“If, and only if” in mathematics

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1 Why should we consider the converses of our results more seriously?

Although mathematics, and sciences in general, owe a lot to our predecessors, it appears that in mathematics some of them are also responsible, unintentionally, for our possible negligences with regard to the forms of the statements of some results. Most of the results in Euclidean geometry, number theory, and on the subject of inequalities, and so on, in the literature are of the form “if P then Q.” In particular, almost all the exercises and problems in the elementary textbooks on Euclidean geometry are of this form (see, e.g., [1, 6, 7, 18]). In the results presented in this form, Q is not asserted, only that Q is a necessary consequence of P. This style of thinking (called here briefly *pq-style*), which refers generally to thinking in the form “if P then Q” for presenting results, is somehow deep-seated in some of us for a long time. Perhaps, from the time of our learning secondary school mathematics. Later, it has also led some of us on, and made us so overwhelmed with this kind of thinking, that we are still generally attempting to draw necessary conclusions.

This is perhaps why Peirce, defining mathematics, begins his lengthy article [20], with “Mathematics is the science which draws necessary conclusions.” And more explicitly, Bertrand Russell, in [23, Chapter 1], essentially says: “Pure Mathematics is the class of all propositions of the form ‘p implies q,’ where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants.” At the same time, there are also some results in the literature which, even though not stated in the “if, and only if” form, are, in fact, intrinsically of this form. However, the actual forms are not emphasized on.

For example, in a triangle with two different angles we can state that an angle is bigger if, and only if, it has a shorter bisector, or equivalently if, and only if, the altitude which goes through the vertex of that angle is shorter, and finally if, and only if, the median which goes through the vertex of that angle is shorter (for the latter, less familiar fact, see the expression of the lengths of medians in terms of the lengths of the sides in [1, Theorem 106]). Most of the related books which contain some of these results state them in pq-style, see for example [1, Theorem 114], [7, Lemma 1.512], [6, Exercise 7, p. 9]. If we were used to presenting our results in the “if, and only if” style, perhaps the notorious old query concerning a direct proof of the Steiner–Lehmus theorem could not stay open in the literature for such a long time. Some students and their teachers, throughout the world, still have some trouble with this question, see [4, 13], for more details. Or, if the Morley’s trisector theorem had been stated in the “if, and only if” style, perhaps this theorem would not have seemed mysterious to so many authors, see [12, Corollary 1] for a statement of the theorem in the latter form, and also see the recently published books on the philosophy of mathematics, [21, pp. 41–43, and p. 46] and [17, p. 253]. Even in most elementary textbooks on geometry, the fact that Euclid’s fifth axiom is valid if, and only if, the sum of the angles in any triangle is 180°, is not emphasized on. It goes without saying that this fact could have helped the kids in high school to learn at an early age that in non-Euclidean geometry this sum is certainly not 180°, even without knowing anything about this geometry. Or, after Andrew Wiles’s proof of Fermat’s Last Theorem (FLT), we could state the theorem as: The equation $x^n + y^n = z^n$, where x, y, z, and $n > 0$ are integers, has nontrivial solutions if, and only if, $n = 2$ (note that, in the case $n \geq 2$ the solutions with $xyz = 0$ are considered trivial, and all the solutions in case $n = 1$ may be considered trivial, too). In this form, FLT also gives a unique characterization of 2, which is not its evenness. However, apparently everyone still prefers to state FLT in the pq-form.

In what follows we will define a concept of “good complete” theorem (briefly, *gc-theorem*), not only for Euclidean geometry, but also for all parts of mathematics, wherever possible. Before this, let us explain the rationale for the “converse” behind our initial question. Clearly, the converse of a mathematical result is not always necessarily true. In that case, the author may provide counterexamples and then, by studying the properties of these counterexamples, she/he might be motivated to add some extra properties to the statement of the original result, in order to get an eligible new result whose converse is also true. This might look as a drawback that causes a loss. However, the author may present the original result without any converse, separately, as an immediate corollary of the above new result, and thus easily overcome the loss (note, certainly the proof of the original result is a part of the proof of the above new result, hence the above corollary, which is in fact the original result, is indeed immediate). We should also emphasize that if the converse of a result in pq-style holds, then stating it the “if, and only if” style not only does not cause a loss, on the contrary, it is all about gaining advantages. Of course, we admit that turning a result into “if, and only if” form is not always easy, and even providing counterexamples in some particular cases may remain an open problem for years. An example is provided by FLT in the above form. However, we should admit that when a result is stated in the “if, and only if” form, a better understanding of the result is reached. Even more, in this case, each part of the result might give a clue for the proof of the other part.

