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Twisted Hodge diamonds give rise to non-Fourier–Mukai functors

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Abstract. We apply computations of twisted Hodge diamonds to construct an infinite number of non-Fourier–Mukai functors with well-behaved target and source spaces.

To accomplish this, we first study the characteristic morphism introduced in Buchweitz and Flenner [Adv. Math. 217 (2008), 205–242] in order to control it for tilting bundles. Then, we continue by applying twisted Hodge diamonds of hypersurfaces embedded in projective space to compute the Hochschild dimension of these spaces. This allows us to compute the kernel of the embedding into the projective space in Hochschild cohomology. Finally, we use the above computations to apply the construction in Rizzardo, Van den Bergh, and Neeman [Invent. Math. 216 (2019), 927–1004] of non-Fourier–Mukai functors and verify that the constructed functors indeed cannot be Fourier–Mukai for odd-dimensional quadrics.

Using this approach, we prove that there are a large number of Hochschild cohomology classes that can be used for the construction of Rizzardo, Van den Bergh, and Neeman [Invent. Math. 216 (2019), 927–1004]. Furthermore, our results allow the application of computer-based calculations to construct candidate functors for arbitrary degree hypersurfaces in arbitrary high dimensions. Verifying that these are not Fourier–Mukai still requires the existence of a tilting bundle.

In particular, we prove that there is at least one non-Fourier–Mukai functor for every odd-dimensional smooth quadric.

1. Introduction

1.1. Background and results

The concept of Fourier–Mukai functors generalizes the idea of a correspondence to the categorical level.

Definition 1.1. A functor $f : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ between bounded derived categories of schemes is called Fourier–Mukai if there exists an object $M \in \mathcal{D}^b(Y \times X)$ such that

$$f \cong \Phi_M := \mathrm{R}\pi_{Y,*}\Big(M \overset{L}{\otimes} \mathrm{L}\pi_X^*(\_)\Big).$$

In this case, M is called the Fourier–Mukai kernel.

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In particular, these functors can be understood geometrically as Φ_M admits a complete characterization by $M \in \mathcal{D}^b(Y \times X)$.

It also turns out that most functorial constructions done in algebraic geometry are Fourier–Mukai. This means that understanding the property of being Fourier–Mukai, respectively, of not being Fourier–Mukai, is essential for understanding which functors between derived categories of sheaves may arise from geometric constructions and which do not. Another indicator of the geometric nature of Fourier–Mukai functors are the following results by V. Orlov and B. Toën.

Theorem ([18]). *Let X and Y be smooth projective schemes. Then, every fully faithful exact functor $\Psi_M : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is a Fourier–Mukai functor for some Fourier–Mukai kernel $M \in \mathcal{D}^b(X \times Y)$.*

Theorem ([25]). *Let X and Y be smooth projective schemes. Then, a functor*

$$\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$$

is precisely Fourier–Mukai if it is induced by a dg-functors between the canonical dg-enhancements.

The above results show that a lot of functors between derived categories of smooth projective schemes are Fourier–Mukai. So, Bondal, Larsen, and Lunts [4] conjectured nearly 20 years ago that every exact functor between such derived categories admits a description as a Fourier–Mukai functor.

This conjecture was disproven fifteen years later when A. Rizzardo, M. Van den Bergh, and A. Neeman [22] constructed the first non-Fourier–Mukai functor

$$\Psi_\eta : \mathcal{D}^b(Q_3) \hookrightarrow \mathcal{D}^b(\mathbb{P}^4),$$

where Q_3 denotes the smooth three-dimensional quadric in $\mathbb{P}^4$. Shortly thereafter, V. Vologodsky constructed in a note [26] another class of non-Fourier–Mukai functors over a field of characteristic $p > 0$. However, Vologodsky's functor turns out to be liftable to a $\mathbb{Z}_p$-linear dg-level, whereas the example from [22] can be proven to not even have a lift to the spectral level if one works over the rational numbers.

In this work, we generalize the result from [22] to higher dimensions. In particular, we will work over a field of characteristic zero in order to show that even in the nicest possible case there is an abundance of non-Fourier–Mukai functors.

We then verify that in the case of a smooth odd-dimensional quadric we can apply our result to get a non-Fourier–Mukai functors in arbitrary high dimensions.

Theorem. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth odd-dimensional quadric for $k > 2$. Then, we have an exact functor*

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b(\mathbb{P}^n)$$

that cannot be Fourier–Mukai.

1.2. Proof strategy

Generally, we follow the ideas from [22]. In order to conclude that we can construct more non-Fourier–Mukai functors, we include auxiliary results on the kernel of the pushforward in Hochschild cohomology. Furthermore, we will use more general objects, degrees, and indices. We need to do this as the proof in [22] is very specialized to the three-dimensional quadric, and one needs to take care when generalizing their strategy to a more general setting.

Recall that the construction in [22] proceeds in two steps:

(1) First, the authors construct a prototypical non-Fourier–Mukai functor between not necessarily geometric dg-categories.

(2) Second, using the behavior of Hochschild cohomology under embeddings, this functor is turned into a geometric functor.

More precisely, in step (1), [22] constructs a functor

$$L : \mathcal{D}^b(X) \to \mathcal{D}_\infty(\mathcal{X}_\eta)$$

for a smooth scheme X and $\eta \in \mathrm{HH}^{\geq \dim X+3}$, where $\mathcal{D}_\infty(\mathcal{X}_\eta)$ is the derived category of an $\mathcal{A}_\infty$-category arising as infinitesimal deformation in the η-direction.

In step (2), the construction of L is turned into a geometric one. In [22], this is achieved by showing that the canonical $\eta \in \mathrm{HH}^{2\dim Q_3}(Q_3, \omega_{Q_3}^{\otimes 2})$ is annihilated by the embedding $Q_3 \hookrightarrow \mathbb{P}^4$, which allows the passing from the algebraic world to the geometric world. The authors then define Ψ_η to be L composed with the pushforward into the geometric category $\mathcal{D}^b(\mathbb{P}^4)$.

Although the construction in [22] is very general, it has two major drawbacks:

The first is that although L is constructed to be prototypical non-dg, it is not obvious that the composition with the pushforward is again non-Fourier–Mukai. One usually handles these complications by applying an inductive obstruction theory that gets unwieldy quickly as one needs to keep track of inductively chosen lifts. Indeed, [22] only gives a single example of a non-Fourier–Mukai functor although the construction given in steps (1) and (2) is very general in nature.

We are able to solve this issue by restricting to Hochschild cohomology classes in degree $\dim(X) + 3$; this leads to the first obstruction vanishing, and so, we do not need to control the previous lifts in order to conclude that the pushed forward obstruction does not vanish.

The second drawback is that the results in [22] rely heavily on the existence of a tilting bundle in order to conclude that the prototypical functor L cannot be dg. Furthermore, in [20], T. Raedschelders, A. Rizzardo, and M. Van den Bergh construct an infinite amount of non-Fourier–Mukai functors using the prototypical L mentioned above. However, to do this, they apply a geometrification result by Orlov and hence lose control over the target space. In particular, the above mentioned geometrification result relies even more on the existence of a tilting bundle. Although our concrete examples still require the existence of a tilting bundle, we study the naturality of the characteristic morphism, which might in the future allow results using more general generators. In particular, we phrase our main result such that a non-vanishing characteristic morphism suffices, which is guaranteed for tilting objects.

Altogether this work improves on the construction from [22] to prove the existence of non-Fourier–Mukai functors,

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b(\mathbb{P}^{n+1}),$$

for Q a smooth quadric in arbitrary high dimension.

Furthermore, one can use our results to calculate the dimensions of choices for constructing candidate non-Fourier–Mukai functors as entries in twisted Hodge diamonds. For instance, if one wants to deform a smooth degree 6 hypersurface $f : X \hookrightarrow \mathbb{P}^{n+1}$ along the Hochschild cohomology of $\mathcal{O}_X(-8)$ in a way that might gives rise to a non-Fourier–Mukai functor, we may pick an η in a 20993-dimensional space:

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
2996 & & 20993 & & 15267 & & 917 & & 0 & & 0 \\
 & 1575 & & 0 & & 0 & & 0 & & 0 & \\
 & & 5775 & & 0 & & 0 & & 0 & & \\
 & & & 10395 & & 0 & & 0 & & & \\
 & & & & 9002 & & 0 & & & & \\
 & & & & & 2996. & & & & &
\end{array}$$

Notation

Throughout this work, we consider $\Bbbk$ to be a field of characteristic zero, and all schemes, algebras, $\mathcal{A}_\infty$-categories, and dg-categories are considered to be over $\Bbbk$. We will assume that all $\mathcal{A}_\infty$-structures are strictly unital and graded cohomologically.

Furthermore, the bounded derived category of coherent sheaves over a scheme X will be denoted by $\mathcal{D}^b(X)$ or $\mathcal{D}^b(\mathrm{coh}(X))$ depending on the context; wherever we need to pass to the category $\mathcal{D}^b_{\mathrm{coh}\,X}(\mathrm{Qch}(X))$ using [10, Proposition 3.5] we will indicate this. We denote the derived category of modules over a $\Bbbk$-linear category $\mathcal{X}$ as $\mathcal{D}(\mathcal{X})$ and also use the same notation for dg-categories. We refer to the dg-category of $\mathcal{A}_\infty$-modules over an $\mathcal{A}_\infty$-algebra $\mathcal{X}_\eta$ with homotopic maps identified by $\mathcal{D}_\infty(\mathcal{X}_\eta)$; this is often also referred to as the derived category of $\mathcal{X}_\eta$-modules.

The change of rings functor associated to a $\Bbbk$-linear functor $f : \mathcal{X} \to \mathcal{Y}$ will be referred to by $f_* : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ in order to be compatible with the notation for schemes. Also, wherever applicable, functors are intended as derived.

2. Preliminaries: $\mathcal{A}_\infty$ deformations of schemes and objects

For the general notion of $\mathcal{A}_\infty$-structures, we refer to [13], for an English reference consult [12] or [8].

We recall the following results from [22], which we will need to construct our candidate functors. In particular, we refer the interested reader to [22] for a more in-depth approach.

Definition 2.1. Let $\mathcal{X}$ be a $\Bbbk$-linear category. An $\mathcal{X}$-bimodule $\mathcal{M}$ is called $\Bbbk$-central if the $\Bbbk$-action induced by the left $\mathcal{X}$-action coincides with the $\Bbbk$-action induced by the right $\mathcal{X}$-action.

Definition 2.2. Let $\mathcal{X}$ be a small $\Bbbk$-linear category, and let $\mathcal{M}$ be a $\Bbbk$-central $\mathcal{X}$-bimodule. The Hochschild complex $\mathrm{C}^*(\mathcal{X},\mathcal{M})$ is defined as

$$\mathrm{C}^n(\mathcal{X},\mathcal{M}) := \prod_{X_0,\dots,X_n\in\mathrm{obj}(\mathcal{X})} \mathrm{Hom}(\mathcal{X}(X_0,X_1)\otimes_\Bbbk\cdots\otimes_\Bbbk\mathcal{X}(X_{n-1},X_n),\mathcal{M}(X_0,X_n))$$

with differential given by

$$\begin{aligned} df(x_1\otimes\cdots\otimes x_{n+1}) := x_1 f(x_2\otimes\cdots\otimes x_{n+1}) &\\ +\sum_{i=1}^{n}(-1)^i f(x_1\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_{n+1})&\\ +(-1)^{n+1} f(x_1\otimes\cdots\otimes x_n)x_{n+1}.& \end{aligned}$$

The Hochschild cohomology $\mathrm{HH}^*(\mathcal{X},\mathcal{M})$ is the cohomology of $\mathrm{C}^*(\mathcal{X},\mathcal{M})$.

Definition 2.3. Let $\mathcal{X}$ be an $\mathcal{A}_\infty$-category, and let Γ be a $\Bbbk$-algebra. Then, we define the $\mathcal{A}_\infty$-category $\mathcal{X}\otimes_\Bbbk\Gamma$ to consist of the same objects as $\mathcal{X}$ and morphism spaces given by

$$(\mathcal{X}\otimes_\Bbbk\Gamma)(a,b) := \mathcal{X}(a,b)\otimes_\Bbbk\Gamma,$$

with higher composition morphisms given by

$$\mathrm{m}_{i,\mathcal{X}\otimes_\Bbbk\Gamma}((x_1\otimes_\Bbbk\gamma_1),\dots,(x_i\otimes_\Bbbk\gamma_i)) := \mathrm{m}_{i,\mathcal{X}}(x_1,\dots,x_i)\otimes_\Bbbk\gamma_1\cdots\gamma_i$$

for composable arrows $x_j\otimes\gamma_j\in(\mathcal{X}\otimes_\Bbbk\Gamma)(a_{j-1},a_j)$.

Observe that there is no sign arising as we are considering Γ to be a $\Bbbk$-linear algebra, and so, all γ_i are in degree 0.

We now will define a version of $\mathcal{X}$ deformed along a Hochschild cocycle η. For a more in-depth discussion of this construction, we refer to [22, Section 6].

Definition 2.4. Let $\mathcal{X}$ be a small $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, and let $\eta\in\mathrm{C}^{\geq 3}(\mathcal{X},\mathcal{M})$ such that $d\eta=0$.

We define the $\mathcal{A}_\infty$-category $\mathcal{X}_\eta$ to have the same objects as $\mathcal{X}$, morphism spaces given by

$$\mathcal{X}_\eta(a,b) := \mathcal{X}(a,b) \oplus \mathcal{M}(a,b)[n-2]$$

and non-zero composition morphisms

$$\begin{aligned} \mathrm{m}_2((x,m),(x',m')) &:= (xx', xm' + mx') \\ \mathrm{m}_n((x_1,m_1),\ldots,(x_n,m_n)) &:= (0, \eta(x_1,\ldots,x_n)) \end{aligned}$$

for composable arrows $x_1,\ldots,x_n$.

The category $\mathcal{X}_\eta$ comes with a canonical $\Bbbk$-linear functor $\pi : \mathcal{X}_\eta \to \mathcal{X}$ acting by the identity on objects and on morphisms by

$$\begin{aligned} \pi : \mathcal{X}_\eta(a,b) = \mathcal{X}(a,b) \oplus \mathcal{M}(a,b)[n-2] &\to \mathcal{X}(a,b) \\ (\varphi,\psi) &\mapsto \varphi. \end{aligned}$$

Proposition 2.5 ([22, Lemma 6.1.1]). *Let $\mathcal{X}$ be a $\Bbbk$-linear category, let $\mathcal{M}$ be a $\Bbbk$-central $\mathcal{X}$-bimodule, and let $\eta,\mu \in \mathrm{C}^n(\mathcal{X},\mathcal{M})$ with $n \geq 3$ such that $\overline{\eta} = \overline{\mu} \in \mathrm{HH}^n(\mathcal{X},\mathcal{M})$. Then, we have*

$$\mathcal{X}_\eta \cong \mathcal{X}_\mu.$$

Remark 2.6 ([22, Section 6.2]). Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, and Γ a $\Bbbk$-algebra. Then, we define the morphism of Hochschild complexes

$$\begin{aligned} \mathrm{C}^*(\mathcal{X},\mathcal{M}) &\to \mathrm{C}^*(\mathcal{X}\otimes\Gamma,\mathcal{M}\otimes\Gamma) \\ \eta &\mapsto \eta \cup \mathrm{Id} \end{aligned}$$

to send a Hochschild cocycle η in degree n to the degree n morphism

$$\begin{aligned} \eta \cup \mathrm{Id} : (\mathcal{X}\otimes\Gamma)^{\otimes n} &\to (\mathcal{M}\otimes\Gamma) \\ (a_1\otimes\gamma_1)\otimes\cdots\otimes(a_n\otimes\gamma_n) &\mapsto \eta(a_1\otimes\cdots\otimes a_n)\otimes(\gamma_1\cdots\gamma_n). \end{aligned}$$

In particular, we get a morphism

$$\begin{aligned} \mathrm{HH}^*(\mathcal{X},\mathcal{M}) &\to \mathrm{HH}^*(\mathcal{X}\otimes\Gamma,\mathcal{M}\otimes\Gamma) \\ \eta &\mapsto \eta \cup \mathrm{Id}\,. \end{aligned}$$

One can compute that this morphism is compatible with deformations, i.e.,

$$\mathcal{X}_\eta \otimes \Gamma \cong (\mathcal{X}\otimes\Gamma)_{\eta\cup 1}.$$

Definition 2.7 ([22, Section 6.4]). Let $\mathcal{X}$ be a small $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, $\eta \in \mathrm{HH}^*(\mathcal{X},\mathcal{M})$, and let $U \in \mathcal{X}$-mod. A colift of U to $\mathcal{X}_\eta$ is a pair (V,ϕ), where $V \in \mathcal{D}_\infty(X_\eta)$ and ϕ is an isomorphism of graded $\mathrm{H}^*(\mathcal{X}_\eta)$-modules:

$$V \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\mathrm{H}^*(\mathcal{X}_\eta), U).$$

Although we will later discuss the geometric characteristic morphism in depth, we recall the next proposition using the algebraic characteristic morphism from [22] here, as it introduces obstructions against the existence of colifts. Later in Section 5, we will compare the geometric and algebraic characteristic morphisms.

Proposition 2.8. *Assume that $\mathcal{M}$ is an invertible $\Bbbk$-central $\mathcal{X}$-bimodule and $\mathcal{X}_\eta$ is as in Definition 2.4. Then, we have that the object $U \in \mathcal{D}(\mathcal{X})$ has a colift if and only if $c_U(\eta) = 0$, where c_U is the (algebraic) characteristic morphism*

$$c_U : \mathrm{HH}^*(\mathcal{X}, M) \to \mathrm{Ext}^*(U, M \otimes U),$$

obtained by interpreting $\eta \in \mathrm{HH}^*(\mathcal{X}, M)$ as a degree n morphism $\mathcal{X} \to M$ in $\mathcal{D}(\mathcal{X} \otimes \mathcal{X}^{\mathrm{op}})$ and applying $\_ \otimes U$.

Proof. This is a combination of [22, Lemma 6.4.1] and [22, Lemma 6.3.1]. ■

3. Equivariant sheaves and the characteristic morphism

In this section, we define Γ-equivariant sheaves on a scheme X for a $\Bbbk$-algebra Γ. We will use this in order to study the (geometric) Γ-equivariant characteristic morphism.

3.1. Equivariant sheaves and Fourier–Mukai functors

In this section, we introduce equivariant sheaves and prove that the equivariant structure is compatible with Fourier–Mukai functors. In particular, we can use this later to get a contradiction to being Fourier–Mukai.

Definition 3.1 ([15, Section 4]). Let Γ be a $\Bbbk$-algebra and $\mathcal{C}$ a $\Bbbk$-linear category. Then, we define the category $\mathcal{C}_\Gamma$ to consist of objects

$$\mathrm{obj}(\mathcal{C}_\Gamma) := (M, \psi : \Gamma \to \mathrm{End}_{\mathcal{C}}(M)),$$

where $\mathcal{M} \in \mathcal{C}$ and ψ is a morphism of $\Bbbk$-algebras, and morphisms

$$\mathcal{C}_\Gamma((M,\psi),(N,\varphi)) := \{\alpha \in \mathcal{C}(F,G) \mid \alpha \circ \psi(\gamma) = \varphi(\gamma) \circ \alpha \in \mathcal{C}(M,N)\ \forall \gamma \in \Gamma\}.$$

We will mostly denote (T,φ) by T if the action is clear from context to avoid clumsy notation.

Example 3.2. We give a few examples to illustrate Definition 3.1:

- Since $\mathcal{C}$ is required to be $\Bbbk$-linear, we have that $\mathrm{End}_{\mathcal{C}}(\_)$ comes with a canonical $\Bbbk$-action, and so, we have
$$\mathcal{C}_\Bbbk \cong \mathcal{C}.$$
- Let $\mathcal{C}$ be a $\Bbbk$-linear category and $M \in \mathcal{C}$; then we have canonically
$$M = (M, \mathrm{Id}) \in \mathcal{C}_{\mathrm{End}_{\mathcal{C}}(M)}.$$

- Let $F : \mathcal{C} \to \mathcal{C}'$ be a $\Bbbk$-linear functor between $\Bbbk$-linear categories, and let Γ be a (possibly non-commutative) $\Bbbk$-algebra. Then, we can extend F canonically to a functor

$$F : \mathcal{C}_\Gamma \to \mathcal{C}'_\Gamma$$
$$M \mapsto FM := (FM, F \circ \gamma) \in \mathcal{C}'_\Gamma.$$

- Consider the point $* = \operatorname{Spec}(\Bbbk)$ and a $\Bbbk$-algebra Γ. Then, $(M, \varphi) \in \operatorname{coh}(*)_\Gamma$ consists of $M \in \operatorname{coh}(*) \cong \operatorname{Vect}_\Bbbk$ and a $\Bbbk$-algebra morphism $\varphi : \Gamma \to \operatorname{End}_\Bbbk(M)$, which means that

$$\operatorname{coh}(*)_\Gamma \cong \Gamma\text{-mod}.$$

- Let $T \in \operatorname{coh}(X)$ be tilting for X smooth projective, and set $\Gamma := \operatorname{End}_X(T)$. Then, we have

$$\mathcal{D}^b(X) \cong \mathcal{D}^b(\Gamma) \cong \mathcal{D}^b(\operatorname{coh}(*)_\Gamma).$$

  We will prove in Lemma 3.19 that this equivalence is compatible with products of schemes under mild conditions.

