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The matching problem between functional shapes via a BV penalty term: A Γ-convergence result

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Abstract. The matching problem often arises in image processing and involves finding a correspondence between similar objects. In particular, variational matching models optimize suitable energies that evaluate the dissimilarity between the current shape and the relative template. A penalty term often appears in the energy to constrain the regularity of the solution. To perform numerical computation, a discrete version of the energy is defined. Then, the question of consistency between the continuous and discrete solutions arises. This paper proves a Γ-convergence result for the discrete energy to the continuous one. In particular, we highlight some geometric properties that must be guaranteed in the discretization process to ensure the convergence of minimizers. We prove the result in the framework introduced in the 2017 paper of Charlier et al., which studies the matching problem between geometric structures carrying on a signal (fshapes). The matching energy is defined for L^2 signals and evaluates the difference between fshapes in terms of the varifold norm. This paper maintains a dual attachment term, but we consider a BV penalty term in place of the original L^2 norm.

1. Introduction

Context and previous work. This paper discusses some theoretical aspects of the matching problem, which has several applications in image processing. The matching problem seeks a bijection between a current surface (or curve) and a target one by minimizing a dissimilarity function called the matching energy. Such an error function is the sum of two terms: the penalty term defining the regularity of the optimal solution and the attachment term estimating the dissimilarity between current and target surfaces.

The solution to the matching problem defines the geometric transformation that links the two surfaces, enabling their comparison or transformation to a standard template. Moreover, this variational approach is implemented in the discrete framework using the stepwise descent algorithm, which provides the evolution related to the mentioned geometric transformation. The image processing community broadly studies the matching problem to define robust methods for shape analysis or image registration [7,11,13,24,25].

We point out, in particular, the increasing use of tools from geometric measure theory (currents, varifolds) for the definition of matching energies [11, 12, 19, 28]. F. Almegren

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has developed the theory of varifolds [4], afterward generalized by W. K. Allard [3], to generalize the notion of a manifold in the framework of measure theory. Varfiold theory considers a shape as a rectifiable measure and enables the definition of attachment terms in the sense of measures. This theoretical framework is suitable for obtaining compactness and existence results, but the computation of weak metrics remains a challenging problem.

Nowadays, new developments in non-invasive acquisition techniques, such as Magnetic Resonance Imaging (MRI) or Optical Coherence Tomography (OCT), provide geometric-functional data for several diseases (e.g., cortical thickness in the study of Alzheimer's disease or thickness of retina layers for the evolution of glaucoma). Then, new methods are needed to deal with data containing geometrical and functional information [20, 21] to describe anatomical variability and produce statistical analysis for related diseases.

In this context, [11] introduces a new framework to study the matching problem of surfaces equipped with a signal (functional shapes, or fshapes). The matching energy proposed in [11] considers the L^2 norm of the signal as a penalty term and a varifold-type attachment term. To overcome the difficulties linked to the discretization of the weak varifold norm, the authors develop the model in the framework of Reproducing Kernel Hilbert Spaces (RKHS), which allows them to compute a varifold norm via the dual representation theorem in Hilbert spaces.

As usual, for minimization problems on surfaces, the question of error estimates for discrete solutions arises. The definition of a suitable discrete framework is central to ensuring that discrete solutions give a good approximation of continuous ones. In particular, the approximation quality should improve when the triangulation approximating the surface is finer (for example, in the context of finite elements, when the diameter of the triangles of the triangulation is sufficiently small).

Unfortunately, this is not true in general, and a very famous example called the Schwartz lantern (see [9, Section 3.9]) shows that a cylinder can be approximated (in the Hausdorff topology) by a sequence of triangulations whose areas diverge. This example shows that the discretization process of a surface can highly affect the quality of the optimal solution. We finally note that, although we focus on the fshapes matching problem, the previous remarks hold for all variational problems on surfaces.

This paper aims to determine the conditions ensuring that the discrete solution represents a consistent (in terms of geometry) and close (in terms of energy) approximation of the continuous optimal solution. In particular, we prove that when the discretization process satisfies some geometrical conditions, the discrete solution is a good approximation of the continuous one. The proof is given in the framework of Γ-convergence theory, proving in particular that the discrete minimizers converge to the continuous one as the diameter of the triangles in the triangulation goes to zero. Error estimates are well established for Euclidean finite elements. However, to our knowledge, this kind of result has not been established for variational problems on surfaces and represents the main contribution of this paper.

Contributions. We call a functional shape (fshape) any couple (X, f) composed of a smooth surface X with boundary and a signal f defined on X.

A functional shape defines a varifold by considering the measure $\mu_{(X,f)} = \mathcal{H}^2 \llcorner X \otimes \delta_{T_X(x)} \otimes \delta_{f(x)}$. We study the matching problem between two fixed surfaces X, Y to optimize the signal f on X with respect to a target signal g defined on Y. The optimal signal is found by minimizing the following matching energy:

$$E(f) = \|f\|_{BV(X)} + \|\mu_{(X,f)} - \mu_{(Y,g)}\|^2,$$

where $\|.\|$ denotes a dual norm, and the minimum is taken on $BV(X)$.

This formulation differs slightly from the original model presented in [11]. First, we consider a BV penalty term instead of the L^2 norm, proving the result for classic non-regular metrics. The regularity of f can strongly influence the behavior of the optimal solution. Standard L^2 signals allow one to work with smooth norms, but accumulation or oscillation phenomena may appear in the optimal solution. For this reason, gradient-dependent norms are increasingly used in image processing to guarantee more realistic optimal configurations. Furthermore, we prove that our results generalize to penalty terms L^2 or H^1.

The other main difference concerns the definition of the dual norm. This work considers a standard dual norm instead of the RKHS-based norm proposed in [11]. This choice provides the result in a more general context and allows it to be read even without expertise in RKHS theory (Remarks 3.5 and 4.3 show how to adapt our results to the RKHS-based norm). Finally, the proof is given in the case of fixed geometry (X is fixed), and the generalization to the optimization problem concerning (X, f) will be addressed in further work.

The first problem we address concerns the definition of admissible triangulation to obtain a geometrically consistent approximation of surfaces. To ensure a complete discretization of the surface in a neighborhood of its boundary, we consider a larger triangulation whose excess part of the projection of X has a small area (Definition 5.3). This allows us to overcome the bijection problems at the boundary of X.

Moreover, as explained above, the proximity in terms of Hausdorff distance does not guarantee a consistent surface approximation. The cited example of the Schwartz lantern shows that when approximating a cylinder with triangulated surfaces, the area of the triangulations can even diverge depending on the geometric properties of the triangles. In particular, [23] shows that convergence of areas can be ensured by a condition on the angle between the normal vectors at corresponding points (via the projection) of the surface and the triangulation. This additional requirement is added to the definition of admissible triangulations to guarantee a convenient approximation of the related surface (see Lemma 5.5 and Hypothesis 2).

Once the set of admissible triangulations is defined, we recall the discrete version of the problem, following [11]. Section 6 describes in detail how to set up the problem in the setting of finite elements via the projection of the triangulations on the surface. Then, the continuous problem can be approximated by a sequence of discrete problems defined on some triangulations whose triangle diameter goes to zero. The triangle diameter is also the parameter indexing the family of discrete problems, which ensures the convergence results for fine triangulations.

The main result proves that discrete solutions approximate continuous minimizers. The result is proven using Γ-convergence theory, which is a notion of convergence of functionals introduced by E. De Giorgi [8, 15], allowing us to justify the transition from discrete to continuous problems. Γ-convergence guarantees, in particular, the convergence of the discrete minimizers to the continuous one as the diameter of the triangles goes to zero (see Theorems 7.3 and 7.4).

We finally point out that, beyond the specifics of the matching problem for functional shapes, the quality problem for discrete approximations concerns many numerical problems defined on surfaces. This work shows how the approximation quality depends on the discretization process used for the numerical approach. Moreover, Γ-convergence is the proper framework to establish the consistency of numerical results. Then, our approach can generalize to other problems with promising theoretical and numerical results.

Structure of the paper. In Section 2, we present the theory of BV functions on manifolds with boundary and adapt the classical results of approximation and compactness. In Section 3, we recall the framework of functional varifolds and the link with [11]. In Section 4, we set up the continuous problem and prove the related existence result. Section 5 is dedicated to the definition of admissible triangulations (Definition 5.3) and the geometric conditions to guarantee the areas convergence (Hypothesis 2 and Proposition 5.6). Section 6 defines discrete operators on triangulations in the framework of finite elements. Finally, in Section 7, we consider the discrete problem and prove the Γ-convergence result (see Theorems 7.3 and 7.4).

2. BV functions on manifolds

In this section, we introduce the central notions and results about BV functions on manifolds. To develop a much larger setting beyond our framework, a similar definition is given in [6].

Although this work concerns approximation results on surfaces, we present here the BV theory on a slightly more general framework, where X is a general compact d-dimensional manifold (that is not supposed to be a finite 2D compact submanifold on $\mathbb{R}^3$).

The definition of BV functions depends on introducing a divergence operator (or, equivalently, a volume form) and a local notion of length. A Riemannian structure provides these two things.

Let X denote an oriented smooth (at least C^1) compact d-dimensional Riemannian manifold, possibly with boundary denoted ∂X, and let vol_X be the associated Riemannian volume form. The boundary ∂X is supposed to be a C^1 compact $(d-1)$-dimensional manifold and we have $\mathrm{vol}_X(\partial X) = 0$. Finally, let us denote

$$X_0 = X \setminus \partial X,$$

which is a C^1 manifold without boundary. When X is without boundary, the previous construction gives $X_0 = X$. We say that $f \in L^1(X)$ is a function of bounded variation on X if

$$|D_X f|(X) = \sup\left\{ \int_X f \,\mathrm{div}_X(u)\mathrm{vol}_X \mid u \in \chi_c^1(X_0),\ \|u\|_\infty \leq 1 \right\} < \infty,$$

where $\chi_c^1(X_0)$ denotes the set of C^1 vector fields $u : X \to TX$ on X compactly supported in X_0 and where div_X is the divergence operator on X defined by

$$\mathrm{div}_X(u) = \sum_{i=1}^{d} g(e_i, du(e_i)),$$

where $(e_1, \cdots, e_d)$ is an orthonormal frame on TX. Here $\|u\|_\infty = \sup_{x\in X} g_x(u(x), u(x))^{\frac{1}{2}}$ where g is the metric tensor associated with the Riemannian structure.

We recall the integration by parts formula

$$\int_X f \,\mathrm{div}_X(u)\mathrm{vol}_X = -\int_X u(f)\mathrm{vol}_X = -\int_X g(\nabla f, u)\mathrm{vol}_X, \tag{2.1}$$

where $h \in C_c^1(X_0)$ and $u(f)$ denotes the derivative of f along the vector fields u that is defined by $[u(f)](x) = d_x f(u(x))$ for any $x \in X_0$. We retrieve the usual submanifold setting when considering the metric induced on the submanifold by the ambient space $\mathbb{R}^3$.

The functional space $BV(X)$ endowed with the norm

$$\|f\|_{BV(X)} = \|f\|_{L^1(X)} + |D_X f|(X)$$

is a Banach space.

Definition 2.1. The space $BV(X)$ can also be equipped with the following notions of convergence, both weaker than the norm convergence:

(1) *Weak-$*$ topology*. Let $\{f_h\}_h \subset BV(X)$ and $f \in BV(X)$. We say that the sequence $\{f_h\}_h$ weakly-$*$ converges in $BV(X)$ to f if

$$f_h \xrightarrow{L^1(X)} f \quad \text{and} \quad D_X f_h \overset{*}{\rightharpoonup} D_X f \quad \text{as} \quad h \to \infty,$$

where $\overset{*}{\rightharpoonup}$ denotes the weak convergence in the space of measures on X.

(2) *Strict topology*. Let $\{f_h\}_h \subset BV(X)$ and $f \in BV(X)$. We say that the sequence $\{f_h\}_h$ strictly converges to f in $BV(X)$ if

$$f_h \xrightarrow{L^1(X)} f \quad \text{and} \quad |D_X f_h|(X) \longrightarrow |D_X f|(X) \quad \text{as} \quad h \to \infty.$$

Remark that

$$d(f,g) = \|f - g\|_{L^1(X)} + ||D_X f|(X) - |D_X g|(X)|$$

is a distance in $BV(X)$ inducing the strict convergence.

We recall the main compactness result for functions of bounded variation extended to the manifold framework.

