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Almost minimizers for the thin obstacle problem with variable coefficients

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Abstract. We study almost minimizers for the thin obstacle problem with variable Hölder continuous coefficients and zero thin obstacle, and establish their $C^{1,\beta}$ regularity on the either side of the thin space. Under an additional assumption of quasisymmetry, we establish the optimal growth of almost minimizers as well as the regularity of the regular set and a structural theorem on the singular set. The proofs are based on the generalization of Weiss- and Almgren-type monotonicity formulas for almost minimizers established earlier in the case of constant coefficients.

Contents

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1. Introduction and main results

1.1. The thin obstacle (or Signorini) problem with variable coefficients

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and Π a smooth hypersurface (the *thin space*), that splits D into two subdomains $D^\pm$: $D \setminus \Pi = D^+ \cup D^-$. Let $\psi\colon \Pi \to \mathbb{R}$ be a certain (smooth) function (the *thin obstacle*) and $g\colon \partial D \to \mathbb{R}$ (the *boundary values*). Let also $A(x) = (a_{ij}(x))$ be an $n \times n$ symmetric uniformly elliptic matrix, α-Hölder continuous as a function of $x \in D$, for some $0 < \alpha < 1$, with ellipticity constants $0 < \lambda \leq 1 \leq \Lambda < \infty$:

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^n.$$

Then consider the minimizer U of the energy functional

$$\mathcal{J}_{A,D}(V) = \int_D \langle A(x)\nabla V, \nabla V\rangle dx,$$

over a closed convex set $\mathfrak{K}_{\psi,g}(D,\Pi) \subset W^{1,2}(D)$ defined by

$$\mathfrak{K}_{\psi,g}(D,\Pi) := \{V \in W^{1,2}(D) \mid V = g \text{ on } \partial D,\ V \geq \psi \text{ on } \Pi \cap D\}.$$

Because of the unilateral constraint on the thin space Π, the problem is known as the *thin obstacle problem*. Away from Π, the minimizer solves a uniformly elliptic divergence form equation with variable coefficients

$$\operatorname{div}(A(x)\nabla U) = 0 \quad \text{in } D^+ \cup D^-.$$

On the thin space, the minimizers satisfy

$$\begin{aligned} &U \geq \psi, \quad \langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle \geq 0,\\ &(U - \psi)(\langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle) = 0 \quad \text{on } D \cap \Pi, \end{aligned}$$

in a certain weak sense, where $\nu^\pm$ are the exterior normals to $D^\pm$ on Π and $\langle A\nabla U, \nu^\pm\rangle$ are understood as the limits from inside $D^\pm$. These are known as the *Signorini complementarity conditions* and therefore the problem is often referred to as the *Signorini problem* with variable coefficients (or *A-Signorini problem*, for short). One of the main objects of the study is the *free boundary*

$$\Gamma(U) = \partial_\Pi\{x \in \Pi \mid U(x) = \psi(x)\} \cap D,$$

which separates the *coincidence set* $\{U = \psi\}$ from the *noncoincidence set* $\{U > \psi\}$ in $D \cap \Pi$. The set $\Gamma(U)$ is also called a *thin* free boundary as it lives in Π and is expected to be of codimension two with respect to the domain D.

These types of problems go back to the original Signorini problem in elastostatics [41], but also appear in many applications ranging from math biology (semipermeable membranes) to boundary heat control [21] or more recently in math finance, with connection to the obstacle problem for the fractional Laplacian, through the Caffarelli–Silvestre extension [8]. The presence of the free boundary makes the problem particularly challenging and while the $C^{1,\beta}$ regularity of the minimizers (on either side of the thin space) was known already in [9, 32, 43], the study of the free boundary became possible only after the breakthrough work of [3] on the optimal $C^{1,1/2}$ regularity of the minimizers. Since then there has been a significant effort in the literature to understand the structure and regularity properties of the free boundary in many different settings including equations with variable coefficients, problems for the fractional Laplacian, as well as the time-dependent problems, see e.g. [4–7, 10–13, 18, 23–28, 33–37, 39, 40, 42], and many others.

1.2. Almost minimizers

The approach we take in this paper is by considering the so-called almost minimizers of the functional $\mathcal{J}_{A,D}$ in the sense of Anzellotti [2]. For this we need a *gauge* function $\omega\colon (0, r_0) \to [0, \infty)$, $r_0 > 0$, which is a nondecreasing function with $\omega(0+) = 0$, as well as a family $\{E_r(x_0)\}_{0<r<r_0}$ of open sets for any $x_0 \in D$, comparable to balls centered at x_0 (in what comes next, we will take it to be a family of ellipsoids).

Definition 1.1 (Almost minimizers). We say U is an *almost minimizer for the A-Signorini problem* in D if $U \in W^{1,2}_{\mathrm{loc}}(D)$, $U \geq \psi$ on $D \cap \Pi$, and for any $E_r(x_0) \Subset D$ with $0 < r < r_0$, we have

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle \leq (1 + \omega(r)) \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle, \tag{1.1}$$

for any competitor function $V \in \mathfrak{K}_{\psi,U}(E_r(x_0), \Pi)$, i.e., V satisfying

$$V = U \quad \text{on } \partial E_r(x_0), \qquad V \geq \psi \quad \text{on } E_r(x_0) \cap \Pi.$$

In fact, observing that for $x, x_0 \in D$, and $\xi \in \mathbb{R}^n$, $\xi \neq 0$

$$(1 - C|x - x_0|^\alpha) \leq \frac{\langle A(x_0)\xi, \xi\rangle}{\langle A(x)\xi, \xi\rangle} \leq (1 + C|x - x_0|^\alpha),$$

with C depending on the ellipticity of A and $\|A\|_{C^{0,\alpha}(D)}$, we can rewrite (1.1) in the form *with frozen coefficients*

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq (1 + \omega(r)) \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle, \tag{1.2}$$

by replacing the gauge $\omega(r)$ with $C(\omega(r) + r^\alpha)$ if necessary.

An example of an almost minimizer is given in Appendix. Generally, we view almost minimizers as perturbations of minimizers in a certain sense, but in the case of variable coefficients there are even some advantages of treating minimizers themselves as almost minimizers, particularly in the sense of frozen coefficients (1.2).

Almost minimizers for the Signorini problem have already been studied in [31] in the case $A(x) \equiv I$, where their $C^{1,\beta}$-regularity (on either side of the thin space) has been established and a number of technical tools such as Weiss- and Almgren-type monotonicity formulas were proved. In combination with the epiperimetric and log-epiperimetric inequalities these tools allowed to establish the optimal growth and prove the $C^{1,\gamma}$-regularity of the regular set and a structural theorem on the singular set. The aim of this paper is to extend these results to the variable coefficient case. It is noteworthy that the results that we obtain (see Theorems I–IV below) for almost minimizers improve even on some of the results available for the minimizers. For example, we only need the coefficients $A(x)$ to be $C^{0,\alpha}$ with arbitrary $0 < \alpha < 1$ in order to study the free boundary, compared to $W^{1,p}$, $p > n$, in [36] or $C^{0,\alpha}$, $1/2 < \alpha < 1$, in [40] for the regular part of the free boundary and $C^{0,1}$ in [27] for the singular set.

A related notion of almost minimizers has been considered recently in [30] for the obstacle problem for the fractional Laplacian, with the help of the Caffarelli–Silvestre extension. While the $C^{1,\beta}$ regularity of almost minimizers holds for the fractional orders $1/2 \leq s < 1$, the study of the free boundary still remains open.

Almost minimizers have been studied also for other free boundary problems, particularly Alt–Caffarelli-type (or Bernoulli-type) problems [15, 16, 19], their thin counterpart [20], as well as the variable coefficient versions [14, 17]. We have to mention that the Signorini problem is quite different from Alt–Caffarelli-type problems, as the solutions may grow at different rates near the free boundary (such as $3/2$, 2, $7/2$, 4, $\ldots$, powers of the distance), as opposed to a specific rate in Alt–Caffarelli-type problems (linear in the classical case and the square root of the distance in the thin counterpart). Therefore, it is quite important that the almost minimizing property that we impose for the Signorini problem is *multiplicative*, to allow the capture of all possible rates, while the almost minimizing property in the Alt–Caffarelli-type problems can be also imposed in an *additive* way, see [14].

1.3. Main results

Since we are interested in local regularity results, we will assume that $D = B_1$, the unit ball in $\mathbb{R}^n$, and that

$$\Pi = \mathbb{R}^{n-1} \times \{0\}$$

after a local diffeomorphism. In this paper, we will consider only the case when the thin obstacle is identically zero: $\psi \equiv 0$.

Further, we will assume $r_0 = 1$ in Definition 1.1 and take $\{E_r(x_0)\}$ to be the family of ellipsoids associated with the positive symmetric matrix $A(x_0)$:

$$E_r(x_0) := A^{1/2}(x_0)(B_r) + x_0.$$

By the ellipticity of $A(x_0)$, we have

$$B_{\lambda^{1/2}r}(x_0) \subset E_r(x_0) \subset B_{\Lambda^{1/2}r}(x_0).$$

To simplify the tracking of the constants, we will assume that there is $M > 0$ such that

$$\|A\|_{C^{0,\alpha}(B_1)} \leq M, \quad \lambda^{-1}, \Lambda \leq M, \quad \omega(r) \leq M r^\alpha, \quad 0 < \alpha < 1. \tag{1.3}$$

Then we can go between almost minimizing properties (1.1) and (1.2) by changing M if necessary.

Then our first result is as follows.

Theorem I ($C^{1,\beta}$-regularity of almost minimizers). *Let $U \in W^{1,2}(B_1)$ be an almost minimizer for the A-Signorini problem in B_1, under the assumptions above. Then, $U \in C^{1,\beta}_{\mathrm{loc}}(B_1^{\pm} \cup B_1')$ for $\beta = \beta(\alpha, n) \in (0,1)$ and*

$$\|U\|_{C^{1,\beta}(K)} \le C\|U\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1^{\pm} \cup B_1'$ and $C = C(n, \alpha, M, K)$.

The proof is obtained by using Morrey and Campanato space estimates, following the original idea of Anzellotti [2] that was successfully used in the constant coefficient case of our problem in [31]. We explicitly mention, however, that in the above theorem we do not require the even symmetry of the almost minimizer in the x_n-variable, so Theorem I extends the corresponding result in [31] also in that respect.

To state our results related to the free boundary, we need to assume the following quasisymmetry condition. For $x_0 \in B_1' = B_1 \cap \Pi$, let

$$P_{x_0} = I - 2\frac{A(x_0)e_n \otimes e_n}{a_{nn}(x_0)}$$

be a matrix corresponding to the reflection with respect to Π in the conormal direction $A(x_0)e_n$ at x_0. Note that $P_{x_0}x = x$ for any $x \in \Pi$ and $P_{x_0}E_r(x_0) = E_r(x_0)$. Then, for a function U in B_1 define

$$U^*_{x_0}(x) := \frac{U(x) + U(P_{x_0}x)}{2}.$$

Note that $U^*_{x_0}$ may not be defined in all of B_1, but is defined in any ellipsoid $E_r(x_0)$ as long as it is contained in B_1. Note also that $U = U^*_{x_0}$ on Π.

Definition 1.2 (Quasisymmetry). We say that $U \in W^{1,2}(B_1)$ is A-*quasisymmetric* with respect to Π, if there is a constant Q such that

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \le Q \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle,$$

for any ellipsoid $E_r(x_0) \Subset B_1$ centered at any $x_0 \in B_1'$.

We will assume $Q \le M$ throughout the paper, in addition to (1.3).

Note that when $A(x) \equiv I$ and U is even in x_n, then it is automatically quasisymmetric in the sense of the above definition. The quasisymmetry condition will also hold for even minimizers if e_n is an eigenvector of $A(x_0)$ for any $x_0 \in B_1'$, i.e., when

$$a_{in}(x_0) = 0, \quad \text{for } i = 1, \dots, n-1,\ x_0 \in B_1'.$$

This condition is typically imposed in the existing literature and can be satisfied with an application of a local $C^{1,\alpha}$-diffeomorphism that preserves Π, see [28, 40, 44]. The reason for a quasisymmetry condition is that the growth rate of the symmetrization $U^*_{x_0}$ over the ellipsoids $E_r(x_0)$ captures that of $U = U^*_{x_0}$ on the thin space Π at $x_0 \in \Gamma(U)$, while in the nonsymmetric case there could be a mismatch in those rates caused by the odd component of U, vanishing on Π.

More specifically, the growth rate of U on Π at $x_0 \in \Gamma(U)$ is determined by the following quantity

$$N^A(r, U^*_{x_0}, x_0) := \frac{r \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle}{\int_{\partial E_r(x_0)} (U^*_{x_0})^2 \mu_{x_0}(x - x_0)},$$

which is a version of *Almgren's frequency functional* [1] written in the geometric terms determined by $A(x_0)$, where $\mu_{x_0}(z) = \frac{|A^{-1/2}(x_0)z|}{|A^{-1}(x_0)z|}$ is the *conformal factor*. As in the constant coefficient case, this quantity is of paramount importance for the classification of free boundary points.

Theorem II (Monotonicity of the truncated frequency). *Let U be as in Theorem I and assume additionally that U is A-quasisymmetric with respect to Π. Then for any $\kappa_0 \geq 2$, there is $b = b(n, \alpha, M, \kappa_0)$ such that the* truncated frequency

$$r \mapsto \widehat{N}^A_{\kappa_0}(r, U^*_{x_0}, x_0) := \min\left\{\frac{1}{1 - br^\alpha} N^A(r, U^*_{x_0}, x_0), \kappa_0\right\}$$

is monotone increasing for $x_0 \in B'_{1/2} \cap \Gamma(u)$, and $0 < r < r_0(n, \alpha, M, \kappa_0)$. Moreover, if we define

$$\kappa(x_0) := \widehat{N}^A_{\kappa_0}(0+, U^*_{x_0}, x_0),$$

the frequency of U at x_0*, then we have that either*

$$\kappa(x_0) = 3/2 \quad \textit{or} \quad \kappa(x_0) \geq 2.$$

The monotonicity of the truncated frequency follows from that of a one-parametric family of the so-called Weiss-type energy functionals $\{W^A_\kappa\}_{0<\kappa<\kappa_0}$, see Section 7, which also play a fundamental role in the analysis of the free boundary.

The theorem above gives the following decomposition of the free boundary

$$\Gamma(U) = \Gamma_{3/2}(U) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(U),$$

where

$$\Gamma_\kappa(U) := \{x_0 \in \Gamma(U) \mid \kappa(x_0) = \kappa\}.$$

The set $\Gamma_{3/2}(U)$, where the frequency is minimal is known as the *regular set* and is also denoted by $\mathcal{R}(U)$.

Theorem III (Regularity of the regular set). *Let U be as in Theorem II. Then $\mathcal{R}(U)$ is a relatively open subset of the free boundary $\Gamma(U)$ and is an $(n-2)$-dimensional manifold of class $C^{1,\gamma}$.*

Finally, we state our main result for the so-called *singular set*. A free boundary point $x_0 \in \Gamma(U)$ is called *singular* if the *coincidence set* $\Lambda(U) := \{x \in B'_1 : U(x) = 0\}$ has H^{n-1}-density zero at x_0, i.e.,

$$\lim_{r\to 0+} \frac{H^{n-1}(\Lambda(U) \cap B'_r(x_0))}{H^{n-1}(B'_r)} = 0.$$

We denote the set of all singular points by $\Sigma(U)$ and call it the *singular set*. It can be shown that if $\kappa(x_0) < \kappa_0$, then $x_0 \in \Sigma(U)$ if and only if $\kappa(x_0) = 2m$, $m \in \mathbb{N}$ (see Proposition 12.2). For such values of κ, we then define

$$\Sigma_\kappa(U) := \Gamma_\kappa(U).$$

Theorem IV (Structure of the singular set). *Let U be as in Theorem II. Then, for any $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, $\Sigma_\kappa(U)$ is contained in a countable union of $(n-2)$-dimensional manifolds of class $C^{1,\log}$.*

A more refined version of this result is given in Theorem 12.8.

Theorems III and IV follow by establishing the uniqueness and continuous dependence of *almost homogeneous blowups* with Hölder modulus of continuity in the case of regular free boundary points and a logarithmic one in the case of the singular points. These follow from optimal growth and rotation estimates which are based on the use of Weiss-type monotonicity formulas in conjunction with the so-called *epiperimetric* [26] and log-*epiperimetric* [11] inequalities for the solutions of the Signorini problem.

1.3.1. Proofs of Theorems I–IV. While we don't give formal proofs of the theorems above in the main body of the paper, they are contained in the following results proved there:

- Theorem I is essentially the same as Theorem 5.2.
- Theorem II follows by combining Theorem 7.2 and Corollary 11.4.
- The statement of Theorem III is contained in that of Theorem 11.7.
- The statement of Theorem IV is contained in that of Theorem 12.8.

1.4. Notation

We use the following notation throughout the paper.

$\mathbb{R}^n$ stands for the n-dimensional Euclidean space. The points of $\mathbb{R}^n$ are denoted by $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We often identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in \mathbb{R}^{n-1} \times \{0\}$. $\mathbb{R}^n_\pm$ stand for open halfspaces $\{x \in \mathbb{R}^n \mid \pm x_n > 0\}$.

For $\xi, \eta \in \mathbb{R}^n$, the standard inner product is denoted by $\langle \xi, \eta\rangle$. Thus, $|\xi|^2 = \langle \xi, \xi\rangle$, where $|\xi|$ is the Euclidean norm of ξ.

For $x \in \mathbb{R}^n$, $r > 0$, we denote

$$B_r(x) := \{y \in \mathbb{R}^n \mid |x - y| < r\}, \quad \text{ball in } \mathbb{R}^n,$$

$$B_r^{\pm}(x') := B_r(x',0) \cap \{\pm x_n > 0\}, \quad \text{half-ball in } \mathbb{R}^n,$$
$$B_r'(x') := B_r(x',0) \cap \{x_n = 0\}, \quad \text{ball in } \mathbb{R}^{n-1}, \text{ or thin ball.}$$

We typically drop the center from the notation if it is the origin. Thus, $B_r := B_r(0)$, $B_r' := B_r'(0)$, etc.

For a function f in $\mathbb{R}^n$, ∇f denotes its gradient (in the classical or weak sense)

$$\nabla f := (\partial_{x_1} f, \partial_{x_2} f, \dots, \partial_{x_n} f),$$

where $\partial_{x_i} f$ are the partial derivatives in the variables x_i, $i = 1, 2, \dots, n$.

In integrals, we often drop the variable and the measure of integration if it is with respect to the Lebesgue measure or the surface measure. Thus,

$$\int_{B_r} f = \int_{B_r} f(x)dx, \quad \int_{\partial B_r} f = \int_{\partial B_r} f(x)dS_x,$$

where S_x stands for the surface measure.

If E is a set of positive and finite Lebesgue measure, we indicate by $\langle f \rangle_E$ the integral mean value of a function u over E. That is,

$$\langle f \rangle_E := \fint_E f = \frac{1}{|E|} \int_E f.$$

2. Coordinate transformations

In order to use the results available for almost minimizers in the case of $A \equiv I$, proved in [31], in this section we describe a "deskewing procedure" or coordinate transformations to straighten $A(x_0)$, $x_0 \in B_1$.

For the notational convenience, we will denote

$$\mathfrak{a}_{x_0} = A^{1/2}(x_0), \quad x_0 \in B_1$$

so that

$$\langle A(x_0)\xi, \xi \rangle = |\mathfrak{a}_{x_0}\xi|^2, \quad \xi \in \mathbb{R}^n.$$

Then $\mathfrak{a}_{x_0}$ is a symmetric positive definite matrix, with eigenvalues between $\lambda^{1/2}$ and $\Lambda^{1/2}$ and the mapping $x_0 \mapsto \mathfrak{a}_{x_0}$ is α-Hölder continuous for $x_0 \in B_1$. For every $x_0 \in B_1$, we define an affine transformation T_{x_0} by

$$T_{x_0}(x) = \mathfrak{a}_{x_0}^{-1}(x - x_0).$$

Note that $T_{x_0}^{-1}(y) = \mathfrak{a}_{x_0} y + x_0$. Then for the ellipsoids $E_r(x_0)$, we have

$$E_r(x_0) = T_{x_0}^{-1}(B_r) = \mathfrak{a}_{x_0} B_r + x_0, \quad T_{x_0}(E_r(x_0)) = B_r.$$

Further, we let

$$\Pi_{x_0} := T_{x_0}(\Pi).$$

Then Π_{x_0} is a hyperplane parallel to a linear subspace $\mathfrak{a}_{x_0}^{-1}\Pi$ spanned by the vectors $\mathfrak{a}_{x_0}^{-1} e_1$, $\mathfrak{a}_{x_0}^{-1} e_2$, …, $\mathfrak{a}_{x_0}^{-1} e_{n-1}$ and with a normal $\mathfrak{a}_{x_0} e_n$. Generally, this hyperplane will be tilted with respect to Π, unless $\mathfrak{a}_{x_0} e_n$ is a multiple of e_n, or equivalently that e_n is an eigenvector of the matrix $A(x_0)$, or that $a_{in}(x_0) = 0$ for $i = 1, \dots, n-1$ for its entries. To rectify that, we construct a family of orthogonal transformations O_{x_0}, $x_0 \in B_1$, by applying the Gram–Schmidt process to the ordered basis $\{\mathfrak{a}_{x_0}^{-1} e_1, \mathfrak{a}_{x_0}^{-1} e_2, \dots, \mathfrak{a}_{x_0}^{-1} e_{n-1}\}$ of $\mathfrak{a}_{x_0}^{-1}\Pi$. Namely, let

$$\begin{aligned}
e_1^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1} e_1}{|\mathfrak{a}_{x_0}^{-1} e_1|},\\
e_2^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1} e_2 - \langle \mathfrak{a}_{x_0}^{-1} e_2, e_1^{x_0}\rangle e_1^{x_0}}{|\mathfrak{a}_{x_0}^{-1} e_2 - \langle \mathfrak{a}_{x_0}^{-1} e_2, e_1^{x_0}\rangle e_1^{x_0}|},\\
e_3^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1} e_3 - \langle \mathfrak{a}_{x_0}^{-1} e_3, e_1^{x_0}\rangle e_1^{x_0} - \langle \mathfrak{a}_{x_0}^{-1} e_3, e_2^{x_0}\rangle e_2^{x_0}}{|\mathfrak{a}_{x_0}^{-1} e_3 - \langle \mathfrak{a}_{x_0}^{-1} e_3, e_1^{x_0}\rangle e_1^{x_0} - \langle \mathfrak{a}_{x_0}^{-1} e_3, e_2^{x_0}\rangle e_2^{x_0}|}\\
&\ \vdots
\end{aligned}$$

Moreover, letting

$$e_n^{x_0} := \frac{\mathfrak{a}_{x_0} e_n}{|\mathfrak{a}_{x_0} e_n|},$$

we obtain an ordered orthonormal basis $\{e_1^{x_0}, \dots, e_{n-1}^{x_0}, e_n^{x_0}\}$ of $\mathbb{R}^n$. Then consider the rotation O_{x_0} of $\mathbb{R}^n$ that takes the standard basis $\{e_1, e_2, \dots, e_n\}$ to the one above, i.e.,

$$O_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n, \quad O_{x_0}(e_i) = e_i^{x_0},\ i = 1, 2, \dots, n.$$

Note that the Gram–Schmidt process above guarantees that $x_0 \mapsto O_{x_0}$ is α-Hölder continuous. We also have that by construction

$$O_{x_0}^{-1} \mathfrak{a}_{x_0}^{-1} \Pi = \Pi.$$

In particular, when $x_0 \in \Pi$, we have $\Pi_{x_0} = \mathfrak{a}_{x_0}^{-1}\Pi$ and therefore

$$O_{x_0}^{-1}(\Pi_{x_0}) = \Pi.$$

Because of this property, we also define the modifications of the matrices $\mathfrak{a}_{x_0}$ and the transformations T_{x_0} as follows:

$$\bar{\mathfrak{a}}_{x_0} = \mathfrak{a}_{x_0} O_{x_0}, \quad \bar{T}_{x_0} = O_{x_0}^{-1} \circ T_{x_0},$$

so that $\bar{T}_{x_0}(x) = \bar{\mathfrak{a}}_{x_0}^{-1}(x - x_0)$. Since O_{x_0} is a rotation, we still have

$$E_r(x_0) = \bar{T}_{x_0}^{-1}(B_r), \quad \bar{T}_{x_0}(E_r(x_0)) = B_r,$$

see Figure 1.

