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A minimization problem with free boundary and its application to inverse scattering problems

Pu-Zhao Kow, Mikko Salo, and Henrik Shahgholian

Abstract. We study a minimization problem with free boundary, resulting in hybrid quadrature domains for the Helmholtz equation, as well as some applications to inverse scattering problems.

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1. Introduction

Motivated by questions in inverse scattering theory, the article [23] introduced the notion of quadrature domains for the Helmholtz operator $\Delta + k^2$ with $k > 0$, also called k-quadrature domains. Given any $\mu \in \mathcal{E}'(\mathbb{R}^n)$, a bounded open set $D \subset \mathbb{R}^n$ is called a *k-quadrature domain with respect to* μ if $\mu \in \mathcal{E}'(D)$ and

$$\int_D w(x)\,dx = \langle \mu, w \rangle,$$

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for all $w \in L^1(D)$ satisfying $(\Delta + k^2)w = 0$ in D. The case $k = 0$ corresponds to classical quadrature domains for harmonic functions. As a consequence of a mean value theorem for the Helmholtz equation (which goes back to H. Weber; see, e.g., [23, Proposition A.6] or [6, p. 289]), balls are always k-quadrature domains with μ being a multiple of the Dirac delta function. The work [23] gave further examples of k-quadrature domains including cardioid-type domains in the plane, implemented a partial balayage procedure to construct such domains, and showed that such domains may be non-scattering domains for certain incident waves. The results were based on the following PDE characterization (see [23, Proposition 2.1]): a bounded open set $D \subset \mathbb{R}^n$ is a k-quadrature domain for $\mu \in \mathcal{E}'(D)$ if and only if there is $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying (an obstacle-like free boundary problem)

$$\begin{cases} (\Delta + k^2)u = \chi_D - \mu & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases} \tag{1.1}$$

In this work we study k-quadrature domains in the presence of densities both on D and ∂D. Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$. If we only have one density $h \geq 0$ on D, we may look for bounded domains D for which $\mu \in \mathcal{E}'(D)$ and

$$\int_D w(x)h(x)\,dx = \langle \mu, w \rangle$$

for all $w \in L^1(D)$ solving $(\Delta + k^2)w = 0$ in D. Such a set D could be called a weighted k-quadrature domain. More generally, if we also have a density $g \geq 0$ on ∂D, we consider the following Bernoulli-type free boundary problem generalizing (1.1):

$$\begin{cases} (\Delta + k^2)u = h - \mu & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ |\nabla u| = g & \text{on } \partial D, \end{cases} \tag{1.2}$$

where the Bernoulli condition $|\nabla u| = g$ is in a very weak sense; see Proposition B.4 or [18, Theorem 2.3]. Given any $\mu \in \mathcal{E}'(\mathbb{R}^n)$, a bounded domain D for which $\mu \in \mathcal{E}'(D)$ and (1.2) has a solution u will be called a *hybrid k-quadrature domain*. The main theme of this paper is to study such domains. We will establish the existence of hybrid k-quadrature domains for suitable μ via a minimization problem. We will also give examples of such domains with real-analytic boundary, and show that hybrid k-quadrature domains may be non-scattering domains in the presence of certain boundary sources. We will closely follow [18], which studied the case $k = 0$. It turns out that many of our results can be reduced to the situation in [18], but certain parts will require modifications. Even though part of the treatment is very similar to [18], we will try to give enough details so that readers who are not experts on this topic can also follow the presentation.

1.1. Minimization problem

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ (with $n \geq 2$). Let $C_c^\infty(\Omega)$ consist of $C^\infty(\mathbb{R}^n)$ functions which are supported in Ω, and denote by $H_0^1(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$-norm. We define the set[1]

$$\mathbb{K}(\Omega) := \{u \in H_0^1(\Omega) \mid u \geq 0\},$$

and for each $u \in H_0^1(\Omega)$, we define

$$\{u > 0\} = \{x_0 \in \Omega \mid \text{there exists non-negative } \phi \in C_c^\infty(\Omega) \text{ with } \phi(x_0) > 0 \text{ such that } u \geq \phi \text{ in } \Omega\}.$$

Let $\lambda \in \mathbb{R}$. For given functions $f, g \in L^\infty(\mathbb{R}^n)$ with $g \geq 0$, we define the functional

$$\mathcal{J}_{f,g,\lambda,\Omega}(u) := \int_\Omega (|\nabla u(x)|^2 - \lambda |u(x)|^2 - 2f(x)u(x) + g^2(x)\chi_{\{u>0\}})\,dx, \tag{1.3}$$

which is well-defined for all $u \in H_0^1(\Omega)$. The main purpose of this paper is to study the following minimization problem:

$$\text{minimize} \quad \mathcal{J}_{f,g,\lambda,\Omega}(u) \quad \text{subject to } u \in \mathbb{K}(\Omega). \tag{1.4}$$

It is easy to see that

$$\inf_{u \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(u) \leq \mathcal{J}_{f,g,\lambda,\Omega}(0) = 0. \tag{1.5}$$

We will show that there exists a minimizer of (1.4) when Ω is a bounded Lipschitz domain and $-\infty < \lambda < \lambda^*(\Omega)$ (Proposition 3.6). Here $\lambda^*(\Omega)$ is the *fundamental tone* of Ω, defined by

$$\lambda^*(\Omega) := \inf_{\phi \in C_c^\infty(\Omega), \phi \not\equiv 0} \frac{\|\nabla \phi\|_{L^2(\Omega)}^2}{\|\phi\|_{L^2(\Omega)}^2}.$$

It is well known that $\lambda^*(\Omega)$ is the infimum of the Dirichlet spectrum of $-\Delta$ on Ω. In addition, when Ω is C^1, we will show that there exists a countable set $Z \subset (-\infty, \lambda^*(\Omega))$ such that the minimizer of (1.4) is unique for all $\lambda \in (-\infty, \lambda^*(\Omega)) \setminus Z$ (Proposition 5.4). The functional $\mathcal{J}_{f,g,\lambda,\Omega}$ is unbounded below in $\mathbb{K}(\Omega)$ when $\lambda > \lambda^*(\Omega)$ (Lemma 3.1), which shows the non-existence of a global minimizer of (1.4) for $\lambda > \lambda^*(\Omega)$.

1.2. Quadrature domains via minimization

Given two non-negative functions h and g in $\mathbb{R}^n$ ($n \geq 2$), and a positive measure μ with compact support in $\mathbb{R}^n$, we wish to find a bounded domain D with Hausdorff $(n-1)$-dimensional boundary ∂D containing supp (μ) such that the potential $\Psi_k * \mu$ (see Definition 1.1) for any fundamental solution Ψ_k of $-(\Delta + k^2)$ agrees outside D with that of

[1] In particular, the inequality $u \geq 0$ in the definition of $\mathbb{K}(\Omega)$ can be interpreted in the almost everywhere pointwise sense; see, for example, [22, Definition II.5.1].

the measure

$$\sigma := h\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor\partial D, \tag{1.6}$$

where $\mathcal{L}^n\lfloor D$ and $\mathcal{H}^{n-1}\lfloor\partial D$ denote the Lebesgue measure restricted to D and the $(n-1)$-dimensional Hausdorff measure on ∂D, respectively.

We now introduce a hybrid version of a quadrature domain in the following definition:

Definition 1.1. Let $k > 0$ and let D be a bounded open set in $\mathbb{R}^n$ with the boundary ∂D having finite $(n-1)$-dimensional Hausdorff measure. Let σ be the measure given by (1.6). The set $D \subset \mathbb{R}^n$ is called a *hybrid k-quadrature domain*, corresponding to distribution $\mu \in \mathcal{E}'(D)$ (with $\mathrm{supp}\,(\mu) \subset D$) and density $(g,h) \in L^\infty(\partial D) \times L^\infty(D)$, if

$$\big(\Psi_k * (g\mathcal{H}^{n-1}\lfloor\partial D)\big)(x) \quad \text{is well-defined pointwise for all } x \in \partial D \tag{1.7}$$

and

$$\Psi_k * \mu = \Psi_k * \sigma \quad \text{in } \mathbb{R}^n \setminus D, \tag{1.8}$$

for all fundamental solutions Ψ_k of the Helmholtz operator $-(\Delta + k^2)$, that is, $-(\Delta + k^2)\Psi_k = \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Remark 1.2. In general, condition (1.7) does not follow from standard elliptic regularity results. Some extra assumptions (see, e.g., Theorem 1.5) are required to ensure (1.7) holds.

Remark 1.3. Let Ψ_k be any fundamental solution for $-(\Delta + k^2)$ in $\mathbb{R}^n$. Given any bounded Lipschitz domain D, let $\gamma : H^1_{\mathrm{loc}}(\mathbb{R}^n) \to H^{1/2}(\partial D)$ be the trace operator. The adjoint γ^* of γ is defined by

$$\langle \gamma^*\psi, \phi\rangle = \langle \psi, \gamma\phi\rangle_{\partial D} \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n). \tag{1.9}$$

In particular, γ^* maps $H^{-\frac{1}{2}}(\partial D)$ to $H^{-1}(\mathbb{R}^n)$. Using [26, Theorem 6.11], we can define the single-layer potential by

$$\mathsf{SL} : H^{-\frac{1}{2}}(\partial D) \to H^1_{\mathrm{loc}}(\mathbb{R}^n), \quad \mathsf{SL}(g) := \Psi_k * (\gamma^* g). \tag{1.10}$$

In this case, (1.8) simply reads as

$$\Psi_k * \mu = \Psi_k * (h\chi_D) + \mathsf{SL}(g).$$

Remark 1.4. When D is a hybrid k-quadrature domain with $g \equiv 0$, using an L^1 -density theorem (see [23, Proposition 2.4]) and a similar argument as in [23, Theorem 1.2], we know that

$$\langle \mu, w\rangle = \int_D w(x)h(x)\,dx$$

for all $w \in L^1(D)$ such that $(\Delta + k^2)w = 0$ in D. In this case, if (1.8) is true for one fundamental solution, then it is true for all fundamental solutions.

We have the following theorem (see Theorem 7.6 for a more detailed statement):

Theorem 1.5. *Let $n \geq 2$, and assume h and g are sufficiently regular. If μ is a non-negative measure on $\mathbb{R}^n$ with mass concentrated near a point and $R > 0$, then for each sufficiently small $k > 0$, there exists a bounded open domain D in $\mathbb{R}^n$ with the boundary ∂D having finite $(n-1)$-dimensional Hausdorff measure such that (1.7) holds, which is a hybrid k-quadrature domain corresponding to distribution μ and density (g, h) satisfying $\overline{D} \subset B_{\beta k^{-1}}$. In particular, when $g > 0$ is Hölder continuous in $\overline{B_R}$, there exists a portion $E \subset \partial D$ with $\mathcal{H}^{n-1}(\partial D \setminus E) = 0$ such that E is locally $C^{1,\alpha}$. In the case when $n = 2$, we can even choose $E = \partial D$.*

Remark 1.6. The hybrid k-quadrature domain constructed in Theorem 1.5 can be represented by $D = \{u_* > 0\}$, where u_* is a minimizer of $\mathcal{J}_{f,g,k^2,B_R}$ in $\mathbb{K}(B_R)$ with $f = \mu - h\chi_D$ when μ is bounded (for general μ, we consider some suitable mollifiers). Since the minimizer is unique for k outside a countable set, so is the constructed domain; see Proposition 5.4. See also Proposition 7.5 for the case when μ is bounded.

1.3. Real-analytic quadrature domains

We can construct examples of hybrid k-quadrature domains using the Cauchy–Kowalevski theorem. Let D be a bounded domain in $\mathbb{R}^n$ with real-analytic boundary. Let g be real-analytic on a neighborhood of ∂D with $g > 0$ on ∂D. For each $k \geq 0$, there exists a bounded positive measure μ_1 with $\operatorname{supp}(\mu_1) \subset D$ such that D is a hybrid k-quadrature domain corresponding to μ_1 with density $(g, 0)$. Moreover, if $0 \leq k < j_{\frac{n-2}{2},1} R^{-1}$ (where $j_{\alpha,1}$ is the first positive zero of the Bessel function J_α), $\overline{D} \subset B_R$, and if h is a non-negative integrable function near $\overline{D}$ which is real-analytic near ∂D, then D is a hybrid k-quadrature domain corresponding to some measure μ_2 with density $(0, h)$. The proofs follow easily by solving suitable Cauchy problems near ∂D by the Cauchy–Kowalevski theorem, and defining μ_1 and μ_2 in terms of the obtained solutions. For the details, see Appendix D.

Organization

We first discuss the application to inverse problems in Section 2. Then, we prove the existence of global minimizers of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ in Section 3. We study the relation between local minimizers and partial differential equations in Section 4. Next, we study the local minimizers in Section 5 and Section 6. With these ingredients at hand, we prove Theorem 1.5 in Section 7. For the reader's convenience, we add several appendices to make the paper self-contained. In Appendix A we recall a few facts about functions of bounded variation and sets with finite perimeter. Appendix B provides detailed statements and proofs of results analogous to [18, Section 2]. We then exhibit the detailed proof of Lemma 7.2 in Appendix C. Appendix D discusses examples of hybrid k-quadrature domains with real-analytic boundary. Finally, we give some remarks on null k-quadrature domains in Appendix E.

2. Applications to inverse scattering problems

We say that a solution u of $(\Delta + k_0^2)u = 0$ in $\mathbb{R}^n \setminus \overline{B_R}$ (for some $R > 0$) is *outgoing* if it satisfies the following Sommerfeld radiation condition:

$$\lim_{|x|\to\infty} |x|^{\frac{n-1}{2}} (\partial_r u - ik_0 u) = 0 \quad \text{uniformly in all directions} \quad \hat{x} = \frac{x}{|x|} \in \mathcal{S}^{n-1},$$

where ∂_r denotes the radial derivative. There exists a unique $u^\infty \in L^2(\mathcal{S}^{d-1})$, which is called the far-field pattern of u, such that

$$u(x) = \gamma_{n,k_0} \frac{e^{ik_0|x|}}{|x|^{\frac{n-1}{2}}} u^\infty(\hat{x}) + \mathcal{O}(|x|^{-\frac{n+1}{2}}) \quad \text{as} \quad |x| \to \infty,$$

uniformly in all directions $\hat{x} \in \mathcal{S}^{d-1}$, where we make the choice (as in [33, Section 1.2.3])

$$\gamma_{n,k_0} = \frac{e^{-\frac{(n-3)\pi i}{4}}}{2(2\pi)^{\frac{n-1}{2}}} k_0^{\frac{n-3}{2}}.$$

For $n = 2$, we have $\gamma_{2,k_0} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{8\pi k_0}}$, while when $n = 3$, we have $\gamma_{3,k_0} = \frac{1}{4\pi}$.

Let D be a bounded domain in $\mathbb{R}^n$, which represents a penetrable obstacle with contrast $\rho \in L^\infty(D)$ satisfying $|\rho| \geq c > 0$ almost everywhere near ∂D. When one probes the obstacle (D, ρ) using an incident wave u_0 satisfying $(\Delta + k_0^2)u_0 = 0$ in $\mathbb{R}^n$, it produces an outgoing scattered field $u_{\rm sc}$ solving

$$(\Delta + k_0^2 + \rho\chi_D)(u_0 + u_{\rm sc}) = 0 \quad \text{in } \mathbb{R}^n.$$

We say that the obstacle (D, ρ) is non-scattering with respect to the incident field u_0 and the wave number k_0 if the far-field pattern $u_{\rm sc}^\infty$ of the corresponding scattered field $u^{\rm sc}$ vanishes identically. Using the Rellich uniqueness theorem [5, 19], we know that $u_{\rm sc}^\infty \equiv 0$ if and only if $u_{\rm sc} = 0$ in $\mathbb{R}^n \setminus \overline{B_R}$ for some $R > 0$, therefore, this definition coincides with [23, Definition 1.8]. The next theorem extends [23, Corollary 1.9]. We remind the readers that there are some significant differences between 0-quadrature domains and k-quadrature domains; see Appendix E for more details.

Theorem 2.1. *Let D be a bounded hybrid k-quadrature domain, corresponding to distribution $\mu \in \mathcal{E}'(D)$ and density $(0, h)$ with $h \in L^\infty(D)$ and $|h| \geq c > 0$ near ∂D. Assume that there exist a wave number $k_0 \geq 0$ (which may differ from k) and*

$$\text{a solution } u_0 \text{ of } (\Delta + k_0^2)u_0 = 0 \text{ in } \mathbb{R}^n \text{ such that } u_0 > 0 \text{ on } \partial D. \tag{2.1}$$

Then, there exists a contrast $\rho \in L^\infty(D)$ satisfying $|\rho| \geq c > 0$ almost everywhere near ∂D such that (D, ρ) is non-scattering with respect to the incident field u_0 and the wave number k_0.

Remark 2.2. Using the result in [24] (see also [30]), we know that (2.1) holds at least when

(A) D is Lipschitz so that $\mathbb{R}^n \setminus \overline{D}$ is connected and k_0^2 is not a Dirichlet eigenvalue of $-\Delta$ in D; or

(B) D is a compact set contained in a bounded Lipschitz domain Ω such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $|\Omega| \le |B_r|$, where $r = j_{\frac{n-2}{2},1} k_0^{-1}$.

Proof of Theorem 2.1. Following the same ideas in [23, Theorem 1.2 and Remark 1.3], one can show that there is a neighborhood U of ∂D in $\mathbb{R}^n$ and a distribution $u \in \mathcal{D}'(U)$ satisfying

$$\begin{cases} (\Delta + k_0^2)u = ((k_0^2 - k^2)u + h)\chi_D & \text{in } U, \\ u = |\nabla u| = 0 & \text{in } U \setminus D. \end{cases}$$

By elliptic regularity, one has $u \in C^{1,\alpha}(U)$. The function $v_0 = u + u_0$ satisfies

$$\begin{cases} (\Delta + k_0^2 + \rho_0 \chi_D)v_0 = (\rho_0 v_0 + (k_0^2 - k^2)u + h)\chi_D & \text{in } U, \\ v_0|_{U\setminus D} = u_0|_{U\setminus D}. \end{cases}$$

By choosing $\rho_0 = -\frac{(k_0^2-k^2)u+h}{v_0}$ near ∂D, one can verify that $v_0 \in C^{1,\alpha}(U)$ and $|\rho_0| \ge c > 0$ near ∂D. Hence, the theorem follows by applying the next lemma (Lemma 2.3) with $g \equiv 0$. ■

To investigate the case when g is non-trivial, we need the following technical lemma, which is a refinement of [30, Lemma 2.3]:

Lemma 2.3. *Let $k_0 \ge 0$, let D be a bounded open set in $\mathbb{R}^n$ with the boundary ∂D having finite $(n-1)$-dimensional Hausdorff measure, and let $g \in L^\infty(\partial D)$. Given any u_0 as in (2.1), any open neighborhood U of ∂D in $\mathbb{R}^n$, any $\rho_0 \in L^\infty(U)$ with $|\rho_0| \ge c > 0$ near ∂D, and any $v_0 \in C^{0,1}_{\rm loc}(U)$ such that*

$$\begin{cases} (\Delta + k_0^2 + \rho_0 \chi_D)v_0 = g\mathcal{H}^{n-1}\lfloor \partial D & \text{in } U, \\ v_0|_{U\setminus D} = u_0|_{U\setminus D}, \end{cases} \tag{2.2}$$

there exist $\rho \in L^\infty(\mathbb{R}^n)$ and $v \in C^{0,1}_{\rm loc}(\mathbb{R}^n)$, with $\rho = \rho_0$ and $v = v_0$ near ∂D, such that

$$\begin{cases} (\Delta + k_0^2 + \rho \chi_D)v = g\mathcal{H}^{n-1}\lfloor \partial D & \text{in } \mathbb{R}^n, \\ v|_{\mathbb{R}^n\setminus D} = u_0|_{\mathbb{R}^n\setminus D}. \end{cases} \tag{2.3}$$

Proof. By (2.1), (2.2), and continuity of v_0, one sees that v_0 is positive in some neighborhood $U' \subset U$ of ∂D. Choose $\psi \in C_c^\infty(U')$ such that $0 \le \psi \le 1$ and $\psi = 1$ near ∂D, and define

$$v = \begin{cases} v_0\psi + (1-\psi) & \text{in } D, \\ u_0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

Then, $v \in C^{0,1}_{\rm loc}(\mathbb{R}^n)$ is positive near $\overline{D}$ and satisfies $v = v_0$ near ∂D. We observe that the function defined by

$$\rho = \begin{cases} -\dfrac{(\Delta + k_0^2)v}{v} & \text{in } D, \\ \psi\rho_0 & \text{in } \mathbb{R}^n \setminus D, \end{cases} \tag{2.4}$$

is L^∞ in D and satisfies $\rho = \rho_0$ near ∂D, which implies that $\rho \in L^\infty(\mathbb{R}^n)$. From (2.4), it is not difficult to see that

$$(\Delta + k_0^2 + \rho\chi_D)v = 0 = g\mathcal{H}^{n-1}\lfloor\partial D \quad \text{in } D. \tag{2.5a}$$

Since $v = v_0$ and $\rho = \rho_0$ near ∂D, from (2.2), we see that

$$(\Delta + k_0^2 + \rho\chi_D)v = g\mathcal{H}^{n-1}\lfloor\partial D \quad \text{near } \partial D. \tag{2.5b}$$

Since $v = u_0$ in $\mathbb{R}^n \setminus \overline{D}$, we also have

$$(\Delta + k_0^2 + \rho\chi_D)v = (\Delta + k_0^2)v = 0 = g\mathcal{H}^{n-1}\lfloor\partial D \quad \text{in } \mathbb{R}^n \setminus \overline{D}. \tag{2.5c}$$

By combining (2.5a), (2.5b), and (2.5c), we conclude that (2.3) holds. ■

We are now ready to prove the following theorem:

Theorem 2.4. *Let $D = \{u_* > 0\}$ be the hybrid k-quadrature domain constructed in Theorem 7.6 or Theorem 1.5 corresponding to density (g, h). Given any u_0 as in (2.1), there exist $\rho \in L^\infty(\mathbb{R}^n)$ with $|\rho| \ge c > 0$ near ∂D and $u_{\rho,g} \in C^{0,1}_{\rm loc}(\mathbb{R}^n)$ such that*

$$\begin{cases} (\Delta + k_0^2 + \rho\chi_D)u_{\rho,g} = g\mathcal{H}^{n-1}\lfloor\partial D & \text{in } \mathbb{R}^n, \\ u_{\rho,g} = u_0 & \text{in } \mathbb{R}^n \setminus D. \end{cases} \tag{2.6}$$

Proof. There exists an open neighborhood U of ∂D in $\mathbb{R}^n$ such that

$$\begin{cases} (\Delta + k_0^2)u_* = ((k_0^2 - k^2)u_* + h)\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor\partial D & \text{in } U, \\ u_*|_{U\setminus D} = 0. \end{cases}$$

Since the function $v_0 = u_* + u_0$ satisfies

$$(\Delta + k_0^2 + \rho_0\chi_D)v_0 = (\rho_0 v_0 + (k_0^2 - k^2)u_* + h)\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor\partial D,$$

by choosing $\rho_0 = -\frac{(k_0^2-k^2)u_*+h}{v_0}$ near ∂D, one can verify that $v_0 \in C^{0,1}(U')$, $\rho_0 \in L^\infty(U')$ with $|\rho_0| \ge c > 0$ in U' and

$$\begin{cases} (\Delta + k_0^2 + \rho_0\chi_D)v_0 = g\mathcal{H}^{n-1}\lfloor\partial D & \text{in } U', \\ v_0|_{U'\setminus D} = u_0|_{U'\setminus D}, \end{cases}$$

for some open neighborhood U' of ∂D in U. Finally, we conclude Theorem 2.4 using Lemma 2.3. ■