  We will use the next specific version of the second example throughout this work:

- Let $\mathcal{F} \in \operatorname{coh}(X)$ be a coherent sheaf on a scheme. Then, we have canonically

$$F = (\mathcal{F}, \mathrm{Id}) \in \operatorname{coh}(X)_{\operatorname{End}_X(\mathcal{F})}.$$

Remark 3.3. The categories $\mathcal{D}(\mathcal{C}_\Gamma)$ and $\mathcal{D}(\mathcal{C})_\Gamma$ may seem very similar in notion; however, they do not coincide. An object in $M \in \mathcal{D}(\mathcal{C}_\Gamma)$ can be interpreted as a complex of equivariant objects; i.e., it admits an action in every degree and a differential that is compatible with these actions. On the other hand, an object in $\mathcal{D}(\mathcal{C})_\Gamma$ can be interpreted as a complex of sheaves together with an action on the whole complex that suffices the relations given by the Γ-action up to homotopy. The difference between these two notions essentially boils down to the difference between commutative diagrams up to homotopy not coinciding with homotopy commutative diagrams, which also led to the development of derivators [9]. For some more information on this interplay, we refer to [21].

By the above discussions, there is a canonical forgetful functor

$$\pi : D(\mathcal{C}_\Gamma) \to \mathcal{D}(\mathcal{C})_\Gamma$$
$$\overline{(M^\cdot, \varphi^\cdot)} \mapsto (\bar{M}^\cdot, \overline{\varphi}^\cdot),$$

where we denote by $\overline{(\_)}$ an equivalence class of $(\_)$. One can think of the above functor as forgetting that Γ acts on every degree separately. However, this functor is neither essentially injective nor surjective in general, which we will use later.

Remark 3.4 ([10, Remark 2.51]). Let $f : \mathcal{C} \to \mathcal{C}'$ be a left or right exact functor between abelian categories. Recall that an object $M \in \mathcal{A}$ is called f-adapted if $\mathrm{R}^i f(M) \cong 0$, respectively, $\mathrm{L}^i f(M) \cong 0$, for $i > 0$.

Lemma 3.5. *Let $f : \mathcal{C} \to \mathcal{C}'$ be a right or left exact functor between abelian $\mathbb{k}$-linear categories such that $\mathcal{C}$ has enough f-adapted objects, and let Γ be a $\mathbb{k}$-algebra. Then, the canonical functor*

$$\begin{aligned} f : \mathcal{D}^{\natural}(\mathcal{C}_\Gamma) &\to \mathcal{D}^{\natural}(\mathcal{C}')_\Gamma \\ \overline{(M, \psi)} &\mapsto (\bar{M}, \overline{f \circ \psi}) \end{aligned}$$

admits a lift

$$f_\Gamma : \mathcal{D}^{\natural}(\mathcal{C}_\Gamma) \to \mathcal{D}^{\natural}(\mathcal{C}'_\Gamma),$$

with $\natural \in \{b, +, -, \}$. In the case, $\natural = b$, respectively, $\natural = -$ for left exact and $\natural = +$ for right exact functors, we assume that every $M \in \mathcal{C}$ admits a bounded f-adapted resolution.

Proof. We have by Example 3.2 a canonical functor

$$\begin{aligned} f_\Gamma : \mathcal{C}_\Gamma &\to \mathcal{C}'_\Gamma \\ (M, \psi) &\mapsto (fM, f \circ \psi). \end{aligned}$$

Now, as $\mathcal{C}_\Gamma$ and $\mathcal{C}'_\Gamma$ are abelian with kernels and cokernels computed on objects, we get that f_Γ has the same exactness as f, and as cohomology also is computed on M only, we get that every f-adapted object is also f_Γ adapted.

Since we have enough f-adapted objects, we may consider for $M \in \mathcal{D}^{\natural}(\mathcal{C})$ an f-adapted replacement, which by assumption is also finite for $\natural = b$, respectively, if f is left exact and $\natural = -$ or f being right exact and $\natural = +$. In particular, we may invoke [27, Theorem 10.5.9] in order to find a well-defined derived functor:

$$\begin{aligned} f_\Gamma : \mathcal{D}^{\natural}(\mathcal{C}_\Gamma) &\to \mathcal{D}^{\natural}(\mathcal{C}'_\Gamma) \\ \overline{(M, \psi)^{\cdot}} &\mapsto \overline{(fM, f \circ \psi)^{\cdot}}. \end{aligned}$$

Furthermore, [27, Theorem 10.5.9] allows us to freely use f-adapted resolutions to compute f_Γ on the derived category; i.e., we will assume from now on that every $M^{\cdot}$ is f-adapted.

Recall the functor from Remark 3.3:

$$\begin{aligned} \pi : \mathcal{D}^{\natural}(\mathcal{C}'_\Gamma) &\to \mathcal{D}^{\natural}(\mathcal{C}')_\Gamma \\ \overline{(F, \psi)^{\cdot}} &\mapsto (\overline{F^{\cdot}}, \overline{\psi^{\cdot}}). \end{aligned}$$

We now just need to verify that the diagram

$$\begin{tikzcd} \mathcal{D}^{\natural}(\mathcal{C}_\Gamma) \arrow[r, "f_\Gamma"] \arrow[dr, "f"'] & \mathcal{D}^{\natural}(\mathcal{C}'_\Gamma) \arrow[d, "\pi"] \\ & \mathcal{D}^{\natural}(\mathcal{C}')_\Gamma \end{tikzcd}$$

commutes.

We indeed get

$$\begin{aligned}\pi \circ f_\Gamma \overline{(M, f \circ \psi)^\cdot} &= \pi \overline{(fM, f \circ \psi)^\cdot} \\ &= (\overline{fM^\cdot}, \overline{f \circ \psi^\cdot}) \\ &= f(\overline{M^\cdot}, \overline{\psi}^\cdot)\end{aligned}$$

as claimed. ■

We will drop the Γ in f_Γ if it is clear from context, respectively, from the target or source categories.

Lemma 3.6. *Let $f : X \to Y$ be a morphism of finite-dimensional noetherian $\Bbbk$-schemes, Γ a $\Bbbk$-algebra, and let $M \in \mathcal{D}^b(X)$. Then, we have the following:*

- *If f is proper, then the functor $f_* : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(Y))_\Gamma$ admits a canonical lift:*
$$f_{*,\Gamma} : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma).$$
- *If f is flat, then the functor $f^* : \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X))_\Gamma$ admits a canonical lift:*
$$f^*_\Gamma : \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma).$$
- *If X is regular, then the functor $M \otimes \_ : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X))_\Gamma$ admits a canonical lift:*
$$M \otimes_\Gamma \_ : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma).$$

Proof. We check the cases separately.

f_*. By Lemma 3.5, it suffices to show that every coherent sheaf M admits an f_*-adapted finite resolution in $\mathrm{coh}(X)$. By [10, Theorem 3.22], the object M admits an f_*-adapted resolution of finite length of quasi-coherent sheaves. By [10, Theorem 3.23], these quasi-coherent sheaves can be picked to be coherent for f proper.

So, we can find by Lemma 3.5 a lift:

$$f_{*,\Gamma} : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma).$$

f^*. As f is flat, f^* is exact and does not need to be derived. In particular, we get by Lemma 3.5 immediately a lift:

$$f^*_\Gamma : \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma).$$

$M \otimes (\_)$. By [10, Proposition 3.26], we have that every $\mathcal{F} \in \mathrm{coh}(X)$ admits a bounded locally free resolution, which is in particular $M \otimes (\_)$ adapted. So, we get by Lemma 3.5 that $M \otimes (\_)$ admits the lift

$$M \otimes_\Gamma (\_) : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$$

as claimed. ■

Corollary 3.7. *Let $f : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a Fourier–Mukai functor between finite-dimensional smooth projective $\Bbbk$-schemes, and let Γ be a $\Bbbk$-algebra. Then, we have that the induced functor*

$$f : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(Y))_\Gamma$$

admits a lift:

$$f_\Gamma : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(Y)_\Gamma).$$

Proof. Observe first that X and Y being smooth projective immediately gives that

$$\begin{aligned} \pi_1 : X \times Y &\to X \text{ is proper,} \\ \pi_2 : X \times Y &\to Y \text{ is flat} \\ \text{and} \quad & X \times Y \text{ is regular.} \end{aligned}$$

As f is a Fourier–Mukai functor, it has the form $\pi_{1,*}(M \otimes \pi_2^*(\_))$ for some $M \in \mathcal{D}^b(Y \times X)$. So, we get by Lemma 3.6 that $\pi_{1,*}$, π_2^*, and $M \otimes \_$ admit canonical lifts. In particular, f admits the canonical lift

$$f_\Gamma := \pi_{1,*,\Gamma}(M \otimes_\Gamma \pi_{2,\Gamma}^*(\_))$$

as claimed. ■

3.2. Hochschild cohomology and the characteristic morphism

As we want to study the characteristic morphism, we start by recalling the definition of the (geometric) Hochschild cohomology.

Definition 3.8 ([24]). Let X be a separated scheme and M a sheaf on X. Then, the Hochschild cohomology of X with coefficients in M is given by

$$\mathrm{HH}^*(X, M) := \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \Delta_* M),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding.

For the definition of the (geometric) characteristic morphism below, we follow [14] and [6, Section 3.3].

Definition 3.9. Let X, Y be regular schemes, Γ a $\Bbbk$-algebra, and let $M, T \in \mathrm{coh}(X)$. Then, the (geometric) characteristic morphism is defined to be

$$\begin{aligned} c_T(M) : \mathrm{HH}^*(X, M) = \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \Delta_* M) &\to \mathrm{Ext}^*_X(T, M \otimes T) \\ (\alpha : \mathcal{O}_\Delta \to \Sigma^n \Delta_* M) &\mapsto \Big(T \xrightarrow{\pi_{1*}(\alpha \otimes \pi_2^* \mathrm{Id})} \Sigma^n M \otimes T\Big), \end{aligned}$$

where we use $T \cong \pi_{1*}(\mathcal{O}_\Delta \otimes \pi_2^* T)$ and $M \otimes T \cong \pi_{1*}(\Delta_* \Sigma^n M \otimes \pi_2^* T)$. If we have a Γ-action on T, i.e., $(T, \varphi) \in \mathrm{coh}(X)_\Gamma$, there also exists a Γ-equivariant characteristic morphism:

$$\begin{aligned} c_{T,\Gamma}(M) : \mathrm{HH}^*(X, M) = \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) &\to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M \otimes T) \\ \alpha : (\mathcal{O}_\Delta \to \Sigma^n \Delta_* M) &\mapsto \left(T \xrightarrow{\pi_{1*}(\alpha \otimes \pi_2^* \mathrm{Id})} \Sigma^n M \otimes T\right), \end{aligned}$$

where we consider $M \otimes T$ as an object in $\mathrm{coh}(X)_\Gamma$ via the functor $M \otimes (\_)$, i.e.,

$$\begin{aligned} \psi : \Gamma &\to \mathrm{End}(M \otimes T) \\ \gamma &\mapsto \mathrm{Id} \otimes \varphi(\gamma). \end{aligned}$$

To study the characteristic morphism for special T, we will define the following functor realizing the characteristic morphism on a categorical level.

Definition 3.10. Let X, Y be projective schemes, and let $T = (T, \varphi) \in \mathrm{coh}(Y)_\Gamma$. Then, we define the functor

$$\begin{aligned} C_T^X : D^b(X \times Y) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \\ M &\mapsto (\pi_{1*}(M \otimes \pi_2^* T), \gamma \mapsto \pi_{1*}(\mathrm{Id} \otimes \pi_2^* \varphi(\gamma))) \\ (\alpha : M \to N) &\mapsto C_T^X(\alpha) = \pi_{1*}(\alpha \otimes \pi_2^* T). \end{aligned}$$

Remark 3.11. One can think of the functor C_T^X to send an object $M \in \mathcal{D}^b(X \times Y)$ to the image of T under the Fourier–Mukai functor with kernel M, equipped with the action induced by $\Phi_{M,\Gamma}$, i.e.,

$$\begin{aligned} C_T^X : D^b(X \times Y) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \\ M &\mapsto \Phi_M(T). \end{aligned}$$

The functor C_T^X allows us to compute $c_{T,\Gamma}$ on a categorical level.

Proposition 3.12. *Let X be a scheme, and let $T \in \mathrm{coh}(X)_\Gamma$, and consider*

$$C_T^X : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma).$$

Then, we have that the equivariant characteristic morphism $c_{T,\Gamma}(M)$ is given by evaluating the functor C_T^X on the morphism space $\mathrm{Ext}_{X\times X}(\mathcal{O}_\Delta, \Delta_* M)$:

$$c_{T,\Gamma}(M) = C_T^X : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M \otimes T).$$

Proof. By Definition 3.10, we have

$$\begin{aligned} C_T^X : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) &\to \mathrm{Ext}_{\mathrm{coh}(X)_\Gamma}(C_T^X(\mathcal{O}_\Delta), C_T^X(\Delta_* M)) \\ \alpha &\mapsto \pi_{1*}(\alpha \otimes \pi_2^* T). \end{aligned}$$

We now have $C_T^X(\mathcal{O}_\Delta) \cong T$ and $C_T^X(\Delta_* M) \cong M \times T$. So, the above turns by Definition 3.9 into

$$c_{T,\Gamma}(M) : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M \otimes T)$$

as claimed. ∎

Definition 3.13 ([3, Section 2.1]). Let $\mathcal{C}$ be a pointed category, i.e., a category admitting a zero object. An object $G \in \mathcal{C}$ is a generator if $\mathcal{C}(G, M) = 0$ implies $M = 0$.

Let $\mathcal{T}$ be a pointed graded category. An object $G \in \mathcal{T}$ is called a generator if

$$\mathcal{T}(G, M[i]) = 0$$

for all $i \in \mathbb{Z}$ implies that $M = 0$.

Remark 3.14. In a pointed category $\mathcal{C}$, the equation

$$\mathcal{C}(M, N) = 0$$

for two objects $M, N \in \mathcal{C}$, this means that the only morphism between M and N is the unique morphism factoring over 0.

Furthermore, we have for an object $M \in \mathcal{C}$ that if

$$\mathcal{C}(M, M) = 0$$

means that $M \cong 0$ as in that case $0 = \mathrm{Id}$, and so, the unique morphisms $0 \to M$ and $M \to 0$ define isomorphisms.

Proposition 3.15. *Let $f_* : \mathcal{C} \to \mathcal{D}$ be a faithful functor between pointed categories or graded functor between pointed graded categories with a left adjoint $f^* : \mathcal{D} \to \mathcal{C}$, and let $T \in \mathcal{D}$ be a generator. Then, f^*T is a generator.*

Proof. We cover the case of a graded functor. Observe that the same argument holds for pointed categories by ignoring $[i]$.

Let M be such that $\mathcal{C}(f^*T[i], M) = 0$. Then, we have

$$\mathcal{C}(f^*T[i], M) = \mathcal{D}(T[i], f_*M) = 0;$$

in particular, $f_*M \cong 0$. Now, $\mathcal{D}(f_*M, f_*M) = 0$, and so, $\mathcal{C}(M, M) = 0$. This can only hold if $M \cong 0$, and so, f^*T is a generator. ∎

Proposition 3.16. *Let $\mathcal{C}$ be a $\Bbbk$-linear category or pointed graded category that admits a generator G, and let Γ be a $\Bbbk$-algebra. Then,*

$$\left(G \otimes \Gamma, \psi : \gamma' \mapsto \left(G \otimes \Gamma \xrightarrow{g\otimes\gamma \mapsto g\otimes\gamma'\gamma} G \otimes \Gamma\right)\right)$$

defines a generator of $\mathcal{C}_\Gamma$, where we denote by $G \otimes \Gamma$ the sheaf arising by tensoring locally with the $\Bbbk$-algebra Γ as $\Bbbk$-vector spaces and acting exclusively on Γ.

Proof. Let $(X, \varphi) \in C_\Gamma$, and let $f : G \to X$ be a morphism. Then, we have the following morphism in $\mathcal{C}_\Gamma$:

$$\begin{aligned} \hat{f} : G \otimes \Gamma &\to X \\ g \otimes \gamma &\mapsto \varphi(\gamma) \circ f(g). \end{aligned}$$

This indeed defines a morphism in $\mathcal{C}_\Gamma$ as

$$\begin{aligned}\varphi(\gamma') \circ \hat{f}(g \otimes \gamma) &= \varphi(\gamma') \circ \varphi(\gamma) \circ f(g)\\ &= \varphi(\gamma'\gamma) \circ f(g)\\ &= \hat{f}(g \otimes \gamma'\gamma)\\ &= \hat{f} \circ \psi(\gamma')(g \otimes \gamma).\end{aligned}$$

We can compute that if $\hat{f}$ vanishes, then f has to vanish as well since

$$\begin{aligned}0 &= \hat{f}(g \otimes \mathrm{Id})\\ &= \varphi(\mathrm{Id}) \circ f(g)\\ &= \mathrm{Id} \circ f(g)\\ &= f(g).\end{aligned}$$

This means that if the morphism space $\mathcal{C}_\Gamma(G \otimes \Gamma, (X, \varphi))$ vanishes, then also $\mathcal{C}(G, X)$ vanishes. Observe that the discussion so far did not assume G to be a generator.

We continue again by considering the graded pointed case. For the pointed case, it again suffices to ignore the shift $[i]$.

Now, assume that

$$\mathcal{C}_\Gamma(G \otimes \Gamma[i], (X, \varphi)) = 0.$$

Then, we have by the above discussion that

$$\mathcal{C}(G[i], X) = 0.$$

As G is a generator, we get that X has to be a zero object. And so, (X, φ) has to be a zero object as well. In particular, we get that $G \otimes \Gamma$ is indeed a generator of $\mathcal{C}_\Gamma$. ■

Remark 3.17. Proposition 3.16 is a consequence of $M \otimes \Gamma$ being the free object in $\mathcal{C}_\Gamma$ over M.

Remark 3.18. Recall that an object $\mathcal{T}$ in an abelian category $\mathcal{A}$ is called tilting if $\mathcal{T}$ is a generator in $\mathcal{D}(\mathcal{A})$ and $\mathrm{Ext}^i(T, T) \cong 0$ for all $i > 0$.

Lemma 3.19. *Let X, Y be smooth projective schemes such that X admits a generator $G \in \mathcal{D}^b(X)$ with* $\mathrm{RHom}^i(G, G)$ *finite dimensional for all i; let Y be such that it admits a tilting object $T \in \mathrm{coh}(Y)$, and set $\Gamma := \mathrm{End}(T)$. Then,*

$$C_T^X : \mathcal{D}^b(X \times Y) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$$

is an equivalence of derived categories.

Proof. Throughout this proof, we denote by $T^D := \mathrm{R}\mathcal{H}\mathrm{om}_Y(T, \mathcal{O}_Y)$ the dual of T and by $\pi_S : S \to \mathrm{Spec}(\mathbb{k})$ the unique projection from a scheme S to the point $\mathrm{Spec}(\mathbb{k})$. Observe that by [10, Proposition 3.26] we have $T^D \in \mathcal{D}^b(Y)$ as smooth schemes, which are in particular regular. Furthermore, we will use the following diagram for flat base change twice:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathrm{Spec}(\Bbbk). & \end{tikzcd}$$

Since T is tilting, it is a generator of $\mathcal{D}^b(Y)$ and T^D is generating $\mathcal{D}^b(Y)$ by [22, Lemma 8.9.1]. So, we get that $G \boxtimes T^D$ generates $\mathcal{D}^b(X \times Y)$ by [3, Lemma 3.4.1]. Furthermore, we have by [7, Paragraph 1.10] that $\Gamma = \mathrm{End}_Y(T)$ is finite dimensional.

We first show that $C_T^X(G \boxtimes T^D)$ is isomorphic to $G \otimes \Gamma$:

$$\begin{aligned} C_T^X(G \boxtimes T^D) &= \pi_{1*}((G \boxtimes T^D) \otimes \pi_2^* T) && \text{definition of } C_T^X \\ &\cong \pi_{1*}(\pi_1^* G \otimes \pi_2^* T^D \otimes \pi_2^* T) && \text{definition of } \boxtimes \\ &\cong \pi_{1*}(\pi_1^* G \otimes \pi_2^*(T \otimes T^D)) && [10, (3.12)] \\ &\cong \pi_{1*}(\pi_1^* G \otimes \pi_2^* \mathrm{R}\mathcal{H}\mathrm{om}_Y(T, T)) && \text{definition of } T^D \\ &\cong G \otimes \pi_{1*}\pi_2^*(\mathrm{R}\mathcal{H}\mathrm{om}_Y(T, T)) && [10, (3.11)] \\ &\cong G \otimes \pi_X^* \pi_{Y,*} \mathrm{R}\mathcal{H}\mathrm{om}_Y(T, T) && \text{flat base change} \\ &\cong G \otimes \Gamma. && T \text{ has no higher Ext-groups} \end{aligned}$$

The above computation is compatible with the Γ-action as all isomorphisms involved are natural isomorphism. In particular, replacing $\pi_2^* T$ by $\pi_2^* \gamma$ yields multiplication with γ in Γ.

As by Proposition 3.16, $G \otimes \Gamma$ is a generator for $\mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$, the functor C_T^X sends a generator to a generator. So, it suffices to prove that

$$\mathrm{RHom}_{X \times Y}(G \boxtimes T^D, G \boxtimes T^D) \xrightarrow{C_X^T} \mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}(C_T^X(G \boxtimes T^D), C_T^X(G \boxtimes T^D))$$

is an isomorphism.