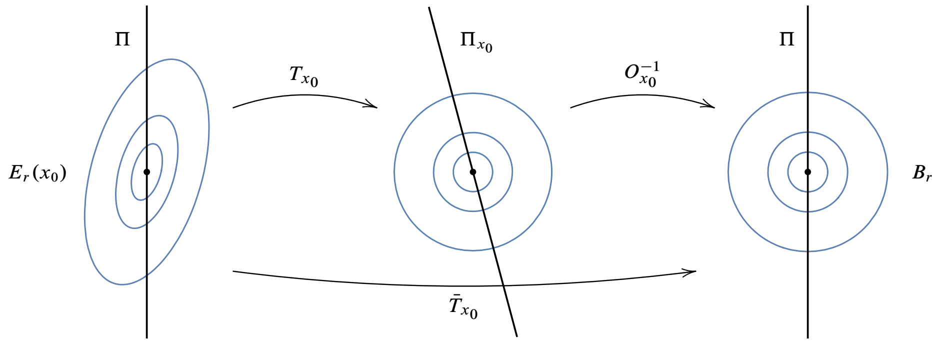

Figure 1. Deskewing: coordinate transformations T_{x_0}, $O_{x_0}^{-1}$, $\bar{T}_{x_0}$.

Next, for a function $U\colon B_1 \to \mathbb{R}$ and a point $x_0 \in B_1$, we define its "deskewed" version at x_0 by

$$u_{x_0} = U \circ \bar{T}_{x_0}^{-1}.$$

As we will see, if U is an almost minimizer, the transformed function u_{x_0} will satisfy an almost minimizing property with the identity matrix I at the origin. Before we state and prove that fact, we need the following basic change of variable formulas:

$$\int_{E_r(x_0)} U^2 = \det \mathfrak{a}_{x_0} \int_{B_r} u_{x_0}^2 \tag{2.1}$$

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle = \det \mathfrak{a}_{x_0} \int_{B_r} |\nabla u_{x_0}|^2 \tag{2.2}$$

$$\int_{\partial E_r(x_0)} U^2 \mu_{x_0}(x - x_0) = \det \mathfrak{a}_{x_0} \int_{\partial B_r} u_{x_0}^2, \tag{2.3}$$

with the conformal factor

$$\mu_{x_0}(z) := \frac{|\mathfrak{a}_{x_0}^{-1} z|}{|A^{-1}(x_0) z|}. \tag{2.4}$$

We also have the following modified version of (2.2).

$$\int_{E_r(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_r(x_0)}|^2 = \det \mathfrak{a}_{x_0} \int_{B_r} |\nabla u_{x_0} - \langle \nabla u_{x_0}\rangle_{B_r}|^2. \tag{2.5}$$

While (2.1)–(2.2) and (2.5) are clear, let us give more details on (2.3). If we let $f(x) := |\mathfrak{a}_{x_0}^{-1}(x - x_0)|$, then $\{f = t\} = \partial E_t(x_0)$, $t > 0$, and by the coarea formula

$$\int_{E_r(x_0)} U^2 dx = \int_0^r \int_{\partial E_t(x_0)} \frac{U^2}{|\nabla f(x)|} dS_x dt.$$

Using now that $1/|\nabla f(x)| = \frac{|\mathfrak{a}_{x_0}^{-1}(x - x_0)|}{|A^{-1}(x_0)(x - x_0)|} = \mu_{x_0}(x - x_0)$ and then differentiating (2.1), we obtain (2.3).

We will also need the following estimate for the conformal factor μ_{x_0}:

$$\lambda^{1/2} \le \mu_{x_0}(z) \le \Lambda^{1/2}.$$

Indeed, if $y = A^{-1}(x_0)z$, then

$$\mu_{x_0}(z) = \frac{|\mathfrak{a}_{x_0} y|}{|y|} \in [\lambda^{1/2}, \Lambda^{1/2}].$$

Definition 2.1 (Almost Signorini property at a point). A function $u \in W^{1,2}(B_R)$ satisfies the *almost Signorini property at* 0 in B_R if

$$\int_{B_r} |\nabla u|^2 \le (1+\omega(r)) \int_{B_r} |\nabla v|^2,$$

for all $0 < r < R$ and $v \in \mathfrak{K}_{0,u}(B_r, \Pi)$.

Lemma 2.2. *Suppose U is an almost minimizer of the A-Signorini problem in B_1. Let $x_0 \in B'_1$ be such that $E_R(x_0) \subset B_1$. Then $u_{x_0} = U \circ \bar{T}_{x_0}^{-1}$ satisfies the almost Signorini property at* 0 *in B_R.*

Proof. Let V be the energy minimizer of $\int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle$ on $\mathfrak{K}_{0,U}(E_r(x_0), \Pi)$, $0 < r < R$. Then $v_{x_0} = V \circ \bar{T}_{x_0}^{-1}$ is the energy minimizer of $\int_{B_r} |\nabla v_{x_0}|^2$ on $\mathfrak{K}_{0,u_{x_0}}(B_r, \Pi)$. Moreover, by (2.2),

$$\begin{aligned}\int_{B_r} |\nabla u_{x_0}|^2 &= \det \mathfrak{a}_{x_0}^{-1} \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \\ &\le (1+\omega(r)) \det \mathfrak{a}_{x_0}^{-1} \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle \\ &= (1+\omega(r)) \int_{B_r} |\nabla v_{x_0}|^2.\end{aligned}$$

This completes the proof. ■

3. Almost A-harmonic functions

We start our analysis of almost minimizers in the absence of the thin obstacle. We call such functions almost A-harmonic functions. In this section, we establish their $C^{1,\alpha/2}$ regularity (Theorem 3.6). A similar result has already been proved by Anzellotti [2], but for almost minimizers over balls $\{B_r(x_0)\}$ rather than ellipsoids $\{E_r(x_0)\}$; nevertheless, the proofs are similar. The proofs in this section also illustrate how we are going to use the results available for "deskewed" functions $u_{x_0} = U \circ \bar{T}_{x_0}^{-1}$ to infer the corresponding results for almost minimizers U.

Definition 3.1 (Almost A-harmonic functions). We say that U is an *almost A-harmonic function* in B_1 if $U \in W^{1,2}(B_1)$ and

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle \le (1+\omega(r)) \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle,$$

whenever $E_r(x_0) \Subset B_1$ and $V \in \mathfrak{K}_U(E_r(x_0)) := U + W_0^{1,2}(E_r(x_0))$.

Note that similarly to the case of A-Signorini problem, we can write the almost minimizing property above in the form with frozen coefficients

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \le (1+\omega(r)) \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle.$$

Definition 3.2 (Almost harmonic property at a point). A function $u \in W^{1,2}(B_R)$ satisfies *almost harmonic property at* 0 in B_R if

$$\int_{B_r} |\nabla u|^2 \le (1+\omega(r)) \int_{B_r} |\nabla v|^2,$$

for all $0 < r < R$ and $v \in \mathfrak{K}_u(B_r)$.

Lemma 3.3. *If U is an almost A-harmonic function in B_1 and $x_0 \in B_1$ with $E_R(x_0) \subset B_1$, then u_{x_0} satisfies the almost harmonic property at* 0 *in B_R.*

Proof. The proof is similar to that of Lemma 2.2. ■

Proposition 3.4 (cf. [31, Proposition 2.3]). *Let U be an almost A-harmonic function in B_1. Then for any $B_r(x_0) \Subset B_1$ and $0 < \rho < r$, we have*

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right] \int_{B_r(x_0)} |\nabla U|^2, \tag{3.1}$$

$$\int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}|^2 \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2 + C r^\alpha \int_{B_r(x_0)} |\nabla U|^2, \tag{3.2}$$

with $C = C(n, \alpha, M)$.

Proof. Since u_{x_0} satisfies the almost harmonic property at 0, if h is the harmonic replacement of u_{x_0} in B_r (i.e., h is harmonic in B_r with $h = u_{x_0}$ on ∂B_r), then

$$\int_{B_r} |\nabla u_{x_0}|^2 \le (1 + M r^\alpha) \int_{B_r} |\nabla h|^2.$$

This is enough to repeat the arguments in [31, Proposition 2.3], to obtain

$$\int_{B_\rho} |\nabla u_{x_0}|^2 \le 2\left[\left(\frac{\rho}{r}\right)^n + M r^\alpha\right] \int_{B_r} |\nabla u_{x_0}|^2,$$

$$\int_{B_\rho} |\nabla u_{x_0} - \langle \nabla u_{x_0} \rangle_{B_\rho}|^2 \le 9\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r} |\nabla u_{x_0} - \langle \nabla u_{x_0} \rangle_{B_r}|^2 + 24 M r^\alpha \int_{B_r} |\nabla u_{x_0}|^2.$$

Then, by the change of variables in formulas (2.2) and (2.5), we have

$$\int_{E_\rho(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \le 2\Big[\Big(\frac{\rho}{r}\Big)^n + M r^\alpha\Big] \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle, \tag{3.3}$$

$$\int_{E_\rho(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_\rho(x_0)}|^2 \le 9\Big(\frac{\rho}{r}\Big)^{n+2} \int_{E_r(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_r(x_0)}|^2 + 24 M r^\alpha \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle. \tag{3.4}$$

To show now that (3.3)–(3.4) imply (3.1)–(3.2), we first consider the case

$$0 < \rho < (\lambda/\Lambda)^{1/2} r.$$

Then, using the inclusions

$$B_\rho(x_0) \subset E_{\lambda^{-1/2}\rho}(x_0) \subset E_{\Lambda^{-1/2} r}(x_0) \subset B_r(x_0),$$

applying (3.3)–(3.4) with $\lambda^{-1/2}\rho$ and $\Lambda^{-1/2} r$ in place of ρ and r, and using the ellipticity of $A(x_0)$, we obtain (3.1)–(3.2) in this case.

In the remaining case

$$(\lambda/\Lambda)^{1/2} r \le \rho \le r,$$

the inequalities (3.1)–(3.2) hold readily, as

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}|^2 &\le \int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2 \\ &\le \Big(\frac{\Lambda}{\lambda}\Big)^{\frac{n+2}{2}} \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2. \qquad \blacksquare\end{aligned}$$

We now recall a useful lemma, the proof of which can be found e.g. in [29].

Lemma 3.5. *Let $r_0 > 0$ be a positive number and φ: $(0, r_0) \to (0, \infty)$ a nondecreasing function. Let a, β, and γ be such that $a > 0$, $\gamma > \beta > 0$. There exist two positive numbers $\varepsilon = \varepsilon(a, \gamma, \beta)$, $c = c(a, \gamma, \beta)$ such that if*

$$\varphi(\rho) \le a\Big[\Big(\frac{\rho}{r}\Big)^\gamma + \varepsilon\Big]\varphi(r) + b\, r^\beta,$$

for all ρ, r with $0 < \rho \le r < r_0$, where $b \ge 0$, then one also has, still for $0 < \rho < r < r_0$,

$$\varphi(\rho) \le c\Big[\Big(\frac{\rho}{r}\Big)^\beta \varphi(r) + b\rho^\beta\Big].$$

Theorem 3.6. *Let U be an almost A-harmonic function in B_1. Then $U \in C^{1,\alpha/2}(B_1)$ with*

$$\|U\|_{C^{1,\alpha/2}(K)} \le C\|U\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1$, with $C = C(n, \alpha, M, K)$.

Proof. Let $K \Subset B_1$ and $x_0 \in \widetilde{K} := \{y \in B_1 \mid \operatorname{dist}(y, \partial B_1) \ge r_0\}$, where $r_0 = \frac{1}{2}\operatorname{dist}(K, \partial B_1)$. For $\sigma \in (0, 1)$, a direct application of Lemma 3.5 to (3.1) gives

$$\int_{B_r(x_0)} |\nabla U|^2 \le C\|\nabla U\|^2_{L^2(B_1)} r^{n-2+2\sigma},$$

for any $0 < r < r_0$, with C depending on n, α, σ, M, K. Combining this with (3.2) also gives

$$\int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}|^2 \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2 + C\|\nabla U\|^2_{L^2(B_1)} r^{n-2+2\sigma+\alpha}, \tag{3.5}$$

for any $0 < \rho < r < r_0$. If we take $\sigma \in (0, 1)$ such that $\alpha' := \frac{-2+2\sigma+\alpha}{2} > 0$, then Lemma 3.5 produces

$$\int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}|^2 \le C\|\nabla U\|^2_{L^2(B_1)} \rho^{n+2\alpha'}$$

and this readily implies $\nabla U \in C^{0,\alpha'}(\widetilde{K})$. Now we know that ∇U is bounded in $\widetilde{K}$, and thus $\int_{B_r(x_0)} |\nabla U|^2 \le C\|\nabla U\|^2_{L^2(B_1)} r^n$. Plugging this in the last term of (3.2) and repeating the arguments above, we conclude that $U \in C^{1,\alpha/2}$. ■

4. Almost Lipschitz regularity of almost minimizers

In this section, we make the first step towards the regularity of almost minimizers for the A-Signorini problem and show that they are almost Lipschitz, i.e., $C^{0,\sigma}$ for every $0 < \sigma < 1$ (Theorem 4.3). The proof is based on the Morrey space embedding, similar to the case of almost A-harmonic functions, as well as the case of almost minimizers with $A = I$, treated in [31]. We want to emphasize, however, that the results on almost Lipschitz and $C^{1,\beta}$ regularity of almost minimizers (in the next section) do not require any symmetry condition that was imposed in [31].

We start with an auxiliary result on the solutions of the Signorini problem.

Proposition 4.1. *Let h be a solution of the Signorini problem in B_1. Then*

$$\int_{B_\rho} |\nabla h|^2 \le \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h|^2, \quad 0 < \rho < R < 1. \tag{4.1}$$

Proof. The difference of this proposition from [31, Proposition 3.2] is that $h(y)$ is not assumed to be even symmetric in y_n-variable. To circumvent that, we decompose h into the sum of even and odd functions in y_n, i.e.,

$$\begin{aligned} h(y', y_n) &= \frac{h(y', y_n) + h(y', -y_n)}{2} + \frac{h(y', y_n) - h(y', -y_n)}{2} \\ &=: h^*(y', y_n) + h^\sharp(y', y_n). \end{aligned} \tag{4.2}$$

It is easy to see that h^* is a solution of the Signorini problem, even in y_n-variable, and $h^\sharp$ is a harmonic function, odd in y_n-variable.

Then both $|\nabla h^*|^2$ and $|\nabla h^\sharp|^2$ are subharmonic functions in B_1 (see [31, Proposition 3.2] for h^*), which implies that for $0 < \rho < R < 1$

$$\begin{aligned} \int_{B_\rho} |\nabla h^*|^2 &\le \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h^*|^2, \\ \int_{B_\rho} |\nabla h^\sharp|^2 &\le \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h^\sharp|^2. \end{aligned}$$

Now observing that $\int_{B_t} |\nabla h|^2 = \int_{B_t} \left(|\nabla h^*|^2 + |\nabla h^\sharp|^2\right)$, for $0 < t \le R$, we obtain (4.1). ∎

Proposition 4.2 (cf. [31, Proposition 3.3]). *Let U be an almost minimizer for the A-Signorini problem in B_1, and $B_R(x_0) \Subset B_1$. Then, there is $C_1 = C_1(n, M) > 1$ such that*

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C_1 \left[\left(\frac{\rho}{R}\right)^n + R^\alpha\right] \int_{B_R(x_0)} |\nabla U|^2, \quad 0 < \rho < R. \tag{4.3}$$

Proof. *Case 1.* Suppose $x_0 \in B_1'$. Note that u_{x_0} satisfies the Signorini property at 0 in B_r with $r = \Lambda^{-1/2} R$. If h is the Signorini replacement of u_{x_0} in B_r (that is, h solves the Signorini problem in B_r with thin obstacle 0 on Π and boundary values $h = u_{x_0}$ on ∂B_r), then h satisfies

$$\int_{B_r} \langle \nabla h, \nabla (v - h) \rangle \ge 0,$$

for any $v \in \mathfrak{K}_{0,u_{x_0}}(B_r, \Pi)$, which easily follows from the standard first variation argument. Plugging in $v = u_{x_0}$, we obtain

$$\int_{B_r} \langle \nabla h, \nabla u_{x_0} \rangle \ge \int_{B_r} |\nabla h|^2.$$

Then it follows that

$$\begin{aligned} \int_{B_r} |\nabla (u_{x_0} - h)|^2 &= \int_{B_r} \left(|\nabla u_{x_0}|^2 + |\nabla h|^2 - 2\langle \nabla u_{x_0}, \nabla h \rangle\right) \le \int_{B_r} |\nabla u_{x_0}|^2 - \int_{B_r} |\nabla h|^2 \\ &\le (1 + M r^\alpha) \int_{B_r} |\nabla h|^2 - \int_{B_r} |\nabla h|^2 \le M r^\alpha \int_{B_r} |\nabla u_{x_0}|^2, \end{aligned}$$

where in the last inequality we have used that h is the energy minimizer of the Dirichlet integral in $\mathfrak{K}_{0,u_{x_0}}(B_r, \Pi)$. Then, for $\rho \le r$, we have

$$\begin{aligned}\int_{B_\rho} |\nabla u_{x_0}|^2 &\le 2\int_{B_\rho} |\nabla h|^2 + 2\int_{B_\rho} |\nabla(u_{x_0} - h)|^2\\ &\le 2\Big(\frac{\rho}{r}\Big)^n \int_{B_r} |\nabla h|^2 + 2Mr^\alpha \int_{B_r} |\nabla u_{x_0}|^2\\ &\le C\bigg[\Big(\frac{\rho}{r}\Big)^n + r^\alpha\bigg]\int_{B_r} |\nabla u_{x_0}|^2.\end{aligned}$$

Now, we transform back from u_{x_0} to U as we did in Proposition 3.4 to obtain (4.3) in this case.

Case 2. Now consider the case $x_0 \in B_1^+$. If $\rho \ge r/4$, then we simply have

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le 4^n\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |\nabla U|^2.$$

Thus, we may assume $\rho < r/4$. Then, let $d := \operatorname{dist}(x_0, B_1') > 0$ and let $x_1 \in \partial B_d(x_0) \cap B_1'$.

Case 2.1. If $\rho \ge d$, then we use $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$ and the result of Case 1 to write

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla U|^2 \le \int_{B_{2\rho}(x_1)} |\nabla U|^2 &\le C\bigg[\Big(\frac{2\rho}{r/2}\Big)^n + (r/2)^\alpha\bigg]\int_{B_{r/2}(x_1)} |\nabla U|^2\\ &\le C\bigg[\Big(\frac{\rho}{r}\Big)^n + r^\alpha\bigg]\int_{B_r(x_0)} |\nabla U|^2.\end{aligned}$$

Case 2.2. Suppose now $d > \rho$. If $d > r$, then $B_r(x_0) \Subset B_1^+$. Since U is almost harmonic in B_1^+, we can apply Proposition 3.4 to obtain

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C\bigg[\Big(\frac{\rho}{r}\Big)^n + r^\alpha\bigg]\int_{B_r(x_0)} |\nabla U|^2.$$

Thus, we may assume $d \le r$. Then we note that $B_d(x_0) \subset B_1^+$ and by a limiting argument from the previous estimate, we obtain

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C\bigg[\Big(\frac{\rho}{d}\Big)^n + r^\alpha\bigg]\int_{B_d(x_0)} |\nabla U|^2.$$

To estimate $\int_{B_d(x_0)} |\nabla U|^2$ in the right-hand side of the above inequality, we further consider the two subcases.