Theorem 2.2 (Compactness). *Let X be a smooth manifold and $\{f_h\}_h$ be a sequence in $BV(X)$ such that $\|f_h\|_{BV(X)}$ is uniformly bounded. Then, $\{f_h\}_h$ is relatively compact in $BV(X)$ with respect to the weak-$*$ convergence.*

Proof. The proof consists in adapting the one given for the Euclidean case (see [5, Theorem 3.23]) to a manifold with a boundary.

As X is compact, we can consider a finite atlas $\{(U_i, \varphi_i)\}_{i=1,\dots,n}$ where $\{U_i\}_{i=1,\dots,n}$ is a finite open cover of X (note that $\{U_i \cap X_0\}_{i=1,\dots,n}$ is a finite open cover of X_0) and $\varphi_i : U_i \to V_i$ is at least a C^1 diffeomorphism onto an open $V_i \subset \mathbb{R}^k$ (k is the dimension of the manifold). Without loss of generality, we can suppose that each V_i is a Lipschitz domain, and we can also consider a partition of unity $\{\eta_i\}_{i=1,\dots,n}$ on X subordinate to the cover $\{U_i\}_{i=1,\dots,n}$.

Now, by [6, Proposition 2.2], as $\{f_h\eta_i\}_h \subset BV(U_i \cap X_0)$, we have $\{(f_h\eta_i) \circ \varphi_i^{-1}\}_h \subset BV(\varphi_i(U_i \cap X_0))$ and we can apply the classical theorem (see [5, Theorem 3.23]) to $\{(f_h\eta_i) \circ \varphi_i^{-1}\}_h$ for any local chart. The extension argument used in the classical compactness theorem allows us to define a subsequence converging in $L^1(V_i)$.

Then, via at most n extractions of subsequences, we can define a subsequence (not relabeled) such that, for every i, $\{f_h\eta_i\}_h$ weakly*-converges to some $f_i \in BV(U_i)$, so that $\{f_h\}$ weakly*-converges to $f = \sum_{i=1}^n f_i$ in $BV(X)$. ∎

We now establish an approximation result of BV functions by smooth functions. Interestingly, even if there exists several density results using smoothing through the heat kernel semigroup for a geodesically complete Riemannian manifold (see [10, 17, 22]), to the best of our knowledge, an approximation result in the case of a compact manifold with boundary does not seem to be available. We give below such a result.

Theorem 2.3. *Let X be an orientable Riemannian compact manifold with boundary ∂X and let $X_0 = X \setminus \partial X$. For any $f \in BV(X)$, there exists a sequence $\{f_h\}_h \subset C^1(X_0)$ such that*

$$f_h \to f{\restriction_{X_0}} \quad \text{in } L^1(X_0, \mathbb{R}) \qquad \text{and} \qquad |D_X f|(X) = \lim_{n\to\infty} \int_{X_0} |\nabla f_h| \mathrm{vol}_X.$$

Proof. The proof shares similar ideas to the proof of the classical approximation result in the Euclidean case (see [5, Theorem 3.9]). Furthermore, it is based on the parameterization of the function by the local charts. When considering the problem in local charts, introducing a non-constant volume term introduces several new elements. The proof is detailed in the appendix. ∎

In the following, we return to the case of a submanifold of $\mathbb{R}^3$ and show how the previous result can be improved. To achieve this goal, we need to introduce an extension operator to be able to expand the manifold and its signal in a consistent way.

Let X denote a C^p ($p \geq 2$) compact oriented 2-dimensional submanifold of $\mathbb{R}^3$ with non-empty boundary denoted by ∂X. We denote by $\boldsymbol{n}_{X_0}$ the C^{p-1} vector field of the positively oriented normal along $X_0 = X \setminus \partial X$. Note that $\boldsymbol{n}_{X_0}$ can be continuously extended on X and denoted in that case by $\boldsymbol{n}_X$. For any $x \in \partial X$, we can define the unit vector $\nu(x)$ pointing outward and orthogonal to $n_X(x)$ and ∂X. For sufficiently small $r > 0$, we consider the open subset

$$\mathcal{N}_r(X_0) = \left\{ x + t\boldsymbol{n}_{X_0}(x) \mid x \in X_0, |t| < r \right\}$$

and the C^{p-1} diffeomorphism

$$\psi_{X_0} : X_0 \times]-r, r[\to \mathcal{N}_r(X_0), \quad (x, t) \mapsto x + t\boldsymbol{n}_{X_0}(x)$$

so that, considering its inverse $\psi_{X_0}^{-1} = (\pi_{X_0}, t_{X_0})$, we have

$$\forall z \in \mathcal{N}_r(X_0), \quad z = \pi_{X_0}(z) + t_{X_0}(z)\boldsymbol{n}_{X_0}(\pi_{X_0}(z)),$$

where $\pi_{X_0}(z)$ is the orthonormal projection of z on X_0 and $|t_{X_0}(z)|$ is the distance of z to X_0.

Now, there exists $\eta_0 > 0$ small enough such that the mapping

$$\mathrm{Ext} : \partial X \times]-\eta_0, \eta_0[\times]-r, r[\to \mathbb{R}^3,$$

$$\mathrm{Ext}(x, s, t) = \begin{cases} x_0 + t\boldsymbol{n}_{X_0}(x_0) \text{ with } x_0 \doteq \pi_{X_0}(x + s\nu(x)) & \text{if } s < 0, \\ x + s\nu(x) + t\boldsymbol{n}_X(x) & \text{otherwise} \end{cases} \tag{2.2}$$

is well defined and is a C^{p-1} diffeomorphism on an open neighborhood of ∂X in $\mathbb{R}^3$ (see Figure 1).

Moreover, Ext maps $\partial X \times \{0\} \times \{0\}$ to the boundary ∂X of X, and for $0 < \eta < \eta_0$,

$$X^{-\eta} = X \setminus \mathrm{Ext}(\partial X \times]-\eta, 0] \times \{0\})$$

is a compact C^{p-1} submanifold of X, whereas

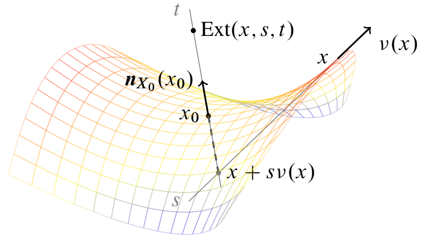
$$X^{+\eta} = X \cup \mathrm{Ext}(\partial X \times]0, +\eta] \times \{0\}) \tag{2.3}$$

is a C^{p-1} compact 2-dimensional submanifold of $\mathbb{R}^3$ extending X along its boundary.

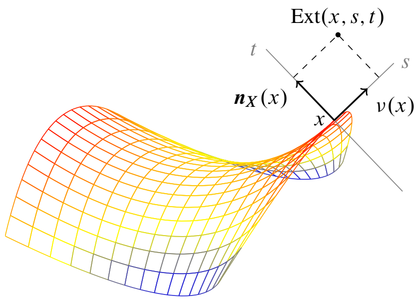
We can now prove the following approximation result:

Theorem 2.4. *Let X be a C^p ($p \geq 2$) compact oriented 2D submanifold of $\mathbb{R}^3$ with non-empty boundary denoted by ∂X. Let $f \in BV(X)$ and let $\varepsilon > 0$. Then, there exists $\eta > 0$ and $\widetilde{f} \in C^{p-1}(X^{+\eta})$ such that*

$$\int_X |f - \widetilde{f}| \, d\mathcal{H}^2 + ||D_X f|(X) - |D_X \widetilde{f}|(X)| \leq \varepsilon$$



(a) $s < 0$.



(b) $s \geq 0$.

Figure 1. The map Ext.

and

$$\int_{X^{+\eta}\setminus X^{-\eta}} |\tilde{f}| + |\nabla_{X^{+\eta}} \tilde{f}| \, d\mathcal{H}^2 \leq \varepsilon.$$

Proof. By Theorem 2.3, there exists $f' \in C^{p-1}(X_0)$ such that

$$\int_X |f - f'| \, d\mathcal{H}^2 + ||D_X f|(X) - |D_X f'|(X)| \leq \varepsilon.$$

Moreover, there exists $0 < \eta' < \eta_0$ such that

$$\int_{X\setminus X^{-\eta'}} |f'| + |\nabla_X f'| \, d\mathcal{H}^2 \leq \varepsilon.$$

Now, consider for $\eta = \eta'/3$ the function $f'' : X^{+\eta} \to \mathbb{R}$ defined as

$$f''(z) = \begin{cases} f'(z) & \text{if } z \in X^{-\eta}, \\ f'(\mathrm{Ext}(x, -2\eta - s, 0)) & \text{if } z \in X^{+\eta} \setminus X^{-\eta} \text{ and where } z = \mathrm{Ext}(x, s, 0). \end{cases}$$

Letting $\delta > 0$, we can easily check that for η' small enough,

$$\int_{X^{+\eta}\setminus X^{-\eta}} |f''| + |\nabla_X f''| \, d\mathcal{H}^2 \leq (1+\delta) \times 2 \int_{X\setminus X^{-\eta'}} |f'| + |\nabla_X f'| \, d\mathcal{H}^2,$$

since by considering the change of variable induced by Ext, we notice that $(x, s) \to \pi_X(x + s\nu(x))$ has a determinant converging to 1 when (x, s) converges to a point on $\partial X \times \{0\}$ with $s < 0$.

Moreover, we verify that

$$f''\restriction_{X^{-\eta}} \in C^{p-1}(X^{-\eta}) \quad \text{and} \quad f''\restriction_{\overline{X^{+\eta}\setminus X^{-\eta}}} \in C^{p-1}(\overline{X^{+\eta} \setminus X^{-\eta}}),$$

where the intersection $X^{-\eta} \cap \overline{X^{+\eta} \setminus X^{-\eta}} = \partial X^{-\eta}$ is a C^{p-1} 1-dimensional submanifold. Applying a smoothing in the vicinity of $\partial X^{-\eta}$, we can obtain $\tilde{f} \in C^1(X^{+\eta}, \mathbb{R})$ such that $\tilde{f} = f''$ on $X^{-\eta}$ and

$$\int_{X^{+\eta}\setminus X^{-\eta}} |\tilde{f}| + |\nabla_X \tilde{f}| \, d\mathcal{H}^2 \leq \int_{X^{+\eta}\setminus X^{-\eta}} |f''| + |\nabla_X f''| \, d\mathcal{H}^2 + \varepsilon$$

so that, choosing $\delta = 1$ and η' small enough, we have

$$\int_{X^{+\eta}\setminus X^{-\eta}} |\tilde{f}| + |\nabla_X \tilde{f}| \, d\mathcal{H}^2 \leq 5\varepsilon,$$

and we immediately get

$$\begin{aligned} &\int_X |f - \tilde{f}| \, d\mathcal{H}^2 + ||D_X f|(X) - |D_X \tilde{f}|(X)| \\ &\quad \leq \int_X |f - f'| \, d\mathcal{H}^2 + ||D_X f|(X) - |D_X f'|(X)| + 5\varepsilon \leq 6\varepsilon, \end{aligned}$$

thus proving the result. ■

Remark 2.5 (L^2 and H^1 norms). Theorems 2.3 and 2.4 can be established, by similar arguments, for the L^2 and H^1 norms. We refer to [18] for an introduction to Sobolev spaces on manifolds.

3. Functional varifolds

This section presents the main definitions and results for functional varifolds as introduced in [11]. We recall that this framework is a generalization of the varifold theory in [29] to spaces with a scalar component; this allows the representation of geometric shapes equipped with a signal in the framework of measure theory. We refer to [11,29] for a more detailed presentation of the respective theories.

The functional approach considers mathematical objects, called fshapes, containing geometrical and functional components. As detailed below, the role of fshapes in the functional varifolds framework is the same as that of manifolds in the varifold theory; they allow one to define measures supported on surfaces and are the bridge between geometry and measure theory.

Now, we work with smooth surfaces and denote by X a generic 2-manifold satisfying the following properties:

Hypothesis 1. X denotes a 2-submanifold of $\mathbb{R}^3$ (surface) with boundary ∂X and which is smooth (C^2, at least), oriented, connected, and compact. The smoothness assumption implies that its interior, denoted by X_0, is a 2-dimensional smooth manifold, and ∂X is a 1-dimensional manifold of the same regularity.