To do that, we first compute the source and target spaces:

$$\begin{aligned} &\mathrm{RHom}_{X \times Y}(G \boxtimes T^D, G \boxtimes T^D) \\ &\cong \mathrm{RHom}_{X \times Y}(\pi_1^* G \otimes \pi_2^* T^D, \pi_1^* G \otimes \pi_2^* T^D) && \text{definition of } \boxtimes \\ &\cong \mathrm{RHom}_{X \times Y}(\pi_1^* G, \mathrm{R}\mathcal{H}\mathrm{om}_{X \times Y}(\pi_2^* T^D, \pi_1^* G \otimes \pi_2^* T^D)) && [10, (3.14)] \\ &\cong \mathrm{RHom}_{X \times Y}(\pi_1^* G, \pi_1^* G \otimes \mathrm{R}\mathcal{H}\mathrm{om}_{X \times Y}(\pi_2^* T^D, \pi_2^* T^D)) && [10, (3.13)] \end{aligned}$$

$$
\begin{aligned}
&\cong \Gamma_{X\times Y} \mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_1^* G, \pi_1^* G \otimes \mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_2^* T^D, \pi_2^* T^D)) && [10, \text{p. } 85]\\
&\cong \pi_{\Bbbk,*} \mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_1^* G, \pi_1^* G \otimes \mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_2^* T^D, \pi_2^* T^D)) && \Gamma_{X\times Y} \cong \pi_{X\times Y,*}\\
&\cong \pi_{X\times Y,*}(\mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_1^* G, \pi_1^* G) \otimes \mathrm{R}\mathcal{H}\mathrm{om}_{X\times Y}(\pi_2^* T^D, \pi_2^* T^D)) && [10, (3.13)]\\
&\cong \pi_{X\times Y,*}(\pi_1^* \mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_2^* \mathrm{R}\mathcal{H}\mathrm{om}_Y(T^D, T^D)) && [10, (3.13)]\\
&\cong \pi_{X,*} \circ \pi_{1,*}(\pi_1^* \mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_2^* \mathrm{R}\mathcal{H}\mathrm{om}_Y(T^D, T^D)) && \pi_{X\times Y} = \pi_X \circ \pi_1\\
&\cong \pi_{X,*} \circ (\pi_{1,*}(\pi_1^* \mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_2^* \mathrm{R}\mathcal{H}\mathrm{om}_Y(T^D, T^D))) && \circ \text{ is associative}\\
&\cong \pi_{X,*}(\mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_{1,*}\pi_2^* \mathrm{R}\mathcal{H}\mathrm{om}_Y(T^D, T^D)) && [10, (3.11)]\\
&\cong \pi_{X,*}(\mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_X^* \pi_{Y,*} \mathrm{R}\mathcal{H}\mathrm{om}_Y(T^D, T^D)) && \text{flat base change}\\
&\cong \pi_{X,*}(\mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \pi_X^* \Gamma^{\mathrm{op}}) && [10, \text{p. } 85]\\
&\cong \pi_{X,*} \mathrm{R}\mathcal{H}\mathrm{om}_X(G, G) \otimes \Gamma^{\mathrm{op}} && [10, (3.11)]\\
&\cong \mathrm{RHom}_X(G, G) \otimes \Gamma^{\mathrm{op}}. && [10, \text{p. } 85]
\end{aligned}
$$

Now, for $\mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}(G \otimes \Gamma, G \otimes \Gamma)$, we have

$$
\begin{aligned}
\mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}(G \otimes \Gamma, G \otimes \Gamma) &= \mathrm{RHom}_{\mathrm{coh}(X)}(G, G) \otimes \mathrm{RHom}_{\Gamma\text{-mod}}(\Gamma, \Gamma)\\
&\cong \mathrm{RHom}_X(G, G) \otimes \Gamma^{\mathrm{op}}.
\end{aligned}
$$

As the two spaces are isomorphic and in particular degree-wise isomorphic, it suffices to prove bijectivity on $\mathrm{RHom}^i_{X\times Y}(G \boxtimes T^D, G \boxtimes T^D)$. Since

$$
\mathrm{RHom}^i_{X\times Y}(G \boxtimes T^D, G \boxtimes T^D) \cong \mathrm{RHom}^i_X(G, G) \otimes \Gamma,
$$

we know that $\mathrm{RHom}^i_{X\times Y}(G \boxtimes T^D, G \boxtimes T^D)$ is finite dimensional as tensor product of finite-dimensional vector spaces. So, it suffices to check that C_T^X is surjective. For this, let

$$
\alpha \otimes \beta \in \mathrm{RHom}^i_{\mathrm{coh}(X)_\Gamma}(G \otimes \Gamma, G \otimes \Gamma) \cong \mathrm{RHom}^i(G, G) \otimes \Gamma^{\mathrm{op}}.
$$

Then, we can pick $\alpha \boxtimes \beta \in \mathrm{RHom}^i_{X\times Y}(G \boxtimes T^D, G \boxtimes T^D)$ and get

$$
\begin{aligned}
C_T^X(\alpha \boxtimes \beta) &\cong \pi_{1,*}(\alpha \boxtimes \beta \otimes \pi_2^* \mathrm{Id}_T)\\
&\cong \pi_{1,*}(\alpha \boxtimes \beta)\\
&\cong \alpha \otimes \beta \in \mathrm{RHom}^i_{\mathrm{coh}(X)_\Gamma}(G \otimes \Gamma, G \otimes \Gamma).
\end{aligned}
$$

This means that C_T^X is surjective on the generating set of morphisms of the form $\alpha \otimes \beta$. In particular, C_T^X is surjective and an isomorphism as it is surjective between vector spaces of the same dimension which finishes the proof. ∎

Lemma 3.20. *Let $f : X \to Y$ be a proper morphism of schemes, Γ a $\Bbbk$-algebra, and let $T \in \mathrm{coh}(Y)_\Gamma$. We have $f^*T \in \mathrm{coh}(X)_\Gamma$. Consider the two functors*

$$
\begin{aligned}
C_{f^*T}^X : \mathcal{D}^b(X \times X) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)\\
M &\mapsto (\pi_{1*}(M \otimes \pi_2^* f^* T))\\
(\alpha : M \to N) &\mapsto \pi_{1*}(\alpha \otimes \pi_2^* f^* T)
\end{aligned}
$$

and

$$\begin{aligned} C_T^X \circ (\mathrm{Id} \times f)_* : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(X \times Y) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \\ M \mapsto (\mathrm{Id} \times f)_* M &\mapsto (\pi_{1*}(\mathrm{Id} \times f)_* M \otimes \pi_2^* T) \\ (\alpha : M \to N) \mapsto (\mathrm{Id} \times f)_* \alpha &\mapsto \pi_{1*}((\mathrm{Id} \times f)_* \alpha \otimes \pi_2^* T). \end{aligned}$$

Then, we have a natural isomorphism $C_{f^*T}^X \cong C_T^X \circ (\mathrm{Id} \times f)_*$.

Proof. Observe that $\mathrm{Id} \times f$ is proper as product of proper morphisms, and so, by [10, Theorem 3.23],

$$(\mathrm{Id} \times f)_* : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(X \times Y)$$

is well defined. We will use the following two commutative diagrams in order to construct the isomorphism

$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{Id} \times f"] \arrow[d, "\pi_1"'] & X \times Y \arrow[d, "\pi_1'"] \\ X \arrow[r, "\mathrm{Id}"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\mathrm{Id} \times f"] \arrow[d, "\pi_2"'] & X \times Y \arrow[d, "\pi_2'"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where we distinguish between the projections from $X \times X$ and $X \times Y$ in order to avoid confusion. This means that in this notation

$$C_T^X = \pi'_{1*}((\_) \otimes \pi_2'^* T) \quad \text{and} \quad C_{f^*T}^X = \pi_{1*}((\_) \otimes \pi_2^* f^* T).$$

On objects and morphisms we have the following sequence of natural isomorphisms:

$$\begin{aligned} C_{f^*T}^X(\_) &= \pi_{1*}((\_) \otimes \pi_2^* f^* T) && \text{Definition 3.10} \\ &\cong \pi'_{1*}(\mathrm{Id} \times f)_*((\_) \otimes (\mathrm{Id} \times f)^* \pi_2'^* T) && \begin{gathered} \pi_1 = \pi_1' \circ (\mathrm{Id} \times f) \\ f \circ \pi_2 = \pi_2' \circ (\mathrm{Id} \times f) \end{gathered} \\ &\cong \pi'_{1*}((\mathrm{Id} \times f)_*(\_) \otimes \pi_2'^* T) && \text{projection formula} \\ &= C_T^X \circ (\mathrm{Id} \times f)_*(\_). && \text{Definition 3.10} \end{aligned}$$

Both functors also induce the same Γ-action as we get analogously:

$$\begin{aligned} \pi_{1*}(\mathrm{Id} \otimes \pi_2^* f^* \gamma) &\cong \pi'_{1*}(\mathrm{Id} \times f)_*(\mathrm{Id} \otimes (\mathrm{Id} \times f)^* \pi_2'^* \gamma) && \begin{gathered} \pi_1 = \pi_1' \circ (\mathrm{Id} \times f) \\ f \circ \pi_2 = \pi_2' \circ (\mathrm{Id} \times f) \end{gathered} \\ &\cong \pi'_{1*}((\mathrm{Id} \times f)_* \mathrm{Id} \otimes \pi_2'^* \gamma). && \text{projection formula} \end{aligned}$$

This means that the actions match up along the same natural isomorphisms, and so,

$$C_{f^*T}^X \cong C_T^X \circ (\mathrm{Id} \times f)_*$$

as claimed. ∎

Remark 3.21. Lemma 3.20 above can be interpreted very naturally using Remark 3.11. As C_T^X sends an M to the image of T under the Fourier–Mukai functor Φ_M and we have by [10, Exercise 5.12]

$$\Phi_{(\mathrm{Id}\times f)_*M} \cong \Phi_M \circ f^*.$$

In particular, the two functors $C_{f^*T}^X$ and $C_T^X \circ (f \times \mathrm{Id})_*$ should be isomorphic.

Proposition 3.22. *Let $f : X \to Y$ be a proper morphism of schemes and $T \in \mathrm{coh}(Y)_\Gamma$. Then, we have*

$$c_{f^*T,\Gamma}(M) = C_T^X \circ (\mathrm{Id} \times f)_* : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(f^*T, M \otimes f^*T).$$

Proof. By Proposition 3.12, we have

$$c_{f^*T,\Gamma}(M) = C_{f^*T}^X : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M \otimes T),$$

and by Lemma 3.20, we get

$$\begin{aligned} c_{f^*T,\Gamma}(M) = C_{f^*T}^X = C_T^X \circ (\mathrm{Id} \times f)_* &: \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \\ &\to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(f^*T, M \otimes f^*T) \end{aligned}$$

as claimed. ∎

Remark 3.23. The above result could be used to compute the injectivity of the characteristic morphism if one can find an $(f, \mathrm{Id}) : X \times X \to X \times Y$ that is injective on $\mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M)$ such that Y admits a tilting bundle. However, the existence of such a morphism is not straightforward. In particular, a closed immersion of a divisor

$$f : X \hookrightarrow \mathbb{P}^n$$

is in general not injective on $\mathrm{Ext}^i_X(\_, \_)$ as by the Grothendieck–Serre spectral sequence there might be correction terms arising in degrees $i > 1$.

4. Twisted Hodge diamonds give kernels in Hochschild cohomology

We will show how twisted Hodge diamonds, and in particular their interior, can be used to understand the pushforward of Hochschild cohomology under the closed embedding of a smooth projective hypersurface of degree d.

Throughout this chapter, we will follow Brückmann's paper "Zur Kohomologie von projektiven Hyperflächen" [5] for computations.

Definition 4.1. Let X be a projective scheme of dimension n, and let $\mathcal{O}_X(1)$ be a very ample line bundle. Then, we define the twisted Hodge numbers of X to be

$$\mathrm{h}_p^{i,j}(X) := \dim \mathrm{H}^j(X, \Omega_X^i(p)).$$

Similarly to the ordinary Hodge numbers, the twisted Hodge numbers can be arranged in a twisted Hodge diamond:

$$\begin{array}{ccc} & \mathrm{h}_p^{n,n}(X) & \\ & \diagup \quad \diagdown & \\ \mathrm{h}_p^{n,0}(X) & & \mathrm{h}_p^{0,n}(X) \\ & \diagdown \quad \diagup & \\ & \mathrm{h}_p^{0,0}(X). & \end{array}$$

We will drop the X if the space is clear from context.

Lemma 4.2. *Let X be a smooth projective scheme of dimension n with canonical sheaf of the form $\mathcal{O}_X(t)$. Then, we have*

$$\mathrm{HH}^m(X, \mathcal{O}_X(p)) \cong \bigoplus_{i=0}^{n} \mathrm{H}^{i-m+n}(X, \Omega_X^i(t-p)).$$

In particular, this gives

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}_{t-p}^{i,i-m+n}(X).$$

Proof. We compute, using $\omega_X \cong \mathcal{O}_X(t)$ and the Hochschild–Kostant–Rosenberg (HKR) isomorphism [24]:

$$\begin{aligned} \mathrm{HH}^m(X, \mathcal{O}_X(p)) &\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{m-i}(\Omega_X^i, \mathcal{O}_X(p)) && \text{HKR} \\ &\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{n-m+i}(\mathcal{O}_X(p), \Omega_X^i(t))^* && \text{Serre duality} \\ &\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{n-m+i}(\mathcal{O}_X, \Omega_X^i(t-p))^* && \text{twisting on both sides} \\ &\cong \bigoplus_{i=0}^{n} \mathrm{H}^{n-m+i}(X, \Omega_X^i(t-p))^* && \mathrm{Ext}_X^j(\mathcal{O}_X, \_) \cong \mathrm{H}^j(X, \_) \end{aligned}$$

Applying dimension on both sides gives

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}_{t-p}^{i,i-m+n}(X)$$

as desired. ∎

Remark 4.3. By Lemma 4.2, one can compute $\dim \mathrm{HH}^m(X, \mathcal{O}_X(p))$ as the sum over the mth column in the $t-p$ twisted Hodge diamond:

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{h}^{n,n}_{t-p} \arrow[dl, dash] \arrow[ddrr, dash] & & \\
 & \mathrm{h}^{n,2n-m}_{t-p} \arrow[dl, dash] \arrow[dd, dash] & & & \\
\mathrm{h}^{n,0}_{t-p} \arrow[dr, dash] & & & & \mathrm{h}^{0,n}_{t-p} \arrow[ddll, dash] \\
 & \mathrm{h}^{m-n,0}_{t-p} \arrow[dr, dash] & & & \\
 & & \mathrm{h}^{0,0}_{t-p} & &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}[column sep=small]
 & & \mathrm{h}^{n,n}_{t-p} \arrow[ddll, dash] \arrow[dr, dash] & & \\
 & & & \mathrm{h}^{m,n}_{t-p} \arrow[dr, dash] \arrow[dd, dash] & \\
\mathrm{h}^{n,0}_{t-p} \arrow[ddrr, dash] & & & & \mathrm{h}^{0,n}_{t-p} \arrow[dl, dash] \\
 & & & \mathrm{h}^{0,n-m}_{t-p} \arrow[dl, dash] & \\
 & & \mathrm{h}^{0,0}_{t-p}. & &
\end{tikzcd}$$

4.1. The Hochschild cohomology of a smooth hypersurface

We will use the computations in [5] and Remark 4.3 to compute the Hochschild cohomology of X.

Lemma 4.4. *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth degree d hypersurface. Then,*

$$\mathrm{h}^{i,j}_p(X) = 0$$

if (i,j) is not of the form $(i,0)$, (i,n), $(i,n-i)$, (i,i), with $0 \leq i \leq n$. And we have for (i,i)

$$\mathrm{h}^{i,i}_p(X) = \delta_{p,0} \quad \text{if } i \notin \left\{0, \frac{n}{2}, n\right\}.$$

Moreover, we get

$$\mathrm{h}^{i,n-i}_p(X) = \sum_{\mu=0}^{n+2} (-1)^{\mu} \binom{n+2}{\mu} \binom{-p+id-(\mu-1)(d-1)}{n+1} + \delta_{p,0}\delta_{i,n-i}. \tag{1}$$

Proof. First of all, we can assume that $0 \leq i, j \leq n$ as outside of that range we have $\Omega^i_X(p) = 0$, respectively, $\mathrm{H}^j(X, \Omega^i_X(p)) = 0$ for dimension reasons.

By [5, Satz 2, (42), (40), (38), and (39)], we have for $0 < i < n$

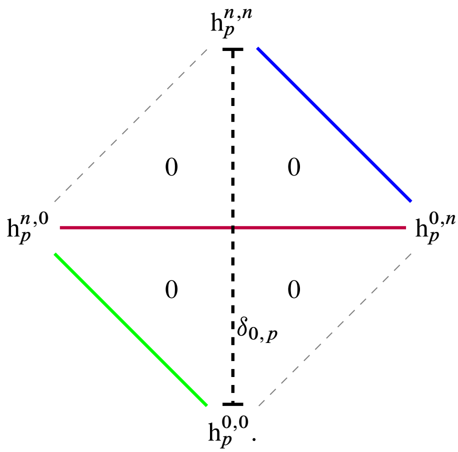
$$\mathrm{h}^{i,j}_p(X) = \begin{cases} \binom{-p-1}{n-i}\binom{-p+1+i}{1+i} + \sum_{\mu=1}^{n-i+1}(-1)^{\mu}\binom{n+2}{\mu}\binom{-p-\mu(d-1)+i}{n+1} & \text{if } j = n \\ \sum_{\mu=0}^{n+2}(-1)^{\mu}\binom{n+2}{\mu}\binom{-p+id-(\mu-1)(d-1)}{n+1} + \delta_{p,0}\delta_{i,j} & \text{if } i+j = n \\ \binom{p-1}{i}\binom{p+n+1-i}{n+1-i} + \sum_{\mu=1}^{i+1}(-1)^{\mu}\binom{n+2}{\mu}\binom{p+n-\mu(d-1)-i}{n+1} & \text{if } j = 0 \\ \delta_{p,0} & \text{if } i = j \notin \{0, n\} \\ 0 & \text{else.} \end{cases}$$

So, only the cases for $i \in \{0, n\}$ remain. Now, [5, Lemma 5] gives for $j \notin \{0, n\}$

$$\mathrm{h}_p^{0,j}(X) = 0 = \mathrm{h}_p^{n,j}(X),$$

which finishes the claim. ∎

Remark 4.5. By Lemma 4.4, the p-twisted Hodge diamond of a smooth degree d hypersurface has the shape



In particular, the only non-trivial entries appear along the indicated lines. More precisely, we have along the blue line the values for $\mathrm{h}^{i,n}(X)$, along the red line the values for $\mathrm{h}^{i,n-i}(X)$, and along the green line the values for $\mathrm{h}^{i,0}(X)$. Furthermore, the dashed line disappears if $p \neq 0$ as these are the Kronecker deltas $\delta_{p,0}$.

Proposition 4.6. *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth degree d hypersurface. Then, the following formulas hold:*

$$\begin{aligned}
\mathrm{h}_p^{i,0}(X) &= \mathrm{h}_{-p}^{n-i,n}(X) && \\
\mathrm{h}_p^{i,n-i}(X) &= \mathrm{h}_{p-d}^{i-1,n+1-i}(X) && i \notin \{0,1,n\},\ p \neq 0 \\
\mathrm{h}_{p-d}^{i,n+1}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i,n+1}(\mathbb{P}^{n+1}) &= \mathrm{h}_p^{i,n}(X) + \mathrm{h}_{p-d}^{i-1,n}(X) && i \notin \{0,1,n\} \\
\mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}) - \mathrm{h}_{p-d}^{i,0}(\mathbb{P}^{n+1}) &= \mathrm{h}_p^{i,0}(X) + \mathrm{h}_{p-d}^{i-1,0}(X) && i \notin \{0,1,n\}.
\end{aligned}$$

Proof. We compute for the first equation:

$$\begin{aligned}
\mathrm{h}_p^{i,0}(X) &= \dim \mathrm{H}^0(X, \Omega_X^i(p)) && \text{definition} \\
&= \dim \mathrm{Ext}^0(\mathcal{O}_X, \Omega_X^i(p)) && \mathrm{Ext}^*(\mathcal{O}_X, _) \cong \mathrm{H}^*(X, _) \\
&= \dim \mathrm{Ext}^n(\Omega_X^i(p), \Omega_X^n) && \text{Serre duality} \\
&= \dim \mathrm{Ext}^n(\mathcal{O}_X, \Omega_X^{n-i}(-p)) && \\
&= \dim \mathrm{H}^n(X, \Omega_X^{n-i}(-p)) && \mathrm{Ext}^*(\mathcal{O}_X, _) \cong \mathrm{H}^*(X, _) \\
&= \mathrm{h}_p^{n-i,0}(X). && \text{definition}
\end{aligned}$$

For the second equation, we have by (1) the following identity:

$$\begin{aligned}
\mathrm{h}_p^{i,n-i}(X) &= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+id-(\mu-1)(d-1)}{n+1} \\
&= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+d-d+id-(\mu-1)(d-1)}{n+1} \\
&= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+d+(i-1)d-(\mu-1)(d-1)}{n+1} \\
&= \mathrm{h}_{p-d}^{i-1,n+1-i}(X).
\end{aligned}$$

And for the last two, Brückmann gives the formula [5, (31)], which together with [5, Satz 2] gives both

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}(X) &= \mathrm{h}_{p-d}^{i+1,n}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i+1,n}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i+1,n}(X), \\
\mathrm{h}_p^{i,0}(X) &= \mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}) - \mathrm{h}_{p-d}^{i-1,0}(X).
\end{aligned}$$

After rearranging, these are

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}(X) + \mathrm{h}_p^{i+1,n}(X) &= \mathrm{h}_{p-d}^{i+1,n}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i+1,n}(\mathbb{P}^{n+1}), \\
\mathrm{h}_p^{i,0}(X) + \mathrm{h}_{p-d}^{i-1,0}(X) &= \mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}).
\end{aligned}$$

Index shifting in the first equation gives

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}(\mathbb{P}^{n+1}) - \mathrm{h}_p^{i,n}(\mathbb{P}^{n+1}) &= \mathrm{h}_p^{i,n}(X) + \mathrm{h}_{p-d}^{i-1,n}(X), \\
\mathrm{h}_p^{i,0}(\mathbb{P}^{n+1}) - \mathrm{h}_{p-d}^{i,0}(\mathbb{P}^{n+1}) &= \mathrm{h}_p^{i,0}(X) + \mathrm{h}_{p-d}^{i-1,0}(X)
\end{aligned}$$

as claimed. ■

We can use Lemma 4.4 together with Lemma 4.2 to compute the dimensions of $\mathrm{HH}^m(X, \mathcal{O}_X(p))$.