We will now discuss how Theorem 2.4 can be interpreted as a non-scattering result. It is easy to see that the function $w_{\rho,g} := u_{\rho,g} - u_0$ satisfies

$$\begin{cases} (\Delta + k_0^2) w_{\rho,g} = -\rho u_{\rho,g} \mathcal{L}^n \lfloor D + g \mathcal{H}^{n-1} \lfloor \partial D & \text{in } \mathbb{R}^n, \\ w_{\rho,g} = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

Since D is bounded, $w_{\rho,g} \in \mathcal{E}'(\mathbb{R}^n)$. Let Ψ_{k_0} be any fundamental solution for $-(\Delta + k_0^2)$ in $\mathbb{R}^n$. By the properties of convolution for distributions, we have

$$\begin{aligned} w_{\rho,g} &= \delta_0 * w_{\rho,g} = -(\Delta + k_0^2)\Psi_{k_0} * w_{\rho,g} = -\Psi_{k_0} * (\Delta + k_0^2) w_{\rho,g} \\ &= \Psi_{k_0} * (\rho u_{\rho,g} \chi_D) - \Psi_{k_0} * (g \mathcal{H}^{n-1} \lfloor \partial D), \end{aligned}$$

that is,

$$u_{\rho,g} = u_0 + \Psi_{k_0} * (\rho u_{\rho,g} \chi_D) - \Psi_{k_0} * (g \mathcal{H}^{n-1} \lfloor \partial D). \tag{2.7}$$

When ∂D above is Lipschitz (by Theorem 7.6, this is true, for example, when $n = 2$), the outer unit normal vector ν on ∂D is $\mathcal{H}^{n-1}$-almost everywhere well-defined in the sense of [13, Theorem 5.8.1]. Now let γ^* be the adjoint of the trace operator on ∂D as in (1.9). In this case, we can write (2.7) as

$$u_{\rho,g} = u_0 + \Psi_{k_0} * (\rho u_{\rho,g} \chi_D) - \mathsf{SL}\,(g),$$

where $\mathsf{SL}\,(g)$ is the single-layer potential as in (1.10). Since $\rho u_{\rho,g} \chi_D \in L^\infty(\mathbb{R}^n)$, one sees that $u_0 + \Psi_{k_0} * (\rho u_{\rho,g} \chi_D) \in C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Consequently, by using the jump relations of the layer potential in [26, Theorem 6.11], we have

$$\partial_\nu^+ u_{\rho,g} - \partial_\nu^- u_{\rho,g} = g \quad \text{in the } H^{-\frac{1}{2}}(\partial D)\text{-sense}. \tag{2.8a}$$

Here ∂_ν^- (resp. ∂_ν^+) denotes the normal derivative from the interior (resp. exterior) of D. Obviously, $u_{\rho,g} \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies

$$(\Delta + k_0^2 + \rho) u_{\rho,g} = 0 \quad \text{in } D, \tag{2.8b}$$

$$u_{\rho,g}|_{\mathbb{R}^n \setminus \overline{D}} = u_0|_{\mathbb{R}^n \setminus \overline{D}}. \tag{2.8c}$$

By (2.8a)–(2.8c), we can interpret $g\mathcal{H}^{n-1} \lfloor \partial D$ in (2.6) as a *non-radiating surface source* with respect to the incident field u_0 and potential $\rho \in L^\infty(D)$. In other words, the obstacle D is non-scattering with respect to both the contrast ρ in D and surface source g on ∂D. We could formally also write equation (2.6) as

$$\begin{cases} (\Delta + k_0^2 + (\rho \chi_D + \tilde{g} \mathcal{H}^{n-1} \lfloor \partial D)) u_{\rho,g} = 0 & \text{in } \mathbb{R}^n, \\ u_{\rho,g} = u_0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

where $\tilde{g} = g / u_{\rho,g}$ on ∂D, which would correspond to a *non-scattering domain with singular contrast*. See also [25] for a discussion about surface sources on Lipschitz surfaces.

We now discuss the case when the background medium is anisotropic and inhomogeneous. Let $m > \frac{n}{2}$ be an integer; let $\rho \in C^{m-1,1}_{\rm loc}(\mathbb{R}^n)$; and let $A \in (C^{m,1}_{\rm loc}(\mathbb{R}^n))^{n\times n}_{\rm sym}$ satisfy the uniform ellipticity condition, that is, there exists a constant $c_0 > 0$ such that

$$\xi \cdot A(x)\xi \geq c_0|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^n. \tag{2.9}$$

Let $u_0 \in H^1_{\rm loc}(\mathbb{R}^n)$ satisfy

$$(\nabla \cdot A\nabla + k_0^2\rho)u_0 = 0 \quad \text{in } \mathbb{R}^n. \tag{2.10}$$

By using [15, Theorem 8.10] and Sobolev embedding, one sees that

$$u_0 \in H^{m+2}_{\rm loc}(\mathbb{R}^n) \subset C^2_{\rm loc}(\mathbb{R}^n).$$

Definition 2.5. We say that the isotropic homogeneous penetrable obstacle D (which is a bounded domain in $\mathbb{R}^n$) is non-scattering with respect to some external source $\mu \in \mathcal{E}'(D)$ and the incident field u_0 as in (2.10), if there exists a $u^{\rm to}$, which is in $H^2_{\rm loc}$ near $\mathbb{R}^n \setminus D$, such that

$$(\nabla \cdot \widetilde{A}\nabla + k_0^2\widetilde{\rho})u^{\rm to} = -\mu \text{ in } \mathbb{R}^n, \quad u^{\rm to}|_{\mathbb{R}^n\setminus D} = u_0|_{\mathbb{R}^n\setminus D}, \tag{2.11}$$

where $\widetilde{A} := A\chi_{\mathbb{R}^n\setminus D} + \mathrm{Id}\chi_D$ and $\widetilde{\rho} := \rho\chi_{\mathbb{R}^n\setminus D} + 1\chi_D$.

By writing $u^{\rm sc} := u^{\rm to} - u_0$ in $\mathbb{R}^n$, one observes that, in the case when D is a bounded Lipschitz domain in $\mathbb{R}^n$, (2.11) is equivalent to the following transmission problem:

$$\begin{cases} (\Delta + k_0^2)u^{\rm sc} = -\mu + h & \text{in } D, \\ u^{\rm sc}|_{\mathbb{R}^n\setminus D} = 0, \quad \partial_\nu^- u^{\rm sc}|_{\partial D} = -g, \end{cases} \tag{2.12}$$

where

$$g := -\nu \cdot (A - \mathrm{Id})\nabla u_0|_{\partial D} \in L^\infty(\partial D), \quad h = -(\Delta + k_0^2)u_0 \in L^\infty(D). \tag{2.13}$$

Here we used that $\partial_\nu^- u^{\rm sc}|_{\partial D} = \partial_\nu^-(u^{\rm to} - u_0)|_{\partial D} = \widetilde{A}\nabla^- u^{\rm to} \cdot \nu|_{\partial D} - \partial_\nu u_0|_{\partial D} = A\nabla^+ u_0 \cdot \nu|_{\partial D} - \partial_\nu u_0|_{\partial D}$, since $u^{\rm to}$ is $H^2_{\rm loc}$ near ∂D. Based on the above observation, we are now able to prove the following theorem:

Theorem 2.6. *Let $m > \frac{n}{2}$ be an integer, let $\rho \in C^{m-1,1}_{\rm loc}(\mathbb{R}^n)$, let $A \in (C^{m,1}_{\rm loc}(\mathbb{R}^n))^{n\times n}_{\rm sym}$ satisfy the uniform ellipticity condition in (2.9), and let u_0 be an incident field as in (2.10). If D is a bounded Lipschitz domain in $\mathbb{R}^n$ such that it is a hybrid k-quadrature domain corresponding to distribution $\mu \in \mathcal{E}'(D)$ and density (g, h) as in (2.13), then there exists a total field $u^{\rm to}$ satisfying (2.11).*

Proof. Let Ψ_{k_0} be any fundamental solution for $-(\Delta + k_0^2)$ in $\mathbb{R}^n$, and define

$$u^{\rm sc} := \Psi_{k_0} * \mu - \Psi_{k_0} * (h\chi_D) - \mathsf{SL}(g) \quad \text{in } \mathbb{R}^n.$$

Since D is a hybrid k-quadrature domain, by Remark 1.3, we know that $u^{\rm sc}|_{\mathbb{R}^n\setminus D} = 0$. Since $u^{\rm sc} \in \mathcal{E}'(\mathbb{R}^n)$ and $\Psi_{k_0} * \mu - \Psi_{k_0} * (h\chi_D)$ is C^1 near ∂D, similarly to (2.8a), one sees that $u^{\rm sc}$ satisfies (2.12). By using the equivalence of (2.11) and (2.12), we conclude the theorem. ∎

3. Existence of minimizers

We first show the boundedness of the functional $\mathcal{J}_{f,g,\lambda,\Omega}$ given in (1.3).

Lemma 3.1. *Let $f, g \in L^\infty(\mathbb{R}^n)$ with $g \geq 0$, and let $|\Omega| < \infty$. If $-\infty < \lambda < \lambda^*(\Omega)$, then $\mathcal{J}_{f,g,\lambda,\Omega}$ is coercive in $H_0^1(\Omega)$. If $\lambda > \lambda^*(\Omega)$, then $\mathcal{J}_{f,g,\lambda,\Omega}$ is unbounded from below in $\mathbb{K}(\Omega)$.*

Remark 3.2. Here we allow $\operatorname{supp}(f_+)$ to be unbounded.

Proof of Lemma 3.1. If $\lambda < \lambda^*(\Omega)$, then there exists a $\gamma > 0$ such that

$$\int_\Omega (|\nabla u|^2 - \lambda |u|^2)\, dx \geq \gamma \|u\|_{H^1(\Omega)}^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Observe that

$$\left| 2 \int_\Omega f u \right| \leq 2\|u\|_{L^2(\Omega)} |\Omega|^{\frac{1}{2}} \|f\|_{L^\infty(\Omega)} \leq \varepsilon \|u\|_{L^2(\Omega)}^2 + \varepsilon^{-1} |\Omega| \|f\|_{L^\infty(\Omega)}^2$$

for all $\varepsilon > 0$. Consequently, we have

$$\begin{aligned}\mathcal{J}_{f,g,\lambda,\Omega}(u) &= \int_\Omega (|\nabla u|^2 - \lambda |u|^2)\, dx - 2\int_\Omega f u\, dx + \int_\Omega g^2 \chi_{\{u>0\}}\, dx \\ &\geq \gamma \|u\|_{H^1(\Omega)}^2 - \varepsilon \|u\|_{L^2(\Omega)}^2 - \varepsilon^{-1} |\Omega| \|f\|_{L^\infty(\Omega)}^2.\end{aligned}$$

Choosing $\varepsilon = \frac{\gamma}{2}$, we reach

$$\mathcal{J}_{f,g,\lambda,\Omega}(u) \geq \frac{\gamma}{2} \|u\|_{H^1(\Omega)}^2 - \frac{2|\Omega| \|f\|_{L^\infty(\Omega)}^2}{\gamma} \quad \text{for all } u \in H_0^1(\Omega). \tag{3.1}$$

This proves the claim for $\lambda < \lambda^*(\Omega)$.

We now consider the case when $\lambda > \lambda^*(\Omega)$. There exists $u \in C_c^\infty(\Omega)$ so that $\|\nabla u\|_{L^2(\Omega)}^2 < \lambda \|u\|_{L^2(\Omega)}^2$. Moreover, $|u| \in H_0^1(\Omega)$ and

$$\|\nabla |u|\|_{L^2(\Omega)}^2 = \|\nabla u\|_{L^2(\Omega)}^2 < \lambda \|u\|_{L^2(\Omega)}^2.$$

Therefore, we know that $t|u| \in \mathbb{K}(\Omega)$ for all $t \geq 0$. Hence, we know that

$$\begin{aligned}\mathcal{J}_{f,g,\lambda,\Omega}(t|u|) &= t^2(\|\nabla |u|\|_{L^2(\Omega)}^2 - \lambda \|u\|_{L^2(\Omega)}^2) - 2t \int_\Omega f|u|\, dx + \int_\Omega g^2 \chi_{\{|u|>0\}}\, dx \\ &\leq t^2 \overbrace{(\|\nabla |u|\|_{L^2(\Omega)}^2 - \lambda \|u\|_{L^2(\Omega)}^2)}^{<0} - 2t \overbrace{\int_\Omega f|u|\, dx}^{>0} + \overbrace{|\operatorname{supp}(u)| \|g\|_{L^\infty(\Omega)}^2}^{<\infty},\end{aligned}$$

which implies

$$\limsup_{t \to \infty} \mathcal{J}_{f,g,\lambda,\Omega}(t|u|) = -\infty;$$

thus, the second claim of the lemma is proved. ∎

Remark 3.3. If $u \in \mathbb{K}(\Omega)$, by observing that

$$\begin{aligned}\mathcal{J}_{f,g,\lambda,\Omega}(u) &= \int_\Omega (|\nabla u|^2 - \lambda |u|^2)\,dx - 2\int_\Omega f_+ u\,dx + \overbrace{2\int_\Omega f_- u\,dx}^{\geq 0} \\ &\quad + \int_\Omega |g|^2 \chi_{\{u>0\}}\,dx \\ &\geq \gamma \|u\|_{H^1(\Omega)}^2 - \varepsilon \|u\|_{L^2(\Omega)}^2 - \varepsilon^{-1}|\Omega| \|f_+\|_{L^\infty(\Omega)}^2,\end{aligned}$$

we can obtain

$$\mathcal{J}_{f,g,\lambda,\Omega}(u) \geq \frac{\gamma}{2}\|u\|_{H^1(\Omega)}^2 - \frac{2|\Omega| \|f_+\|_{L^\infty(\Omega)}^2}{\gamma} \quad \text{for all } u \in \mathbb{K}(\Omega). \tag{3.2}$$

Note that (3.2) is a refinement of (3.1) for functions in $\mathbb{K}(\Omega)$.

Using Lemma 3.1 and following the proof of [12, Section 8.2, Theorem 1], we have the following lemma:

Lemma 3.4. *Let $f, g \in L^\infty(\mathbb{R}^n)$ with $g \geq 0$. Assume that Ω is bounded with Lipschitz boundary and $-\infty < \lambda < \lambda^*(\Omega)$. Then, $\mathcal{J}_{f,g,\lambda,\Omega}$ is weakly lower semi-continuous on $H_0^1(\Omega)$, that is,*

$$\mathcal{J}_{f,g,\lambda,\Omega}(u) \leq \liminf_{j\to\infty} \mathcal{J}_{f,g,\lambda,\Omega}(u_j)$$

whenever $\{u_j\}_{j=1}^\infty \cup \{u\} \subset H_0^1(\Omega)$ satisfies

$$\begin{cases} u_j \to u & \text{weakly in } L^2(\Omega), \\ \nabla u_j \to \nabla u & \text{weakly in } L^2(\Omega). \end{cases}$$

Remark 3.5. Here we remind the readers that the proof of Lemma 3.4 involves the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, which follows from the Rellich–Kondrachov theorem as long as there is a bounded extension operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^n)$, which is true, for example, for Lipschitz domains.

Using Lemma 3.4 and following the proof of [12, Section 8.2], we have the following proposition:

Proposition 3.6. *Let $f, g \in L^\infty(\mathbb{R}^n)$ with $g \geq 0$. Assume further that Ω is a bounded open set with Lipschitz boundary and $-\infty < \lambda < \lambda^*(\Omega)$. Then, there exists a global minimizer $u_* \in \mathbb{K}(\Omega)$ of the functional $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, that is,*

$$\mathcal{J}_{f,g,\lambda,\Omega}(u_*) = \min_{u\in\mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(u).$$

Remark 3.7. Let $u_0 \in \mathbb{K}(\Omega)$ be any global minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. Using (3.2), we see that

$$0 = \mathcal{J}_{f,g,\lambda,\Omega}(0) \geq \mathcal{J}_{f,g,\lambda,\Omega}(u_0) \geq \frac{\gamma}{2}\|u_0\|^2_{H^1(\Omega)} - \frac{2|\Omega|\|f_+\|^2_{L^\infty(\Omega)}}{\gamma},$$

which shows that $\|u_0\|_{H^1(\Omega)} \leq C$ for some constant C independent of u_0. In particular, the set of minimizers of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ is compact in $L^2(\Omega)$.

Remark 3.8. From (3.2), we know that if $f \leq 0$ in Ω, then the minimum is zero and attained only by $u = 0$.

Remark 3.9. If the set $\{f > 0\} \cap \{g = 0\}$ has non-empty interior in Ω, then $\mathcal{J}_{f,g,\lambda,\Omega}(t\phi)$ is negative for any non-trivial $\phi \in C_c^\infty(\{f > 0\} \cap \{g = 0\} \cap \Omega)$ with $t > 0$ sufficiently small. Consequently, we have

$$\inf_{u \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(u) < 0,$$

and then all minimizers are non-trivial.

4. The Euler–Lagrange equation

In order to generalize some of our results, we introduce the following definition:

Definition 4.1. Let Ω be an open set in $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. A function $u_* \in \mathbb{K}(\Omega)$ is called a *local minimizer* of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ if there exists $\varepsilon > 0$ such that

$$\mathcal{J}_{f,g,\lambda,\Omega}(u_*) \leq \mathcal{J}_{f,g,\lambda,\Omega}(u)$$

for all $u \in \mathbb{K}(\Omega)$ with

$$\int_\Omega (|\nabla(u - u_*)|^2 + |\chi_{\{u>0\}} - \chi_{\{u_*>0\}}|)\, dx < \varepsilon. \tag{4.1}$$

Clearly, each (global) minimizer is also a local minimizer. We first prove the next proposition, which is an extension of [18, Lemma 2.2]. In Proposition B.4, we give an extension of [18, Theorem 2.3].

Proposition 4.2. *Let $f, g \in L^\infty(\mathbb{R}^n)$ be such that $g \geq 0$. Let Ω be an open set in $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. If $u_* \in \mathbb{K}(\Omega)$ is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then*[2]

$$\Delta u_* \geq -(f + \lambda u_*)_+ \quad \text{in } \Omega, \tag{4.2a}$$

$$\Delta u_* = -(f + \lambda u_*) \quad \text{in } \{u_* > 0\}, \tag{4.2b}$$

$$\Delta u_* \leq -(f + \lambda u_*) \quad \text{in } \Omega \setminus \operatorname{supp}(g). \tag{4.2c}$$

[2]When $\lambda \geq 0$, (4.2a) implies that $(\Delta + \lambda)u_* \geq -f_+$ in Ω.

Proof. Let $0 \le \phi \in C_c^\infty(\Omega)$. For each $\varepsilon > 0$, we define $v_\varepsilon := (u_* - \varepsilon\phi)_+$. Since $u_* \in \mathbb{K}(\Omega)$, we know that

$$v_\varepsilon \in \mathbb{K}(\Omega) \quad \text{and} \quad 0 \le v_\varepsilon \le u_* \quad \text{in } \Omega.$$

Since $u_* \in \mathbb{K}(\Omega)$ is a local minimizer,

$$\mathcal{J}_{f,g,\lambda,\Omega}(u_*) \le \mathcal{J}_{f,g,\lambda,\Omega}(v_\varepsilon) \quad \text{for all sufficiently small } \varepsilon > 0. \tag{4.3}$$

We observe that

$$\begin{aligned}
&\mathcal{J}_{f,g,\lambda,\Omega}(v_\varepsilon) - \mathcal{J}_{f,g,\lambda,\Omega}(u_*) = \int_\Omega (|\nabla v_\varepsilon|^2 - \lambda|v_\varepsilon|^2)\,dx - \int_\Omega (|\nabla u_*|^2 - \lambda|u_*|^2)\,dx \\
&\qquad - 2\int_\Omega f(v_\varepsilon - u_*)\,dx + \int_\Omega g^2(\chi_{\{v_\varepsilon>0\}} - \chi_{\{u_*>0\}})\,dx \\
&\quad = \int_{\{v_\varepsilon>0\}} |\nabla(u_* - \varepsilon\phi)|^2\,dx - \int_\Omega |\nabla u_*|^2\,dx \\
&\qquad - \lambda\Big(\int_{\{v_\varepsilon>0\}} |u_* - \varepsilon\phi|^2\,dx - \int_\Omega |u_*|^2\,dx\Big) \\
&\qquad + 2\varepsilon\int_{\{v_\varepsilon>0\}} f\phi\,dx + 2\int_{\{v_\varepsilon=0\}} f u_*\,dx - \int_{\{v_\varepsilon=0\}\cap\{u_*>0\}} |g|^2\,dx \\
&\quad = -2\varepsilon\int_{\{v_\varepsilon>0\}} \nabla u_*\cdot\nabla\phi\,dx + \varepsilon^2\int_{\{v_\varepsilon>0\}} |\nabla\phi|^2\,dx \overbrace{- \int_{\{v_\varepsilon=0\}} |\nabla u_*|^2\,dx}^{\le 0} \\
&\qquad + \overbrace{2\varepsilon\int_{\{v_\varepsilon>0\}} (f + \lambda u_*)\phi\,dx}^{\le 2\varepsilon\int_{\{v_\varepsilon>0\}}(f+\lambda u_*)_+\phi\,dx} + \overbrace{2\int_{\{v_\varepsilon=0\}} (f + \lambda u_*)u_*\,dx}^{\le 2\int_{\{v_\varepsilon=0\}}(f+\lambda u_*)_+u_*\,dx} \\
&\qquad \overbrace{-\lambda\varepsilon^2\int_{\{v_\varepsilon>0\}} |\phi|^2\,dx - \int_{\{v_\varepsilon=0\}\cap\{u_*>0\}} |g|^2\,dx}^{\le 0} \\
&\quad \le -2\varepsilon\int_{\{v_\varepsilon>0\}} \nabla u_*\cdot\nabla\phi\,dx + \varepsilon^2\int_{\{v_\varepsilon>0\}} |\nabla\phi|^2\,dx \\
&\qquad + 2\varepsilon\int_{\{v_\varepsilon>0\}} (f + \lambda u_*)_+\phi\,dx + \overbrace{2\int_{\{v_\varepsilon=0\}} (f + \lambda u_*)_+u_*\,dx}^{\le 2\varepsilon\int_{\{v_\varepsilon=0\}}(f+\lambda u_*)_+\phi\,dx} \\
&\quad \le -2\varepsilon\int_{\{v_\varepsilon>0\}} \nabla u_*\cdot\nabla\phi\,dx + 2\varepsilon\int_\Omega (f + \lambda u_*)_+\phi\,dx + \varepsilon^2\int_{\{v_\varepsilon>0\}} |\nabla\phi|^2\,dx.
\end{aligned}$$

This implies that

$$\limsup_{\varepsilon\searrow 0} \frac{1}{2\varepsilon}(\mathcal{J}_{f,g,\lambda,\Omega}(v_\varepsilon) - \mathcal{J}_{f,g,\lambda,\Omega}(u_*)) \le -\int_\Omega \nabla u_*\cdot\nabla\phi\,dx + \int_\Omega (f + \lambda u_*)_+\phi\,dx.$$

Combining this inequality with (4.3), we conclude with (4.2a).