Case 2.2.1. If $r/4 \le d$, then

$$\int_{B_d(x_0)} |\nabla U|^2 \le 4^n\Big(\frac{d}{r}\Big)^n \int_{B_r(x_0)} |\nabla U|^2,$$

which immediately implies (4.3).

Case 2.2.2. It remains to consider the case $\rho < d < r/4$. Using Case 1 again, we have

$$\int_{B_d(x_0)} |\nabla U|^2 \le \int_{B_{2d}(x_1)} |\nabla U|^2 \le C\left[\left(\frac{2d}{r/2}\right)^n + (r/2)^\alpha\right]\int_{B_{r/2}(x_1)} |\nabla U|^2$$
$$\le C\left[\left(\frac{d}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla U|^2,$$

which also implies (4.3). This concludes the proof of the proposition. ■

As we have seen in [31], Proposition 4.2 implies the almost Lipschitz regularity of almost minimizers.

Theorem 4.3. *Let U be an almost minimizer for the A-Signorini problem in B_1. Then $U \in C^{0,\sigma}(B_1)$ for all $0 < \sigma < 1$. Moreover, for any $K \Subset B_1$,*

$$\|U\|_{C^{0,\sigma}(K)} \le C\|U\|_{W^{1,2}(B_1)},$$

with $C = C(n, \alpha, M, \sigma, K)$.

Proof. The proof is essentially identical to that of [31, Theorem 3.1]. Let $K \Subset B_1$ and $x_0 \in K$. Take $r_0 = r_0(n,\alpha,M,\sigma,K) > 0$ such that $r_0 < \operatorname{dist}(K, \partial B_1)$ and $r_0^\alpha \le \varepsilon(C_1, n, n + 2\sigma - 2)$, where $\varepsilon = \varepsilon(C_1, n, n + 2\sigma - 2)$ is as in Lemma 3.5 and $C_1 = C_1(n, M)$ is as in Proposition 4.2. Then for all $0 < \rho < r < r_0$, by Proposition 4.2,

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C_1\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla U|^2.$$

By Lemma 3.5, we get

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C(n, M, \sigma)\left(\frac{\rho}{r}\right)^{n+2\sigma-2}\int_{B_r(x_0)} |\nabla U|^2.$$

Taking $r \nearrow r_0$, we conclude that

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C(n, \alpha, M, \sigma, K)\|\nabla U\|^2_{L^2(B_1)}\rho^{n+2\sigma-2}. \tag{4.4}$$

By the Morrey space embedding [29, Corollary 3.2], we obtain $U \in C^{0,\sigma}(K)$ with

$$\|U\|_{C^{0,\sigma}(K)} \le C(n, \alpha, M, \sigma, K)\|U\|_{W^{1,2}(B_1)}, \tag{4.5}$$

completing the proof. ■

5. $C^{1,\beta}$ regularity of almost minimizers

In this section we prove $C^{1,\beta}$ regularity of the almost minimizers for the A-Signorini problem (Theorem 5.2). While we take advantage of the results available for the even symmetric almost minimizers with $A = I$ in [31], removing the symmetry condition requires new additional steps, combined with "deskewing" arguments to generalize to the variable coefficient case.

We start again with an auxiliary result for the solutions of the Signorini problem.

Proposition 5.1. *Let h be a solution of the Signorini problem in B_r, $0 < r < 1$. Define*

$$\widehat{\nabla h} := \begin{cases} \nabla h(y', y_n) & \text{for } y_n \geq 0, \\ \nabla h(y', -y_n) & \text{for } y_n < 0, \end{cases}$$

the even extension of ∇h from B_r^+ to B_r. Then for $0 < \alpha < 1$, there are $C_1 = C_1(n, \alpha)$, $C_2 = C_2(n, \alpha)$ such that for all $0 < \rho \leq s \leq (3/4)r$,

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 \leq C_1 \Big(\frac{\rho}{s}\Big)^{n+\alpha} \int_{B_s} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_s}|^2 + C_2 \Big(\int_{B_r} h^2\Big) \frac{s^{n+1}}{r^{n+3}}. \quad (5.1)$$

Proof. This proposition differs from [31, Proposition 4.4] only by not requiring $h(y)$ to be even in the y_n-variable. As in the proof of Proposition 4.1 we split h into its even and odd parts

$$h(y) = h^*(y) + h^\sharp(y), \quad y \in B_r.$$

Recall that h^* is still a solution of the Signorini problem in B_r, but now even in y_n and $h^\sharp$ is a harmonic function in B_r, odd in y_n. Then, by [31, Proposition 4.4], we have

$$\int_{B_\rho} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_\rho}|^2 \leq C_1 \Big(\frac{\rho}{s}\Big)^{n+\alpha} \int_{B_s} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_s}|^2 + C_2 \Big(\int_{B_r} (h^*)^2\Big) \frac{s^{n+1}}{r^{n+3}}. \quad (5.2)$$

Now we need a similar estimate for $h^\sharp$. Since $h^\sharp$ is harmonic, by the standard interior estimates, we have

$$\sup_{B_{(3/4)r}} |D^2 h^\sharp| \leq \frac{C(n)}{r^2} \Big(\frac{1}{r^n} \int_{B_r} (h^\sharp)^2\Big)^{1/2}.$$

Thus, taking the averages on B_ρ^+, we will therefore have

$$\begin{aligned} \int_{B_\rho^+} |\nabla h^\sharp - \langle \nabla h^\sharp \rangle_{B_\rho^+}|^2 &\leq C(n) \Big(\sup_{B_\rho} |D^2 h^\sharp|\Big)^2 \rho^{n+2} \leq C(n) \Big(\int_{B_r} (h^\sharp)^2\Big) \frac{\rho^{n+2}}{r^{n+4}} \\ &\leq C(n) \Big(\int_{B_r} (h^\sharp)^2\Big) \frac{s^{n+1}}{r^{n+3}}, \quad 0 < \rho < s \leq (3/4)r, \end{aligned}$$

which can be rewritten as

$$\int_{B_\rho} |\widehat{\nabla h^\sharp} - \langle \widehat{\nabla h^\sharp}\rangle_{B_\rho}|^2 \le C(n)\Big(\int_{B_r} (h^\sharp)^2\Big)\frac{s^{n+1}}{r^{n+3}}. \tag{5.3}$$

Now using that $\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_\rho} = [\widehat{\nabla h^*} - \langle\widehat{\nabla h^*}\rangle_{B_\rho}] + [\widehat{\nabla h^\sharp} - \langle\widehat{\nabla h^\sharp}\rangle_{B_\rho}]$ in B_ρ, we deduce from (5.3) that

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_\rho}|^2 &\le 2\int_{B_\rho} |\widehat{\nabla h^*} - \langle\widehat{\nabla h^*}\rangle_{B_\rho}|^2 + 2\int_{B_\rho} |\widehat{\nabla h^\sharp} - \langle\widehat{\nabla h^\sharp}\rangle_{B_\rho}|^2\\ &\le 2\int_{B_\rho} |\widehat{\nabla h^*} - \langle\widehat{\nabla h^*}\rangle_{B_\rho}|^2 + C(n)\Big(\int_{B_r}(h^\sharp)^2\Big)\frac{s^{n+1}}{r^{n+3}}.\end{aligned} \tag{5.4}$$

Similarly, representing $\widehat{\nabla h^*} - \langle\widehat{\nabla h^*}\rangle_{B_s} = [\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_s}] - [\widehat{\nabla h^\sharp} - \langle\widehat{\nabla h^\sharp}\rangle_{B_s}]$ in B_s, we deduce from (5.3) (by taking $\rho = s$) that

$$\int_{B_s} |\widehat{\nabla h^*} - \langle\widehat{\nabla h^*}\rangle_{B_s}|^2 \le 2\int_{B_s} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_s}|^2 + C(n)\Big(\int_{B_r}(h^\sharp)^2\Big)\frac{s^{n+1}}{r^{n+3}}. \tag{5.5}$$

Hence, combining (5.2)–(5.5), and using that both $\int_{B_r}(h^*)^2$ and $\int_{B_r}(h^\sharp)^2$ cannot exceed $\int_{B_r} h^2$, we obtain the claimed estimate (5.1). ∎

Theorem 5.2. *Let U be an almost minimizer of the A-Signorini problem in B_1. Then*

$$U \in C^{1,\beta}(B_1^\pm \cup B_1') \quad \textit{with } \beta = \frac{\alpha}{4(2n+\alpha)}.$$

Moreover, for any $K \Subset B_1^\pm \cup B_1'$, we have

$$\|U\|_{C^{1,\beta}(K)} \le C(n,\alpha,M,K)\|U\|_{W^{1,2}(B_1)}. \tag{5.6}$$

Proof. Let K be a ball centered at 0. Fix a small $r_0 = r_0(n,\alpha,M,K) > 0$ to be determined later. In particular, we will ask $r_1 := r_0^{\frac{2n}{2n+\alpha}}\Lambda^{1/2} \le (1/2)\operatorname{dist}(K,\partial B_1)$, which implies that

$$\widetilde{K} := \{y\in B_1 : \operatorname{dist}(y,K)\le r_1\} \Subset B_1.$$

Define

$$\widehat{\nabla U}(y', y_n) := \begin{cases}\nabla U(y',y_n) & \text{for } y_n \ge 0,\\ \nabla U(y',-y_n) & \text{for } y_n < 0.\end{cases}$$

Our goal is to show that for $x_0 \in K$, $0<\rho<r<r_0$,

$$\begin{aligned}\int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle\widehat{\nabla U}\rangle_{B_\rho(x_0)}|^2 &\le C(n,\alpha,M)\Big(\frac{\rho}{r}\Big)^{n+\alpha}\int_{B_r(x_0)} |\widehat{\nabla U} - \langle\widehat{\nabla U}\rangle_{B_r(x_0)}|^2\\ &\quad + C(n,\alpha,M,K)\|U\|^2_{W^{1,2}(B_1)} r^{n+2\beta}.\end{aligned} \tag{5.7}$$

Case 1. Suppose $x_0 \in K \cap B_1'$. For given $0 < r < r_0$, we denote $\alpha' := 1 - \frac{\alpha}{8n} \in (0,1)$, $R := r^{\frac{2n}{2n+\alpha}}$. We then consider two cases:

$$\sup_{\partial E_R(x_0)} |U| \le C_3(\Lambda^{1/2}R)^{\alpha'} \quad \text{and} \quad \sup_{\partial E_R(x_0)} |U| > C_3(\Lambda^{1/2}R)^{\alpha'},$$

where $C_3 = 2[U]_{0,\alpha',\widetilde{K}} = 2\sup_{\substack{y,z\in\widetilde{K}\\ y\neq z}} \frac{|U(y)-U(z)|}{|y-z|^{\alpha'}}$.

Case 1.1. Assume that $\sup_{\partial E_R(x_0)} |U| \le C_3(\Lambda^{1/2}R)^{\alpha'}$. Then u_{x_0} satisfies almost Signorini property at 0 in B_R with

$$\sup_{\partial B_R} |u_{x_0}| \le C_3(\Lambda^{1/2}R)^{\alpha'}.$$

Let h be the Signorini replacement of u_{x_0} in B_R. If we define

$$\widehat{\nabla u_{x_0}}(y', y_n) := \begin{cases} \nabla u_{x_0}(y', y_n) & \text{for } y_n \ge 0, \\ \nabla u_{x_0}(y', -y_n) & \text{for } y_n < 0, \end{cases}$$

and

$$\widehat{\nabla} h(y', y_n) := \begin{cases} \nabla h(y', y_n) & \text{for } y_n \ge 0, \\ \nabla h(y', -y_n) & \text{for } y_n < 0, \end{cases}$$

then we have

$$\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_\rho}|^2 \le 3\int_{B_\rho} |\widehat{\nabla} h - \langle \widehat{\nabla} h\rangle_{B_\rho}|^2 + 6\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla} h|^2, \tag{5.8}$$

$$\int_{B_r} |\widehat{\nabla} h - \langle \widehat{\nabla} h\rangle_{B_r}|^2 \le 3\int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + 6\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla} h|^2. \tag{5.9}$$

Note that if $r_0 \le (3/4)^{\frac{2n+\alpha}{\alpha}}$, then $r < (3/4)R$, thus by Proposition 5.1, the Signorini replacement h satisfies, for $0 < \rho < r$,

$$\int_{B_\rho} |\widehat{\nabla} h - \langle \widehat{\nabla} h\rangle_{B_\rho}|^2 \le C(n,\alpha)\Big(\frac{\rho}{r}\Big)^{n+\alpha} \int_{B_r} |\widehat{\nabla} h - \langle \widehat{\nabla} h\rangle_{B_r}|^2 + C(n,\alpha)\Big(\sup_{\partial B_R} h^2\Big)\frac{r^{n+1}}{R^3}.$$

Combining the above three inequalities, we obtain

$$\begin{aligned} \int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_\rho}|^2 &\le C(n,\alpha)\Big(\frac{\rho}{r}\Big)^{n+\alpha} \int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 \\ &\quad + C(n,\alpha)\Big(\sup_{\partial B_R} h^2\Big)\frac{r^{n+1}}{R^3} + C(n,\alpha)\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla} h|^2. \end{aligned} \tag{5.10}$$

Let us estimate the last term in the right-hand side of (5.10). Take $\delta = \delta(n,\alpha,M,K) > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\delta^\alpha \le \varepsilon = \varepsilon(C_1, n, n+2\alpha'-2)$, where $C_1 = C_1(n,M)$ is as in Proposition 4.2 and ε is as in Lemma 3.5. If $r_0 \le \left(\Lambda^{-1/2}\delta\right)^{\frac{2n+\alpha}{2n}}$, then $\Lambda^{1/2}R < \delta$, thus, by following the proof of Theorem 4.3 up to (4.4), we have

$$\int_{B_{\Lambda^{1/2}R}(x_0)} |\nabla U|^2 \le C(n,\alpha,M,K)\|\nabla U\|^2_{L^2(B_1)}\left(\Lambda^{1/2}R\right)^{n+2\alpha'-2}.$$

It follows that

$$\begin{aligned}\int_{E_R(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle &\le \Lambda \int_{B_{\Lambda^{1/2}R}(x_0)} |\nabla U|^2 \\ &\le C\|\nabla U\|^2_{L^2(B_1)} R^{n+2\alpha'-2}.\end{aligned}$$

Then by the change of variables in (2.2), we have

$$\int_{B_R} |\nabla u_{x_0}|^2 \le C\|\nabla U\|^2_{L^2(B_1)} R^{n+2\alpha'-2}. \tag{5.11}$$

Now we can estimate the third term in the right-hand side of (5.10):

$$\begin{aligned}\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 &= 2\int_{B_r^+} |\nabla u_{x_0} - \nabla h|^2 \\ &\le 2\int_{B_R} |\nabla u_{x_0} - \nabla h|^2 \le 2\left(\int_{B_R} |\nabla u_{x_0}|^2 - \int_{B_R} |\nabla h|^2\right) \\ &\le 2MR^\alpha \int_{B_R} |\nabla h|^2 \le 2MR^\alpha \int_{B_R} |\nabla u_{x_0}|^2 \\ &\le C\|\nabla U\|^2_{L^2(B_1)} R^{n+\alpha+2\alpha'-2} \\ &= C\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}.\end{aligned} \tag{5.12}$$

To estimate the second term in the right-hand side of (5.10), we observe that

$$\sup_{\partial B_R} h^2 = \sup_{\partial B_R} u_{x_0}^2 = \sup_{\partial E_R(x_0)} U^2 \le C_3^2(\Lambda^{1/2}R)^{2\alpha'}.$$

Note that by (4.5), $C_3 \le C(n,\alpha,M,K)\|U\|_{W^{1,2}(B_1)}$. Thus,

$$\left(\sup_{\partial B_R} h^2\right)\frac{r^{n+1}}{R^3} \le C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.$$

Now (5.10) becomes

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_\rho}|^2 &\le C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r} |\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_r}|^2 \\ &\quad + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.\end{aligned} \tag{5.13}$$

We now want to deduce (5.7) from (5.13). The complication here is that the mapping $\bar{T}_{x_0}^{-1}$ does not preserve the even symmetry with respect to the thin plane, since the conormal direction $A(x_0)e_n$ might be different from the normal direction e_n to Π at x_0. To address this issue, by using the even symmetry of $\widehat{\nabla u_{x_0}}$, we rewrite (5.13) in terms of halfballs $B_r^+ = B_r \cap \mathbb{R}^n_+$

$$\int_{B_\rho^+} |\nabla u_{x_0} - \langle \nabla u_{x_0} \rangle_{B_\rho^+}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r^+} |\nabla u_{x_0} - \langle \nabla u_{x_0} \rangle_{B_r^+}|^2 + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \tag{5.14}$$

Similarly, if we denote $E_r^+(x_0) = E_r(x_0) \cap \mathbb{R}^n_+$, then using that $\bar{T}_{x_0}(E_t^+(x_0)) = B_t^+$, $t > 0$, (5.14) becomes

$$\int_{E_\rho^+(x_0)} |\mathfrak{a}_{x_0} \nabla U - \langle \mathfrak{a}_{x_0} \nabla U \rangle_{E_\rho^+(x_0)}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha} \int_{E_r^+(x_0)} |\mathfrak{a}_{x_0} \nabla U - \langle \mathfrak{a}_{x_0} \nabla U \rangle_{E_r^+(x_0)}|^2 + C \det \mathfrak{a}_{x_0} \|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.$$

Repeating the argument that (3.4) implies (3.2) in the proof of Proposition 3.4, we have

$$\int_{B_\rho^+(x_0)} |\nabla U - \langle \nabla U \rangle_{B_\rho^+(x_0)}|^2 \leq C\left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r^+(x_0)} |\nabla U - \langle \nabla U \rangle_{B_r^+(x_0)}|^2 + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \tag{5.15}$$

Then by the even symmetry of $\widehat{\nabla U}$, (5.15) implies (5.7).

Case 1.2. Now we assume that $\sup_{\partial E_R(x_0)} |U| > C_3(\Lambda^{1/2}R)^{\alpha'}$. By the choice of $C_3 = 2[U]_{0,\alpha',\widetilde{K}}$, we have either

$$U \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0), \text{ or}$$
$$U \leq -(C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0).$$

However, from $U \geq 0$ on B_1', the only possibility is

$$U \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0).$$

Consequently,

$$u_{x_0} \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } B_R.$$

If we let h again be the Signorini replacement of u_{x_0} in B_R, then the positivity of $h = u_{x_0} > 0$ on ∂B_R and superharmonicity of h in B_R give that $h > 0$ in B_R, and hence h is harmonic in B_R. Thus,

$$\int_{B_\rho} |\nabla h - \langle \nabla h \rangle_{B_\rho}|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h \rangle_{B_r}|^2, \quad 0 < \rho < r.$$

We next decompose $h = h^* + h^\sharp$ in B_R as in (4.2). Note that since both h and $h^\sharp$ are harmonic, h^* must be harmonic as well. Then we have

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_\rho}|^2 &\le 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 6\int_{B_\rho} |\widehat{\nabla h} - \nabla h|^2\\ &= 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 6\int_{B_\rho^-} \left(|2\nabla_{y'} h^\sharp|^2 + |2\partial_{y_n} h^*|^2\right)\\ &= 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 12\int_{B_\rho} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right),\end{aligned}$$

and similarly,

$$\int_{B_r} |\nabla h - \langle \nabla h\rangle_{B_r}|^2 \le 3\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_r}|^2 + 12\int_{B_r} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right).$$

Combining the above three inequalities, we have that for all $0 < \rho < r$

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_\rho}|^2 \le 3\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_r}|^2 + 48\int_{B_r} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right). \tag{5.16}$$

Now, note that if $r_0 \le (1/2)^{\frac{2n+\alpha}{\alpha}}$, then $r \le R/2$. By the harmonicity of both h^* and $h^\sharp$ in B_R, we have

$$\begin{aligned}\sup_{B_{R/2}} |D^2 h^*| + \sup_{B_{R/2}} |D^2 h^\sharp| &\le \frac{C(n)}{R}\Big(\sup_{B_{(3/4)R}} |\nabla h^*| + \sup_{B_{(3/4)R}} |\nabla h^\sharp|\Big)\\ &\le \frac{C(n)}{R^{1+\frac{n}{2}}}\Big(\int_{B_R} |\nabla h^*|^2 + \int_{B_R} |\nabla h^\sharp|^2\Big)^{1/2}\\ &= \frac{C(n)}{R^{1+\frac{n}{2}}}\Big(\int_{B_R} |\nabla h|^2\Big)^{1/2} \le \frac{C(n)}{R^{1+\frac{n}{2}}}\Big(\int_{B_R} |\nabla u_{x_0}|^2\Big)^{1/2}\\ &\le C(n,\alpha,M,K)\|\nabla U\|_{L^2(B_1)} R^{\alpha'-2},\end{aligned}$$

where the last inequality follows from (5.11). Also, note that $\nabla_{y'} h^\sharp = \partial_{y_n} h^* = 0$ on $B'_{R/2}$. Thus, for $y = (y', y_n) \in B_r$, we have

$$\begin{aligned}|\nabla_{y'} h^\sharp| + |\partial_{y_n} h^*| &\le |y_n|\Big(\sup_{B_{R/2}} |D^2 h^*| + \sup_{B_{R/2}} |D^2 h^\sharp|\Big)\\ &\le C\|\nabla U\|_{L^2(B_1)} r R^{\alpha'-2}\\ &= C\|\nabla U\|_{L^2(B_1)} r^{1+\frac{2n}{2n+\alpha}(\alpha'-2)},\end{aligned}$$

with $C = (n, \alpha, M, K)$. Hence, it follows that

$$\begin{aligned}\int_{B_r} |\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2 &\le C\|\nabla U\|^2_{L^2(B_1)} r^{n+2+\frac{4n}{2n+\alpha}(\alpha'-2)}\\ &\le C\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.\end{aligned} \tag{5.17}$$

Combining (5.16) and (5.17), we obtain

$$\int_{B_\rho} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_\rho}|^2 \le 3\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_r}|^2 + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \tag{5.18}$$

Note that (5.12) was induced in Case 1.1 without the use of the assumption $\sup_{\partial E_r(x_0)}|U| \le C_3\left(\Lambda^{1/2}R\right)^{\alpha'}$, so it is also valid in this case. Finally, (5.8), (5.9), (5.12) and (5.18) give

$$\begin{aligned}
\int_{B_\rho}|\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_\rho}|^2 &\le 3\int_{B_\rho} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_\rho}|^2 + 6\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 9\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle\widehat{\nabla h}\rangle_{B_r}|^2 + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + 6\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 27\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r}|\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_r}|^2 + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + 60\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 27\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r}|\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_r}|^2 + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-1/2)} \\
&\le 27\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r}|\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_r}|^2 + C\,\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.
\end{aligned}$$

As we have seen in Case 1.1, this implies (5.7). This completes the proof of (5.7) when $x_0 \in K \cap B_1'$.