Corollary 4.7. *Let X be a smooth n-dimensional hypersurface of degree d, and let $t = d-n-2$. Then, we have*

$$\begin{aligned}
&\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) \\
&= \begin{cases}
\mathrm{h}_{t-p}^{0,n}(X) & \text{for } m=0 \\
\mathrm{h}_{t-p}^{m-n,0}(X) + \mathrm{h}_{t-p}^{\frac{m}{2},n-\frac{m}{2}}(X) + \mathrm{h}_{t-p}^{m,n}(X) + (n-2)\delta_{t,p}\delta_{m,n} & \text{for } 0<m<2n \text{ even} \\
\mathrm{h}_{t-p}^{m-n,0}(X) + \mathrm{h}_{t-p}^{m,n}(X) + (n-1)\delta_{t,p}\delta_{m,n} & \text{for } 0<m<2n \text{ odd} \\
\mathrm{h}_{t-p}^{n,0}(X) & \text{for } m=2n \\
0 & \text{else.}
\end{cases}
\end{aligned}$$

Proof. Observe that we have $\mathcal{O}_X(t) \cong \omega_X$ and that we can assume $0 \le m \le 2n$ for dimension reasons. By Lemma 4.2, we have

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}_{t-p}^{i,i-m+n}(X).$$

So, we can use Lemma 4.4 to compute every summand. In particular, we get for $n = 0$ and $n = 2n$

$$\begin{aligned} \dim \mathrm{HH}^0(X, \mathcal{O}_X(p)) &= \mathrm{h}_{t-p}^{0,n}(X), \\ \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) &= \mathrm{h}_{t-p}^{n,0}(X) \end{aligned}$$

as only one summand appears.

Now, for $0 < m < 2n$, we can use Lemma 4.4 to get for $m \neq n$

$$\begin{aligned} \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) &= \sum_{i}^{m-n} \mathrm{h}_{t-p}^{i,i-n+m}(X) \\ &= \sum_{i-j=m-n} \mathrm{h}_{t-p}^{i,j}(X) \\ &= \begin{cases} \mathrm{h}_{t-p}^{m-n,0}(X) + \mathrm{h}_{t-p}^{\frac{2n-m}{2},\frac{m}{2}}(X) + \mathrm{h}_{t-p}^{m,n}(X) & \text{for } m \text{ even} \\ \mathrm{h}_{t-p}^{m-n,0}(X) + \mathrm{h}_{t-p}^{m,n}(X) & \text{for } m \text{ odd.} \end{cases} \end{aligned}$$

For $m = n$, all the above calculations still hold; however, we get for all (i,i) with $0 < i < n$ and $(i,i) \neq (\frac{n}{2}, \frac{n}{2})$ an additional $\delta_{0,t-p} = \delta_{t,p}$, which means that

$$\begin{aligned} &\dim \mathrm{HH}^n(X, \mathcal{O}_X(p)) \\ &= \begin{cases} \mathrm{h}_{t-p}^{0,0}(X) + \mathrm{h}_{t-p}^{\frac{n}{2},\frac{n}{2}}(X) + \mathrm{h}_{t-p}^{n,n}(X) + (n-2)\delta_{t,p}\delta_{m,n} & \text{for } m \text{ even} \\ \mathrm{h}_{t-p}^{0,0}(X) + \mathrm{h}_{t-p}^{n,n}(X) + (n-1)\delta_{t,p}\delta_{m,n} & \text{for } m \text{ odd} \end{cases} \end{aligned}$$

as claimed. ■

4.2. The Hochschild cohomology of the direct image

Since we want to control the pushforward in Hochschild cohomology, we will use computations by [5] to understand the Hochschild dimensions of the direct image of a line bundle under a smooth embedding.

Lemma 4.8. *Let $f : X \hookrightarrow Y$ be an embedding of a smooth n-dimensional degree d hypersurface, and assume that $\omega_X \cong \mathcal{O}(t)$. Then, we have*

$$\mathrm{HH}^m(Y, f_*\mathcal{O}_X(p)) \cong \bigoplus_{i=0}^{\dim Y} \mathrm{H}^{n-m+i}(X, f^*\Omega_Y^i(t-p)).$$

Proof. We can compute, using $\omega_X \cong \mathcal{O}_X(t)$ and the Hochschild–Kostant–Rosenberg isomorphism (HKR) [24]:

$$\begin{aligned}
\mathrm{HH}^m(Y, f_*\mathcal{O}_X(p)) &\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_Y^{m-i}(\Omega_Y^i, f_*\mathcal{O}_X(p)) && \text{HKR}\\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{m-i}(f^*\Omega_Y^i, \mathcal{O}_X(p)) && f^* \dashv f_*\\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{n-m+i}(\mathcal{O}_X(p), f^*\Omega_Y^i(t))^* && \text{Serre duality}\\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{n-m+i}(\mathcal{O}_X, f^*\Omega_Y^i(t-p))^* && \text{twisting on both sides}\\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{H}^{n-m+i}(X, f^*\Omega_Y^i(t-p))^* && \mathrm{Ext}_X^j(\mathcal{O}_X, _) \cong \mathrm{H}^j(X, _)
\end{aligned}$$

as desired. ∎

Lemma 4.9. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be a closed embedding of a smooth degree d hypersurface. Then, we have for $(i, j) \notin \{(0,0), (0,n), (0,n+1), (n,1)(n,n), (n,n+1)\}$*

$$\dim \mathrm{H}^j(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(p)) = \begin{cases} \mathrm{h}_p^{i,0}(X) + \mathrm{h}_{p-d}^{i-1,0}(X) & \textit{if } j=0\\ \mathrm{h}_p^{i,n}(X) + \mathrm{h}_{p-d}^{i-1,n}(X) & \textit{if } j=n\\ \delta_{p,0} & \textit{if } i=j\notin\{0,n\}\\ \delta_{p,d} & \textit{if } i-1=j\notin\{0,n\}\\ 0 & \textit{else.}\end{cases}$$

Moreover, we get

$$\begin{aligned}
\dim \mathrm{H}^0(X, f^*\Omega^n_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_p^{n,0}(X) + \mathrm{h}_{p-d}^{n-1,0}(X) - \mathrm{h}_{p-d}^{n-1,1}(X)\\
\dim \mathrm{H}^n(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_p^{1,n}(X) + \mathrm{h}_{p-d}^{0,n}(X) - \mathrm{h}_p^{1,n-1}(X)\\
\dim \mathrm{H}^0(X, f^*\Omega^0_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_p^{0,0}(X)\\
\dim \mathrm{H}^n(X, f^*\Omega^0_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_p^{0,n}(X)\\
\dim \mathrm{H}^0(X, f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_{p-d}^{n,0}(X)\\
\dim \mathrm{H}^n(X, f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(p)) &= \mathrm{h}_{p-d}^{n,n}(X).
\end{aligned}$$

Proof. First, observe that $\Omega^i_{\mathbb{P}^{n+1}} = 0$ for $i > n+1$ and $i < 0$. In particular, we can assume that $0 \le i \le n+1$, and for dimension reasons, we can additionally assume that $0 \le j \le n$.

Furthermore, [5, Lemmas 5 and 6] give for $0 < j < n$

$$\dim \mathrm{H}^j(X, f^*\Omega^0(p)) = 0,$$
$$\dim \mathrm{H}^j(X, f^*\Omega^{n+1}(p)) = 0.$$

By [5, Satz 1, Lemma 6, (21), and (25)], we have for $0 < i < n+1$

$$\dim \mathrm{H}^j(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(p)) = \begin{cases} \mathrm{h}^{i,0}_p(\mathbb{P}^{n+1}) - \mathrm{h}^{i,0}_{p-d}(\mathbb{P}^{n+1}) & \text{if } j = 0 \\ \mathrm{h}^{i,n+1}_p(\mathbb{P}^{n+1}) - \mathrm{h}^{i,n+1}_{d-p}(\mathbb{P}^{n+1}) & \text{if } j = n \\ \delta_{p,0}\delta_{i,j} + \delta_{p,d}\delta_{i-1,j} & \text{if } j \notin \{0, n\}, \end{cases}$$

which gives after applying Proposition 4.6 for $i \notin \{1, n\}$

$$\dim \mathrm{H}^j(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(p)) = \begin{cases} \mathrm{h}^{i,0}_p(X) + \mathrm{h}^{i-1,0}_{p-d}(X) & \text{if j=0} \\ \mathrm{h}^{i,n}_p(X) + \mathrm{h}^{i-1,n}_{d-p}(X) & \text{if } j = n \\ \delta_{p,0}\delta_{i,j} + \delta_{p,d}\delta_{i-1,j} & \text{if } j \notin \{0, n\}. \end{cases}$$

Now, we will consider the special cases.

We start with the case of $i \in \{1, n\}$. By the discussion above, we have

$$\dim \mathrm{H}^0(X, f^*\Omega^n_{\mathbb{P}^{n+1}}(p)) = \mathrm{h}^{n,0}_p(\mathbb{P}^{n+1}) - \mathrm{h}^{n,0}_{p-d}(\mathbb{P}^{n+1}),$$
$$\dim \mathrm{H}^n(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(p)) = \mathrm{h}^{1,n+1}_p(\mathbb{P}^{n+1}) - \mathrm{h}^{1,n+1}_{p-d}(\mathbb{P}^{n+1}).$$

This turns, using [5, (31), (33), Satz 2 and Lemma 5] into

$$\dim \mathrm{H}^0(X, f^*\Omega^n_{\mathbb{P}^{n+1}}(p)) = \mathrm{h}^{n,0}_p(X) + \mathrm{h}^{n-1,0}_{p-d}(X) - \mathrm{h}^{n-1,1}_{p-d}(X),$$
$$\dim \mathrm{H}^n(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(p)) = \mathrm{h}^{1,n}_p(X) + \mathrm{h}^{0,n}_{p-d}(X) - \mathrm{h}^{1,n-1}_p(X).$$

So, only the cases for $i = 0$ and $i = n+1$ remain.

For $i = 0$, we have $f^*\Omega^0_{\mathbb{P}^{n+1}}(p) \cong \mathcal{O}_X(p)$, so we can apply Lemma 4.1 to get

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(p)) = \mathrm{h}^{0,0}_p(X),$$
$$\dim \mathrm{H}^n(X, \mathcal{O}_X(p)) = \mathrm{h}^{0,n}_p(X).$$

Now, for $i = n+1$, we have

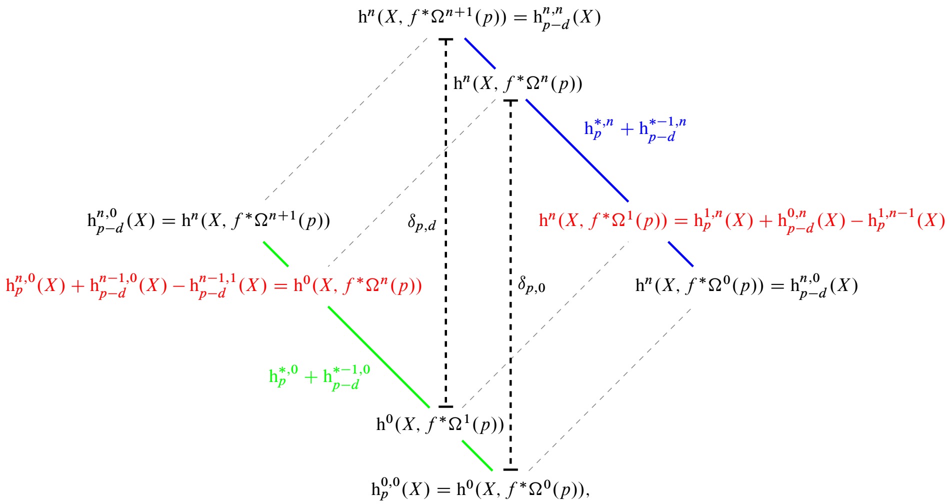
$$\begin{aligned} f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(p) &\cong f^*\mathcal{O}_X(p-n-2) \\ &\cong \mathcal{O}_X(d-n-2+p-d) \\ &\cong \Omega^n_X(p-d). \end{aligned}$$

And so, we get by Definition 4.1

$$\dim \mathrm{H}^0(X, \Omega^n_X(p-d)) = \mathrm{h}^{n,0}_{p-d}(X),$$
$$\dim \mathrm{H}^n(X, \Omega^n_X(p-d)) = \mathrm{h}^{n,n}_{p-d}(X)$$

as claimed. ∎

Remark 4.10. If we arrange the computation of the cohomology dimensions from Lemma 4.9 analogously to a twisted Hodge diamond, we get that it is of the shape



where apart from the two special cases

$$h^0(X, f^*\Omega^n_{\mathbb{P}^{n+1}}(p)) = h^{n,0}_p(X) + h^{n-1,0}_{p,d}(X) - h^{n-1,1}_d(X),$$
$$h^n(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(p)) = h^{1,n}_p(X) + h^{0,n}_{p-d}(X) - h^{1,n-1}_{p-d}(X),$$

the only non-trivial entries are along the indicated lines. There we have

$$h^0(X, f^*\Omega^i_{\mathbb{P}^{n+1}}) = h^{*,0}_p(X) + h^{*-1,0}_{p-d}(X),$$
$$h^n(X, f^*\Omega^i_{\mathbb{P}^{n+1}}) = h^{*,n}_p(X) + h^{*-1,n}_{p-d}(X),$$

and along the two vertical diagonals, we have

$$h^i(X, f^*\Omega^i_{\mathbb{P}^{n+1}}) = \delta_{p,0},$$
$$h^i(X, f^*\Omega^{i+1}_{\mathbb{P}^{n+1}}) = \delta_{p,d}.$$

Observe that this has the shape of the p and $p-d$ twisted Hodge diamond for X laid on top of each other with the interior middle line removed.

Since we will focus on the case $p > d$, we will be able to ignore the dashed lines.

Proposition 4.11. *Let $f: X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth n-dimensional hypersurface of degree d, and set $t = d - n - 2$. Then, we have for $m \notin \{1, 2n\}$*

$$\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) = h^{0,m-n}_{t-p}(X) + h^{0,m-n-1}_{t-p-d}(X) + h^{n,m}_{t-p}(X) + h^{n,m-1}_{t-p-d}(X), \\ + (n-1)(\delta_{d,p}\delta_{m,n+1} + \delta_{0,p}\delta_{m,n})$$

and for $m = 1$, $m = 2n$

$$\dim \mathrm{HH}^1(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) = \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X),$$
$$\dim \mathrm{HH}^{2n}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) = \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p-d}(X).$$

Proof. We will compute the cases separately using

$$\mathcal{O}_X(t) \cong \omega_X.$$

We can use Lemma 4.8 to get for $m \notin \{1, 2n\}$

$$\begin{aligned}
\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \sum_{i=0}^{n} \dim \mathrm{H}^{n-m+i}(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)) \\
&= \sum_{i-j=m-n} \dim \mathrm{H}^j(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)) \\
&= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) \\
&\quad + (n-1)(\delta_{d,p}\delta_{m,n+1} + \delta_{0,p}\delta_{m,n}).
\end{aligned}$$

For $m = 1$, we similarly get

$$\begin{aligned}
\dim \mathrm{HH}^1(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \sum_{i=0}^{n+1} \dim \mathrm{H}^{n-1+i}(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)) \\
&= \dim \mathrm{H}^n(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(t-p)) \\
&= \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X).
\end{aligned}$$

And for $m = 2n$, we get

$$\begin{aligned}
\dim \mathrm{HH}^{2n}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \sum_{i=0}^{n+1} \dim \mathrm{H}^{-n-1+i}(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)) \\
&= \dim \mathrm{H}^0(X, f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(t-p)) \\
&= \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p-d}(X)
\end{aligned}$$

which finishes the claim. ∎

Remark 4.12. Since we will be able to assume that $p \notin \{0, d\}$ in the next section, we will exclude these cases. However, all of the following proofs and arguments still hold in these cases; one just needs to keep track of the Kronecker deltas in $\dim \mathrm{HH}^n(X, \mathcal{O}_X(p))$.

Proposition 4.13. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth n-dimensional hypersurface of degree d, and let*

$$t = d - n - 2.$$

Then, we have for all $p \in \mathbb{Z}$ such that $t - p \notin \{0, d\}$ that $\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))$ *is given by*

$$\begin{cases} \dim \mathrm{HH}^0(X, \mathcal{O}_X(p)) & m = 0 \\ \dim \mathrm{HH}^1(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^0(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{1,n-1}_{t-p}(X) & m = 1 \\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m}{2}}_{t-p}(X) & 1 < m < 2n \text{ even} \\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m+1}{2}, n-\frac{m+1}{2}}_{t-p}(X) & 1 < m < 2n \text{ odd} \\ \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{2n-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{n-1,1}_{t-p-d}(X) & m = 2n \\ \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p+d)) & m = 2n+1 \\ 0 & \text{else.} \end{cases}$$

Proof. For dimension reasons, we immediately get $\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) = 0$ for $m < 0$, respectively, $2n + 1 < m$. Now, for the computations.

$m = 0$. We compute

$$\begin{aligned} \dim \mathrm{HH}^0(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \mathrm{h}^{0,n}_{t-p}(X) && \text{Proposition 4.11} \\ &= \dim \mathrm{HH}^0(X, \mathcal{O}_X(p)). && \text{Corollary 4.7} \end{aligned}$$

$m = 1$. We get by Proposition 4.11 and Corollary 4.7

$$\begin{aligned} \dim \mathrm{HH}^1(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X) \\ &= \dim \mathrm{HH}^1(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^0(X, \mathcal{O}_X(p+d)) \\ &\quad - \mathrm{h}^{1,n-1}_{t-p}(X). \end{aligned}$$

$1 < m < 2n$. In this case, we have by Corollary 4.7 and Proposition 4.11

$$\begin{aligned} &\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) \\ &= \begin{cases} \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m}{2}}_{t-p}(X) & m \text{ even} \\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m+1}{2}}_{t-p}(X) & m \text{ odd.} \end{cases} \end{aligned}$$

$m = 2n$. Here, we get by Proposition 4.11 and Corollary 4.7

$$\begin{aligned} \dim \mathrm{HH}^{2n}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p}(X) \\ &= \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{2n-1}(X, \mathcal{O}_X(p+d)) \\ &\quad - \mathrm{h}^{n-1,1}_{t-p-d}(X). \end{aligned}$$

$m = 2n + 1$. We compute

$$\begin{aligned} \dim \mathrm{HH}^{2n+1}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) &= \mathrm{h}^{n,0}_{t-p-d}(X) && \text{Proposition 4.11} \\ &= \dim \mathrm{HH}^n(X, \mathcal{O}_X(p+d)) && \text{Corollary 4.7.} \end{aligned}$$

So, we covered all cases, and the statement holds. ∎

Proposition 4.14 ([22, Proposition 9.5.1]). *Consider the embedding of a smooth n-dimensional degree d hypersurface*

$$X \overset{f}{\hookrightarrow} \mathbb{P}^{n+1}.$$

Then, we have a long exact sequence of the form

$$\cdots \to \mathrm{HH}^{i-2}(X, \mathcal{O}_X(p+d)) \to \mathrm{HH}^i(X, \mathcal{O}_X(p)) \xrightarrow{f_*} \mathrm{HH}^i(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) \to \cdots.$$

Theorem 4.15. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth degree d hypersurface, and set*

$$t = d - n - 2.$$

Then, we have for all $p \in \mathbb{Z}$ such that $t - p \notin \{0, d\}$

$$\dim \ker(f_* : \mathrm{HH}^m(X, \mathcal{O}_X(p)) \to \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))) = \begin{cases} \mathrm{h}^{\frac{m}{2}, n-\frac{m}{2}}_{t-p}(X) & 0 < m < 2n \text{ even} \\ \mathrm{h}^{n-1,1}_{t-p-d}(X) & m = 2n \\ 0 & \text{else.} \end{cases}$$

Proof. For dimension reasons, we may assume that

$$0 \le m \le 2n.$$

In the diagrams for this proof, we will denote $\mathcal{O}_X$ by $\mathcal{O}$ and $\mathbb{P}^{n+1}$ by $\mathbb{P}$ in order to avoid clumsy notation.

We will proceed by induction over l with $2l = m$ using the long exact sequence from Proposition 4.14:

$$\cdots \to \mathrm{HH}^{m-2}(X, \mathcal{O}_X(p+d)) \to \mathrm{HH}^m(X, \mathcal{O}_X(p)) \xrightarrow{f_*} \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) \to \cdots.$$

This way, we can cover the odd case $2l - 1$ and even case $2l$ in the induction step simultaneously.

We will start with $l = 1$ as induction start and include the case of $m = 0$ to cover the cases for $m = 0, 1, 2$.

