Groups Geom. Dyn. 18 (2024), 939–962
DOI 10.4171/GGD/758

Twisted conjugacy in SL_n and GL_n over subrings of $\overline{\mathbb{F}_p}(t)$

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Abstract. Let $\varphi\colon G \to G$ be an automorphism of an infinite group G. One has an equivalence relation $\sim_\varphi$ on G defined as $x \sim_\varphi y$ if there exists a $z \in G$ such that $y = zx\varphi(z^{-1})$. The equivalence classes are called φ-twisted conjugacy classes, and the set $G/{\sim_\varphi}$ of equivalence classes is denoted by $\mathcal{R}(\varphi)$. The cardinality $R(\varphi)$ of $\mathcal{R}(\varphi)$ is called the Reidemeister number of φ. We write $R(\varphi) = \infty$ when $\mathcal{R}(\varphi)$ is infinite. We say that G has the R_∞-property if $R(\varphi) = \infty$ for every automorphism φ of G. We show that the groups $G = \mathrm{GL}_n(R), \mathrm{SL}_n(R)$ have the R_∞-property for all $n \geq 3$ when $F[t] \subset R \subsetneq F(t)$, where F is a subfield of $\overline{\mathbb{F}_p}$. When $n \geq 4$, we show that any subgroup $H \subset \mathrm{GL}_n(R)$ that contains $\mathrm{SL}_n(R)$ also has the R_∞-property.

1. Introduction

Given an endomorphism $\varphi\colon G \to G$ of a group G, one has the φ-twisted conjugacy action of G on itself defined by $g.x = gx\varphi(g^{-1})$. The orbits of this action are called the φ-twisted conjugacy classes. The cardinality of the orbit space $\mathcal{R}(\varphi)$ is called the Reidemeister number of φ and is denoted by $R(\varphi)$. When the orbit space is infinite, we write $R(\varphi) = \infty$. One says that G has the R_∞-property if $R(\varphi) = \infty$ for every automorphism φ of G.

The notion of Reidemeister number originated in Nielsen fixed point theory in the 1930s. There was renewed interest in Reidemeister numbers of endomorphisms of groups since the work of Fel'shtyn and Hill [7]. The paper of Levitt and Lustig [21] showed that non-elementary torsionless Gromov hyperbolic groups have the R_∞-property. Fel'shtyn [6] extended this to all non-elementary hyperbolic groups. Gonçalves and Wong [14] found examples of groups having exponential growth which do not have the R_∞-property, disproving a conjecture of Fel'shtyn and Hill and also provided examples to show that R_∞-property is not a '(coarse) geometric property'. They also obtained, in [15], applications of twisted conjugacy to nilmanifolds. The work of Gonçalves and Wong, motivated by the work of Fel'shtyn, Hill, Levitt, Lustig, gave impetus to the problem of classification of groups according to whether or not they have the R_∞-property. The appellation R_∞ seems to have been due to Taback and Wong [38]. Many classes of groups have been

Mathematics Subject Classification 2020: 20F28 (primary); 20G35 (secondary).
Keywords: twisted conjugacy, general linear group, special linear group, automorphism group, Reidemeister number.

classified according to whether or not they have the R_∞-property. This is an interesting classification problem because there is no specific method or procedure to decide whether a given group or class of groups have the R_∞-property. The tools needed are to be developed depending on the class of groups under consideration and have often required results and techniques from different branches of mathematics. We mention as examples geometric group theory [21], combinatorial group theory [13], Bass–Serre theory [25], Σ-theory [11], dynamics of PL-homeomorphisms of the intervals, number theory and Σ-theory [16], algebraic groups and Lie theory [27], etc.

In this paper, we address this classification problem for the class of general and special linear groups, $\mathrm{GL}_n(R)$, $\mathrm{SL}_n(R)$, $n \geq 3$, over an integral domain R such that $F[t] \subset R \subsetneq F(t)$, where F is a subfield of $\overline{\mathbb{F}_p}$, p a prime. We consider only the case $n \geq 3$ here. The groups $\mathrm{GL}_2(R)$, $\mathrm{SL}_2(R)$ required an entirely different approach, and it was shown in [25] that $\mathrm{GL}_2(\mathbb{F}_q[t])$, $\mathrm{GL}_2(\mathbb{F}_q[t, t^{-1}])$, $\mathrm{SL}_2(\mathbb{F}_q[t])$ have the R_∞-property.

We now state our main results.

Theorem 1.1. *Let $n \geq 3$ and let F be a subfield of $\overline{\mathbb{F}_p}$. Let R be an integral domain such that $F[t] \subset R \subsetneq F(t)$. Then the groups $\mathrm{GL}_n(R)$, $\mathrm{SL}_n(R)$ have the R_∞-property.*

Theorem 1.2. *Let R be as in Theorem 1.1. Suppose that H is any group such that $\mathrm{SL}_n(R) \subset H \subset \mathrm{GL}_n(R)$. Then H has the R_∞-property in the following cases:*

(a) $n \geq 4$,

(b) $n = 3$ *and* $\det(H) \subset F^\times$.

We have not been able to decide whether the R_∞-property holds for H as in Theorem 1.2 when $n = 3$ and $\det(H)$ is not contained in $F^\times$. See Remark 5.4 (i).

Our proof of Theorem 1.1 relies heavily on some classical results concerning general and special linear groups over integral domains which are not fields, namely:

(i) the description, due to O'Meara [32], of the automorphisms of $\mathrm{SL}_n(R)$ and $\mathrm{GL}_n(R)$, $n \geq 3$, which is valid for any integral domain R that is not a field,

(ii) the group $\mathrm{SL}_n(R)$, $n \geq 3$, is perfect and is generated by elementary matrices when R is a Euclidean domain, see [35, Theorem 2.3.2],

(iii) some dynamical properties of the action of the group $\mathrm{Aut}(R)$ of automorphisms of the ring R on $\mathrm{SL}_n(R)$, and, the crucial construction of certain sequence of elements $\{x_m\}$ in $\mathrm{SL}_n(R)$ which are used in showing $R(\varphi) = \infty$ for many automorphisms φ of considered in our main theorems.

Some of our observations in Section 3.3 concerning $\mathrm{Aut}(R)$ and its action on $\mathrm{SL}_n(R)$ may be of independent interest.

We shall prove Theorem 1.1 first for the case when F is a finite field $\mathbb{F}_q$, in Section 4.2. In this case, we prove that every outer automorphism $[\varphi]$ of $G(R) = \mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$ is represented by a convenient automorphism φ using O'Meara's theorem. The proof in this case crucially uses the observation that when φ is induced by an automorphism of the ring R, it has finite order. We also show that, in this case, the fixed subgroup of φ

contains a subgroup isomorphic to $\mathrm{SL}_n(\mathbb{F}_p[t])$. In the more general case, it is not true that the outer automorphisms are represented by finite order automorphisms. (For example, automorphisms of G induced by a Frobenius automorphism of the ring R are not of finite order when F is infinite.) Nevertheless, any automorphism that is induced by an automorphism φ of the ring R restricts to an automorphism of $G_q = G \cap \mathrm{GL}_n(R_q)$, where $R_q = R \cap \mathbb{F}_q(t)$ for some $q = p^e$. This allows us to extend our proof technique in case when F is a finite field to the more general case when $F \subset \overline{\mathbb{F}_q}$. The proof of Theorem 1.1 in the general case is completed in Section 4.3.

Although the automorphism group of H as in Theorem 1.2 is not known to us, any automorphism θ of H restricts to an automorphism θ' of $\mathrm{SL}_n(R)$. After choosing a convenient representative of the outer automorphism class of θ, we find a subsequence of the sequence $\{x_m\}$ of elements of $\mathrm{SL}_n(R)$ that are in pairwise distinct θ'-twisted conjugacy classes and show that they remain in pairwise distinct θ-twisted conjugacy classes in H, under the hypotheses of the theorem. The proof is given in Section 5.

Let $\mathcal{P}$ be the set of all prime ideals of $F[t]$ and let $S \subset \mathcal{P}$. Denote by $R_\mathfrak{p}$ the localization of R at the prime ideal $\mathfrak{p} \in \mathcal{P}$. Let $K = F(t)$ and let R_S denote the subring $R_S = \bigcap_{\mathfrak{p} \in \mathcal{P} \setminus S} R_\mathfrak{p}$ of K.

Let R be any ring such that $F[t] \subset R \subsetneq K$. It can be seen, using the Euclidean algorithm, that if $f(t)/g(t) \in R$, where $f(t), g(t) \in F[t]$ have no common factors, then $1/g(t) \in R$. So R is generated over $F[t]$ by a (possibly infinite) set of reciprocals of irreducible polynomials in $F[t]$. It follows that $R = R_S$, where $S \subset \mathcal{P}$ consists of prime ideals $\mathfrak{p}$ generated by irreducible polynomials which are *not* invertible in R. Suppose that $F = \mathbb{F}_q$, S is finite, and $R = R_S$, then $\mathrm{SL}_n(R)$ is an S-arithmetic subgroup of $\mathrm{SL}_n(K)$ and is a lattice in $\mathrm{SL}_n(K_\infty) \times \prod_{\mathfrak{p} \in S} \mathrm{SL}_n(K_\mathfrak{p})$ when embedded diagonally, where $K_\mathfrak{p}$ is the $\mathfrak{p}$-adic completion of K for $\mathfrak{p} \in \mathcal{P}$ and $K_\infty = \mathbb{F}_q((t^{-1}))$. See [23, p. 63, (3.2.5)]. It was shown in [27] that any irreducible lattice in a connected non-compact semisimple real Lie group having rank at least 2 and with finite centre has the R_∞-property. In the case of rank 1 lattices, the R_∞-property follows from the works of Levitt–Lustig [21] and Fel'shtyn [6]. The present work may be viewed as a first step in classifying, according to the R_∞-property, lattices in semisimple algebraic groups over complete local fields of positive characteristics.

Recently, Garge and Mitra [10] have obtained results analogous to those of Theorem 1.1 for the classical groups $\mathrm{SO}_n(R)$, $n \geq 4$, $\mathrm{Sp}_{2n}(R)$, $n > 1$, for any ring R such that $F[t] \subset R \subsetneq F(t)$, $F \subset \overline{\mathbb{F}_p}$, $p \geq 3$.

Our main theorems lend further evidence to the expectation that all non-amenable S-arithmetic groups should have the R_∞-property.

Lang [20] has shown that if G is a linear connected algebraic group over an algebraically closed field F of characteristic $p > 0$ and $\rho\colon G(F) \to G(F)$ is a Frobenius endomorphism, then the map $g \mapsto g^{-1}\rho(g)$ is surjective. See also [37] for a more general result. Thus, as has been observed in [9], every element is ρ-twisted conjugate to $1 \in G$. Note that when F is algebraically closed, the Frobenius endomorphism is an *automorphism* of the underlying *abstract* group $G(F)$. So G does not have the R_∞-property. However, restricting oneself only to *algebraic* automorphisms of a linear algebraic group G over any algebraically closed field K, Bhunia and Bose [2] have shown that any connected non-solvable linear algebraic group has the *algebraic* R_∞-property, more precisely, $R(\varphi) = \infty$ for any K-automorphism of the algebraic group G. See also [1] for further results on algebraic R_∞-property.