Definition 3.1 (fshape). We define a functional shape (fshape) as a couple (X, f) where X is a surface satisfying Hypothesis 1 and $f \in BV(X)$ is a signal defined on X.

Of course, the penalty term in the matching problem defines the regularity of the signal. In the following, we consider fshapes endowed with an L^2 or H^1 signal when studying the problem with L^2 or H^1 penalty terms, respectively.

Similarly to the classical theory of varifolds, we can define a functional varfiold via the dual of $C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$, which denotes the closure, with respect to the C^1 norm, of the set of C^1 functions with compact support on $\mathbb{R}^3 \times G(3,2) \times \mathbb{R}$. We denote by $G(3,2)$ the Grassmannian of the non-oriented 2-dimensional linear subspaces of $\mathbb{R}^3$.

Definition 3.2 (fvarifolds). A 2-dimensional functional varifold (fvarifold) is any operator μ belonging to $(C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R}))'$, the dual space of $C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$, containing all bounded linear real-valued maps on $C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$. In what follows, we consider the following norm for fvarifolds:

$$\|\mu\| = \sup\big\{\mu(\varphi) \mid \|\varphi\|_{C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})} \leq 1\big\}. \tag{3.1}$$

We can associate an fvarifold to a given fshape (X, f) by considering the measure

$$\mu_{(X,f)} = \mathcal{H}^2 \llcorner X \otimes \delta_{T_X(x)} \otimes \delta_{f(x)},$$

that acts as a linear functional in the following way:

$$\mu_{(X,f)}(\varphi) = \int_X \varphi(x, T_x X, f(x)) d\mathcal{H}^2(x), \quad \forall \varphi \in C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R}),$$

where $T_X(x)$ denotes the tangent space to X at x and $\mathcal{H}^2$ is the 2-dimensional Hausdorff (or volume) measure.

Finally, we prove some lemmas that will be useful in what follows concerning some convergence properties for fvarifolds supported on fshapes. In particular, the convergence to the null fvarifold depends on the geometric component.

Lemma 3.3. *Let $\{X_h\}$ be a sequence of surfaces such that $\mathcal{H}^2(X_h) \to 0$. Then, $\mu_{(X_h,f_h)}$ converges to the null fvarifold for every sequence $\{f_h\}$ of signals.*

Proof. For every h and for every $C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$ with $\|\varphi\|_{C_0(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})} \leq 1$, we obtain that $|\mu_{(X_h,f_h)}(\varphi)| \leq \mathcal{H}^2(X_h)$, which implies that $\|\mu_{(X_h,f_h)}\|$ converges to zero. ■

The following lemma links the convergence of signals and the weak-$*$ convergence of fvarifolds:

Lemma 3.4. *If $f_h \to f$ in $L^1(X)$, then $\mu_{(X,f_h)} \overset{*}{\rightharpoonup} \mu_{(X,f)}$ and $\|\mu_{(X,f_h)} - \mu_{(X,f)}\| \to 0$.*

Proof. For every $\varphi \in C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$ with $\|\varphi\|_{C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})} \leq 1$, we get

$$\begin{aligned}|\mu_{(X,f_h)}(\varphi) - \mu_{(X,f)}(\varphi)| &\leq \int_X |\varphi(x, T_x X, f_h(x)) - \varphi(x, T_x X, f(x))| d\mathcal{H}^2(x) \\ &\leq \|f_h - f\|_{L^1(X)},\end{aligned}$$

which proves the result by taking the supremum and using the L^1 convergence. ∎

Remark 3.5 (The choice of the fvarifold norm). This work uses the dual norm as a distance between fvarifolds. However, in [11], another metric for fvarifolds is defined by using the framework of Reproducing Kernel Hilbert Spaces (RKHS) [31]. We refer to [11] for a general definition, and we recall that their framework allows them to define the following dual product for fvarifolds supported on fshapes:

$$\begin{aligned}&\langle \mu_{(X,f)}, \mu_{(Y,g)} \rangle_{W'} \\ &\quad = \int_X \int_Y k_e(x,y) k_t(T_x X, T_y Y) k_f(f(x), g(y))\, d\mathcal{H}^2(x) d\mathcal{H}^2(y),\end{aligned} \tag{3.2}$$

with

$$k_e(x,y) = e^{-\frac{\|x_1 - x_2\|^2}{\sigma_e^2}}, \quad k_t(T_1, T_2) = e^{-\frac{2(1 - \langle \boldsymbol{n}_{T_1}, \boldsymbol{n}_{T_2} \rangle^2)}{\sigma_t^2}}, \quad k_f(a,b) = e^{-\frac{|f_1 - f_2|^2}{\sigma_f^2}},$$

where $\sigma_e, \sigma_t, \sigma_f$ are three positive constants and $\boldsymbol{n}_T$ represents the unit normal vector to T. In particular, W' denotes the dual space of W, the RKHS associated with the kernel $k_e \otimes k_t \otimes k_f$ (see [11, Propositions 2 and 4]). As W is continuously embedded into $C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$, its dual norm defines a metric for fvarifolds.

The choice of this metric has many advantages. It allows them to prove an existence result for the matching problem with L^2 signals (see Section 4). Moreover, in numerical applications, using the Gaussian kernel allows the localization of the matching at a given scale.

In this work, we decided to consider the matching problem defined for BV or H^1 signals and use the standard dual norm for fvarifolds. This framework is strong enough to prove the existence of optimal solutions and a Γ-convergence result. Throughout the paper, we will demonstrate how to adapt the results to the original framework introduced in [11].

4. Existence of optimal solutions for the matching problem

In this section, we define the matching energy between two fshapes and prove an existence result for the optimal solution.

Let be X a surface and (Y, g) a target fshape. We consider the following energy defined for a generic fshape (X, f):

$$E(f) = \|f\|_{BV(X)} + \frac{1}{2}\|\mu_{(X,f)} - \mu_{(Y,g)}\|^2,$$

and we aim to solve the minimization problem

$$\inf_{f \in BV(X)} E(f).$$

We recall that we optimize only with respect to the signal, which implies that the initial and optimal configurations have the same geometric support. However, the geometry is taken into account in the attachment term.

Theorem 4.1. *Let (Y, g) a given fshape and X a 2-manifold satisfying Hypothesis 1. Then, there exists at least one solution to the problem*

$$\inf_{f \in BV(X)} E(f).$$

Proof. Let $\{f_h\}_h$ be a minimizing sequence belonging to $BV(X)$. We can suppose that $\|f_h\|_{BV(X)}$ is uniformly bounded and, by Theorem 2.2, $\{f_h\}_h$ converges (up to a subsequence) to some $f \in BV(X)$ with respect to the weak-$*$ topology. The result follows by remarking that the fvarifold norm is continuous with respect to the L^1 topology (Lemma 3.4) and that the BV norm is lower semicontinuous with respect to the L^1 topology. ∎

Remark 4.2 (The H^1 model). We get the same result if we consider signals belonging to $H^1(X)$ instead of $BV(X)$. It follows from the fact that the unity ball of $H^1(X)$ is compact with respect to the weak topology. Moreover, $H^1(X)$ is compactly embedded in $L^2(X)$, which implies (up to a subsequence) the L^1 convergence of the minimizing sequence.

Remark 4.3 (L^2 model in [11]). The L^2 model is defined (see [11, part I]) via the following matching energy:

$$E(f) = \frac{\gamma_f}{2}\|f\|^2_{L^2(X,\mathbb{R})} + \frac{\gamma_W}{2}\|\mu_{(X,f)} - \mu_{(Y,g)}\|^2_{W'},$$

where γ_f, γ_W are two positive constants and the fvarivold norm is induced by (3.2). We point out that, for every minimizing sequence $\{f_h\}_h$, a bound on $E(f_h)$ guarantees only the L^2 weak compactness for the signals, which is not enough to get the semicontinuity of the fvarifold term (a result similar to Lemma 3.4 holds for the W' norm with respect to the almost everywhere convergence of the signals). This justifies the choice of the RKHS-based metric described in Remark 3.5, whose properties play a central role in the existence result given below.

Theorem 4.4 ([11, Proposition 7]). *Let X, Y be two finite volume bounded 2-rectifiable subsets of $\mathbb{R}^3$. Let us assume that W is continuously embedded in $C_0^2(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$ and $g \in L^2(Y)$.*

If the ratio γ_f/γ_W is large enough, then there exists at least one solution to the minimization problem

$$\inf_{f \in L^2(X)} E(f)$$

and every minimizer belongs to $L^\infty(X)$. Moreover, if X is a C^p surface and $W \hookrightarrow C_0^m(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$ with $m \geq \max\{p, 2\}$, then every minimizer belongs to $C^{p-1}(X)$.

Finally, there exists a constant $C > 0$ (independent of X and Y) such that every minimizer satisfies

$$\|f\|_{L^\infty(X)} \leq C \frac{\gamma_W}{\gamma_f} (\mathcal{H}^2(X) + \mathcal{H}^2(Y)). \tag{4.1}$$

The proof is based on the relaxation of the energy to the class $\mathcal{M}^X$, defined as follows:

Definition 4.5. $\mathcal{M}^X$ is the class of the Borel finite measures ν on $\mathbb{R}^3 \times G(3,2) \times \mathbb{R}$ such that

$$\int \varphi(x, V) d\nu(x, V, f) = \int_X \varphi(x, T_x X) d\mathcal{H}^2(x), \quad \forall \varphi \in C_c(\mathbb{R}^3 \times G(3,2)).$$

Note that $\mu_{(X,f)} \in \mathcal{M}^X$ for every fshape (X, f).

Then, the energy E can be relaxed to the functional

$$\widetilde{E} : \mathcal{M}^X \to \mathbb{R}, \quad \widetilde{E}(\nu) = \frac{\gamma_f}{2} \int |f|^2 d\nu + \frac{\gamma_W}{2} \|\nu - \mu_{(Y,g)}\|^2,$$

and it holds that

$$E(f) = \widetilde{E}(\mu_{(X,f)}). \tag{4.2}$$

The relaxed energy provides compactness for minimizing sequences in the space of measures. It can be shown that the minimizing measure ν^* of $\widetilde{E}$ is associated with an fshape, so that $\nu^* = \mu_{(X,f^*)}$ for some $f^* \in L^2(X)$. The proof relies on the Implicit Function Theorem, which needs the hypothesis on γ_f/γ_W. We refer to [11, Proposition 7 and Lemma 2] for more details.

However, the L^2 model has two main issues. Firstly, the existence result depends on the weights used to define the energy. Moreover, the L^2 penalty does not prevent some oscillating configurations for the optimal signal. For this reason, we introduced a penalty on the signal derivative that justifies the BV and H^1 models studied in this work.

5. Surfaces and triangulations

This section is dedicated to defining suitable triangulations of surfaces to ensure a good approximation of continuous functions by their discretization.

As explained in the introduction, the quality of approximation and the error estimates highly depend on the geometric properties of triangulations. The famous example of the Schwartz lantern (see [9, Section 3.9]) exhibits a sequence of polyhedral surfaces converging to the cylinder in the Hausdorff metric and whose areas diverge. This example points out that triangulations must verify some specific hypothesis to represent a geometrically consistent surface approximation.

Section 5.1 recalls some general facts about triangulations and defines the set of admissible triangulations in Definition 5.3. In Section 5.2, we give a sufficient condition (Hypothesis 2), ensuring the convergence of areas for admissible triangulations. Finally, in Section 5.3, we define the sampling method for signals and the error estimate for Sobolev norms.

5.1. Triangulations of a surface

As X satisfies Hypothesis 1, following [1, 26, 27], we give the following definition of triangulation:

Definition 5.1 (Triangulations). A triangulation $\mathcal{T}$ is a 2-dimensional manifold (with boundary) consisting of a finite set $\Delta_{\mathcal{T}}$ of affine triangles such that:

(1) any point $p \in \mathcal{T}$ lies in at least one triangle $T \in \Delta_{\mathcal{T}}$;

(2) each point $p \in \mathcal{T}$ has a neighborhood that intersects only finitely many triangles of $\Delta_{\mathcal{T}}$;

(3) the intersection of any two non-identical triangles $T, T' \in \Delta_{\mathcal{T}}$ is either empty or consists of a common vertex or edge.