There are some results in our field of expertise (topology and algebra) which are stated in the latter form, where none of the parts can be proved completely without invoking something from the other part, see for example [16, proof of Theorem 1.3]. This means that no part could exist separately. There are also some results which can be stated in the “if and only if” form, in which case they need no proof at all, but if stated in pq-form one runs into difficulties, see [12, Problem 2, Corollary 4], and [12, Problem 1, Corollary 3] for some examples of this kind. Let us recall briefly that [12, Problem 2] was originally stated in pq-style, and I believe that this form of the problem might have misled Terence Tao to explain and justify, in three pages, his trigonometric solution of the problem, and to give up any hope for presenting a possible geometric solution of the problem; see the comment preceding [12, Problem 2]. Whereas if we state this problem in “if, and only if” form, as in [12, Corollary 4], then by invoking a simple geometric fact we notice that the latter corollary needs no proof, a fortiori, the latter problem needs no solution at all. However, see the proof of the corollary in [12], which is given there to justify our latter claim. It is folklore that the fuller statement of any result in mathematics is easier to understand and be proved than a restricted one, let alone if this fuller statement is also stated properly in the “if, and only if” form.

In my opinion, in general presenting a result properly in “if, and only if” form causes no serious drawbacks or any losses, as far as mathematics is concerned, except possibly that we have to devote much time to either guessing a proper converse, or to finding appropriate counterexamples in case the converse is not true. It is also possible that this will slow down the speed of our publications – but this is not a significant loss for most people, and I am one of them, for this is not a loss in terms of mathematics. So one should generally attempt to examine seriously the question of the validity of converse results.

In what follows we give some appropriate examples to show that when the converse of a result fails, we might be able to resolve it in the way described above. To start, we all know that if a triangle $\triangle ABC$ is isosceles with apex A, then in $\triangle ABC$ the altitude, the bisector, and the median that go through the vertex A coincide. The converse is also obvious. We may recast the previous fact by asserting that the bisector of the angle A is also the bisector of the angle between the altitude and the median of this triangle that go through the vertex A. However, in this case, unfortunately, the converse is not true in general (note that the right triangles also have the latter property and may serve as counterexamples). Now let us consider a converse for the latter case and present the next interesting and useful theorem in the “if, and only if” form to justify our claim that considering the converses of our results is important.

Theorem 1.1. *In any triangle $\triangle ABC$, the bisector of the angle $\angle A$ is also the bisector of the angle between the altitude and the median which go through the vertex A, if and only if, the triangle $\triangle ABC$ is either isosceles with apex A, or is a right triangle with $\angle A = 90^\circ$.*

We can easily provide a proof by invoking the well-known fact that in every triangle $\triangle ABC$ the bisector of the angle $\angle A$ and the perpendicular bisector of the side BC either coincide, or meet each other on the circumcircle of the triangle. Is it not interesting that, by just considering a converse for the previous recast statement of an obvious fact we have obtained a nontrivial result which shows that isosceles triangles and right triangles have, in fact, a common characterization? A fact which seems to have been overlooked in the literature, see [12, Corollary 3] for another new common characterization of this kind. Despite their common characterizations, the following new fact in the “if, and only if” form which shows that a certain kind of these two triangles cannot coexist gives a new natural geometric proof of the irrationality of $\sqrt{2}$, see [5, 19] for various proofs of this irrationality. Should we not admit that we owe this to thinking in the “if, and only if” style?