The R_∞-property for linear groups over fields of characteristic zero was considered by Nasybullov [29,30] and Fel'shtyn and Nasybullov [9], culminating in the result that a Chevalley group of classical type over an algebraically closed field F of characteristic zero has the R_∞-property if and only if F has finite transcendence degree over $\mathbb{Q}$. When F is any field of characteristic zero having infinite transcendence degree over $\mathbb{Q}$, Fel'shtyn and Nasybullov [9] have shown that any Chevalley group over F associated to an irreducible system has the R_∞-property.

There is also vast body of work on nilpotent groups, abelian groups, polycyclic groups etc. See, for example, [4,5,15,33,38].

Jabara [19, Theorem C] has shown that a finitely generated linear group that admits a finite order automorphism φ with finite Reidemeister number is necessarily virtually solvable. His proof makes use of the deep work of Hrushovski, Kropholler, Lubotzky, and Shalev [18]. It should be noted that typically the groups $G = \mathrm{SL}_n(R), \mathrm{GL}_n(R)$ with R as in our main theorem are not finitely generated. See Section 3.1 for more details on finite generation of G. When G is finitely generated, it is residually finite; see [22, Proposition 7.11, Chapter III]. When an automorphism φ of G has finite order, then by Jabara's theorem, we have $R(\varphi) = \infty$. However, even when G is finitely generated, not every outer automorphism is represented by a finite order automorphism.

2. Some basic results on twisted conjugacy

Let $\varphi: G \to G$ be an automorphism of a group G. We shall denote by $[x]_\varphi$ the φ-twisted conjugacy class of x and by $\mathcal{R}(\varphi)$ the set of all φ-twisted conjugacy classes in G.

We collect here some basic results concerning twisted conjugacy and the R_∞-property which are relevant for our purposes. Let G be an infinite group (not necessarily finitely generated) and let $K \subset G$ be a normal subgroup. Let $\eta: G \to H$ be the canonical quotient map where $H = G/K$. Suppose that $\varphi: G \to G$ is an automorphism such that $\varphi(K) = K$ so that we have the following diagram in which the rows are exact and isomorphisms $\varphi', \bar{\varphi}$ are induced by φ:

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & H & \to & 1 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \bar{\varphi} & & \\ 1 & \to & K & \to & G & \to & H & \to & 1 \end{array} \tag{1}$$

We recall that a subgroup K of G is *characteristic* in G if every automorphism of G restricts to an automorphism of K.

The following two lemmas are well known. We refer the reader to [26, Lemmas 2.1 and 2.2] for a proof of Lemma 2.1.

Lemma 2.1. *Suppose that $\varphi: G \to G$ is an automorphism of an infinite group such that the rows in (1) are exact and the homomorphisms φ', $\overline{\varphi}$ are isomorphisms. Then*

(i) *If $R(\overline{\varphi}) = \infty$, then $R(\varphi) = \infty$. In particular, if K is characteristic in G, then G has the R_∞-property if H does.*

(ii) *Suppose that H is finite. Then $R(\varphi) = \infty$ if $R(\varphi') = \infty$. In particular, if K is characteristic and has finite index in G, then G has the R_∞-property if K does.* ∎

Let $g \in G$ and let $\iota_g: G \to G$ denote the inner automorphism $x \mapsto gxg^{-1}$.
Proof of the following lemma can be found in [8, §3].

Lemma 2.2. *If $\varphi: G \to G$ is any automorphism, then $[x]_{\iota_g \circ \varphi} \mapsto [xg]_\varphi$ defines a bijection $\mathcal{R}(\iota_g \circ \varphi) \to \mathcal{R}(\varphi)$. In particular, $R(\iota_g \circ \varphi) = R(\varphi)$.* ∎

The following lemma can be proved along the same lines as [12, Lemma 2.3]. For the sake of completeness, we give the proof.

Lemma 2.3. *Let $\theta: G \to G$ be a finite order automorphism. Let $r = o(\theta)$. Suppose that $[x]_\theta = [y]_\theta$. Then*

(i) *$\prod_{0 \le j < r} \theta^j(x)$ and $\prod_{0 \le j < r} \theta^j(y)$ are conjugates in G.*

(ii) *Further, if $x, y \in \mathrm{Fix}(\theta)$, then x^r and y^r are conjugates in G.*

Proof. Suppose that $y = zx\theta(z^{-1})$ for some $z \in G$. Applying θ^j to both sides of the equality, we obtain that $\theta^j(y) = \theta^j(z)\theta^j(x)\theta^{j+1}(z^{-1})$. Now taking product in order as we vary j in $\{0, 1, \ldots, r-1\}$, we obtain that

$$\prod_{0 \le j < r} \theta^j(y) = \prod_{0 \le j < r} \theta^j(z)\theta^j(x)\theta^{j+1}(z^{-1}) = z\Big(\prod_{0 \le j < r} \theta^j(x)\Big)\theta^r(z^{-1}).$$

Since $\theta^r = \mathrm{id}$, assertion (i) follows. Assertion (ii) is a special case of the first. ∎

3. Automorphisms of $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$

In this section, we recall some properties of the groups $\mathrm{SL}_n(R)$ and $\mathrm{GL}_n(R)$ as well as their automorphisms where R is an integral domain, but not a field, with $F[t] \subset R \subsetneq F(t)$ and F being a field. Up to the end of Section 3.3, we do not assume that F is a subfield of $\overline{\mathbb{F}_p}$. The reader is referred to the book by Hahn and O'Meara [17] for detailed study of classical groups over integral domains.

3.1. Linear groups over subrings of $F(t)$

Since $F[t]$ is a PID, any extension ring R that is contained in $F(t)$ is a localization of $F[t]$ by some multiplicatively closed subset of $F[t]$. Moreover, such a ring R is a Euclidean domain since any localization of a Euclidean domain is again a Euclidean domain; see [36, Proposition 7].

Let $E_n(R) \subset \mathrm{GL}_n(R)$ denote the subgroup generated by the elementary matrices $e_{ij}(\lambda)$, $\lambda \in R$, $1 \le i, j \le n$, $i \ne j$. By definition, $e_{ij}(\lambda)$ is the matrix whose diagonal entries are 1, the (i, j)-th entry is λ and all other entries are zero.

One has the obvious inclusions $E_n(R) \subset \mathrm{SL}_n(R) \subset \mathrm{GL}_n(R)$. As R is a Euclidean domain, we have $E_n(R) = \mathrm{SL}_n(R)$ for $n \ge 2$; see [35, Theorem 2.3.2]. When $n \ge 3$, we also have $E_n(R) = [E_n(R), E_n(R)]$; this follows from the observation that the commutator $[e_{ik}(x), e_{kj}(1)]$ equals $e_{ij}(x)$ for all $x \in R$ provided i, j, k are all distinct. An immediate consequence of this is recorded below as a proposition for easy reference.

Proposition 3.1. *Let R be any Euclidean domain and let $n \ge 3$. Then $\mathrm{SL}_n(R)$ is perfect and it equals the derived group $[H, H]$, where H is any subgroup of $\mathrm{GL}_n(R)$ that contains $\mathrm{SL}_n(R)$.*

Proof. Since $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] \subset \mathrm{SL}_n(R) = E_n(R) = [E_n(R), E_n(R)]$, we see that for any subgroup $H \subset \mathrm{GL}_n(R)$ that contains $\mathrm{SL}_n(R)$ we have $[H, H] = \mathrm{SL}_n(R)$. ∎

Let $G(R) = \mathrm{SL}_n(R)$ or $\mathrm{GL}_n(R)$, $n \ge 3$. Suppose that $G(R)$ is finitely generated, then it is residually finite; see [22, Proposition 7.11, Chapter III]. It is easily seen that G is also not virtually solvable. (This is also implied by the above proposition.) If $\varphi\colon G(R) \to G(R)$ is a finite order automorphism, then by Jabara's theorem [19, Theorem C], we have $R(\varphi) = \infty$.

We give below examples of rings R for which $G(R)$ are not finitely generated. See also [17, §4.3].

Example 3.2. (i) If $R^\times$ is not a finitely generated group, then $\mathrm{GL}_n(R)$ is not finitely generated. This is because the determinant map $\det\colon \mathrm{GL}_n(R) \to R^\times$ is a surjective homomorphism of groups. We claim that $R^\times$ is finitely generated as a group if and only if R is finitely generated as a ring. Note that $R = F[t][1/b(t); b(t) \in \mathcal{B}]$, where $\mathcal{B} \subset F[t]$ is the set of all monic irreducible polynomials that are invertible in R. It follows that R is finitely generated as a ring if and only if F is finite and $\mathcal{B}$ is finite. On the other hand, the group $R^\times$ is isomorphic to $F^\times \times U$, where U is a free abelian group of rank equal to the cardinality of the set $\mathcal{B}$. (See Theorem 5.1 below.) In particular, $R^\times$ is finitely generated if and only if $F^\times$ is finite and $\mathcal{B}$ is finite. This proves our claim. Thus $\mathrm{GL}_n(R)$ is not finitely generated if R is not finitely generated as a ring.

(ii) If R is not finitely generated as a ring, then $\mathrm{SL}_n(R)$ is also not finitely generated as a group. Indeed, suppose that $g_1, \dots, g_k$ generate $\mathrm{SL}_n(R)$. Write $g_r = (g_{r;i,j}(t)) \in \mathrm{SL}_n(R)$, where $g_{r;i,j}(t) \in R$. Let $E \subset F$ be the subfield generated by the coefficients of the rational functions $\{g_{r;i,j}\}$ in $F(t)$, and let $A = E[g_{r;i,j}; 1 \le i, j \le n, 1 \le r \le k] \subset R$. Then E is a *finite* subfield (since $F \subset \overline{\mathbb{F}_p}$) and A is a finitely generated subring of R. Moreover, $\mathrm{SL}_n(R) = \mathrm{SL}_n(A)$. This implies that $R = A$ and so R is finitely generated as a ring.

(iii) Nagao [28] has shown that $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not finitely generated.

That the converse also holds was proved by O'Meara [31], which we state below. His description of $R \subset \mathbb{F}_q(t)$ was in terms of valuations of $\mathbb{F}_q(t)$. We have reformulated it using localization.

Suppose that $\mathcal{B}$ is a finite set of irreducible polynomials in $\mathbb{F}_q[t]$. Let

$$R = \mathbb{F}_q[t][1/b(t); b(t) \in \mathcal{B}] = \mathbb{F}_q[t][1/f(t)],$$

where $f(t) = \prod_{b(t)\in\mathcal{B}} b(t)$. Then $\mathrm{SL}_n(R)$, $n \geq 3$, is finitely generated. When $\mathcal{B} = \emptyset$, $f(t)$ is to be regarded as $1 \in \mathbb{F}_p$.