In what follows, we denote by T a generic triangle of $\Delta_{\mathcal{T}}$ and by $\partial\mathcal{T}$ the boundary of the manifold $\mathcal{T}$. The vertices of $\mathcal{T}$ are the vertices of the triangles belonging to $\Delta_{\mathcal{T}}$. The diameter of $\mathcal{T}$ is defined as follows:

$$\mathrm{diam}_{\mathcal{T}} = \max_{T \in \Delta_{\mathcal{T}}} \left\{\mathrm{diam}(T)\right\}. \tag{5.1}$$

We also assume that every triangulation is regular, which means that

$$\frac{h_T}{\rho_T} \leq C, \quad \forall\, T \in \Delta_{\mathcal{T}} \tag{5.2}$$

for some $C > 0$, where h_T is the diameter of T and ρ_T is the diameter of the sphere inscribed in T.

We define the distance function to X as follows:

$$\forall x \in \mathbb{R}^3, \quad d_X(x) = d(x, X) = \inf_{y \in X} |x - y|.$$

For every $x \in \mathbb{R}^3$, we call (if it exists) the projection of x on X every point $\pi_X(x) \in X$ such that $d_X(x) = |x - \pi_X(x)|$.

We also recall that the Hausdorff distance between two surfaces $X, Y \subset \mathbb{R}^3$ is defined as

$$d_{\mathcal{H}}(X,Y) = \max\Big\{\sup_{x\in X} d_Y(x), \sup_{y\in Y} d_X(y)\Big\}.$$

Definition 5.2 (Tubular neighborhoods). Let X be a surface satisfying Hypothesis 1. We denote by $U_r(X)$ the subset of $\mathbb{R}^3$ of the form

$$U_r = \big\{x \in \mathbb{R}^3 \mid d_X(x) < r\big\},$$

such that every point $x \in U_r$ admits a unique projection $\pi_X(x) \in X$. We refer to [16] for the proof of the existence of such a tubular neighborhood U_r for some $r > 0$.

To guarantee the injectivity of the projection, we introduce the following tubular neighborhood:

$$\mathcal{N}_r(X) = \big\{x + t\boldsymbol{n}_X(x) \mid t \in]-r, r[,\ x \in X\big\} \subset U_r,$$

where we denote by $\boldsymbol{n}_X(x)$ the unit normal vector to X at x. Then, we have $x \in \mathcal{N}_r(X)$ if and only if $d_X(x) < r$ and

$$x = \pi_X(x) + d_X(x)\boldsymbol{n}_X(\pi_X(x)).$$

Then, we can split every triangulation into the following two parts to isolate the set $\mathcal{T}^{\text{in}}$ of points that can be bijectively projected onto the surface:

$$\mathcal{T}^{\text{in}} = \mathcal{T} \cap \mathcal{N}_r(X) \quad \text{and} \quad \mathcal{T}^{\text{out}} = \mathcal{T} \cap \mathcal{N}_r(X)^c.$$

Generally, a triangulation is not in bijection with a surface with a smooth boundary through the normal projection. The bijectivity can fail close to ∂X because of the curvature of ∂X, as depicted in Figure 2a. Locally, the normal projection of ∂X on a hyperplane must not be a line, and it is impossible to project ∂X on the edge of a triangle.

So, we introduce a new class of admissible triangulations to ensure the projection bijectivity between triangulation and surface except on a small part close to the boundary:

Definition 5.3 (h-admissible triangulations for a surface). Let $h > 0$. We say that a triangulation $\mathcal{T}$ is h-admissible for the surface X if the following properties hold:

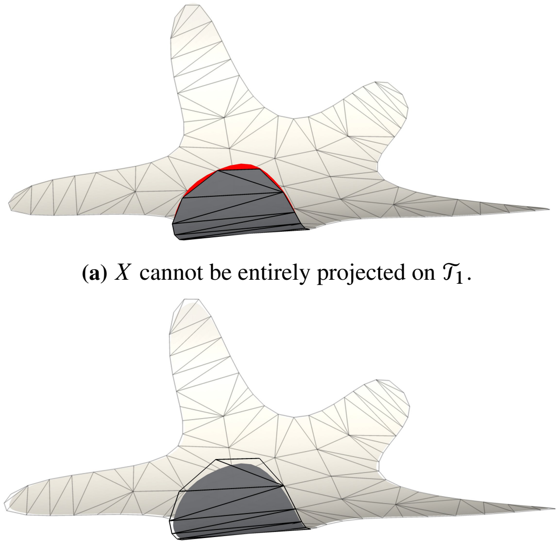
(i) $\mathcal{T}$ lies in $\mathcal{N}_h(X^{+\eta})$ for some $\eta > 0$, where $X^{+\eta}$ is an extension of X defined in (2.3);

(ii) $\mathcal{T}^{\text{in}} = \mathcal{T} \cap \mathcal{N}_h(X)$ and X are in one-to-one correspondence through π_X;

(iii) $\mathcal{H}^2(\mathcal{T}^{\text{out}}) = O(h)$;

(iv) $\text{diam}_{\mathcal{T}} = O(h)$, where $\text{diam}_{\mathcal{T}}$ is defined in (5.1).

This definition introduces a discretization framework based on triangulations larger than the corresponding surface but converging to it with respect to the Hausdorff distance (as $h \to 0$). Condition (i) means that $\mathcal{T}$ is adapted to discretize an extended surface



(a) X cannot be entirely projected on $\mathcal{T}_1$.

(b) An h-admissible triangulation $\mathcal{T}_2$ for X.

Figure 2. The surface X is a bent smooth star (solid grey), and two triangulations (black lines) are illustrated. Figure 2a: the triangulation $\mathcal{T}_1$ is not in one-to-one correspondence with X through the projection map (for instance, the part of the smooth star in red exceeds the triangulation). Figure 2b: the subset $\mathcal{T}_2^{\text{in}}$ of $\mathcal{T}_2$ is in one-to-one correspondence with X.

and, because of (ii), X can be completely projected onto the triangulation. However, the part $\mathcal{T}^{\text{out}}$, corresponding to the triangles along the boundary, is assumed to be small via condition (iii). Finally, condition (iv) allows us to identify the family parameter with the triangulation diameter in order to describe finer triangulations as $h \to 0$. Figure 2 shows an example of admissible and inadmissible triangulations.

We point out that the vertices of triangulations are not, as usual, a set of sampled points on the surface. This approach corresponds better to the data acquisition routines used in image processing. For instance, biomedical images (OCT, functional MRI) are often modified (segmentation, deblurring, denoising) to improve their quality for numerical experiments. Then, the data do not correspond to a sampled version of the imaged objects, but instead, they represent an approximation, often noisy, of the natural structures.

5.2. Convergence of areas

As pointed out in the previous section, if $\mathcal{T}$ is an h-admissible triangulation of X for some $h > 0$, then $d_{\mathcal{H}}(X, \mathcal{T}) = O(h)$. However, this does not guarantee a similar error estimate for the respective areas.

In this section, we introduce some geometric conditions to ensure geometrically consistent triangulations and avoid pathological cases, such as the already cited Schwartz lantern (see [9, Section 3.9]).

Definition 5.4 (Angle between normals). Let $h > 0$. Assume that $\mathcal{T}$ is an h-admissible triangulation for X. For every $x \in \mathcal{T}^{\text{in}}$, we define the angle α_x as follows:

- if x belongs to the interior of some triangle, then α_x is the angle belonging to $[0, \pi/2]$ between the two normals $\boldsymbol{n}_X(\pi_X(x))$ and $\boldsymbol{n}_{\mathcal{T}}(x)$;
- if x belongs to an edge of a triangle, then α_x is the biggest angle belonging to $[0, \pi/2]$ between $\boldsymbol{n}_X(\pi_X(x))$ and the normals of the triangles containing x.

In what follows, we set

$$\alpha_{\max} = \sup_{x \in \mathcal{T}^{\text{in}}} \alpha_x.$$

Lemma 5.5. *Let $h > 0$ and $\mathcal{T}$ be an h-admissible triangulation for X. We have*

$$|\mathcal{H}^2(X) - \mathcal{H}^2(\mathcal{T}^{\text{in}})| = O(\alpha_{\max}^2 + d_{\mathcal{H}}(X, \mathcal{T}))$$

as $\alpha_{\max}^2, d_{\mathcal{H}}(X, \mathcal{T}) \to 0$, where $\alpha_{\max}$ is introduced in Definition 5.4.

Proof. For every $x \in X$, we consider on the tangent space $T_x X$ the basis $\mathcal{B}(x) = \{e^1(x), e^2(x)\}$ given by the two principal directions. We denote by $\kappa_1(x)$ and $\kappa_2(x)$ the principal curvatures of X at x. Similarly, for every $x \in X$, we can consider the basis $\widetilde{\mathcal{B}}(x) = \{e^1(x), e^2(x), \boldsymbol{n}_X(x)\}$ for $\mathbb{R}^3$.

Consider the differential $D\pi_X : \mathbb{R}^3 \to T_{\pi_X(x)}X$ of π_X. Note that, for every $x \in \mathcal{N}_h(X)$, we have $D\pi_X(x)(v) = 0$ for every variation v in the direction $\boldsymbol{n}_X(\pi_X(x))$ orthogonal to $T_{\pi_X(x)}X$. So, we should consider the tangential variations to calculate the projection's Jacobian.

In [30] it is proved that for any $x \in U_h(X)$ and v parallel to $T_{\pi_X(x)}X$, we have

$$D\pi_X(x)(v) = \left(I_{\restriction T_{\pi_X(x)}X} - \varepsilon_x d_X(x) D\boldsymbol{n}_X(\pi_X(x))\right)^{-1} v,$$

where the matrix of $D\pi_X(x)$ written with respect to the basis $\widetilde{\mathcal{B}}(\pi_X(x))$ and $\mathcal{B}(\pi_X(x))$ is

$$D\pi_X(x) = \begin{pmatrix} \frac{1}{1+d_X(x)\varepsilon_x\kappa_1(\pi_X(x))} & 0 & 0 \\ 0 & \frac{1}{1+d_X(x)\varepsilon_x\kappa_2(\pi_X(x))} & 0 \end{pmatrix}$$

and $\varepsilon_x = \langle \frac{\pi_X(x)-x}{\|\pi_X(x)-x\|}, \boldsymbol{n}_X(\pi_X(x)) \rangle \in \{-1, +1\}$.

Let now $A \subset \mathcal{T}^{\text{in}}$ be a subset of a triangle of $\mathcal{T}$. This implies that the Jacobian of the projection on X restricted to A is given by

$$\det(D_{\restriction A}\pi_X)(x) = \frac{\cos\alpha_x}{(1 + d_X(x)\varepsilon_x\kappa_1(\pi_X(x)))(1 + d_X(x)\varepsilon_x\kappa_2(\pi_X(x)))}, \quad \forall\, x \in A.$$

We have $\cos\alpha_x = 1 + O(\alpha_{\max}^2)$. Moreover, as the principal curvatures are uniformly bounded and $d_X(x) = O(d_{\mathcal{H}}(X, \mathcal{T}))$, we get

$$\begin{aligned}\det(D_{\restriction_A}\pi_X)(x) &= (1 + O(\alpha_{\max}^2))(1 + O(d_{\mathcal{H}}(X, \mathcal{T}))) \\ &= 1 + O(\alpha_{\max}^2 + d_{\mathcal{H}}(X, \mathcal{T}))\end{aligned} \tag{5.3}$$

as $\alpha_{\max}^2, d_{\mathcal{H}}(X, \mathcal{T}) \to 0$. We can conclude by performing the usual change of variables in the area formula. ∎

Because of Lemma 5.5, we introduce the following set of assumptions:

Hypothesis 2. Let $\{\mathcal{T}_h\}_h$ be a sequence of triangulations indexed by a parameter $h \to 0$ such that

(i) for any $h > 0$, the triangulation $\mathcal{T}_h$ is h-admissible for X;

(ii) the sequence $\alpha_{\max}^h = O(h)$ as $h \to 0$, where $\alpha_{\max}^h$ is the angle defined in Definition 5.4 for $\mathcal{T}_h$.

We point out that, because of Definition 5.3, we get $d_{\mathcal{H}}(X, \mathcal{T}_h) = O(h)$, which implies that, under the assumption of Hypothesis 2, the area error computed in Lemma 5.5 is also $O(h)$. Then, Hypothesis 2 implies the geometric consistency of the approximation and helps avoid pathological cases such as the Schwartz lantern.