If $\operatorname{supp}(\phi) \subset \{u_* > 0\}$, we define $v_\varepsilon^\pm = u_* \pm \varepsilon\phi \in \mathbb{K}(\Omega)$ for small $\varepsilon > 0$. Note that $\{v_\varepsilon^\pm > 0\} = \{u_* > 0\}$ for small $\varepsilon > 0$. Therefore, for small $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathcal{J}_{f,g,\lambda,\Omega}(v_\varepsilon^\pm) - \mathcal{J}_{f,g,\lambda,\Omega}(u_*) \\
&\quad = \int_{\{u_*>0\}} (|\nabla v_\varepsilon^\pm|^2 - \lambda |v_\varepsilon^\pm|^2)\,dx - \int_{\{u_*>0\}} (|\nabla u_*|^2 - \lambda |u_*|^2)\,dx \\
&\qquad - 2\int_{\{u_*>0\}} f(v_\varepsilon^\pm - u_*)\,dx \\
&\quad = \pm 2\varepsilon\Big(\int_{\{u_*>0\}} \nabla u_* \cdot \nabla\phi\,dx - \lambda \int_{\{u_*>0\}} u_*\phi\,dx\Big) \mp 2\varepsilon \int_{\{u_*>0\}} f\phi\,dx \\
&\qquad + \varepsilon^2\Big(\int_{\{u_*>0\}} |\nabla\phi|^2\,dx - \lambda \int_{\{u_*>0\}} |\phi|^2\,dx\Big).
\end{aligned}$$

Using (4.3) and dividing both sides of the inequality above by ε and letting $\varepsilon \to 0$, we conclude with (4.2b).

If $\operatorname{supp}(\phi) \cap \operatorname{supp}(g) = \emptyset$, taking $\tilde{v}_\varepsilon = u_* + \varepsilon\phi$, we have

$$\begin{aligned}
&\mathcal{J}_{f,g,\lambda,\Omega}(\tilde{v}_\varepsilon) - \mathcal{J}_{f,g,\lambda,\Omega}(u_*) \\
&\quad = \int_\Omega (|\nabla \tilde{v}_\varepsilon|^2 - \lambda|\tilde{v}_\varepsilon|^2)\,dx - \int_\Omega (|\nabla u_*|^2 - \lambda|u_*|^2)\,dx - 2\int_\Omega f(\tilde{v}_\varepsilon - u_*)\,dx \\
&\quad = 2\varepsilon\Big(\int_{\Omega\setminus\operatorname{supp}(g)} \nabla u_* \cdot \nabla\phi\,dx - \lambda\int_{\Omega\setminus\operatorname{supp}(g)} u_*\phi\,dx\Big) - 2\varepsilon\int_{\Omega\setminus\operatorname{supp}(g)} f\phi\,dx \\
&\qquad + \varepsilon^2\Big(\int_\Omega |\nabla\phi|^2\,dx - \lambda\int_\Omega |\phi|^2\,dx\Big).
\end{aligned}$$

Using (4.3) and dividing both sides of the inequality above by ε and letting $\varepsilon \to 0$, we conclude with (4.2c). ■

Remark 4.3. We now give some observations when $\{u_* > 0\}$ has Lipschitz boundary. Using integration by parts on the Lipschitz domain $\{u_* > 0\}$ and from (4.2b), we have

$$\mathcal{J}_{f,g,\lambda,\Omega}(u_*) = \int_\Omega (g^2\chi_{\{u_*>0\}} - fu_*)\,dx. \tag{4.4}$$

If u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ with $\mathcal{J}_{f,g,\lambda,\Omega}(u_*) < 0$, using (4.4) we know that

$$\int_{\{u_*>0\}\cap\{f>0\}} fu_*\,dx \equiv \int_\Omega fu_*\,dx > \int_\Omega g^2\chi_{\{u_*>0\}}\,dx \geq 0,$$

which immediately implies $|\{u_* > 0\} \cap \{f > 0\}| > 0$. If we additionally assume that $g \equiv 0$ and $\lambda \geq 0$, from (4.2c), we know that $(\Delta + \lambda)u_* + f \leq 0$ in Ω. From this, we know that

$$\Delta u_* \leq -f - \lambda u_* \leq 0 \quad \text{in } \{f > 0\} \cap \Omega,$$

because $u_* \geq 0$ in Ω. Using the strong minimum principle for super-solutions (as formulated in [15, Theorem 8.19]), we know that $u_* > 0$ in $\{f > 0\} \cap \Omega$.

5. Comparison of minimizers

The main purpose of this section is to prove the L^∞-regularity of the minimizers. We first prove the following comparison principle, which is analogous to [18, Lemma 1.1]:

Proposition 5.1. *For $f_m, g_m \in L^\infty(\mathbb{R}^n)$ with $g_m \geq 0$ for each $m = 1, 2$, suppose*

$$f_1 \leq f_2, \quad g_1 \geq g_2 \geq 0 \quad \text{in } \mathbb{R}^n. \tag{5.1}$$

Assume further that $-\infty < \lambda_1 \leq \lambda_2 < +\infty$ and that $\Omega_m \subset \mathbb{R}^n$ are open sets satisfying $\Omega_1 \subset \Omega_2$. For each $u_m \in \mathbb{K}(\Omega_m)$, we define $v := \min\{u_1, u_2\}$ and $w := \max\{u_1, u_2\}$. Then, $v \in \mathbb{K}(\Omega_1)$, $w \in \mathbb{K}(\Omega_2)$ satisfy

$$\mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}(v) + \mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}(w) \leq \mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}(u_1) + \mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}(u_2).$$

We also have the following statements:

(1) *If u_1 is a (global) minimizer of $\mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}$ in $\mathbb{K}(\Omega_1)$, then*

$$\mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}(w) \leq \mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}(u_2).$$

(2) *If u_2 is a (global) minimizer of $\mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}$ in $\mathbb{K}(\Omega_2)$, then*

$$\mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}(v) \leq \mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}(u_1).$$

(3) *If each u_m is a (global) minimizer of $\mathcal{J}_{f_m,g_m,\lambda_m,\Omega_m}$ in $\mathbb{K}(\Omega_m)$, then v is a (global) minimizer of $\mathcal{J}_{f_1,g_1,\lambda_1,\Omega_1}$ in $\mathbb{K}(\Omega_1)$, while w is a (global) minimizer of $\mathcal{J}_{f_2,g_2,\lambda_2,\Omega_2}$ in $\mathbb{K}(\Omega_2)$.*

Remark 5.2. When $\Omega_1 = \Omega_2 = \Omega$, we only need to assume (5.1) in Ω.

Proof of Proposition 5.1. We first show that, if $\Phi(t)$ is a non-decreasing function of $t \in [0, \infty)$ and $h_1 \leq h_2$ in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} (h_1\Phi(u_1) + h_2\Phi(u_2))\,dx \leq \int_{\mathbb{R}^n} (h_1\Phi(v) + h_2\Phi(w))\,dx. \tag{5.2}$$

Indeed, by definition, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} h_1(\Phi(u_1) - \Phi(v))\,dx &= \int_{\{u_1 \geq u_2\}} h_1(\Phi(u_1) - \Phi(u_2))\,dx \\
&\leq \int_{\{u_1 \geq u_2\}} h_2(\Phi(u_1) - \Phi(u_2))\,dx \\
&= \int_{\mathbb{R}^n} h_2(\Phi(w) - \Phi(u_2))\,dx,
\end{aligned}$$

where we used that $\Phi(u_1) - \Phi(u_2) \geq 0$ in $\{u_1 \geq u_2\}$. Hence, we conclude (5.2).

From (5.2), we find that

$$\int_{\mathbb{R}^n} (f_1 u_1 + f_2 u_2)\, dx \le \int_{\mathbb{R}^n} (f_1 v + f_2 w)\, dx \quad (\text{choosing } h_j = f_j \text{ and } \Phi(t) = t),$$
$$\int_{\mathbb{R}^n} (k_1^2 u_1^2 + k_2^2 u_2^2)\, dx \le \int_{\mathbb{R}^n} (k_1^2 v^2 + k_2^2 w^2)\, dx \quad (\text{choosing } h_j = k_j^2 \text{ and } \Phi(t) = t^2).$$

On the other hand, choosing $h_j = -g_j^2$ and

$$\Phi(t) = \begin{cases} 0 & t \le 0, \\ 1 & t > 0, \end{cases}$$

we obtain

$$\int_{\mathbb{R}^n} (g_1^2 \chi_{\{u_1>0\}} + g_2^2 \chi_{\{u_2>0\}})\, dx \ge \int_{\mathbb{R}^n} (g_1^2 \chi_{\{v>0\}} + g_2^2 \chi_{\{w>0\}})\, dx.$$

By observing that

$$\int_{\mathbb{R}^n} (|\nabla u_1|^2 + |\nabla u_2|^2)\, dx = \int_{\mathbb{R}^n} (|\nabla v|^2 + |\nabla w|^2)\, dx,$$

we conclude the proof by putting these inequalities together. ■

We now prove the following lemma using Propositions 3.6 and 5.1:

Lemma 5.3. *Let $f, g \in L^\infty(\mathbb{R}^n)$ be such that $g \ge 0$. Let Ω be a bounded domain with C^1 boundary. Assume that u_0 is a non-trivial (global) minimizer of $\mathcal{J}_{f,g,\lambda_0,\Omega}$ in $\mathbb{K}(\Omega)$ with $-\infty < \lambda_0 < \lambda^*(\Omega)$. Then, for each λ with $\lambda_0 < \lambda < \lambda^*(\Omega)$, there exists a non-trivial (global) minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. In addition, any (global) minimizer u_* of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ satisfies*

$$u_* \ge u_0 \quad \text{in } \Omega, \tag{5.3a}$$
$$u_* > u_0 \quad \text{in } \{u_0 > 0\}. \tag{5.3b}$$

Proof. We first show existence of non-trivial minimizer. Using Proposition 3.6, there exists a (global) minimizer u_* of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. Note that

$$\inf_{u \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(u) \le \mathcal{J}_{f,g,\lambda,\Omega}(u_0) = \mathcal{J}_{f,g,\lambda_0,\Omega}(u_0) - (\lambda - \lambda_0)\|u_0\|_{L^2(\Omega)}^2.$$

Since u_0 is non-trivial, $\|u_0\|_{L^2(\Omega)}^2 > 0$. Therefore, using (1.5), we have

$$\inf_{u \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(u) < 0,$$

which shows that 0 is not a minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}(u)$ in $\mathbb{K}(\Omega)$. Consequently, $u_* \not\equiv 0$.

To prove (5.3a), we let u_* be any (global) minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}(u)$ in $\mathbb{K}(\Omega)$. Using Proposition 5.1, we know that $w := \max\{u_0, u_*\}$ is also a (global) minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}(u)$ in $\mathbb{K}(\Omega)$. Using (4.2b) in Proposition 4.2, we know that

$$\begin{aligned} \Delta u_0 &= -f - \lambda_0 u_0 && \text{in } \{u_0 > 0\}, \\ \Delta w &= -f - \lambda w && \text{in } \{w > 0\}. \end{aligned}$$

Note that $u_0 = w$ in $\{u_0 > u_*\}$ and $\{u_0 > u_*\} \subset \{w > 0\} \cap \{u_0 > 0\}$; then, restricting the above two identities in $\{u_0 > u_*\}$ yields

$$\Delta w + \lambda w = \Delta w + \lambda_0 w = -f \quad \text{in } \{u_0 > u_*\}.$$

Since $\lambda_0 < \lambda$,

$$w = u_0 = 0 \quad \text{in } \{u_0 > u_*\};$$

hence, we know that $|\{u_0 > u_*\}| = 0$, which allows us to conclude with (5.3a).

Finally, to prove (5.3b), we define $v = u_* - u_0 \geq 0$ in B_R and note that

$$\begin{aligned} \Delta v &= -(\lambda - \lambda_0) v \leq 0 && \text{in } \{u_0 > 0\}, \\ v &\geq 0 && \text{on } \partial\{u_0 > 0\}. \end{aligned}$$

Using the strong minimum principle for super-solutions (as formulated in [15, Theorem 8.19]), we know that $u > u_0$ in $\{u_0 > 0\}$, because $u \not\equiv u_0$, which allows us to conclude with (5.3b). ∎

We now prove the minimizer is unique for all except countably many λ.

Proposition 5.4. *We assume that Ω is bounded with C^1 boundary and $-\infty < \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ with $g \geq 0$. Then, there exist smallest and largest (in the pointwise sense) minimizers of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. We define the functions*

$$\begin{aligned} m(\lambda) &:= \min\Big\{\|u\|_{L^2(\Omega)} \;\Big|\; \mathcal{J}_{f,g,\lambda,\Omega}(u) = \inf_{v \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(v)\Big\} \\ &\qquad \textit{for all } -\infty < \lambda < \lambda^*(\Omega), \\ M(\lambda) &:= \max\Big\{\|u\|_{L^2(\Omega)} \;\Big|\; \mathcal{J}_{f,g,\lambda,\Omega}(u) = \inf_{v \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(v)\Big\} \\ &\qquad \textit{for all } -\infty < \lambda < \lambda^*(\Omega). \end{aligned}$$

Then, the functions

$$m : (-\infty, \lambda^*(\Omega)) \to \mathbb{R}, \quad M : (-\infty, \lambda^*(\Omega)) \to \mathbb{R}$$

are strictly increasing. Moreover, we have

$$M(\lambda - \varepsilon) < m(\lambda) \quad \textit{for all } -\infty < \lambda < \lambda^*(\Omega) \textit{ and } \varepsilon > 0. \tag{5.4}$$

Consequently, there exists a countable set $Z \subset (-\infty, \lambda^*(\Omega))$ such that the minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$ is unique for all $\lambda \in (-\infty, \lambda^*(\Omega)) \setminus Z$.

Proof. From Remark 3.7, we know that the set of minimizers is compact in $L^2(\Omega)$. Since $L^2(\Omega)$ is separable, there exists a countable dense set in the set of minimizers. Taking the pointwise supremum, as well as the pointwise infimum, in this countable set produces two new minimizers. This proves the first part of the proposition, and hence, the functions m and M are well-defined.

The strict monotonicity of m and M follows from Lemma 5.3. Choosing $\lambda_0 = \lambda - \varepsilon$ in Lemma 5.3, we also know that any minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ is larger than all minimizers of $\mathcal{J}_{f,g,\lambda-\varepsilon,\Omega}$ (in particular, the largest one), and we conclude with (5.4). The final claim follows from (5.4) and the fact that monotone functions are continuous except for a countable set of jump discontinuities. ∎

Lemma 5.5. *We assume that Ω is bounded with C^1 boundary. Let $f, g \in L^\infty(\Omega)$ with $g \geq 0$, and define*

$$\Phi(\lambda) := \inf_{v \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(v) \quad \textit{for all } \lambda \in (-\infty, \lambda^*(\Omega)).$$

Then, $\Phi : (-\infty, \lambda^*(\Omega)) \to (-\infty, 0]$ is concave and, hence, continuous. In addition, if $\Phi(\lambda_0) < 0$, then it is strictly decreasing near $\lambda = \lambda_0$.

Proof. Fix $-\infty < \lambda_0 < \lambda^*(\Omega)$ and let $u_0 \in \mathbb{K}(\Omega)$ be such that

$$\mathcal{J}_{f,g,\lambda_0,\Omega}(u_0) = \inf_{v \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda_0,\Omega}(v) \equiv \Phi(\lambda_0).$$

For each $\lambda \in (-\infty, \lambda^*(\Omega))$, we have

$$\begin{aligned} \Phi(\lambda) &= \inf_{v \in \mathbb{K}(\Omega)} \mathcal{J}_{f,g,\lambda,\Omega}(v) \leq \mathcal{J}_{f,g,\lambda,\Omega}(u_0) = \mathcal{J}_{f,g,\lambda_0,\Omega}(u_0) - (\lambda - \lambda_0)\|u_0\|_{L^2(\Omega)}^2 \\ &= \Phi(\lambda_0) - (\lambda - \lambda_0)\|u_0\|_{L^2(\Omega)}^2, \end{aligned}$$

which proves the claimed concavity. In addition, if $\Phi(\lambda_0) < 0$, then $u_0 \not\equiv 0$ (since $\mathcal{J}_{f,g,\lambda_0,\Omega}(0) = 0$), which implies that the function is strictly decreasing near $\lambda = \lambda_0$. ∎

We now prove the L^∞-regularity of the minimizers.

Proposition 5.6. *We assume that Ω is bounded with C^1 boundary and $-\infty < \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ with $g \geq 0$. Let $v \in H_0^1(\Omega)$ be the unique solution of $(\Delta + \lambda)v = -1$ in Ω. If $u_* \in \mathbb{K}(\Omega)$ is a global minimizer $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then*

$$0 \leq u_*(x) \leq \|f_+\|_{L^\infty(\Omega)} v(x) \quad \textit{for all } x \in \Omega.$$

Remark 5.7. Using [15, Theorem 8.15], we also know that $v \in L^\infty(\Omega)$. Therefore, we know that $\|u_*\|_{L^\infty(\Omega)} \leq C(\lambda, \Omega)\|f_+\|_{L^\infty(\Omega)}$.

Proof of Proposition 5.6. In view of Remark 3.8, we only need to consider the case when $\|f_+\|_{L^\infty(\Omega)} > 0$. We define $v_0 := \|f_+\|_{L^\infty(\Omega)} v$, which is a minimizer of $\mathcal{J}_{\|f_+\|_{L^\infty(\Omega)},0,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. Using Proposition 5.1, we know that $\max\{u_*, v_0\}$ is also a minimizer of $\mathcal{J}_{\|f_+\|_{L^\infty(\Omega)},0,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. By (4.2a) and (4.2c) in Proposition 4.2, it follows that both v_0 and $\max\{u_*, v_0\}$ satisfy

$$\begin{cases} (\Delta + \lambda)u = -\|f_+\|_{L^\infty(\Omega)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Since $-\infty < \lambda < \lambda^*(\Omega)$, the solution of (5.5) is unique. The uniqueness of solution of (5.5) implies

$$v_0 = \max\{u_*, v_0\} \quad \text{in } \Omega,$$

which concludes the pointwise bound. ∎

6. Some properties of local minimizers

We shall study the regularity of local minimizers and obtain some consequences for the case when $\lambda \geq 0$.

Lemma 6.1. *Let Ω be an open set in $\mathbb{R}^n$, and let $\lambda \geq 0$. Let $f, g \in L^\infty(\Omega)$ be such that $g \geq 0$. If u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then it is also a local minimizer of $\mathcal{J}_{f+\lambda u_*,g,0,\Omega}$ in $\mathbb{K}(\Omega)$.*

Proof. Write $\tilde{f} = f + \lambda u_*$. For each $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} \mathcal{J}_{\tilde{f},g,0,\Omega}(v) &= \int_\Omega (|\nabla v|^2 - 2\tilde{f}v + g^2\chi_{\{v>0\}})\,dx = \mathcal{J}_{f,g,\lambda,\Omega}(v) + \lambda \int_\Omega (v^2 - 2u_*v)\,dx \\ &= \mathcal{J}_{f,g,\lambda,\Omega}(v) + \lambda \int_\Omega (v - u_*)^2\,dx - \lambda \int_\Omega u_*^2\,dx. \end{aligned}$$

Since $\lambda \geq 0$, we see that

$$\mathcal{J}_{\tilde{f},g,0,\Omega}(v) \geq \mathcal{J}_{f,g,\lambda,\Omega}(u_*) - \lambda \int_\Omega u_*^2\,dx = \mathcal{J}_{\tilde{f},g,0,\Omega}(u_*), \tag{6.1}$$

and the equality holds in (6.1) if and only if $v = u_*$. Hence, we conclude the proof. ∎

With this lemma at hand, one can prove that the minimizer u_* is Lipschitz continuous, as well as some results analogous to those in [18, Sections 2 and 5], by using the corresponding results in [18] where one just replaces $f \in L^\infty(\Omega)$ with $f + \lambda u_* \in L^\infty(\Omega)$ (see Proposition 5.6). This works, since the proofs in [18] only rely on variations of u_* locally. The detailed statements and proofs can be found in Appendix B. Here we highlight some results which we will use later. The following proposition concerns the PDE characterization of the minimizer u_*:

Proposition 6.2. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \le \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ be such that $g \ge 0$ and $g^2 \in W^{1,1}(\Omega)$. Suppose that u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. If $\partial\{g > 0\} \cap \Omega \neq \emptyset$, we further assume that there exists $0 < \alpha \le 1$ such that g is C^α near $\partial\{g > 0\} \cap \Omega$ and $\mathcal{H}^{n-1+\alpha}(\partial\{g > 0\} \cap \Omega) = 0$. We assume that $\overline{\{u_* > 0\}} \subset \Omega$. Then, $\{u_* > 0\}$ has locally finite perimeter in $\{g > 0\}$,*

$$\mathcal{H}^{n-1}((\partial\{u_* > 0\} \setminus \partial_{\mathrm{red}}\{u_* > 0\}) \cap \{g > 0\}) = 0, \tag{6.2}$$

and

$$(\Delta + \lambda)u_* + f\mathcal{L}^n\lfloor\{u_* > 0\} = g\mathcal{H}^{n-1}\lfloor\partial\{u_* > 0\} = g\mathcal{H}^{n-1}\lfloor\partial_{\mathrm{red}}\{u_* > 0\}. \tag{6.3}$$

The following proposition concerns the regularity of the reduced free boundary $\partial_{\mathrm{red}}\{u_* > 0\}$:

Proposition 6.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \le \lambda < \lambda^*(\Omega)$. Let f, g, and u_* be functions given in Proposition 6.2. If there exists a ball $B_r(x_0) \subset \Omega$ such that g is Hölder continuous and satisfies $g \ge \text{constant} > 0$ in $B_r(x_0)$, then $\partial_{\mathrm{red}}\{u_* > 0\}$ is locally $C^{1,\alpha}$ in such a ball $B_r(x_0)$, and in the case when $n = 2$, we even have $\partial_{\mathrm{red}}\{u_* > 0\} = \partial\{u_* > 0\}$.*

Remark 6.4. If $g > 0$ is Hölder continuous in Ω, together with (6.2), we then know that $\partial_{\mathrm{red}}\{u_* > 0\}$ is locally $C^{1,\alpha}$ with $\mathcal{H}^{n-1}(\partial\{u_* > 0\} \setminus \partial_{\mathrm{red}}\{u_* > 0\}) = 0$.

7. Relation with hybrid quadrature domains

We now obtain the following simple lemma:

Lemma 7.1. *Suppose the assumptions in Proposition 6.2 hold. We write $\lambda = k^2$. If we further assume that $\overline{\{u_* > 0\}} \subset \Omega$ and $f = \mu - h\chi_{\{u_*>0\}} \in L^\infty(\Omega)$ for some $\mu \in \mathcal{E}'(\{u_* > 0\})$ and $h \in L^\infty(\{u_* > 0\})$, then $(\Psi_k * (g\mathcal{H}^{n-1}\lfloor\partial\{u_* > 0\}))(x)$ is pointwise well-defined for all $x \in \partial\{u_* > 0\}$. Moreover, we also know that $\{u_* > 0\}$ is a hybrid k-quadrature domain (Definition 1.1), corresponding to distribution μ and density (g, h).*

Proof. Let $\Psi_k \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ be any fundamental solution of the Helmholtz operator $-(\Delta + k^2)$ and let $D = \{u_* > 0\}$. By the properties of convolution for distributions and by (6.3), we have

$$\begin{aligned} u_* &= \delta_0 * u_* = -(\Delta + k^2)\Psi_k * u_* \\ &= -\Psi_k * (\Delta + k^2)u_* = \Psi_k * (f\mathcal{L}^n\lfloor D - g\mathcal{H}^{n-1}\lfloor\partial D). \end{aligned} \tag{7.1}$$

By using the fact $u_* \in C^{0,1}_{\mathrm{loc}}(\Omega)$ (see Appendix B) and the assumption $\overline{\{u_* > 0\}} \subset \Omega$, from (7.1) we conclude the first result. The second result immediately follows from the observation $u_* = 0$ in $\mathbb{R}^n \setminus \{u_* > 0\}$. ∎

Finally, we want to show that there exists some μ so that

$$\operatorname{supp}(\mu) \subset \{u_* > 0\} \quad \text{and} \quad \overline{\{u_* > 0\}} \subset \Omega .$$

We first study a particular radially symmetric case (the case when $\lambda = 0$ was considered in [18, Lemma 1.2]).