See the recent paper of Bux, Köhl, and Witzel [3] for further finiteness properties of the groups $G(R)$ in the more general setting of S-arithmetic groups over global function fields.

Notations. We denote by $\delta(a_1, \ldots, a_n) \in \mathrm{GL}_n(R)$ the diagonal matrix whose (i,i)-entry equals $a_i \in R^\times$, $1 \leq i \leq n$. Also, we denote by $\delta(A_1, \ldots, A_k)$ the block diagonal matrix whose j-th diagonal block is equal to A_j. We denote by $h(a)$ the matrix $\delta(I_{n-1}, a) \in \mathrm{GL}_n(R)$, where $a \in R^\times$.

3.2. Theorem of O'Meara

Let $n \geq 3$ and let $G(R)$ be one of the groups $\mathrm{SL}_n(R)$, $\mathrm{GL}_n(R)$, where R is any (commutative) integral domain which is not a field. Let K be the fraction field of R. An automorphism $\varphi\colon G(R) \to G(R)$ is called *standard* if it is in the subgroup generated by the following four types of automorphisms:

(i) *Conjugation*: Conjugation by $g \in \mathrm{GL}_n(K)$, denoted by $\iota_g\colon G(R) \to G(R)$, is defined as $x \mapsto gxg^{-1}$. It is inner if $g \in G(R)$.

(ii) *Ring automorphism*: These are automorphisms of $G(R)$ induced by automorphisms of the ring R. We make no distinction in the notation between the automorphism of the ring R and the induced automorphism of $G(R)$.

(iii) *Homothety*: Recall that a homomorphism $\mu = \mu_\chi\colon G(R) \to G(R)$ is a *homothety* if there is a character $\chi\colon G(R) \to R^\times$ such that $\mu_\chi(x) = \chi(x)x$. Since $x, \chi(x)x \in G(R)$, it follows that $\chi(x)I_n \in G(R)$. Being a scalar matrix, $\chi(x)I_n$ belongs to the centre of the group $G(R)$. A homothety μ_χ fails to be injective if and only if there exists a central element $zI_n \in G(R)$ other than I_n such that $\chi(zI_n) = z^{-1}$.

Suppose μ_χ is an automorphism of $G(R)$. Observe that, since $\mathrm{SL}_n(R) = [G(R), G(R)]$ is characteristic in $G(R)$, μ_χ restricts to the identity automorphism on $\mathrm{SL}_n(R)$. In particular, $\mu_\chi = \mathrm{id}$ when $G(R) = \mathrm{SL}_n(R)$.

(iv) *Contragredient*: The contragredient automorphism $\varepsilon\colon G(R) \to G(R)$ is defined as $x \mapsto {}^t x^{-1}$ $\forall x \in G(R)$. Evidently, it is an involution.

O'Meara [32] has shown that for any integral domain R which is not a field, any automorphism of $G(R)$ is standard, provided $n \geq 3$.

Theorem 3.3 (O'Meara [32]). *Let R be an integral domain with fraction field $K \neq R$, and let $G(R) = \mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$, where $n \geq 3$. Then any automorphism $\varphi\colon G(R) \to G(R)$ can be expressed as follows:*

$$\varphi = \mu_\chi \circ \rho \circ \iota_g \quad \text{or} \quad \varphi = \mu_\chi \circ \rho \circ \iota_g \circ \varepsilon,$$

where μ_χ is a homothety automorphism corresponding to a character $\chi\colon G(R) \to R^\times$, $g \in \mathrm{GL}_n(K)$, $\rho\colon G(R) \to G(R)$ is induced by a ring automorphism denoted by the same symbol $\rho\colon R \to R$ and ε is the contragredient $x \mapsto {}^t x^{-1}$.

Remark 3.4. O'Meara's theorem is stated in the module-theoretic language. The expression for an automorphism [32] involves a semilinear automorphism Φ_g of $M = R^n$, which corresponds to the composition $\rho \circ \iota_g$, where $\rho\colon K \to K$ is a field automorphism and $g\colon K^n \to K^n$ is a K-linear isomorphism. Theorem B from [32, §5] states that a semilinear automorphism of K^n yields an automorphism of M if and only if $g(M) = \mathfrak{a}.M$ for an invertible fractional ideal $\mathfrak{a}$ of R and $\rho(R) = R$. When R is a PID (as in our case), every fractional ideal is invertible and moreover $\mathfrak{a} = \lambda R$ for some $\lambda \in K^\times$.

Theorem C from [32, §5] is the analogous statement when the contragredient is involved. Thus, when R is a subring of $\overline{\mathbb{F}_p}(t)$ which is not a field, the element g in Theorem 3.3 may be taken to be in $\mathrm{GL}_n(R)$.

The following corollary is immediate from Theorem 3.3 and the above remark.

Corollary 3.5. (i) *Any outer automorphism of $\mathrm{SL}_n(R)$ is represented by one of the following automorphisms: $\iota_{h(\alpha)} \circ \rho \circ \eta$, where $\eta \in \langle\varepsilon\rangle$, $\alpha \in R^\times$ and $h(\alpha)$ denotes $\delta(I_{n-1}, \alpha) \in \mathrm{GL}_n(R)$.*

(ii) *Any outer automorphism of $\mathrm{GL}_n(R)$ is represented by one of the following automorphisms: $\mu_\chi \circ \rho \circ \eta$, where $\eta \in \langle\varepsilon\rangle$ and χ is a suitable character $\chi\colon \mathrm{GL}_n(R) \to R^\times$.*

(iii) *Every automorphism of $\mathrm{SL}_n(R)$ extends to an automorphism of $\mathrm{GL}_n(R)$.* ■

3.3. Commutation relations

The ring automorphisms ρ, the conjugations ι_g, the homothety automorphisms μ_χ and the contragredient automorphism ε satisfy certain commutation relations that are stated in the lemma below. These relations will be used in the sequel.

Lemma 3.6. *Let $G = \mathrm{SL}_n(R)$, or $\mathrm{GL}_n(R)$, where R is an integral domain which is not a field. With the above notations, we have*

(i) $\rho \circ \iota_g = \iota_{\rho(g)} \circ \rho$, $\mu_\chi \circ \iota_g = \iota_g \circ \mu_\chi$,

(ii) $\varepsilon \circ \rho = \rho \circ \varepsilon$,

(iii) $\varepsilon \circ \iota_g = \iota_{\varepsilon(g)} \circ \varepsilon$,

(iv) $\mu_\chi \circ \rho = \rho \circ \mu_\eta$, *where* $\eta = \rho_0^{-1} \circ \chi \circ \rho\colon G \to R^\times$. *Here* $\rho_0\colon R^\times \to R^\times$ *is the restriction of* $\rho\colon R \to R$ *to the group* $R^\times$ *of all units of* R,

(v) $\mu_{-\chi} \circ \varepsilon = \varepsilon \circ \mu_{\chi\circ\varepsilon}$, *where* $-\chi(g) = \chi(g)^{-1}\ \forall g \in G$.

Proof. Proofs of parts (i)–(iii) are straightforward and omitted; we only prove (iv) and (v).

(iv) For any $g \in G$, we have

$$\begin{aligned}\rho(\mu_\eta(g)) &= \rho(\eta(g)g) = \rho(\eta(g)I_n)\rho(g) = \rho(\rho_0^{-1}(\chi(\rho(g)))I_n)\rho(g)\\ &= \chi(\rho(g))I_n\rho(g) = \mu_\chi(\rho(g)),\end{aligned}$$

which proves the assertion.

(v) Let $g \in G$. Then, writing $\eta := \chi \circ \varepsilon$, we have $\eta(g) = \chi({}^tg^{-1})$ and so $\mu_\eta(g) = \chi({}^tg^{-1})I_ng$. Applying ε to both sides, we obtain

$$\begin{aligned}\varepsilon(\mu_\eta(g)) &= \varepsilon(\chi({}^tg^{-1})I_ng) = \varepsilon(\chi({}^tg^{-1})I_n){}^tg^{-1} = \chi({}^tg^{-1})^{-1}I_n{}^tg^{-1}\\ &= \chi(\varepsilon(g))^{-1}\varepsilon(g) = \mu_{-\chi}(\varepsilon(g)).\end{aligned}$$

This completes the proof. ∎

As an application of the commutation relations, we have the following lemma, which will be needed in the sequel.

Lemma 3.7. *Let $\theta_j = \iota_{g_j} \circ \rho_j \circ \varepsilon$, where $g_j \in \mathrm{GL}_n(R)$, $j = 1, 2$. Then $\theta_1 \circ \theta_2 = \iota_g \circ \rho$ where $g = g_1\rho_1({}^tg_2^{-1})$ and $\rho = \rho_1 \circ \rho_2$.*

Proof. We have

$$\begin{aligned}\theta_1 \circ \theta_2 &= \iota_{g_1} \circ \rho_1 \circ \varepsilon \circ \iota_{g_2} \circ \rho_2 \circ \varepsilon\\ &= \iota_{g_1} \circ \rho_1 \circ \iota_{{}^tg_2^{-1}} \circ \varepsilon \circ \rho_2 \circ \varepsilon \qquad \text{(by Lemma 3.6 (iii))}\\ &= \iota_{g_1} \circ \iota_{\rho_1({}^tg_2^{-1})} \circ \rho_1 \circ \rho_2 \qquad \text{(using Lemma 3.6 (i) and } \varepsilon^2 = \mathrm{id})\\ &= \iota_g \circ \rho,\end{aligned}$$

as asserted. ∎

Remark 3.8. We remark that if $a \in R^\times \setminus F^\times$, the automorphism $\iota_{h(a)}$ is an automorphism of $\mathrm{SL}_n(R)$ of infinite order.

3.4. Ring automorphisms

Let R be a ring such that $F[t] \subset R \subsetneq K := F(t)$, where F is a subfield of $\overline{\mathbb{F}_p}$. Any ring automorphism $\rho\colon R \to R$ extends to an automorphism $\widetilde{\rho}$ of K. Since $\widetilde{\rho}$ preserves the set of all elements of finite order in $K^\times$, it follows that ρ restricts to an automorphism of F. Since $F \subset \overline{\mathbb{F}_p}$, $\rho|_F$ is a *Frobenius automorphism* $x \mapsto x^{p^r}$. The value of r may depend on $x \in F$, when F is infinite. For example, when ρ is the inverse of the standard Frobenius automorphism $x \mapsto x^p$, we have $\rho(x) = x^{p^{r-1}}$, where $o(x) = p^r - 1$. Note that any automorphism φ of F extends to a unique automorphism $\widetilde{\varphi}$ of K by setting $\widetilde{\varphi}(t) = t$. Evidently, $\widetilde{\varphi}$ restricts to an automorphism of R, again denoted by $\widetilde{\varphi}$. We shall refer to such a $\widetilde{\varphi} \in \mathrm{Aut}(R)$ as a *Frobenius automorphism*. If $\rho\colon R \to R$ is any automorphism, then $\psi^{-1} \circ \rho =: \rho_1$ is an F-automorphism of R, where ψ is the Frobenius automorphism of R defined by $\rho|_F \in \mathrm{Aut}(F)$. (By an *$F$-automorphism* of R we mean an F-algebra automorphism of R.)

Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PGL}(2, F)$. Let $M_\gamma \in \mathrm{Aut}_F(K)$ be the *Möbius automorphism* that defined as $M_\gamma(t) = \frac{at+b}{ct+d}$. It is convenient to write γ to also denote the Möbius automorphism M_γ. When γ is a Möbius automorphism of K that stabilizes R, we say that $\gamma|_R$ is a Möbius automorphism and again denote this by γ. A Möbius automorphism is also referred to as a *Möbius transformation.*

Let $\varphi\colon F \to F$ be an automorphism. We have the induced automorphism, again denoted by φ, on $\mathrm{PGL}(2, F)$. Thus we obtain an action of $\Phi = \mathrm{Aut}(F)$ on $\mathrm{PGL}(2, F)$.

We shall denote by $\mathrm{Aut}_F(R)$ the group of F-automorphisms of R.

Lemma 3.9. (i) *Any F-automorphism of $K = F(t)$ is a Möbius transformation*

$$\gamma\colon K \to K.$$

The group $\mathrm{Aut}(K)$ *is a semidirect product*

$$\mathrm{Aut}_F(K) \rtimes \Phi \cong \mathrm{PGL}(2, F) \rtimes \Phi,$$

where $\Phi = \mathrm{Aut}(F)$ *is the group of all automorphisms of F.*

(ii) *The group* $\mathrm{Aut}(R)$ *is a semidirect product* $\mathrm{Aut}_F(R) \rtimes \Phi \subset \mathrm{Aut}(K)$. *In particular, any automorphism $\rho\colon R \to R$ can be expressed uniquely as $\rho = \rho_1 \circ \varphi$, where φ is a Frobenius automorphism and ρ_1 is a Möbius automorphism of R.*

Proof. (i) Let $\varphi \in \Phi$ and let $\widetilde{\varphi}$ be its extension to R (or to K) that fixes t. We have the commutation relation

$$\widetilde{\varphi} \circ \gamma = \varphi(\gamma) \circ \widetilde{\varphi},$$

where γ is a Möbius automorphism. To see this, we need only note that, writing $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have that $\widetilde{\varphi}(t) = t$ and

$$\widetilde{\varphi} \circ \gamma(t) = \widetilde{\varphi}\Big(\frac{at+b}{ct+d}\Big) = \frac{\varphi(a)t + \varphi(b)}{\varphi(c)t + \varphi(d)} = \varphi(\gamma)(\widetilde{\varphi}(t)),$$

and that $\widetilde{\varphi} \circ \gamma(\lambda) = \widetilde{\varphi}(\lambda) = \varphi(\lambda)\ \forall \lambda \in F$ as γ is an F-automorphism.

We have a split exact sequence

$$1 \to \mathrm{Aut}_F(K) \to \mathrm{Aut}(K) \to \Phi \to 1,$$

where $\mathrm{Aut}(K) \to \Phi$ is the restriction to F. The splitting $\Phi \to \mathrm{Aut}(K)$ is given by $\varphi \mapsto \widetilde{\varphi}$ (in the above notation). So, to complete the proof, it suffices to show that any F-automorphism $\sigma\colon K \to K$ is a Möbius automorphism. Let $\tau\colon K \to K$ be the inverse of σ.

Let $\sigma(t) = f(t)/g(t)$ and let $\tau(t) = h(t)/k(t)$, where $f(t), g(t), h(t), k(t) \in F[t]$ and $\gcd(f(t), g(t)) = 1 = \gcd(h(t), k(t))$. We need to show that $\deg(f), \deg(g) < 2$.

Since $t \mapsto 1/t$ defines an automorphism, we may assume without loss of generality that $m := \deg(f) \geq \deg(g) =: n$.

Suppose $n = 0$. Then $g(t) \in F$ and $\sigma(t) \in F[t]$ is a polynomial of degree at least 1. If $\deg f(t) \geq 2$, then it is readily seen that t is not in the image of σ. Hence, $\deg f(t) = 1$, and so σ is a Möbius transformation.

By the same argument, if $\deg k(t) = 0$, then τ is a Möbius transformation, which again implies that the same is true for σ as well. Similarly, $\deg h(t) = 0$ implies that τ and σ are Möbius transformations.

So, assume that $n \geq 1$, $\deg h(t) \cdot \deg k(t) \neq 0$ and $\max\{\deg h(t), \deg k(t)\} \geq 2$.

Let $f(t) = \sum_{0 \leq j \leq m} f_j t^j$ and let $g(t) = \sum_{0 \leq i \leq n} g_i t^i$, where $f_j, g_i \in F$. We have that

$$t = \tau(\sigma(t)) = \tau\Big(\frac{f(t)}{g(t)}\Big) = \frac{f(\tau(t))}{g(\tau(t))} = \frac{f(h(t)/k(t))}{g(h(t)/k(t))} = \frac{\sum f_j h^j k^{-j}}{\sum g_i h^i k^{-i}}.$$

This leads to the following equality in $F[t]$:

$$t\Big(\sum_{0 \leq i \leq n} g_i h^i k^{m-i}\Big) = \sum_{0 \leq j \leq m} f_j h^j k^{m-j}.$$

Since $\gcd(f(t), g(t)) = 1 = \gcd(h(t), k(t))$, at most one of $f(t)$, $g(t)$ is divisible by t and the same assertion holds for $h(t), k(t)$. If $f_0 g_0 \neq 0$, then t divides the left-hand side of the above to a higher power than the right-hand side. If $g_0 = 0$, then $f_0 \neq 0$ in which case, $h(t)$ divides the left-hand side, but not the right-hand side. It remains to consider the case $f_0 = 0$. Suppose that $f_0 = 0$. Then $g_0 \neq 0$ and $h(t)$ divides the right-hand side but not $\sum_{0 \leq i \leq n} g_i h^i k^{m-i}$. It follows that $h(t) = at$ for some $a \in F^\times$ and so $\deg k(t) \geq 2$ since $\max\{\deg h(t), \deg k(t)\} \geq 2$. Comparing the degrees of the polynomials on both sides, we obtain that $1 + m \deg k(t) = j + (m - j) \deg k(t)$, where j is the largest positive integer such that t^j divides $f(t)$. This is a contradiction. Thus we obtain a contradiction in all cases, and we are led to the conclusion that σ is a Möbius transformation.

(ii) As already noted, every automorphism of F extends to a Frobenius automorphism of R. Since any F-automorphism of R extends uniquely to an automorphism of K, the assertion follows from (i). ∎

Remark 3.10. (i) Let $K_q = \mathbb{F}_q(t)$ for each finite subfield $\mathbb{F}_q \subset F$ and let $R_q = R \cap K_q$. Then $R = \bigcup_q R_q$ where the union is over those values of q such that $\mathbb{F}_q \subset F$. Indeed, any $f(t) \in K = F(t)$ has all its coefficient in a finite subfield of F as $F \subset \overline{\mathbb{F}_q}$. It follows that $K = \bigcup_q K_q$ whence $R = \bigcup_q R_q$.

(ii) When $\varphi \in \mathrm{Aut}(R)$ is a Frobenius automorphism, it is clear that φ restricts to an automorphism φ_q of R_q for any q. By Lemma 3.9 applied to $F = \mathbb{F}_q$, we see that $\mathrm{Aut}(K_q)$ is a finite group and so is $\mathrm{Aut}(R_q)$.

(iii) When F is infinite, the Frobenius automorphisms of R are of infinite order and hence the induced ring automorphism of $G(R)$ is also of infinite order.

Corollary 3.11. *Let $F[t] \subset R \subsetneq K = F(t)$, where $F \subset \overline{\mathbb{F}_q}$. Let $G(R) = \mathrm{GL}(n, R)$ or $\mathrm{SL}(n, R)$.*

(i) *If $\rho = \rho_1 \circ \varphi$, where ρ_1 is a Frobenius automorphism of R and $\varphi \in \mathrm{PGL}(2, \mathbb{F}_q)$ a Möbius transformation, then $\rho_q := \rho|_{R_q}$ induces an automorphism $\rho_q : G(R_q) \to G(R_q)$ of finite order.*

(ii) *If $g \in G(R)$, and $\rho \in \mathrm{Aut}(R)$, then the orbit of g under the action of the cyclic group $\langle\rho\rangle \subset \mathrm{Aut}(G(R))$ is finite.* ■

Since $\mathbb{F}_q[t] \subset R_q$ for any q, it follows that if R_q is a field, then $R_q = \mathbb{F}_q(t)$. Since $R = \bigcup_q R_q$ and since R is not a field, there exists a q such that R_q is not a field and so R_ℓ is also not a field if $\mathbb{F}_q \subset \mathbb{F}_\ell$.

We set

$$\mathfrak{R}_q := \{\rho \in \mathrm{Aut}(R) \mid \rho(R_q) = R_q\}.$$

We have the restriction homomorphism $\mathfrak{R}_q \to \mathrm{Aut}(R_q)$.

Corollary 3.12. *Let q be such that $R_q = R \cap \mathbb{F}_q(t)$ is not a field. There exists an $s \in R_q \setminus F$ such that $\rho(s) = s$ for all $\rho \in \mathfrak{R}_q$. Also, the subgroup $\mathrm{SL}_n(\mathbb{F}_p[s]) \subset G(R)$ is element-wise fixed by all $\rho \in \mathfrak{R}_q$.*

Proof. Let $\rho \in \mathrm{Aut}(R)$. Write $\rho = \rho_1 \circ \varphi$, where φ is a Frobenius automorphism of R and ρ_1 is a Möbius transformation. Then ρ restricts to an automorphism of R_q if and only if $\rho_1 \in \mathrm{PGL}(2, \mathbb{F}_q)$. Similarly, $\rho(G(R_q)) = G(R_q)$ if and only if $\rho_1 \in \mathrm{PGL}(2, \mathbb{F}_q)$.

By our hypothesis on R and on q, there exists an element $f \in R_q$ which is a non-unit in R_q. (It is possible to choose f to be in $\mathbb{F}_q[t]$ but this is not relevant for our purposes.) Recall that $\mathrm{Aut}(R_q)$ is finite. Set

$$s := \prod_{\rho \in \mathrm{Aut}(R_q)} \rho(f).$$

Clearly, $s \in R_q$ and is fixed under all automorphisms in $\mathrm{Aut}(R_q)$. As the action of $\mathfrak{R}_q$ on R_q factors through $\mathrm{Aut}(R_q)$, we see that s is fixed under all automorphisms of $\mathfrak{R}_q$.

The element s is non-zero and is a non-unit since $f(t)$ is not invertible. In particular, it is not in F. So $\mathbb{F}_p[s]$ is isomorphic to a polynomial algebra. It is clear that $\mathrm{SL}_n(\mathbb{F}_p[s])$ is element-wise fixed by $\mathfrak{R}_q$. ■

4. Proof of Theorem 1.1

We shall first prove Theorem 1.1 for the case when F is a finite field in Section 4.2. The proof in more general case when $F \subset \overline{\mathbb{F}_p}$ will be given in Section 4.3. We shall construct, in Lemma 4.1 below a sequence of elements $\{x_k\}_{k \geq 1}$ in $\mathrm{SL}_n(R)$ which will play a crucial role in our proofs of the main theorems.