We compute all the dimensions in the long exact sequence in Proposition 4.14 using Proposition 4.13 and proceed by diagram chase. Consider the following diagram, where we denote the spaces on the left and their dimensions on the right. We will use the arrows on the right-hand side to indicate that the dimensions to the right of their tail are the dimensions to the left of their tip:

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[dll, dashed] \\
\mathrm{HH}^0(X, \mathcal{O}(p)) \arrow[d, "f_*"] & 0 & + & \dim \mathrm{HH}^0(X, \mathcal{O}(p)) \arrow[dll, dashed] \\
\mathrm{HH}^0(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & \dim \mathrm{HH}^0(X, \mathcal{O}(p)) & + & 0 \arrow[dll, dashed] \\
\mathrm{HH}^{-1}(X, \mathcal{O}(p+d)) \arrow[d] & 0 & + & 0 \arrow[dll, dashed] \\
\mathrm{HH}^1(X, \mathcal{O}(p)) \arrow[d, "f_*"] & 0 & + & \dim \mathrm{HH}^1(X, \mathcal{O}(p)) \arrow[dll, dashed] \\
\mathrm{HH}^1(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & \dim \mathrm{HH}^1(X, \mathcal{O}(p)) & + & \dim \mathrm{HH}^0(X, \mathcal{O}(p+d)) - \mathrm{h}^{1,n-1}_{t-p} \arrow[dll, dashed] \\
\mathrm{HH}^0(X, \mathcal{O}(p+d)) \arrow[d] & \dim \mathrm{HH}^0(X, \mathcal{O}(p+d)) - \mathrm{h}^{1,n-1}_{t-p} & + & \mathrm{h}^{1,n-1}_{t-p} \arrow[dll, dashed] \\
\mathrm{HH}^2(X, \mathcal{O}(p)) & \mathrm{h}^{1,n-1}_{t-p} & - & \mathrm{h}^{1,n-1}_{t-p} + \dim \mathrm{HH}^2(X, \mathcal{O}(p)).
\end{tikzcd}$$

By the above diagram chase, we get that the image of the last arrow on the left has dimension $\mathrm{h}^{1,n-1}_{t-p}(X)$. So, by the exactness of the sequence from Proposition 4.14, we get that this is also the dimension of the kernel of

$$f_* : \mathrm{HH}^2(X, \mathcal{O}_X(p)) \to \mathrm{HH}^2(\mathbb{P}^n, f_*\mathcal{O}_X(p)).$$

And so, we get

$$\begin{aligned}
\dim \ker(f_* : \mathrm{HH}^0(X, \mathcal{O}_X(p)) \to \mathrm{HH}^0(\mathbb{P}^n, f_*\mathcal{O}_X(p))) &= 0, \\
\dim \ker(f_* : \mathrm{HH}^1(X, \mathcal{O}_X(p)) \to \mathrm{HH}^1(\mathbb{P}^n, f_*\mathcal{O}_X(p))) &= 0, \\
\dim \ker(f_* : \mathrm{HH}^2(X, \mathcal{O}_X(p)) \to \mathrm{HH}^2(\mathbb{P}^n, f_*\mathcal{O}_X(p))) &= \mathrm{h}^{1,n-1}_{t-p}(X)
\end{aligned}$$

as expected.

For the induction step, we will cover the cases $m = 2l - 1$ and $m = 2l$ simultaneously. Assume that

$$\dim \ker(f_* : \mathrm{HH}^{2l-2}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{2l-2}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) = \mathrm{h}^{n-l+1,l-1}_{t-p}(X).$$

We compute again the dimensions in the long exact sequence from Proposition 4.14 using our computations in Proposition 4.13. We write the long exact sequence on the left and the dimensions on the right. We draw the arrows on the right-hand side from left to right to indicate that the dimensions to the right of their tail are the dimensions to the left of their tip:

$$\begin{tikzcd}
\ker(f_*)^{2l-2} \arrow[d, hook] & \mathrm{h}_{t-p}^{l-1,n-l+1} \arrow[d, dashed] \\
\mathrm{HH}^{2l-2}(X,\mathcal{O}(p)) \arrow[d, "f_*"] & \mathrm{h}_{t-p}^{l-1,n-l+1} - \mathrm{h}_{t-p}^{l-1,n-l+1} + \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p)) \arrow[d, dashed] \\
\mathrm{HH}^{2l-2}(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & -\mathrm{h}_{t-p}^{l-1,n-l+1} + \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p)) + \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p+d)) \arrow[d, dashed] \\
\mathrm{HH}^{2l-3}(X,\mathcal{O}(p+d)) \arrow[d] & \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p+d)) + 0 \arrow[d, dashed] \\
\mathrm{HH}^{2l-1}(X,\mathcal{O}(p)) \arrow[d, "f_*"] & 0 + \dim \mathrm{HH}^{2l-1}(X,\mathcal{O}(p)) \arrow[d, dashed] \\
\mathrm{HH}^{2l-1}(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & \dim \mathrm{HH}^{2l-1}(X,\mathcal{O}(p)) + \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p+d)) - \mathrm{h}_{t-p}^{l-1,n-l+1} \arrow[d, dashed] \\
\mathrm{HH}^{2l-2}(X,\mathcal{O}(p-d)) \arrow[d] & \dim \mathrm{HH}^{2l-2}(X,\mathcal{O}(p+d)) - \mathrm{h}_{t-p}^{l,n-l} + \mathrm{h}_{t-p}^{l,n-l}(X) \arrow[d, dashed] \\
\mathrm{HH}^{2l}(X,\mathcal{O}(p)) & \mathrm{h}_{t-p}^{l,n-l} - \mathrm{h}_{t-p}^{l,n-l} + \dim \mathrm{HH}^{2l}(X,\mathcal{O}(p)).
\end{tikzcd}$$

By the exactness of the sequence, this means that

$$\dim \ker(f_* : \mathrm{HH}^{2l-1}(X,\mathcal{O}_X(p)) \to \mathrm{HH}^{2l-1}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) = 0,$$
$$\dim \ker(f_* : \mathrm{HH}^{2l}(X,\mathcal{O}_X(p)) \to \mathrm{HH}^{2l}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) = \mathrm{h}_{t-p}^{n-l,l}(X).$$

Now, finally, for the case of $l = n$.
By the above induction, we have

$$\dim \ker(f_* : \mathrm{HH}^{2n-2}(X,\mathcal{O}_X(p)) \to \mathrm{HH}^{2n-2}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) \\ = \mathrm{h}_{t-p}^{1,n-1}(X).$$

We apply again diagram chase along long exact sequence from Proposition 4.14 using the computations in Proposition 4.13. We continue to write the long exact sequence on the left and the dimensions on the right. The diagonal arrows on the right again symbolize that the dimensions to the right of their tail are the dimensions of the kernel to the left of their tip:

$$
\begin{tikzcd}
\ker(f_*)^{2n-2} \arrow[d, hook] & & & \mathrm{h}^{n-1,1}_{t-p} \arrow[dll, dashed] \\
\mathrm{HH}^{2n-2}(X,\mathcal{O}(p)) \arrow[d, "f_*"] & \mathrm{h}^{n-1,1}_{t-p} & - & \mathrm{h}^{n-1,1}_{t-p} + \dim \mathrm{HH}^{2n-2}(X,\mathcal{O}(p)) \arrow[dll, dashed] \\
\mathrm{HH}^{2n-2}(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & -\mathrm{h}^{n-1,1}_{t-p} + \dim \mathrm{HH}^{2n-2}(X,\mathcal{O}(p)) & + & \dim \mathrm{HH}^{2n-3}(X,\mathcal{O}(p+d)) \arrow[dll, dashed] \\
\mathrm{HH}^{2n-3}(X,\mathcal{O}(p+d)) \arrow[d] & \dim \mathrm{HH}^{2n-2}(X,\mathcal{O}(p+d)) & + & 0 \arrow[dll, dashed] \\
\mathrm{HH}^{2n-1}(X,\mathcal{O}(p)) \arrow[d, "f_*"] & 0 & + & \dim \mathrm{HH}^{2n-1}(X,\mathcal{O}(p)) \arrow[dll, dashed] \\
\mathrm{HH}^{2n-1}(\mathbb{P}, f_*\mathcal{O}(p)) \arrow[d] & \dim \mathrm{HH}^{2n-1}(X,\mathcal{O}(p)) & + & \dim \mathrm{HH}^{2n-2}(X,\mathcal{O}(p+d)) - \mathrm{h}^{n-1,1}_{t-p} \arrow[dll, dashed] \\
\mathrm{HH}^{2n-2}(X,\mathcal{O}(p-d)) \arrow[d] & \dim \mathrm{HH}^{2n-2}(X,\mathcal{O}(p+d)) - \mathrm{h}^{n-1,1}_{t-p-d} & + & \mathrm{h}^{n-1,1}_{t-p-d} \arrow[dll, dashed] \\
\mathrm{HH}^{2n}(X,\mathcal{O}(p)) & \mathrm{h}^{n-1,1}_{t-p-d} & + & \dim \mathrm{HH}^{2n}(X,\mathcal{O}(p)) - \mathrm{h}^{n-1,1}_{t-p-d}.
\end{tikzcd}
$$

This diagram gives us

$$
\begin{aligned}
&\dim \ker(f_* : \mathrm{HH}^{2n-1}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{2n-1}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) = 0, \\
&\dim \ker(f_* : \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{2n}(\mathbb{P}^n, f_*\mathcal{O}_X(p))) = \mathrm{h}^{n-1,1}_{t-p-d}(X).
\end{aligned}
$$

So, we covered the case for $m = 2n$ and we are done as for $m < 0$ and $m > 0$ the source space is trivial. ∎

We now finally state the following in order to guarantee the existence of non-trivial kernels of pushforwards of Hochschild cohomology.

Proposition 4.16. *Let $f : X \hookrightarrow \mathbb{P}^{2k}$ be an embedding of a smooth odd-dimensional degree $d > 1$ hypersurface of dimension $n = 2k - 1$ for $k > 2$, and let $p = -kd - d$. Then, we have*

$$
\ker(\mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))) \cong \Bbbk.
$$

Proof. By Theorem 4.15, we have

$$
\dim \ker(\mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))) = \mathrm{h}^{k+1,k-2}_{t-p}(X)
$$

with $t = d - n - 2 = d - 2k - 1$.

So, it suffices to compute that $\mathrm{h}^{k+1,k-2}_{t-p}(X) = 1$, with $t - p = kd + 2d - 2k - 1$. By (1), this is

$$
\begin{aligned}
&\mathrm{h}^{k+1,k-2}_{t-p}(X)\\
&= \sum_{\mu=0}^{2k-1+2} (-1)^{\mu} \binom{2k-1+2}{\mu} \binom{-kd-2d+2k+1+(k+1)d-(\mu-1)(d-1)}{2k-1+1}\\
&= \sum_{\mu=0}^{2k+1} (-1)^{\mu} \binom{2k+1}{\mu} \binom{-kd-2d+2k+1+kd+d-\mu d+d+\mu-1}{2k}\\
&= \sum_{\mu=0}^{2k+1} (-1)^{\mu} \binom{2k+1}{\mu} \binom{2k-\mu d+\mu}{2k}\\
&= \binom{2k+1}{0} \binom{2k}{2k}\\
&= 1.
\end{aligned}
$$

Here, we used that for $\mu > 1$ we have $2k - \mu d + \mu < 2k$ as $d > 1$, which means that the terms $\binom{2k+1}{\mu}\binom{2k+\mu d+\mu}{2k}$ vanish for $\mu \geq 1$.

So, we get

$$\dim\ker(\mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))) = 1$$

as claimed. ∎

4.3. Examples

We collect a few examples of twisted Hodge diamonds that were computed using the Sage package by Pieter Belmans and Piet Glas [1].

The first two examples illustrate the general shape as given in Lemma 4.4, and the third will be an explicit example of Proposition 4.16.

Example 4.17. Let $f : X \hookrightarrow \mathbb{P}^6$ be a smooth degree 7 hypersurface; then, the 8-twisted Hodge diamond is

$$
\begin{array}{c}
0\\
0 \quad 0\\
0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0\\
2996 \quad 20993 \quad 15267 \quad 917 \quad 0 \quad 0\\
1575 \quad 0 \quad 0 \quad 0 \quad 0\\
5775 \quad 0 \quad 0 \quad 0\\
10395 \quad 0 \quad 0\\
9002 \quad 0\\
2996.
\end{array}
$$

And so, we have by Theorem 4.15, since $t - 8 = -8$,

$$\dim \ker(f_* : \mathrm{HH}^4(X, \mathcal{O}_X(-8)) \to \mathrm{HH}^4(X, f_*\mathcal{O}_X(-8))) = 917,$$

$$\dim \ker(f_* : \mathrm{HH}^6(X, \mathcal{O}_X(-8)) \to \mathrm{HH}^6(X, f_*\mathcal{O}_X(-8))) = 15267,$$

$$\dim \ker(f_* : \mathrm{HH}^8(X, \mathcal{O}_X(-8)) \to \mathrm{HH}^8(X, f_*\mathcal{O}_X(-8))) = 20993.$$

Example 4.18. Let $f : X \hookrightarrow \mathbb{P}^8$ be a smooth degree 5 hypersurface, then the -7-twisted Hodge diamond is

$$\begin{array}{c}
6390 \\
0 \quad 20511 \\
0 \quad 0 \quad 25704 \\
0 \quad 0 \quad 0 \quad 16840 \\
0 \quad 0 \quad 0 \quad 0 \quad 4950 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 720 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 36 \\
0 \quad 0 \quad 0 \quad 486 \quad 13051 \quad 30276 \quad 8451 \quad 165 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \\
0 \quad 0 \\
0.
\end{array}$$

And so, we have by Theorem 4.15, since $t + 7 = 3$,

$$\dim \ker(f_* : \mathrm{HH}^2(X, \mathcal{O}_X(3)) \to \mathrm{HH}^2(X, f_*\mathcal{O}_X(3))) = 8451,$$

$$\dim \ker(f_* : \mathrm{HH}^4(X, \mathcal{O}_X(3)) \to \mathrm{HH}^4(X, f_*\mathcal{O}_X(3))) = 15267,$$

$$\dim \ker(f_* : \mathrm{HH}^6(X, \mathcal{O}_X(3)) \to \mathrm{HH}^6(X, f_*\mathcal{O}_X(3))) = 13051,$$

$$\dim \ker(f_* : \mathrm{HH}^8(X, \mathcal{O}_X(3)) \to \mathrm{HH}^8(X, f_*\mathcal{O}_X(3))) = 486.$$

The next example illustrates a case of Proposition 4.16:

Example 4.19. Let

$$f : X \hookrightarrow \mathbb{P}^{10}$$

be a smooth degree 5 hypersurface, and consider $\mathcal{O}_X(-30)$. Then, we can compute, using Theorem 4.15,

$$\dim \ker(f_* : \mathrm{HH}^m(X, \mathcal{O}_X(-30)) \to \mathrm{HH}^m(X, f_*\mathcal{O}_X(-30))).$$

To do this, we need to compute the $t - p = 24$ twisted Hodge diamond:

$$\begin{array}{c}
0 \\
0 \quad 0 \\
0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
11979044 \quad 100298 \quad 2882 \quad {\color{red}1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
107439618 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
523445109 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
1580020794 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
3149538513 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
4236318471 \quad 0 \quad 0 \quad 0 \quad 0 \\
3815626243 \quad 0 \quad 0 \quad 0 \\
2209626573 \quad 0 \quad 0 \\
744650346 \quad 0 \\
111098130.
\end{array}$$

And as expected, by Proposition 4.16, we get

$$\dim \ker(f_* : \mathrm{HH}^{12}(X, \mathcal{O}_X(-30)) \to \mathrm{HH}^{12}(X, f_*\mathcal{O}_X(-30))) = 1.$$

5. Non-trivial kernel in Hochschild cohomology give non-Fourier–Mukai functors

In this section, we follow the ideas from [22] to construct candidate non-Fourier–Mukai functors for hypersurfaces of arbitrary degree. We then verify that under assumptions on the characteristic morphisms and some concentrated Ext-groups these indeed cannot be Fourier–Mukai. We finish the chapter by computing that these assumptions are satisfied when the source category is the derived category of an odd-dimensional quadric, which gives concrete non-Fourier–Mukai functors between well-behaved spaces in arbitrary high dimensions.

Since we follow the approach from [22], we will consider functors of a similar form:

$$\Psi_\eta : \mathcal{D}^b(X) \xrightarrow{L} \mathcal{D}^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta^{dg}) \xrightarrow{\psi_{\mathcal{X},\eta,*}} \mathcal{D}^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta) \xrightarrow{\tilde{f}_*} \mathcal{D}^b(\mathbb{P}^{n+1}), \tag{2}$$

where $\mathcal{X}_\eta^{dg}$ denotes the dg-hull of $\mathcal{X}_\eta$ and $\psi_{\mathcal{X},\eta,*}$ is the induced comparison functor.

5.1. Constructing candidate non-Fourier–Mukai functors

We start by collecting a few results from [22], which are central for our construction. We refer the interested reader to [22] for an in-depth discussion.

In order to apply Definition 5.1, Lemmas 5.2, 5.3, and 5.4, we assume that every quasi-projective scheme X comes equipped with an open affine cover

$$X = \bigcup_{i=0}^{m} U_i .$$

The following construction was originally introduced by W. Lowen and M. Van den Bergh in [16].

Definition 5.1 ([20, Definition 4.2]). Let $X = \bigcup_{i=1}^{m} U_i$ be an open affine covering of a quasi-projective scheme. Consider for $I \subset \{1, \dots, m\}$ the sets $U_I := \bigcap_{i \in I} U_i$ indexed by $I \in \mathcal{I} := \mathrm{P}(X) \setminus \emptyset$. Then, $\mathfrak{X}$ is the category with objects $\mathcal{I}$ and morphisms:

$$\mathfrak{X}(I, J) := \begin{cases} \mathcal{O}_X(U_J) & I \subset J \\ 0 & \text{else,} \end{cases}$$

where composition is induced by composing with the restriction morphism.

Roughly $\mathfrak{X}$-mod acts as the category of presheaves associated to an affine covering. This means that it comes with the following useful properties.

Lemma 5.2 ([16]). *Let X be quasi-projective. Then, there is a fully faithful embedding*

$$w : \mathcal{D}(\mathrm{Qch}\, X) \xrightarrow{\sim} \mathcal{D}_{w\,\mathrm{Qch}(X)}(\mathfrak{X}) \hookrightarrow \mathcal{D}(\mathfrak{X})$$

and a fully faithful embedding

$$W : \Delta_* \mathcal{D}(\mathrm{Qch}\, X) \to \mathcal{D}(\mathfrak{X} \otimes_{\Bbbk} \mathfrak{X}^{\mathrm{op}}),$$

where $\Delta_*(\mathrm{Qch}\, X)$ is the essential image of the direct image of the diagonal embedding $\Delta : X \to X \times X$. In particular, we have for quasi-coherent M

$$\mathrm{HH}^*(X, M) \cong \mathrm{HH}^*(\mathfrak{X}, WM).$$

Proof. The construction of w can be found in [22, (8.5)]. The functor W gets constructed in the following paragraph of [22]. For the Hochschild cohomology comparison, we can use that W is a fully faithful embedding to get

$$W : \mathrm{HH}^*(X, M) := \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \Delta_* M) \xrightarrow{\sim} \mathrm{Ext}^*_{\mathfrak{X} \otimes \mathfrak{X}^{op}}(\mathfrak{X}, WM) =: \mathrm{HH}^*(\mathfrak{X}, WM)$$

as desired. ∎

Lemma 5.3. *Let X be quasi-projective, and let Γ be a $\Bbbk$-algebra. Then, there is an embedding*

$$w : \mathcal{D}(\mathrm{coh}(X)_\Gamma) \hookrightarrow \mathcal{D}(\mathfrak{X} \otimes \Gamma).$$

Proof. By [22, Section 8.5], we have an embedding

$$w : \mathcal{D}(\mathrm{Qch}(X)_\Gamma) \hookrightarrow \mathcal{D}(\mathcal{X} \otimes \Gamma).$$

There is also a canonical embedding

$$\mathcal{D}(\mathrm{coh}(X)_\Gamma) \hookrightarrow \mathcal{D}(\mathrm{Qch}(X)_\Gamma).$$

In particular, we get the desired embedding by composition. ■

Lemma 5.4 ([22, Section 8.7]). *Let $f : X \to Y$ be a closed embedding of quasi-projective schemes. Then, we have an induced functor*

$$\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$$

such that the diagram

$$\begin{tikzcd} \mathcal{D}(X) \arrow[r, "f_*"] \arrow[d, "w"'] & \mathcal{D}(Y) \arrow[d, "w"] \\ \mathcal{D}(\mathcal{X}) \arrow[r, "\mathfrak{f}_*"'] & \mathcal{D}(\mathcal{Y}) \end{tikzcd}$$

commutes.

Proof. The construction of $\mathfrak{f}$ is done in [22, Section 8.7]. By [10, Proposition 3.5], we have an inclusion $\mathcal{D}(X) \hookrightarrow \mathcal{D}(\mathrm{Qch}(X))$. So, we can restrict the diagram from [22, Lemma 8.7.1] to $\mathcal{D}(X)$. ■

We use the following construction from [22] as the core of our candidate functors.

Proposition 5.5. *Let X be smooth projective of dimension n, and let $\eta \in \mathrm{HH}^{\geq n+3}(X, M)$. Then, there exists an exact functor*

$$\mathcal{D}^b(X) \xrightarrow{L} \mathcal{D}^b_{w\,\mathrm{coh}(X)}(\mathcal{X}^{dg}_\eta)$$

such that $\underline{\mathrm{RHom}}_{\mathcal{X}_\eta}(\mathcal{X}, L(\_)) \cong w$.

Proof. First, observe that by Lemma 5.2 we have an isomorphism

$$\mathrm{HH}^*(X, M) \cong \mathrm{HH}^*(\mathcal{X}, WM),$$

and so, we may consider $\eta \in \mathrm{HH}^{\geq n+3}(\mathcal{X}, WM)$.

By [22, Lemma 10.1] and since $\mathrm{Qch}(X)$ has global dimension n and $\mathrm{H}^i\,\mathcal{X}_\eta$ vanishes in the right degrees, we can apply [22, Proposition 5.3.1] with $\mathcal{A} = w\,\mathrm{Qch}(X)$ and $\mathfrak{c} = \mathcal{X}_\eta$ to get a functor

$$L' : \mathcal{D}^b(\mathrm{Qch}(X)) \cong \mathcal{D}^b(w\,\mathrm{Qch}\,X) \to \mathcal{D}^b_{w\,\mathrm{Qch}(X)}(\mathcal{X}^{dg}_\eta).$$

Now, we can use [10, Proposition 3.5] to turn this into a functor:

$$L : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b_{\mathrm{coh}(X)}(\mathrm{Qch}(X)) \xrightarrow{L'} \mathcal{D}^b(\mathcal{X}^{dg}_\eta)$$

with the desired property.

Finally, by [22, Corollary 10.4], we know that the essential image of this functor is contained in $\mathcal{D}^b_{w\,\mathrm{coh}\,X}(\mathcal{X}^{dg}_\eta)$. ■

We will also use the following notation from [22] for $\tilde{f}$.