Lemma 7.2. *Let $\Omega = B_R$ with $R > 0$ and $0 < \lambda < \lambda^*(B_R) \equiv j_{\frac{n-2}{2},1}^2 R^{-2}$. Suppose that*

$$f = a\chi_{B_{r_1}} - b \quad \text{with} \quad a > b > 0 \quad \text{and} \quad 0 < r_1 < R,$$

and let g be a radially non-decreasing function g with $g = 0$ in $\overline{B_{r_1}}$. Then, there exists $u_* \in \mathbb{K}(B_R)$ such that

$$\mathcal{J}_{f,g,\lambda,B_R}(u_*) = \inf_{v \in \mathbb{K}(B_R)} \mathcal{J}_{f,g,\lambda,B_R}(v) < 0.$$

Moreover, the following hold:

(1) *Any global minimizer u_* of $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$ is continuous, radially symmetric, and radially non-increasing, and satisfies*

$$\overline{B_{r_1}} \subset \{u_* > 0\}.$$

(2) *If we set*

$$R' = \max\left\{\rho \in (r_1, R] \mid \frac{b}{a} - \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{\rho^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda}\rho)} \leq 0\right\} > r_1, \tag{7.2}$$

then u_* has support in the ball $\overline{B_{R'}}$. In particular, $R' < R$, whenever

$$\frac{b}{a} > \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} R)}.$$

Proof. The existence of minimizers was established in Proposition 3.6. Since

$$B_{r_1} \subset \{f > 0\} \cap \{g = 0\} \cap B_R,$$

from Remark 3.9, we know that all minimizers are non-trivial.

Step 1: Rearrangement. Given any $u \in \mathbb{K}(B_R)$, let u^{rad} denote its radially symmetric decreasing rearrangement, that is,

$$u^{\text{rad}}(x) := \int_0^\infty \chi_{\{u>t\}^{\text{rad}}}(x)\, dt,$$

where $A^{\text{rad}} = \{x \in \mathbb{R}^n \mid \omega_n |x|^n < |A|\}$ for any measurable set $A \subset \mathbb{R}^n$. Then, $u^{\text{rad}} \in \mathbb{K}(B_R)$ and

$$\int_{B_R} |u^{\text{rad}}|^2\, dx = \int_{B_R} |u|^2\, dx, \tag{7.3a}$$

$$\int_{B_R} |\nabla u^{\mathrm{rad}}|^2 \, dx \leq \int_{B_R} |\nabla u|^2 \, dx, \tag{7.3b}$$

$$\int_{B_R} f u^{\mathrm{rad}} \, dx \geq \int_{B_R} f u \, dx, \tag{7.3c}$$

$$\int_{B_R} |g|^2 \chi_{\{u^{\mathrm{rad}}>0\}} \, dx \leq \int_{B_R} |g|^2 \chi_{\{u>0\}} \, dx. \tag{7.3d}$$

Here (7.3b) is the classical Pólya–Szegő inequality [3, Theorem 1.1], and (7.3c) and (7.3d) follow by the fact that f is non-increasing and g is non-decreasing as functions of $r = |x|$. It follows that

$$\mathcal{J}_{f,g,\lambda,B_R}(u^{\mathrm{rad}}) \leq \mathcal{J}_{f,g,\lambda,B_R}(u). \tag{7.4}$$

We define

$$\mathbb{K}^{\mathrm{rad}}(B_R) = \{u \in \mathbb{K}(B_R) \mid u = u^{\mathrm{rad}}\}.$$

Using (7.4), there exists $u_*^{\mathrm{rad}} \in \mathbb{K}^{\mathrm{rad}}(B_R)$ such that

$$\mathcal{J}_{f,g,\lambda,B_R}(u_*^{\mathrm{rad}}) = \inf_{v \in \mathbb{K}(B_R)} \mathcal{J}_{f,g,\lambda,B_R}(v). \tag{7.5}$$

Step 2: Minimizers in $\mathbb{K}^{\mathrm{rad}}(B_R)$. Let $\widetilde{u} \in \mathbb{K}^{\mathrm{rad}}(B_R)$ be any function such that

$$\mathcal{J}_{f,g,\lambda,B_R}(\widetilde{u}) = \inf_{v \in \mathbb{K}(B_R)} \mathcal{J}_{f,g,\lambda,B_R}(v).$$

From (4.2b) in Proposition 4.2, we know that $\widetilde{u}$ satisfies the equation

$$(\Delta + \lambda)\widetilde{u} + f = 0 \quad \text{in } \{\widetilde{u} > 0\}.$$

In polar coordinates, the above equation reads as

$$|\widetilde{u}(0)| < \infty, \quad \widetilde{u}''(r) + \frac{n-1}{r}\widetilde{u}'(r) + \lambda\widetilde{u}(r) + a\chi_{\{r<r_1\}} - b = 0 \quad \text{for } r \in (0,\rho), \tag{7.6a}$$

with $\widetilde{u}'(r) \leq 0$ for all $r \in (0,\rho)$ and $\widetilde{u}(r) = 0$ for all $r \geq \rho$, where $\rho \in (0, R]$. In addition, one has (see Proposition B.4)

$$\widetilde{u}'(\rho) = -g(\rho) \leq 0, \tag{7.6b}$$

and $\widetilde{u}$ is the unique solution of ODE system (7.6a)–(7.6b).

We now compute an explicit formula for $\widetilde{u}$. Let u be the unique solution of

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + \lambda u(r) + a\chi_{\{r<r_1\}} - b = 0 & \text{for } r \in (\rho,\infty), \\ u(\rho) = \widetilde{u}(\rho), \quad u'(\rho) = \widetilde{u}'(\rho). \end{cases}$$

By defining $u|_{(0,\rho)} = \widetilde{u}$, one sees that $u \in C^1_{\mathrm{loc}}(\mathbb{R})$ and

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda u(r) + a\chi_{\{r<r_1\}} - b = 0 \quad \text{for } r \in (0,\infty). \tag{7.7}$$

By direct computations (see Appendix C for details), one sees that the general solution of (7.7) is

$$u(r) = \frac{b-a}{\lambda} + c_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) + \chi_{\{r>r_1\}}\Big[\frac{a}{\lambda} + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} r^{\frac{2-n}{2}} \big(Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) J_{\frac{n-2}{2}}(\sqrt{\lambda} r) - J_{\frac{n}{2}}(\sqrt{\lambda} r_1) Y_{\frac{n-2}{2}}(\sqrt{\lambda} r)\big)\Big] \tag{7.8}$$

with $c_1 \in \mathbb{R}$. Since $u = \tilde{u}$ is positive and decreasing near 0, we have $c_1 > 0$. By direct computations (see Appendix C for details), one sees that there exists a zero $\rho_0 \in (0, R]$ of u such that

$$u \text{ is positive and non-increasing on } (0, \rho_0); \tag{7.9}$$

therefore, $\rho_0 = \rho$, where ρ is the constant given in (7.6a). We now impose the boundary condition $u'(\rho) = -g(\rho)$. Using assumptions on g, direct computations (see Appendix C for details) yield

$$\rho \in (r_1, R'), \quad \text{where } R' \text{ is given in (7.2).} \tag{7.10}$$

From this, we conclude that $\overline{B_{r_1}} \subset \{\tilde{u} > 0\}$ as well as $\operatorname{supp}(\tilde{u}) = \overline{B_\rho} \subset B_{R'}$.

Step 3: All minimizers belong to $\mathbb{K}^{\mathrm{rad}}(B_R)$. Let $u_* \in \mathbb{K}(B_R)$ be a minimizer of $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$. Using (7.4), we see that its radially symmetric decreasing rearrangement $u_*^{\mathrm{rad}} \in \mathbb{K}^{\mathrm{rad}}(B_R)$ satisfies (7.5), that is, u_*^{rad} is one of our radial solutions, and we have

$$\int_{B_R} |\nabla u_*^{\mathrm{rad}}|^2 \, dx = \int_{B_R} |\nabla u_*|^2 \, dx.$$

Since the radial solutions are radially strictly decreasing on the positivity set, we deduce that u_*^{rad} is strictly decreasing on $(0, \rho)$ with

$$\operatorname{supp}(u_*^{\mathrm{rad}}) = \overline{B_\rho}.$$

Therefore, from [3, Theorem 1.1], we know that

$$u_*(x) = u_*^{\mathrm{rad}}(x - x_0) \quad \text{for some } x_0.$$

Now, by way of contradiction, suppose that $x_0 \neq 0$. Since u_*^{rad} satisfies (7.5), Proposition 5.1 tells us that $w = \max\{u_*, u_*^{\mathrm{rad}}\}$ does also, but w is not radially decreasing around some x_0, which contradicts the minimality of u_*. ■

Remark 7.3. If $r_1 = R$, from the general solution and the boundary condition $\tilde{u}(\rho) = 0$, we know that

$$\tilde{u}(r) = \frac{b-a}{k^2}\left(1 - \frac{r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r)}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)}\right)\chi_{\{r<\rho\}} \quad \text{for some } \rho \in [0, R].$$

Since $\{\widetilde{u} > 0\}$ is a Lipschitz domain, using Remark 4.3, we compute that

$$\begin{aligned}
\mathcal{J}_{f,g,\lambda,B_R}(\widetilde{u}) &= \int_{B_\rho} g^2\,dx - (a-b)\int_{B_\rho} \widetilde{u}\,dx \\
&= \int_{B_\rho} g^2\,dx + \frac{(a-b)^2}{\lambda}\Big(|B_\rho| - \frac{1}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(k\rho)}|B_1|\int_0^\rho J_{\frac{n-2}{2}}(kr) r^{\frac{n}{2}}\,dr\Big) \\
&= \int_{B_\rho} g^2\,dx + \frac{(a-b)^2}{\lambda}\Big(|B_\rho| - \frac{1}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(k\rho)}|B_1|\frac{\rho^{\frac{n}{2}} J_{\frac{n}{2}}(k\rho)}{k}\Big) \\
&= \int_{B_\rho} g^2\,dx + \frac{(a-b)^2}{\lambda}|B_\rho|\Big(1 - \frac{J_{\frac{n}{2}}(k\rho)}{k\rho J_{\frac{n-2}{2}}(k\rho)}\Big) \\
&= \int_{B_\rho} g^2\,dx + \frac{(a-b)^2}{\lambda}|B_\rho|\Big(1 - \frac{\Gamma(\frac{n}{2})}{2\Gamma(1+\frac{n}{2})}\Big) \\
&= \int_{B_\rho} g^2\,dx + \frac{(a-b)^2}{\lambda}|B_\rho|(n-1)\frac{\Gamma(\frac{n}{2})}{2\Gamma(1+\frac{n}{2})}.
\end{aligned}$$

Since $a > b$, from (1.5), we conclude that $\rho = 0$, that is, $\widetilde{u} \equiv 0$ in B_R. Since all minimizers belong to $\mathbb{K}^{\mathrm{rad}}(B_R)$, in this case each minimizer of $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$ must be trivial.

Combining Lemma 7.2 with the comparison principle (Proposition 5.1), we have the following proposition:

Proposition 7.4. *Let $\Omega = B_R$ with $R > 0$ and $0 < \lambda < \lambda^*(B_R) \equiv j^2_{\frac{n-2}{2},1} R^{-2}$. Suppose that $f = \mu - h$ with*

$$a_0\chi_{B_{r_1}} \le \mu(x) \le a\chi_{B_{r_2}}, \quad b \le h(x) \le b_0 \quad \text{for all } x \in B_R \tag{7.11}$$

for some constants r_1, r_2, a, a_0, b, b_0 satisfying

$$\begin{gathered}
0 < b \le b_0 < a_0 \le a, \quad 0 < r_1 \le r_2 < R, \\
\frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_2)} > \frac{b_0}{a_0} \ge \frac{b}{a} > \frac{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_2)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} R)}.
\end{gathered} \tag{7.12}$$

We also assume that $g \in L^\infty(\mathbb{R}^n)$ with

$$g = 0 \quad \text{in } \operatorname{supp}(\mu) \equiv \overline{B_{r_1}}.$$

There exists u_* such that

$$\mathcal{J}_{f,g,\lambda,B_R}(u_*) = \inf_{u\in\mathbb{K}(B_R)} \mathcal{J}_{f,g,\lambda,B_R}(u_*).$$

Moreover, each minimizer u_* of $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$ satisfies

$$\operatorname{supp}(\mu) \subset B_{R_0'} \subset \{u_* > 0\} \quad \text{and} \quad \operatorname{supp}(u_*) \subset B_R$$

for some $R_0' > 0$.

Proof. Since $0 < \lambda < j_{\frac{n-2}{2},1}^2 R^{-2}$, $r \mapsto r^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r)$ is monotonically increasing on $(0, R)$. Then, we have

$$\frac{b_0}{a_0} > \frac{b}{a} > \frac{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_2)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} R)} \geq \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} R)}.$$

By (7.11) and (7.12), we know that $\mu - h = f \leq \tilde{f} = a \chi_{B_{r_2}} - b$. Let u and $\tilde{u}$ be the respective minimizers of $\mathcal{J}_{f,g,\lambda,B_R}$ and $\mathcal{J}_{\tilde{f},0,\lambda,B_R}$ in $\mathbb{K}(B_R)$. Using Proposition 5.1, we know that $\max\{u, \tilde{u}\}$ minimizes $\mathcal{J}_{\tilde{f},g,\lambda,B_R}$. By Lemma 7.2, we know that

$$\mathrm{supp}\,(u) \subset \mathrm{supp}\,(\max\{u, \tilde{u}\}) \subset \overline{B_{R'}} \subset B_R,$$

with

$$R' = \max\left\{\rho \in (0, R] \mid \frac{b}{a} - \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_2)}{\rho^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} \rho)} \leq 0\right\} > r_2.$$

On the other hand, by (7.11) and (7.12), we know that $\mu - h = f \geq \tilde{f}_0 = a_0 \chi_{B_{r_1}} - b_0$. Let u_0 and $\tilde{u}_0$ be minimizers of $\mathcal{J}_{f,g,\lambda,B_R}$ and $\mathcal{J}_{\tilde{f}_0,\tilde{g}_0,\lambda,B_R}$ in $\mathbb{K}(B_R)$, respectively, where

$$\tilde{g}_0 = \|g\|_{L^\infty(\mathbb{R}^n)} \chi_{\mathbb{R}^n \setminus \overline{B_{r_1}}}.$$

Using Proposition 5.1, we know that $\max\{u_0, \tilde{u}_0\}$ minimizes $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$. By choosing u_0 to be the largest (pointwise) minimizer of $\mathcal{J}_{f,g,\lambda,B_R}$ in $\mathbb{K}(B_R)$, we have

$$u_0 \geq \max\{u_0, \tilde{u}_0\} \quad \text{in } B_R,$$

which implies $u_0 \geq \tilde{u}_0$ in B_R. By Lemma 7.2, we know that $\tilde{u}_0 > 0$ in $B_{R_0'}$ with

$$R_0' = \max\left\{\rho \in (0, R] \mid \frac{b_0}{a_0} - \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{\rho^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} \rho)} \leq 0\right\} > r_1.$$

Since we have

$$\frac{b_0}{a_0} - \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_2)} < 0,$$

we have $R_0' > r_2$, which implies that

$$\mathrm{supp}\,(\mu) \equiv \overline{B_{r_2}} \subset B_{R_0'} \subset \{u_* > 0\}.$$

This completes the proof of the proposition. ∎

Combining Proposition 6.2, Proposition 6.3, Lemma 7.1, and Proposition 7.4 with $\lambda = k^2$ and $f = \mu - h\chi_D$, we arrive at the following theorem (with $D = \{u_* > 0\}$):

Proposition 7.5. *Let* $0 < k < j_{\frac{n-2}{2},1} R^{-1}$,

$$a_0 \chi_{B_{r_1}} \leq \mu(x) \leq a \chi_{B_{r_2}}, \quad b \leq h(x) \leq b_0 \quad \text{for all } x \in B_R$$

for some constants r_1, r_2, a, a_0, b, b_0 *with* $0 < b \leq b_0 < a_0 \leq a$ *and* $0 < r_1 \leq r_2 < R$ *satisfying*[3]

$$\frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(kr_1)}{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(kr_2)} > \frac{b_0}{a_0} \geq \frac{b}{a} > \frac{r_2^{\frac{n}{2}} J_{\frac{n}{2}}(kr_2)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(kR)}, \tag{7.13}$$

such that $g \in L^\infty(B_R)$ *with* $g \geq 0$ *and* $g^2 \in W^{1,1}(B_R)$. *If* $\partial\{g > 0\} \cap B_R \neq \emptyset$, *we further assume that there exists* $0 < \alpha \leq 1$ *such that* g *is* C^α *near* $\partial\{g > 0\} \cap B_R$ *and* $\mathcal{H}^{n-1+\alpha}(\partial\{g > 0\} \cap B_R) = 0$. *Then, there exists a bounded open domain* D *in* $\mathbb{R}^n$ *with the boundary* ∂D *having finite* $(n-1)$*-dimensional Hausdorff measure such that*

$$\big(\Psi_k * (g\mathcal{H}^{n-1} \lfloor \partial D)\big)(x) \quad \text{is pointwise well-defined for all } x \in \partial D$$

for all fundamental solutions Ψ_k *of the Helmholtz operator* $-(\Delta + k^2)$. *The set* D *is a hybrid* k*-quadrature domain* D, *corresponding to distribution* μ *and density* (g, h), *with* $\overline{D} \subset B_R$. *Moreover, there exists* $u_* \in C^{0,1}_{\rm loc}(B_{\beta k^{-1}})$ *such that* $D = \{u_* > 0\}$ *and*

$$(\Delta + k^2) u_* = -\mu + h\mathcal{L}^n \lfloor D + g\mathcal{H}^{n-1} \lfloor \partial D.$$

If in addition we assume that $g > 0$ *is Hölder continuous in* $\overline{B_R}$, *then* $\partial_{\rm red} D$ *is locally* $C^{1,\alpha'}$ *with* $\mathcal{H}^{n-1}(\partial D \setminus \partial_{\rm red} D) = 0$. *In the case when* $n = 2$, *we even have* $\partial D = \partial_{\rm red} D$.

Finally, we want to generalize Proposition 7.5 for unbounded non-negative measures μ. Assume that μ satisfies

$$\mu = 0 \quad \text{outside } B_\varepsilon \tag{7.14}$$

for some parameter $\varepsilon > 0$. We define

$$\phi_{2\varepsilon} := (c^{\rm MVT}_{n,k,2\varepsilon})^{-1} \chi_{B_{2\varepsilon}} \quad \text{with} \quad c^{\rm MVT}_{n,k,2\varepsilon} := (2\pi)^{\frac{n}{2}} k^{-\frac{n}{2}} (2\varepsilon)^{\frac{n}{2}} J_{\frac{n}{2}}(2k\varepsilon).$$

It is easy to see that

$$\mu * \phi_{2\varepsilon} \text{ is supported in } B_{3\varepsilon} \text{ and } \mu * \phi_{2\varepsilon}(x) = (c^{\rm MVT}_{n,k,2\varepsilon})^{-1} \mu(B_{2\varepsilon}(x)) \text{ for all } x \in B_{3\varepsilon}.$$

Thus, we see that

$$\mu * \phi_{2\varepsilon}(x) \begin{cases} \leq (c^{\rm MVT}_{n,k,2\varepsilon})^{-1} \mu(\mathbb{R}^n) & \text{for all } x \in B_{3\varepsilon}, \\ = (c^{\rm MVT}_{n,k,2\varepsilon})^{-1} \mu(\mathbb{R}^n) & \text{for all } x \in B_\varepsilon, \end{cases}$$

[3] Since $t \mapsto t^{\frac{n}{2}} J_{\frac{n}{2}}(t)$ is strictly increasing on $[0, j_{\frac{n-2}{2},1}]$, the second condition of (7.13) implies $b_0 < a_0$.

and thus,

$$(c^{\mathrm{MVT}}_{n,k,2\varepsilon})^{-1}\mu(\mathbb{R}^n)\chi_{B_\varepsilon} \le \mu * \phi_{2\varepsilon} \le (c^{\mathrm{MVT}}_{n,k,2\varepsilon})^{-1}\mu(\mathbb{R}^n)\chi_{B_{3\varepsilon}}.$$

We choose $r_1 = \varepsilon$ and $r_2 = 3\varepsilon$, as well as

$$a_0 = a = (c^{\mathrm{MVT}}_{n,k,2\varepsilon})^{-1}\mu(\mathbb{R}^n).$$

Then, (7.13) is equivalent to

$$\frac{(\varepsilon)^{\frac{n}{2}} J_{\frac{n}{2}}(k\varepsilon)}{(3\varepsilon)^{\frac{n}{2}} J_{\frac{n}{2}}(3k\varepsilon)} > \frac{b_0}{(c^{\mathrm{MVT}}_{n,k,2\varepsilon})^{-1}\mu(\mathbb{R}^n)}, \tag{7.15a}$$

$$\frac{b}{(c^{\mathrm{MVT}}_{n,k,2\varepsilon})^{-1}\mu(\mathbb{R}^n)} > \frac{(3\varepsilon)^{\frac{n}{2}} J_{\frac{n}{2}}(3k\varepsilon)}{R^{\frac{n}{2}} J_{\frac{n}{2}}(kR)}. \tag{7.15b}$$

We can write (7.15a) as

$$\mu(\mathbb{R}^n) > 3^{\frac{n}{2}} \frac{J_{\frac{n}{2}}(3k\varepsilon)}{J_{\frac{n}{2}}(k\varepsilon)} \frac{b_0}{(c^{\mathrm{MVT}}_{n,k,\delta})^{-1}(2\varepsilon)^n}(2\varepsilon)^n.$$

Using [23, (8.19)], we know that

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} \ge \frac{1}{(c^{\mathrm{MVT}}_{n,k,\delta})^{-1}(2\varepsilon)^n}.$$

Hence, (7.15a) is fulfilled, provided

$$\mu(\mathbb{R}^n) > C_n b_0 \varepsilon^n \quad \text{with} \quad C_n \ge 2^n \frac{(3\pi)^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} \frac{J_{\frac{n}{2}}(j_{\frac{n-2}{2},1})}{J_{\frac{n}{2}}(j_{\frac{n-2}{2},1}/3)}. \tag{7.16}$$

Using the definition of $c^{\mathrm{MVT}}_{n,k,2\varepsilon}$, we now write (7.15b) as

$$k^n < \left(\frac{4\pi}{3}\right)^{\frac{n}{2}} \frac{1}{\mu(\mathbb{R}^n)} b(kR)^{\frac{n}{2}} J_{\frac{n}{2}}(kR) \frac{J_{\frac{n}{2}}(2k\varepsilon)}{J_{\frac{n}{2}}(3k\varepsilon)}.$$

We now fix any parameter $0 < \beta < j_{\frac{n-2}{2},1}$ and we choose $R = \beta k^{-1}$. If

$$k < \min\left\{\frac{1}{3}, \left(C_{n,\beta}\frac{b}{\mu(\mathbb{R}^n)}\right)^{\frac{1}{n}}\right\} \quad \text{with} \quad C_{n,\beta} = \left(\frac{4\pi}{3}\right)^{\frac{n}{2}} \beta^{\frac{n}{2}} J_{\frac{n}{2}}(\beta) \frac{J_{\frac{n}{2}}(\frac{2}{3} j_{\frac{n-2}{2},1})}{J_{\frac{n}{2}}(j_{\frac{n-2}{2},1})}, \tag{7.17}$$

then (7.15b) holds. The above discussion is valid for $0 < \varepsilon < \beta$.