When $F = \mathbb{F}_q$, we shall use the notation A, instead of R, for a ring which is such that $\mathbb{F}_q[t] \subset A \subsetneq \mathbb{F}_q(t)$.

4.1. A crucial lemma

Consider the automorphism $\rho\colon G(A) \to G(A)$ induced by a ring automorphism $\rho\colon A \to A$ where $\mathbb{F}_q[t] \subset A \subsetneq \mathbb{F}_q(t)$. Let $S = \mathbb{F}_p[s] \subset A$ be as in Corollary 3.12 applied to the ring A. Then S is contained in the subring $\mathrm{Fix}(\rho) \subset A$ and the group $G(S) \subset G(A)$ is element-wise fixed by ρ.

Set

$$x_m := e_{12}(s^m)e_{21}(-s^m) = \begin{pmatrix} 1-s^{2m} & s^m \\ -s^m & 1 \end{pmatrix} \in \mathrm{SL}_2(S).$$

We observe that $\rho(x_m) = x_m = e_{12}(s^m)\varepsilon(e_{12}(s^m))$ and that the x_m satisfy the polynomial $X^2 + (s^{2m} - 2)X + I_2 = 0$. We regard x_m also as an element of $\mathrm{SL}_n(S)$ for $n \geq 3$ by identifying it with the block diagonal matrix $\delta(x_m, I_{n-2})$. These elements will play an important role in the our proofs as they will be shown to be in pairwise distinct φ-twisted conjugacy classes for many automorphisms. The following lemma will play a crucial role in our proof.

Lemma 4.1. *Let A be a ring such that $\mathbb{F}_q[t] \subset A \subsetneq \mathbb{F}_q(t)$. Fix $r \geq 1$. With notations as above, the elements $x_m = e_{12}(s^m)e_{21}(-s^m) \in \mathrm{SL}_n(A)$ are such that $\mathrm{tr}(x_m^r)$, $m \geq 1$, are pairwise distinct.*

Proof. Let $u \in A$ be a non-unit and set

$$x = x(u) := e_{12}(u)e_{21}(-u) = \begin{pmatrix} 1-u^2 & u \\ -u & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p[u]).$$

We see that $\mathrm{tr}(x) = 2 - u^2$, $\mathrm{tr}(x^2) = 2 - 4u^2 + u^4$. As the characteristic polynomial of x is $X^2 - (2-u^2)X + 1$, we obtain the relation $\mathrm{tr}(x^r) = (2-u^2)\mathrm{tr}(x^{r-1}) - \mathrm{tr}(x^{r-2})$ for any $r \geq 3$. It follows by induction that $\mathrm{tr}(x^r)$ is a polynomial in u of degree $2r$ with leading coefficient $(-1)^r \in \mathbb{F}_p$.

The above statement still holds when $x(u)$ is identified with the matrix $\delta(x(u), I_{n-2})$ and viewed as an element of $\mathrm{SL}_n(\mathbb{F}_p[u])$, $n \geq 3$. Applying this to the elements $x_m \in \mathrm{SL}_n(A)$ defined above, $\mathrm{tr}(x_m^r) \in S = \mathbb{F}_p[s] \subset A$ is a polynomial in s of degree $2rm$. Hence $\mathrm{tr}(x_m^r)$, $m \geq 1$, are pairwise distinct. ∎

4.2. Proof of Theorem 1.1 for a finite field F

We shall now prove Theorem 1.1 when $F = \mathbb{F}_q$ and $n \geq 3$. We shall use A instead of R to denote a ring such that $\mathbb{F}_q[t] \subset A \subsetneq \mathbb{F}_q(t)$.

Let $G = \mathrm{GL}_n(A)$ or $\mathrm{SL}_n(A)$. To show that G has the R_∞-property, it suffices to show that $R(\varphi) = \infty$ for $\varphi \in \mathcal{S}$, where $\mathcal{S}$ is a complete set of representatives of the outer automorphisms of G. We take $\mathcal{S}$ to be the set consisting of representative automorphisms as in Corollary 3.5.

We are now ready to prove Theorem 1.1 when F is a finite field.

Proof. The proof will depend on the type of automorphism as listed in Corollary 3.5. The symbol ρ will always denote an automorphism of G induced by a ring automorphism of A, ε, the contragredient, ι_g, the conjugation by a $g \in \mathrm{GL}_n(A)$, etc.

First we consider the case when $G = \mathrm{GL}_n(A)$. We treat separately the cases of a ring automorphism ρ and the automorphism $\rho \circ \varepsilon$.

Type ρ: Note that $x_m \in \mathrm{Fix}(\rho)$. Taking $r = o(\rho)$ in Lemma 4.1, we see, according to Lemma 2.3, that the x_m^r, $m \geq 1$, are in pairwise distinct ρ-twisted conjugacy classes and so $R(\rho) = \infty$.

Type $\rho \circ \varepsilon$: Let $\theta = \rho \circ \varepsilon$. Since $\rho \circ \varepsilon = \varepsilon \circ \rho$ and since $\varepsilon^2 = \mathrm{id}$, we have $\theta^2 = \rho^2$. We shall show that $e_{12}(s^k)$, $e_{12}(s^m)$ are not in the same θ-twisted conjugacy class if $m > k \geq 1$.

Suppose $e_{12}(s^m) = ze_{12}(s^k)\theta(z^{-1})$. Applying θ to both sides, we obtain $e_{21}(-s^m) = \theta(z)e_{21}(-s^k)\theta^2(z^{-1})$. Multiplying the two equations and using $\theta^2 = \rho^2$, we obtain that

$$x_m = e_{12}(s^m)e_{21}(-s^m) = ze_{12}(s^k)e_{21}(-s^k)\rho^2(z^{-1}) = zx_k\rho^2(z^{-1}).$$

That is, x_k, x_m are in the same ρ^2-twisted conjugacy class. By what has been shown in the case of type ρ, this is a contradiction. It follows that $R(\theta) = \infty$.

Type $\mu_\chi \circ \rho$: Let $\theta = \mu_\chi \circ \rho$ and let $r = o(\rho)$. We claim that the elements $x_m^r \in \mathrm{SL}_n(A)$, $m \geq 1$, are in pairwise distinct θ-twisted conjugacy classes.

Suppose that $x_k = zx_m\theta(z^{-1})$ for some $z \in \mathrm{GL}_n(A)$. Note that $\theta(z^{-1}) = \rho(z^{-1})u$, where $u := \chi(\rho(z^{-1}))I_n$. Thus $x_k = zx_m\rho(z^{-1})u$. Since u is in the centre of $\mathrm{GL}_n(A)$, for any $j \geq 1$, applying θ repeatedly, we obtain that for any $j \geq 0$, $x_k = \theta^j(x_k) = \rho^j(z)x_m\rho^{j+1}(z^{-1})u_j$ for a suitable scalar matrix u_j in $\mathrm{GL}_n(A)$. Setting $r := o(\rho)$, we are led to the equation $x_k^r = zx_m^r z^{-1}v$ for some scalar matrix $v \in \mathrm{GL}_n(A)$. Writing $v = aI_n$ and taking determinants on the both sides of the above equation, we obtain that $a^n = 1$, i.e., a is a torsion element in A and hence v is a scalar matrix in $\mathrm{GL}_n(\mathbb{F}_q)$. Now, we take trace on both sides of the equation $x_k^r = zx_m^r z^{-1}v$ and get $\mathrm{tr}(x_k^r) = a\,\mathrm{tr}(x_m^r) \in \mathbb{F}_q[s]$. This contradicts Lemma 4.1 as the degree of $\mathrm{tr}(x_j^r)$ as a polynomial in s equals $2jr$. This shows that $R(\mu_\chi \circ \rho) = \infty$.

Type $\mu_\chi \circ \rho \circ \varepsilon$: The proof that $R(\mu_\chi \circ \rho \circ \varepsilon) = \infty$ uses $e_{12}(s^m) \in \mathrm{SL}_n(A)$ and is similar to the proof for the type $\rho \circ \varepsilon$, just as the above proof for $\mu_\chi \circ \rho$ parallels the proof for type ρ.

This completes the proof that $\mathrm{GL}_n(A)$ has the R_∞-property for $n \geq 3$.

It remains to consider the case of the automorphisms of $\mathrm{SL}_n(A)$ as in Corollary 3.5 (i).

Type $\iota_h \circ \rho$: Consider an automorphism φ of $\mathrm{SL}_n(A)$ of the form $\varphi = \iota_h \circ \rho$ with $h = h(a) \in H$ with $a \in A^\times$ as in Corollary 3.5 (i). Suppose that k and m are distinct but $x_k = zx_m\varphi(z^{-1}) = zx_m h\rho(z^{-1})h^{-1}$. So $x_k h$ and $x_m h$ are ρ-twisted conjugates in $\mathrm{GL}_n(A)$. We apply Lemma 2.3 (i) to ρ. Setting $r = o(\rho)$, we obtain that $\prod_{0\leq j<r} \rho^j(x_k h)$ and $\prod_{0\leq j<r} \rho^j(x_m h)$ are conjugates in $\mathrm{GL}_n(A)$. Since x_k and $\rho^j(h) = h(\rho^j(a))$ commute, we obtain that $\prod_{0\leq j<r} \rho^j(x_k h) = x_k^r h(\prod \rho^j(a))$ and the same holds when k is replaced

by m. Now, $\mathrm{tr}(x_k^r h(\prod \rho^j(a))) = n - 3 + \prod \rho^j(a) + \mathrm{tr}(x_k^r)$ and so $\mathrm{tr}(\prod_{0 \le j < r} \rho^j(x_k h)) = \mathrm{tr}(\prod_{0 \le j < r} \rho^j(x_m h))$ implies $\mathrm{tr}(x_k^r) = \mathrm{tr}(x_m^r)$, contradicting Lemma 4.1.

Type $\iota_h \circ \rho \circ \varepsilon$: Finally, it remains to consider automorphisms of $\mathrm{SL}_n(A)$ of the form $\psi := \iota_h \circ \rho \circ \varepsilon$ with $h = h(a)$, $a \in A^\times$, as in Corollary 2.3. We assert that $e_{12}(s^m)$, $m \ge 1$, are in pairwise distinct ψ-twisted conjugacy classes. Suppose that there exist positive integers k, m such that

$$e_{12}(s^m) = z e_{12}(s^k) \psi(z^{-1}).$$

Applying ψ to both sides of this equation, we obtain

$$e_{21}(-s^m) = \psi(z) e_{21}(-s^m) \psi^2(z^{-1}).$$

Multiplying the two sides of the two equations and using $x_m = e_{12}(s^m) e_{21}(-s^m)$, we have

$$x_m = z x_k \psi^2(z^{-1}).$$

Since, by Lemma 3.7, $\psi^2 = \iota_{h\rho(h^{-1})} \circ \rho^2 = \iota_{h(a\rho(a^{-1}))} \circ \rho^2$, it is of the previous type, namely, type $\iota_h \circ \rho$. So x_m, x_k are not ψ^2-twisted conjugates unless $m = k$. Therefore, we conclude that $R(\psi) = \infty$, completing the proof. ∎

4.3. Completion of the proof of Theorem 1.1

Let R be a ring such that $F[t] \subset R \subsetneq K := F(t)$, where F is a subfield of $\overline{\mathbb{F}_p}$ and t is a variable.