Theorem 1.2. *$\sqrt{2}$ is irrational if, and only if, there exists no isosceles right triangle with rational sides.*

The proof of the theorem is evident in view of the fact that all the isosceles right triangles are similar to each other and, in particular, they are all similar to the one with the side lengths $1, 1, \sqrt{2}$. However, to infer from this theorem that $\sqrt{2}$ is indeed irrational, we may prove the second part of the statement of the theorem without invoking the irrationality of $\sqrt{2}$. To this end, just notice that if there is such a triangle, then by multiplying the sides by an appropriate integer we get isosceles right triangles with integer sides. Hence, there is the smallest isosceles right triangle among the latter ones (i.e., with the integer sides). But the bisector of the right angle in this smallest triangle divides into two congruent smaller isosceles right triangles and still with the integer sides (note, the length of the hypotenuse is even in all the isosceles right triangles with integer sides), which is a contradiction.

Also, see [14, Introduction] for the non-familiar converse of an obvious fact which leads us to some useful nontrivial and overlooked results in the "if, and only if" form in elementary number theory.

Definition 1.3. A theorem is called a *gc-theorem*, if its statement is of the form of "if, and only if" with no part that consists of facts that are too obvious to mention, unless some part is just the definition of a mathematical object.

Although, generally there should not be any restriction to certain kinds of methods (except the correctness of the methods) for the proofs of mathematical results, in my opinion in the case of gc-theorems, the proof of at least one part of the theorem should be motivated.

Needless to say, if a definition contains a statement that is a personal opinion, then one should provide enough explanations to justify this opinion. Accordingly, in what follows, some comments are given to clarify the definition as much as possible, and make it unambiguous. By considering almost any significant result of the form "if, and only if" (or, if not of this form, it can be easily turned into this form) in the literature, which might be regarded naturally by anyone as a gc-theorem, one immediately notices that the proofs of at least one part of such a result is usually motivated. We should emphasize that a proof is motivated if each step in the proof can be done by some reasoning and if it is also free of any kind of *deus ex machina*; see [12, p. 298], for some non-motivated proofs, and see [21, pp. 41–43] for a discussion of a motivated proof of Morley's trisector theorem. An appropriate example of latter kind is Hilbert's weak Nullstellensatz, which asserts that every maximal ideal M of $R = K[x_1, \ldots, x_n]$, where K is an algebraically closed field, is of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ for some point $x = (a_1, \ldots, a_n)$ in K^n. But, we know that this theorem of Hilbert is nothing but a generalization of the fact that over any field F, every non-constant polynomial has a root if, and only if, F is algebraically closed. Therefore, the weak Nullstellensatz can be easily turned into a gc-theorem, by claiming that M is of the above form if, and only if, K is algebraically closed (note, every non-constant polynomial in R is contained in a maximal ideal).

We should bear in mind that most authors are usually motivated by the existing results in the literature to obtain new results, whether in the form of gc-theorems or not. And in some cases they might also use proofs similar to those of the existing results. When the proofs of the two parts of an "if, and only if"-style result are simultaneously non-motivated and are also too complicated, it seems that the result may be artificially created. For example, see [11, Theorem 3], for such an artificial result, which is given there for some purpose. One can easily notice that the proof of the latter theorem is not motivated, and it is extremely difficult, if not impossible, to be given by anyone who does not already know the side lengths of the required triangles in the statement of the theorem. Incidentally, the statement of this artificial theorem can be easily turned into "if, and only if" form. We must admit that we know of no useful such results in the literature. In a nutshell, if a result in the form of "if, and only if" is to be considered as a genuine gc-theorem, it is reasonable to expect that at least one part of it should be stated in a natural way (i.e., not described in a convoluted way) with a rational motivated method for its proof. Can we call an "if, and only if" result a good complete theorem (i.e., a gc-theorem), if the proof of each part of the result is non-motivated and consisting of, say, more than 100 pages? Should we not naturally ask in this case, how on earth, could anyone have arrived at the idea of guessing such a result? Fortunately, there do not seem to be such results in the literature yet (at least, not to my knowledge). Let us, as a last remark in this regard, recall Fermat's Last Theorem, in the "if, and only if" form suggested above. Is it not true that the short, simple, and motivated proof of its first part, i.e., the case $n = 2$, had a major role in motivating Fermat to further work on the theorem and also for his correct guessing of the statement of its converse? And should we not emphasize that it was this motivated and simple proof of the first part that attracted the attention of so many mathematicians and students alike (not to mention the mathematical cranks) to the theorem, when it was still unresolved? Finally, it is this motivated and simple proof of the first part of FLT, and the simplicity of its statement, that is the source of its popularity, even among the high school kids.

