrix}, x > 0, y \in \mathbf{R} \right\}.$$

The corresponding matrices are then

$$\rho_X(c_1) = \begin{pmatrix} X_1 & 0 \\ X_1 + X_2^{-1} X_3 & {}^t X_1^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{x} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & \frac{4}{x} & 1 & 0 \\ -\frac{4}{x} & 2 & -\frac{4}{x} & 1 \end{pmatrix},$$

$$\rho_X(c_3) = \begin{pmatrix} {}^t X_3^{-1} & -{}^t X_3^{-1} - X_1^{-1} X_2 \\ 0 & X_3 \end{pmatrix} = \begin{pmatrix} 1-y & x & -2 & -x - \frac{y^2}{x} \\ -\frac{y^2}{x} & 1+y & x + \frac{y^2}{x} & -2 \\ 0 & 0 & 1+y & \frac{y^2}{x} \\ 0 & 0 & -x & 1-y \end{pmatrix}.$$

The above formulas allow us to consider ρ_X as a representation of Γ with coefficients in the ring $\mathbf{R}[x, \frac{1}{x}, y]$. Now for every $\alpha \in \mathbf{R}$, we consider the ring morphism of $\mathbf{R}[x, \frac{1}{x}, y]$ into the Hahn field $\mathcal{H}(\mathbf{R})$ defined by sending x to x and y to x^α . In this way, we obtain for every $\alpha \in \mathbf{R}$ a representation $\rho_\alpha: \Gamma \rightarrow \text{Sp}(4, \mathcal{H}(\mathbf{R}))$. It is easy to verify that the triple $X = (X_1, X_2, X_3)$ with $y = x^\alpha$ is in $R_{\mathcal{H}(\mathbf{R})}$ and it follows from Proposition 8.2 that ρ_α is maximal framed. For the computation of the length function L , it is not difficult to see that if $g \in \text{Sp}(4, \mathbf{F})$, where \mathbf{F} is real closed non-Archimedean with an order compatible valuation, then

$$L(g) = -v(T(g)),$$

where $T(g) = (\text{tr}g)^2 - \text{tr}g^2 - 4$. In our case, we obtain

$$T(\rho_\alpha(c_1^{-1}c_3)) = 4(4x^2 + 32x^{-4+4\alpha} + (18 + 8x^{2\alpha}) + 4x^{-2}(16 + 12x^{2\alpha} + x^{4\alpha})),$$

$$T(\rho_\alpha(c_1^{-1}c_2)) = 4(50 + 4x^2 + 8x^{2\alpha} + 4x^{-2}(16 + 12x^{2\alpha} + x^{4\alpha})),$$

therefore

$$v(T(\rho_\alpha(c_1^{-1}c_3))) = \min(-2, -4 + 4\alpha, -2 + 2\alpha) = \begin{cases} -2, & \alpha \geq \frac{1}{2}, \\ -4 + 4\alpha, & \alpha \leq \frac{1}{2}, \end{cases}$$

$$v(T(\rho_\alpha(c_1^{-1}c_2))) = \min(-2, -2 + 4\alpha, -2 + 2\alpha) = \begin{cases} -2, & \alpha \geq 0, \\ -2 + 4\alpha, & \alpha \leq 0. \end{cases}$$

We deduce that for $\alpha \leq 1/2$, the length functions $\gamma \mapsto L(\rho_\alpha(\gamma))$ are distinct even when considered up to positive scaling. It is easy to verify that

$$\mathbf{Q}(\rho_\alpha) = \begin{cases} \mathbf{Q}(x^\alpha) & \text{if } \alpha \in \mathbf{Q} \setminus \{0\}, \\ \mathbf{Q}(x) & \text{if } \alpha = 0, \\ \mathbf{Q}(x, x^\alpha) & \text{if } \alpha \notin \mathbf{Q}. \end{cases}$$

As a result, the image of the valuation is

$$v(\mathbf{Q}(\rho_\alpha)) = \begin{cases} \alpha \mathbf{Z} & \text{if } \alpha \in \mathbf{Q} \setminus \{0\}, \\ \mathbf{Z} & \text{if } \alpha = 0, \\ \mathbf{Z} + \alpha \mathbf{Z} & \text{if } \alpha \notin \mathbf{Q}, \end{cases}$$

which implies by Theorem 1.4 that the geodesic current corresponding to ρ_α is a multicurve if $\alpha \in \mathbf{Q}$.

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Marc Burger

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland;
burger@math.ethz.ch

Alessandra Iozzi

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland;
iozzi@math.ethz.ch

Anne Parreau

Institut Fourier, Université Grenoble Alpes, CS 40700, 38058 Grenoble Cedex 09, France;
anne.parreau@univ-grenoble-alpes.fr

Maria Beatrice Pozzetti

Institute for Mathematics, Heidelberg University, Im Neuenheimerfeld 205, 69120 Heidelberg, Germany;
pozzetti@mathi.uni-heidelberg.de