The following result, generalizing [23, Corollary 5] to surfaces with smooth boundaries, holds:

Proposition 5.6 (Convergence of the area). *Let X be a surface satisfying Hypothesis 1 and $\{\mathcal{T}_h\}_h$ a sequence of triangulation satisfying Hypothesis 2. Then, we have* $\lim_{h\to 0} \mathcal{H}^2(\mathcal{T}_h) = \mathcal{H}^2(X)$.

Proof. The proof follows from Lemma 5.5 and Definition 5.3(iii). ∎

5.3. From the triangulation to the surface

This section defines how to carry a signal from the triangulation to the surface.

Definition 5.7 (Projection). For every function f defined on an h-admissible triangulation $\mathcal{T}$ for X ($h > 0$), we define the projection of f onto X by

$$f^\ell : X \to \mathbb{R}, \quad f^\ell(x) = f(\pi_X^{-1}(x)), \quad x \in X.$$

We point out that the function f^ℓ carries on X the signal defined on $\mathcal{T}^{\text{in}}$.

Proposition 5.8. *Let $h > 0$ and $\mathcal{T}$ be an admissible h-triangulation for X. Then, for every $f \in W^{1,\infty}(\mathcal{T}, \mathbb{R})$, we have*

$$\begin{aligned}\|f^\ell\|_{L^p(X)} &= \|f\|_{L^p(\mathcal{T}^{\text{in}})} + O(\alpha_{\max}^2 + d_{\mathcal{H}}(X, \mathcal{T})), \\ \|\nabla_X f^\ell\|_{L^p(X)} &= \|\nabla_{\mathcal{T}^{\text{in}}} f\|_{L^p(\mathcal{T}^{\text{in}})} + O(\alpha_{\max}^2 + d_{\mathcal{H}}(X, \mathcal{T}))\end{aligned}$$

for every $p \in [1, \infty]$, as $\alpha_{\max}^2, d_{\mathcal{H}}(X, \mathcal{T}) \to 0$.

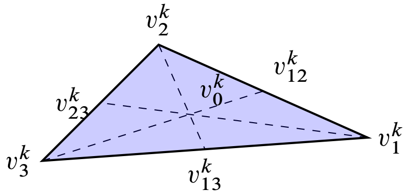


Figure 3. Labels of various points in the triangle T_k.

Proof. The first equality is proved by performing the change of variables $y = \pi_X(x)$ and using Lemma 5.5. The second relationship is proved by the same arguments and applying the chain rule. ∎

We point out that, for triangulations satisfying Hypothesis 2, the previous proposition generalizes to surfaces with boundary the Sobolev error estimates for Euclidean finite elements.

6. Discretization

This section aims to write down the discretized matching energy to compare the discrete problem defined on triangulations to the continuous one defined on the original surfaces.

For clarity, we detail the definition of all discrete norms and functionals in the framework of finite elements. Although it may seem redundant to a reader accustomed to this kind of formalization, we prefer to define the several operators to prevent misinterpretation of notations.

6.1. Notations

Let $\mathcal{T}$ be a triangulation in the sense of Definition 5.1. We denote by N_v, N_e, and N_t the number of vertices, edges, and triangles of $\mathcal{T}$, respectively. The family $\{v_i\}_{i=1,\ldots,N_v}$ denotes the vertices of the triangulation.

For every $k = 1, \ldots, N_t$, we denote by $\{v_i^k\}_{i=1,2,3}$ the vertices of the triangle $T_k \subset \mathcal{T}$, $\{v_{ij}^k\}_{1 \le i < j \le 3}$ the center of the edge linking v_i^k to v_j^k, and $v_0^k = \frac{1}{3} \sum_{i=1}^{3} v_i^k$ the center of mass of the triangle (see Figure 3). Analogously, we denote by $\{f_i^k\}_{0 \le i \le 3}$ the values of the function f at location $\{v_i^k\}_{0 \le i \le 3}$ of T_k.

6.2. P_0 and P_1 triangular finite elements

Let us start with the following definition:

Definition 6.1. For a given triangulation $\mathcal{T}$, we denote by $\mathbb{P}_0(\mathcal{T})$ (resp. $\mathbb{P}_1(\mathcal{T})$) the set of functions that are constant on the interior of each triangle and null on their edges (resp. the set of the continuous functions that are affine on each triangle).

The elements of $\mathbb{P}_0(\mathcal{T})$ (resp. $\mathbb{P}_1(\mathcal{T})$) are completely described by their values $\{f_0^k\}_k$ at the center of mass $\{v_0^k\}_k$ (resp. their values $\{\{f_i^k\}_{1\le i\le 3}\}_k$ at vertices $\{\{v_i^k\}_{1\le i\le 3}\}_k$) of the triangulation. Note that for each triangle $T_k \in \mathcal{T}$,

$$\forall f \in \mathbb{P}_1(\mathcal{T}), \quad f_0^k = f(v_0^k) = f\Big(\sum_{i=1}^{3} v_i^k/3\Big) = \sum_{i=1}^{3} f(v_i^k)/3 = \sum_{i=1}^{3} f_i^k/3.$$

On the other hand, if $f \in \mathbb{P}_0(\mathcal{T})$, as f is null on the edges of each triangle, we cannot compute f_0^k by the values at the vertices, and we can only set

$$\forall f \in \mathbb{P}_0(\mathcal{T}), \quad f_0^k = f(v_0^k).$$

We denote by p_0 the L^2 projection of $\mathbb{P}_1(\mathcal{T})$ on $\mathbb{P}_0(\mathcal{T})$ where, for $f \in \mathbb{P}_1(\mathcal{T})$, the function $p_0(f)$ is the unique element of $\mathbb{P}_0(\mathcal{T})$ such that

$$p_0(f)(v_0^k) = f(v_0^k). \tag{6.1}$$

In other words, the operator p_0 replaces the affine approximation of a signal on each triangle with a constant approximation using the value at the center of mass.

A basis for $\mathbb{P}_1(\mathcal{T})$ (barycentric basis) is given by the family $\{\varphi\}_{i=1,\cdots,N_v}$ with $\varphi_i \in \mathbb{P}_1(\mathcal{T})$ and $\varphi_i(v_j) = \delta_{ij}$ (Kronecker's delta), for every $i, j = 1, \cdots, N_v$. Then, every $f \in \mathbb{P}_1(\mathcal{T})$ can be written as

$$\forall x \in \mathcal{T}, \quad f(x) = \sum_{j=1}^{N_v} f_j \varphi_j(x), \quad f_j = f(v_j).$$

Remark that there exists a bijection between $\mathbb{P}_1(\mathcal{T})$ and $\mathbb{R}^{N_v}$, defined by the following operator:

$$P_1 : (f_1, \dots, f_{N_v}) \in \mathbb{R}^{N_v} \mapsto f = \sum_{j=1}^{N_v} f_j\, \varphi_j \in \mathbb{P}_1(\mathcal{T}). \tag{6.2}$$

6.3. Discrete operators for finite elements

For every $k = 1, \dots, N_t$, the area of the triangle T_k is denoted by $|T_k|$ and is equal to $\frac{1}{2}\|\boldsymbol{n}_{T_k}\|$, where $\boldsymbol{n}_{T_k} = (v_2^k - v_1^k) \wedge (v_3^k - v_1^k)$.

6.3.1. Discrete functional norms. Depending on how the continuous signal is discretized, various methods may be used to compute the norm of the discrete signal.

L^p norm of P_0 finite elements. Let $p \ge 1$ and $f : \mathcal{T} \to \mathbb{R}$ be a function in $L^p(\mathcal{T}, \mathbb{R})$. The pth power of the discrete L^p norm of f is simply defined as

$$L_0^p[f, \mathcal{T}] = \sum_{k=1}^{N_t} |T_k||f_0^k|^p. \tag{6.3}$$

Formula (6.3) is exact for signals that are (almost everywhere) constant on each triangle, so that we have $L_0^p[f, \mathcal{T}] = \|f\|_{L^p(\mathcal{T})}^p$ for any $f \in \mathbb{P}_0(\mathcal{T})$.

L^p norm of P_1 finite elements. Since in this work we are concerned with L^p norms with $p = 1, 2$, we need to define the discrete operators via an exact formula on piecewise quadratic polynomials. We use the following approximation by the evaluation at the midpoints (see [2, p. 178–179]):

$$L_1^p[f, \mathcal{T}] = \frac{1}{3} \sum_{k=1}^{N_t} |T_k| (\|f_{12}^k\|^p + \|f_{13}^k\|^p + \|f_{23}^k\|^p), \tag{6.4}$$

where for a piecewise linear signal $f \in \mathbb{P}_1(\mathcal{T})$, we have $f_{ij}^k = \frac{1}{2}(f_i^k + f_j^k)$ for any $k = 1, \ldots, N_t$ and $1 \leq i < j \leq 3$.

Formula (6.4) is exact if $p = 2$ (i.e., $L_1^2[f, \mathcal{T}] = \|f\|_{L^2(\mathcal{T})}^2$), since the function $f^2\restriction_{T_k}$ is a polynomial of degree 2 for any $k = 1, \ldots, N_t$. Formula (6.4) is also exact if $p = 1$ and f has constant sign on each triangle (i.e., we have $L_1^1[f, \mathcal{T}] = \|f\|_{L^1(\mathcal{T})}$ if $f \geq 0$ or $f \leq 0$). Suppose the signal $f \in \mathbb{P}_1(\mathcal{T})$ has a changing sign on a given triangle T_k. In that case, the function to integrate is not a polynomial on the entire triangle, and the formula is not exact anymore. Then, the triangle is decomposed into several sub-triangles with constant sign signal and the operator in (6.4) is computed as the sum of the respective operators computed (exactly) on each sub-triangle. This is equivalent to performing a local triangulation refinement to ensure the signal has a constant sign on each triangle.

We suppose now that $f \in \mathbb{P}_1(\mathcal{T})$ and we define the discrete operators corresponding to the H^1 and BV norms. For every $T_k \in \mathcal{T}$ and for every $f \in \mathbb{P}_1(\mathcal{T})$, the gradient of f on T_k can be computed by

$$[\nabla_{\mathcal{T}} f]_{T_k} = \frac{e_2^k \wedge e_3^k}{\|e_2^k \wedge e_3^k\|^2} \wedge (f_1^k e_1^k + f_2^k e_2^k + f_3^k e_3^k), \tag{6.5}$$

where

$$e_1^k = v_3^k - v_2^k, \quad e_2^k = v_1^k - v_3^k, \quad e_3^k = v_2^k - v_1^k.$$

The previous relationship can be stated by writing f in the barycentric coordinates system and recalling that the gradients of the basis elements are perpendicular to the triangle edges.

In this framework, the gradient $\nabla_{\mathcal{T}} f$ is constant on each triangle and, by convention, is null on the edges. We can now define the discrete operators of the gradient norms that are exact for every $f \in \mathbb{P}_1$.

Total variation. The total variation of $f \in \mathbb{P}_1$ on $\mathcal{T}$ is given by

$$V[f, \mathcal{T}] = \sum_{k=1}^{N_t} |T_k| \|[\nabla_{\mathcal{T}} f]_{T_k}\|. \tag{6.6}$$

H^1 norm. For $f \in \mathbb{P}_1(\mathcal{T})$, the square of the L^2 norm of the gradient is given by

$$H[f, \mathcal{T}] = \sum_{k=1}^{N_t} |T_k| \|[\nabla_{\mathcal{T}} f]_{T_k}\|^2.$$

Remark that $H[f, \mathcal{T}] = L_0^2[\|\nabla_{\mathcal{T}} f\|, \mathcal{T}]$ as $f \in \mathbb{P}_1(\mathcal{T})$, and by formula (6.5), we have $\|\nabla_{\mathcal{T}} f\| \in \mathbb{P}_0(\mathcal{T})$.

6.3.2. Discrete fvarifold norm. The definition of discrete fvarifolds can be posed by considering an fvarifold associated to $(\mathcal{T}, f)$. We point out that $(\mathcal{T}, f)$ does not verify the conditions (smoothness) defining fshapes; however, the fshape definition given in Section 3 remains consistent with the properties of $\mathcal{T}$.