We should emphasize that not all the results in the "if, and only if" form are genuine gc-theorems. For example, Fermat's little theorem in the form: "A prime number p divides $a^{p-1} - 1$, where a is a natural number, if, and only if $(a, p) = 1$," and similarly Wilson's theorem (i.e., an integer p is prime if, and only if, p divides $(p-1)! + 1$) are not genuine gc-theorems, because one part of their statements is too obvious (note that although these theorems are very useful, they are rarely used to recognize prime numbers). Also, the Steiner–Lehmus theorem in the form: "In a triangle, angle bisectors are equal if, and only if, they bisect equal angles" may not be considered as a gc-theorem either, for the same reason. Since the obviousness of a result might differ from individual to individual, in order to make the statement of our previous definition more precise, I suggest that although the form of results similar to the previous theorems might be improved whenever possible, in my opinion it is still preferable to formulate these three non-genuine gc-theorems in pq-style, and perhaps refer to them as *almost gc-theorems*.

For example, for the converse of Fermat's theorem one may claim that if n divides $a^{n-1} - 1$ and $n - 1$ is the smallest among the natural numbers m with the property that n divides $a^m - 1$, then n must be prime (note, if n is not prime, then $\phi(n) < n - 1$, where $\phi(n)$ is Euler's totient function). However, since there is no to place the phrase "if, and only if" between the statement of Fermat's theorem and the latter statement for its converse, we cannot get a gc-theorem by just combining the two statements. Otherwise, we could present Fermat's theorem as a genuine gc-theorem, too. As for the Steiner–Lehmus theorem, we have already stated it as a gc-theorem, namely, in a triangle with two different angles say, an angle is bigger if, and only if, it has a shorter bisector. We should emphasize that in this form only one direction needs a proof; the reader is referred to [4] and [1, Theorem 114] for the same proof of the theorem, which uses the similarity concept. In [4], the reader may also consult some other different proofs of this theorem, see also [7, Lemmas 1.511, 1.512] and [6, p. 420] for the history and more proofs. In particular, the notorious old query of finding a direct proof of this theorem is discussed in [4]; see also [13, last paragraph] to see a contrast between a very short proof of the theorem in its latter form and its possible formal, too lengthy, direct proof. It goes without saying that if a statement of a theorem consists of several equivalent statements, then the result can be considered as a gc-theorem. However, we should also emphasize that it seems that by inserting the phrase "if, and only if" between any two statements, A and B, say, with the same truth-value, we might get a result which looks like a gc-theorem. But here we must remind ourselves that by the equivalence of these kinds of statements we should mean that a proof of B must be deduced from A and vice versa, see [14, last theorem]. Therefore, this prevents us from interpreting any such artificial gc-theorems as genuine ones. We should also remind the reader that there are many important mathematical results that are mutually non-artificially equivalent, in the sense explained above (e.g., the axiom of choice, Zorn's lemma, Tychonoff's theorem, the fact that every set of independent vectors in a vector space can be extended to a basis of the space, the fact that every ring with unity has a maximal ideal). Indeed, it is well known that all these statements are equivalent, see [10]; for another example of a gc-theorem of this kind, see [14, last theorem].