Case 2. The extension of (5.7) to general $x_0 \in K$ follows from the combination of Case 1 and (3.5). The argument is the same as Case 2 in the proof of Theorem 4.6 in [31].

Thus, the estimate (5.7) holds in all possible cases.

To complete the proof of the theorem, we now apply Lemma 3.5 to the estimate (5.7) to obtain for $0 < \rho < r < r_0$

$$\int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle\widehat{\nabla U}\rangle_{B_\rho(x_0)}|^2 \le C\Big[\Big(\frac{\rho}{r}\Big)^{n+2\beta} \int_{B_r(x_0)} |\widehat{\nabla U} - \langle\widehat{\nabla U}\rangle_{B_r(x_0)}|^2 + \|U\|^2_{W^{1,2}(B_1)}\rho^{n+2\beta}\Big].$$

Taking $r \nearrow r_0 = r_0(n,\alpha,M,K)$, we have

$$\int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle\widehat{\nabla U}\rangle_{B_\rho(x_0)}|^2 \le C\,\|U\|^2_{W^{1,2}(B_1)}\rho^{n+2\beta},$$

with $C = C(n, \alpha, M, K)$. Then by the Campanato space embedding this readily implies that $\widehat{\nabla U} \in C^{0,\beta}(K)$ with

$$\|\widehat{\nabla U}\|_{C^{0,\beta}(K)} \le C\|U\|_{W^{1,2}(B_1)}.$$

Since $\widehat{\nabla U} = \nabla U$ in $B_1^+ \cup B_1'$, we therefore conclude that

$$U \in C^{1,\beta}(K \cap (B_1^+ \cup B_1')),$$

and combining with the bound in Theorem 4.3, we also deduce that

$$\|U\|_{C^{1,\beta}(K\cap(B_1^+\cup B_1'))} \le C(n, \alpha, M, K)\|U\|_{W^{1,2}(B_1)}.$$

To see the $C^{1,\beta}$ regularity of U in $B_1^- \cup B_1'$, we simply observe that the function $U(y', -y_n)$ is also an almost minimizer of the Signorini problem with the appropriately modified coefficient matrix A. ∎

6. Quasisymmetric almost minimizers

In the study of the free boundary in the Signorini problem, the even symmetry of the minimizer with respect to the thin space plays a crucial role. The even symmetry guarantees that the growth rate of the minimizer u over "thick" balls $B_r(x_0) \subset \mathbb{R}^n$ matches the growth rate over thin balls $B_r'(x_0) \subset \Pi$. This allows to use tools such as Almgren's monotonicity formula (see the next section) to classify the free boundary points. Without even symmetry, minimizers may have an odd component, vanishing on the thin space Π that may create a mismatch of growth rates on the thick and thin spaces.

In the case of minimizers of the Signorini problem (with $A = I$) or harmonic functions, it is easy to see that the even symmetrization

$$u^*(x) = \frac{u(x', x_n) + u(x', -x_n)}{2}$$

is still a minimizer. Unfortunately, the even symmetrization may destroy the almost minimizing property, as well as the minimizing property with variable coefficients, as can be seen from the following simple example.

Example 6.1. Let $u\colon (-1, 1) \to \mathbb{R}$ be defined by $u(x) = x + x^2/4$. Then u is an almost harmonic function in $(-1, 1)$ with a gauge function $\omega(r) = C(\alpha)r^\alpha$ for $0 < \alpha < 1$. In fact, u is a minimizer of the energy functional

$$\int (1 + x/2)^{-1}(v')^2$$

with a Lipschitz function $A(x) = (1 + x/2)^{-1}$ in $(-1, 1)$. On the other hand, the even symmetrization

$$u^*(x) = \frac{u(x) + u(-x)}{2} = \frac{x^2}{4}$$

is not almost harmonic for any gauge function $\omega(r)$. Indeed, for any small $\delta > 0$, if we take a competitor $v = \delta^2/4$ in $(-\delta, \delta)$, then it satisfies $\int_{-\delta}^{\delta} |v'|^2 = 0$ and if u^* were almost harmonic, we would have that $\int_{-\delta}^{\delta} |(u^*)'|^2 = 0$ as well, implying that u^* is constant in $(-\delta, \delta)$, a contradiction.

To overcome this difficulty, we need to impose the *A-quasisymmetry* condition on almost minimizers U, that we have already stated in Definition 1.2. In this section, we give more details on quasisymmetric almost minimizers.

Recall that for each $x_0 \in B_1'$, we defined a reflection matrix P_{x_0} by

$$P_{x_0} = I - 2\frac{A(x_0)e_n \otimes e_n}{a_{nn}(x_0)}.$$

From the ellipticity of A, we have $a_{nn}(x_0) \geq \lambda$, thus P_{x_0} is well-defined. Note that $P_{x_0}^2 = I$. Besides, $P_{x_0}\big|_\Pi = I\big|_\Pi$ and $P_{x_0}E_r(x_0) = E_r(x_0)$. We then define the "skewed" even/odd symmetrizations of the almost minimizer U in B_1 by

$$U_{x_0}^*(x) := \frac{U(x) + U(P_{x_0}x)}{2},$$
$$U_{x_0}^\sharp(x) := \frac{U(x) - U(P_{x_0}x)}{2}.$$

Note that $U_{x_0}^*$ and $U_{x_0}^\sharp$ may not be defined in all of B_1, but are defined in any ellipsoid

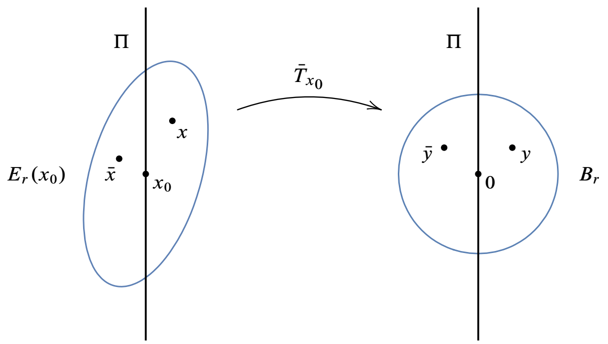

Figure 2. Reflection P_{x_0}: here $\bar{x} = P_{x_0}x$, $y = \bar{T}_{x_0}(x)$, and $\bar{y} = (y', -y_n) = \bar{T}_{x_0}(\bar{x})$

$E_r(x_0)$ as long as it is contained in B_1. Note also that $U = U_{x_0}^*$ and $U_{x_0}^\sharp = 0$ on Π. Further, we note that transformed with $\bar{T}_{x_0}$, P_{x_0} becomes an even reflection with respect to Π, i.e.,

$$\bar{T}_{x_0} \circ P_{x_0} \circ \bar{T}_{x_0}^{-1}(y) = (y', -y_n),$$

see Figure 2. Therefore, denoting

$$u^*_{x_0}(y) := \frac{u_{x_0}(y', y_n) + u_{x_0}(y', -y_n)}{2},$$
$$u^\sharp_{x_0}(y) := \frac{u_{x_0}(y', y_n) - u_{x_0}(y', -y_n)}{2},$$

the even/odd symmetrizations of u_{x_0} about Π, we will have

$$U^*_{x_0} \circ \bar{T}^{-1}_{x_0} = u^*_{x_0}, \qquad U^\sharp_{x_0} \circ \bar{T}^{-1}_{x_0} = u^\sharp_{x_0}.$$

We also observe that the symmetries of $u^*_{x_0}$ and $u^\sharp_{x_0}$ imply the following decompositions

$$\int_{B_r} u^2_{x_0} = \int_{B_r} (u^*_{x_0})^2 + \int_{B_r} (u^\sharp_{x_0})^2,$$
$$\int_{B_r} |\nabla u_{x_0}|^2 = \int_{B_r} |\nabla u^*_{x_0}|^2 + \int_{B_r} |\nabla u^\sharp_{x_0}|^2,$$

which after a change of variables, can also be written as

$$\int_{E_r(x_0)} U^2 = \int_{E_r(x_0)} (U^*_{x_0})^2 + \int_{E_r(x_0)} (U^\sharp_{x_0})^2, \tag{6.1}$$

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle = \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle + \int_{E_r(x_0)} \langle A(x_0)\nabla U^\sharp_{x_0}, \nabla U^\sharp_{x_0}\rangle. \tag{6.2}$$

We now recall that by Definition 1.2, $U \in W^{1,2}(B_1)$ is called A-*quasisymmetric* if there is a constant $Q > 0$ such that

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \le Q \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle, \tag{6.3}$$

whenever $E_r(x_0) \Subset B_1$ and $x_0 \in B'_1$. By the uniform ellipticity of A, (6.3) is equivalent to

$$\int_{E_r(x_0)} |\nabla U|^2 \le Q \int_{E_r(x_0)} |\nabla U^*_{x_0}|^2,$$

by changing Q to $Q(\Lambda/\lambda)$, if necessary. Besides, using (6.2), (6.3) is also equivalent to

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U^\sharp_{x_0}, \nabla U^\sharp_{x_0}\rangle \le C \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle, \tag{6.4}$$

with some $C = C(Q)$.

Lemma 6.2. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, with constant $Q > 0$. Then there are $r_1 = r_1(n, \alpha, M, Q) > 0$ and $M_1 = M_1(n, M, Q) > 0$ such that*

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle \le (1 + M_1 r^\alpha) \int_{E_r(x_0)} \langle A(x_0)\nabla W, \nabla W\rangle, \tag{6.5}$$

whenever $E_r(x_0) \Subset B_1$, $x_0 \in B'_1$, $0 < r < r_1$, and $W \in \mathfrak{K}_{0,U^*_{x_0}}(E_r(x_0), \Pi)$.

Remark 6.3. Since we are interested in local results, in what follows, we will assume without loss of generality that $r_1 = 1$ and $M_1 = M$.

Proof. Let V be the energy minimizer of

$$\int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle \quad \text{on } \mathfrak{K}_{0,U}(E_r(x_0), \Pi).$$

Then $v_{x_0} = V \circ \bar{T}_{x_0}^{-1}$ is the energy minimizer of

$$\int_{B_r} |\nabla v_{x_0}|^2 \quad \text{on } \mathfrak{K}_{0,u_{x_0}}(B_r, \Pi).$$

Note that $v_{x_0}^*$ is a solution of the Signorini problem, even in y_n, with $v_{x_0}^* = u_{x_0}^*$ on ∂B_r. Similarly, $v_{x_0}^\sharp$ is a harmonic function, odd in y_n, with $v_{x_0}^\sharp = u_{x_0}^\sharp$ on ∂B_r. Thus, $v_{x_0}^*$ is the energy minimizer of

$$\int_{B_r} |\nabla v_{x_0}^*|^2 \quad \text{on } \mathfrak{K}_{0,u_{x_0}^*}(B_r, \Pi),$$

and so $V_{x_0}^*$ is the energy minimizer of

$$\int_{E_r(x_0)} \langle A(x_0)\nabla V_{x_0}^*, \nabla V_{x_0}^*\rangle \quad \text{on } \mathfrak{K}_{0,U_{x_0}^*}(E_r(x_0), \Pi).$$

Thus, to show (6.5), it is enough to show

$$\int_{B_r} |\nabla u_{x_0}^*|^2 \le (1 + M_1 r^\alpha) \int_{B_r} |\nabla v_{x_0}^*|^2.$$

To this end, we first observe that the quasisymmetry of U implies the quasisymmetry of u_{x_0}:

$$\int_{B_r} |\nabla u_{x_0}^\sharp|^2 \le C \int_{B_r} |\nabla u_{x_0}^*|^2.$$

Using this, together with the symmetry of $u_{x_0}^*$, $u_{x_0}^\sharp$, $v_{x_0}^*$ and $v_{x_0}^\sharp$, we have

$$\begin{aligned}
\int_{B_r} |\nabla u_{x_0}^*|^2 &= \int_{B_r} |\nabla u_{x_0}|^2 - \int_{B_r} |\nabla u_{x_0}^\sharp|^2 \\
&\le (1 + M r^\alpha) \int_{B_r} |\nabla v_{x_0}|^2 - \int_{B_r} |\nabla u_{x_0}^\sharp|^2 \\
&= (1 + M r^\alpha) \int_{B_r} |\nabla v_{x_0}^*|^2 + (1 + M r^\alpha) \int_{B_r} |\nabla v_{x_0}^\sharp|^2 - \int_{B_r} |\nabla u_{x_0}^\sharp|^2 \\
&\le (1 + M r^\alpha) \int_{B_r} |\nabla v_{x_0}^*|^2 + M r^\alpha \int_{B_r} |\nabla u_{x_0}^\sharp|^2 \\
&\le (1 + M r^\alpha) \int_{B_r} |\nabla v_{x_0}^*|^2 + C M r^\alpha \int_{B_r} |\nabla u_{x_0}^*|^2.
\end{aligned}$$

Therefore,

$$\int_{B_r} |\nabla u^*_{x_0}|^2 \le \frac{1+Mr^\alpha}{1-CMr^\alpha}\int_{B_r} |\nabla v^*_{x_0}|^2 \le (1+M_1 r^\alpha)\int_{B_r} |\nabla v^*_{x_0}|^2,$$

for $0 < r < r_1 = (2CM)^{-1/\alpha}$, as desired. ∎

Remark 6.4. If U satisfies the following *weak quasisymmetry* with order $-\gamma$:

$$\int_{E_r(x_0)} |\nabla U|^2 \le Q\, r^{-\gamma} \int_{E_r(x_0)} |\nabla U^*_{x_0}|^2,$$

whenever $E_r(x_0) \Subset B_1$, $x_0 \in B'_1$ for some $0 < \gamma < \alpha$, then it is easy to see from the proof of Lemma 6.2 that $U^*_{x_0}$ satisfies (6.5), but with $\alpha - \gamma > 0$ instead of α.

Theorem 6.5. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Then for $x_0 \in B'_{1/2}$ and $0 < r \le (1/2)\Lambda^{-1/2}$, we have $U^*_{x_0} \in C^{1,\beta}(E^\pm_r(x_0) \cup E'_r(x_0))$ with $\beta = \frac{\alpha}{4(2n+\alpha)}$. Moreover,*

$$\|U^*_{x_0}\|_{C^{1,\beta}(K)} \le C(n,\alpha,M,K,r)\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))},$$

for any $K \Subset E^\pm_r(x_0) \cup E'_r(x_0)$. Similarly, $u^*_{x_0} \in C^{1,\beta}(B^\pm_r \cup B'_r)$ with

$$\|u^*_{x_0}\|_{C^{1,\beta}(K)} \le C(n,\alpha,M,K,r)\|u^*_{x_0}\|_{W^{1,2}(B_r)},$$

for any $K \Subset B^\pm_r \cup B'_r$.

Proof. From Theorem 5.2, we have $U \in C^{1,\beta}(B^\pm_1 \cup B'_1)$, which immediately gives $U^*_{x_0} \in C^{1,\beta}(E^\pm_r(x_0) \cup E'_r(x_0))$, by using the inclusion $E_r(x_0) \subset B_{\Lambda^{1/2}r}(x_0) \subset B_1$. Thus, for

$$\widehat{\nabla U^*_{x_0}}(x', x_n) := \begin{cases} \nabla U^*_{x_0}(x', x_n) & \text{if } x_n \ge 0, \\ \nabla U^*_{x_0}(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

we have $\widehat{\nabla U^*_{x_0}} \in C^{0,\beta}(E_r(x_0))$ with

$$\|\widehat{\nabla U^*_{x_0}}\|_{C^{0,\beta}(K)} \le C(n,\alpha,M,K,r)\|U\|_{W^{1,2}(E_r(x_0))},$$

for any $K \Subset E_r(x_0)$. Hence, it is enough to show that

$$\|U\|_{W^{1,2}(E_r(x_0))} \le C\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))}.$$

Now, note that by (6.1)–(6.2), we readily have

$$\|U\|_{W^{1,2}(E_r(x_0))} \le C\left(\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))} + \|U^\sharp_{x_0}\|_{W^{1,2}(E_r(x_0))}\right),$$

and thus, it will suffice to show that

$$\|U^\sharp_{x_0}\|_{W^{1,2}(E_r(x_0))} \le C\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))}.$$

By the symmetry again,

$$\langle U^\sharp_{x_0}\rangle_{E_r(x_0)} = \langle u^\sharp_{x_0}\rangle_{B_r} = 0,$$

thus, by Poincare's inequality,

$$\|U^\sharp_{x_0}\|_{L^2(E_r(x_0))} \le C(n,M)r\|\nabla U^\sharp_{x_0}\|_{L^2(E_r(x_0))}.$$

Finally, by the quasisymmetry of U, we have

$$\|\nabla U^\sharp_{x_0}\|_{L^2(E_r(x_0))} \le C\|\nabla U^*_{x_0}\|_{L^2(E_r(x_0))},$$

see (6.4). This completes the proof of the theorem for $U^*_{x_0}$.

Applying the affine transformation $\bar{T}_{x_0}$, we obtain the part of the theorem for $u^*_{x_0}$. ■

We complete this section with a version of Signorini's complementarity condition that will play an important role in the analysis of the free boundary.

Lemma 6.6 (Complementarity condition). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0 \in B'_{1/2}$. Then $u^*_{x_0}$ satisfies the following* complementarity condition

$$u^*_{x_0}(\partial^+_{y_n} u^*_{x_0}) = 0 \quad \text{on } B'_{R_0}, \qquad R_0 = (1/2)\Lambda^{-1/2},$$

where $\partial^+_{y_n} u^*_{x_0}$ on B'_{R_0} is computed as the limit from inside $B^+_{R_0}$. Moreover, if $x_0 \in \Gamma(U)$, then

$$u^*_{x_0}(0) = 0 \quad \text{and} \quad |\widehat{\nabla u^*_{x_0}}(0)| = 0.$$

Proof. Let $y_0 \in B'_{R_0}$ be such that $u^*_{x_0}(y_0) > 0$. Then we need to show that $\partial^+_{y_n} u^*_{x_0}(y_0) = 0$. Since $u_{x_0} = u^*_{x_0}$ on Π, we have $u_{x_0}(y_0) > 0$ and by continuity $u_{x_0} > 0$ in a small ball $B_\delta(y_0)$. Then $U > 0$ in $\Omega = \bar{T}^{-1}_{x_0}(B_\delta(y_0))$. We claim now that U is almost A-harmonic in Ω. Indeed, if $E_r(y) \Subset \Omega$ (not necessarily with $y \in B'_1$) and V is the $A(y)$-harmonic replacement of U on $E_r(y)$ (i.e., $\operatorname{div}(A(y)\nabla V) = 0$ in $E_r(y)$ with $V = U$ on $\partial E_r(y)$), then since $V = U > 0$ on $\partial E_r(y)$, by the minimum principle $V > 0$ on $\overline{E_r(y)}$. This means that $V \in \mathfrak{K}_{0,U}(E_r(y), \Pi)$ and therefore we must have

$$\int_{E_r(y)} \langle A(y)\nabla U, \nabla U\rangle \le (1+\omega(r)) \int_{E_r(y)} \langle A(y)\nabla V, \nabla V\rangle,$$

which also implies that U is an almost A-harmonic function in Ω. Hence, $U \in C^{1,\alpha/2}(\Omega)$ by Theorem 3.6, implying also that $u_{x_0} \in C^{1,\alpha/2}(B_\delta(y_0))$. Consequently, also $u^*_{x_0} \in C^{1,\alpha/2}(B_\delta(y_0))$ and by even symmetry in the y_n-variable, we therefore conclude that $\partial^+_{y_n} u^*_{x_0}(y_0) = 0$.

The second part of the lemma now follows by the $C^{1,\beta}$ regularity and the complementarity condition. ■

7. Weiss- and Almgren-type monotonicity formulas

In this section we introduce two technical tools: Weiss- and Almgren-type monotonicity formulas, that will play a fundamental role in the analysis of the free boundary. In fact, the proofs of these formulas follow immediately from the case $A \equiv I$, following the deskewing procedure.

To proceed, we fix a constant $\kappa_0 > 0$. We can take it as large as we want, however, some constants in what follows, will depend on κ_0. Then for $0 < \kappa < \kappa_0$, we consider the Weiss-type energy functional introduced in [31]:

$$W_\kappa(t, v, x_0) := \frac{e^{at^\alpha}}{t^{n+2\kappa-2}}\left[\int_{B_t(x_0)} |\nabla v|^2 - \kappa \frac{1 - bt^\alpha}{t} \int_{\partial B_t(x_0)} v^2\right],$$

with

$$a = a_\kappa = \frac{M(n + 2\kappa - 2)}{\alpha}, \quad b = \frac{M(n + 2\kappa_0)}{\alpha}.$$

(The formula in [31] corresponds to the case $M = 1$.) Based on that, we define an appropriate version of Weiss' functional for our problem. For a function V in $E_r(x_0)$, let

$$W_\kappa^A(t, V, x_0) := \frac{e^{at^\alpha}}{t^{n+2\kappa-2}}\left[\int_{E_t(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle - \kappa \frac{1 - bt^\alpha}{t} \int_{\partial E_t(x_0)} V^2 \mu_{x_0}(x - x_0)\right],$$

for $0 < t < r$, with a, b same as above, where the weight μ_{x_0} is as in (2.4). Note that by the change of variables in formulas (2.1)–(2.3), we have

$$W_\kappa^A(t, V, x_0) := \det \mathfrak{a}_{x_0} W_\kappa(t, v_{x_0}, 0), \quad v_{x_0} = V \circ \bar{T}_{x_0}^{-1}.$$

Let now U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and $x_0 \in B'_{1/2}$. By Lemma 6.2, $U_{x_0}^*$ satisfies the almost A-Signorini property at x_0 in $E_{(1/2)\Lambda^{-1/2}}(x_0)$. Thus, $u_{x_0}^*$ also satisfies the almost Signorini property at 0 in $B_{(1/2)\Lambda^{-1/2}}$. By using this observation, we then have the following Weiss-type monotonicity formulas for $U_{x_0}^*$ and $u_{x_0}^*$.