Let $G(R)$ denote one of the groups $\mathrm{GL}_n(R)$, $\mathrm{SL}_n(R)$, $n \ge 3$. The proof of Theorem 1.1 for $F \subset \overline{\mathbb{F}_p}$ is similar to the special case when $F = \mathbb{F}_q$. In fact, most of the proof in the general case reduces to the special case. For this reason, we shall omit most of the details.

Let $\rho \in \mathrm{Aut}(R)$. As noted in Lemma 3.9, we have $\rho = \rho_1 \circ \varphi$, where ρ_1 is a Möbius transformation and φ is a Frobenius automorphism. Also, ρ_1 is defined over $\mathbb{F}_q$ for some q.

By our choice of q, if $\mathbb{F}_q \subset \mathbb{F}_\ell \subset F$, we see that $R_\ell := R \cap \mathbb{F}_\ell(t) \subset F(t)$ is stable under ρ. Also, since R is not a field, we may (and do) assume that R_ℓ is not a field. Recall that $R = \bigcup_\ell R_\ell$ where the union is over such values of ℓ that R_ℓ is not a field.

Consequently, the groups $G_\ell := G(R_\ell) \in \{\mathrm{GL}_n(R_\ell), \mathrm{SL}_n(R_\ell)\}$ are stable under ρ, and Theorem 1.1 holds for each of them. We observe that $G(R)$ is the union of the groups G_ℓ.

We are now ready to prove Theorem 1.1 in the general case.

Proof of Theorem 1.1. As observed already, with notations from Corollary 3.5, we need only to consider the automorphisms $\varphi = \mu_\chi \circ \rho$, $\mu_\chi \circ \rho \circ \varepsilon$ when $G(R) = \mathrm{GL}_n(R)$, and, when $G = \mathrm{SL}_n(R)$, the automorphisms $\varphi = \iota_h \circ \rho$, $\iota_h \circ \rho \circ \varepsilon$, where $h = h(a) = \delta(I_{n-1}, a)$ in $\mathrm{GL}_n(R)$ and $a \in R^\times$.

Let $\rho \in \mathrm{Aut}(R)$ be defined over $\mathbb{F}_q$ for some q. Let $s \in R_q$ be as in Corollary 3.12 and $x_m = e_{12}(s^m) e_{21}(-s^m) \in G_q = \mathrm{GL}_n(R_q)$, where $m \ge 1$ and $R_q = R \cap \mathbb{F}_q(t)$, be as in Lemma 4.1. Then $x_m \in \mathrm{Fix}(\rho)$. Suppose that there exists an element $z \in G$ such that $x_k = z x_m \rho(z^{-1})$ with $k \ne m$. There exists $\ell = q^d = p^{de}$, where $q = p^e$, a sufficiently large power of q such that $\mathbb{F}_\ell \subset F$ and $x_k, x_m, z \in G_\ell$. Then $\rho^N|_{G_\ell} = \mathrm{id}$, where $N := de$. This implies, by Lemma 2.3, that x_m^N and x_k^N are conjugates in G_ℓ. This contradicts Lemma 4.1, and we conclude that $R(\rho) = \infty$. The proof for $\rho \circ \varepsilon$ is similar to the proof of the corresponding type of automorphism in Theorem 1.1 for $n \geq 3$ given in Section 4.2.

Now let $\varphi = \mu_\chi \circ \rho$.

Suppose that $x_k \sim_\varphi x_m$ for some $k \neq m$. Let $z \in G$ such that $x_k = z x_m \varphi(z^{-1}) = z x_m \rho(z^{-1}) u I_n$, where $u = \chi(\rho(z^{-1})) \in R^\times$. Then there exists an $\ell = q^d = p^{de}$ for a sufficiently large d so that $z \in G_\ell$. Then $\rho^{de}|_{G_\ell} = \mathrm{id}$. Applying ρ repeatedly to both sides of this equation, we obtain

$$x_k = \rho^j(z) x_m \rho^{j+1}(z^{-1}) \cdot u_j I_n$$

for $j \geq 1$ for suitable $u_j \in R^\times$. Multiplying these equations in order for $0 \leq j < de$ and using the fact that $\rho^{de}(z^{-1}) = z^{-1}$, we obtain that

$$x_k^{de} = z x_m^{de} z^{-1} \cdot v I_n$$

for some $v \in R^\times$. First taking determinant on both sides of the equation, we observe that v is a torsion element of R and so $v \in F^\times$. Now taking trace on both sides, we get $\mathrm{tr}(x_k^{de}) = v\,\mathrm{tr}(x_m^{de})$. This is a contradiction since $v \in F$ and the degrees of traces of x_k^{de}, x_m^{de} as polynomials in s are $2kde$, $2mde$, respectively, which are unequal if $k \neq m$. Hence we conclude that $R(\varphi) = \infty$ in this case.

The proof is similar when $\varphi = \mu_\chi \circ \rho \circ \varepsilon$. This completes the proof when $G = \mathrm{GL}_n(R)$.

When $G = \mathrm{SL}_n(R)$, we need to show that $R(\varphi) = \infty$ when $\varphi = \iota_h \circ \rho$, $\iota_h \circ \rho \circ \varepsilon$, where $h = h(a) = \delta(I_{n-1}, a)$, $a \in R^\times$. We choose $q = p^e$ so that ρ restricts to G_q and hence to G_ℓ for all $\ell = q^r$. We choose ℓ so that R_ℓ is not a field. The rest of the proof is as in the proof of Theorem 1.1 for the automorphisms of $\mathrm{SL}_n(R_\ell)$ of the corresponding types, given in Section 4.2. The details are left to the reader. ∎

5. Proof of Theorem 1.2

We begin by describing the multiplicative group $R^\times$ of all invertible elements of R and the action of $\mathrm{Aut}(R)$ on it. Since R is a localization of the polynomial algebra $F[t]$, R is a Euclidean domain. See [36, Proposition 7]. In particular, R is a unique factorization domain.

Let $\mathcal{B}$ be the set of all monic irreducible polynomials in $F[t]$ which are invertible in R. Then $R = F[t][1/b(t); b(t) \in \mathcal{B}]$. The following result, which is an analogue of the Dirichlet unit theorem, is perhaps well known to experts. The case when $\mathcal{B}$ is finite and $F = \mathbb{F}_q$ is treated in [34, Proposition 14.2]. We give an elementary proof as we could not find a reference.

Theorem 5.1. (i) *Any $f(t) \in R^\times \setminus F$ has a unique factorization $f(t) = af_1^{n_1} \cdots f_k^{n_k}$ with elements $f_j \in \mathcal{B}$, n_j non-zero integers, and $a \in F^\times$.*

(ii) *$R^\times$ is isomorphic to $F^\times \times U$, where U is a free abelian group with basis $\mathcal{B}$.*

(iii) *Any automorphism of the group $R^\times$ preserves $F^\times$ and induces an automorphism of $R^\times/F^\times \cong U$.*

Proof. Let $f(t) \in R^\times$. Write that $f(t) = g(t)/h(t)$, where $g(t), h(t) \in F[t]$ with $\gcd(g(t), h(t)) = 1$. We claim that $g(t), h(t) \in R^\times$. It suffices to show that $g(t) \in R^\times$ since $h(t) = g(t)/f(t)$. Since $\gcd(g(t), h(t)) = 1$, we have $u(t), v(t) \in F[t]$ such that $g(t)u(t) + h(t)v(t) = 1$. Multiplying both sides by $f(t)$, we obtain that

$$f(t) = f(t)g(t)u(t) + f(t)h(t)v(t) = g(t)(f(t)u(t) + v(t)).$$

Since $f(t) \in R^\times$, we conclude that $g(t)$ – and hence $h(t)$ – are units in R. The asserted factorization of $f(t) = g(t) \cdot (h(t))^{-1}$ follows immediately by factoring $g(t)$ and $h(t)$ into irreducible polynomials. Since $g(t)$, $h(t)$ are units in R, so are their irreducible factors. This proves (i). Now (ii) follows immediately from (i). Assertion (iii) follows since $F^\times$ equals the torsion subgroup of $R^\times$ as $F \subset \overline{\mathbb{F}_p}$. ∎

There is a bijective correspondence between subgroups $D \subset R^\times$ and subgroups $H(D) \subset \mathrm{GL}_n(R)$ that contain $\mathrm{SL}_n(R)$, where $H(D) = \{g \in \mathrm{GL}_n(R) \mid \det g \in D\}$.

Recall from Proposition 3.1 that $\mathrm{SL}_n(R) = [H(D), H(D)] = [\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$. Hence $\mathrm{SL}_n(R)$ is characteristic in $H(D)$ for all $D \subset R^\times$. Recall that $h(c)$ denotes the diagonal matrix $\delta(I_{n-1}, c) \in \mathrm{GL}_n(R)$ for $c \in R^\times$. Setting

$$h(D) := \{h(c) \in H(D) \mid c \in D\} \cong D,$$

we have

$$H(D) = \mathrm{SL}_n(R) \cdot h(D) \cong \mathrm{SL}_n(R) \rtimes D.$$

Lemma 5.2. *Suppose that $g \in H = H(D)$ commutes with every element of $\mathrm{SL}_{n-1}(R)$. Then $g = ah(b) = \delta(aI_{n-1}, ab)$ for some $a, b \in R^\times$ with $\det(g) = a^n b \in D$.*

Proof. Suppose that g is not a diagonal matrix, say $g = (g_{ij})$ with $g_{km} \neq 0$, where $k \neq m$. Then $e_{1k}(1)ge_{1k}(-1) \neq g \neq e_{1m}(1)ge_{1m}(-1)$. Since at least one of k, m is less than n, we get a contradiction. Now suppose that g is a diagonal matrix $g = \delta(a_1, \ldots, a_n)$. If $a_i \neq a_j$ for some $i < j < n$, then $ge_{ij}(1)g^{-1} \neq e_{ij}(1)$ and the lemma follows. ∎

Let $H = H(D)$ and let $\theta \colon H \to H$ be an automorphism. Then θ restricts to an automorphism $\theta' \colon \mathrm{SL}_n(R) \to \mathrm{SL}_n(R)$. Replacing θ by $\theta \circ \iota_g$ for a suitable $g \in \mathrm{SL}_n(R)$ if necessary, we may (and do) assume that $\theta' = \iota_h \circ \rho \circ \eta$, where $h = h(a) = \delta(I_{n-1}, a)$, $a \in R^\times$, $\rho \colon \mathrm{SL}_n(R) \to \mathrm{SL}_n(R)$ is induced by an automorphism $\rho \colon R \to R$ of the ring R and η belongs to the cyclic group $\langle \varepsilon \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then $\theta'(\mathrm{SL}_{n-1}(R)) = \mathrm{SL}_{n-1}(R)$. By Corollary 3.5 (iii), θ' extends to an automorphism $\varphi \colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$. Any extension φ

equals $\mu_\chi \circ \iota_{h(a)} \circ \rho \circ \eta$, where μ_χ is a homothety automorphism associated to a character $\chi\colon \mathrm{GL}_n(R) \to R^\times$. It is not clear that $\theta \in \mathrm{Aut}(H)$ admits an extension $\varphi\colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$. When it does, following proposition guarantees that $R(\theta) = \infty$.