Fortunately, many of the important results in the literature are genuine gc-theorems. We should remind the reader that the biconditional logical connective phrase "if, and only if" is used commonly enough in mathematics that it has its own abbreviation "iff." Apparently this abbreviation appeared first in John L. Kelly's 1955 book *General Topology*, where its invention is credited to Paul R. Halmos, see the last four lines in the preface of this book. However, in the literature there are still thousands of useful theorems, propositions, corollaries, lemmas and problems, which are not even in the "if, and only if" form, let alone gc-theorems. In what follows we like to consider some of these results and turn them into gc-theorems. Naturally, their selection is somewhat personal. We may deal with many important results, whether elementary or not, and try to turn them into gc-theorems. However, because of the scope of this essay only a few apparently non-elementary cases will be treated, to demonstrate that the literature abounds with non-elementary non-gc-theorems, too. Let us also emphasize that our discussion here is not intended to be a pq-style vs. gc-theorems argument: rather, our aim is to show that among the aforementioned important and useful non-gc-theorems many are eligible to be presented as gc-theorems.

2 Some gc-theorems in topology, algebra, and analysis

There are thousands of useful non-gc-theorems in the three fields listed above, where most of them are eligible to be reconsidered as gc-theorems, see also [8, Section 6.1]. As prototypes, in the following subsections I mention only three of them (one from each field, in reverse order), which are also in the textbooks.

2.1 When does an infinite set in $\mathbb{R}^n$ have a limit point?

Without further ado, we believe the following classical and very important non-gc-theorem is a good candidate to begin with, for it can, immediately, be turned into a gc-theorem which settles the question in the title.

Theorem (Bolzano–Weierstrass). *Every infinite bounded subset of $\mathbb{R}^n$ has a limit point.*

This result appears in every introductory textbook on analysis, see, e.g., [2, Theorems 3.13, 3.29] and [22, Theorem 2.42]. It seems that the existence of some unbounded countable subsets such as $\mathbb{N}$ or $\mathbb{Z}$ in $\mathbb{R}$ is responsible for the above boundedness constraint. At the same time, the set of rational numbers, the set of irrational numbers, and many other infinite sets in $\mathbb{R}$, which are not bounded, have limit points, but not directly as a consequence of the above theorem. However, with a little thought we may restate and record the above theorem as follows, which takes care of these sets, too.

Theorem 2.1 (Bolzano–Weierstrass). *Let $A \subseteq \mathbb{R}^n$. Then A has a limit point if, and only if, A is either uncountable, or it has a countably infinite bounded subset (or equivalently, if, and only if, A has the latter property).*

Proof. The proof for the case $n = 1$, which follows, can be imitated word-for-word for the general case. If A has a limit point, say $x \in \mathbb{R}$, then let $x \in (a, b)$, where $a, b \in \mathbb{R}$. Obviously, $A \cap (a, b)$ is an infinite set that contains an infinite countable subset of A which is clearly bounded. For the converse, we may only show that if A is uncountable without a limit point, we get a contradiction. Put $B = \{(r, s) : r, s \in \mathbb{Q}\}$. Since A has no limit point, for each $a \in A$ the set $F_a = \{G \in B : G \cap A = \{a\}\}$ is nonempty. Now, by the Axiom of Choice, for each $a \in A$ we can choose an element $G_a \in F_a$ and put $E = \{G_a : G_a \in F_a\}$. Obviously, $E \subseteq B$, hence E is countable. But the function $f : A \to E$, where $f(a) = G_a$ for each $a \in A$, is clearly one-to-one, which implies that A is countable too, a contradiction. ■

Let us comment on what one might lose if we do not care about the converses of our results. Sure, the set theoretic distinction of cardinality was not, perhaps, on the mind of anybody in those days. At the same time, we believe that, had the above two outstanding mathematicians of their time cared about a converse for their result, they could have easily, somehow, formulated one. And then, as a consequence, they could have had some understanding of the concept of the cardinality of a set before Cantor, and even they could have become the inventors of set theory instead of Cantor.