Then, with an abuse of language, we call $(\mathcal{T}, f)$ a discrete fshape, and the related fvarifold supported on it is defined as follows:

$$\mu_{(\mathcal{T},f)} = \mathcal{H}^2 \llcorner \mathcal{T} \otimes \delta_{T_{\mathcal{T}}(x)} \otimes \delta_{f(x)}.$$

Moreover, we approximate such an fvarifold by a discrete operator $\boldsymbol{\mu}_{(\mathcal{T},f)}$ to simplify the computation. Such an approximation is set in the same way in both the P_0 and P_1 frameworks:

$$\boldsymbol{\mu}_{(\mathcal{T},f)} = \sum_{k=1}^{N_t} |T_k| \delta_{(v_0^k, V_k, f_0^k)},$$

where

$$v_0^k = \frac{1}{3}(v_1^k + v_2^k + v_3^k), \quad f_0^k = f(v_0^k), \quad V_k = \mathrm{Span}\{v_2^k - v_1^k, v_3^k - v_1^k\}.$$

The discrete fvarifold norm is defined by

$$\mathrm{Var}[\mu_{(\mathcal{T},f)}] = \|\boldsymbol{\mu}_{(\mathcal{T},f)}\|. \tag{6.7}$$

We point out that by formula (6.1), we have

$$\boldsymbol{\mu}_{(\mathcal{T},f)} = \boldsymbol{\mu}_{(\mathcal{T},p_0(f))}, \quad \forall f \in \mathbb{P}_1(\mathcal{T}).$$

We end this section with a technical lemma that will be useful in the next section. It proves an error estimate, with respect to the fvarifold norm, between a discrete signal f (belonging to P_0 or P_1) and its projection f^l on the surface (see Definition 5.7).

Lemma 6.2. *Let X be a surface satisfying Hypothesis 1 and $\mathcal{T}$ an h-admissible triangulation for X satisfying Hypothesis 2. Then, we have*

$$\sup_{f \in \mathbb{P}_0(\mathcal{T})} \|\boldsymbol{\mu}_{(\mathcal{T},f)} - \mu_{(X,f^\ell)}\| = O(h), \tag{6.8}$$

and

$$\forall f \in \mathbb{P}_1(\mathcal{T}), \quad \|\boldsymbol{\mu}_{(\mathcal{T},f)} - \mu_{(X,f^\ell)}\| = O(h)(1 + \|\nabla f\|_{L^1(\mathcal{T})}). \tag{6.9}$$

Proof. Let $f \in \mathbb{P}_0(\mathcal{T})$. By the change of variables $x = \pi_X(y)$, because of Hypothesis 2 and formula (5.3), for every $f \in \mathbb{P}_0(\mathcal{T})$ and $\varphi \in C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$, we get

$$\left| \int_X \varphi(x, T_x X, f^\ell(x)) d\mathcal{H}^2(x) - \int_{\mathcal{T}^{\mathrm{in}}} \varphi(y, T_y \mathcal{T}^{\mathrm{in}}, f(y)) d\mathcal{H}^2(y) \right|$$

$$\leq \|\varphi\|_{L^\infty(\mathbb{R}^3\times G(3,2)\times\mathbb{R})}\, O(h).$$

Moreover, because of Definition 5.3(iii) and Lemma 3.3, $\mu_{(\mathcal{T}^{\text{out}},f)}$ converges towards the null fvarifold as $h \to 0$. Then,

$$\left|\int \varphi d(\mu_{(\mathcal{T},f)} - \mu_{(X,f^\ell)})\right| \leq \|\varphi\|_{L^\infty(\mathbb{R}^3\times G(3,2)\times\mathbb{R})} O(h).$$

Now, if $f \in \mathbb{P}_0(\mathcal{T})$, then $f = f_0^k$ on the interior of every triangle T_k. Then, for every function $\varphi \in C_0^1(\mathbb{R}^3 \times G(3,2) \times \mathbb{R})$, we get

$$\begin{aligned}
&|(\mu_{(\mathcal{T},f)} - \boldsymbol{\mu}_{(\mathcal{T},f)})(\varphi)| \\
&\quad \leq \sum_{k=1}^{N_t} \int_{T_k} |\varphi(x, T_x\mathcal{T}, f(x)) - \varphi(v_0^k, V_k, f_0^k)|\, d\mathcal{H}^2(x) \\
&\quad \leq \sum_{k=1}^{N_t} \|\varphi\|_{C^1(\mathbb{R}^3\times G(3,2)\times\mathbb{R})} \int_{T_k} \|(x, V_k, f(x)) - (v_0^k, V_k, f_0^k)\|_{\mathbb{R}^3\times G(3,2)\times\mathbb{R}}\, d\mathcal{H}^2(x) \\
&\quad \leq \sum_{k=1}^{N_t} |T_k| \|\varphi\|_{C^1(\mathbb{R}^3\times G(3,2)\times\mathbb{R})} \mathrm{diam}(T_k) \leq \mathcal{H}^2(\mathcal{T}) \|\varphi\|_{C^1(\mathbb{R}^3\times G(3,2)\times\mathbb{R})} O(h),
\end{aligned}$$

where N_t denotes the number of triangles contained in $\mathcal{T}$ and V_k the tangent space to T_k. So, by (3.1), we obtain

$$\|\mu_{(\mathcal{T},f)} - \boldsymbol{\mu}_{(\mathcal{T},f)}\| \leq O(h)\mathcal{H}^2(\mathcal{T}),$$

and, by the triangle inequality, we get $\|\mu_{(X,f^\ell)} - \boldsymbol{\mu}_{(\mathcal{T},f)}\| \leq O(h)$, which proves (6.8).

If $f \in \mathbb{P}_1(\mathcal{T})$, the proof is similar. The bound for $|(\mu_{(\mathcal{T},f)} - \boldsymbol{\mu}_{(\mathcal{T},f)})(\varphi)|$ depends on f because

$$\forall\, x \in T_k, \quad |f(x) - f_0^k| \leq \|\nabla f\|_{L^1(T_k)} \mathrm{diam}(T_k)$$

and, by the same arguments, we get

$$\|\boldsymbol{\mu}_{(\mathcal{T},f)} - \mu_{(X,f^\ell)}\| \leq O(h)(1 + \|\nabla f\|_{L^1(\mathcal{T})}),$$

which proves (6.9). ■

6.4. Discretization of continuous signals

Finally, we present the discretization process (on admissible triangulations) of a signal defined on a surface. Let X be a surface satisfying Hypothesis 1 and $\mathcal{T}$ be an h-admissible triangulation for X (for some $h > 0$), and $f \in BV(X)$ (resp. $H^1(X)$, $L^2(X)$).

According to Definition 5.3(i), by using the map Ext defined in (2.2), we can extend the manifold X to a larger suitable manifold $X^{+\eta}$ such that $\mathcal{T} \subset \mathcal{N}_h(X^{+\eta})$. Moreover, because of Theorem 2.4, the signal f can be extended to a signal $\widetilde{f}$ defined on $X^{+\eta}$ which is $W^{1,\infty}(X^{+\eta})$ with a small norm on $X^{+\eta} \setminus X^{-\eta}$.

Let T be a triangle contained in $\mathcal{T}$. We set the following discretization process:

- P_0 elements: as every P_0 element is null on each edge of T, we define the piecewise (almost everywhere) constant f_h corresponding to f on T as

$$f_h = \begin{cases} 0 & \text{on the edges of } T, \\ \frac{1}{3}\big(\tilde{f}(\pi_{X+\eta}(v_1)) + \tilde{f}(\pi_{X+\eta}(v_2)) + \tilde{f}(\pi_{X+\eta}(v_3))\big) & \text{otherwise,} \end{cases}$$

  where v_1, v_2, v_3 denote the three vertices of T.

- P_1 elements: the piecewise linear f_h corresponding to f on T is defined as

$$f_h = P_1\big(\tilde{f}(\pi_{X+\eta}(v_1)), \tilde{f}(\pi_{X+\eta}(v_2)), \tilde{f}(\pi_{X+\eta}(v_3))\big),$$

  where v_1, v_2, v_3 denote the three vertices of T and P_1 is defined by (6.2).

This process defines a discrete signal f_h on the triangulation whose different norms can be computed by the operators introduced above. We note that these operators represent an approximation of their continuous counterparts, which motivates the study of the relationship between the discrete and continuous minimizers discussed in the next section.

7. Discrete problem and Γ-convergence results

This section aims to show the Γ-convergence of the discretized problems to the continuous one when the triangulation is fine enough. The main consequence is that the discrete optimal solution computed on triangulations represents a good approximation of the optimum computed on the surface (Theorem 7.4). The proof is provided under the geometric assumptions established in Hypothesis 2. This implies, in particular, that the area error (see Lemma 5.5) is $O(h)$, ensuring the geometric consistency of the approximation process.

Let X, Y be two surfaces satisfying Hypothesis 1 and $g \in BV(Y)$. We denote by $\{\mathcal{T}_h\}_h$ and $\{\mathcal{Y}_h\}_h$ two sequences of h-admissible triangulations of X and Y, respectively, satisfying Hypothesis 2. We recall that the control on the angle between continuous and discrete corresponding normal vectors prevents pathological cases like the Schwartz lantern. We finally denote by $\{g_h\}_h$ the discretization of g on the sequence of triangulations $\{\mathcal{Y}_h\}_h$ obtained by the discretization process described in Section 6.4.

We point out that h denotes the family parameter and the triangulation diameter (the maximum diameter of the triangles contained in $\mathcal{T}_h$). Then, the Γ-convergence results hold for the typical case of increasingly fine triangulations. Finally, we note that regularity condition (5.2) is fundamental in our proof to ensure a global error estimate in the finite elements framework.

Then, for every h, the discrete energy is defined by

$$E_h : \mathbb{P}_1(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\},$$
$$E_h(f_h) = \big(L_1^1[f_h, \mathcal{T}_h] + V[f_h, \mathcal{T}_h]\big) + \mathrm{Var}[\mu_{(\mathcal{T}_h, f_h)} - \mu_{(\mathcal{Y}_h, g_h)}]^2, \tag{7.1}$$

where the operators L_1^1, V, Var are defined in (6.4), (6.6), and (6.7), respectively. For every $h > 0$, the optimal signal on $\mathcal{T}_h$ is defined by the following problem:

$$\inf_{f_h \in \mathbb{P}_1(\mathcal{T}_h)} E_h(f_h). \tag{7.2}$$

The main goal of this section is to prove that the minimizers of E_h converge to the minimizers of the continuous problem

$$\inf_{f \in BV(X)} E(f), \quad E(f) = \|f\|_{BV(X)} + \|\mu_{(X,f)} - \mu_{(Y,g)}\|^2. \tag{7.3}$$

First of all, we prove the existence result for the discrete problem.

Proposition 7.1. *For every $h > 0$, there exists at least one solution to problem* (7.2).

Proof. For every $f \in \mathbb{P}_1(\mathcal{T}_h)$, $\|\nabla_{\mathcal{T}_h} f\|_{L^1(\mathcal{T}_h)} = V[f, \mathcal{T}_h]$ and $L_1^1[f, \mathcal{T}_h] = \|f\|_{L^1(\mathcal{T}_h)}$. Then, every minimizing sequence is bounded in BV so that it weakly-$*$ converges in BV (up to a subsequence) to some function belonging to $\mathbb{P}_1(\mathcal{T}_h)$. The result follows from Lemma 3.4 and the lower semicontinuity of the total variation with respect to the weak-$*$ convergence. ∎

As the discrete and continuous energies are not defined in the same space, we introduce a suitable topology to compare the two spaces and generalize the classical definition of Γ-convergence (see [8]).

Definition 7.2 (S-topology and Γ-convergence). Let X be a surface satisfying Hypothesis 1, $f \in BV(X)$, and $\{\mathcal{T}_h\}_h$ be a sequence of admissible triangulations for X satisfying Hypothesis 2. In this definition, $\{f_h\}_h$ denotes a sequence of functions such that $f_h \in \mathbb{P}_1(\mathcal{T}_h)$ for every $h > 0$.

- We say that $\{f_h\}_h$ converges to f with respect to the S-topology ($f_h \overset{S}{\rightharpoonup} f$) if and only if
$$\lim_{h \to 0} \|f_h^\ell - f\|_{L^1(X)},$$
where for any $h > 0$, f_h^ℓ is the projection of f_h onto X (see Definition 5.7).
- We say that $(E_h)_h$ Γ-converges to E if the following conditions hold:
  (i) *Lower bound*: for every $f \in BV(X)$ and for every sequence $\{f_h\}_h$ such that $f_h \overset{S}{\rightharpoonup} f$, we have
$$E(f) \le \liminf_{h \to 0} E_h(f_h);$$

(ii) *Upper bound*: for every $f \in BV(X)$, there exists a sequence $\{f_h\}_h$ such that $f_h \overset{S}{\rightharpoonup} f$ and

$$E(f) \geq \limsup_{h \to 0} E_h(f_h).$$

We can now prove the main result of this work.