Proposition 5.6 ([22, Proposition 7.2.6]). *Let $f : \mathcal{P}^{n+1} \to \mathcal{X}$ be a functor of $\Bbbk$-linear categories, and let $\eta \in \mathrm{HH}^k(\mathcal{X}, \mathcal{M})$ such that*

$$f_* \eta = 0.$$

Then, there exists an $\mathcal{A}_\infty$-functor $\tilde{f}$ making the diagram

$$\begin{tikzcd} & \mathcal{X}_\eta \arrow[dl, "\pi"'] & \\ \mathcal{X} & & \mathcal{P}^{n+1} \arrow[ll, "f"] \arrow[ul, dashed, "\tilde{f}"'] \end{tikzcd}$$

commute. In particular, we have

$$\pi \circ \tilde{f} = f.$$

Now, we construct a candidate functor Ψ_η for $\eta \in \mathrm{HH}^{\geq n+3}(X, \mathcal{O}_X(p))$.

Construction 5.7. Let $X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth n-dimensional scheme with $n \geq 3$, and let

$$0 \neq \eta \in \ker(f_* : \mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))).$$

Then, a functor of the form (2) is constructed to be

$$\Psi_\eta : \mathcal{D}^b(\mathrm{coh}(X)) \xrightarrow{L} \mathcal{D}^b_{w\,\mathrm{coh}(X)}(\mathcal{X}^{dg}_\eta) \xrightarrow{\psi_{\mathcal{X}_\eta,*}} \mathcal{D}^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta)$$
$$\xrightarrow{\tilde{f}_*} \mathcal{D}^b_{w\,\mathrm{coh}(\mathbb{P}^{n+1})}(\mathcal{P}^{n+1}) \cong \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^{n+1})),$$

where we have the functor L by Proposition 5.5, $\psi_{\mathcal{X}_\eta,*}$ is the functor constructed in [22, Section D.1], and $\tilde{f}_*$ exists by Proposition 5.6.

Corollary 5.8. *Let $f : X \to \mathbb{P}^{n+1}$ be the embedding of a degree d hypersurface, and let $m > n+2$; then, we have a $\mathrm{h}_p^{\frac{m}{2}, n-\frac{m}{2}}(X)$-dimensional space of choices to construct a candidate functor*

$$\Psi_\eta : \mathcal{D}^b(X) \to \mathcal{D}^b(\mathbb{P}^{n+1}).$$

Proof. In order for Construction 5.7 to work, we need

$$0 \neq \eta \in \ker(f_* : \mathrm{HH}^m(X, \mathcal{O}_X(p)) \to \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))).$$

By Theorem 4.15, $\ker(f_* : \mathrm{HH}^m(X, \mathcal{O}_X(p)) \to \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)))$ has dimension $\mathrm{h}_p^{\frac{m}{2}, n-\frac{m}{2}}(X)$ which finishes the claim. ∎

Now, we can state our main theorem, which we will prove throughout Section 5.2.

Theorem 5.9. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be an embedding of a smooth degree d hypersurface of dimension $n \geq 3$, and let*

$$0 \neq \eta \in \ker(f_* : \mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)))$$

such that there exists a $\Bbbk$-algebra Γ and $G \in \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$ with

$$\begin{aligned} &c_{G,\Gamma}(\eta) \neq 0 \\ &\mathrm{Ext}^i_X(G(-p), G) = 0 && \textit{for } i \neq n \\ &\mathrm{Ext}^{n-1}_X(G, G(p+d)) \cong \mathrm{Ext}^{n-2}_X(G, G(p+d)) \cong 0. \end{aligned}$$

Then, we have that the functor

$$\Psi_\eta : \mathcal{D}^b(\mathrm{coh}(X)) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^{n+1}))$$

is well defined and not a Fourier–Mukai functor.

Remark 5.10. By the same proof as [22, Proposition B.2.1], the functors ψ_η do not admit a lift to the spectral level in the case of $\Bbbk = \mathbb{Q}$.

5.2. Proving Theorem 5.9

We fix for the rest of this section an embedding of a smooth degree d hypersurface $f : X \hookrightarrow \mathbb{P}^{n+1}$, a non-vanishing Hochschild cohomology class $\eta \in \mathrm{HH}^{n+3}(X, \mathcal{O}(p))$, such that $f_*\eta = 0$, Γ a $\Bbbk$-algebra, and $G \in \mathcal{D}(\mathrm{coh}(X)_\Gamma)$ such that

$$c_{G,\Gamma}(\eta) \neq 0 \tag{I}$$

$$\mathrm{Ext}^i_X(G(-p), G) = 0 \qquad \text{for } i \neq n \tag{II}$$

$$\mathrm{Ext}^{n-1}_X(G, G(p+d)) \cong \mathrm{Ext}^{n-2}(G, G(p+d)) \cong 0. \tag{III}$$

Observe first that by Construction 5.7 Ψ_η is well defined and even unique up to a choice of $\tilde{f}$. So, we may focus for the rest of this section on verifying that Ψ_η cannot be Fourier–Mukai.

We follow mostly the ideas from [22].

We start by recalling Lemma 5.11 from [22] in order to have obstructions against lifts of $\tilde{\mathcal{G}}$ to $\mathcal{D}(\mathcal{X}_\eta \otimes \Gamma)$. These obstructions and their naturality will be later used in order to conclude that Ψ_η cannot be Fourier–Mukai.

Lemma 5.11 ([22, Lemma 7.3.1]). (1) *Let $\mathcal{X}$ be a dg-category, Γ a $\Bbbk$-algebra, and let $\mathcal{G} \in \mathcal{D}(\mathcal{X})_\Gamma$. Then, there is a sequence of obstructions*

$$o_{i+2}(\mathcal{G}) \in \mathrm{HH}^{i+2}(\Gamma, \mathrm{Ext}^{-i}_{\mathcal{X}}(\mathcal{G}, \mathcal{G}))$$

for $i \geq 1$ such that $\mathcal{G}$ lifts to an object in $\mathcal{D}(\mathcal{X} \otimes_\Bbbk \Gamma)$ if and only if all obstructions vanish. More precisely, $o_{i+1}(\mathcal{G})$ is only defined if $o_3(\mathcal{G}), \ldots, o_i(\mathcal{G})$ vanish, and it depends on choices.

(2) *If $f : \mathcal{Y} \to \mathcal{X}$ is a dg-functor and $f_* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ is the corresponding change of rings functor, then after having made choices for $\mathcal{G}$ we may make corresponding choices for $f_*(\mathcal{G})$ in such a way that*

$$f_*(o_{i+2}(\mathcal{G})) = o_{i+2}(f_*(\mathcal{G})).$$

We now use the assumptions on G to prove that the negative part of $\mathrm{Ext}^*_{\mathcal{X}_\eta}(LG, LG)$ is concentrated in degree -1, which allows us to control which $\mathcal{A}_\infty$-obstruction does not vanish. This obstruction we will then push forward to prove that Ψ_η cannot be Fourier–Mukai. In order to avoid clumsy notation, we start by setting

$$\mathcal{G} := wG \in \mathcal{X}\text{-mod and } \widetilde{\mathcal{G}} := L(G). \tag{3}$$

Remark 5.12. We have by [22, Section D.1] an equivalence $\psi_{\mathcal{X}_\eta} : \mathcal{X}^{dg}_\eta \xrightarrow{\sim} \mathcal{X}_\eta$ and by Definition 2.4 a canonical functor $\pi : \mathcal{X}_\eta \to \mathcal{X}$. So, we will denote the functor

$$\psi^{-1}_{\mathcal{X}_\eta,*} \circ \pi_* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}_\infty(\mathcal{X}_\eta) \to \mathcal{D}(\mathcal{X}^{dg}_\eta)$$

simply by π_* and

$$\psi_{\mathcal{X}_\eta,*} \circ \pi^* : \mathcal{D}(\mathcal{X}^{dg}_\eta) \to \mathcal{D}_\infty(\mathcal{X}_\eta) \to \mathcal{D}(\mathcal{X})$$

by π^* to avoid clumsy and confusing notation.

Definition 5.13. Consider the distinguished triangle in $\mathcal{D}(\mathcal{X}^{dg}_\eta)$ [22, Lemma 10.3]:

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\gamma} \Sigma\mathcal{G}, \tag{4}$$

where $\mathcal{G}$ is considered as an $\mathcal{X}^{dg}_\eta$-module via $\pi_* : \mathcal{D}^b(\mathcal{X}) \to \mathcal{D}(\mathcal{X}^{dg}_\eta)$. Then, define the morphism φ by

$$\varphi : \mathrm{Ext}^{n+1+i}_X(G(-p), G) \to \mathrm{Ext}^i_{\mathcal{X}^{dg}_\eta}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}})$$
$$(g : \Sigma^{-n-1-i}G(-p) \to G) \mapsto \alpha \circ \pi_*(w(g)) \circ \Sigma^{-i}\beta : (\Sigma^{-i}\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}).$$

Lemma 5.14. *For $i < 0$, the morphism*

$$\varphi : \mathrm{Ext}^{n+1+i}_X(G(-p), G) \cong \mathrm{Ext}^i_{\mathcal{X}^{dg}_\eta}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}})$$

is an isomorphism.

Proof. We will check that for $i < 0$ the morphisms involved in the definition of

$$\varphi : \mathrm{Ext}_X^{n+1+i}(G(-p), G) \to \mathrm{Ext}^i_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}})$$
$$(g : \Sigma^{-n-1-i} G(-p) \to G) \mapsto (\alpha \circ \pi_*)(w(g)) \circ \Sigma^{-i}\beta : (\Sigma^{-i}\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}})$$

are isomorphisms.

w. By Lemma 5.2, $w : \mathcal{D}^b(X) \to \mathcal{D}^b(\mathcal{X})$ is a fully faithful embedding; in particular, using $\mathcal{G} = wG$ (3), we have

$$w : \mathrm{Ext}_X^{n+1+i}(G, G) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{X}}^{n+1+i}(\mathcal{G}, \mathcal{G}).$$

$\alpha \circ \pi_*(\_)$. We have by [22, Corollary 5.3.2] an adjunction:

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \cong \mathrm{RHom}_{\mathcal{X}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \mathcal{G}).$$

This isomorphism can be computed explicitly to be

$$\alpha \circ \pi_* : \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \cong \mathrm{RHom}_{\mathcal{X}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \mathcal{G});$$

see [22, (11.6)].

$\_ \circ \beta$. Consider the distinguished triangle (4) in $\mathcal{D}(\mathcal{X}_\eta^{dg})$:

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p).$$

Apply $\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\_, \widetilde{\mathcal{G}})$ to get the distinguished triangle:

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \xrightarrow{\_\circ\beta} \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \to \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\mathcal{G}, \widetilde{\mathcal{G}}).$$

Now, we may use [22, Corollary 5.3.2] and Proposition 5.1,

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\mathcal{G}, \widetilde{\mathcal{G}}) \cong \mathrm{RHom}_{\mathcal{X}_\eta}(\mathcal{G}, \widetilde{\mathcal{G}}) \cong \mathrm{RHom}_{\mathcal{X}}(\mathcal{G}, \mathcal{G}) \cong \mathrm{RHom}_X(G, G),$$

to get

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \xrightarrow{\_\circ\beta} \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \to \mathrm{RHom}_X(G, G).$$

Applying H^i turns this into the long exact sequence:

$$\cdots \to \mathrm{Ext}_X^{i-1}(G, G) \to \mathrm{Ext}^{n+1+i}_{\mathcal{X}_\eta^{dg}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \xrightarrow{\_\circ\beta} \mathrm{Ext}^i_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \to \cdots.$$

And as G is a sheaf on X, specializing to $i < 0$ yields the long exact sequence:

$$\cdots \to 0 \to \mathrm{Ext}^{n+1+i}_{\mathcal{X}_\eta^{dg}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \xrightarrow{\_\circ\beta} \mathrm{Ext}^i_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \to 0 \to \cdots.$$

In particular,

$$\_ \circ \beta : \mathrm{Ext}^{n+1+i}_{\mathcal{X}_\eta^{dg}}(\mathcal{G} \otimes w\mathcal{O}_X(-p), \widetilde{\mathcal{G}}) \xrightarrow{\sim} \mathrm{Ext}^{i}_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}})$$

is an isomorphism for $i < 0$.

So altogether, we get that

$$\begin{aligned} \varphi : \mathrm{Ext}^{n+1+i}_X(G(-p), G) &\xrightarrow{\sim} \mathrm{Ext}^{i}_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \\ (g : \Sigma^{-n-1-i} G(-p) \to G) &\mapsto (\alpha \circ \pi_*(w(g)) \circ \Sigma^{-i}\beta : \Sigma^{-i}\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}) \end{aligned}$$

is indeed an isomorphism for $i < 0$ as it is a composition of isomorphisms. ∎

Corollary 5.15. *Let $p < -n-1$ and $i > 1$. Then,* $\mathrm{Ext}^{-i}_{\mathcal{X}_\eta}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) = 0$.

Proof. By (II), we have that $\mathrm{Ext}^*_X(G(-p), G)$ is concentrated in degree n, and so, we have by Lemma 5.14

$$\mathrm{Ext}^{-i}_{\mathcal{X}_\eta^{dg}}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \cong \mathrm{Ext}^{n+1-i}_X(G(-p), G) \cong 0$$

for $i > 1$.

And since we have a quasi-equivalence $\mathcal{X}_\eta \cong \mathcal{X}_\eta^{dg}$, we get

$$\mathrm{Ext}^{-i}_{\mathcal{X}_\eta}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \cong \mathrm{Ext}^{n+1-i}_X(G(-p), G)$$

as claimed. ∎

Definition 5.16 ([22, Lemma 11.4]). Let $\mathcal{X}$ be a $\Bbbk$-linear category, Γ a $\Bbbk$-algebra, and let $\mathcal{M}$ be a $\Bbbk$-central $\mathcal{X}$-bimodule. Then, we have for a Γ-equivariant $\mathcal{X}$-module $\mathcal{G}$, i.e., $\mathcal{G} \in \mathcal{X}\text{-mod}_\Gamma$, the (algebraic) Γ-equivariant characteristic morphism

$$\begin{aligned} c_{\mathcal{G},\Gamma} : \mathrm{HH}^*(\mathcal{X}, \mathcal{M}) = \mathrm{Ext}^*_{\mathcal{X} \otimes \mathcal{X}^{\mathrm{op}}}(\mathcal{X}, \mathcal{M}) &\to \mathrm{Ext}^*_{\mathcal{X} \otimes \Gamma}(\mathcal{G}, \mathcal{G} \otimes \mathcal{M}) \\ \eta &\mapsto \mathcal{G} \otimes_{\mathcal{X}} \eta. \end{aligned}$$

Observe that this morphism factors naturally as

$$c_{\mathcal{G},\Gamma} : \mathrm{HH}^*(\mathcal{X}, \mathcal{M}) \xrightarrow{\eta \mapsto \eta \cup 1} \mathrm{HH}^*(\mathcal{X} \otimes \Gamma, \mathcal{M} \otimes \Gamma) \xrightarrow{c_{\mathcal{G}}} \mathrm{Ext}^*_{\mathcal{X} \otimes \Gamma}(\mathcal{G}, \mathcal{G} \otimes \mathcal{M}),$$

where $c_{\mathcal{G}} : \mathrm{HH}^*(\mathcal{X} \otimes \Gamma, \mathcal{M} \otimes \Gamma) \to \mathrm{Ext}^*_{\mathcal{X} \otimes \Gamma}(\mathcal{G}, \mathcal{G} \otimes \mathcal{M})$ is the (algebraic) characteristic morphism for $\mathcal{G} \in \mathcal{D}(\mathcal{X} \otimes \Gamma)$; see Proposition 2.8.

Lemma 5.17. *There is a commutative diagram:*

$$\begin{tikzcd} \mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}(G, G(p)) \arrow[d, "\wr"] \\ \mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) \arrow[r, "c_{\mathcal{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X} \otimes \Gamma}(\mathcal{G}, \mathcal{G} \otimes w\mathcal{O}_X(p)), \end{tikzcd}$$

where $c_{G,\Gamma}$ is the (geometric) equivariant characteristic morphism discussed in Section 3 and $c_{\mathscr{G},\Gamma}$ is the (algebraic) characteristic morphism from Definition 5.16.

Proof. By [22, (8.13)], we have the commutative diagram

$$\begin{tikzcd}
\Delta_*\mathscr{D}(X) \arrow[r, "\pi_{1,*}(_\otimes\pi_2^*G)"] \arrow[d, "W", "\wr"'] & \mathscr{D}(\mathrm{coh}(X)_\Gamma) \arrow[d, "w"', "\wr"] \\
\mathscr{D}(\mathcal{X}\otimes\mathcal{X}^{\mathrm{op}}) \arrow[r, "\mathscr{G}\otimes_{\mathcal{X}}_"'] & \mathscr{D}(\mathcal{X}\otimes\Gamma),
\end{tikzcd}$$

where we denote by $\Delta_*\mathscr{D}(X)\subset\mathscr{D}(X\times X)$ the essential image of the direct image along the diagonal embedding $\Delta : X\to X\times X$.

Considering the induced diagram on morphism spaces for

$$\mathrm{HH}^{n+3}(X,\mathcal{O}(p)) = \mathrm{Ext}^{n+3}_{X\times X}(\mathcal{O}_\Delta,\mathcal{O}_\Delta(p)) = \mathrm{Ext}^{n+3}_{\Delta_*\mathscr{D}(X)}(\mathcal{O}_\Delta,\mathcal{O}_\Delta(p))$$

gives that the diagram

$$\begin{tikzcd}
\mathrm{HH}^{n+3}(X,\mathcal{O}_X(p)) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}(G,G(p)) \arrow[d, "\wr"] \\
\mathrm{HH}^{n+3}(\mathcal{X},w\mathcal{O}_X(p)) \arrow[r, "c_{\mathscr{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathscr{G},\mathscr{G}\otimes w\mathcal{O}_X(p))
\end{tikzcd}$$

commutes. ■

Since $G\in\mathscr{D}^b(\mathrm{coh}(X)_\Gamma)$, we get a Γ-action on $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ via the functors w and L, i.e., $\widetilde{\mathscr{G}}\in\mathscr{D}(\mathcal{X}_\eta^{dg})_\Gamma$. So, Lemma 5.11 gives well-defined obstructions against $\widetilde{\mathscr{G}}\in\mathscr{D}_\infty(\mathcal{X}_\eta)_\Gamma\cong\mathscr{D}(\mathcal{X}_\eta^{dg})_\Gamma$ admitting a lift to an $\mathcal{A}_\infty$-module in $\mathscr{D}_\infty(\mathcal{X}_\eta\otimes\Gamma)$:

$$o_i(\widetilde{\mathscr{G}})\in\mathrm{HH}^i(\Gamma,\mathrm{Ext}^{2-i}_{\mathcal{X}_\eta}(\widetilde{\mathscr{G}},\widetilde{\mathscr{G}}))\quad\text{for } i>2.$$

Remark 5.18. The next lemma will use the obstruction obtained from the equivariant characteristic morphism (I) in order to conclude that the first $\mathcal{A}_\infty$-obstruction against an equivariant lift of $\widetilde{\mathscr{G}}$ cannot vanish. We do this by observing that a colift of $\mathscr{G}$ to $\mathcal{X}_\eta$ would also give an equivariant lift of $\widetilde{\mathscr{G}}$. The control of $o_3(\widetilde{\mathscr{G}})$ is necessary as we want to push forward the obstruction from $\mathcal{X}_\eta$ to $\mathcal{P}^{n+1}$ which cannot be done with the obstruction arising by the characteristic morphism.

Lemma 5.19. *We have*

$$0 \neq o_3(\widetilde{\mathscr{G}}) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathcal{X}_\eta}(\widetilde{\mathscr{G}}, \widetilde{\mathscr{G}})).$$

Proof. Assume that $o_3(\widetilde{\mathscr{G}})$ vanishes. Then, by Corollary 5.15, $\mathrm{Ext}^{-i}_{\mathcal{X}_\eta}(\widetilde{\mathscr{G}}, \widetilde{\mathscr{G}}) = 0$ for $i > 1$, and so

$$o_i(\widetilde{\mathscr{G}}) \in \mathrm{HH}^i(\Gamma, \mathrm{Ext}^{2-i}_{\mathcal{X}_\eta}(\widetilde{\mathscr{G}}, \widetilde{\mathscr{G}})) = 0$$

for all $i > 2$.

So, $\widetilde{\mathscr{G}}$ would admit a lift, i.e., an object

$$\widehat{\mathscr{G}} \in \mathcal{D}(\mathcal{X}^{\mathrm{dg}}_\eta \otimes \Gamma) \cong \mathcal{D}_\infty(\mathcal{X}_\eta \otimes \Gamma)$$

with $\widehat{\mathscr{G}} \cong \widetilde{\mathscr{G}}$ in $\mathcal{D}(\mathcal{X}_\eta)_\Gamma$.

Consider the triangle (4) in $\mathcal{D}_\infty(\mathcal{X}_\eta) \cong \mathcal{D}(\mathcal{X}^{dg}_\eta)$

$$\mathscr{G} \to \widetilde{\mathscr{G}} \cong \widehat{\mathscr{G}} \to \Sigma^{n+1}\mathscr{G} \otimes w\mathcal{O}_X(-p) \to \Sigma\mathscr{G},$$

where we use the shorthand $\mathscr{G}$ for $\pi_*\mathscr{G}$. This gives

$$\mathrm{H}^*(\widehat{G}) \cong \mathscr{G} \oplus \Sigma^{n+1}\mathscr{G} \otimes w\mathcal{O}_X(-p).$$

By the construction of the triangle (4) in [22, Section 10], the above isomorphism is compatible with the $\mathcal{X}_\eta$-action. So, by Definition 2.7, $\widehat{\mathscr{G}}$ is a colift of $\mathscr{G} \in \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma)$ to $\mathcal{D}_\infty((\mathcal{X}_\eta \otimes_{\Bbbk} \Gamma)_{\eta\cup 1})$.