Proposition 5.3. *Let $\theta \in \mathrm{Aut}(H)$ and suppose that $\theta = \varphi|_H$ for some $\varphi \in \mathrm{Aut}(\mathrm{GL}_n(R))$. Then $R(\theta) = \infty$.*

Proof. We keep the above notation. By the previous discussion, we need only consider the case $\theta' = \iota_{h(a)} \circ \rho \circ \varepsilon$ or $\iota_{h(a)} \circ \rho$ in $\mathrm{Aut}(\mathrm{SL}_n(R))$. First suppose that $\theta' = \iota_{h(a)} \circ \rho$. Choose a non-zero, non-unit element $s \in R$ fixed by ρ. Such an element exists by Corollary 3.12. Choose q so that $s \in R_q = R \cap \mathbb{F}_q(t)$. Let r be the order of the automorphism $\rho|_{R_q}$. Since the commutation relation $\mu_\chi \circ \iota_h = \iota_h \circ \mu_\chi$ holds by Lemma 3.6 (i), we have $\varphi = \mu_\chi \circ \iota_{h(a)} \circ \rho = \iota_{h(a)} \circ \psi$, where $\psi := \mu_\chi \circ \rho$.

Consider the elements $x_m = e_{12}(s^m)e_{21}(-s^m)$, $m \geq 1$. We claim that x_k and x_m are in distinct θ-twisted conjugacy classes if $k \neq m$. Assume that $[x_m]_\theta = [x_k]_\theta$. It follows that $[x_k]_\varphi = [x_m]_\varphi$. This implies that $[x_k h(a)]_\psi = [x_m h(a)]_\psi$. Proceeding as in the case of type $\mu_\chi \circ \rho$ in the proof of Theorem 1.1 for finite fields, we obtain that

$$x_m^r h(u) = z x_m^r v h(u) z^{-1},$$

where $v \in R^\times$, $u = \prod_{0 \leq j < r} \rho^j(a)$. Taking determinants on both sides, we obtain that $u = v^n u$ and so $v^n = 1$. Raising to the n-th power, we obtain $x_m^{rn} h(u^n) = z x_m^{rn} h(u^n) z^{-1}$. Taking trace on both sides of the last equality, we obtain that $\mathrm{tr}(x_m^{rn}) = \mathrm{tr}(x_k^{rn})$. This contradicts Lemma 4.1 if $k \neq m$. Hence $R(\psi) = \infty$.

The proof in the case when $\psi = \mu_\chi \circ \rho \circ \varepsilon$ is similar and omitted. ■

Write $\rho = \rho_1 \circ \varphi$, where ρ_1 is an F-automorphism and φ is a Frobenius automorphism. We choose q so that ρ_1 is defined over $\mathbb{F}_q$ and that $a \in \mathbb{F}_q$. We further assume that $R_q = R \cap \mathbb{F}_q(t)$ is not a field. All these conditions can be met so long as q is sufficiently large (and $\mathbb{F}_q \subset F$).

Recall that $\mathfrak{R}_q \subset \mathrm{Aut}(R)$ denotes the subgroup $\{\rho \in \mathrm{Aut}(R) \mid \rho(R_q) = R_q\}$. As noted in Remark 3.10 (ii), if $\rho \in \mathfrak{R}_q$, then $\rho|_{R_q}$ has finite order and so the same is true for the induced automorphism of $\mathrm{SL}_n(R_q)$.

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\theta \in \mathrm{Aut}(H)$. Replacing θ by $\iota_g \circ \theta$ if necessary, we may assume that $\theta' = \theta|_{\mathrm{SL}_n(R)}$ equals $\iota_{h(a)} \circ \rho$ or $\iota_{h(a)} \circ \rho \circ \varepsilon$, where $a \in R^\times$, $\rho \in \mathrm{Aut}(\mathrm{SL}_n(R))$ is a ring automorphism $\rho\colon R \to R$ and ε is the contragredient. We first consider the case $\theta' = \iota_h \circ \rho$, where $h = h(a)$. We choose q so that R_q is not a field and ρ restricts to an automorphism of R_q.

Let $x_m \in \mathrm{SL}_n(R_q) \subset H$, $m \geq 1$, be as in Lemma 4.1. Suppose that $m > k \geq 1$ and that

$$x_m = z x_k \theta(z^{-1}). \tag{2}$$

Write $z = yh(c)$, where $c = \det(z) \in D$ and $y = zh(c)^{-1} \in \mathrm{SL}_n(R)$. Substituting $z = yh(c)$, we obtain

$$\theta(z) = \theta'(y)\theta(h(c)) = h(a)\rho(y)h(a^{-1})\theta(h(c)).$$

Since $h(c)$ commutes with the elements of $\mathrm{SL}_{n-1}(R)$, Lemma 5.2 implies that $\theta(h(c)) = b_1h(c_1)$ for some elements b_1 and c_1 in $R^\times$. Taking determinants on both sides of equation (2), we obtain $\det(z) = \det(\theta(z))$ which implies $\theta(h(c)) = b_1h(b_1^{-n}\rho(c))$.

Since $\det(x_m) = \det(x_k) = 1 = \det(y)$, (2) implies that $c = \det(z) = \det(\theta(z)) = b_1^n.b_1^{-n}\rho(c)$ and so $\rho(c) = c$. It follows that $\theta(h(c)) = b_1h(b_1^{-n}c)$.

Since x_k commutes with $h(b)$ for all $b \in R^\times$, we get

$$\begin{aligned} zx_k\theta(z^{-1}) &= yh(c)x_kb_1^{-1}h(b_1^nc^{-1})\theta'(y^{-1}) \\ &= yh(c)x_kb_1^{-1}h(b_1^nc^{-1})h(a)\rho(y)h(a^{-1}) \\ &= yx_kb_1^{-1}h(ab_1^n)\rho(y)h(a^{-1}). \end{aligned}$$

Hence we obtain that

$$x_mh(a) = b_1^{-1}yh(ab_1^n)x_k\rho(y^{-1}).$$

Applying ρ^j to both sides and using

$$\rho(x_k) = x_k, \quad \rho(x_m) = x_m, \quad \rho(h(u)) = h(\rho(u)),$$

we get

$$x_mh(\rho^j(a)) = \rho^j(b_1^{-1})\rho^j(y)h(\rho^j(ab_1^n))x_k\rho^{j+1}(y^{-1}).$$

Replacing $\mathbb{F}_q$ by a finite extension field contained in F if necessary, we may (and do) assume that c, a, b_1 are all in R_q. Multiplying these successively as j varies from 0 to $r-1$, where $r := o(\rho|_{R_q})$, we obtain that

$$x_m^r \prod_{0\le j<r} h(\rho^j(a)) = \prod_{0\le j<r} \rho^j(b_1^{-1})y\Big(\prod_{0\le j<r} h(\rho^j(ab_1^n))\Big)x_k^r y^{-1} \tag{3}$$

since $\rho(x_k) = x_k$, $\rho(x_m) = x_m$.

To simplify notations, we set

$$\beta := \prod_{0\le j<r} \rho^j(b_1^{-1}), \quad u := \prod_{0\le j<r} \rho^j(a), \quad \text{and} \quad v := \prod_{0\le j<r} \rho^j(ab_1^n) = \beta^{-n}u \in R_q.$$

Equation (3) says that the matrices $x_m^rh(u)$ and $\beta x_k^rh(v)$ are similar since βI_n and y commute. Therefore, their characteristic polynomials are equal. Note that these are block diagonal matrices with block sizes $2, 1, \ldots, 1$. Recall the definition

$$x_m = e_{12}(s^m)e_{21}(-s^m) = \delta(A_m, I_{n-2}), \quad \text{where } A_m = \begin{pmatrix} 1-s^{2m} & s^m \\ -s^m & 1 \end{pmatrix}.$$

The characteristic polynomials of $x_m^r h(u)$ and $\beta x_k^r h(v)$ are $(X^2-\lambda_m X+1)(X-1)^{n-3}\times (X-u)$ and $(X^2-\beta\lambda_k X+\beta^2)(X-\beta)^{n-3}(X-\beta v)$, respectively, where $\lambda_m=\mathrm{tr}(A_m^r)$. We have

$$(X^2-\lambda_m X+1)(X-1)^{n-3}(X-u)=(X^2-\beta\lambda_k X+\beta^2)(X-\beta)^{n-3}(X-\beta v).$$

If at least one of $(X^2-\beta\lambda_k+\beta^2)$ or $(X^2-\lambda_m X+1)$ is irreducible (in R), then so is the other and the two must be equal. Any root in R of the polynomial on the one side must also occur as a root of the polynomial on the other side with the *same multiplicity*. The rest of the proof will make repeated use of this observation.

Suppose $X^2-\lambda_m X+1$ is irreducible. Then $X^2-\lambda_m X+1=X^2-\lambda_k\beta X+\beta^2$, and so, by comparing coefficients of the polynomials, we must have $\beta=\pm 1$ and $\lambda_m=\pm\lambda_k$. The last equality contradicts Lemma 4.1. It follows that $X^2-\lambda_m X+1$ (and hence $X^2-\lambda_k\beta X+\beta^2$) have their roots in R.

Let $\alpha_k^{\pm 1},\alpha_m^{\pm 1}$ be the eigenvalues of A_k, A_m, respectively. We do not assume that these eigenvalues are in R. However, by what was noted above, $\beta\alpha_k^r$, $\beta\alpha_k^{-r}$, α_m^r, α_m^{-r} are all in R. Since $\beta\in R^\times$, we have $\alpha_k^r,\alpha_k^{-r}\in R$. By Lemma 4.1, $\alpha_k^r\neq\alpha_k^{-r}$ and $\alpha_m^r\neq\alpha_m^{-r}$.

Suppose that $\beta\in R^\times$ is a torsion element. Say, $\beta^\ell=1$. Then we may raise to the ℓ-th power both sides of equation (3) and obtain the same equation in which r is replaced by $r\ell$ and β by 1. Consequently, we obtain that traces of $A_m^{r\ell}$ and $A_k^{r\ell}$ are equal, contradicting Lemma 4.1. So, we must have that $\beta\notin F^\times$.