Using Proposition 7.5 on $\mu * \phi_{2\varepsilon}$, we then know that there exists a hybrid k-quadrature domain D, corresponding to the distribution $\mu * \phi_{2\varepsilon}$ and density (g, h), with $\overline{D} \subset B_R$. Using the mean value theorem for the Helmholtz equation [23, Appendix A], we have

$$\langle \mu * \phi_{2\varepsilon}, w\rangle = \langle \mu, w * \phi_{2\varepsilon}\rangle = \langle \mu, w\rangle$$

for all w satisfying $(\Delta + k^2)w = 0$ in D. Hence, such a D is indeed also a hybrid k-quadrature domain D, corresponding to distribution μ and density (g, h). We now conclude the above discussions in the following theorem (cf. Theorem 1.5):

Theorem 7.6. *Fix parameters* $0 < b \leq b_0$ *and* $0 < \beta < j_{\frac{n-2}{2},1}$. *Let* $0 < \varepsilon < \beta$,

$$b \leq h(x) \leq b_0 \quad \text{for all } x \in B_{\beta k^{-1}},$$

and $g \in L^\infty(B_{\beta k^{-1}})$ *with* $g \geq 0$ *and* $g^2 \in W^{1,1}(B_{\beta k^{-1}})$. *If* $\partial\{g > 0\} \cap B_R \neq \emptyset$, *we further assume that there exists* $0 < \alpha \leq 1$ *such that* g *is* C^α *near* $\partial\{g > 0\} \cap B_R$ *and* $\mathcal{H}^{n-1+\alpha}(\partial\{g > 0\} \cap B_R) = 0$. *If* μ *is a non-negative measure satisfying* (7.14) *and* (7.16), *then for each* k *that satisfies* (7.17), *there exists a bounded open domain* D *in* $\mathbb{R}^n$ *with the boundary* ∂D *having finite* $(n-1)$*-dimensional Hausdorff measure such that*

$$\big(\Psi_k * (g\mathcal{H}^{n-1}\lfloor \partial D)\big)(x) \quad \text{is pointwise well-defined for all } x \in \partial D$$

for all fundamental solutions Ψ_k *of the Helmholtz operator* $-(\Delta + k^2)$. *This domain* D *is a hybrid* k*-quadrature domain corresponding to distribution* μ *and density* (g, h) *and it satisfies* $\overline{D} \subset B_{\beta k^{-1}}$. *Moreover, there exists a non-negative function* $u_* \in C^{0,1}_{\mathrm{loc}}(B_{\beta k^{-1}})$ *such that* $D = \{u_* > 0\}$ *and*

$$(\Delta + k^2)u_* = -\tilde{\mu} + h\mathcal{L}^n\lfloor D + g\mathcal{H}^{n-1}\lfloor \partial D$$

for some non-negative $\tilde{\mu} \in L^\infty(D) \cap \mathcal{E}'(D)$. *If we additionally assume that* $g > 0$ *is Hölder continuous in* $\overline{B_R}$, *then* $\partial_{\mathrm{red}} D$ *is locally* $C^{1,\alpha'}$ *with* $\mathcal{H}^{n-1}(\partial D \setminus \partial_{\mathrm{red}} D) = 0$. *In the case when* $n = 2$, *we even have* $\partial D = \partial_{\mathrm{red}} D$.

A. Functions of bounded variation and sets with finite perimeter

We recall a few facts about functions of bounded variation and sets with finite perimeter. Here we refer to the monographs [13, 16] for more details. The following definition can be found in [16, Definition 1.6]:

Definition A.1. Let E be a Borel set and Ω an open set in $\mathbb{R}^n$. We define the *perimeter* of E in E_0 as

$$\mathcal{P}(E, E_0) := \int_{E_0} |\nabla \chi_E| \, dx \equiv \sup_{\phi \in (C_c^1(\Omega))^n, |\phi(x)| \leq 1} \int_{E_0} \nabla \cdot \phi \, dx.$$

We say that E is a *Caccioppoli set*, if E has *locally finite perimeter*, that is,

$$\mathcal{P}(E, K) < \infty \quad \text{for every compact set } K \text{ in } \mathbb{R}^n.$$

In other words, the function χ_E has locally bounded variation in $\mathbb{R}^n$; see [13, Section 5.1].

The following definition can be found in [16, Definition 3.3] (this concept was introduced by De Giorgi [7]; see also [13, Section 5.7]):

Definition A.2. Assuming that E is a Caccioppoli set, we define the *reduced boundary* $\partial_{\mathrm{red}} E$ of E by the set of points $x \in \mathbb{R}^n$ for which the following hold:

(1) $\int_{B_r(x)} |\nabla \chi_E| \, dx > 0$ for all $r > 0$;

(2) the limit $\nu_E(x) := \lim_{r\to 0} \nu_E^r(x)$ exists, where
$$\nu_E^r(x) := -\frac{\int_{B_r(x)} \nabla \chi_E \, dx}{\int_{B_r(x)} |\nabla \chi_E| \, dx},$$
and $|\nu_E(x)| = 1$.

From the Besicovitch differentiation of measures, it follows that $\nu_E(x)$ exists and $|\nu_E(x)| = 1$ for $|\nabla \chi_E|$-almost all $x \in \mathbb{R}^n$, and furthermore, that
$$\nabla \chi_E = -\nu_E |\nabla \chi_E|.$$
Using [16, Theorem 4.4], we indeed know that
$$|\nabla \chi_E| = \mathcal{H}^{n-1} \lfloor \partial_{\mathrm{red}} E \quad \text{and} \quad \partial_{\mathrm{red}} E \text{ is a dense subset of } \partial E.$$
Thus, we have
$$\mathcal{P}(E, E_0) = \mathcal{H}^{n-1}(E_0 \cap \partial_{\mathrm{red}} E) \quad \text{and} \quad \nabla \chi_E = -\nu_E \mathcal{H}^{n-1} \lfloor \partial_{\mathrm{red}} E, \tag{A.1}$$
and then we immediately have the following generalized Gauss–Green theorem:
$$\int_E \nabla \cdot \phi \, dx = \int_{\partial_{\mathrm{mes}} E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \quad \text{for all } \phi \in (C_c^1(\mathbb{R}^n))^n;$$
see also [13, Section 5.8, Theorem 1].

Remark A.3. If ∂E is a C^1 hypersurface, then $\partial_{\mathrm{red}} E = \partial E$ and $\nu_E(x)$ is the unit outward normal vector to ∂E at x; however, if ∂E is Lipschitz, $\partial_{\mathrm{red}} E$ is in general strictly contained in ∂E; see [16, Remark 3.4] for details. Therefore, we also refer to ν_E as the *measure theoretic outward unit normal vector* of E on $\partial_{\mathrm{red}} E$.

From [13, Section 5.8, Lemma 1], we also know that
$$\partial_{\mathrm{red}} E \subset \partial_{\mathrm{mes}} E \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_{\mathrm{mes}} E \setminus \partial_{\mathrm{red}} E) = 0. \tag{A.2}$$
Combining (A.1) and (A.2), we then know that

> E is a Caccioppoli set if and only if $\mathcal{H}^{n-1}(\partial_{\mathrm{mes}} E \cap K) < \infty$ for each compact set K in $\mathbb{R}^n$;

see also [13, Section 5.11, Theorem 1]. We also recall [13, Section 2.3, Theorem 2] regarding the Hausdorff measure below.

Lemma A.4. *Let $0 < s < n$. If $\mathcal{H}^s(E) < \infty$, then*
$$\frac{1}{2^s} \le \limsup_{r\to 0} \frac{\mathcal{H}^s(B_r(x) \cap E)}{\omega_s r^s} \le 1 \quad \textit{for } \mathcal{H}^s\textit{-a.e. } x \in E,$$
where $\omega_s = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$.

B. Further properties of local minimizers

In this appendix we provide detailed statements and proofs which are analogous to results in [18, Section 2].

The following lemma concerns the growth rate of the integral-mean of minimizers:

Lemma B.1. *Let Ω be an open set in $\mathbb{R}^n$ and let $\lambda \geq 0$. Let $f, g \in L^\infty(\Omega)$ be such that $g \geq 0$. If u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then there is an $r_0 > 0$ such that for any $B_r(x_0)$ with $0 < r < r_0$ and $\overline{B_r(x_0)} \subset \Omega$, we have*

$$\begin{aligned} &\frac{1}{r} \fint_{\partial B_r(x_0)} u_* \, dS > 2^n \Big(\frac{r}{n} \|(f + \lambda u_*)_-\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))} \Big) \\ &\qquad \Longrightarrow \; B_r(x_0) \subset \{u_* > 0\} \text{ and } u_* \text{ is continuous in } B_r(x_0), \end{aligned} \tag{B.1}$$

where $\fint_{\partial B_r(x_0)} = \frac{1}{\mathcal{H}^{n-1}(\partial B_r)} \int_{\partial B_r(x_0)}$ denotes the average integral.

Proof. Let u_* be a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. Using Lemma 6.1, we know that u_* is also a local minimizer of $\mathcal{J}_{\tilde{f},g,0,\Omega}$ in $\mathbb{K}(\Omega)$ with $\tilde{f} = f + \lambda u_*$. Without loss of generality, we may assume that $x_0 = 0$, and we write $B_r = B_r(x_0)$. Define $v \in H^1(\Omega)$ by

$$\begin{cases} \Delta v = -\tilde{f} & \text{in } B_r, \\ v = u_* & \text{in } \Omega \setminus \overline{B_r}. \end{cases} \tag{B.2}$$

It is easy to see that $v \in C(B_r)$. In particular, using elliptic regularity and Sobolev embedding, we know that

$$v \in \bigcap_{1<p<\infty} W^{2,p}_{\mathrm{loc}}(B_r) \subset \bigcap_{0<\alpha<1} C^{1,\alpha}_{\mathrm{loc}}(B_r). \tag{B.3}$$

It is easy to compute

$$\begin{aligned} &\mathcal{J}_{\tilde{f},g,0,\Omega}(u_*) - \mathcal{J}_{\tilde{f},g,0,\Omega}(v) \\ &\quad = \int_\Omega (|\nabla u_*|^2 - |\nabla v|^2) \, dx - 2 \int_\Omega \tilde{f}(u_* - v) \, dx + \int_\Omega g^2 (\chi_{\{u_*>0\}} - \chi_{\{v>0\}}) \, dx \\ &\quad = \int_\Omega (|\nabla u_*|^2 - |\nabla v|^2) \, dx + 2 \int_\Omega \Delta v (u_* - v) \, dx - \int_{B_r} g^2 (\chi_{\{v>0\} \cap \{u_*=0\}}) \, dx \\ &\quad = \int_\Omega (|\nabla u_*|^2 - |\nabla v|^2) \, dx - 2 \int_\Omega \nabla v \cdot \nabla u_* \, dx + 2 \int_\Omega |\nabla v|^2 \, dx \\ &\qquad - \int_{B_r} g^2 (\chi_{\{v>0\} \cap \{u_*=0\}}) \, dx \\ &\quad = \int_{B_r} |\nabla (u_* - v)|^2 \, dx - \int_{B_r} g^2 (\chi_{\{v>0\} \cap \{u_*=0\}}) \, dx \\ &\quad \geq \int_{B_r} |\nabla (u_* - v)|^2 \, dx - |\{u = 0\} \cap B_r| \sup_{B_r} g^2. \end{aligned} \tag{B.4}$$

Next, we want to show that $v \in \mathbb{K}(\Omega)$. Applying [18, Lemma 2.4(a)] to v, we show that

$$v(x) \geq r^n \frac{r - |x|}{(r + |x|)^{n-1}} \Big(\frac{1}{r^2} \fint_{\partial B_r} u_* \, dS - \frac{2^{n-1} M}{n} \Big) \quad \text{for all } x \in B_r, \tag{B.5}$$

where $M = \| \widetilde{f}_- \|_{L^\infty(B_r)}$. The assumption in (B.1) implies

$$\frac{1}{r^2} \fint_{\partial B_r} u_* \, dS \geq \frac{2^n M}{n};$$

then, we have

$$\begin{aligned} v(x) &\geq \frac{r^n}{2} \frac{r - |x|}{(r + |x|)^{n-1}} \frac{1}{r^2} \fint_{\partial B_r} u_* \, dS \\ &\geq 2^{-n} (r - |x|) \frac{1}{r} \fint_{\partial B_r} u_* \, dS \quad \text{for all } x \in B_r, \end{aligned} \tag{B.6}$$

which shows that $v \in \mathbb{K}(\Omega)$. Since u_* is a local minimizer, by choosing $r_0 > 0$ sufficiently small, we know that

$$\mathcal{J}_{\widetilde{f},g,0,\Omega}(u_*) \leq \mathcal{J}_{\widetilde{f},g,0,\Omega}(v),$$

and hence, from (B.4), we know that

$$\int_{B_r} |\nabla (u_* - v)|^2 \, dx \leq |\{u_* = 0\} \cap B_r| \sup_{B_r} g^2 \quad \text{for each } 0 < r < r_0. \tag{B.7}$$

To estimate the left-hand side of (B.7), from (B.6), we have

$$\chi_{\{u_*=0\}} \Big(\frac{1}{r} \fint_{\partial B_r} u_* \, dS \Big)^2 (r - |x|)^2 \leq 2^{2n} \chi_{\{u_*=0\}} |v(x)|^2 \quad \text{for all } x \in B_r. \tag{B.8}$$

For each $0 \neq x \in B_r$, writing $\hat{x} = x/|x|$, note that

$$\begin{aligned} \chi_{\{u_*=0\}} |v(x)|^2 &= \chi_{\{u_*=0\}} \Big(\int_{|x|}^r \partial_{|z|} (u_* - v)(s\hat{x}) \, ds \Big)^2 \\ &\leq \chi_{\{u_*=0\}} \Big(\int_{|x|}^r 1^2 \, ds \Big) \Big(\int_{|x|}^r |\partial_{|z|} (u_* - v)(s\hat{x})|^2 \, ds \Big) \\ &\leq \chi_{\{u_*=0\}} (r - |x|) \Big(\int_{|x|}^r |\nabla (u_* - v)(s\hat{x})|^2 \, ds \Big), \end{aligned}$$

and hence, from (B.8), we have

$$\begin{aligned} &(r - |x|) \chi_{\{u_*=0\}} \Big(\frac{1}{r} \fint_{\partial B_r} u_* \, dS \Big)^2 \\ &\quad \leq 2^{2n} \chi_{\{u_*=0\}} \Big(\int_{|x|}^r |\nabla (u_* - v)(s\hat{x})|^2 \, ds \Big) \quad \text{for all } 0 \neq x \in B_r. \end{aligned} \tag{B.9}$$

We consider $\theta \in \mathcal{S}^{n-1}$ such that $u_*(s\theta) = 0$ for some $0 < s < r$. For such θ, we can define

$$s_\theta := \inf\{0 < s < r \mid u_*(s\theta) = 0\}.$$

Then, (B.9) implies

$$\begin{aligned}(r - s_\theta)\chi_{\{u_*=0\}\cap B_r}(s_\theta\theta)\Big(\frac{1}{r}\fint_{\partial B_r} u_*\,dS\Big)^2 \\ \le 2^{2n}\chi_{\{u_*=0\}\cap B_r}(s_\theta\theta)\int_{s_\theta}^{r} |\nabla(u_* - v)(s\theta)|^2\,ds \quad \text{for all } \theta \in \mathcal{S}^{n-1}.\end{aligned} \tag{B.10}$$

Integrating (B.10) over $\theta \in \mathcal{S}^{n-1}$, we reach

$$|\{u_* = 0\} \cap B_r|\Big(\frac{1}{r}\fint_{\partial B_r} u_*\,dS\Big)^2 \le 2^{2n}\int_{B_r} |\nabla(u_* - v)|^2\,dx. \tag{B.11}$$

Combining (B.7) and (B.11), we reach

$$|\{u_* = 0\} \cap B_r|\Big(\frac{1}{r}\fint_{\partial B_r} u_*\,dS\Big)^2 \le 2^{2n}|\{u = 0\} \cap B_r|\sup_{B_r} g^2.$$

The assumption in (B.1) implies

$$\frac{1}{r}\fint_{\partial B_r(x_0)} u_*\,dS > 2^n \sup_{B_r(x_0)} g;$$

then, we necessarily have

$$|\{u_* = 0\} \cap B_r| = 0. \tag{B.12}$$

From (B.7), we know that

$$\int_{B_r} |\nabla(u_* - v)|^2\,dx = 0,$$

and thus, we also showed that $u_* = v \in C(B_r)$. Using (B.3) and (B.12), we conclude that $B_r \subset \{u_* > 0\}$. ∎

The following proposition concerns the continuity of the local minimizers:

Proposition B.2. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \le \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ be such that $g \ge 0$. If u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then $u_* \in C(\Omega)$ and there exists a constant C_n such that*

$$u_*(x) \le C_n\mathsf{d}(x)\big(\|g\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} + \|f + \lambda u_*\|_{L^\infty(B_{2\mathsf{d}(x)}(x))}\mathsf{d}(x)\big), \tag{B.13}$$

for all $x \in \Omega$ near $\partial\{u_* > 0\}$, where $\mathsf{d}(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \{u_* > 0\})$.

Remark B.3. The assumptions on Ω ensure that $u_* \in L^\infty(\Omega)$; see Proposition 5.6. From this, we know that

$$\|f + \lambda u_*\|_{L^\infty(\Omega)} \le C(\lambda, \Omega)\|f\|_{L^\infty(\Omega)}.$$

Proof of Proposition B.2. Let u_* be a local minimizer $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. Using Lemma 6.1, we know that u_* is also a local minimizer of $\mathcal{J}_{\tilde{f},g,0,\Omega}$ in $\mathbb{K}(\Omega)$ with $\tilde{f} = f + \lambda u_*$. Using Proposition 5.6, we know that $\tilde{f}$ is bounded.

We first show that $\{u_* > 0\}$ is an open set and that u_* is continuous in it. Let $x \in \{u_* > 0\}$. Using (4.2a) of Proposition 4.2, we know that

$$\Delta u_* \geq -\tilde{f}_+ \geq -\|\tilde{f}_+\|_{L^\infty(B_r(x))} \quad \text{in } B_r(x)$$

for all $r > 0$ whenever $B_r(x) \subset \Omega$. Therefore, using [18, Lemma 2.4(b)], we know that

$$u_*(y+x) \leq r^n \frac{r+|y|}{(r-|y|)^{n-1}} \left(\frac{1}{r^2} \fint_{\partial B_r(x)} u_* \, dS + \frac{\|\tilde{f}_+\|_{L^\infty(B_r(x))}}{2n} \right) \quad \text{for all } y \in B_r(0). \tag{B.14}$$

Choosing $y = 0$ in (B.14), we have

$$u_*(x) \leq \fint_{\partial B_r(x)} u_* \, dS + r^2 \frac{\|\tilde{f}_+\|_{L^\infty(B_r(x))}}{2n} \quad \text{for all } r > 0 \text{ with } B_r(x) \subset \Omega,$$

that is,

$$\frac{1}{r} \fint_{\partial B_r(x)} u_* \, dS \geq \frac{1}{r} u_*(x) - r \frac{\|\tilde{f}_+\|_{L^\infty(B_r(x))}}{2n} \quad \text{for all } r > 0 \text{ with } B_r(x) \subset \Omega. \tag{B.15}$$

Since $u_*(x_0) > 0$, we can choose $r_0 > 0$ sufficiently small such that

$$\frac{1}{r} u_*(x) - r \frac{\|\tilde{f}_+\|_{L^\infty(B_r(x))}}{2n} > 2^n \left(\frac{r}{n} \|\tilde{f}_-\|_{L^\infty(B_r(x))} + \|g\|_{L^\infty(B_r(x))} \right) \quad \text{for all } 0 < r \leq r_0, \tag{B.16}$$

and hence, (B.1) is satisfied for all $0 < r < r_0$. Therefore, Lemma B.1 implies that $B_{r_0}(x) \subset \{u_* > 0\}$ and u_* is continuous in $B_{r_0}(x)$, which shows that

$$\{u_* > 0\} \text{ is an open set and } u_* \text{ is continuous in } \{u_* > 0\}. \tag{B.17}$$

To prove (B.13), we only need to show (B.13) for $x \in \{u_* > 0\}$. Clearly, (B.16) cannot hold when $r = 2\mathsf{d}(x)$, otherwise using the same argument will show that $B_{2\mathsf{d}(x)}(x) \subset \{u_* > 0\}$, which contradicts the fact that $B_{\mathsf{d}(x)}(x)$ touches $\partial\{u_* > 0\}$. Thus, we have

$$\frac{1}{2\mathsf{d}(x)} u_*(x) - 2\mathsf{d}(x) \frac{\|\tilde{f}_+\|_{L^\infty(B_{2\mathsf{d}(x)}(x))}}{2n} \leq 2^n \left(\frac{2\mathsf{d}(x)}{n} \|\tilde{f}_-\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} + \|g\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} \right)$$

for all $x \in \{u_* > 0\}$ near $\partial\{u_* > 0\}$, which implies (B.13). Combining (B.17) and (B.13), we know that $u_* \in C(\Omega)$. ∎

Using Sard's theorem and the coarea formula, one can show that $|\nabla u_*| = g$ in some suitable sense. This is stated in the next proposition.

Proposition B.4. *Let $f, g \in L^\infty(\mathbb{R}^n)$ be such that $g \geq 0$. Let Ω be an open set in $\mathbb{R}^n$ and let $\lambda \geq 0$. Let $u_* \in \mathbb{K}(\Omega)$ be a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. If $g^2 \in W^{1,1}(\Omega)$, then (for a suitable sequence of ε_j)*

$$\lim_{\varepsilon_j \searrow 0} \int_{\partial\{u>\varepsilon_j\}} (|\nabla u_*|^2 - g^2)\eta \cdot \nu_{\varepsilon_j} \, dS = 0,$$

for all $\eta \in (C_c^\infty(B_R))^n$. Here ν_{ε_j} denotes the outward normal vector of $\partial\{u_* > \varepsilon_j\}$.