Theorem 7.1 (Weiss-type monotonicity formula). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Suppose $x_0 \in B'_{1/2}$ and $U(x_0) = 0$. Let $0 < \kappa < \kappa_0$ with a fixed $\kappa_0 > 0$. Then, for $0 < t < t_0 = t_0(n, \alpha, \kappa_0, M)$,*

$$\frac{d}{dt} W_\kappa(t, u_{x_0}^*, 0) \geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial B_t} \left(\partial_\nu u_{x_0}^* - \frac{\kappa(1 - bt^\alpha)}{t} u_{x_0}^*\right)^2,$$

$$\frac{d}{dt} W_\kappa^A(t, U_{x_0}^*, x_0) \geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial E_t(x_0)} \left(\langle \mathfrak{a}_{x_0} \nabla U_{x_0}^*, \nu\rangle - \frac{\kappa(1 - bt^\alpha)}{t} U_{x_0}^*\right)^2 \mu_{x_0}(x - x_0).$$

In particular, $W_\kappa(t, u_{x_0}^*, 0)$ and $W_\kappa^A(t, U_{x_0}^*, x_0)$ are nondecreasing in t for $0 < t < t_0$.

Proof. We note that the proof of [31, Theorem 5.1] for the monotonicity of $W_\kappa(t, v, x_0)$ requires the function v to be an almost minimizer for the Signorini problem with $v(x_0) = 0$ for the monotonicity of its energy. However, it is not hard to see that the almost minimizing property of v is used only when it is compared with the κ-homogeneous replacement w of v on balls centered at the given point x_0 to obtain

$$\int_{B_t(x_0)} |\nabla w|^2 \geq \frac{1}{1+t^\alpha} \int_{B_t(x_0)} |\nabla v|^2,$$

see [31, (5.2)]. This means that the argument in the proof of [31, Theorem 5.1] also works in our case as long as $u^*_{x_0}(0) = U(x_0) = 0$ and implies the part of the theorem for $u^*_{x_0}$. We note that the constants a_κ and b in our case will have an additional factor of M, as we work with $\omega(r) = Mr^\alpha$ rather than $\omega(r) = r^\alpha$ in our case, but this change of the constants can be easily traced.

The part of the theorem for $U^*_{x_0}$ follows by a change of variables. ■

The families of monotonicity formulas $\{W_\kappa\}_{0<\kappa<\kappa_0}$ and $\{W^A_\kappa\}_{0<\kappa<\kappa_0}$ have an important feature that their intervals of monotonicity and the constant b can be taken the same for all $0 < \kappa < \kappa_0$. Because of that, their monotonicity indirectly implies that of another important quantity that we describe below. Namely, recall that for a function v in $B_r(x_0)$, *Almgren's frequency* of v at x_0 is defined as

$$N(t, v, x_0) := \frac{t \int_{B_t(x_0)} |\nabla v|^2}{\int_{\partial B_t(x_0)} v^2}, \quad 0 < t < r.$$

Note that this quantity is well-defined when v has an almost Signorini property at x_0 and $x_0 \in \Gamma(v)$, since vanishing of $\int_{\partial B_t(x_0)} v^2$ for any $t > 0$ would imply vanishing of v in $B_t(x_0)$ by taking 0 as a competitor and consequently that $x_0 \notin \Gamma(v)$.

Next consider a modification of N, which we call the *truncated frequency*:

$$\widehat{N}_{\kappa_0}(t, v, x_0) := \min\left\{\frac{1}{1-bt^\alpha} N(t, v, x_0), \kappa_0\right\},$$

where b is as in Weiss-type monotonicity formulas for $\kappa < \kappa_0$. We next define the appropriate version of N, $\widehat{N}_{\kappa_0}$ in our setting. For a function V in $E_r(x_0)$, we define

$$N^A(t, V, x_0) := N(t, v_{x_0}, 0),$$
$$\widehat{N}^A_{\kappa_0}(t, V, x_0) := \widehat{N}_{\kappa_0}(t, v_{x_0}, 0),$$

for $0 < t < r$, where $v_{x_0} = V \circ \bar{T}^{-1}_{x_0}$. More explicitly, we have

$$N^A(t, V, x_0) := \frac{t \int_{E_t(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle}{\int_{\partial E_t(x_0)} V^2 \mu_{x_0}(x - x_0)},$$
$$\widehat{N}^A_{\kappa_0}(t, V, x_0) := \min\left\{\frac{1}{1-bt^\alpha} N^A(t, V, x_0), \kappa_0\right\}.$$

As observed in [31, Theorem 5.4], the Weiss-type monotonicity formula implies the following monotonicity of $\widehat{N}^A_{\kappa_0}$.

Theorem 7.2 (Almgren-type monotonicity formula). *Let U, κ_0, and t_0 be as in Theorem 7.1, and $x_0 \in B'_{1/2}$ a free boundary point. Then*

$$t \mapsto \widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(t, u^*_{x_0}, 0)$$

is nondecreasing for $0 < t < t_0$.

Definition 7.3 (Almgren's frequency at free boundary point). For an A-quasisymmetric almost minimizer U of the A-Signorini problem in B_1 and $x_0 \in \Gamma(U)$ let

$$\kappa(x_0) := \widehat{N}^A_{\kappa_0}(0+, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(0+, u^*_{x_0}, 0).$$

We call $\kappa(x_0)$ the *Almgren's frequency* at x_0.

Remark 7.4. Note that even though the monotonicity of the truncated frequency is stated in Theorem 7.2 only for $x_0 \in B'_{1/2} \cap \Gamma(U)$, by a simple recentering and a scaling argument, it will be monotone also at all $x_0 \in \Gamma(U)$, but for a possibly shorter interval of values $0 < t < t_0(x_0)$ depending on x_0. Thus, $\kappa(x_0)$ exists at all $x_0 \in \Gamma(U)$.

Further, note that when $\kappa(x_0) < \kappa_0$, then $\widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \frac{1}{1-bt^\alpha} N^A(t, U^*_{x_0}, x_0)$ for small t and therefore

$$\kappa(x_0) = N^A(0+, U^*_{x_0}, x_0),$$

which means that it will not change if we replace κ_0 with a larger value.

8. Almgren rescalings and blowups

Our analysis of the free boundary is based on the analysis of blowups, which are the limits of rescalings of the solutions at free boundary points. In Signorini problem, there are a few types of rescalings that use different normalizations. In this section, we look at so-called Almgren rescalings and blowups that play well with the Almgren frequency formula.

Let $V \in W^{1,2}(B_1)$ and $x_0 \in B'_{1/2}$ be a free boundary point. For small $r > 0$ define the *Almgren rescaling* of V at x_0 by

$$V^A_{x_0,r}(x) := \frac{V(rx + x_0)}{\left(\frac{1}{r^{n-1}} \int_{\partial E_r(x_0)} V^2 \mu_{x_0}(x - x_0)\right)^{1/2}}.$$

The Almgren rescalings have the following normalization and scaling properties

$$\begin{aligned} &\|V^A_{x_0,r}\|_{L^2(\mathfrak{a}_{x_0}\partial B_1)} = 1, \\ &N^{A(x_0)}(\rho, V^A_{x_0,r}, 0) = N^A(\rho r, V, x_0). \end{aligned}$$

Here $N^{A(x_0)}$ denotes Almgren's frequency for a constant matrix $A(x_0)$. Thus, we also have $N^A(r, V, x_0) = N^{A(x_0)}(r, V, x_0)$. Note that when $A = I$, then

$$V^I_{x_0,r} = \frac{V(rx + x_0)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} V^2\right)^{1/2}}$$

is same as the Almgren rescaling in [31], and satisfies

$$\|V^I_{x_0,r}\|_{L^2(\partial B_1)} = 1,$$
$$N(\rho, V^I_{x_0,r}, 0) = N(\rho r, V, x_0).$$

We will call the limits of $V^A_{x_0,r}$ over any subsequence $r = r_j \to 0+$ *Almgren blowups* of V at x_0 and denote them by $V^A_{x_0,0}$.

By using a change of variables, we can express Almgren rescalings of V in terms of those of $v_{x_0} = V \circ \bar{T}^{-1}_{x_0}$ and vice versa. Namely, we have

$$(v_{x_0})^I_r(y) = (\det \mathfrak{a}_{x_0})^{1/2} V^A_{x_0,r}(\bar{\mathfrak{a}}_{x_0} y),$$

wherever they are defined. Applied to the particular case $V = U^*_{x_0}$, we have

$$(u^*_{x_0})^I_r(y) = (\det \mathfrak{a}_{x_0})^{1/2} (U^*_{x_0})^A_{x_0,r}(\bar{\mathfrak{a}}_{x_0} y).$$

Proposition 8.1 (Existence of Almgren blowups). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0 \in B'_{1/2} \cap \Gamma(U)$ be such that $\kappa(x_0) < \kappa_0$. Then, every sequence of Almgren rescalings $(U^*_{x_0})^A_{x_0,t_j}$, with $t_j \to 0+$, contains a subsequence, still denoted t_j such that for a function $(U^*_{x_0})^A_{x_0,0} \in C^1_{\mathrm{loc}}(\mathfrak{a}_{x_0}(B^\pm_1 \cup B'_1))$*

$$(U^*_{x_0})^A_{x_0,t_j} \to (U^*_{x_0})^A_{x_0,0} \quad \text{in } C^1_{\mathrm{loc}}(\mathfrak{a}_{x_0}(B^\pm_1 \cup B'_1)).$$

Moreover, $(U^*_{x_0})^A_{x_0,0}$ extends to a nonzero solution of the $A(x_0)$-Signorini problem in $\mathbb{R}^n$, $(U^*_{x_0})^A_{x_0,0}(x) = (U^*_{x_0})^A_{x_0,0}(P_{x_0} x)$, and it is homogeneous of degree $\kappa(x_0)$ in $\mathbb{R}^n$.

Similarly, every sequence of Almgren rescalings $(u^*_{x_0})^I_{t_j}$, with $t_j \to 0+$ contains a subsequence, still denoted t_j such that for a function $(u^*_{x_0})^I_0 \in C^1_{\mathrm{loc}}(B^\pm_1 \cup B'_1)$

$$(u^*_{x_0})^I_{t_j} \to (u^*_{x_0})^I_0 \quad \text{in } C^1_{\mathrm{loc}}(B^\pm_1 \cup B'_1).$$

Moreover, $(u^*_{x_0})^I_0$ extends to a nonzero solution of the Signorini problem in $\mathbb{R}^n$, even in y_n, and it is homogeneous of degree $\kappa(x_0)$ in $\mathbb{R}^n$.

Proof. Step 1. Since $\kappa(x_0) < \kappa_0$, we must have $N(t, u^*_{x_0}, 0) < \kappa_0$ for small $t > 0$. Then, for such t

$$\int_{B_1} |\nabla (u^*_{x_0})^I_t|^2 = N(1, (u^*_{x_0})^I_t, 0) = N(t, u^*_{x_0}, 0) \le \kappa_0,$$

and combined with the normalization $\int_{\partial B_1}\big((u^*_{x_0})^I_t\big)^2=1$, we see that the family $(u^*_{x_0})^I_t$ is bounded in $W^{1,2}(B_1)$, for small $t>0$. Hence, for any sequence $t_j\to 0+$, there is a function $(u^*_{x_0})^I_0\in W^{1,2}(B_1)$ such that, over a subsequence,

$$(u^*_{x_0})^I_{t_j}\to (u^*_{x_0})^I_0 \qquad \text{weakly in } W^{1,2}(B_1),$$
$$(u^*_{x_0})^I_{t_j}\to (u^*_{x_0})^I_0 \qquad \text{strongly in } L^2(\partial B_1).$$

In particular, $\int_{\partial B_1}\big((u^*_{x_0})^I_0\big)^2=1$, implying that $(u^*_{x_0})^I_0\not\equiv 0$ in B_1.

Step 2. For $0<t<1$ and $x\in B_{1/(2t)}(x_0)$, let

$$U_{x_0,t}(x)=U(x_0+t(x-x_0)),\quad A_{x_0,t}(x)=A(x_0+t(x-x_0)).$$

Then by a simple scaling argument, we have that $U_{x_0,t}$ is an almost minimizer of the $A_{x_0,t}$-Signorini problem in $B_{1/(2t)}(x_0)$ with a gauge function $\mu_t(r)=(tr)^\alpha\le r^\alpha$. In particular, for any $R>0$, we will have that $U_{x_0,t}\in C^{1,\beta}(E^\pm_R(x_0)\cup E'_R(x_0))$ for $0<t<t(R,M)$ with

$$\|U_{x_0,t}\|_{C^{1,\beta}(K)}\le C\|U_{x_0,t}\|_{W^{1,2}(E_R(x_0))},$$

with $C=C(n,\alpha,M,R,K)$, for any $K\Subset E^\pm_R(x_0)\cup E'_R(x_0)$. Then, arguing as in the proof of Theorem 6.5, by using the quasisymmetry of U, we obtain that

$$\|(U_{x_0,t})^*_{x_0}\|_{C^{1,\beta}(K)}\le C\|(U_{x_0,t})^*_{x_0}\|_{W^{1,2}(E_R(x_0))},$$

where

$$(U_{x_0,t})^*_{x_0}(x)=\frac{U_{x_0,t}(x)+U_{x_0,t}(P_{x_0}x)}{2}.$$

Next, observing that $(u^*_{x_0})^I_t$ is a positive constant multiple of $(U_{x_0,t})^*_{x_0}\circ\bar{T}^{-1}_{x_0}$, we obtain that

$$\|(u^*_{x_0})^I_t\|_{C^{1,\beta}(K)}\le C\|(u^*_{x_0})^I_t\|_{W^{1,2}(B_R)},$$

for any $K\Subset B^\pm_R\cup B'_R$. Taking $R=1$, combined with the boundedness of $(u^*_{x_0})^I_t$ in $W^{1,2}(B_1)$ for small $t>0$, it follows that up to a subsequence,

$$(u^*_{x_0})^I_{t_j}\to (u^*_{x_0})^I_0 \quad \text{in } C^1_{\mathrm{loc}}(B^\pm_1\cup B'_1).$$

Step 3. Next, we claim that the blowup $(u^*_{x_0})^I_0$ is a solution of the Signorini problem in B_1. Indeed, fix $0<R<1$, and for each t_j let h_{t_j} be the Signorini replacement of $(u^*_{x_0})^I_{t_j}$ in B_R. Then a first variation argument gives (see [31, (3.2)])

$$\int_{B_R}\langle\nabla h_{t_j},\nabla((u^*_{x_0})^I_{t_j}-h_{t_j})\rangle\ge 0.$$

Since $(u^*_{x_0})^I_{t_j}$ has an almost Signorini property at 0 with a gauge function $r\mapsto C(t_jr)^\alpha$, it follows that

$$\int_{B_R}|\nabla((u^*_{x_0})^I_{t_j}-h_{t_j})|^2\le C(Rt_j)^\alpha\int_{B_R}|\nabla(u^*_{x_0})^I_{t_j}|^2.$$

This implies that $h_{t_j} \to (u^*_{x_0})^I_0$ weakly in $W^{1,2}(B_R)$. On the other hand, by the boundedness of the sequence h_{t_j} in $W^{1,2}(B_R)$, we have also boundedness in $C^{1,1/2}$ norm locally in $(B^\pm_R \cup B'_R)$ and hence, over a subsequence, $h_{t_j} \to (u^*_{x_0})^I_0$ in $C^1_{\mathrm{loc}}(B^\pm_R \cup B'_R)$. By this convergence, we then conclude that $(u^*_{x_0})^I_0$ satisfies

$$\Delta(u^*_{x_0})^I_0 = 0 \quad \text{in } B_R \setminus B'_R$$
$$(u^*_{x_0})^I_0 \geq 0, \quad -\partial^+_{y_n}(u^*_{x_0})^I_0 \geq 0, \quad (u^*_{x_0})^I_0 \partial^+_{y_n}(u^*_{x_0})^I_0 = 0 \quad \text{on } B'_R,$$

and hence, by letting $R \to 1$, $(u^*_{x_0})^I_0$ itself solves the Signorini problem in B_1.

Step 4. Recall now that the blowup $(u^*_{x_0})^I_0$ is nonzero in B_1. In particular, $\int_{\partial B_r}((u^*_{x_0})^I_0)^2 > 0$ for any $0 < r < 1$, otherwise we would have that $(u^*_{x_0})^I_0$ is identically zero on ∂B_r and consequently also on B_r. Using this fact, combined with C^1_{loc} convergence in $B^\pm_1 \cup B'_1$, we have that for any $0 < r < 1$

$$\begin{aligned} N(r, (u^*_{x_0})^I_0, 0) &= \lim_{t_j \to 0} N(r, (u^*_{x_0})^I_{t_j}, 0) \\ &= \lim_{t_j \to 0} N(rt_j, u^*_{x_0}, 0) \\ &= N(0+, u^*_{x_0}, 0) \\ &= \kappa(x_0). \end{aligned}$$

Thus, Almgren's frequency of $(u^*_{x_0})^I_0$ is constant $\kappa(x_0)$ on $0 < r < 1$ which is possible only if $(u^*_{x_0})^I_0$ is a $\kappa(x_0)$-homogeneous solution of the Signorini problem in B_1, see [38, Theorem 9.4]. Finally, by using the homogeneity, we readily extend $(u^*_{x_0})^I_0$ to a solution of the Signorini problem in all of $\mathbb{R}^n$. This completes the proof for $(u^*_{x_0})^I_0$.

The corresponding result for $(U^*_{x_0})^A_{x_0,t_j}$ follows now by a change of variables. ■

With Proposition 8.1 at hand, we can repeat the argument in the proof of [31, Lemma 6.2] with $u^*_{x_0}$ to obtain the following, which is possible since $u^*_{x_0}$ satisfies the complementarity condition and an Almgren-type monotonicity formula with a blowup as a nonzero solution of the Signorini problem.

Lemma 8.2 (Minimal frequency). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. If $x_0 \in B'_{1/2} \cap \Gamma(U)$, then*

$$\kappa(x_0) \geq \frac{3}{2}.$$

Consequently, we also have

$$\widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(t, u^*_{x_0}, 0) \geq 3/2 \quad \textit{for } 0 < t < t_0.$$

Lemma 8.2 readily gives the following (see [31, Corollary 6.3]).

Corollary 8.3. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and x_0 a free boundary point. Then*

$$W^A_{3/2}(t, U^*_{x_0}, x_0) = \det \mathfrak{a}_{x_0} W_{3/2}(t, u^*_{x_0}, 0) \geq 0, \quad \textit{for } 0 < t < t_0.$$

9. Growth estimates

The first result in this section (Lemma 9.1) provides growth estimates for the quasisymmetric almost minimizers near free boundary points x_0 with $\kappa(x_0) \geq \kappa$. Such estimates were obtained in [31, Lemma 7.1] in the case $A \equiv I$ as a consequence of Weiss-type monotonicity formulas. However, they contain an unwanted logarithmic term that creates difficulties in the blowup analysis of the problem.

The next two results (Lemmas 9.2 and 9.3) remove the logarithmic term from these estimates for $\kappa = 3/2$, by establishing first a growth rate for $W_{3/2}$. (Recall that $\kappa(x_0) \geq 3/2$ at every free boundary point x_0, by Lemma 8.2.) These are analogous to [31, Lemmas 7.3, 7.4] in the case $A \equiv I$ and follow from the so-called epiperimetric inequality for $\kappa = 3/2$ (see e.g. [31, Theorem 7.2]). Later, in Section 12, we remove the logarithmic term also in the case $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, see Lemma 12.3.

The results in this section are stated in terms of both $u^*_{x_0}$ and $U^*_{x_0}$, as we need both forms in the subsequent arguments. We note that the estimates for $u^*_{x_0}$ follow directly from [31, Lemmas 7.1, 7.3, 7.4] and the ones for $U^*_{x_0}$ are obtained by using the deskewing procedure, and therefore we skip all proofs in this section.

In the estimates below, as well as in the rest of the paper, we use the notation

$$R_0 := (1/2)\Lambda^{-1/2},$$

which is the radius of the largest ball B_{R_0}, where $u^*_{x_0}$ is guaranteed to exist for any $x_0 \in B'_{1/2}$ for an almost minimizer U in B_1.

Lemma 9.1 (Weak growth estimate). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and $x_0 \in B'_{1/2} \cap \Gamma(U)$. If*

$$\kappa(x_0) \geq \kappa$$

for some $\kappa \leq \kappa_0$, then

$$\int_{\partial B_t} (u^*_{x_0})^2 \leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} \Big(\log \frac{1}{t} \Big) t^{n+2\kappa-1},$$

$$\int_{B_t} |\nabla u^*_{x_0}|^2 \leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} \Big(\log \frac{1}{t} \Big) t^{n+2\kappa-2},$$

$$\int_{\partial E_t(x_0)} (U^*_{x_0})^2 \leq C \|U\|^2_{W^{1,2}(B_1)} \Big(\log \frac{1}{t} \Big) t^{n+2\kappa-1},$$

$$\int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 \leq C \|U\|^2_{W^{1,2}(B_1)} \Big(\log \frac{1}{t} \Big) t^{n+2\kappa-2},$$

for $0 < t < t_0 = t_0(n, \alpha, M, \kappa_0)$ and $C = C(n, \alpha, M, \kappa_0)$.