2.2 Rings in which every maximal ideal is generated by an idempotent

Let us first emphasize Cohen's theorem, which says that a commutative ring R is Noetherian (i.e., all ideals in R are finitely generated) if, and only if, every prime ideal in R is finitely generated, is a very useful gc-theorem in commutative ring theory, see, e.g., [3, Theorem 12–6]. Incidentally, it is well known that if in this theorem prime ideals are replaced by maximal ideals, its assertion is no longer true, see, e.g., [9]. However, as a corollary to this theorem it is proved in [3] that if every maximal ideal of R is generated by an idempotent, then R is Noetherian too. Clearly, this corollary is not a gc-theorem. In fact, a much stronger result holds if we try to make it into a gc-theorem. We may say that: A commutative ring R is a finite direct product of fields (in particular, it has only a finite number of ideals, so a fortiori it is Artinian, which in turn implies it is Noetherian too) if, and only if, every maximal ideal in R is generated by an idempotent. Its proof is as easy as the proof of the corollary. Indeed, since every maximal ideal of R is a direct summand, we readily infer that the sum of its minimal ideals (i.e., the socle of R) cannot be contained in a maximal ideal, M say. For if $M = eR$, where e is idempotent, then $(1-e)R$ is a minimal ideal contained in the socle of R, which is not in M. Hence, the socle of R must be equal to R, and we are done. We should remind the reader that a generalization of the latter gc-theorem with the same proof holds in more general (not necessarily commutative) rings, see [15].

2.3 What are the topological spaces, in which closed sets and boundary sets coincide?

Let us bring to the attention of the reader that, in [24, Exercise 3B], it is asked to show that any closed subset of the plane $\mathbb{R}^2$ is the boundary of some set in $\mathbb{R}^2$ (note that therein boundary is called frontier). This is a good question, because the boundary of a set is always closed, but the converse is not necessarily true in every topological space (e.g., in a discrete space). However, we may ask what is so special about $\mathbb{R}^2$? Is this claim not true, for example, in $\mathbb{R}^n$? Therefore, we may first pose an appropriate question. What are the spaces X with the property that for any closed subset $F \subseteq X$ there is a subset $A \subseteq X$ with $F = b(A)$, where $b(A)$ denotes the boundary of A. Now, $b(A) = \overline{A} \setminus A^\circ$, where $\overline{A}$ is the closure of A and A° is the interior of A. Hence, since X is closed, we must have $X = b(Y) = \overline{Y} \setminus Y^\circ$ for some subset Y of X. This implies that $Y^\circ = \emptyset$ and $\overline{Y} = X$. So we have a clue, i.e., there must exist a dense subset with empty interior. This leads us to state a new gc-theorem which characterizes the closed sets, and hence the open sets, in any topological space X, a characterization which seems to have been overlooked in the literature.

Theorem 2.2. *Let X be a topological space. Then $F \subseteq X$ is a closed subset, if, and only if, for each dense subset $Y \subseteq X$ there exists a subset $F_Y \subseteq X$ such that $F = \overline{F_Y}$ with $F_Y^\circ = F^\circ \cap Y^\circ$.*

A proof of this theorem, which depends on the simple and well-known fact that $\overline{G \cap Y} = \overline{G}$, where Y (resp., G) is any dense (resp., open) set in X, is not hard and is left to the reader, see also [8, p. 28].

The following immediate corollary, which is also a gc-theorem, settles the above question in $\mathbb{R}^n$ (note that $\mathbb{Q}^n$ is a dense subset with empty interior in $\mathbb{R}^n$).

Corollary 2.3. *Let X be a topological space. Then every closed subset of X is the boundary of a subset of X if, and only if, X has a dense subset with empty interior.*

Let us conclude this article with two comments:

(1) Not only is the pq-style responsible for the overlooking of some of the converses related to useful non-gc-theorems in the literature, but also it is sometimes, equally responsible, for our inveterate tendency to overlook some obvious useful facts; see [12, Corollaries 1, 3, 4, Bisector Proposition] for some of these facts.
(2) We may also search the literature, especially the textbooks, to look for some interesting results which are non-gc-theorems. And then do our best to present them, if possible, in the form of gc-theorems. This would help to substantially reduce the number of non-gc-theorems in the literature, or at least in the textbooks, in the future. In particular, the mathematics education students who may follow the latter comment, could provide suitable materials for writing good dissertations.

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