Proof. Let $\eta \in (C_c^\infty(\Omega))^n$ and $\varepsilon > 0$ be small. We define $u_\varepsilon \in \mathbb{K}(\Omega)$ by $u_\varepsilon(\tau_\varepsilon(x)) = u_*(x)$, where $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$. From equation (4.2b) in Proposition 4.2, it is easy to see that $u_* \in C^1(\{u_* > 0\})$. Note that

$$\begin{aligned} 0 &\leq \mathcal{J}_{f,g,\lambda,\Omega}(u_\varepsilon) - \mathcal{J}_{f,g,\lambda,\Omega}(u_*) \\ &= \int_{\{u_*>0\}} \Big(|\nabla u_*|^2 |\nabla \tau_\varepsilon|^{-2} - \lambda |u_*|^2 - 2(f \circ \tau_\varepsilon) u_* + g^2 \circ \tau_\varepsilon\Big) \det(\nabla \tau_\varepsilon) \, dx \\ &\qquad - \int_{\{u_*>0\}} (|\nabla u_*|^2 - \lambda |u_*|^2 - 2 f u_* + g^2) \, dx \\ &= \varepsilon \int_{\{u_*>0\}} (|\nabla u_*|^2 - g^2) \nabla \cdot \eta \, dx \\ &\qquad + \varepsilon \int_{\{u_*>0\}} (-2\nabla u_* \cdot (\nabla \eta) \nabla u_* + \nabla(g^2) \cdot \eta) \, dx + o(\varepsilon), \end{aligned}$$

where

$$\lim_{\varepsilon \searrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

We denote by $\otimes : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ the juxtaposition operator defined by $a \otimes b := ab^T$ for all $a, b \in \mathbb{R}^n$. Dividing both sides of the inequality above by ε and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} 0 &= \int_{\{u_*>0\}} \nabla \cdot ((|\nabla u_*|^2 + g^2)\eta - 2(\eta \otimes \nabla u_*) \nabla u_* \, dx \\ &= \lim_{\varepsilon \searrow 0} \int_{\{u_*>\varepsilon\}} \nabla \cdot ((|\nabla u_*|^2 + g^2)\eta - 2(\eta \otimes \nabla u_*) \nabla u_*) \, dx \\ &= \lim_{\varepsilon \searrow 0} \int_{\partial\{u_*>\varepsilon\}} \Big((|\nabla u_*|^2 + g^2)\eta \cdot \nu_\varepsilon - 2\overbrace{((\eta \otimes \nabla u_*) \nabla u_*) \cdot \nu_\varepsilon}^{=|\nabla u_*|^2 \eta \cdot \nu_\varepsilon}\Big) \, dS \\ &= \lim_{\varepsilon \searrow 0} \int_{\partial\{u_*>\varepsilon\}} (-|\nabla u_*|^2 + g^2)\eta \cdot \nu_\varepsilon \, dS, \end{aligned}$$

which conclude our proposition. ∎

We now show that the local minimizer is Lipschitz.

Proposition B.5. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \le \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ be such that $g \ge 0$ and $g^2 \in W^{1,1}(\Omega)$. If u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, then $u_* \in C^{0,1}_{\rm loc}(\Omega)$ and there exists a constant C_n such that*

$$|\nabla u_*(x)| \le C_n\big(\|g\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} + \|f + \lambda u_*\|_{L^\infty(B_{2\mathsf{d}(x)}(x))}\mathsf{d}(x)\big), \tag{B.18}$$

for all $x \in \Omega$ near $\partial\{u_* > 0\}$.

Proof. It is easy to see that $|\nabla u_*(x)| = 0$ for all $x \in \Omega \setminus \overline{\{u_* > 0\}}$. Since $g^2 \in W^{1,1}(\Omega)$ and $g \in C(\Omega)$, we know that $|\nabla u_*(x)| = g(x) = \|g\|_{L^\infty(B_{\mathsf{d}(x)}(x))}$ for all $x \in \partial\{u_* > 0\}$. It is remains to show (B.18) holds for $x \in \{u_* > 0\}$.

Since $B_{\mathsf{d}(x)}(x) \subset \{u_* > 0\}$, using equation (4.2b) of Proposition 4.2, we have $|\Delta u_*| = |\tilde{f}| \le \|\tilde{f}\|_{L^\infty(B_{\mathsf{d}(x)}(x))}$ in $B_{\mathsf{d}(x)}(x)$. Therefore, using [18, Lemma 2.4(d)], we know that

$$|\nabla u_*(x)| \le n\Big(\frac{1}{\mathsf{d}(x)} \fint_{\partial B_{\mathsf{d}(x)}(x)} u_*\,dS + \frac{\|\tilde{f}\|_{L^\infty(B_{\mathsf{d}(x)}(x))}}{n+1}\mathsf{d}(x)\Big). \tag{B.19}$$

Since $B_{\mathsf{d}(x)+\varepsilon}(x) \not\subset \{u_* > 0\}$, using Lemma B.1, we know that[4]

$$\begin{aligned}\frac{1}{\mathsf{d}(x)+\varepsilon} \fint_{\partial B_{\mathsf{d}(x)+\varepsilon}(x)} u_*\,dS &\le 2^n\Big(\frac{\mathsf{d}(x)+\varepsilon}{n}\|\tilde{f}_-\|_{L^\infty(B_{\mathsf{d}(x)+\varepsilon}(x))} + \|g\|_{L^\infty(B_{\mathsf{d}(x)+\varepsilon}(x))}\Big) \\ &\le 2^n\Big(\frac{\mathsf{d}(x)+\varepsilon}{n}\|\tilde{f}_-\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} + \|g\|_{L^\infty(B_{2\mathsf{d}(x)}(x))}\Big)\end{aligned}$$

for all sufficiently small $\varepsilon > 0$. Using the continuity of u_*, taking $\varepsilon \to 0_+$ yields

$$\frac{1}{\mathsf{d}(x)} \fint_{\partial B_{\mathsf{d}(x)}(x)} u_*\,dS \le 2^n\Big(\frac{\mathsf{d}(x)}{n}\|\tilde{f}_-\|_{L^\infty(B_{2\mathsf{d}(x)}(x))} + \|g\|_{L^\infty(B_{2\mathsf{d}(x)}(x))}\Big). \tag{B.20}$$

Combining (B.19) and (B.20), we conclude that equation (B.18) holds for all $x \in \{u_* > 0\}$ near $\partial\{u_* > 0\}$. ∎

The following lemma gives a sufficient condition in terms of mean averages to ensure the local vanishing property of a local minimizer:

Lemma B.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \le \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ be such that $g \ge 0$. Suppose that u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. If there exists an open set Ω' with $\overline{\Omega'} \subset \Omega$ such that*

$$g \ge c > 0 \quad \text{in } \Omega',$$

[4] If $f, g \in C(\Omega)$, we can improve (B.18) by replacing $2\mathsf{d}$ by d.

then there exists a constant $C > 0$ *such that for any sufficiently small ball* $B_r(x_0) \subset \Omega'$, *we have*

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u_* \le C \implies u_* = 0 \quad \text{in } B_{\frac{r}{4}}(x_0). \tag{B.21}$$

The constant C *depends only on* $\inf_{B_r(x_0)} g$, $r\|(f + \lambda u_*)_+\|_{L^\infty(B_r(x_0))}$ *and* n. *Moreover,* C *is positive whenever* $\inf_{B_r(x_0)} g > 0$ *and* $r\|(f + \lambda u_*)_+\|_{L^\infty(B_r(x_0))}$ *is sufficiently small.*

Remark B.7. If $g \ge c > 0$ in a neighborhood of a point $x_0 \in \partial\{u_* > 0\}$, then

$$u_*(x) \ge C \operatorname{dist}(x, \mathbb{R}^n \setminus \{u_* > 0\}) \tag{B.22}$$

for some constant $C > 0$. We only need to show (B.22) for $x \in \{u_* > 0\}$ near x_0. Write $r = \operatorname{dist}(x, \mathbb{R}^n \setminus \{u_* > 0\})$. In particular, from the contra-positive statement of (B.21), we know that

$$\frac{1}{r} \fint_{\partial B_r(x)} u_* > C_1$$

for some constant $C_1 > 0$. Using (4.2b) in Proposition 4.2, we have

$$\Delta u_* = -\tilde{f} \le C_2 := \|\tilde{f}\|_{L^\infty(B_r(x))} \quad \text{in } B_r(x) \subset \{u_* > 0\}$$

with $\tilde{f} = f + \lambda u_*$. Using [18, Lemma 2.4(a)], we have

$$u_*(x) \ge \fint_{\partial B_r} u - \frac{2^{n-1} C_2}{n} r^2 \ge C_1 r - \frac{2^{n-1} C_2}{n} r^2 \ge C r,$$

provided $r = \operatorname{dist}(x, \mathbb{R}^n \setminus \{u_* > 0\})$ is sufficiently small.

Proof of Lemma B.6. Without loss of generality, we may assume that $x_0 = 0$. From equation (B.14), we have

$$u_*(y) \le C_1 \fint_{\partial B_r} u_* \, dS + C_2 r^2 \|\tilde{f}_+\|_{L^\infty(B_r)} \quad \text{for all } y \in B_{\frac{r}{2}} \tag{B.23}$$

for some absolute constants C_1 and C_2, with $\tilde{f} = f + \lambda u_*$. We define $m := \inf_{B_r} g$, $M := \|\tilde{f}_+\|_{L^\infty(B_r)}$, and

$$\mathcal{J}_r(v) := \int_{B_{\frac{r}{2}}} (|\nabla v|^2 - 2\tilde{f} v + g^2 \chi_{\{v>0\}}) \, dx,$$

$$\tilde{\mathcal{J}}_r(v) := \int_{B_{\frac{r}{2}}} (|\nabla v|^2 - 2M v + m^2 \chi_{\{v>0\}}) \, dx.$$

Now given a constant $\beta > 0$, we consider the problem of minimizing $\tilde{\mathcal{J}}_r$ over

$$\mathbb{K}_\beta = \left\{ v \in H^1(B_{\frac{r}{2}}) \mid v \ge 0 \text{ in } B_{\frac{r}{2}} \text{ and } v = \beta \text{ on } \partial B_{\frac{r}{2}} \right\}.$$

Following similar arguments as in [18, Conclusion of Lemma 2.8], we know that there exists a sufficiently small constant $\beta_0 = \beta_0(r,m,M)$ such that if

$$0 < \frac{r}{2} < \frac{nm}{M} \quad \text{and} \quad 0 < \beta \le \beta_0,$$

then the largest minimizer v_β of $\tilde{\mathcal{J}}_r$ in $\mathbb{K}_\beta$ satisfies

$$v_\beta = 0 \quad \text{in } B_{\frac{r}{4}}. \tag{B.24}$$

From [18, (2.13)], we have

$$\beta_0(r,m,M) = r\beta_0(1,m,rM).$$

Next, we let v_{β_0} be the largest minimizer of $\tilde{\mathcal{J}}_r$ in $\mathbb{K}_{\beta_0}$ and let

$$w = \begin{cases} \min\{u_*, v_{\beta_0}\} & \text{in } B_{\frac{r}{2}}, \\ u_* & \text{outside } B_{\frac{r}{2}}. \end{cases}$$

If

$$C_1 \frac{1}{r} \fint_{\partial B_r(x_0)} u_* < 2\beta_0(1,m,rM),$$

then for each sufficiently small r, we have (because $\beta_0(1,m,0) > 0$ when $m > 0$ and $\beta_0(1,m,\cdot)$ is continuous)

$$C_1 \frac{1}{r} \fint_{\partial B_r(x_0)} u_* + C_2 rM < \beta_0(1,m,rM).$$

Consequently, from (B.23), we have

$$u_* < \beta_0(r,m,M) \equiv r\beta_0(1,m,rM) \quad \text{on } \partial B_{\frac{r}{2}}.$$

Since we consider small $r > 0$, we know that $w \in \mathbb{K}(\Omega)$ and it is close to u_* in the sense of (4.1). Since u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$, by Lemma 6.1, we know that it is also a local minimizer of $\mathcal{J}_{\tilde{f},g,0,\Omega}$ in $\mathbb{K}(\Omega)$. Hence, we know that

$$\mathcal{J}_r(u_*) \le \mathcal{J}_r(w) \equiv \mathcal{J}_r(\min\{u_*, v_{\beta_0}\}). \tag{B.25}$$

Since v_{β_0} is a minimizer of $\tilde{\mathcal{J}}_r$ in $\mathbb{K}_{\beta_0}$, then

$$\tilde{\mathcal{J}}_r(v_{\beta_0}) \le \tilde{\mathcal{J}}_r(\max\{u_*, v_{\beta_0}\}). \tag{B.26}$$

Combining (B.25) and (B.26), we have

$$\mathcal{J}_r(u_*) + \tilde{\mathcal{J}}_r(v_{\beta_0}) \le \mathcal{J}_r(\min\{u_*, v_{\beta_0}\}) + \tilde{\mathcal{J}}_r(\max\{u_*, v_{\beta_0}\}). \tag{B.27}$$

Following the proof of Proposition 5.1, we can show that

$$\mathcal{J}_r(\min\{u_1,u_2\}) + \tilde{\mathcal{J}}_r(\max\{u_1,u_2\}) \le \mathcal{J}_r(u_1) + \tilde{\mathcal{J}}_r(u_2) \tag{B.28}$$

for any $u_1, u_2 \in H^1(B_{\frac{r}{2}})$ with $u_1 \geq 0$ and $u_2 \geq 0$. Combining (B.27) and (B.28), we obtain

$$\mathcal{J}_r(u_*) + \widetilde{\mathcal{J}}_r(v_{\beta_0}) = \mathcal{J}_r(\min\{u_*, v_{\beta_0}\}) + \widetilde{\mathcal{J}}_r(\max\{u_*, v_{\beta_0}\}).$$

Since v_{β_0} is the largest minimizer of $\widetilde{\mathcal{J}}_r$ in $\mathbb{K}_{\beta_0}$, we have $\max\{u_*, v_{\beta_0}\} \leq v_{\beta_0}$, which implies $0 \leq u_* \leq v_{\beta_0}$ in $B_{\frac{r}{2}}$. Combining this with (B.24), we conclude with (B.21) with $C = 2C_1^{-1}\beta_0(1, m, rM)$. ∎

The next lemma concerns the density of the boundary of $\{u_* > 0\}$.

Lemma B.8. *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary and $0 \leq \lambda < \lambda^*(\Omega)$. Let $f, g \in L^\infty(\Omega)$ be such that $g \geq 0$ and $g^2 \in H^{1,1}(\Omega)$. Suppose that u_* is a local minimizer of $\mathcal{J}_{f,g,\lambda,\Omega}$ in $\mathbb{K}(\Omega)$. If $g \geq c > 0$ in a neighborhood of a point $x_0 \in \partial\{u_* > 0\}$, then there exist constants c_1 and c_2 such that*

$$0 < c_1 \leq \frac{|B_r(x_0) \cap \{u_* > 0\}|}{|B_r(x_0)|} \leq c_2 < 1 \quad \textit{for all sufficiently small } r > 0. \tag{B.29}$$

Remark B.9. For each (Lebesgue) measurable set $E \subset \mathbb{R}^n$, it is well known that

$$\lim_{r \to 0} \frac{|B_r(x_0) \cap E|}{|B_r(x_0)|} = \begin{cases} 1 & \text{for a.e. } x \in E, \\ 0 & \text{for a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Therefore, the *measure theoretic boundary* $\partial_{\mathrm{mes}} E$ of E is defined to be the set of points $x \in \mathbb{R}^n$ such that both E and $\mathbb{R}^n \setminus E$ have positive upper Lebesgue density at x. In particular, $\partial_{\mathrm{mes}} E \subset \partial E$. As an immediate corollary of Lemma B.8, we know that

$$\partial\{u_* > 0\} \cap \{g > 0\} = \partial_{\mathrm{mes}}\{u_* > 0\} \cap \{g > 0\}.$$

Proof of Lemma B.8. Without loss of generality, we may assume that $x_0 = 0$. Using Remark B.7, we know that there exist a point $y \in \partial B_{\frac{r}{2}}$ and a constant $c > 0$ such that $u_*(y) \geq cr$. Using (B.15) on $B_{\kappa r}(y) \subset \Omega$, provided $\kappa > 0$ is small, we have

$$\frac{1}{\kappa r} \fint_{\partial B_{\kappa r}(y)} u_* \, dS \geq \frac{1}{\kappa r} u_*(y) - \kappa r \frac{\|\widetilde{f}_+\|_{L^\infty(B_{\kappa r}(y))}}{2n} \geq \frac{c}{\kappa}.$$

Using Lemma B.1, we know that $B_{\kappa r}(y) \subset \{u_* > 0\} \cap B_r$ for sufficiently small $\kappa > 0$, and hence,

$$\frac{|B_r \cap \{u_* > 0\}|}{|B_r|} \geq \frac{|B_{\kappa r}(y)|}{|B_r|} = \frac{|B_{\kappa r}|}{|B_r|} = \kappa^n, \tag{B.30}$$

which proves the lower bound of (B.29) with $c_1 = \kappa^n$.

Combining (B.30) with Lemma B.6, we know that

$$\fint_{\partial B_r} u_* \geq cr. \tag{B.31}$$

Now let v be as in (B.2). From (B.7) and Poincaré's inequality, we have

$$|B_r \cap \{u_* = 0\}| \geq c \int_{B_r} |\nabla(u_* - v)|^2 \, dx \geq \frac{c}{r^2} \int_{B_r} |u_* - v|^2 \, dx \tag{B.32}$$

for each sufficiently small $r > 0$. By restricting (B.5) on $B_{\kappa r}$, we obtain

$$v(y) \geq \frac{(1-\kappa)}{(1+\kappa)^{n-1}} \Big(\fint_{\partial B_r} u_* \, dS - Cr^2 \Big) \quad \text{for all } y \in B_{\kappa r}. \tag{B.33}$$

Using Proposition B.5 (which required $g^2 \in W^{1,1}(\Omega)$), we know that u_* is Lipschitz, and hence, we know that

$$u_*(y) \leq C\kappa r \quad \text{for all } y \in B_{\kappa r}, \tag{B.34}$$

because $0 \in \partial\{u_* > 0\}$. Combining (B.34) with (B.33) as well as (B.31), for all sufficiently small $r > 0$, we have

$$(v - u_*)(y) \geq \frac{(1-\kappa)}{(1+\kappa)^{n-1}} \fint_{\partial B_r} u_* \, dS - C\kappa r - Cr^2 \geq (c - C\kappa) r \geq c' r \quad \text{for all } y \in B_{\kappa r},$$

provided $\kappa > 0$ is sufficiently small. Hence,

$$|(v - u_*)| = v - u_* \geq c' r \quad \text{in } B_{\kappa r}.$$

Therefore, from (B.32), we have

$$|B_r \cap \{u_* = 0\}| \geq \frac{c}{r^2} \int_{B_{\kappa r}} |u_* - v|^2 \, dx \geq \frac{c}{r^2} |B_{\kappa r}| (c')^2 r^2 = c'' |B_r| \kappa^n,$$

and consequently,

$$\frac{|B_r \cap \{u_* > 0\}|}{|B_r|} = 1 - \frac{|B_r \cap \{u_* = 0\}|}{|B_r|} \leq 1 - \frac{c''|B_r|\kappa^n}{|B_r|} = 1 - c''\kappa^n,$$

which proves the upper bound of (B.29) with $c_2 = 1 - c''\kappa^n$. ∎

Proof of Proposition 6.2. Here we only prove the result when $\partial\{g > 0\} \cap \Omega \neq \emptyset$, as the case when $\partial\{g > 0\} \cap \Omega = \emptyset$ can be easily proved using the same idea by omitting some paragraphs.

Step 1: Initialization. Using Proposition 5.6, we know that $\tilde{f} = f + \lambda u_* \in L^\infty(\Omega)$. Using (4.2a) in Proposition 4.2, we know that Δu_* is a signed Radon measure in Ω and $\Delta u_* \geq -\tilde{f} \geq -\|\tilde{f}\|_{L^\infty(\Omega)}$ in Ω. Then, we see that

$$\Delta\Big(u_* + \frac{\|\tilde{f}\|_{L^\infty(\Omega)}}{2n}|x|^2\Big) = \Delta u_* + \|\tilde{f}\|_{L^\infty(\Omega)} \geq 0 \quad \text{in } \Omega.$$

Since $u_* \geq 0$ in Ω, using [18, Lemma 2.16], we know that $\Delta u_* \geq 0$ in $\Omega \setminus \{u_* > 0\}$. We now define

$$\rho := \Delta u_* + \tilde{f} \chi_{\{u_* > 0\}}.$$

Using (4.2b) in Proposition 4.2, we know that

$$\rho = 0 \quad \text{in } \{u_* > 0\}. \tag{B.35}$$

Clearly, $\rho = 0$ in the open set $\Omega \setminus \overline{\{u_* > 0\}}$; therefore, we know that

$$\rho := \Delta u_* + \tilde{f}\chi_{\{u_*>0\}} \text{ is a non-negative Radon measure supported on } \partial\{u_* > 0\}.$$

Furthermore, from (4.2c), we know that

$$\begin{aligned} &\rho := \Delta u_* + \tilde{f}\chi_{\{u_*>0\}} \text{ is a non-negative Radon measure} \\ &\text{supported on } \partial\{u_* > 0\} \cap \{g > 0\}. \end{aligned} \tag{B.36}$$

For each $x_0 \in \Omega$, we estimate

$$\left| \int_{B_r(x_0)} \Delta u_* \, dx \right| \le \int_{\partial B_r(x_0)} |\nabla u_*| \, dS \le C r^{n-1} \sup_{\partial B_r(x)} |\nabla u_*| \le C r^{n-1} \tag{B.37}$$

for all sufficiently small $r > 0$, where the last inequality follows from (B.18) in Proposition B.5 and the assumption $\overline{\{u_* > 0\}} \subset \Omega$. This shows that Δu_*, as well as λ, is absolutely continuous with respect to $\mathcal{H}^{n-1}$.

Step 2: Showing that $\rho = 0$ *on* $\partial\{u_* > 0\} \setminus \{g > 0\}$. For each $x_0 \in \partial\{u_* > 0\} \setminus \{g > 0\}$, using (B.18) in Proposition B.5, we have[5]

$$\left| \int_{B_r(x_0)} \Delta u_* \, dx \right| \le C r^{n-1+\alpha} \quad \text{for all sufficiently small } r > 0,$$

which shows that Δu_*, as well as ρ, is absolutely continuous with respect to $\mathcal{H}^{n-1+\alpha}$ on $\partial\{u_* > 0\} \setminus \{g > 0\}$. Using the assumption $\mathcal{H}^{n-1+\alpha}(\partial\{g > 0\} \cap \Omega) = 0$, we know that

$$\rho = \Delta u_* = 0 \quad \text{on } \partial\{u_* > 0\} \cap \partial\{g > 0\}. \tag{B.38}$$

On the other hand, using (4.2a) and (4.2c) in Proposition 4.2, we know that

$$-(f + \lambda u_*)_+ \le \Delta u_* \le -(f + \lambda u_*) \quad \text{in } \Omega \setminus \overline{\{g > 0\}},$$

and thus, $\Delta u_* \in L^\infty(\Omega \setminus \overline{\{g > 0\}})$. Using [22, Chapter II, Lemma A.4], we know that

$$\rho = \Delta u_* = 0 \quad \text{on } \partial\{u_* > 0\} \setminus \overline{\{g > 0\}}. \tag{B.39}$$

Combining (B.38) and (B.39), we know that

$$\rho = 0 \quad \text{on } \partial\{u_* > 0\} \setminus \{g > 0\}.$$

Next, we want to study the behavior of ρ on $\partial\{u_* > 0\} \cap \{g > 0\}$.

[5] In particular, when $g \equiv 0$ in Ω (i.e., $G \cap \Omega = \emptyset$), we even can choose $\alpha = 1$.

Step 3: Proving $\{u_* > 0\}$ has locally finite perimeter in $\{g > 0\}$ and (6.2). We first show that there exists a constant $C > 0$, depending on $\inf_{B_r(x_0)} g$, such that

$$\int_{B_r(x_0)} \Delta u_* \, dx \geq C r^{n-1} \tag{B.40}$$

for all sufficiently small $r > 0$ and for all $x_0 \in \partial\{u_* > 0\} \cap \{g > 0\}$.