Lemma 9.2. *Let U and x_0 be as above. Then, there exists $\delta = \delta(n, \alpha) > 0$ such that*

$$0 \le W_{3/2}(t, u^*_{x_0}, 0) \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^{\delta},$$
$$0 \le W^A_{3/2}(t, U^*_{x_0}, x_0) \le C \|U\|^2_{W^{1,2}(B_1)} t^{\delta},$$

for $0 < t < t_0 = t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$.

Lemma 9.3 (Optimal growth estimate). *Let U and x_0 be as above. Then,*

$$\int_{\partial B_t} (u^*_{x_0})^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^{n+2},$$
$$\int_{B_t} |\nabla u^*_{x_0}|^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^{n+1},$$
$$\int_{\partial E_t(x_0)} (U^*_{x_0})^2 \le C \|U\|^2_{W^{1,2}(B_1)} t^{n+2},$$
$$\int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 \le C \|U\|^2_{W^{1,2}(B_1)} t^{n+1},$$

for $0 < t < t_0 = t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$.

10. 3/2-almost homogeneous rescalings and blowups

In this section we study another kind of rescalings and blowups that will play a fundamental role in the analysis of regular free boundary points where $\kappa(x_0) = 3/2$ (see the next section), namely, 3/2-almost homogeneous blowups. The main result that we prove in this section is the uniqueness and Hölder continuous dependence of such blowups at a free boundary point x_0 (Lemma 10.3).

For a function v in B_1 and $x_0 \in B'_{1/2}$, we define the 3/2-*almost homogeneous rescalings* of v at x_0 by

$$v^{\phi}_{x_0,t}(x) = \frac{v(tx + x_0)}{\phi(t)}, \quad \phi(t) = e^{-(\frac{3b}{2\alpha})t^{\alpha}} t^{3/2},$$

with b as in the Weiss-type monotonicity formulas $W^A_{3/2}$ and $W_{3/2}$. When $x_0 = 0$, we simply write $v^{\phi}_{0,t} = v^{\phi}_t$.

The name is explained by the fact that

$$\lim_{t \to 0} \frac{\phi(t)}{t^{3/2}} = 1,$$

and the reason to look at such rescalings instead of 3/2-homogeneous rescalings (that would correspond to $\phi(t) = t^{3/2}$) is how they play well with the Weiss-type monotonicity formulas $W^A_{3/2}$ and $W_{3/2}$.

Now, if U is an A-quasisymmetric almost minimizer and $x_0 \in B'_{1/2} \cap \Gamma(U)$, then for any fixed $R > 1$, if $t = t_j > 0$ is small, then by Lemma 9.3,

$$\int_{B_R} |\nabla (u^*_{x_0})^\phi_t|^2 = \frac{e^{\frac{3b}{\alpha}t^\alpha}}{t^{n+1}} \int_{B_{Rt}} |\nabla u^*_{x_0}|^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} R^{n+1}, \tag{10.1}$$
$$\int_{\partial B_R} ((u^*_{x_0})^\phi_t)^2 = \frac{e^{\frac{3b}{\alpha}t^\alpha}}{t^{n+2}} \int_{\partial B_{Rt}} (u^*_{x_0})^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} R^{n+2},$$

with $C = C(n, \alpha, M)$, $R_0 = (1/2)\Lambda^{-1/2}$. Therefore, $(u^*_{x_0})^\phi_{t_j}$ is a bounded sequence in $W^{1,2}(B_R)$. Next, arguing as in the proof of Proposition 8.1, we will have that

$$\|\widehat{\nabla (u^*_{x_0})^\phi_t}\|_{C^{0,\beta}(K)} \le C \|(u^*_{x_0})^\phi_t\|_{W^{1,2}(B_R)}, \tag{10.2}$$

with $C = C(n, \alpha, M, R, K)$ for $K \Subset B_R$. Thus, by letting $R \to \infty$ and using Cantor's diagonal argument, we can conclude that over a subsequence $t = t_j \to 0+$,

$$(u^*_{x_0})^\phi_{t_j} \to (u^*_{x_0})^\phi_0 \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

We call such $(u^*_{x_0})^\phi_0$ a 3/2-*homogeneous blowup* of $u^*_{x_0}$ at 0. (We may skip the "almost" modifier here as the limit is the same as for 3/2-homogeneous rescalings.) Furthermore, from the relation

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0} y),$$

we also conclude that for any sequence $t_j \to 0+$, there is a subsequence, still denoted by t_j, such that

$$(U^*_{x_0})^\phi_{x_0,t_j} \to (U^*_{x_0})^\phi_{x_0,0} \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

Apriori, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ may depend on the sequence $t_j \to 0+$. However, this does not happen in the case of 3/2-homogeneous blowups. We start with what we call a rotation estimate for rescalings.

Lemma 10.1 (Rotation estimate). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, $x_0 \in B'_{1/2}$ a free boundary point, and δ as in Lemma 9.2. Then,*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_s| \le C \|u^*_{x_0}\|_{W^{1,2}(B_{R_0})} t^{\delta/2},$$
$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,s}| \le C \|U\|_{W^{1,2}(B_1)} t^{\delta/2},$$

for $s < t < t_0 = t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$.

Proof. This is an analogue of [31, Lemma 8.2], which follows from the computation done in the proof of [31, Lemma 7.1], the growth estimate for $W_{3/2}$ in [31, Lemma 7.3] and a dyadic argument. The analogues of those results in our case are stated in Lemma 9.1

and 9.2. This proves the lemma for $u^*_{x_0}$. The estimate for $(U^*_{x_0})^\phi_{x_0,t}$ then follows from the equality

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0} y), \quad y \in B_{R_0/t}. \qquad \blacksquare$$

The uniqueness of $3/2$-homogeneous blowup now follows.

Lemma 10.2. *Let $(U^*_{x_0})^\phi_{x_0,0}$ and $(u^*_{x_0})^\phi_0$ be blowups of $(U^*_{x_0})^\phi_{x_0,t}$ and $(u^*_{x_0})^\phi_t$, respectively, at a free boundary point $x_0 \in B'_{1/2}$. Then,*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_0| \le C \|u^*_{x_0}\|_{W^{1,2}(B_{R_0})} t^{\delta/2},$$

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,0}| \le C \|U\|_{W^{1,2}(B_1)} t^{\delta/2},$$

for $0 < t < t_0(n,\alpha,M)$ and $C = C(n,\alpha,M)$, where $\delta = \delta(n,\alpha) > 0$ is as in Lemma 10.1. In particular, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ are unique.

Proof. If $(u^*_{x_0})^\phi_0$ is the limit of $(u^*_{x_0})^\phi_{t_j}$ for $t_j \to 0$, then the first part of the lemma follows immediately from Lemma 10.1, by taking $s = t_j \to 0$ and passing to the limit.

To see the uniqueness of blowups, we observe that $(u^*_{x_0})^\phi_0$ is a solution of the Signorini problem in B_1, by arguing as in the proof of Proposition 8.1 for Almgren blowups. Now, if v_0 is another blowup, over a possibly different sequence $t'_j \to 0$, then passing to the limit in the first part of the lemma we will have

$$\int_{\partial B_1} |v_0 - (u^*_{x_0})^\phi_0|^2 = 0,$$

implying that both v_0 and $(u^*_{x_0})^\phi_0$ are solutions of the Signorini problem in B_1 with the same boundary values on ∂B_1. By the uniqueness of such solutions, we have $v_0 = (u^*_{x_0})^\phi_0$ in B_1. The equality propagates to all of $\mathbb{R}^n$ by the unique continuation of harmonic functions in $\mathbb{R}^n_\pm$. This completes the proof for $u^*_{x_0}$. An analogous argument holds for $U^*_{x_0}$ using the equalities

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0} y), \quad y \in B_{R_0/t},$$
$$(u^*_{x_0})^\phi_0(y) = (U^*_{x_0})^\phi_{x_0,0}(\bar{\mathfrak{a}}_{x_0} y), \quad y \in \mathbb{R}^n. \qquad \blacksquare$$

The rotation estimate for rescalings implies not only the uniqueness of blowups and the convergence rate to blowups, but also the continuous dependence of blowups on a free boundary point.

Lemma 10.3 (Continuous dependence of blowups). *There exists $\rho = \rho(n,\alpha,M) > 0$ such that if $x_0, y_0 \in B'_\rho$ are free boundary points of U, then*

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C|x_0 - y_0|^\gamma, \tag{10.3}$$

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_0 - (u^*_{y_0})^\phi_0| \le C|x_0 - y_0|^\gamma, \tag{10.4}$$

$$\int_{\partial B_1'} |(u_{x_0}^*)_0^\phi - (u_{y_0}^*)_0^\phi| \le C|x_0 - y_0|^\gamma, \tag{10.5}$$

with $C = C(n, \alpha, M, \|U\|_{W^{1,2}(B_1)})$, $\gamma = \gamma(n, \alpha, M) > 0$.

Proof. Step 1. Let $d = |x_0 - y_0|$ and $d^\tau \le r \le 2d^\tau$ with $\tau = \tau(\alpha) \in (0, 1)$ to be determined later.

Next note that we can incorporate the weight $\mu_{x_0}/\det \mathfrak{a}_{x_0}$ with μ_{x_0} as in (2.4) in the integral on the left-hand side of (10.3) because of the bounds

$$\left(\frac{\lambda}{\Lambda}\right)^{1/2} \le \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \le \left(\frac{\Lambda}{\lambda}\right)^{1/2}.$$

Then, by using Lemma 10.2, we have

$$\begin{aligned}
&\int_{\mathfrak{a}_{x_0}\partial B_1} |(U_{x_0}^*)_{x_0,0}^\phi - (U_{y_0}^*)_{y_0,0}^\phi| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\quad\le \int_{\mathfrak{a}_{x_0}\partial B_1} \Big(\big|(U_{x_0}^*)_{x_0,0}^\phi - (U_{x_0}^*)_{x_0,r}^\phi\big| + \big|(U_{x_0}^*)_{x_0,r}^\phi - (U_{x_0}^*)_{y_0,r}^\phi\big| \\
&\qquad + \big|(U_{x_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,r}^\phi\big| + \big|(U_{y_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,0}^\phi\big| \Big) \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\qquad + \int_{\mathfrak{a}_{y_0}\partial B_1} |(U_{y_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,0}^\phi| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}} \\
&\qquad - \int_{\mathfrak{a}_{y_0}\partial B_1} |(U_{y_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,0}^\phi| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}} \\
&\quad\le 2Cr^{\delta/2} + \mathrm{I}_r + \mathrm{II}_r + \mathrm{III}_r \\
&\quad\le Cd^{\tau\delta/2} + \mathrm{I}_r + \mathrm{II}_r + \mathrm{III}_r,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U_{x_0}^*)_{x_0,r}^\phi - (U_{x_0}^*)_{y_0,r}^\phi| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}}, \\
\mathrm{II}_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U_{x_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,r}^\phi| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}}, \\
\mathrm{III}_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U_{y_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,0}^\phi| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\quad - \int_{\mathfrak{a}_{y_0}\partial B_1} |(U_{y_0}^*)_{y_0,r}^\phi - (U_{y_0}^*)_{y_0,0}^\phi| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}}.
\end{aligned}$$

Step 2. By the definition of the almost homogeneous rescalings, we have

$$\mathrm{I}_r \le \frac{C}{d^{\tau(n+1/2)}} \int_{\mathfrak{a}_{x_0}\partial B_r} |U_{x_0}^*(z + x_0) - U_{x_0}^*(z + y_0)| dS_z.$$

This gives

$$\begin{aligned}
\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau} \mathrm{I}_r\,dr &\le \frac{C}{d^{\tau(n+3/2)}}\int_{d^\tau}^{2d^\tau}\int_{\mathfrak{a}_{x_0}\partial B_r}\big|U^*_{x_0}(z+x_0)-U^*_{x_0}(z+y_0)\big|dS_z dr\\
&\le \frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})}\big|U^*_{x_0}(z+x_0)-U^*_{x_0}(z+y_0)\big|dz\\
&= \frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})}\left|\int_0^1\frac{d}{ds}\Big[U^*_{x_0}(z+x_0(1-s)+y_0 s)\Big]ds\right|dz\\
&\le \frac{C}{d^{\tau(n+3/2)}}|x_0-y_0|\int_0^1\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})}\big|\nabla U^*_{x_0}(z+x_0(1-s)+y_0 s)\big|dzds\\
&\le \frac{C}{d^{\tau(n+3/2)-1}}\int_0^1\int_{\mathfrak{a}_{x_0}B_{2d^\tau}+[x_0(1-s)+y_0 s]}\big|\nabla U^*_{x_0}\big|dzds.
\end{aligned}$$

Notice that the last integral is taken over

$$\begin{aligned}
\mathfrak{a}_{x_0}B_{2d^\tau}+[x_0(1-s)+y_0 s] &= \mathfrak{a}_{x_0}[B_{2d^\tau}+s\mathfrak{a}_{x_0}^{-1}(y_0-x_0)]+x_0\\
&\subset \mathfrak{a}_{x_0}B_{2d^\tau+\lambda^{-1/2}d}+x_0\subset E_{3d^\tau}(x_0),
\end{aligned}$$

if $\rho=\rho(n,\alpha,M)$ is small so that $(2\rho)^{1-\tau}\le\lambda^{1/2}$ which readily implies $d^{1-\tau}\le\lambda^{1/2}$. Thus,

$$\begin{aligned}
\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau}\mathrm{I}_r\,dr &\le \frac{C}{d^{\tau(n+3/2)-1}}\int_0^1\int_{E_{3d^\tau}(x_0)}|\nabla U^*_{x_0}|dzds\\
&\le \frac{C}{d^{\tau(n/2+3/2)-1}}\left(\int_{E_{3d^\tau}(x_0)}|\nabla U^*_{x_0}|^2\right)^{1/2}\\
&\le C\|U\|_{W^{1,2}(B_1)}d^{1-\tau},
\end{aligned}$$

where the third inequality follows from Lemma 9.3.

Step 3. By the definition of rescalings and symmetrizations, we have

$$\begin{aligned}
\mathrm{II}_r &\le \frac{C}{d^{\tau(n+1/2)}}\int_{\mathfrak{a}_{x_0}\partial B_r+y_0}|U^*_{x_0}(z)-U^*_{y_0}(z)|dS_z\\
&\le \frac{C}{d^{\tau(n+1/2)}}\int_{\mathfrak{a}_{x_0}\partial B_r+y_0}|U(P_{x_0}z)-U(P_{y_0}z)|dS_z.
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau}\mathrm{II}_r\,dr &\le \frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}|U(P_{x_0}z)-U(P_{y_0}z)|dz\\
&\le \frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}\int_0^1\left|\frac{d}{ds}[U([(1-s)P_{x_0}+sP_{y_0}]z)]\right|dsdz\\
&\le \frac{C|P_{x_0}-P_{y_0}|}{d^{\tau(n+3/2)}}\int_0^1\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}|\nabla U([(1-s)P_{x_0}+sP_{y_0}]z)|dzds.
\end{aligned}$$

Now we do the change of variables

$$y = [(1-s)P_{x_0} + sP_{y_0}]z.$$

Since P_{x_0} and P_{y_0} are upper-triangular matrices with diagonal entries $1, 1, \dots, 1, -1$, so is $(1-s)P_{x_0} + sP_{y_0}$. Thus,

$$\left|\det[(1-s)P_{x_0} + sP_{y_0}]\right| = 1.$$

Moreover, $y \in [(1-s)P_{x_0} + sP_{y_0}](\mathfrak{a}_{x_0} B_{2d^\tau} + y_0)$. Since

$$\mathfrak{a}_{x_0} B_{2d^\tau} + y_0 \subset \mathfrak{a}_{y_0} B_{2(\Lambda/\lambda)^{1/2}d^\tau} + y_0 = E_{2(\Lambda/\lambda)^{1/2}d^\tau}(y_0),$$

we have

$$P_{y_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) \subset P_{y_0} E_{2(\Lambda/\lambda)^{1/2}d^\tau}(y_0) = E_{2(\Lambda/\lambda)^{1/2}d^\tau}(y_0).$$

Similarly, since

$$\begin{aligned}\mathfrak{a}_{x_0} B_{2d^\tau} + y_0 &= E_{2d^\tau}(x_0) + (y_0 - x_0) \subset B_{2\Lambda^{1/2}d^\tau}(x_0) + (y_0 - x_0)\\ &\subset B_{4\Lambda^{1/2}d^\tau}(x_0) \subset E_{4(\Lambda/\lambda)^{1/2}d^\tau}(x_0),\end{aligned}$$

we have

$$P_{x_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) \subset E_{4(\Lambda/\lambda)^{1/2}d^\tau}(x_0).$$

Thus,

$$\begin{aligned}y &\in (1-s)P_{x_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) + sP_{y_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0)\\ &\subset (1-s)E_{4(\Lambda/\lambda)^{1/2}d^\tau}(x_0) + sE_{2(\Lambda/\lambda)^{1/2}d^\tau}(y_0)\\ &\subset B_{6(\Lambda/\lambda^{1/2})d^\tau} + x_0 + s(y_0 - x_0)\\ &\subset B_{7(\Lambda/\lambda^{1/2})d^\tau} + x_0 \subset E_{7(\Lambda/\lambda)d^\tau}(x_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau} \mathrm{II}_r\, dr &\le \frac{C}{d^{\tau(n+3/2)-\alpha}} \int_0^1 \int_{E_{7(\Lambda/\lambda)d^\tau}(x_0)} |\nabla U| dz ds\\ &\le \frac{C}{d^{\tau(n/2+3/2)-\alpha}} \left(\int_{E_{7(\Lambda/\lambda)d^\tau}(x_0)} |\nabla U|^2\right)^{1/2}\\ &\le \frac{C}{d^{\tau(n/2+3/2)-\alpha}} \left(\int_{E_{7(\Lambda/\lambda)d^\tau}(x_0)} |\nabla U_{x_0}^*|^2\right)^{1/2}\\ &\le C\|U\|_{W^{1,2}(B_1)} d^{\alpha-\tau},\end{aligned}$$

for small ρ, where the third inequality follows from the quasisymmetry property and the last inequality from Lemma 9.3.

Step 4. By the change of variables, we have

$$
\begin{aligned}
\mathrm{III}_r &= \int_{\partial B_1} |(U^*_{y_0})^{\phi}_{y_0,r}(\mathfrak{a}_{x_0} z) - (U^*_{y_0})^{\phi}_{y_0,0}(\mathfrak{a}_{x_0} z)| - \int_{\partial B_1} |(U^*_{y_0})^{\phi}_{y_0,r}(\mathfrak{a}_{y_0} z) - (U^*_{y_0})^{\phi}_{y_0,0}(\mathfrak{a}_{y_0} z)| \\
&\le \int_{\partial B_1} |(U^*_{y_0})^{\phi}_{y_0,r}(\mathfrak{a}_{x_0} z) - (U^*_{y_0})^{\phi}_{y_0,r}(\mathfrak{a}_{y_0} z)| + \int_{\partial B_1} |(U^*_{y_0})^{\phi}_{y_0,0}(\mathfrak{a}_{x_0} z) - (U^*_{y_0})^{\phi}_{y_0,0}(\mathfrak{a}_{y_0} z)| \\
&\le C\Big(\|\nabla (U^*_{y_0})^{\phi}_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})} + \|\nabla (U^*_{y_0})^{\phi}_{y_0,0}\|_{L^\infty(B_{\Lambda^{1/2}})}\Big)|\mathfrak{a}_{x_0} - \mathfrak{a}_{y_0}|,
\end{aligned}
$$

where we have used the fact that both $\mathfrak{a}_{x_0} z$ and $\mathfrak{a}_{y_0} z$ are contained in $\overline{B_{\Lambda^{1/2}}}$ for $z \in \partial B_1$. To estimate the gradients of rescalings we first observe that by the inclusion $B_{r\Lambda^{1/2}}(y_0) \subset E_{r(\Lambda/\lambda)^{1/2}}(y_0) \subset B_{r\Lambda/\lambda^{1/2}}(y_0)$, we have

$$
\|\nabla (U^*_{y_0})^{\phi}_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})} \le \frac{C}{r^{1/2}} \|\nabla U^*_{y_0}\|_{L^\infty(B_{r\Lambda^{1/2}}(y_0))} \le \frac{C}{r^{1/2}} \|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))}.
$$

Let $U_{y_0,r}(x) := U(r(x-y_0)+y_0)$. Then, arguing as in the proof of Proposition 8.1, we have

$$
\|\nabla U_{y_0,r}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}}(y_0))} \le C(n,\alpha,M)\|U_{y_0,r}\|_{W^{1,2}(B_{2\Lambda/\lambda^{1/2}}(y_0))}.
$$

Thus,

$$
\begin{aligned}
\|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))} &= \frac{1}{r}\|\nabla U_{y_0,r}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}}(y_0))} \\
&\le \frac{C}{r}\|U_{y_0,r}\|_{W^{1,2}(B_{2\Lambda/\lambda^{1/2}}(y_0))} \\
&\le \frac{C}{r^{n/2+1}}\|U\|_{L^2(B_{2r\Lambda/\lambda^{1/2}}(y_0))} + \frac{C}{r^{n/2}}\|\nabla U\|_{L^2(B_{2r\Lambda/\lambda^{1/2}}(y_0))} \\
&\le \frac{C}{r^{n/2+1}}\|U^*_{y_0}\|_{L^2(E_{2r\Lambda/\lambda}(y_0))} + \frac{C}{r^{n/2}}\|\nabla U^*_{y_0}\|_{L^2(E_{2r\Lambda/\lambda}(y_0))} \\
&\le C r^{1/2}\|U\|_{W^{1,2}(B_1)},
\end{aligned}
$$

where we have used the inclusion $B_{2r\Lambda/\lambda^{1/2}}(y_0) \subset E_{2r\Lambda/\lambda}(y_0)$ and the quasisymmetry property in the third inequality and Lemma 9.3 in the forth. Therefore,

$$
\|\nabla (U^*_{y_0})^{\phi}_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})} \le \frac{C}{r^{1/2}}\|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))} \le C\|U\|_{W^{1,2}(B_1)}.
$$

Moreover, by C^1_{loc} convergence of $(U^*_{y_0})^{\phi}_{y_0,r}$ to $(U^*_{y_0})^{\phi}_{y_0,0}$, we also have

$$
\|\nabla (U^*_{y_0})^{\phi}_{y_0,0}\|_{L^\infty(B_{\Lambda^{1/2}})} = \lim_{r_j\to 0+}\|\nabla (U^*_{y_0})^{\phi}_{y_0,r_j}\|_{L^\infty(B_{\Lambda^{1/2}})} \le C\|U\|_{W^{1,2}(B_1)}. \tag{10.6}
$$

Therefore,

$$
\begin{aligned}
\mathrm{III}_r &\le C|\mathfrak{a}_{x_0} - \mathfrak{a}_{y_0}|\|U\|_{W^{1,2}(B_1)} \\
&\le C\|U\|_{W^{1,2}(B_1)} d^{\alpha}.
\end{aligned}
$$

Step 5. Now we are ready to prove (10.3). Using the estimates in Steps 2–4 and taking the average over $d^\tau \le r \le 2d^\tau$, we have

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C\|U\|_{W^{1,2}(B_1)}(d^{\tau\delta/2} + d^{1-\tau} + d^{\alpha-\tau} + d^\alpha).$$

If we simply take $\tau = \alpha/2$, then we conclude

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C|x_0 - y_0|^\gamma,$$

with $\gamma = \alpha\delta/4$ and $C = C(n, \alpha, M, \|U\|_{W^{1,2}(B_1)})$.