By Proposition 2.8, the obstruction against such a colift is the image of $\eta \cup 1$ under the characteristic morphism

$$\mathrm{HH}^{n+3}(\mathcal{X} \otimes \Gamma, w\mathcal{O}_X(p) \otimes \Gamma) \to \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathscr{G}, \mathscr{G} \otimes w\mathcal{O}_X(p)).$$

However, this obstruction cannot vanish. As if we consider the equivariant characteristic morphism

$$\begin{aligned} c_{\mathscr{G},\Gamma} : \mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) &\xrightarrow{\mu\mapsto\mu\cup 1} \mathrm{HH}^{n+3}(\mathcal{X} \otimes \Gamma, w\mathcal{O}_X(p) \otimes \Gamma) \\ &\xrightarrow{c_{\mathscr{G}}} \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathscr{G}, \mathscr{G} \otimes w\mathcal{O}_X(p)), \end{aligned}$$

we have the commutative diagram from Lemma 5.17:

$$\begin{tikzcd} \mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}(G, G(p)) \arrow[d, "\wr"] \\ \mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) \arrow[r, "c_{\mathscr{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathscr{G}, \mathscr{G} \otimes w\mathcal{O}_X(p)). \end{tikzcd}$$

By assumption (I), we have that $c_{G,\Gamma}(\eta) \neq 0$. So $c_{\mathscr{G}}(\eta \cup 1) \neq 0$, which means that such a colift of $\widetilde{\mathscr{G}}$ to $(\mathcal{X} \otimes \Gamma)_{\eta\cup 1}$ cannot exist. Now, by the discussion above, this means that a lift of $\widetilde{\mathscr{G}}$ to $\mathcal{D}_\infty(\mathcal{X}_\eta \otimes \Gamma)$ cannot exist, and so, $o_3(\widetilde{\mathscr{G}})$ cannot be zero. ∎

Lemma 5.20. *There is a commutative diagram*

$$\begin{tikzcd} \mathrm{Ext}^n_X(G(-p),G) \arrow[r, "\varphi"] \arrow[d, "f_*"'] & \mathrm{Ext}^{-1}_{\mathcal{X}_\eta}(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}) \arrow[d, "\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}"] \\ \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(G(-p)), f_*(G)) \arrow[r, "\tilde{f}_*\varphi"'] & \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}(\tilde{f}_*(\widetilde{\mathcal{G}}), \tilde{f}_*(\widetilde{\mathcal{G}})), \end{tikzcd} \tag{5}$$

where the lower morphism is given by

$$\begin{aligned} \tilde{f}_*\varphi : \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*G(-p), f_*G) &\to \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}(\tilde{f}_*\widetilde{\mathcal{G}}, \tilde{f}_*\widetilde{\mathcal{G}}) \\ g &\mapsto \tilde{f}_*\alpha \circ w(g) \circ \tilde{f}_*\beta. \end{aligned}$$

Proof. Recall that by Definition 5.13 the morphism φ is given by

$$\begin{aligned} \varphi : \mathrm{Ext}^{n+1+i}_X(G(-p),G) &\to \mathrm{Ext}^{-1}_{\mathcal{X}^{dg}_\eta}(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}) \\ (g : \Sigma^{-n-1-i}G(-p) \to G) &\mapsto (\alpha \circ \pi_*(wg) \circ \beta : \Sigma^i\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}), \end{aligned}$$

where α and β are the first and second morphisms in the distinguished triangle (4) in $\mathcal{D}(\mathcal{X}^{dg}_\eta)$:

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\gamma} \Sigma\mathcal{G}.$$

Applying the exact functor $\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}$ gives the distinguished triangle in $\mathcal{D}(\mathcal{P}^{n+1})$:

$$\tilde{f}_*\mathcal{G} \xrightarrow{\tilde{f}_*\alpha} \tilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\tilde{f}_*\beta} \Sigma^{-n-1}\tilde{f}_*\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\tilde{f}_*\gamma} \Sigma\tilde{f}_*\mathcal{G},$$

which is a shorthand for

$$\tilde{f}_*\pi_*\mathcal{G} \xrightarrow{\tilde{f}_*\alpha} \tilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\tilde{f}_*\beta} \Sigma^{-n-1}\tilde{f}_*\pi_*\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\tilde{f}_*\gamma} \Sigma\tilde{f}_*\pi_*\mathcal{G}.$$

So, we may use $\pi \circ \tilde{f} = f$ to get

$$f_*\mathcal{G} \xrightarrow{\tilde{f}_*\alpha} \tilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\tilde{f}_*\beta} \Sigma^{-n-1}f_*\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\tilde{f}_*\gamma} \Sigma f_*\mathcal{G}. \tag{6}$$

In particular,

$$\begin{aligned} \tilde{f}_*\varphi : \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(w\mathcal{O}_X(p) \otimes \mathcal{G}), f_*(\mathcal{G})) &\to \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}(\tilde{f}_*(\widetilde{\mathcal{G}}), \tilde{f}_*(\widetilde{\mathcal{G}})) \\ g &\mapsto \tilde{f}_*\alpha \circ wg \circ \tilde{f}_*\beta \end{aligned}$$

is well defined.

Now, we compute

$$\begin{aligned}
\tilde{f}_* \circ \varphi(g) &= \widetilde{f_*}(\alpha \circ \pi_*(wg) \circ \beta) && \text{Definition of } \varphi \\
&= \tilde{f}_*\alpha \circ (\tilde{f}_* \circ \pi_*)(wg) \circ \tilde{f}_*\beta && \tilde{f}_* \text{ is a functor} \\
&= \tilde{f}_*\alpha \circ (f_* \circ w)(g) \circ \tilde{f}_*\beta && \tilde{f}_* \circ \pi_*(g) = f_*g \\
&= \tilde{f}_*\alpha \circ (w \circ f_*)(g) \circ \tilde{f}_*\beta && \text{[22, Lemma 8.7.1]} \\
&= \tilde{f}_*\varphi(f_*g), && \text{Definition of } \tilde{f}_*\varphi
\end{aligned}$$

and the diagram indeed commutes. ∎

Corollary 5.21. *The right map in the diagram* (5)

$$\tilde{f}_* \circ \psi_{\eta,*} : \mathrm{Ext}^{-1}_{\mathcal{X}^{dg}_\eta}(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}) \xrightarrow{\sim} \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}(\tilde{f}_*(\widetilde{\mathcal{G}}), \tilde{f}_*(\widetilde{\mathcal{G}}))$$

is an isomorphism.

Proof. Since G, $G(-p)$ are coherent sheaves on X and f_* is exact, we have

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{P}^{n+1}}(f_*(\Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p)), f_*(\mathcal{G})) \\
&\quad \cong \mathrm{Ext}^1_{\mathbb{P}^{n+1}}(f_*(\Sigma^{-n-1}G \otimes \mathcal{O}_X(-p)), f_*(G)) && \text{Lemma 5.2} \\
&\quad \cong \mathrm{Ext}^{n+2}_{\mathbb{P}^{n+1}}(f_*(G(-p)), f_*(G)) && \mathrm{Ext}^i(\Sigma^{-j}_,_) \cong \mathrm{Ext}^{i+j}(_,_) \\
&\quad = 0. && \dim \mathbb{P}^{n+1} = n+1
\end{aligned}$$

So, in the distinguished triangle (6)

$$f_*\mathcal{G} \xrightarrow{\tilde{f}_*\alpha} \tilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\tilde{f}_*\beta} \Sigma^{-n-1} f_*\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\tilde{f}_*\gamma} \Sigma f_*\mathcal{G},$$

$\tilde{f}_*\gamma$ vanishes, and we have

$$\tilde{f}_*(\widetilde{\mathcal{G}}) \cong f_*(\mathcal{G}) \oplus f_*(\Sigma^{-n-1} w\mathcal{O}_X(-p) \otimes_X \mathcal{G})$$

via the splitting morphisms $\tilde{f}_*\alpha$ and $\tilde{f}_*\beta$.

This means that both the top morphism, by Lemma 5.14, and the lower morphism, by splitting, in (5) are isomorphisms. So, by Lemma 5.20, it suffices to prove that

$$f_* : \mathrm{Ext}^n_X(\mathcal{O}_X(-p) \otimes G, G) \to \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(\mathcal{O}_X(-p) \otimes G), f_*(G))$$

is an isomorphism.

As tensoring with $\mathcal{O}_X(p)$ is an autoequivalence, this is equivalent to

$$f_* : \mathrm{Ext}^n_X(G, G(p)) \to \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*G, f_*G(p))$$

being an isomorphism. Consider the long exact sequence associated to a divisor [22, (9.13)]:

$$\cdots \to \mathrm{Ext}^{n-2}_X(G, G(p+d)) \to \mathrm{Ext}^n_X(G, G(p)) \xrightarrow{f_*} \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(G), f_*G(p)) \to \cdots.$$

By assumption (III), we have

$$\mathrm{Ext}_X^{n-2}(G, G(p+d)) \cong 0 \quad \text{and} \quad \mathrm{Ext}_X^{n-1}(G, G(p+d)) \cong 0,$$

so the long exact sequence has the shape

$$\cdots \to 0 \to \mathrm{Ext}_X^n(G, G(p)) \xrightarrow{f_*} \mathrm{Ext}_{\mathbb{P}^{n+1}}^n(f_*(G), f_*G(p)) \to 0 \to \cdots .$$

By exactness, that immediately gives that

$$f_* : \mathrm{Ext}_X^n(G(-p), G) \xrightarrow{\sim} \mathrm{Ext}_{\mathbb{P}^{n+1}}^n(f_*G(-p), f_*G)$$

is an isomorphism, which finishes the proof. ■

Lemma 5.22. *The obstruction $o_3(\Psi_\eta(G)) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G)))$ against lifting to $\mathcal{D}(\mathcal{P}^{n+1} \otimes \Gamma)$ from Lemma 5.11 does not vanish.*

Proof. By part (2) of Lemma 5.11, we have

$$o_3(\Psi(G)) = (\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}) o_3(\widetilde{\mathcal{G}}) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G))).$$

Furthermore, as o_3 is the first obstruction, we do not need to keep track of any choices. So, we can use Corollary 5.21 to get that $\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}$ induces an isomorphism in degree -1, and by Lemma 5.19, we have $0 \neq o_3(\widetilde{\mathcal{G}})$. So altogether,

$$0 \neq (\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}) o_3(\widetilde{\mathcal{G}}) = o_3(\Psi(G)) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G))).$$ ■

Now, we can finally finish the proof of Theorem 5.9.

Proof. Assume that Ψ_η is Fourier–Mukai. Then, by Corollary 3.7, Ψ_η admits a lift

$$\Psi_{\eta,\Gamma} : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^{n+1})_\Gamma).$$

This means that $\Psi_\eta(G) \in \mathcal{D}^b(\mathbb{P}^{n+1})_\Gamma$ has a lift to $\mathcal{D}^b(\mathrm{coh}(\mathbb{P}^{n+1})_\Gamma) \hookrightarrow \mathcal{D}_\infty(\mathcal{P}^{n+1} \otimes_{\Bbbk} \Gamma)$. Since we have by Lemma 5.22 that

$$o_3(\Psi_\eta(G)) \neq 0,$$

such a lift cannot exist.

So, Ψ_η cannot be Fourier-Mukai. ■

5.3. Application: Odd-dimensional quadrics

We will show that the tilting bundle G for an odd-dimensional quadric hypersurface and its endomorphism algebra Γ satisfy the assumptions of Theorem 5.9. For this, we start by recalling that quadrics admit an exceptional sequence, which gives rise to a tilting bundle.

Theorem 5.23 ([2, Corollary 3.2.8]). *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth quadric. Then, Q admits an exceptional sequence:*

$$(S(-2k+1), \mathcal{O}_Q(-2k+2), \dots, \mathcal{O}_Q(-1), \mathcal{O}_Q),$$

where S denotes the spinor bundle.

In particular, we may consider for the embedding of a smooth quadric

$$f : Q \hookrightarrow \mathbb{P}^{2k}$$

the tilting bundle:

$$G := S(-2k+1) \oplus \bigoplus_{l=0}^{-2k+2} \mathcal{O}_Q(-l) \quad \text{and} \quad \Gamma := \mathrm{End}(G).$$

Now, we need to verify the assumptions on the concentration of $\mathrm{Ext}^*_Q(G(-p), G)$ and $\mathrm{Ext}^*_Q(G, G(p+d))$. We will use $p = -2k-2$ and

$$0 \neq \eta \in \ker(f_* : \mathrm{HH}^{n+3}(X, \mathcal{O}_Q(-2k-2)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{2k}, f_*\mathcal{O}_Q(-2k-2)))$$

as we know by Proposition 4.16 that

$$f_* : \mathrm{HH}^{n+3}(Q, \mathcal{O}_Q(-2k-2)) \to \mathrm{HH}^{n+3}(\mathbb{P}^{n+1}, f_*\mathcal{O}_Q(-2k-2))$$

has one-dimensional kernel.

For the Ext-calculations, we will need the following statement which also holds for even quadrics. However, as in the even case we would need to track the different spinor bundles depending on the equivalence class of the dimension modulo four, we will restrict to the odd case for legibility.

Lemma 5.24. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be a smooth odd-dimensional quadric, and let S be the spinor bundle. Then, the following hold:*

(1) *We have for $i \notin \{0, 1, n\}$*

$$\mathrm{Ext}^i_Q(S, S(m)) \cong \mathrm{Ext}^{i-1}_Q(S, S(m+1)).$$

(2) *If $m \leq -1$, we have additionally*

$$\mathrm{Ext}^i_Q(S, S(m)) \cong \mathrm{Ext}^{i-1}_Q(S, S(m+1)),$$
$$\mathrm{Ext}^i_Q(S, S(m)) \cong \mathrm{Ext}^{i-1}_Q(S, S(m+1)).$$

Proof. Consider the short exact sequence [19, Theorem 2.8]

$$0 \mapsto S \mapsto \mathcal{O}_Q^{2^{k+1}} \to S(1) \to 0$$

which gives after applying $\mathrm{Ext}^i_Q(\_, S(m+1))$ the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Ext}^{i-1}_Q(\mathcal{O}_Q^{2^{k+1}}, S(m+1)) \arrow[r] & \mathrm{Ext}^{i-1}_Q(S, S(m+1)) \arrow[dll, out=-10, in=170] \\ \mathrm{Ext}^i_Q(S(1), S(m+1)) \arrow[r] & \mathrm{Ext}^i_Q(\mathcal{O}_Q^{2^{k+1}}, S(m+1)) \arrow[r] & \cdots. \end{tikzcd}$$

In particular, we have

$$\mathrm{Ext}^i_Q(S, S(m)) \cong \mathrm{Ext}^i_Q(S(1), S(m+1)) \cong \mathrm{Ext}^{i+1}_Q(S, S(m-1))$$

if we have for $j \in \{i, i-1\}$

$$\mathrm{Ext}^j(\mathcal{O}_Q^{2^{k+1}}, S(m+1)) \cong \bigoplus_{l=0}^{2^{k+1}} \mathrm{Ext}^j(\mathcal{O}_Q, S(m+1)) \cong \bigoplus_{l=0}^{2^{k+1}} \mathrm{H}^j(X, S(m+1)) = 0.$$

By [19, Theorem 2.3], we have $\mathrm{H}^j(X, S(m+1)) = 0$ for $j \notin \{0, n\}$ which implies 1.

If $m \leq -1$, we have $m+1 \leq 0$, and so, we get by [19, Theorem 2.3]

$$H^0(X, S(m+1)) = 0,$$

which gives 2. ∎

Proposition 5.25. *Let $i \neq 2k-1$. Then, we have*

$$\mathrm{Ext}^*_Q(G(2k+2), G) = 0.$$

Proof. Since G is a sheaf, we may assume that $0 \leq i \leq 2k-2$ for dimension reasons. By definition of G and additivity of Ext, we have

$$\begin{aligned} \mathrm{Ext}^i_Q(G(2k+2), G) &= \mathrm{Ext}^i_Q((S(-2k+1) \\ &\quad \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q(-l)(2k+2)), S(-2k+1) \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q(-l)) \\ &\cong \bigoplus_{h,l=0}^{2k-2} \mathrm{Ext}^i_Q(\mathcal{O}_Q(2k+2-l), \mathcal{O}_Q(-h)) \\ &\quad \oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q(\mathcal{O}_Q(2k+2-l), S(-2k+1)) \\ &\quad \oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q(S(2k+2-2k+1), \mathcal{O}_Q(-l)) \\ &\quad \oplus \mathrm{Ext}^i_Q(S(-2k+1+2k+2), S(-2k+1)). \end{aligned}$$

In particular, we can compute these Ext-groups one by one.

We start with $\mathrm{Ext}^i_Q(\mathcal{O}_Q(2k+2-l), \mathcal{O}_Q(-h))$ for which we get

$$\begin{aligned}
&\mathrm{Ext}^i_Q(\mathcal{O}_Q(2k+2-l), \mathcal{O}_Q(-h)) \\
&\quad \cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, \mathcal{O}_Q(l-h-2k-2)) && \text{twisting on both sides} \\
&\quad \cong \mathrm{H}^i(Q, \mathcal{O}_Q(l-h-2k-2)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q, _) \cong \mathrm{H}^i(Q, _) \\
&\quad \cong 0. && l-h-2k-2 < 0
\end{aligned}$$

Since we have $l-4k-1 \leq 0$, we get

$$\begin{aligned}
&\mathrm{Ext}^i_Q(\mathcal{O}_Q(2k+2-l), S(-2k+1)) \\
&\quad \cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, S(l-4k-1)) && \text{twisting on both sides} \\
&\quad \cong \mathrm{H}^i(Q, S(l-4k-1)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q, _) \cong \mathrm{H}^i(Q, _) \\
&\quad \cong 0. && \text{[19, Theorem 2.3]}
\end{aligned}$$

By [19, Theorem 2.8], we have $S^\vee \cong S(1)$, which we may use to compute

$$\begin{aligned}
\mathrm{Ext}^i_Q(S(3), \mathcal{O}_Q(-l)) &\cong \mathrm{Ext}^i_Q(S, \mathcal{O}_Q(-3-l)) && \text{twisting on both sides} \\
&\cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, S^\vee(-3-l)) && \text{dualizing} \\
&\cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, S(-2-l)) && S^\vee \cong S(1) \\
&\cong \mathrm{H}^i(Q, S(-2-l)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q, _) \cong \mathrm{H}^i(Q, _) \\
&\cong 0. && \text{[19, Theorem 2.3]}
\end{aligned}$$

We may use $i \leq 2k-1$ to get

$$\begin{aligned}
\mathrm{Ext}^i_Q(S(3), S(-2k+1)) &\cong \mathrm{Ext}^i_Q(S, S(-2k-2)) && \text{twisting on both sides} \\
&\cong \mathrm{Ext}^1_Q(S, S(-2k-3+i)) && \text{Lemma 5.24} \\
&\cong \mathrm{Ext}^{-1}_Q(S, S(-2k-1+i)) && \text{Lemma 5.24} \\
&\cong 0. && S \text{ is a sheaf}
\end{aligned}$$

So, every direct summand vanishes, and in particular,

$$\mathrm{Ext}^i_Q(G(2k+2), G) = 0 \text{ for } i \neq n$$

as desired. ∎

Proposition 5.26. *Let $i \notin \{0, 2k-1\}$. Then, we have*

$$\mathrm{Ext}^i_Q(G, G(-2k)) \cong 0.$$

Proof. Since Q has dimension $2k-1$ and G is a sheaf, we may assume that $0 < i < 2k-1$.

By definition of G and additivity of $\mathrm{Ext}^i_Q(\_,\_)$, we have

$$\begin{aligned}\mathrm{Ext}^i_Q(G, G(-2k)) &= \mathrm{Ext}^i_Q\Big(S(-2k+1) \\ &\qquad \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q(-l), S(-4k+1) \oplus \bigoplus_{h=0}^{2k-2} \mathcal{O}_Q(-2k-h)\Big) \\ &\cong \bigoplus_{l,h=0}^{2k-2} \mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), \mathcal{O}_Q(\mathcal{O}_Q(-2k-h))) \\ &\qquad \oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), S(-4k+1)) \\ &\qquad \oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q(S(-2k+1), \mathcal{O}_Q(-2k-l)) \\ &\qquad \oplus \mathrm{Ext}^i_Q(S(-2k+1), S(-4k+1)).\end{aligned}$$

As above, we can compute the cases separately.