Let Φ_y be the (positive) Green function for $-\Delta$ in $B_r(x_0)$ with pole $y \in B_r(x_0)$, that is,

$$\begin{cases} \Delta\Phi_y = -\delta_y & \text{in } B_r(x_0), \\ \Phi_y \geq 0 & \text{in } B_r(x_0), \\ \Phi_y = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Using integration by parts, we can easily see that

$$\int_{B_r(x_0)} \Phi_y \Delta u_* \, dx = -u_*(y) + \int_{\partial B_r(x_0)} u_* \partial_{-\nu} \Phi_y \, dS.$$

Using Lemma B.1, for each sufficiently small $\kappa > 0$, there is a point $y \in \partial B_{\kappa r}(x_0)$ with

$$u_*(y) \geq c\kappa r > 0,$$

and since u_* is Lipschitz, we have

$$u_*(y) \leq C\kappa r \quad \text{and} \quad u_* > 0 \text{ in } B_{c(\kappa)r}(y)$$

for some constant $c(\kappa)$. Hence, we have

$$\int_{B_r(x_0)} \Phi_y \Delta u_* \, dx \geq -u_*(y) + c \fint_{\partial B_r(x_0)} u_* \, dS \geq -C\kappa r + cr \geq c'r, \tag{B.41}$$

which can possibly be done by using a smaller $\kappa > 0$.

On the other hand, using (B.35), we know that $\rho = 0$ in $B_{c(\kappa)r}(y) \subset \{u_* > 0\}$; hence, we have

$$\begin{aligned} &\int_{B_r(x_0)} \Phi_y \Delta u_* \, dx \\ &\quad = \int_{B_r(x_0) \setminus B_{c(\kappa)r}(y)} \Phi_y \lambda \, dx - \int_{B_r(x_0)} \Phi_y \tilde{f} \chi_{\{u_*>0\}} \, dx \\ &\quad = \int_{B_r(x_0) \setminus B_{c(\kappa)r}(y)} \Phi_y \Delta u_* \, dx - \int_{B_{c(\kappa)r}(y)} \Phi_y \tilde{f} \, dx \\ &\quad \leq \|\Phi_y\|_{L^\infty(B_r(x_0) \setminus B_{c(\kappa)r}(y))} \int_{B_r(x_0)} \Delta u_* \, dx + \|\tilde{f}\|_{L^\infty(\Omega)} \int_{B_{c(\kappa)r}(y)} \Phi_y \, dx \end{aligned}$$

$$\le C(\kappa)r^{2-n}\int_{B_r(x_0)}\Delta u_*\,dx + C(\kappa)r^2 \tag{B.42}$$

for all sufficiently small $r > 0$. Combining (B.41) and (B.42), we conclude with (B.40).

Let K be any compact set in $\partial\{u_* > 0\} \cap \{g > 0\}$. We now cover K by the balls $B_r(x_0)$ given in (B.40), and we know that

$$\mathcal{H}^{n-1}(K) \le C\int_K \Delta u_*\,dx \overset{\text{(B.37)}}{<} \infty,$$

which shows that $\{u_* > 0\}$ has locally finite perimeter in $\{g > 0\}$. Combining this with Remark B.9 and (A.2), we conclude with (6.2).

Step 4: Sketching the proof of (6.3). Combining (6.2) and (B.36), we see that

$$\Delta u_* + \tilde{f}\chi_{\{u_*>0\}} = h\mathcal{H}^{n-1}\lfloor\partial_{\mathrm{red}}\{u_* > 0\}$$

for some Borel function $h \ge 0$ on $\partial_{\mathrm{red}}\Omega \cap \{g > 0\}$. It just remains to show

$$h(x_0) = g(x_0) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in \partial_{\mathrm{red}}\{u_* > 0\} \cap \{g > 0\}. \tag{B.43}$$

Despite the fact that the ideas are virtually same as in [1, 4.7–5.5], here we still present the details for the reader's convenience. It is enough to prove (B.43) for those $x_0 \in \partial_{\mathrm{red}}\{u_* > 0\} \cap \{g > 0\}$ which satisfy

$$\limsup_{r\to\infty}\frac{\mathcal{H}^{n-1}(B_r(x_0)\cap\partial\{u_* > 0\})}{\omega_{n-1}r^{n-1}} \le 1 \quad \text{(see Lemma A.4)}, \tag{B.44a}$$

$$\limsup_{r\to\infty} ⨍_{\partial\{u_*>0\}\cap B_r(x)} |h(x) - h(x_0)|\,dS(x) = 0; \tag{B.44b}$$

see [1, Remark 4.9]. Without loss of generality, we assume that

$$x_0 = 0 \quad \text{and} \quad \nu_{\{u_*>0\}}(0) = e_n = (0, \cdots, 0, 1).$$

Define the blow-up sequences

$$u_n(x) = nu_*(n^{-1}x), \quad \tilde{f}_n(x) = n^{-1}\tilde{f}(n^{-1}x),$$
$$g(x) = g(n^{-1}x), \quad h_n(x) = h(n^{-1}x), \quad \Omega_n = \{u_n > 0\}.$$

Note that u, f, g are scaled according to [18, Remark 2.7 (with $\alpha = -1$)]. It is also easy to see that

$$|\tilde{f}_n| \le C/n, \tag{B.45a}$$

$$\int_{B_1}|g(n^{-1}x) - g(0)|\,dx \to 0, \tag{B.45b}$$

$$\int_{\partial\Omega_n\cap B_1}|h(n^{-1}x) - h(0)|\,dS(x) \to 0 \tag{B.45c}$$

as $n \to \infty$. We define the half-space $H := \{x_n < 0\}$. Using [13, Section 5.7.2, Theorem 1], we know that

$$|(\Omega_n \triangle H) \cap B_1| \to 0 \quad \text{as } n \to \infty, \tag{B.46}$$

where $\triangle$ denotes the symmetric difference between the sets. Using Proposition B.5 (together with the assumption $\overline{\{u_* > 0\}} \subset \Omega$) and Remark B.7, we know that $|\nabla u_n| \leq C$. It follows that there exists a Lipschitz continuous limit function $u_0 \geq 0$ such that, for a subsequence,

$$\begin{aligned} u_n &\to u_0 && \text{uniformly in } B_1, \\ \nabla u_n &\to \nabla u_0 && L^\infty(B_1)\text{-weak} * . \end{aligned}$$

Using Remark B.7, we know that $u_n(x) \geq C \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_n)$. Using (4.2b) in Proposition 4.2 and (B.45a), we know that

$$|\Delta u_n| = |\tilde{f}_n| \leq C/n \quad \text{in } \Omega_n.$$

Therefore, we know that u_0 is harmonic in Ω_0 and, in particular, $\Omega_n \cap B_1 \to \Omega_0 \cap B_1$ in measure. Therefore, from (B.46), we have $|\Omega_0 \triangle H| = 0$. Since Ω_0 is open, we conclude that

$$\Omega_0 \subset H \quad \text{and} \quad |H \setminus \Omega_0| = 0.$$

Using [1, Theorem 4.8] together with (B.44a) and (B.44b), we know that

$$\Omega_0 = H \quad \text{and} \quad u_0(x) = h(0)(-x_n)_+.$$

On the other hand, using [1, Lemma 5.4], we know that u_0 is a global minimum of

$$\mathcal{J}_0(v) := \int_B (|\nabla v|^2 + g(0)^2 \chi_{\{v>0\}}) \quad \text{among all } v \in \mathbb{K}(B_1).$$

Therefore, using Proposition B.4, we conclude that $h(0) = g(0)$, and we thus complete the proof. ∎

C. Computations related to Bessel functions

The main purpose of this appendix is to exhibit the details of computation for Lemma 7.2.

Computations of (7.8). Since the solution space of $u''(r) + \frac{n-1}{r}u'(r) + \lambda u(r) = 0$ for $r > 0$ is spanned by

$$r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) \quad \text{and} \quad r^{\frac{2-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{\lambda} r),$$

the solution of (7.6a) must satisfy

$$\begin{aligned} u(r) &= \frac{b-a}{\lambda} + c_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) \quad \text{for } r \in (0, r_1), \\ u(r) &= \frac{b}{\lambda} + c_2 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) + c_3 r^{\frac{2-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{\lambda} r) \quad \text{for } r_1 < r. \end{aligned}$$

So, we know that

$$u'(r) = -c_1\sqrt{\lambda} r^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r) \quad \text{for } r \in (0, r_1),$$
$$u'(r) = -c_2\sqrt{\lambda} r^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r) - c_3\sqrt{\lambda} r^{\frac{2-n}{2}} Y_{\frac{n}{2}}(\sqrt{\lambda} r) \quad \text{for } r_1 < r.$$

Since u is C^1 at r_1, and since $\lambda > 0$ and $r > 0$, we have

$$(c_2 - c_1) J_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) + c_3 Y_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) = -\frac{a r_1^{\frac{n-2}{2}}}{\lambda},$$
$$(c_2 - c_1) J_{\frac{n}{2}}(\sqrt{\lambda} r_1) + c_3 Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) = 0,$$

that is,

$$\begin{pmatrix} J_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) & Y_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) \\ J_{\frac{n}{2}}(\sqrt{\lambda} r_1) & Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) \end{pmatrix} \begin{pmatrix} c_2 - c_1 \\ c_3 \end{pmatrix} = -\frac{a r_1^{\frac{n-2}{2}}}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that

$$\det \begin{pmatrix} J_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) & Y_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) \\ J_{\frac{n}{2}}(\sqrt{\lambda} r_1) & Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) \end{pmatrix} = -\frac{2}{\pi\sqrt{\lambda} r_1},$$

so

$$\begin{pmatrix} c_2 - c_1 \\ c_3 \end{pmatrix} = \frac{\pi\sqrt{\lambda} r_1}{2} \begin{pmatrix} Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) & -Y_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) \\ -J_{\frac{n}{2}}(\sqrt{\lambda} r_1) & J_{\frac{n-2}{2}}(\sqrt{\lambda} r_1) \end{pmatrix} \frac{a r_1^{\frac{n-2}{2}}}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} \begin{pmatrix} Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) \\ -J_{\frac{n}{2}}(\sqrt{\lambda} r_1) \end{pmatrix}.$$

Hence, we know that

$$\begin{aligned} u(r) &= \chi_{\{r<r_1\}}\Big[\frac{b-a}{\lambda} + c_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r)\Big] \\ &\quad + \chi_{\{r>r_1\}}\Big[\frac{b}{\lambda} + c_2 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) + c_3 r^{\frac{2-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{\lambda} r)\Big] \\ &= \chi_{\{r<r_1\}}\Big[\frac{b-a}{\lambda} + c_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r)\Big] \\ &\quad + \chi_{\{r>r_1\}}\Big[\frac{b}{\lambda} + \Big(c_1 + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} Y_{\frac{n}{2}}(\sqrt{\lambda} r_1)\Big) r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) \\ &\quad - \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1) r^{\frac{2-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{\lambda} r)\Big] \\ &= \frac{b-a}{\lambda} + c_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r) \\ &\quad + \chi_{\{r>r_1\}}\Big[\frac{a}{\lambda} + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} r^{\frac{2-n}{2}} \Big(Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) J_{\frac{n-2}{2}}(\sqrt{\lambda} r) - J_{\frac{n}{2}}(\sqrt{\lambda} r_1) Y_{\frac{n-2}{2}}(\sqrt{\lambda} r)\Big)\Big], \end{aligned}$$

which shows (7.8) and

$$u'(r) = -c_1\sqrt{\lambda}r^{\frac{2-n}{2}}J_{\frac{n}{2}}(\sqrt{\lambda}r) - \chi_{\{r>r_1\}}\frac{a\pi r_1^{\frac{n}{2}}}{2}r^{\frac{2-n}{2}}\big(Y_{\frac{n}{2}}(\sqrt{\lambda}r_1)J_{\frac{n}{2}}(\sqrt{\lambda}r) - J_{\frac{n}{2}}(\sqrt{\lambda}r_1)Y_{\frac{n}{2}}(\sqrt{\lambda}r)\big) \tag{C.1}$$

hold. The proof is complete. ■

Proof of (7.9). Since $J_{\frac{n}{2}}(\sqrt{\lambda}r)$ is non-negative on $(0, R)$ by the assumption on λ, we deduce that u' has constant sign on $(0, r_1)$. We see that

$$u'(r) = \overbrace{r^{\frac{2-n}{2}}J_{\frac{n}{2}}(\sqrt{\lambda}r)}^{>0}\Big[-c_1\sqrt{\lambda} - \frac{a\pi r_1^{\frac{n}{2}}}{2}\Big(Y_{\frac{n}{2}}(\sqrt{\lambda}r_1) - J_{\frac{n}{2}}(\sqrt{\lambda}r_1)\frac{Y_{\frac{n}{2}}(\sqrt{\lambda}r)}{J_{\frac{n}{2}}(\sqrt{\lambda}r)}\Big)\Big] \tag{C.2}$$

for all $r > r_1$. Since

$$\frac{\partial}{\partial r}\Big(\frac{Y_{\frac{n}{2}}(\sqrt{\lambda}r)}{J_{\frac{n}{2}}(\sqrt{\lambda}r)}\Big) = \frac{2}{\pi r J_{\frac{n}{2}}(\sqrt{\lambda}r)^2} > 0 \quad \text{for } r \in (0, R) \subset (0, j_{\frac{n-2}{2},1}\lambda^{-\frac{1}{2}}),$$

we deduce that there is at most one point $r' \in (r_1, R)$ where $u'(r') = 0$ and u' is negative on (r_1, r') and positive on (r', R) (not excluding the possibilities that $r' = r_1$ or $r = R$). This implies that u can at most have two zeros in $(0, R]$. Since u has at least one zero in $(0, R]$, we have the following cases:

(1) If u has exactly one zero $0 < \rho_0 \leq R$, then $u'(\rho_0) \leq 0$.

(2) If u has exactly two zeros $0 < \rho_1 < \rho_2 \leq R$, then $u'(\rho_1) \leq 0$ and $u'(\rho_2) \geq 0$. In this case, we choose $\rho_0 = \rho_1$.

In either case, we conclude with (7.9). ■

Proof of (7.10). Plugging $u(\rho) = 0$ into (7.8), we have

$$0 = \frac{b-a}{\lambda} + c_1\rho^{\frac{2-n}{2}}J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) + \chi_{\{r>r_1\}}\Big[\frac{a}{\lambda} + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}}\rho^{\frac{2-n}{2}}\Big(Y_{\frac{n}{2}}(\sqrt{\lambda}r_1)J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) - J_{\frac{n}{2}}(\sqrt{\lambda}r_1)Y_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)\Big)\Big].$$

We now show that $\rho > r_1$. Suppose, on the contrary, that $\rho \leq r_1$. From (7.8), we have

$$c_1 = \frac{a-b}{\lambda}\frac{1}{\rho^{\frac{2-n}{2}}J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} > 0.$$

From (C.1) and (7.9), we know that

$$-g(\rho) = u'(\rho) = -\frac{a-b}{\sqrt{\lambda}}\frac{J_{\frac{n}{2}}(\sqrt{\lambda}\rho)}{J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} < 0;$$

this contradicts the assumption $g = 0$ in $\overline{B_{r_1}}$. Therefore, $\rho > r_1$, and hence, we know that $\{\widetilde{u} > 0\} \supset \overline{B_{r_1}}$.

From (7.8), we have

$$
\begin{aligned}
0 &= \frac{b}{\lambda} + c_1 \rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) \\
&\quad + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} \rho^{\frac{2-n}{2}} \left(Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) - J_{\frac{n}{2}}(\sqrt{\lambda} r_1) Y_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) \right) \\
&= \frac{b}{\lambda} + \rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) \left[c_1 + \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} Y_{\frac{n}{2}}(\sqrt{\lambda} r_1) \right] - \frac{a\pi r_1^{\frac{n}{2}}}{2\sqrt{\lambda}} \rho^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1) Y_{\frac{n-2}{2}}(\sqrt{\lambda}\rho),
\end{aligned}
$$

which implies

$$
-c_1\sqrt{\lambda} = \frac{b}{\sqrt{\lambda}} \frac{1}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} - \frac{a\pi r_1^{\frac{n}{2}}}{2} J_{\frac{n}{2}}(\sqrt{\lambda} r_1) \frac{Y_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)}{J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} + \frac{a\pi r_1^{\frac{n}{2}}}{2} Y_{\frac{n}{2}}(\sqrt{\lambda} r_1).
$$

From (C.2), we have

$$
\begin{aligned}
u'(\rho) &= \rho^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda}\rho) \Bigg[\frac{b}{\sqrt{\lambda}} \frac{1}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} \\
&\qquad - \frac{a\pi r_1^{\frac{n}{2}}}{2} J_{\frac{n}{2}}(\sqrt{\lambda} r_1) \overbrace{\left(\frac{Y_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)}{J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} - \frac{Y_{\frac{n}{2}}(\sqrt{\lambda}\rho)}{J_{\frac{n}{2}}(\sqrt{\lambda}\rho)} \right)}^{= \frac{2}{\pi\sqrt{\lambda}\rho J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) J_{\frac{n}{2}}(\sqrt{\lambda}\rho)}} \Bigg] \\
&= \rho^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda}\rho) \Bigg[\frac{b}{\sqrt{\lambda}} \frac{1}{\rho^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)} \\
&\qquad - \frac{a\pi r_1^{\frac{n}{2}}}{2} J_{\frac{n}{2}}(\sqrt{\lambda} r_1) \frac{2}{\pi\sqrt{\lambda}\rho J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) J_{\frac{n}{2}}(\sqrt{\lambda}\rho)} \Bigg] \\
&= \frac{b\rho^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda}\rho) - a r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{\rho^{\frac{n}{2}} \sqrt{\lambda} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho)}.
\end{aligned}
$$

Since $\rho^{\frac{n}{2}} \sqrt{\lambda} J_{\frac{n-2}{2}}(\sqrt{\lambda}\rho) > 0$, the sign of $u'(\rho)$ is the same as that of

$$
\phi(\rho) \equiv \frac{b}{a} - \frac{r_1^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r_1)}{\rho^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda}\rho)}.
$$

Note that $\phi : [r_1, R] \to \mathbb{R}$ is monotonically increasing and $\phi(r_1) = \frac{b-a}{a} < 0$. Let $R' > r_1$ as in (7.2). Then, $\phi(R') = 0$ and also $u'(R') = 0$. Since u is monotonically decreasing on $(0, R')$, we conclude with (7.10). ■

D. Examples of hybrid k-quadrature domains with real-analytic boundary

In this appendix we construct some examples of hybrid k-quadrature domains using the Cauchy–Kowalevski theorem. We extend [17, Theorems 3.1 and 3.3] in the theorems which follow.

Theorem D.1. *Let D be a bounded domain in $\mathbb{R}^n$ with real-analytic boundary. Let g be real analytic on a neighborhood of ∂D with $g > 0$ on ∂D. For each $k \geq 0$, there exists a bounded positive measure μ_1 with* $\operatorname{supp}(\mu_1) \subset D$ *such that D is a hybrid k-quadrature domain corresponding to μ_1 with density* $(g, 0)$.

Theorem D.2. *Let $0 \leq k < j_{\frac{n-2}{2},1} R^{-1}$ and let D be a bounded domain such that $\overline{D} \subset B_R$ and ∂D is real-analytic. Let h be a non-negative integrable function on a neighborhood of $\overline{D}$, which is real-analytic on some neighborhood of ∂D. There exists a bounded positive measure μ_2 with* $\operatorname{supp}(\mu_2) \subset D$ *such that D is a hybrid k-quadrature domain corresponding to μ_2 with density* $(0, h)$.

The main purpose of this section is to prove Theorems D.1 and D.2.

Proof of Theorem D.1. By the Cauchy–Kowalevski theorem, there exist a neighborhood $\mathcal{N}$ of ∂D and a real-analytic function u satisfying

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } \mathcal{N}, \\ u = 0 & \text{on } \partial D, \\ \partial_\nu u = -g & \text{on } \partial D. \end{cases}$$

Following [17, Proposition 3.2], there exist an open set $W \subset D \cap \mathcal{N}$ and a constant $\varepsilon > 0$ such that

(1) u is positive on W,

(2) $\partial D \subset \partial W$,

(3) $u(x) = \varepsilon$ for all $x \in \partial W \setminus \partial D$,

(4) $\partial_\nu u < 0$ on $\partial W \setminus \partial D$. In particular, ∇u vanishes nowhere on $\partial W \setminus \partial D$, so $\partial W \setminus \partial D$ is real-analytic.

For each $w \in H^1(D)$ with $(\Delta + k^2)w = 0$ in D, we have

$$0 = \int_W (w\Delta u - u\Delta w)\,dx = \int_{\partial W} (w\partial_\nu u - u\partial_\nu w)\,dS,$$

where $dS(x) = d\mathcal{H}^{n-1}(x)$ is the surface measure, and hence,

$$\int_{\partial D} w\partial_\nu u\,dS - \int_{\partial W\setminus\partial D} w\partial_\nu u\,dS = \int_{\partial D} u\partial_\nu w\,dS - \int_{\partial W\setminus\partial D} u\partial_\nu w\,dS.$$

Let D' be the region with $\partial D' = \partial W \setminus \partial D$ (and thus, $\overline{D'} \subset D$); then, we see that

$$-\int_{\partial D} wg\,dS - \int_{\partial D'} w\partial_\nu u\,dS = -\varepsilon \int_{\partial W\setminus\partial D} \partial_\nu w\,dS$$
$$= \varepsilon \int_{D'} \Delta w\,dx = -\varepsilon k^2 \int_D \chi_{D'} w\,dx,$$

that is,

$$\int_{\partial D} wg\,dS = \varepsilon k^2 \int_D \chi_{D'} w\,dx - \int_{\partial D'} w\partial_\nu u\,dS.$$

Finally, we define the measure $\mu_1 := \varepsilon k^2 \mathcal{L}^n\lfloor D' - \partial_\nu u \mathcal{H}^{n-1}\lfloor \partial D'$. Note that $\operatorname{supp}(\mu_1) \subset D$ and μ_1 is positive and bounded; hence, we know that

$$\langle \mu_1, w\rangle = \int_{\partial D} gw\,dS \tag{D.1}$$

for all $w \in H^1(D)$ with $(\Delta + k^2)w = 0$ in D. For each $y \in \mathbb{R}^n \setminus \overline{D}$, choosing $w(x) = \Psi_k(x, y)$, where Ψ is any fundamental solution of $-(\Delta + k^2)$ in (D.1), we reach

$$\Psi_k * \mu_1 = \Psi_k * (g\mathcal{H}^{n-1}\lfloor \partial D) \quad \text{in } \mathbb{R}^n \setminus \overline{D}. \tag{D.2}$$

Since ∂D is real-analytic, the right-hand side of (D.2) is the single-layer potential. Therefore, by continuity of the single-layer potential [26, Theorem 6.11] and since μ_1 is bounded, we conclude our theorem. ■

Proof of Theorem D.2. If h vanishes identically on a neighborhood of ∂D, we have nothing to prove. We now assume that this is not the case.

Let $v \in H_0^1(D)$ be the unique solution of

$$\begin{cases} (\Delta + k^2)v = h & \text{in } D,\\ v = 0 & \text{on } \partial D. \end{cases}$$

Since $h \geq 0$, using the strong maximum principle for the Helmholtz operator (see [23, Appendix]), we know that $v(x) < 0$ for all $x \in D$. By the analyticity theorem for elliptic equations, v extends real-analytically to some neighborhood of ∂D. Using integration by parts, we have

$$\langle \sigma_2, w\rangle = \int_D wh\,dx = \int_D (w\Delta v - v\Delta w)\,dx$$
$$= \int_{\partial D} (w\partial_\nu v - v\partial_\nu w)\,dS = \int_{\partial D} w\partial_\nu v\,dS, \tag{D.3}$$

where σ_2 is the measure given as in (1.6).