Step 6. To prove (10.4), we first observe that from (10.3),

$$\begin{aligned}\int_{\partial B_1} |(u^*_{x_0})^\phi_0(z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}^{-1}_{y_0}\bar{\mathfrak{a}}_{x_0}z)| &= \int_{\partial B_1} |(U^*_{x_0})^\phi_{x_0,0}(\bar{\mathfrak{a}}_{x_0}z) - (U^*_{y_0})^\phi_{y_0,0}(\bar{\mathfrak{a}}_{x_0}z)| \\ &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\ &\le C|x_0 - y_0|^\gamma.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\partial B_1} |(u^*_{y_0})^\phi_0(z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}^{-1}_{y_0}\bar{\mathfrak{a}}_{x_0}z)| &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}^{-1}_{x_0}z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}^{-1}_{y_0}z)| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\ &\le C\|\nabla(u^*_{y_0})^\phi_0\|_{L^\infty(B_{(\Lambda/\lambda)^{1/2}})}|\bar{\mathfrak{a}}^{-1}_{x_0} - \bar{\mathfrak{a}}^{-1}_{y_0}| \\ &\le C\|\nabla(U^*_{y_0})^\phi_{y_0,0}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}})}|x_0 - y_0|^\alpha \\ &\le C\|U\|_{W^{1,2}(B_1)}|x_0 - y_0|^\alpha,\end{aligned}$$

where the last inequality follows from (10.6). (It is easy to see that we can enlarge the domain in (10.6).) Therefore, combining the preceding two estimates, we conclude that

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_0 - (u^*_{y_0})^\phi_0| \le C|x_0 - y_0|^\gamma.$$

Step 7. Finally, (10.4) implies (10.5), by arguing precisely as in [26, Proposition 7.4]. ■

11. Regularity of the regular set

In this section we combine the uniqueness and Hölder continuous dependence of 3/2-homogeneous blowups of the symmetrized almost minimizers $(U^*_{x_0})^\phi_{x_0,0}$ (Lemma 10.3) with a classification of such blowups at so-called regular points (Proposition 11.3) to prove one of the main results of this paper, the $C^{1,\gamma}$ regularity of the regular set (Theorem 11.7).

While some arguments follow directly from those in the case $A \equiv I$ by a coordinate transformation $\bar{T}_{x_0}$, the dependence of these transformations on x_0 creates an additional difficulty.

We start by defining the regular set.

Definition 11.1 (Regular points). For an A-quasisymmetric almost minimizer U for the A-Signorini problem in B_1, we say that a free boundary point x_0 of U is *regular* if

$$\kappa(x_0) = 3/2.$$

We denote the set of all regular points of U by $\mathcal{R}(U)$ and call it the *regular set*.

We explicitly observe here that $3/2 < 2 \le \kappa_0$, so the fact $x_0 \in \mathcal{R}(U)$ is independent of the choice of $\kappa_0 \ge 2$, see Remark 7.4.

The proofs of the following two results (Lemma 11.2 and Proposition 11.3) are established precisely as in [31, Lemma 9.2, Proposition 9.3] for the transformed functions $u^*_{x_0}$. The equivalent statements for $U^*_{x_0}$ are obtained by changing back to the original variables.

Lemma 11.2 (Nondegeneracy at regular points). *Let $x_0 \in B'_{1/2} \cap \mathcal{R}(U)$ for an A-quasi-symmetric almost minimizer U for the A-Signorini problem in B_1. Then, for $\kappa = 3/2$,*

$$\liminf_{t\to 0} \int_{\mathfrak{a}_{x_0}\partial B_1} ((U^*_{x_0})^{\phi}_{x_0,t})^2 \mu_{x_0} = \det \mathfrak{a}_{x_0} \liminf_{t\to 0} \int_{\partial B_1} ((u^*_{x_0})^{\phi}_t)^2 > 0.$$

Proposition 11.3. *If $\kappa(x_0) < 2$, then necessarily $\kappa(x_0) = 3/2$ and*

$$(u^*_{x_0})^{\phi}_0(z) = a_{x_0} \operatorname{Re}(z' \cdot \nu_{x_0} + i|z_n|)^{3/2},$$
$$(U^*_{x_0})^{\phi}_{x_0,0}(x) = a_{x_0} \operatorname{Re}((\bar{\mathfrak{a}}^{-1}_{x_0} x)' \cdot \nu_{x_0} + i|(\bar{\mathfrak{a}}^{-1}_{x_0} x)_n|)^{3/2},$$

for some $a_{x_0} > 0$, $\nu_{x_0} \in \partial B'_1$.

The next two corollaries are obtained by repeating the same arguments as in [31, Corollaries 9.4 and 9.5].

Corollary 11.4 (Almgren's frequency gap). *Let U and x_0 be as in Lemma 11.2. Then either*

$$\kappa(x_0) = 3/2 \quad \textit{or} \quad \kappa(x_0) \ge 2.$$

Corollary 11.5. *The regular set $\mathcal{R}(U)$ is a relatively open subset of the free boundary.*

The combination of Proposition 11.3 and Lemma 10.3 implies the following lemma.

Lemma 11.6. *Let U and x_0 be as in Lemma 11.2. Then there exists $\rho > 0$, depending on x_0 such that $B'_\rho(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$ and if*

$$(u^*_{\bar{x}})^{\phi}_0(z) = a_{\bar{x}} \operatorname{Re}(z' \cdot \nu_{\bar{x}} + i|z_n|)^{3/2}$$

is the unique $3/2$*-homogeneous blowup of* $u^*_{\bar{x}}$ *at* $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$*, then*

$$|a_{\bar{x}} - a_{\bar{y}}| \le C_0 |\bar{x} - \bar{y}|^\gamma,$$
$$|\nu_{\bar{x}} - \nu_{\bar{y}}| \le C_0 |\bar{x} - \bar{y}|^\gamma,$$

for any $\bar{x}, \bar{y} \in B'_\rho(x_0) \cap \Gamma(u)$ *with a constant* C_0 *depending on* x_0*.*

Proof. The proof follows by repeating the argument presented in [26, Lemma 7.5] with $(u^*_{\bar{x}})^\phi_0$, $(u^*_{\bar{y}})^\phi_0$. ■

Now we are ready to prove the main result on the regularity of the regular set.

Theorem 11.7 ($C^{1,\gamma}$ regularity of the regular set). *Let* U *be an* A*-quasisymmetric almost minimizer for the* A*-Signorini problem in* B_1*. Then, if* $x_0 \in B'_{1/2} \cap \mathcal{R}(U)$*, there exists* $\rho > 0$*, depending on* x_0 *such that, after a possible rotation of coordinate axes in* $\mathbb{R}^{n-1}$*, one has* $B'_\rho(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$*, and*

$$B'_\rho(x_0) \cap \Gamma(U) = B'_\rho(x_0) \cap \{x_{n-1} = g(x_1, \dots, x_{n-2})\},$$

for $g \in C^{1,\gamma}(\mathbb{R}^{n-2})$ *with an exponent* $\gamma = \gamma(n, \alpha, M) \in (0, 1)$*.*

Proof. The proof of the theorem is similar to those of in [31, Theorem 9.7] and [26, Theorem 1.2]. However, we provide full details since there are technical differences.

Step 1. By relative openness of $\mathcal{R}(U)$ in $\Gamma(U)$, for small $\rho > 0$ we have $B'_{2\rho}(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$. We then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, $r < r_\varepsilon$, we have that

$$\|(u^*_{\bar{x}})^\phi_r - (u^*_{\bar{x}})^\phi_0\|_{C^1(\overline{B_1^\pm})} < \varepsilon.$$

Assuming the contrary, there is a sequence of points $\bar{x}_j \in B'_\rho(x_0) \cap \Gamma(U)$ and radii $r_j \to 0$ such that

$$\|(u^*_{\bar{x}_j})^\phi_{r_j} - (u^*_{\bar{x}_j})^\phi_0\|_{C^1(\overline{B_1^\pm})} \ge \varepsilon_0,$$

for some $\varepsilon_0 > 0$. Taking a subsequence if necessary, we may assume $\bar{x}_j \to \bar{x}_0 \in \overline{B'_\rho(x_0)} \cap \Gamma(U)$. Using estimates (10.1)–(10.2), we can see that $\nabla(u^*_{\bar{x}_j})^\phi_{r_j}$ are uniformly bounded in $C^{0,\beta}(B_2^\pm \cup B'_2)$. Since $(u^*_{\bar{x}_j})^\phi_{r_j}(0) = 0$, we also have that $(u^*_{\bar{x}_j})^\phi_{r_j}$ is uniformly bounded in $C^{1,\beta}(B_2^\pm \cup B'_2)$. Thus, we may assume that for some w

$$(u^*_{\bar{x}_j})^\phi_{r_j} \to w \quad \text{in } C^1(\overline{B_1^\pm}).$$

By arguing as in the proof of Proposition 8.1, we see that the limit w is a solution of the Signorini problem in B_1. Further, by Lemma 10.2, we have

$$\|(u^*_{\bar{x}_j})^\phi_{r_j} - (u^*_{\bar{x}_j})^\phi_0\|_{L^1(\partial B_1)} \to 0.$$

On the other hand, by Lemma 11.6, we have

$$(u^*_{\bar{x}_j})^\phi_0 \to (u^*_{\bar{x}_0})^\phi_0 \quad \text{in } C^1(\overline{B_1^\pm}),$$

and thus

$$w = (u^*_{\bar{x}_0})^\phi_0 \quad \text{on } \partial B_1.$$

Since both w and $(u^*_{\bar{x}_0})^\phi_0$ are solutions of the Signorini problem, they must coincide also in B_1. Therefore,

$$(u^*_{\bar{x}_j})^\phi_{r_j} \to (u^*_{\bar{x}_0})^\phi_0 \quad \text{in } C^1(\overline{B_1^\pm}),$$

implying also that

$$\|(u^*_{\bar{x}_j})^\phi_{r_j} - (u^*_{\bar{x}_j})^\phi_0\|_{C^1(\overline{B_1^\pm})} \to 0,$$

which contradicts our assumption.

Step 2. For a given $\varepsilon > 0$ and a unit vector $\nu \in \mathbb{R}^{n-1}$ define the cone

$$\mathcal{C}_\varepsilon(\nu) = \{x' \in \mathbb{R}^{n-1} \mid x' \cdot \nu > \varepsilon|x'|\}.$$

By Lemma 11.6, we may assume $a_{\bar{x}} \geq \frac{a_{x_0}}{2}$ for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$ by taking ρ small. For such ρ, we then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, we have

$$\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \subset \{u^*_{\bar{x}}(\cdot, 0) > 0\}.$$

Indeed, denoting $\mathcal{K}_\varepsilon(\nu) = \mathcal{C}_\varepsilon \cap \partial B'_{1/2}$, we have for some universal $C_\varepsilon > 0$

$$\mathcal{K}_\varepsilon(\nu_{\bar{x}}) \Subset \{(u^*_{\bar{x}})^\phi_0(\cdot, 0) > 0\} \cap B'_1 \quad \text{and} \quad (u^*_{\bar{x}})^\phi_0(\cdot, 0) \geq a_{\bar{x}} C_\varepsilon \geq \frac{a_{x_0}}{2} C_\varepsilon \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

Since $\frac{a_{x_0}}{2} C_\varepsilon$ is independent of $\bar{x}$, by Step 1 we can find $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$(u^*_{\bar{x}})^\phi_r(\cdot, 0) > 0 \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

This implies that for $r < 2r_\varepsilon$,

$$u^*_{\bar{x}}(\cdot, 0) > 0 \quad \text{on} \quad r\mathcal{K}_\varepsilon(\nu_{\bar{x}}) = \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap \partial B'_{r/2}.$$

Taking the union over all $r < 2r_\varepsilon$, we obtain

$$u^*_{\bar{x}}(\cdot, 0) > 0 \quad \text{on } \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}.$$

Step 3. We claim that for given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, we have $-\big(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\big) \subset \{u^*_{\bar{x}}(\cdot, 0) = 0\}$.

Indeed, we first note that

$$-\partial^+_{x_n}(u^*_{\bar{x}})^\phi_0 \geq a_{\bar{x}} C_\varepsilon > \left(\frac{a_{x_0}}{2}\right) C_\varepsilon \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}}),$$

for a universal constant $C_\varepsilon > 0$. From Step 1, there exists $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$-\partial^+_{x_n}(u^*_{\bar{x}})^\phi_r(\cdot,0) > 0 \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

By arguing as in Step 2, we obtain

$$-\partial^+_{x_n}u^*_{\bar{x}}(\cdot,0) > 0 \quad \text{on } -\left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right).$$

By the complementarity condition in Lemma 6.6, we therefore conclude that

$$-\left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{-\partial^+_{x_n}u^*_{\bar{x}}(\cdot,0) > 0\} \subset \{u^*_{\bar{x}}(\cdot,0) = 0\}.$$

Step 4. By direct computation, we have

$$\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon} \subset \bar{\mathfrak{a}}_{\bar{x}}\left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right),$$

where

$$\nu^A_{\bar{x}} := \frac{(\bar{\mathfrak{a}}^{-1}_x)^{\mathrm{tr}}\nu_{\bar{x}}}{|(\bar{\mathfrak{a}}^{-1}_x)^{\mathrm{tr}}\nu_{\bar{x}}|}.$$

(Here $(\cdot)^{\mathrm{tr}}$ stands for the transpose of the matrix.) Indeed, if $y' \in \mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}$, then

$$y' \in B'_{\lambda^{1/2}r_\varepsilon} = \bar{\mathfrak{a}}_{\bar{x}}\left(\bar{\mathfrak{a}}^{-1}_{\bar{x}}B'_{\lambda^{1/2}r_\varepsilon}\right) \subset \bar{\mathfrak{a}}_{\bar{x}}B'_{r_\varepsilon},$$

and

$$\begin{aligned}\langle\bar{\mathfrak{a}}^{-1}_x y', \nu_{\bar{x}}\rangle &= \langle y', (\bar{\mathfrak{a}}^{-1}_x)^{\mathrm{tr}}\nu_{\bar{x}}\rangle = \langle y', \nu^A_{\bar{x}}\rangle|(\bar{\mathfrak{a}}^{-1}_x)^{\mathrm{tr}}\nu_{\bar{x}}| \\ &\ge (\Lambda^{1/2}\lambda^{-1/2}\varepsilon|y'|)(\Lambda^{-1/2}) \\ &= \lambda^{-1/2}\varepsilon|y'| \ge \varepsilon|\bar{\mathfrak{a}}^{-1}_{\bar{x}}y'|.\end{aligned}$$

Combining this with Step 2 and Step 3, for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$,

$$\begin{aligned}\bar{x} + \left(\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \bar{x} + \bar{\mathfrak{a}}_{\bar{x}}\left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \\ &\subset \{U^*_{\bar{x}}(\cdot,0) > 0\}, \\ \bar{x} - \left(\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \{U^*_{\bar{x}}(\cdot,0) = 0\}.\end{aligned}$$

Step 5. By rotation in $\mathbb{R}^{n-1}$ we may assume $\nu^A_{x_0} = e_{n-1}$. For any $\varepsilon > 0$, by Lemma 11.6 and the Hölder continuity of A, we can take $\rho_\varepsilon = \rho(x_0, \varepsilon, M)$, possibly smaller than ρ in the previous steps, such that

$$\mathcal{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(e_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon} \subset \mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon},$$

for $\bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(U)$. By Step 4, we also have

$$\bar{x} + \left(\mathcal{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(e_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) \subset \{U(\cdot,0) > 0\},$$

$$\bar{x} - \left(\mathscr{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(e_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) \subset \{U(\cdot,0)=0\}.$$

Now, fixing $\varepsilon = \varepsilon_0$, by the standard arguments, we conclude that there exists a Lipschitz function $g\colon \mathbb{R}^{n-2} \to \mathbb{R}$ with $|\nabla g| \le C_{n,M}/\varepsilon_0$ such that

$$B'_{\rho_{\varepsilon_0}}(x_0) \cap \{U(\cdot,0)=0\} = B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} \le g(x'')\},$$
$$B'_{\rho_{\varepsilon_0}}(x_0) \cap \{U(\cdot,0)>0\} = B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} > g(x'')\}.$$

Step 6. Taking $\varepsilon \to 0$ in Step 5, $\Gamma(U)$ is differentiable at x_0 with normal $\nu^A_{x_0}$. Recentering at any $\bar{x} \in B'_{\rho_{\varepsilon_0}}(x_0) \cap \Gamma(U)$, we see that $\Gamma(U)$ has a normal $\nu^A_{\bar{x}}$ at $\bar{x}$. By noticing that $\bar{x} \mapsto \nu^A_{\bar{x}}$ is $C^{0,\gamma}$, we conclude that the function g in Step 5 is $C^{1,\gamma}$. This completes the proof. ∎

12. Singular points

In this section we study another type of free boundary points for almost minimizers, the so-called singular set $\Sigma(U)$. Because of the machinery developed in the earlier sections, we are able to prove a stratification type result for $\Sigma(U)$ (Theorem 12.8), following a similar approach for the minimizers and almost minimizers with $A = I$.

Definition 12.1 (Singular points). Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. We say that a free boundary point x_0 is *singular* if the coincidence set $\Lambda(U) = \{U(\cdot,0)=0\} \subset B'_1$ has zero H^{n-1}-density at x_0, i.e.,

$$\lim_{r\to 0+} \frac{H^{n-1}\left(\Lambda(U) \cap B'_r(x_0)\right)}{H^{n-1}(B'_r)} = 0.$$

We denote the set of all singular points by $\Sigma(U)$ and call it the *singular set*.

Denote by $\bar{\mathfrak{a}}'_{x_0}$ the $(n-1)\times(n-1)$ submatrix of $\bar{\mathfrak{a}}_{x_0}$ formed by the first $(n-1)$ rows and columns. We then claim that there are constants $C, c > 0$ depending only on n, λ, and Λ such that

$$c \le |\det \bar{\mathfrak{a}}'_{x_0}| \le C. \tag{12.1}$$

Indeed, this follows from the ellipticity of $\mathfrak{a}_{x_0}$ and the invariance of both $\mathbb{R}^{n-1}\times\{0\}$ and $\{0\}\times\mathbb{R}$ under $\bar{\mathfrak{a}}_{x_0}$, since we have

$$|\det \bar{\mathfrak{a}}'_{x_0} (\bar{\mathfrak{a}}_{x_0})_{nn}| = |\det \bar{\mathfrak{a}}_{x_0}| = |\det \mathfrak{a}_{x_0}|$$

and

$$|(\bar{\mathfrak{a}}_{x_0})_{nn}| = |\langle \bar{\mathfrak{a}}_{x_0} e_n, e_n\rangle| = |\bar{\mathfrak{a}}_{x_0} e_n| \in [\lambda^{1/2}, \Lambda^{1/2}].$$

Recall now that for $x_0 \in \Gamma(u)$, $u_{x_0}(y) = U(\bar{\mathfrak{a}}_{x_0} y + x_0)$ and note that $\bar{\mathfrak{a}}'_{x_0} B'_r + x_0 = E'_r(x_0)$. Thus,

$$H^{n-1}(\Lambda(U) \cap E'_r(x_0)) = |\det \bar{\mathfrak{a}}'_{x_0}| H^{n-1}(\Lambda(u^*_{x_0}) \cap B'_r). \tag{12.2}$$

Now, by (12.2) and (12.1), together with $B_{\lambda^{1/2} r}(x_0) \subset E_r(x_0) \subset B_{\Lambda^{1/2} r}(x_0)$, we have

$$\lim_{r\to 0+} \frac{H^{n-1}\left(\Lambda(U) \cap B'_r(x_0)\right)}{H^{n-1}(B'_r)} = 0 \iff \lim_{r\to 0+} \frac{H^{n-1}\left(\Lambda(U) \cap E'_r(x_0)\right)}{H^{n-1}(E'_r(x_0))} = 0$$
$$\iff \lim_{r\to 0+} \frac{H^{n-1}\left(\Lambda(u^*_{x_0}) \cap B'_r\right)}{H^{n-1}(B'_r)} = 0.$$

In terms of Almgren rescalings $(u^*_{x_0})^I_r$, we can rewrite the condition above as

$$\lim_{r\to 0+} H^{n-1}\left(\Lambda((u^*_{x_0})^I_r) \cap B'_1\right) = 0.$$

We then have the following characterization of singular points.

Proposition 12.2 (Characterization of singular points). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0 \in B'_{1/2} \cap \Gamma(U)$ be such that $\kappa(x_0) = \kappa < \kappa_0$. Then the following statements are equivalent.*

(i) $x_0 \in \Sigma(U)$.