Theorem 7.3. *The sequence $\{E_h\}_h$ described in (7.1) Γ-converges to E (see (7.3)) with respect to the S-topology of Definition 7.2.*

Proof. Lower bound. Let $\{f_h\}_h$ be a sequence of functions such that $f_h \in \mathbb{P}_1(\mathcal{T}_h)$ for every $h > 0$ and $f_h \overset{S}{\rightharpoonup} f \in BV(X)$. Thus, we have

$$f_h^\ell \overset{L^1(X)}{\longrightarrow} f,$$

where f_h^ℓ is the projection of f_h onto X (see Definition 5.7). Without loss of generality, we can suppose that

$$\sup_{h>0} E_h(f_h) < \infty.$$

By Proposition 5.8 and Hypothesis 2, we get

$$\|f_h^\ell\|_{BV(X)} \leq \|f_h\|_{BV(\mathcal{T}_h)} + O(h) = L_1^1[f_h, \mathcal{T}_h] + V[f_h, \mathcal{T}_h] + O(h). \tag{7.4}$$

Moreover, by Lemma 6.2, we have

$$\|\boldsymbol{\mu}_{(\mathcal{T}_h, f_h)} - \mu_{(X, f_h^\ell)}\|^2 \overset{h \to 0}{\to} 0,$$

so that, as $f_h^\ell \overset{L^1(X)}{\to} f$, Lemma 3.4 implies

$$\|\mu_{(X, f_h^l)} - \mu_{(X,f)}\|^2 \overset{h \to 0}{\to} 0.$$

Then,

$$\mathrm{Var}[\mu_{(\mathcal{T}_h, f_h)} - \mu_{(\mathcal{Y}_h, g_h)}]^2 = \|\boldsymbol{\mu}_{(\mathcal{T}_h, f_h)} - \boldsymbol{\mu}_{(\mathcal{Y}_h, g_h)}\|^2 \overset{h \to 0}{\to} \|\mu_{(X,f)} - \mu_{(Y,g)}\|^2.$$

Now, as the BV norm is lower semicontinuous with respect to the L^1 convergence, we get

$$E(f) \leq \liminf_{h \to 0} \big(\|f_h^\ell\|_{BV(X)} + \mathrm{Var}[\mu_{(\mathcal{T}_h, f_h)} - \mu_{(\mathcal{Y}_h, g_h)}]^2\big) \leq \liminf_{h \to 0} E_h(f_h).$$

Upper bound. Because of Theorem 2.4, we can assume that $f \in C^1(X)$ and get the general result by a diagonal argument.

The discretization process defines, for every $h > 0$, an extension $X^{+\eta}$ of X to extend f to $\widetilde{f} \in W^{1,\infty}(X^{+\eta})$ (Theorem 2.4) and define the discrete function f_h (see Section 6.4). Of course, the extension $X^{+\eta}$ can depend on h, but in what follows, we prove some estimates for a fixed $h > 0$ and write $X^{+\eta}$. In particular, as $\widetilde{f}$ is defined via a reflection symmetry with respect to the boundary of X (see Theorem 2.4), we have

$$\|\widetilde{f}\|_{W^{1,\infty}(X^{+\eta})} \le \|f\|_{W^{1,\infty}(X)}, \tag{7.5}$$

which implies that $f_h \in W^{1,\infty}(\mathcal{T}_h)$ for every h, and $\sup_h \|f_h\|_{W^{1,\infty}(\mathcal{T}_h)} \le \|f\|_{W^{1,\infty}(X)}$.

The usual estimates for interpolation error, as $h \to 0$, give

$$\|\widetilde{f} \circ \pi_{X^{+\eta}} - f_h\|_{W^{1,1}(\mathcal{T}_h)} \le O(h)\|\widetilde{f} \circ \pi_{X^{+\eta}}\|_{W^{1,\infty}(\mathcal{T}_h)}. \tag{7.6}$$

This can be deduced from [14, Theorem 3.1.6] (applied with $k = s = 0$, $q = m = 1$, $p = \infty$) by considering the sum on all the triangles and by using the fact that the area of every triangle is bounded by $\pi(h/2)^2$. We note, in particular, that these estimates hold for regular triangulations satisfying (5.2).

Now, according to Theorem 2.4, $\widetilde{f}$ coincides with f on $X^{-\eta}$ and its BV norm can be arbitrarily small on $X \setminus X^{-\eta}$. Then, we get

$$\|f - f_h^\ell\|_{W^{1,1}(X)} = \|\widetilde{f} - f_h^\ell\|_{W^{1,1}(X^{-\eta})} + O(h).$$

Moreover, as $\mathcal{T}_h$ satisfies Definition 5.3(ii) , $X^{-\eta}$ can be projected on a subset of $\mathcal{T}_h$, and by using (7.5) and (7.6), we get

$$\|f - f_h^\ell\|_{W^{1,1}(X)} = O(h).$$

This proves, in particular, that $f_h^\ell \to f$ strongly in $L^1(X)$, which means that $f_h \xrightarrow{S} f$. Now, by Proposition 5.8 and Hypothesis 2, we have

$$\|f_h^\ell\|_{L^1(X)} = \|f_h\|_{L^1(\mathcal{T}_h^{\text{in}})} + O(h) = L_1^1[f_h, \mathcal{T}_h] + O(h),$$

so that

$$L_1^1[f_h, \mathcal{T}_h] \to \|f\|_{L^1(X)}^1 \quad \text{as} \quad h \to 0.$$

Similarly, the convergence of the total variation term follows from

$$\|\nabla_X f_h^\ell\|_{L^1(X)} = \|\nabla_{\mathcal{T}_h} f_h\|_{L^1(\mathcal{T}_h^{\text{in}})} + O(h) = V[f_h, \mathcal{T}_h] + O(h).$$

As in the case of the lower bound, the convergence of the fvarifold term follows by Lemmas 6.2 and 3.4, which finally proves

$$E(f) = \lim_{h\to 0} E_h(f_h). \qquad \blacksquare$$

The Γ-convergence result implies, in particular, the convergence of the minimizers.

Theorem 7.4 (Convergence of minimizers). *Let $\{\mathcal{T}_h\}_h$ be a sequence of admissible triangulations for X satisfying Hypothesis 2. Let $\{f_h\}_h$ be a sequence of minimizers of E_h (i.e., $E_h(f_h) = \min_{f\in\mathbb{P}_1(\mathcal{T}_h)} E_h(f)$). Then, $\{f_h^\ell\}_h$ weakly-$*$ converges in $BV(X)$ (up to a subsequence) to a minimizer of E and*

$$\lim_{h\to 0} \min_{f\in\mathbb{P}_1(\mathcal{T}_h)} E_h(f) = \min_{f\in BV(X)} E(f).$$

Proof. We consider the sequence $\{f_h \in \mathbb{P}_1(\mathcal{T}_h)\}_h$ of minimizers of E_h and, without loss of generality, we can also suppose that

$$\sup_{h>0} E_h(f_h) < \infty.$$

Similarly to (7.4), we have

$$\|f_h^\ell\|_{BV(X)} \le L_1^1[f_h, \mathcal{T}_h] + V[f_h, \mathcal{T}_h] + O(h),$$

so that $\{f_h^\ell\}_h$ is uniformly bounded in $BV(X)$. Then, there exists a subsequence (not relabeled) such that $\{f_h^\ell\}_h$ weakly-$*$ converges to $f^\infty \in BV(X)$ and, by Lemmas 6.2 and 3.4, we get

$$\mathrm{Var}_h[\mu_{(\mathcal{T}_h, f_h)} - \mu_{(\mathcal{Y}_h, g_h)}]^2 \to \|\mu_{(X, f^\infty)} - \mu_{(Y,g)}\|^2.$$

Then, by lower semicontinuity, the BV norm with respect to the L^1 topology, we get

$$\min_{f \in BV(X)} E(f) \le E(f^\infty) \le \liminf_{h \to 0} \min_{f \in \mathbb{P}_1(\mathcal{T}_h)} E_h(f).$$

The other inequality follows from the upper bound condition of Γ-convergence applied to a minimizer of E. This proves, in particular, that f^∞ minimizes E. ■

Remark 7.5 (H^1 model). The discrete problem for H^1 signals is defined by the discrete energy

$$E_h(f) = L_1^2[f, \mathcal{T}_h] + H_h[f, \mathcal{T}_h] + \mathrm{Var}_h[\mu_{(\mathcal{T}_h, f)} - \mu_{(\mathcal{Y}_h, g_h)}]^2.$$

Then, Theorems 7.3 and 7.4 still hold in this case, and their proofs can be adapted by considering as S-topology the L^2 convergence of the projection (i.e., $f_h \xrightharpoonup{S} f$ if $f_h^\ell \to f$ strongly in $L^2(X)$) and using the compact embedding in $L^2(X)$ and the compactness of the unit ball with respect to the weak topology of $H^1(X)$.

Remark 7.6 (L^2 model). In this case, the discrete energy is defined on the set of P_0 finite elements by the following function to minimize on $\mathbb{P}_0(\mathcal{T}_h)$:

$$E_h : \mathbb{P}_0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}, \quad E_h(f_h) = \frac{\gamma_f}{2} L_0^2[f_h, \mathcal{T}_h] + \frac{\gamma_W}{2} \mathrm{Var}[\mu_{(\mathcal{T}_h, f_h)} - \mu_{(\mathcal{Y}_h, g_h)}]^2.$$

We must suppose that γ_f / γ_W is large enough to apply Theorem 4.4. We note that in the case of the W'-fvarifold norm (see Remarks 3.5 and 4.3), Lemma 6.2 still holds and Lemma 3.4 can be proved under the hypothesis of almost everywhere convergence of signals.

For L^2 signals, it is sufficient to define the S-topology, $f_h \xrightharpoonup{S} f$, via the following fvarifold convergence:

$$\mu_{(X, f_h^\ell)} \xrightharpoonup{*} \mu_{(X,f)} \quad \text{in } \mathcal{M}^X,$$

where $\mathcal{M}^X$ is defined in Definition 4.5. Because of (4.2), we get the lower semicontinuity for the lower bound of Γ-convergence. The upper bound follows similarly to the BV case, knowing that the discretization process guarantees the strong convergence in L^2.

Concerning the convergence of minimizers, Theorem 4.4 (in particular, (4.1)) implies that the sequence of discrete minimizers $\{f_h\}_h$ is uniformly bounded in L^∞. Then, the sequence of measures $\{\mu_{(X,f_h^\ell)}\}_h \subset \mathcal{M}^X$ is tight and, because of Prokhorov's Theorem, it weakly-$*$ converges (up to a subsequence) to some $\mu_\infty \in \mathcal{M}^X$. Now, because of the lower semicontinuity of $\widetilde{E}$ with respect to the weak-$*$ convergence of measures and the fact that $\widetilde{E}$ is minimized by an fshape associated to a L^2 function (see Remark 4.3), we get

$$\min_{f\in L^2(X)} E(f) \le \widetilde{E}(\mu_\infty) \le \liminf_{h\to 0} \widetilde{E}(\mu_{(X,f_h^\ell)}) \le \liminf_{h\to 0} \min_{f\in \mathbb{P}_0(\mathcal{T}_h)} E_h(f),$$

that gives the needed inequality. The other one follows from the upper bound condition of Γ-convergence. In particular, we get

$$\min_{f\in L^2(X)} E(f) = \widetilde{E}(\mu_\infty)$$

and, from Theorem 4.4, there exists $f_* \in L^2(X)$ such that $\mu_\infty = \mu_{(X,f_*)}$. Thus, $f_h \overset{S}{\rightharpoonup} f_*$.