Again, using the strong maximum principle for the Helmholtz operator, we know that the unique solution u_0 of

$$\begin{cases} (\Delta + k^2)u_0 = 0 & \text{in } D,\\ u_0 = 1 & \text{on } \partial D \end{cases}$$

must satisfy $u_0 > 0$ in $\overline{D}$. By observing that

$$(\Delta + k^2)v = u_0^{-1}\nabla \cdot u_0^2 \nabla (u_0^{-1} v) \quad \text{in } D,$$

we can apply the Hopf maximum principle (see, e.g., [15, Lemma 3.4]) on $u_0^{-1}v$ to ensure $\partial_\nu (u_0^{-1} v) > 0$ on ∂D. Since $v = 0$ on ∂D,

$$\partial_\nu v = u_0 \partial_\nu (u_0^{-1} v) > 0 \quad \text{on } \partial D.$$

Since ∂D is real-analytic, ∂D locally has a representation as $\{y \mid \varphi(y) = 0\}$ for some real-analytic φ with non-vanishing gradient. We may choose φ to be positive outside D, so

$$\partial_\nu v = \nabla v \cdot \frac{\nabla \varphi}{|\nabla \varphi|} \quad \text{on } \partial D,$$

that is, $\partial_\nu v$ can be extended real-analytically and is strictly positive near ∂D. Using Theorem D.1, there exists a positive bounded measure μ_2 with supp $(\mu_2) \subset D$ such that

$$\langle \mu_2, w \rangle = \int_{\partial D} w \partial_\nu v \, dS. \tag{D.4}$$

Combining (D.3) and (D.4), we conclude our theorem. ∎

E. Some remarks on null k-quadrature domains

In this appendix we give some remarks on null k-quadrature domains. They are defined by Definition 1.1 with $g \equiv 0$, $h \equiv 1$, and $\mu \equiv 0$. It was confirmed in [10, 11] that the null 0-quadrature domains in $\mathbb{R}^n$ with $n \geq 2$ must be either

- half-space,
- the complement of an ellipsoid,
- the complement of a paraboloid, or
- the complement of a cylinder with an ellipsoid or a paraboloid as its base;

see also [14, 20, 21, 28] for some classical works.

In addition, it is worth mentioning that in the two-dimensional case, starting from null 0-quadrature domains, we can always construct quadrature domains of positive measure [29, Theorem 11.5]. This motivates us to study null k-quadrature domains for $k > 0$. We also give some remarks showing that they are quite different.

As a consequence of the mean value theorem for the Helmholtz operator $-(\Delta + k^2)$ (see, e.g., [23, Appendix]), it is not difficult to see that a ball is a null k-quadrature domain (i.e., $\mu \equiv 0$) if and only if its radius r satisfies $J_{\frac{n}{2}}(kr) = 0$, where J_α denotes the Bessel function of the first kind.

Following the ideas in [2], one can show the next theorem.

Theorem E.1. *Let $n \geq 2$ be an integer, $k > 0$, and $\theta \in (0, \frac{\pi}{2})$. We consider the conical domain*

$$\Sigma_\theta := \left\{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > -|x| \tan\theta\right\} \quad \textit{(see [2, Figure 1]).}$$

If $w \in L^1(\Sigma_\theta)$ satisfies $(\Delta + k^2)w = 0$ in Σ_θ, then $w \equiv 0$ in Σ_θ.

The notion of "null k-quadrature domains" for $k > 0$ therefore makes no sense for general unbounded sets. Hence, it is natural to ask about the classification of *bounded* null k-quadrature domains. As pointed out in our previous work [23], this problem is equivalent to the well-known Pompeiu's problem [27, 34], which is also equivalent to an obstacle-type free boundary problem [31, 32]. The unanswered question is whether any bounded (Lipschitz) null k-quadrature domain must be a ball. We also remark that (1.2) is related to Schiffer's problem, which asks whether the existence of a non-trivial solution u of

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ |\nabla u| = 1 & \text{on } \partial D \end{cases}$$

implies that such a (Lipschitz) bounded domain D must be a ball. We refer to [4] for the Pompeiu problem for convex domains in $\mathbb{R}^2$, where several equivalent formulations (including the equivalence with the Morera problem) as well as some partial results are given. See also [8] for the case of domains in $\mathbb{R}^2$ which are strictly convex, and [9] for domains in $\mathbb{R}^2$ under different assumptions.

It remains to prove Theorem E.1.

E.1. The case when $n = 2$

We denote by $\mathcal{F}_1$ the one-dimensional Fourier transform given by

$$\mathcal{F}_1\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) e^{-ix\xi}\, dx \quad \text{for all } \xi \in \mathbb{R},$$

which is clearly well-defined for $\varphi \in L^1(\mathbb{R})$ and can be extended for tempered distributions $\varphi \in \mathcal{S}'(\mathbb{R})$.

Lemma E.2. *Given any $\varepsilon > 0$ and $k > 0$, let $v \in L^1(\mathbb{R} \times (-\varepsilon, \infty))$ satisfy $(\Delta + k^2)v = 0$ in $\mathbb{R} \times (-\varepsilon, \infty)$. Then,*

$$v(x, y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} e^{ix\xi} (\mathcal{F}_1 v)(\xi, 0) e^{-y\sqrt{\xi^2-k^2}}\, d\xi$$

for all $(x, y) \in \mathbb{R} \times [0, \infty)$ with

$$\begin{aligned} |(\mathcal{F}_1 v)(\xi, 0)| &\leq \frac{1}{\sqrt{2\pi}} \|v\|_{L^1(\mathbb{R}\times(-\varepsilon,\infty))} \sqrt{\xi^2 - k^2} e^{-\varepsilon\sqrt{\xi^2-k^2}} && \textit{for all } |\xi| > k, \\ (\mathcal{F}_1 v)(\xi, 0) &= 0 && \textit{for all } |\xi| \leq k. \end{aligned}$$

Proof. Since $v \in L^1(\mathbb{R} \times (-\varepsilon, \infty))$, by Fubini's theorem we have $v(\cdot, y) \in L^1(\mathbb{R})$ for almost every $y \in (-\varepsilon, \infty)$, and so $\mathcal{F}_1 v(\cdot, y)$ is continuous and

$$\begin{aligned}\int_{-\varepsilon}^{\infty} \|\mathcal{F}_1 v(\cdot, y)\|_{L^\infty(\mathbb{R})}\, dy &\le \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{\infty} \|v(\cdot, y)\|_{L^1(\mathbb{R})}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \|v\|_{L^1(\mathbb{R}\times(-\varepsilon,\infty))}. \end{aligned} \tag{E.1}$$

By applying the Fourier transform $\mathcal{F}_1$ on $(\Delta + k^2)v = 0$ in $\mathbb{R} \times (-\varepsilon, \infty)$, for almost every $\xi \in \mathbb{R}$, we have

$$\partial_y^2 (\mathcal{F}_1 v)(\xi, y) + (k^2 - \xi^2)(\mathcal{F}_1 v)(\xi, y) \quad \text{in } \mathcal{D}'(-\varepsilon, \infty). \tag{E.2}$$

Since $\mathcal{F}_1 v(\cdot, y) \in L^\infty(\mathbb{R})$, the general solution of ODE (E.2) is given by

$$(\mathcal{F}_1 v)(\xi, y) = \begin{cases} A(\xi) e^{-(y+\varepsilon)\sqrt{\xi^2 - k^2}} & \text{for } |\xi| > k, \\ B_1(\xi) e^{-i(y+\varepsilon)\sqrt{k^2-\xi^2}} + B_2(\xi) e^{i(y+\varepsilon)\sqrt{k^2-\xi^2}} & \text{for } |\xi| \le k, \end{cases} \tag{E.3}$$

for some complex-valued functions A, B_1, B_2. For each $|\xi| < k$, we see that $(\mathcal{F}_1 v)(\xi, \cdot)$ is periodic with respect to variable y. However, from (E.1), we must have

$$(\mathcal{F}_1 v)(\xi, \cdot) = 0 \quad \text{for all } |\xi| < k.$$

When $|\xi| = k$, we have $(\mathcal{F}_1 v)(\xi, y) = B_1(\xi) + B_2(\xi)$. Again, by (E.1), we must have

$$(\mathcal{F}_1 v)(\xi, \cdot) = 0 \quad \text{for all } |\xi| = k.$$

Therefore, we can write (E.3) as

$$(\mathcal{F}_1 v)(\xi, y) = A(\xi) e^{-(y+\varepsilon)\sqrt{\xi^2-k^2}}, \text{ provided } A(\xi) = 0 \text{ for all } |\xi| \le k. \tag{E.4}$$

Plugging (E.4) into (E.1) yields

$$\frac{1}{\sqrt{2\pi}} \|v\|_{L^1(\mathbb{R}\times(-\varepsilon,\infty))} \ge |A(\xi)| \int_{-\varepsilon}^{\infty} e^{-(y+\varepsilon)\sqrt{\xi^2-k^2}}\, dy = \frac{|A(\xi)|}{\sqrt{\xi^2 - k^2}} \quad \text{for all } |\xi| > k.$$

Finally, using the Fourier inversion formula on (E.4), we conclude the proof. ∎

Corollary E.3. *Given any $\ell > 0$, $k > 0$, and $\theta \in (0, \frac{\pi}{2})$, we consider the half-planes*

$$\Pi_{\theta,\varepsilon}^{\pm} := \{(x, y) \mid y > -\ell \mp x \tan\theta\} \equiv \{(x, y) \mid \pm x \sin\theta + y\cos\theta > -\ell\cos\theta\}.$$

If $w_\pm \in L^1(\Pi_\theta^\pm)$ satisfies $(\Delta + k^2) w_\pm = 0$ in $\Pi_\theta^\pm$, then

$$w_\pm(x, y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} e^{i(x\cos\theta \mp y\sin\theta)\xi} \hat{\varphi}_\pm(\xi) e^{-(\pm x\sin\theta + y\cos\theta)\sqrt{\xi^2-k^2}}\, d\xi$$

for all $(x, y) \in \{(x, y) \mid y \ge \mp x \tan\theta\}$, for some function $\hat{\varphi}_\pm$ satisfying

$$\begin{aligned} |\hat{\varphi}_\pm(\xi)| &\le \frac{1}{\sqrt{2\pi}} \|w_\pm\|_{L^1(\Pi_\theta^\pm)} \sqrt{\xi^2 - k^2}\, e^{-\ell\cos\theta\sqrt{\xi^2-k^2}} && \textit{for all } |\xi| > k, \\ \hat{\varphi}_\pm(\xi) &= 0 && \textit{for all } |\xi| \le k. \end{aligned} \tag{E.5}$$

Remark E.4. In particular, we have

$$w_\pm(x,0) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} e^{i(x\cos\theta)\xi} \widehat{\varphi}_\pm(\xi) e^{\mp x \sin\theta \sqrt{\xi^2-k^2}}\, d\xi$$

for all x with $\pm x \geq 0$.

Proof of Corollary E.3. Since the Laplacian is rotation invariant, the function $v_\pm$ is given by

$$w_\pm(x,y) = v_\pm(x\cos\theta \mp y\sin\theta, \pm x\sin\theta + y\cos\theta)$$

and satisfies $v_\pm \in L^1(\mathbb{R} \times (-\ell\cos\theta, \infty))$ with $(\Delta + k^2)v_\pm = 0$ in $\mathbb{R} \times (-\ell\cos\theta, \infty)$. Therefore, using Lemma E.2 with $\varepsilon = \ell\cos\theta$ yields

$$v_\pm(x,y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k} e^{ix\xi} (\mathcal{F}_1 v_\pm)(\xi,0) e^{-y\sqrt{\xi^2-k^2}}\, d\xi$$

with

$$\begin{aligned} |(\mathcal{F}_1 v_\pm)(\xi,0)| &\leq \frac{1}{\sqrt{2\pi}} \|w_\pm\|_{L^1(\Pi_\theta^\pm)} \sqrt{\xi^2 - k^2} e^{-\ell\cos\theta\sqrt{\xi^2-k^2}} && \text{for all } |\xi| > k, \\ (\mathcal{F}_1 v_\pm)(\xi,0) &= 0 && \text{for all } |\xi| \leq k, \end{aligned}$$

because $\|v_\pm\|_{L^1(\mathbb{R}\times(-\ell\cos\theta,\infty))} = \|w_\pm\|_{L^1(\Pi_\theta^\pm)}$, which concludes the proof. ∎

We are now ready to prove Theorem E.1 for the case when $n = 2$.

Proof of Theorem E.1 *when* $n = 2$. For each $\ell > 0$, we define

$$\Sigma_\theta^\ell := \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y + \ell > -|x|\tan\theta\}$$

and it is easy to see that $v(x,y) = w(x, y+\ell)$ satisfies $v \in L^1(\Sigma_\theta^\ell)$ and $(\Delta + k^2)v = 0$ in Σ_θ^ℓ. Since $\mathbb{R} \times (-\ell, \infty) \subset \Sigma_\theta^\ell$, v admits the representation as described in Lemma E.2 with $\varepsilon = \ell$. Since $\mathcal{F}v(\xi,0) = 0$ for all $|\xi| \leq k^2$, to prove our theorem, we only need to show that

$$\mathcal{F}_1 v(\cdot,0) \text{ is analytic in a neighborhood of the real axis.} \tag{E.6}$$

Since $w(\cdot,y) \in L^1(\mathbb{R})$ for almost every $y \in (0,\infty)$, we can choose $\ell > 0$ such that

$$v(\cdot,0) \equiv w(\cdot,\ell) \in L^1(\mathbb{R}),$$

so that we can write

$$\mathcal{F}_1 v(\xi,0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} v|_{\mathbb{R}_+}(x,0) e^{-ix\xi}\, dx + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_-} v|_{\mathbb{R}_-}(x,0) e^{-ix\xi}\, dx.$$

Using Remark E.4 with $w_\pm = v|_{\mathbb{R}_\pm}$, we have

$$\mathcal{F}_1 v(\xi,0) = \frac{1}{2\pi} \sum_\pm \int_{\mathbb{R}_\pm} \Big(\int_{|\eta|>k} e^{i(x\cos\theta)\eta} \widehat{\varphi}_\pm(\eta) e^{-|x|\sin\theta\sqrt{\eta^2-k^2}}\, d\eta \Big) e^{-ix\xi}\, dx, \tag{E.7}$$

where $\widehat{\varphi}_{\mp}$ satisfies (E.5). The integral given by (E.7) is well-defined, since

$$\begin{aligned}
&\int_{\mathbb{R}_\pm} \Big(\int_{|\eta|>k} |e^{i(x\cos\theta)\eta} \widehat{\varphi}_\pm(\eta) e^{-|x|\sin\theta\sqrt{\eta^2-k^2}}| \, d\eta \Big) |e^{-ix\xi}| \, dx \\
&\quad = \int_{\mathbb{R}_\pm} \Big(\int_{|\eta|>k} |\widehat{\varphi}_\pm(\eta)| e^{-|x|\sin\theta\sqrt{\eta^2-k^2}} \, d\eta \Big) \, dx \\
&\quad \le \frac{1}{\sqrt{2\pi}} \|v\|_{L^1(\Sigma_\theta^\ell)} \int_{|\eta|>k} \sqrt{\eta^2-k^2} e^{-\ell\cos\theta\sqrt{\eta^2-k^2}} \Big(\int_{\mathbb{R}_\pm} e^{-|x|\sin\theta\sqrt{\eta^2-k^2}} \, dx \Big) \, d\eta \\
&\quad = \frac{1}{\sqrt{2\pi}} \frac{\|v\|_{L^1(\Sigma_\theta^\ell)}}{\sin\theta} \int_{|\eta|>k} e^{-\ell\cos\theta\sqrt{\eta^2-k^2}} \, d\eta < \infty,
\end{aligned}$$

which holds because $\sin\theta > 0$ and $\cos\theta > 0$. From this, we are also able to use Fubini's theorem on (E.7) to reach

$$\begin{aligned}
\mathcal{F}_1 v(\xi, 0) &= \frac{1}{2\pi} \sum_\pm \int_{|\eta|>k} \Big(\int_{\mathbb{R}_\pm} e^{i(x\cos\theta)\eta} e^{-|x|\sin\theta\sqrt{\eta^2-k^2}} e^{-ix\xi} \, dx \Big) \widehat{\varphi}_\pm(\eta) \, d\eta \\
&= \int_{|\eta|>k} \frac{1}{2\pi} \sum_\pm \Big(\int_{\mathbb{R}_\pm} e^{x(i(\eta\cos\theta-\xi)\mp\sin\theta\sqrt{\eta^2-k^2})} \, dx \Big) \widehat{\varphi}_\pm(\eta) \, d\eta \\
&= \int_{|\eta|>k} F(\eta,\xi) \, d\eta,
\end{aligned}$$

where

$$F(\eta,\xi) = \frac{1}{2\pi} \sum_\pm \frac{\widehat{\varphi}_\pm(\eta)}{\mp i(\eta\cos\theta - \xi) + \sin\theta\sqrt{\eta^2-k^2}}.$$

Finally, following the proof in [2, Theorem 1.1] exactly, we conclude with (E.6), which completes the proof. ∎

E.2. The case when $n > 2$

The result for $n > 2$ can be proved exactly in the same way as in [2, Section 4], and hence, its proof is omitted.

Notation index

$\mathcal{L}^n \lfloor D$ Lebesgue measure restricted to D

$\mathcal{H}^{n-1} \lfloor \partial D$ $(n-1)$-dimensional Hausdorff measure on ∂D

$\partial_{\mathrm{mes}} E$ measure theoretic boundary

$\partial_{\mathrm{red}} E$ reduced boundary

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References

[1] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144 Zbl 0449.35105 MR 618549

[2] A.-S. Bonnet-Ben Dhia, S. Fliss, C. Hazard, and A. Tonnoir, A Rellich type theorem for the Helmholtz equation in a conical domain. *C. R. Math. Acad. Sci. Paris* **354** (2016), no. 1, 27–32 Zbl 1341.35014 MR 3439720

[3] J. E. Brothers and W. P. Ziemer, Minimal rearrangements of Sobolev functions. *J. Reine Angew. Math.* **384** (1988), 153–179 Zbl 0633.46030 MR 929981

[4] L. Brown and J.-P. Kahane, A note on the Pompeiu problem for convex domains. *Math. Ann.* **259** (1982), no. 1, 107–110 Zbl 0464.30035 MR 656655

[5] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*. Appl. Math. Sci. 93, Springer, Cham, 2019 Zbl 1425.35001 MR 3971246

[6] R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. II*. Wiley Classics Lib., John Wiley & Sons, New York, 1989 Zbl 0729.35001 MR 1013360

[7] E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. *Ricerche Mat.* **4** (1955), 95–113 Zbl 0066.29903 MR 74499

[8] J. Deng, Some results on the Schiffer conjecture in R^2. *J. Differential Equations* **253** (2012), no. 8, 2515–2526 Zbl 1253.35084 MR 2950461

[9] P. Ebenfelt, Some results on the Pompeiu problem. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **18** (1993), no. 2, 323–341 Zbl 0793.30034 MR 1234737

[10] S. Eberle, A. Finalli, and G. S. Weiss, Complete classification of global solutions to the obstacle problem. 2022, arXiv:2208.03108

[11] S. Eberle, H. Shahgholian, and G. S. Weiss, On global solutions of the obstacle problem. *Duke Math. J.* **172** (2023), no. 11, 2149–2193 Zbl 1522.35600 MR 4627249

[12] L. C. Evans, *Partial differential equations*. Second edn., Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 2010 Zbl 1194.35001 MR 2597943

[13] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*. Revised edn., Textb. Math., CRC Press, Boca Raton, FL, 2015 Zbl 1310.28001 MR 3409135

[14] A. Friedman and M. Sakai, A characterization of null quadrature domains in $\mathbf{R}^N$. *Indiana Univ. Math. J.* **35** (1986), no. 3, 607–610 Zbl 0615.31001 MR 855176

[15] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Classics Math., Springer, Berlin, 2001 Zbl 1042.35002 MR 1814364

[16] E. Giusti, *Minimal surfaces and functions of bounded variation*. Monogr. Math. 80, Birkhäuser, Basel, 1984 Zbl 0545.49018 MR 775682

[17] B. Gustafsson, M. Sakai, and H. S. Shapiro, On domains in which harmonic functions satisfy generalized mean value properties. *Potential Anal.* **7** (1997), no. 1, 467–484 Zbl 0882.31002 MR 1462581

[18] B. Gustafsson and H. Shahgholian, Existence and geometric properties of solutions of a free boundary problem in potential theory. *J. Reine Angew. Math.* **473** (1996), 137–179 Zbl 0846.31005 MR 1390686

[19] L. Hörmander, Lower bounds at infinity for solutions of differential equations with constant coefficients. *Israel J. Math.* **16** (1973), 103–116 Zbl 0271.35005 MR 340793

[20] L. Karp, On null quadrature domains. *Comput. Methods Funct. Theory* **8** (2008), no. 1–2, 57–72 Zbl 1156.31004 MR 2419460

[21] L. Karp and A. S. Margulis, Null quadrature domains and a free boundary problem for the Laplacian. *Indiana Univ. Math. J.* **61** (2012), no. 2, 859–882 Zbl 1272.35082 MR 3043599

[22] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*. Classics Appl. Math. 31, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000 Zbl 0988.49003 MR 1786735

[23] P.-Z. Kow, S. Larson, M. Salo, and H. Shahgholian, Quadrature domains for the Helmholtz equation with applications to non-scattering phenomena. *Potential Anal.* **60** (2024), no. 1, 387–424 Zbl 07798456 MR 4696043

[24] P.-Z. Kow, M. Salo, and H. Shahgholian, On positivity sets for Helmholtz solutions. *Vietnam J. Math.* **51** (2023), no. 4, 985–994 Zbl 07785099 MR 4669027

[25] P.-Z. Kow and J.-N. Wang, On the characterization of nonradiating sources for the elastic waves in anisotropic inhomogeneous media. *SIAM J. Appl. Math.* **81** (2021), no. 4, 1530–1551 Zbl 1473.35198 MR 4295059

[26] W. McLean, *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000 Zbl 0948.35001 MR 1742312

[27] D. Pompéiu, Sur certains systèmes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique* **188** (1929), 1138–1139 Zbl 55.0139.02

[28] M. Sakai, Null quadrature domains. *J. Analyse Math.* **40** (1981), 144–154 (1982) Zbl 0483.30002 MR 659788

[29] M. Sakai, *Quadrature domains*. Lecture Notes in Math. 934, Springer, Berlin-New York, 1982 Zbl 0483.30001 MR 663007

[30] M. Salo and H. Shahgholian, Free boundary methods and non-scattering phenomena. *Res. Math. Sci.* **8** (2021), no. 4, article no. 58 Zbl 1480.35408 MR 4323345

[31] S. A. Williams, A partial solution of the Pompeiu problem. *Math. Ann.* **223** (1976), no. 2, 183–190 Zbl 0329.35045 MR 414904

[32] S. A. Williams, Analyticity of the boundary for Lipschitz domains without the Pompeiu property. *Indiana Univ. Math. J.* **30** (1981), no. 3, 357–369 Zbl 0439.35046 MR 611225

[33] D. R. Yafaev, *Mathematical scattering theory*. Math. Surveys Monogr. 158, American Mathematical Society, Providence, RI, 2010 Zbl 1197.35006 MR 2598115

[34] L. Zalcman, A bibliographic survey of the Pompeiu problem. In *Approximation by solutions of partial differential equations (Hanstholm, 1991)*, pp. 185–194, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 365, Kluwer Academic Publishers Group, Dordrecht, 1992 Zbl 0830.26005 MR 1168719

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Pu-Zhao Kow
Department of Mathematical Sciences, National Chengchi University, No. 64, Sec. 2, ZhiNan Rd., Wenshan District, 116302 Taipei, Taiwan; pzkow@g.nccu.edu.tw

Mikko Salo
Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, 40014 Jyväskylä, Finland; mikko.j.salo@jyu.fi

Henrik Shahgholian (corresponding author)
Department of Mathematics, KTH Royal Institute of Technology, Lindstedtsvägen 25, 100 44 Stockholm, Sweden; henriksh@kth.se