(ii) *Any Almgren blowup $(u^*_{x_0})^I_0$ of $u^*_{x_0}$ at 0 is a nonzero polynomial from the class*

$$\mathcal{Q}_\kappa = \{q \mid q \text{ is homogeneous polynomial of degree } \kappa \text{ such that}$$
$$\Delta q = 0,\ q(y', 0) \ge 0,\ q(y', y_n) = q(y', -y_n)\}.$$

(iii) *Any Almgren blowup $(U^*_{x_0})^A_{x_0,0}$ of $U^*_{x_0}$ at x_0 is a nonzero polynomial from the class*

$$\mathcal{Q}^{A,x_0}_\kappa = \{p \mid p \text{ is homogeneous polynomial of degree } \kappa \text{ such that}$$
$$\operatorname{div}(A(x_0)\nabla p) = 0,\ p(x', 0) \ge 0,\ p(x) = p(P_{x_0} x)\}.$$

(iv) $\kappa(x_0) = 2m$ *for some* $m \in \mathbb{N}$.

Proof. This is the analogue of [31, Proposition 10.2] in the case $A \equiv I$.

Clearly, (ii) and (iii) are equivalent. By Proposition 8.1, any Almgren blowup $(u^*_{x_0})^I_0$ of $u^*_{x_0}$ at 0 is a nonzero global solution of the Signorini problem, homogeneous of degree κ. Moreover, $(u^*_{x_0})^I_0$ is a C^1_{loc} limit of Almgren rescalings $(u^*_{x_0})^I_{t_j}$ in $\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}$. Since $u^*_{x_0}$ also satisfies the complementarity condition in Lemma 6.6, the equivalence among (i), (ii) and (iv) follows by repeating the arguments in [31, Proposition 10.2]. ∎

In order to proceed with the blowup analysis at singular points, we need to remove the logarithmic term from the growth estimates in Lemma 9.1. This was achieved in [31, Lemma 10.8] in the case $A \equiv I$ by using a bootstrapping argument [31, Lemmas 10.4–10.6, Corollary 10.7], based on the log-epiperimetric inequality of [11]. All the arguments above work directly for $u^*_{x_0}$ (and then for $U^*_{x_0}$, by deskewing) and we obtain the following optimal growth estimate.

Lemma 12.3 (Optimal growth estimate at singular points). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. If $x_0 \in B'_{1/2} \cap \Gamma(U)$ and $\kappa(x_0) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$, then there are t_0 and C, depending on n, α, M, κ, κ_0, $\|U\|_{W^{1,2}(B_1)}$, such that for $0 < t < t_0$,*

$$\int_{\partial B_t} (u^*_{x_0})^2 \le C t^{n+2\kappa-1}, \qquad \int_{B_t} |\nabla u^*_{x_0}|^2 \le C t^{n+2\kappa-2},$$
$$\int_{\partial E_t(x_0)} (U^*_{x_0})^2 \le C t^{n+2\kappa-1}, \quad \int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 \le C t^{n+2\kappa-2}.$$

With this growth estimate at hand, we now proceed as in the beginning of Section 10 but with $\kappa = 2m < \kappa_0$ in place of $\kappa = 3/2$. Namely, for such κ, let

$$\phi(r) = \phi_\kappa(r) := e^{-(\frac{\kappa b}{\alpha})r^\alpha} r^\kappa, \quad 0 < r < t_0,$$

where $b = \frac{M(n+2\kappa_0)}{\alpha}$ is as in Weiss-type monotonicity formula. Then, define the *κ-almost homogeneous rescalings* of a function v at x_0 by

$$v^\phi_{x_0,r}(x) := \frac{v(rx + x_0)}{\phi(r)}.$$

Again, when $x_0 = 0$, we simply write $v^\phi_{0,r} = v^\phi_r$.

The growth estimates in Lemma 12.3 enable us to consider *κ-homogeneous blowups*

$$(u^*_{x_0})^\phi_{t_j} \to (u^*_{x_0})^\phi_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}),$$
$$(U^*_{x_0})^\phi_{x_0,t_j} \to (U^*_{x_0})^\phi_{x_0,0} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}),$$

for $t = t_j \to 0+$, similar to 3/2-homogeneous blowups in Section 10.

Furthermore, the arguments in [31, Proposition 10.10] also go through for $u^*_{x_0}$ (and then for $U^*_{x_0}$, by deskewing), and we obtain the following rotation estimate for almost homogeneous rescalings.

Proposition 12.4 (Rotation estimate). *For U and x_0 as in Lemma 12.3, there exist $C > 0$ and $t_0 > 0$ such that*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_s| \le C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}},$$
$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,s}| \le C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}},$$

for $0 < s < t < t_0$. In particular, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ are unique.

We next show that the rotation estimate as above holds uniformly for $u^*_{x_0}$ replaced with its Almgren rescalings $(u^*_{x_0})^I_r$, $0 < r < 1$. (Note that the objects $\left[(u^*_{x_0})^I_r\right]^\phi_t$ in the proposition below are κ-almost homogeneous rescalings of Almgren rescalings.)

Proposition 12.5. *For U and x_0 as in Lemma 12.3 and $0 < r < 1$, there are $C > 0$ and $t_0 > 0$, independent of r such that*

$$\int_{\partial B_1} \left| \left[(u^*_{x_0})^I_r\right]^\phi_t - \left[(u^*_{x_0})^I_r\right]^\phi_s \right| \le C \left(\log \frac{1}{t} \right)^{-\frac{1}{n-2}},$$

for $0 < s < t < t_0$. In particular, the κ-homogeneous blowup $\left[(u^*_{x_0})^I_r\right]^\phi_0$ is unique.

Proof. We first observe that since $u^*_{x_0}$ has the almost Signorini property at 0, $(u^*_{x_0})^I_r$ also has the almost Signorini property at 0. This implies that $W_\kappa(\rho, (u^*_{x_0})^I_r, 0)$ and $\widehat{N}_{\kappa_0}(\rho, (u^*_{x_0})^I_r, 0)$ are monotone nondecreasing on ρ. Thus,

$$\begin{aligned} \widehat{N}_{\kappa_0}(0+, (u^*_{x_0})^I_r, 0) &= \lim_{\rho\to 0} \widehat{N}_{\kappa_0}(\rho, (u^*_{x_0})^I_r, 0) = \lim_{\rho\to 0} \widehat{N}_{\kappa_0}(\rho r, u^*_{x_0}, 0) \\ &= \kappa(x_0) = \kappa. \end{aligned}$$

Fix $R > 1$. If t is small, then we can argue as in the proof of Proposition 8.1 to obtain that for any $K \Subset B^\pm_R \cup B'_R$,

$$\left\| \left[(u^*_{x_0})^I_r\right]^\phi_t \right\|_{C^{1,\beta}(K)} \le C(n, \alpha, M, R, K) \left\| \left[(u^*_{x_0})^I_r\right]^\phi_t \right\|_{W^{1,2}(B_R)}.$$

Those are all we need to proceed all the arguments with $(u^*_{x_0})^I_r$ as in [31, Lemmas 10.4–10.6, Corollary 10.7, Lemma 10.8, and Proposition 10.10]. This completes the proof. ∎

Once we have Proposition 12.5, we can argue as in [31, Lemma 10.11] to obtain the nondegeneracy for $u^*_{x_0}$, and also for $U^*_{x_0}$.

Lemma 12.6 (Nondegeneracy at singular points). *Let U and x_0 be as in Lemma 12.3. Then*

$$\liminf_{t\to 0} \int_{\partial B_1} ((u^*_{x_0})^\phi_t)^2 = \liminf_{t\to 0} \frac{1}{t^{n+2\kappa-1}} \int_{\partial B_t} (u^*_{x_0})^2 > 0,$$

$$\liminf_{t\to 0} \int_{\alpha_{x_0}\partial B_1} ((U^*_{x_0})^\phi_{x_0,t})^2 = \liminf_{t\to 0} \frac{1}{t^{n+2\kappa-1}} \int_{\partial E_t(x_0)} (U^*_{x_0})^2 > 0.$$

To state our main result on the singular set, we need to introduce certain subsets of $\Sigma(U)$. For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, let

$$\Sigma_\kappa(U) := \{x_0 \in \Sigma(U) \mid \kappa(x_0) = \kappa\} = \Gamma_\kappa(U).$$

Note that the last equality follows from the implication (iv) ⇒ (i) in Proposition 12.2.

Lemma 12.7. *The set $\Sigma_\kappa(U)$ is of topological type F_σ; i.e., it is a countable union of closed sets.*

Proof. For $j \in \mathbb{N}$, $j \geq 2$, let

$$F_j := \left\{ x_0 \in \Sigma_\kappa(U) \cap \overline{B_{1-1/j}} \mid \frac{1}{j} \leq \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial E_\rho(x_0)} (U_{x_0}^*)^2 \leq j \quad \text{for} \quad 0 < \rho < \frac{1}{2j} \right\}.$$

Note that if $x_j \to x_0$, then by the local uniform continuity of U and A,

$$\int_{\partial E_\rho(x_j)} (U_{x_j}^*)^2 \to \int_{\partial E_\rho(x_0)} (U_{x_0}^*)^2.$$

Using this, together with Lemma 12.3, Lemma 12.6 and Lemma 9.1, we can argue as in [31, Lemma 10.12] to prove that $\Sigma_\kappa(U) = \cup_{j=2}^\infty F_j$ and each F_j is closed. ∎

Next, for $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$ and $x_0 \in \Sigma_\kappa(U)$, we define

$$d_{x_0}^{(\kappa)} := \dim\{\xi \in \mathbb{R}^{n-1} \mid \xi \cdot \nabla_{y'} (u_{x_0}^*)_0^\phi (y', 0) \equiv 0 \text{ on } \mathbb{R}^{n-1}\},$$

which has the meaning of the dimension of $\Sigma_\kappa(u_{x_0}^*)$ at 0, and where $(u_{x_0}^*)_0^\phi$ is the unique κ-homogeneous blowup of $u_{x_0}^*$ at 0. We note here that $d_{x_0}^{(\kappa)}$ can only take the values 0, 1, …, $n-2$. Indeed, otherwise $(u_{x_0}^*)_0^\phi$ would vanish identically on Π and consequently on $\mathbb{R}^n$, since it is a solution of the Signorini problem, even symmetric with respect to Π (see [24]). However, that would contradict the nondegeneracy Lemma 12.6. Then, for $d = 0, 1, \ldots, n-2$, let

$$\Sigma_\kappa^d(U) := \{x_0 \in \Sigma_\kappa(U) \mid d_{x_0}^{(\kappa)} = d\}.$$

Theorem 12.8 (Structure of the singular set). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Then for every $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, and $d = 0, 1, \ldots, n-2$, the set $\Sigma_\kappa^d(U)$ is contained in the union of countably many submanifolds of dimension d and class $C^{1,\log}$.*

Proof. We follow the idea in [31, Theorem 10.13]. For $x_0 \in \Sigma_\kappa(U) \cap B'_{1/2}$, let $q_{x_0} \in \mathcal{Q}_\kappa$ denote the unique κ-homogeneous blowup of $u_{x_0}^*$ at 0. By the optimal growth (Lemma 12.3) and the nondegeneracy (Lemma 12.6), we can write

$$q_{x_0} = \eta_{x_0} q_{x_0}^I, \quad \eta_{x_0} > 0, \quad \|q_{x_0}^I\|_{L^2(\partial B_1)} = 1,$$

where $q_{x_0}^I \in \mathcal{Q}_\kappa$ is the corresponding Almgren blowup. If $x_1, x_2 \in \Sigma_\kappa(U) \cap B'_{1/2}$, for $t > 0$, to be chosen below, we can write

$$\begin{aligned} \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} &\leq \|q_{x_1} - (u_{x_1}^*)_t^\phi\|_{L^1(\partial B_1)} + \|(u_{x_1}^*)_t^\phi - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1)} \\ &\quad + \|q_{x_2} - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1} \\ &\leq C\left(\log \frac{1}{t}\right)^{-\frac{1}{n-2}} \|(u_{x_1}^*)_t^\phi - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1)}, \end{aligned} \tag{12.3}$$

where we have used Proposition 12.4 in the second inequality. Moreover, we have

$$\begin{aligned}
\|(u^*_{x_1})^\phi_t-(u^*_{x_2})^\phi_t\|_{L^1(\partial B_1)} &= \frac{1}{2\phi(t)}\int_{\partial B_1}\big|U(t\bar{\mathfrak{a}}_{x_1}y+x_1)+U(P_{x_1}(t\bar{\mathfrak{a}}_{x_1}y+x_1))\\
&\qquad -U(t\bar{\mathfrak{a}}_{x_2}y+x_2)-U(P_{x_2}(t\bar{\mathfrak{a}}_{x_2}y+x_2))\big|\,dS_y\\
&\le \frac{C}{t^\kappa}\int_{\partial B_1}\Big(\big|U(t\bar{\mathfrak{a}}_{x_1}y+x_1)-U(t\bar{\mathfrak{a}}_{x_2}y+x_2)\big|\\
&\qquad +\big|U(P_{x_1}(t\bar{\mathfrak{a}}_{x_1}y+x_1))-U(P_{x_1}(t\bar{\mathfrak{a}}_{x_2}y+x_2))\big|\\
&\qquad +\big|U(P_{x_1}(t\bar{\mathfrak{a}}_{x_2}y+x_2))-U(P_{x_2}(t\bar{\mathfrak{a}}_{x_2}y+x_2))\big|\Big)\,dS_y\\
&\le \frac{C}{t^\kappa}\|\nabla U\|_{L^\infty(B_1)}\left(|\bar{\mathfrak{a}}_{x_1}-\bar{\mathfrak{a}}_{x_2}|+|x_1-x_2|+|P_{x_1}-P_{x_2}|\right)\\
&\le C\frac{|x_1-x_2|^\alpha}{t^\kappa}=C|x_1-x_2|^{\alpha/2}, \qquad (12.4)
\end{aligned}$$

if we choose $t=|x_1-x_2|^{\frac{\alpha}{2\kappa}}$ and have $|x_1-x_2|<(1/4\Lambda^{-1}\lambda^{1/2})^{\frac{2\kappa}{\alpha}}$. Combining (12.3) and (12.4), we obtain

$$\|q_{x_1}-q_{x_2}\|_{L^1(\partial B_1)}\le C\left(\log\frac{1}{|x_1-x_2|}\right)^{-\frac{1}{n-2}}.$$

After this, we can repeat the argument in the proof of [31, Theorem 10.13] to obtain the estimates that for $x_0\in\Sigma_\kappa(U)\cap B'_{1/2}$, there is $\delta=\delta(x_0)>0$ such that

$$\begin{aligned}
|\eta_{x_1}-\eta_{x_2}|&\le C\left(\log\frac{1}{|x_1-x_2|}\right)^{-\frac{1}{2(n-2)}},\\
\|q^I_{x_1}-q^I_{x_2}\|_{L^\infty(B_1)}&\le C\left(\log\frac{1}{|x_1-x_2|}\right)^{-\frac{1}{2(n-2)}},\quad x_1,x_2\in\Sigma_\kappa(U)\cap B_\delta(x_0).
\end{aligned}$$

Now, we also have the similar result for $U^*_{x_0}$. For $x_0\in\Sigma_\kappa(U)\cap B'_{1/2}$, where $\kappa=2m$, $m\in\mathbb{N}$, let $p_{x_0}\in\mathcal{Q}^{A,x_0}_\kappa$ be the unique κ-homogeneous blowup of $U^*_{x_0}$ at x_0. Then we can write

$$p_{x_0}=\eta^A_{x_0}p^A_{x_0},\quad \eta^A_{x_0}>0,\quad \|p^A_{x_0}\|_{L^2(\partial B_1)}=1,$$

where $p^A_{x_0}\in\mathcal{Q}^{A,x_0}_\kappa$ is the corresponding Almgren blowup of $U^*_{x_0}$. Using that

$$q^I_{x_0}(z)=(\det\mathfrak{a}_{x_0})^{1/2}\,p^A_{x_0}(\mathfrak{a}_{x_0}z),\quad q_{x_0}(z)=p_{x_0}(\mathfrak{a}_{x_0}z),$$

together with the ellipticity and Hölder continuity of $\mathfrak{a}_{x_0}$ and the homogeneity of blowups, we easily conclude that for $x_0\in\Sigma_\kappa(U)\cap B'_{1/2}$, there is $\delta=\delta(x_0)>0$ such that

$$\begin{aligned}
|\eta^A_{x_1}-\eta^A_{x_2}|&\le C\left(\log\frac{1}{|x_1-x_2|}\right)^{-\frac{1}{2(n-2)}},\\
\|p^A_{x_1}-p^A_{x_2}\|_{L^\infty(B_1)}&\le C\left(\log\frac{1}{|x_1-x_2|}\right)^{-\frac{1}{2(n-2)}},\quad x_1,x_2\in\Sigma_\kappa(U)\cap B_\delta(x_0).
\end{aligned}$$

Once we have these estimates, as well as Lemma 12.7, we can apply the Whitney Extension Theorem of Fefferman [22] to complete the proof, similarly to that of [11, Theorem 1.7]. ∎

A. Example of almost minimizers

Example A.1. Let U be a solution of the A-Signorini problem in B_1 with velocity field $b \in L^p(B_1)$, $p > n$:

$$\begin{aligned} -\operatorname{div}(A\nabla U) + \langle b(x), \nabla U\rangle = 0 \quad &\text{in } B_1^{\pm},\\ U \geq 0, \quad \langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle \geq 0, \quad U(\langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle) = 0 \quad &\text{on } B_1',\end{aligned}$$

where $\nu^{\pm} = \mp e_n$ and $\langle A\nabla U, \nu^{\pm}\rangle$ on B_1' are understood as the limits from inside $B_1^{\pm}$. We interpret this in the weak sense that U satisfies the variational inequality

$$\int_{B_1} \langle A\nabla U, \nabla(W-U)\rangle + \langle b, \nabla U\rangle(W-U) \geq 0,$$

for any competitor $W \in \mathfrak{K}_{0,U}(B_1, \Pi)$. Then U is an almost minimizer of the A-Signorini problem in B_1 with thin obstacle $\psi = 0$ on $\Pi = \mathbb{R}^{n-1} \times \{0\}$ and a gauge function $\omega(r) = Cr^{1-n/p}$, $C = C(n, p, \lambda, \Lambda)\|b\|^2_{L^p(B_1)}$.

Proof. For any $E_r(x_0) \Subset B_1$ and $W \in \mathfrak{K}_{0,U}(E_r(x_0), \Pi)$, we extend W as equal to U in $B_1 \setminus E_r(x_0)$ to obtain

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla(W-U)\rangle + \langle b, \nabla U\rangle(W-U) \geq 0. \tag{A.1}$$

Let V be the minimizer of the energy functional

$$\int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle \quad \text{on } \mathfrak{K}_{0,U}(E_r(x_0), \Pi).$$

Then it follows from a standard variation argument that V satisfies the variational inequality

$$\int_{E_r(x_0)} \langle A\nabla V, \nabla(W-V)\rangle \geq 0 \quad \text{for any } W \in \mathfrak{K}_{0,U}(E_r(x_0), \Pi). \tag{A.2}$$

Taking $W = U \pm (U-V)^+$ in (A.1) and $W = V + (U-V)^+$ in (A.2), we obtain

$$\int_{E_r(x_0)} \langle A\nabla(U-V)^+, \nabla(U-V)^+\rangle \leq -\int_{E_r(x_0)} \langle b, \nabla U\rangle(U-V)^+.$$

Similarly, taking $W = U + (V-U)^+$ in (A.1) and $W = V \pm (V-U)^+$ in (A.2), we get

$$\int_{E_r(x_0)} \langle A\nabla(V-U)^+, \nabla(V-U)^+\rangle \leq \int_{E_r(x_0)} \langle b, \nabla U\rangle(V-U)^+.$$

These two inequalities give

$$\int_{E_r(x_0)} \langle A\nabla(U-V), \nabla(U-V)\rangle \le \int_{E_r(x_0)} |b||\nabla U||U-V|.$$

Applying Hölder's inequality,

$$\begin{aligned}\int_{E_r(x_0)} |\nabla(U-V)|^2 &\le \lambda^{-1}\int_{E_r(x_0)} \langle A\nabla(U-V), \nabla(U-V)\rangle \\ &\le \lambda^{-1}\|b\|_{L^p(E_r(x_0))}\|\nabla U\|_{L^2(E_r(x_0))}\|U-V\|_{L^{p^*}(E_r(x_0))},\end{aligned}$$

with $p^* = 2p/(p-2)$. Since $U - V \in W_0^{1,2}(E_r(x_0))$ and $\operatorname{diam}(E_r(x_0)) \le 2\Lambda^{1/2}r$, from the Sobolev's inequality,

$$\|U-V\|_{L^{p^*}(E_r(x_0))} \le C(n,p,\lambda,\Lambda)r^{1-n/p}\|\nabla(U-V)\|_{L^2(E_r(x_0))}.$$

Now we have

$$\int_{E_r(x_0)} |\nabla(U-V)|^2 \le Cr^{2(1-n/p)}\int_{E_r(x_0)} |\nabla U|^2, \tag{A.3}$$

with $C = C(n,p,\lambda,\Lambda)\|b\|^2_{L^p(B_1)}$. Thus,

$$\begin{aligned}\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle - \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle &= \int_{E_r(x_0)} \langle A\nabla(U+V), \nabla(U-V)\rangle \\ &\le C\int_{E_r(x_0)} |\nabla(U+V)||\nabla(U-V)| \\ &\le Cr^\gamma \int_{E_r(x_0)} \left(|\nabla U|^2 + |\nabla V|^2\right) + Cr^{-\gamma}\int_{E_r(x_0)} |\nabla(U-V)|^2 \\ &\le Cr^\gamma \int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle + Cr^\gamma \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle \\ &\quad + Cr^{2(1-n/p)-\gamma}\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle,\end{aligned}$$

where we applied Young's inequality and used (A.3) at the end. We choose $\gamma = 1 - n/p$ to complete the proof. ■

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