A. Approximation theorem for BV functions on manifolds

Theorem. *Let X be an orientable Riemannian compact manifold with boundary ∂X and let $X_0 = X \setminus \partial X$. For any $f \in BV(X)$, there exists a sequence $\{f_h\}_h \subset C^1(X_0)$ such that*

$$f_h \to f_{\restriction X_0} \text{ in } L^1(X_0,\mathbb{R}) \quad \text{and} \quad |D_X f|(X) = \lim_{h\to\infty} \int_{X_0} |\nabla f_h| \mathrm{vol}_X .$$

Proof. Let $\{U_i\}_{i\in[\![1,n]\!]}$ be a finite atlas on X_0 and for any $i \in [\![1,n]\!]$, let $\varphi_i : U_i \to V_i$ be a local chart such that $\overline{U_i}$ is compact and φ_i is the restriction to U_i of a C^1 diffeomorphisms from $\overline{U_i} \to \overline{V_i}$ (such an atlas exists, since X is compact). Let $\{\eta_j\}_{j\ge 0}$ be a partition of unity such that $\mathrm{supp}(\eta_j)$ is compact for any $j \ge 0$ and there exists a partition $\{J_i\}_{i\in[\![1,n]\!]}$ of $\mathbb{N}$ for which $\mathrm{supp}(\eta_j) \subset U_i$ for any $j \in J_i$ which is locally finite on any U_i (i.e., for any $x \in U_i$, there exists an open set $U_i(x) \subset U_i$ such that $\mathrm{supp}(\eta_j) \cap U_i(x) = \varnothing$ for any $j \in J_i$, except on a finite number of j's).

For any $j \in J_i$, we consider ε_j such that $d(\varphi_i(\mathrm{supp}(\eta_j)), V_i^c) > \varepsilon_j$ and for $\boldsymbol{\varepsilon} = \{\varepsilon_j\}_{j\ge 0}$, we consider the linear operator $L_{\boldsymbol{\varepsilon}} : BV(X) \to C^1(X_0)$ defined by

$$L_{\boldsymbol{\varepsilon}} f = \sum_{i=1}^n \varphi_i^* \Big(\sum_{j\in J_i} \psi_i^*(f\eta_j) * \rho_{\varepsilon_j} \Big),$$

where $\psi_i : V_i \to U_i$ is the inverse mapping of φ_i. We recall the classical notation of differential geometry for pullbacks where for any function $\ell \in C_c(U_i)$, $\psi_i^*\ell = \ell \circ \psi_i$ and for any $v \in \chi_c^1(U_i)$, $\psi_i^* v = (d\psi_i)^{-1} v \circ \psi_i$. We recall that $\chi_c^1(X_0)$ denotes the set of C^1 vector fields $u : X \to TX$ on X compactly supported in X_0. Eventually, on every V_i, we introduce $\alpha_i dx$, the pullback of $\mathrm{vol}_X\restriction_{U_i}$, by ψ_i on V_i such that for any $\ell \in C_c(U_i)$, we have $\int_{U_i} \ell \mathrm{vol}_X = \int_{V_i} (\psi_i^* \ell)\alpha_i dx$.

Let $\delta > 0$. We can assume that for any $1 \le i \le n$ and any $j \in J_i$, we have ε_j small enough so that

$$\int_{V_i} |\psi_i^*(f\eta_j) * \rho_{\varepsilon_j} - \psi_i^*(f\eta_j)|\alpha_i dx \le \delta 2^{-j}.$$

Since $\{\eta_j\}_{j\ge 0}$ is a partition of unity, we have $f = \sum_{i=1}^n \sum_{j\in J_i} f\eta_j$ and

$$\int_X |L_{\boldsymbol{\varepsilon}} f - f|\mathrm{vol}_X \le \sum_{i=1}^n \int_{V_i} \sum_{j\in V_j} |\psi_i^*(f\eta_j) * \rho_{\varepsilon_j} - \psi_i^*(f\eta_j)|\alpha_i dx \le 2\delta.$$

This first inequality is enough to prove an approximation result in an L^1 sense. We turn now to the control of the total variation part.

Let $u \in \chi_c^1(X_0)$. We have the following decomposition using the integration by parts formula in (2.1) for equality (a) and the classical integration by parts on $\mathbb{R}^d$ for equality (b):

$$\begin{aligned}
\int_X L_{\boldsymbol{\varepsilon}} f \,\mathrm{div}_X(u)\mathrm{vol}_X &= \sum_{i=1}^n \int_X \varphi_i^*\Big(\sum_{j\in J_i} \psi_i^*(f\eta_j) * \rho_{\varepsilon_j}\Big)\mathrm{div}_X(u)\mathrm{vol}_X \\
&\overset{(a)}{=} -\sum_{i=1}^n \int_X u\Big(\varphi_i^*\Big(\sum_{j\in J_i} \psi_i^*(f\eta_j) * \rho_{\varepsilon_j}\Big)\Big)\mathrm{vol}_X \\
&= -\sum_{i=1}^n \int_{V_i} \sum_{j\in J_i} (\psi_i^* u)\Big(\sum_{j\in J_i} \psi_i^*(f\eta_j) * \rho_{\varepsilon_j}\Big)\alpha_i dx \\
&\overset{(b)}{=} \sum_{i=1}^n \int_{V_i} \sum_{j\in J_i} [\psi_i^*(f\eta_j)] * \rho_{\varepsilon_j} \mathrm{div}(\alpha_i \psi_i^* u) dx \\
&= \sum_{i=1}^n \int_{V_i} \sum_{j\in J_i} \psi_i^*(f\eta_j)\mathrm{div}([\alpha_i \psi_i^* u] * \rho_{\varepsilon_j}) dx \\
&= \sum_{i=1}^n \underbrace{\int_{V_i} \sum_{j\in J_i} \psi_i^*(f)\mathrm{div}(\psi_i^* \eta_j([\alpha_i \psi_i^* u] * \rho_{\varepsilon_j})) dx}_{A_{ij}} \\
&\quad - \sum_{i=1}^n \underbrace{\int_{V_i} \sum_{j\in J_i} \psi_i^*(f)([\alpha_i \psi_i^* u] * \rho_{\varepsilon_j})(\psi_i^* \eta_j) dx}_{B_{ij}}
\end{aligned}$$

with

$$A_{ij} = \int_{V_i} \sum_{j \in J_i} \psi_i^*(f)\big[\mathrm{div}(\alpha_i \psi_i^* \eta_j((\psi_i^* u) * \rho_{\varepsilon_j})) + \mathrm{div}(\psi_i^* \eta_j([\alpha_i \psi_i^* u] * \rho_{\varepsilon_j} \\ - \alpha_i[(\psi_i^* u) * \rho_{\varepsilon_j}]))\big] dx.$$

However, for any $\delta > 0$, denoting by $|\cdot|_x$ the norm at $x \in X$ induced by the metric, for $x \in \mathrm{supp}(\eta_j)$ and ε_j small enough, we have

$$\begin{aligned} |\varphi_i^*((\psi_i^* u) * \rho_{\varepsilon_j})|_x(x) &= \Big|(d_x\varphi_i)^{-1}\Big(\int_{V_i} d_{\psi_i(\varphi_i(x)-y)}\varphi_i u(\psi_i(\varphi_i(x)-y))\rho_{\varepsilon_j}(y)dy\Big)\Big|_x \\ &\leq 1 + \Big|(d_x\varphi_i)^{-1}\Big(\int_{V_i}(d_{\psi_i(\varphi_i(x)-y)}\varphi_i - d_x\varphi_i)u(\psi_i(\varphi_i(x)-y))\rho_{\varepsilon_j}(y)dy\Big)\Big|_x \\ &\leq 1 + \delta \end{aligned}$$

uniformly in u such that $\|u\|_\infty \leq 1$ and $j \in J_i$. Hence,

$$\begin{aligned} \sum_{i=1}^n \Big|\int_{V_i} \sum_{j \in J_i} \psi_i^*(f)\mathrm{div}(\alpha_i \psi_i^* \eta_j((\psi_i^* u) * \rho_{\varepsilon_j}))dx\Big| &\leq \sum_i \int_X \sum_{j \in J_i} \eta_j(1+\delta)d|D_X f| \\ &\leq (1+\delta)|D_X f|(X). \end{aligned}$$

Moreover, for ε_j small enough, for $x \in \mathrm{supp}(\eta_j)$, we can assume

$$\begin{aligned} &\Big|\varphi_i^* \frac{[\alpha_i \psi_i^* u] * \rho_{\varepsilon_j} - \alpha_i[(\psi_i^* u) * \rho_{\varepsilon_j}]}{\alpha_i}\Big|_x \\ &= \Big|d_x\varphi_i^{-1} \int_{V_i} \frac{\alpha_i(\varphi_i(x)-y) - \alpha_i(\varphi_i(x))}{\alpha_i(\varphi_i(x))} d_{\psi(\varphi_i(x)-y)}\varphi_i u(\psi_i(\varphi_i(x)-y))\rho_{\varepsilon_j}(y)dy\Big|_x \\ &\leq \delta \end{aligned}$$

so that

$$\begin{aligned} &\sum_{i=1}^n \Big|\int_{V_i} \sum_{j \in J_i} \psi_i^* f \,\mathrm{div}\Big(\alpha_i \psi_i^* \eta_j \frac{[\alpha_i \psi_i^* u] * \rho_{\varepsilon_j} - \alpha_i[(\psi_i^* u) * \rho_{\varepsilon_j}]}{\alpha_i}\Big)dx\Big| \\ &\qquad \leq \sum_i \int_X \sum_{j \in J_i} \eta_j \delta |D_X f| \leq \delta |D_X f|(X). \end{aligned}$$

Thus, we have

$$\sum_{i=1}^n A_{ij} \leq |D_X f|(X)(1+2\delta). \tag{A.1}$$

Let us consider now the B_{ij}'s. We have

$$B_{ij} = \int_{V_i} \Big\langle \int_{V_i} (\alpha_i \psi_i^* u)(x-y)\rho_{\varepsilon_j}(y)dy, (\psi_i^* f \nabla(\psi_i^* \eta_j))(x)\Big\rangle dx$$

$$= \int_{V_i} \Big\langle (\alpha_i \psi_i^* u)(x), \int_{V_j} (\psi_i^* f \nabla(\psi_i^* \eta_j))(x-y) \rho_{\varepsilon_j}(y) dy \Big\rangle dx$$
$$= \underbrace{\int_{V_i} \alpha_i \psi_i^* f \psi_i^* u (\psi_i^* \eta_j) dx}_{B_{ij}^1}$$
$$+ \underbrace{\int_{V_i} \Big\langle (\alpha_i \psi_i^* u)(x), (\psi_i^* f \nabla(\psi_i^* \eta_j)) * \rho_{\varepsilon_j} - \psi_i^* f \nabla(\psi_i^* \eta_j) \Big\rangle dx}_{B_{ij}^2}.$$

Concerning the B_{ij}^1 terms, we have

$$\sum_{i=1}^{n} \sum_{j \in J_j} B_{ij}^1 = \sum_{i=1}^{n} \int_X \sum_{j \in J_j} f u(\eta_j) \mathrm{vol}_X = \int_X f u(1) \mathrm{vol}_X = 0. \tag{A.2}$$

For the B_{ij}^2 terms, let us notice that $\sup_{V_i} |\alpha_i \psi_i^* u| < \infty$ uniformly in u (since $\|u\|_\infty \leq 1$) and $(\psi_i^* f \nabla(\psi_i^* \eta_j)) * \rho_{\varepsilon_j} \to (\psi_i^* f \nabla(\psi_i^* \eta_j))$ in $L^1(\mathbb{R}^d, dx)$, so that for ε_j sufficiently small, we can assume that $|B_{ij}^2| \leq \delta 2^{-(j+1)}$. Summing along the indices, we get

$$\Big| \sum_{i=1} \sum_{j \in J_i} B_{ij}^2 \Big| \leq \delta \tag{A.3}$$

and, with (A.1), (A.2), and (A.3), we get eventually that, for sufficiently small values of the ε_j's, we have

$$\Big| \int_X L_{\boldsymbol{\varepsilon}} f \, \mathrm{div}_X(u) \mathrm{vol}_X \Big| \leq |D_X f|(X)(1 + 2\delta + \delta^2) + \delta$$

uniformly in $u \in \chi_c^2(X_0)$ satisfying $\|u\|_\infty \leq 1$. Taking the supremum over such u, we get

$$|D_X L_{\boldsymbol{\varepsilon}} f|(X) \leq |D_X f|(X)(1 + 2\delta + \delta^2) + \delta.$$

Since δ is arbitrary, we have shown that there exists a sequence $(\boldsymbol{\varepsilon}_k)_{k \geq 0}$ such that $L_{\boldsymbol{\varepsilon}_k} f \in C^\infty(X_0)$ and

$$\limsup_k |D_X L_{\boldsymbol{\varepsilon}_k} f|(X) \leq |D_X f|(X).$$

Thus, the proof is complete. ∎

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