We shall denote a (finite) multiset by an unordered sequence where each element in it is repeated as many times as its multiplicity. We shall write $\lambda^{(r)}$ to denote that λ occurs r times. The multiset of eigenvalues of $x_m^r h(u)$ is $\mathcal{M}=\alpha_m^r,\alpha_m^{-r},1^{(n-3)},u$, and that of $\beta x_k^r h(v)$ is $\mathcal{K}=\beta\alpha_k^r,\beta\alpha_k^{-r},\beta^{(n-3)},\beta v$. The fact that $\mathcal{M}=\mathcal{K}$ will be exploited to force a contradiction. The rest of the proof will involve case considerations, depending on the value of n.

Case 1: Suppose that $n=3$. By hypothesis, $D\subset F^\times$ and so $\det z=c\in F^\times$. It follows that b_1, c_1 are also in $F^\times$ and so $\beta\in F^\times$, contradicting our earlier conclusion that $\beta\notin F^\times$. This shows that x_k, x_m are not θ-twisted conjugates and so $R(\theta)=\infty$.

Case 2: Suppose that $n\geq 5$. Since 1 occurs in $\mathcal{M}$ with multiplicity at least 2, it occurs in $\mathcal{K}$ with the same multiplicity. Since $\beta\alpha_k^r\neq\beta\alpha_k^{-r}$, at most one of them can be equal to 1. So we must have $\beta=1$ or $\beta v=1$. Since $\beta\notin F^\times$, we have that $\beta v=1$.

Since $v=u\beta^{-n}$, we have $u=\beta^{n-1}$. Since 1 occurs in $\mathcal{M}$ with multiplicity at least 2, we must have $\beta\alpha_k^r=1$ or $\beta\alpha_k^{-r}=1$. In either case, we have $\mathcal{K}=1,\beta^2,\beta^{(n-3)},1$. Since $\alpha_m^r.\alpha_m^{-r}=1$, there should be two terms of $\mathcal{K}$ which are reciprocals of each other. This implies that $\beta=1$ or $\beta^2=1$ or $\beta^3=1$ and so $\beta\in F^\times$, a contradiction.

Case 3: Suppose that $n=4$. Since $\beta\neq 1$ and since 1 occurs in $\mathcal{M}$, one of the following must hold: (i) $\beta v=1$, or (ii) $\beta\alpha_k^r=1$, or (iii) $\beta\alpha_k^{-r}=1$.

Subcase (i): Suppose that $\beta v=1$. It follows that $u=\beta^{n-1}=\beta^3$. Since β occurs in $\mathcal{K}$, it should occur in $\mathcal{M}$. Since $\beta\notin F^\times$, we have $u\neq\beta$, and so we must have $\alpha_m^r=\beta$

or $\alpha_m^{-r} = \beta$. In any case, we have $\mathcal{M} = \beta, \beta^{-1}, 1, \beta^3$. It follows that $\{\beta\alpha_k^r, \beta\alpha_k^{-r}\} = \{\beta^3, \beta^{-1}\}$. So we have $\mathrm{tr}(A_k^r) = \alpha_k^r + \alpha_k^{-r} = \beta^2 + \beta^{-2}$ and $\mathrm{tr}(\alpha_m^r) = \alpha_m^r + \alpha_m^{-r} = \beta + \beta^{-1}$. This leads to the relation $(\mathrm{tr}(A_m^r))^2 - \mathrm{tr}(A_k^r) - 2 = 0$. Since $\mathrm{tr}(A_m^r), \mathrm{tr}(A_k^r) \in F[s]$ are polynomials in s of degree $2mr$, $2kr$, respectively, and since $m > k$, the last equation cannot hold. So we are led to the conclusion that $\beta v \neq 1$.

Subcase (ii): Suppose that $\beta\alpha_k^r = 1$. Then $\beta\alpha_k^{-r} = \beta^2$ and $\mathcal{K} = 1, \beta^2, \beta, \beta v$. Since $\alpha_m^r, \alpha_m^{-r} \in \mathcal{K}$ and $\alpha_m^r \neq 1$, the product of some pair of elements of $\{\beta^2, \beta, \beta v\}$ should be equal to 1. If $\beta^3 = 1$, then $\beta \in F^\times$, a contradiction. Then there are two possibilities: $\{\alpha_m^r, \alpha_m^{-r}\} = \{\beta, \beta v\}$ or $\{\alpha_m^r, \alpha_m^{-r}\} = \{\beta^2, \beta v\}$. If $\{\alpha_m^r, \alpha_m^{-r}\} = \{\beta, \beta v\}$, then $\alpha_m^r + \alpha_m^{-r} = \beta + \beta^{-1}$. Since $\beta\alpha_k^r + \beta\alpha_k^{-r} = 1 + \beta^2$, we obtain that $\alpha_k^r + \alpha_k^{-r} = \beta^{-1} + \beta = \alpha_m^r + \alpha_m^{-r}$. Thus $\mathrm{tr}(A_m^r) - \mathrm{tr}(A_k^r) = 0$ in $F[s]$, again a contradiction as $m > k$. If $\{\alpha_m^r, \alpha_m^{-r}\} = \{\beta^2, \beta v\}$, then $\alpha_m^r + \alpha_m^{-r} = \beta^2 + \beta^{-2}$. Again, we have $\alpha_k^r + \alpha_k^{-r} = \beta^{-1} + \beta$. The last two equations imply that $(\mathrm{tr}(A_k^r))^2 - \mathrm{tr}(A_m^r) - 2 = 0$. Since $\mathrm{tr}(A_m^r)$, $\mathrm{tr}(A_k^r) \in F[s]$ are polynomials in s of degree $2mr$, $2kr$, respectively, we have $m = 2k$. This is a contradiction if we choose m, k to be arbitrary *odd* positive integers.

Subcase (iii): Suppose that $\beta\alpha_k^{-r} = 1$. Then $\beta\alpha_k^r = \beta^2$ and so $\mathcal{K} = 1, \beta^2, \beta, \beta v$. Rest of the proof is exactly as in subcase (ii).

This completes the proof in the case when $\theta' = \theta|_{\mathrm{SL}_n(R)} = \iota_h \circ \rho$.

Next consider the case $\theta' = \iota_h \circ \rho \circ \varepsilon$, where $h = h(a)$, $a \in R^\times$, ρ is induced by a ring automorphism $\rho\colon R \to R$, and ε is the contragredient. As in the proof of Theorem 1.1 for the type $\iota_h \circ \rho \circ \varepsilon$, we consider the sequence of elements $\{e_{12}(s^m)\}_{m\geq 1}$ and show that infinitely many terms of the sequence are in pairwise distinct θ-twisted conjugacy classes. Here again, $s \in R$ is a non-zero non-invertible element fixed by ρ.

Suppose that $m > k \geq 1$ and the following holds:

$$e_{12}(s^m) = z e_{12}(s^k)\theta(z^{-1}), \tag{4}$$

where $z = yh(c)$ with $c = \det(z) \in D$ and $y = zh(c)^{-1} \in \mathrm{SL}_n(R)$. Applying θ to both sides of the above equation, we get

$$e_{21}(-s^m) = \theta(z)e_{21}(-s^m)\theta^2(z^{-1}). \tag{5}$$

We multiply each side of equations (4) and (5) to obtain, using $e_{12}(s^m)e_{21}(-s^m) = x_m$,

$$x_m = z x_k \theta^2(z^{-1}).$$

From Lemma 3.7, we have $\varphi := \theta^2 = \iota_{h(a\rho(a^{-1}))} \circ \rho^2$.

Therefore, we obtain that x_m and x_k are φ-twisted conjugates.

Suppose that $n \geq 4$ or $D \subset F^\times$. By what has been established already, the sequence $\{x_k\}_{k\geq 1}$ contains an infinite subsequence whose terms are in pairwise distinct φ-twisted conjugacy classes. Therefore, the sequence $\{e_{12}(s^k)\}_{k\geq 1}$ contains an infinite subsequence whose terms are in pairwise distinct θ-twisted conjugacy classes. This completes the proof of Theorem 1.2. ∎

We end this section with the following remarks.

Remark 5.4. (i) Let $n = 3$. Suppose that $u = \beta v$. Since $v = \beta^{-3}u$, we obtain that $u = \beta^{-2}u$ and so $\beta^2 = 1$, a contradiction. There remains the following four possibilities:

(a) $u = \beta\alpha_k^r, \alpha_m^r = \beta v, \alpha_m^{-r} = \beta\alpha_k^{-r}$,

(b) $u = \beta\alpha_k^{-r}, \alpha_m^r = \beta v, \alpha_m^{-r} = \beta\alpha_k^{r}$,

(c) $u = \beta\alpha_k^r, \alpha_m^r = \beta\alpha_k^{-r}, \alpha_m^{-r} = \beta v$,

(d) $u = \beta\alpha_k^{-r}, \alpha_m^r = \beta\alpha_k^{r}, \alpha_m^{-r} = \beta v$.

Each of these possibilities appears to be consistent with the relation $v = \beta^{-3}u$ and we have not been able to show that $R(\theta) = \infty$ in these cases when $D \subset R^\times$ is arbitrary.

(ii) Suppose that $R^\times$ contains infinitely many irreducible elements of $F[t]$. For example, when F is infinite, let $\mathcal{S}$ be the linear polynomials $t + \lambda, \lambda \in F$. When $F = \mathbb{F}_q$, we take $\mathcal{S}$ to be the set the polynomials of $\{t^{q^r} - t + 1 \mid r \geq 1\}$. For any *proper* infinite subset $\mathcal{B} \subset \mathcal{S}$, we take R to be the localization $F[t][1/f(t); f(t) \in \mathcal{B}]$ so that $F[t] \subset R \subsetneq F(t)$. Then $R^\times = F^\times \times U$, where U is isomorphic to the free abelian group with basis $\mathcal{B}$. (See Theorem 5.1.) For any subset $B \subset \mathcal{B}$, we take $D = D(B) \subset R^\times$ to be the subgroup generated by B. Since $\mathcal{B}$ is countably infinite, there are $\aleph_1$ many such subgroups $D \subset R^\times$ that are pairwise *distinct*, although $D(B)$ and $D(B')$ are isomorphic if B, B' have the same cardinality. The corresponding collection $\mathcal{H} := \{H(D)\}$ of subgroups of $\mathrm{GL}_n(R)$ consists of pairwise distinct subgroups of $\mathrm{GL}_n(R)$. When $n \geq 4$, each of them has the R_∞-property. It seems plausible that there is a subcollection of $\mathcal{H}$ having cardinality $\aleph_1$ whose members of are pairwise non-isomorphic.

Acknowledgements. This project was initiated when both the authors were members of the Institute of Mathematical Sciences (Homi Bhabha Research Institute), Chennai. We are grateful to the referee of an earlier version of this paper for their many valuable comments and suggestions. A weaker version of Theorem 1.1 formed part of the Ph.D. thesis of the first author [24]. We are indebted to the referee of the previous version for a very thorough reading of our paper, for numerous helpful and detailed comments and suggestions for improving the presentation, for pointing out some mathematical inaccuracies, and for pointing out typos.

Funding. The authors gratefully acknowledge the Infosys Foundation for partial support for this project.

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Received 22 February 2022.

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