We start with $\mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), \mathcal{O}_Q(-2k-h))$. For this, we get

$$\begin{aligned}&\mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), \mathcal{O}_Q(-2k-h)) \\ &\quad \cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, \mathcal{O}_Q(l-2k-h)) && \text{twisting on both sides} \\ &\quad \cong \mathrm{H}^i(Q, \mathcal{O}_Q(1-2k-h)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q,\_) \cong \mathrm{H}^i(Q,\_) \\ &\quad \cong 0. && i \notin \{0, 2k-1\}\end{aligned}$$

For $\mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), S(-4k+1))$, we get

$$\begin{aligned}&\mathrm{Ext}^i_Q(\mathcal{O}_Q(-l), S(-4k+1)) \\ &\quad \cong \mathrm{Ext}^i(\mathcal{O}_Q, S(l-4k+1)) && \text{twisting on both sides} \\ &\quad \cong \mathrm{H}^i(Q, S(l-4k+1)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q,\_) \cong \mathrm{H}^i(Q,\_) \\ &\quad \cong 0. && i \notin \{0, 2k-1\} \text{ [19, Theorem 2.3]}\end{aligned}$$

While for $\mathrm{Ext}^i_Q(S(-2k+1), \mathcal{O}_Q(-2k-l))$, one can compute:

$$\begin{aligned}&\mathrm{Ext}^i_Q(S(-2k+1), \mathcal{O}_Q(-2k-l)) \\ &\quad \cong \mathrm{Ext}^i_Q(S, \mathcal{O}_Q(-1-l)) && \text{twisting on both sides} \\ &\quad \cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, S^\vee(-1-l)) && \text{dualizing} \\ &\quad \cong \mathrm{Ext}^i_Q(\mathcal{O}_Q, S(-l)) && \text{[19, Theorem 2.8]} \\ &\quad \cong \mathrm{H}^i(Q, S(-l)) && \mathrm{Ext}^i_Q(\mathcal{O}_Q,\_) \cong \mathrm{H}^i(Q,\_) \\ &\quad = 0. && i \notin \{0, 2k-1\} \text{ [19, Theorem 2.3]}\end{aligned}$$

Finally, for $\operatorname{Ext}^i_Q(S(-2k+1), S(-4k+1))$, we get

$$\begin{aligned}
\operatorname{Ext}^i_Q(S(-2k+1), S(-4k+1)) &\cong \operatorname{Ext}^i_Q(S, S(-2k)) && \text{twisting on both sides}\\
&\cong \operatorname{Ext}^1_Q(S, S(-2k+i-1)) && \text{Lemma 5.24 (1)}\\
&\cong \operatorname{Ext}^0_Q(S, S(-2k-i)) && \text{Lemma 5.24 (2)}\\
&\cong \operatorname{Ext}^{-1}_Q(S, S(-2k+1-i)) && \text{Lemma 5.24 (2)}\\
&= 0, && S \text{ is a sheaf}
\end{aligned}$$

where we used $i < 2k-1$ and so $-2k+i \leq -1$, respectively, $-2k+1+i \leq -1$ for the last two lines.

So, all the direct summands of $\operatorname{Ext}^i_Q(G, G(-2k))$ vanish for $i \notin \{0, 2k-1\}$ as claimed. ■

So altogether, we can now phrase the following Theorem 5.27 which also recovers the result from [22] when specialized to the case $k = 2$.

Theorem 5.27. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth odd-dimensional quadric for $k \geq 2$. Then, we have an exact functor:*

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b(\mathbb{P}^n)$$

that cannot be Fourier–Mukai.

Proof. We want to apply Theorem 5.9.

First of all, we have by Proposition 4.16 for $k > 2$ an

$$0 \neq \eta \in \mathrm{HH}^{2k+2}(Q, \mathcal{O}_Q(-2k-2))$$

that is in the kernel of $f_* : \mathrm{HH}^{n+3}(Q, \mathcal{O}(-2k-2)) \to \mathrm{HH}^{n+3(\mathbb{P}^{2k}, f_*\mathcal{O}(-2k-2))}$.

For $k = 2$, we get that the top Hochschild cohomology is $\mathrm{HH}^{n+3}(Q, \mathcal{O}(-2k-2))$, and so by Theorem 4.15, we have

$$\dim \ker(f_* : \mathrm{HH}^{n+3}(Q, \mathcal{O}(-6)) \to \mathrm{HH}^{n+3}(\mathbb{P}^4, f_*\mathcal{O}(-6))) = \mathrm{h}^{2,1}_1(Q).$$

Using formula (1), we compute

$$\begin{aligned}
\mathrm{h}^{2,1}_1(Q) &= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{-1+4-(\mu-1)(2-1)}{4}\\
&= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{-1+4-(\mu-1)}{4}\\
&= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{4-\mu}{4}\\
&= (-1)^0 \binom{6}{0} \binom{4}{4} = 1,
\end{aligned}$$

where we used that $\binom{4-\mu}{4}$ only can be non-zero if $\mu = 0$. In particular, we get a one-dimensional kernel from which we may pick an $\eta \neq 0$.

We now collect the other assumptions which we verified above.

By Theorem 5.23, Q admits a tilting bundle G, and by Lemma 3.19, we know that for $\Gamma := \mathrm{End}(G)$ the functor $C^{Q}_{G,\Gamma}$ is an equivalence. In particular, we get by Proposition 3.12 $c_{G,\Gamma}(\eta) \neq 0$, which is assumption (I). Now, finally, we need to verify that the corresponding Ext-groups are suitably concentrated, which is verified in Proposition 5.25 for assumption (II) and Proposition 5.26 for assumption (III).

So, we may apply Theorem 5.9 to get a non-Fourier–Mukai functor Ψ_η. ∎

A. Modules over $\Bbbk$-linear categories and $\mathcal{A}_\infty$-modules over $\mathcal{A}_\infty$-categories

In this Appendix, we will recall a few basic facts about modules over $\Bbbk$-linear categories and $\mathcal{A}_\infty$-modules over $\mathcal{A}_\infty$-categories, for a field $\Bbbk$.

A.1. Modules over $\Bbbk$-linear categories

We start by recalling the definition of modules over a $\Bbbk$-linear category and the relationship between those and the classical notion of modules.

The idea of generalizing the notion of modules over rings to categories first was introduced by B. Mitchell [17]. All in all, the idea is that one can interpret a $\Bbbk$-algebra as a $\Bbbk$-linear category with one object, and under that interpretation, a module corresponds to a functor from the $\Bbbk$-linear category to the category of $\Bbbk$-vector-spaces.

Remark A.1. Recall that a $\Bbbk$-linear category $\mathcal{C}$ is a category such that every morphism space $\mathcal{C}(M, N)$ is a $\Bbbk$-vector space and composition defines a $\Bbbk$-linear map

$$\_ \circ \_ : \mathcal{C}(M', M) \otimes \mathcal{C}(M'', M') \to \mathcal{C}(M'', M).$$

Definition A.2 ([17]). Let $\mathcal{X}$ be a small $\Bbbk$-linear category. An $\mathcal{X}$-module is a $\Bbbk$-linear functor

$$\mathcal{M} : \mathcal{X} \to \mathrm{Vect}(\Bbbk).$$

A morphism of $\mathcal{X}$-modules is a natural transformation between two $\mathcal{X}$-modules $\mathcal{N}$ and $\mathcal{M}$:

$$f : \mathcal{N} \to \mathcal{M}.$$

We refer to the category of $\mathcal{X}$-modules by $\mathcal{X}$-mod.

Lemma A.3. *Let $\mathcal{X}$ be a $\Bbbk$-linear category. Then, we have that the category $\mathcal{X}$-mod is a $\Bbbk$-linear abelian category.*

Proof. By Definition A.2, we have $\mathcal{X}\text{-mod} = \mathrm{Fun}_{\Bbbk}(\mathcal{X}, \mathrm{Vect}(\Bbbk))$. In particular, we have immediately a canonical $\Bbbk$-action on the morphism spaces. As kernels and cokernels can be computed objectwise in the target category [27, Exercise A.4.33], we have that $\mathrm{Fun}_{\Bbbk}(\mathcal{X}, \mathrm{Vect}(\Bbbk))$ is also abelian. In particular, we get that $\mathcal{X}$-mod is abelian $\Bbbk$-linear. ■

Remark A.4. Let Γ be a $\Bbbk$-algebra. Then, we have that a classically defined Γ-module M consists of a $\Bbbk$-vector space V together with a $\Bbbk$-algebra morphism $\gamma : \Gamma \to \mathrm{End}(V)$.

On the other hand, if we consider Γ to be a $\Bbbk$-linear category with one object $*$, then $\mathcal{M}$ consists by Definition A.2 also of a vector space $V = \mathcal{M}(*)$ together with a morphism of $\Bbbk$-algebras (a map of morphism spaces):

$$\Gamma \to \mathrm{End}(V) = \mathrm{Vect}(\Bbbk)(\mathcal{M}(*), \mathcal{M}(*)).$$

In particular, in this case, the two notions of modules over Γ coincide.

Similarly, the notion of natural transformation captures in this case precisely the commuting with the Γ action.

Definition A.5 ([17]). Let $\mathcal{X}$ be a $\Bbbk$-linear category. We define the derived category of $\mathcal{X}$-modules (respectively, bounded, bounded below, or bounded above) derived category to be the derived category (respectively, bounded, bounded below, or bounded above derived category) of the abelian category $\mathcal{X}$-mod:

$$\mathcal{D}^{\natural}(\mathcal{X}) := \mathcal{D}^{\natural}(\mathcal{X}\text{-mod})$$

for $\natural \in \{\_, b, -, +\}$.

As we will later define a $\Bbbk$-linear category corresponding to a scheme and then model morphisms of schemes also as functors between $\Bbbk$-linear categories, we will denote the restriction of scalar functors in the following way.

Definition A.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\Bbbk$-linear functor, and let $\mathcal{M}$ be a $\mathcal{Y}$-module. Then, we define the module $f_*\mathcal{M}$ to be the $\mathcal{X}$-module defined by

$$f_*\mathcal{M} := \mathcal{M} \circ f.$$

Remark A.7. We choose the notation f_* over f^* as we will later model the category of sheaves on a projective scheme by modules over a $\Bbbk$-linear category, and under this construction, the functor f_* corresponds to the direct image, and so, the notation turns out to be more consistent and less confusing throughout this work.

Lemma A.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\Bbbk$-linear functor. Then, the assignment*

$$\mathcal{M} \mapsto f_*\mathcal{M}$$

defines a left exact functor $f_* : \mathcal{Y}$-mod $\mapsto \mathcal{X}$-mod.

Proof. As kernels and images are computed on the target category, we do not need to worry about left exactness. It also defines a functor as it is just a precomposition with a functor, and so, it has to be functorial. ■

A.2. $\mathcal{A}_\infty$-Structures

Throughout this section, we follow [12,23]; in particular, we will use the sign conventions from [12]. Although B. Keller only talks about $\mathcal{A}_\infty$-algebras, the sign conventions can also be applied to $\mathcal{A}_\infty$-categories and are equivalent to the sign conventions in the book by P. Seidel which is considering $\mathcal{A}_\infty$-categories throughout. Furthermore, K. Lefèvre–Hasegawa [13] covers the case of $\mathcal{A}_\infty$-categories using the same signs as Keller; however we primarily refer to [23] for the category case, as [13] is in French.

A.3. $\mathcal{A}_\infty$-Categories and their functors

Since we will repeatedly use dg-categories as examples for $\mathcal{A}_\infty$-categories, we recall the definition of a dg-category

Definition A.9. A dg category $\mathcal{C}$ is a category such that we have for all $M, N \in \mathcal{C}$ a chain complex $\mathcal{C}^*(M, N)$ such that the Leibnitz rule holds:

$$d(x \circ y) = dx \circ y + x \circ dy.$$

Definition A.10 ([12, Section 3.1]). Let $n \in \mathbb{N} \cup \{\infty\}$. An $\mathcal{A}_n$-category $\mathcal{X}$ over a field $\Bbbk$ consists of a class of objects $\operatorname{obj}(\mathcal{X})$ and $\mathbb{Z}$-graded $\Bbbk$-vector-spaces as morphism spaces

$$\mathcal{X}(a, b),$$

for $a, b \in \operatorname{obj}(\mathcal{X})$, together with compositions

$$\mathrm{m}_i : \underbrace{\mathcal{X}(a_i, a_{i-1}) \otimes_\Bbbk \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \cdots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{X}(a_i, a_0)$$

of degree $2 - i$ for $1 \le i \le n$ and $a_0, \ldots, a_i \in \operatorname{obj}(\mathcal{X})$ such that

$$\sum_{r+s+t=k} (-1)^{r+st} \mathrm{m}_u \circ (\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}) = 0 \qquad (*_k)$$

holds for all $k \le n$, where $u = r + 1 + t$.

We will sometimes denote $a \in \operatorname{obj}(\mathcal{X})$ by $a \in \mathcal{X}$ to avoid clumsy notation.

Definition A.11 ([23, (2a)]). An $\mathcal{A}_n$-category $\mathcal{X}$ is called unital if every object $a \in \operatorname{obj}(\mathcal{X})$ admits a unit $\mathrm{Id} \in \mathcal{X}(a, a)^0$ such that

$$\begin{aligned}
&\mathrm{m}_1(\mathrm{Id}) = 0\\
&\mathrm{m}_2(x, \mathrm{Id}) = x = \mathrm{m}_2(\mathrm{Id}, x)\\
&\mathrm{m}_i(x_i, \ldots, \mathrm{Id}, \ldots, x_1) = 0 \quad i \neq 2.
\end{aligned}$$

Remark A.12. Observe that the first few incarnations of $(*_k)$ give the following.

$k = 1$. In this case, $(*_1)$ gives

$$\mathrm{m}_1 \circ \mathrm{m}_1 = 0.$$

This means that m_1 defines a differential on $\mathcal{X}(a, b)$.

$k = 2$. Here, $(*_2)$ boils down to

$$\mathrm{m}_1 \circ \mathrm{m}_2 = \mathrm{m}_2(\mathrm{m}_1 \circ \mathrm{Id} + \mathrm{Id} \circ \mathrm{m}_1),$$

which is the Leibnitz rule $d(x \circ y) = dx \circ y + x \circ dy$.

$k = 3$. And $(*_3)$ gives

$$\begin{aligned} &\mathrm{m}_2 \circ (\mathrm{Id} \otimes \mathrm{m}_2 - \mathrm{m}_2 \otimes \mathrm{Id}) \\ &\quad = \mathrm{m}_1 \circ \mathrm{m}_3 + \mathrm{m}_3 \otimes (\mathrm{m}_1 \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{m}_1), \end{aligned}$$

which means that m_2 is associative up to a homotopy given by m_3. More generally, one can think of an $\mathcal{A}_n$-category as a category that is homotopy-associative up to degree n.

By Definition A.10, every $\mathcal{A}_n$ category defines an $\mathcal{A}_m$-category for all $m \leq n$ just by forgetting the higher actions.

Definition A.13 ([23, (1a)]). Let $\mathcal{X}$ be an $\mathcal{A}_n$-category for $n \geq 3$. Then, the category $\mathrm{H}^*(\mathcal{X})$ is the graded $\Bbbk$-linear category consisting of the same objects as $\mathcal{X}$ and morphism spaces

$$\mathrm{H}^*(\mathcal{X})(a, b) := \mathrm{H}^*(\mathcal{X}(a, b)),$$

where we use Remark A.12 to consider $\mathcal{X}(a, b)$ as a chain complex with differential m_1.

The $\Bbbk$-linear category $\mathrm{H}^0(\mathcal{X})$ is the category with the same objects as $\mathcal{X}$ and morphism spaces

$$\mathrm{H}^0(\mathcal{X})(a, b) := \mathrm{H}^0(\mathcal{X}(a, b)).$$

We have by Remark A.12 that $\mathrm{H}^*(\mathcal{X})$ defines a graded $\Bbbk$-linear category and $\mathrm{H}^0(\mathcal{X})$ defines an ordinary $\Bbbk$-linear category.

Definition A.14 ([23, (2a)]). An $\mathcal{A}_n$-category is called homologically unital if $\mathrm{H}^0(\mathcal{X})$ admits a unit morphism $\mathrm{Id} \in \mathrm{H}^0(\mathcal{X})(a, a)$ for all $a \in \mathrm{obj}(\mathcal{X})$.

Definition A.15. An $\mathcal{A}_n$-category is called small if its objects form a set. It is called essentially small if the isomorphism classes of objects form a set.

Definition A.16 ([12, Section 3.1]). An $\mathcal{A}_n$-category is an $\mathcal{A}_n$-algebra if $\mathrm{obj}(\mathcal{X})$ consists of only one object for $n \in \mathbb{N} \cup \{\infty\}$.

Example A.17. There are a few obvious examples of $\mathcal{A}_\infty$-categories:

- Let $\mathcal{X}$ be a $\Bbbk$-linear category; then, it is an $\mathcal{A}_\infty$ category via

$$\mathrm{m}_i = \begin{cases} (\_) \circ (\_) & i = 2, \\ 0 & i \neq 2. \end{cases}$$

- More generally, let $\mathcal{X}$ be a dg-category; then, $\mathcal{X}$ is an $\mathcal{A}_\infty$-category with

$$\mathrm{m}_i = \begin{cases} d & i = 1, \\ (\_) \circ (\_) & i = 2, \\ 0 & i \notin \{1, 2\}. \end{cases}$$

Definition A.18 ([12, Section 3.4]). An $\mathcal{A}_n$-functor between two $\mathcal{A}_n$-categories $f : \mathcal{X} \to \mathcal{Y}$ is given by a map on objects

$$f : \mathrm{obj}(\mathcal{X}) \to \mathrm{obj}(\mathcal{Y})$$

and a set of morphisms

$$\{f_i : \mathcal{X}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \cdots \otimes \mathcal{X}(a_1, a_2) \to \mathcal{Y}(f(a_i), f(a_0))\}$$

of degree $1 - i$ for every $i \leq n$ and $a_i, \ldots, a_0 \in \mathrm{obj}(\mathcal{X})$ such that

$$\begin{aligned} &\sum_{r+s+t=k} (-1)^{r+st} f_u(\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}) \\ &\quad = \sum_{\substack{1 \leq l \leq n \\ k=i_1+\cdots+i_l}} (-1)^m \mathrm{m}_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) \end{aligned} \tag{$**_k$}$$

holds, where $u = r + 1 + t$ and

$$m = (l-1)(i_1 - 1) + (l-2)(i_2 - 1) + \cdots + 2(i_{l-2} - 1) + (i_{l-1} - 1).$$

Remark A.19. Again, we compute the first few incarnations of ($**_k$) as follows.

$k = 1$. In this case, we have

$$f_1 \circ \mathrm{m}_1 = m_1 \circ f_1;$$

in particular, f_1 defines a morphism of chain complexes.

$k = 2$. Here, we get

$$f_1 \circ \mathrm{m}_2 = \mathrm{m}_2 \circ (f_1 \otimes f_1) + \mathrm{m}_1 \circ f_2 + f_2(\mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1),$$

so f_1 commutes with m_2 up to a homotopy given by f_2.

More generally, one can think of an $\mathcal{A}_n$-functor as a functor which commutes with the $\mathcal{A}_n$-structures of the source and target up to homotopies. These higher homotopies are encoded in the higher f_i, for $i > 1$. In particular, on H^* only the f_1 remains.

Definition A.20 ([12, Section 3.1]). An $\mathcal{A}_n$-functor between two unital $\mathcal{A}_n$-algebras is called an $\mathcal{A}_n$-morphism for $n \in \mathbb{N} \cup \{\infty\}$.

Definition A.21 ([12, Section 3.1]). An $\mathcal{A}_\infty$-functor $f : \mathcal{A} \to \mathbb{B}$ is a quasi-equivalence if

$$f : \mathrm{obj}(\mathcal{A})/_{\cong} \to \mathrm{obj}(\mathbb{B})/_{\cong}$$

is surjective and all f_1 induce isomorphisms on cohomology

$$\mathrm{H}^*(f_1) : \mathrm{H}^*(\mathcal{A}(a, a')) \xrightarrow{\sim} \mathrm{H}^*(\mathbb{B}(fa, fa')).$$

Proposition A.22 ([13, Proposition 3.2.1]). *Every homologically unital $\mathcal{A}_\infty$-category is quasi-equivalent to a unital one.*

Definition A.23 ([12, Section 3.4]). A quasi-equivalence between two $\mathcal{A}_\infty$-algebras is called a quasi-isomorphism.

Theorem A.24 ([11]). *Let $\mathcal{X}$ be an $\mathcal{A}_\infty$-category. Then, the cohomology $\mathrm{H}^*(\mathcal{X})$ has an $\mathcal{A}_\infty$-category structure such that*

- $\mathrm{m}_1 = 0$,
- *there is a quasi-equivalence $\mathrm{H}^*\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ lifting the identity on $\mathrm{H}^*\mathcal{X}$.*

Moreover, this structure is unique up to (non-unique) isomorphism of $\mathcal{A}_\infty$-categories.

Remark A.25. From now on, we will assume that the cohomology $\mathrm{H}^*(\mathcal{X})$ of an $\mathcal{A}_\infty$-category is equipped with the $\mathcal{A}_\infty$-structure arising by Theorem A.24 instead of just regarding it as a graded category interpreted as an $\mathcal{A}_\infty$-category. The $\mathcal{A}_\infty$-category constructed in Theorem A.24 is also referred to as the minimal $\mathcal{A}_\infty$-model of $\mathcal{X}$.

A.4. $\mathcal{A}_\infty$-Modules and their functors

Definition A.26 ([12, Section 4.2]). Let $\mathcal{X}$ be a small $\mathcal{A}_n$-category for $n \in \mathbb{N} \cup \{\infty\}$. An $\mathcal{A}_n$-module over $\mathcal{X}$ consists of a $\mathbb{Z}$-graded space

$$\mathcal{M}(a, b)$$

for every pair of objects $a, b \in \operatorname{obj} \mathcal{X}$ and higher composition morphisms

$$\mathrm{m}_i : \underbrace{\mathcal{M}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \cdots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{M}(a_i, a_0)$$

of degree $2 - i$ such that the following equation holds:

$$\sum_{r+s+t=k} (-1)^{r+st} \mathrm{m}_u \circ (\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}) = 0, \qquad (**_k)$$

where depending on the input m_i needs to be considered as the ith higher composition morphism of $\mathcal{X}$ or $\mathcal{M}$.

Remark A.27. We again compute a few incarnations of $(**_k)$ to give some intuition on the modeled structure.

$k = 1$. In this case, we get

$$\mathrm{m}_1^{\mathcal{M}} \circ \mathrm{m}_1^{\mathcal{M}} = 0.$$

So, m_1 defines a differential.

$k = 2$. Here, we get

$$\mathrm{m}_1^{\mathcal{M}} \circ \mathrm{m}_2^{\mathcal{M}} = \mathrm{m}_2^{\mathcal{M}} \circ (\mathrm{m}_1^{\mathcal{M}} \otimes \mathrm{Id}_{\mathcal{M}} + \mathrm{Id}_{\mathcal{M}} \otimes \mathrm{m}_1^{\mathcal{A}}),$$

which means that m_2 suffices the Leibniz rule.

$k = 3$. For this, we get similar to the $\mathcal{A}_\infty$-algebra case that the action of $\mathcal{M}$ induced by m_2 is associative up to a homotopy, which is given by m_3.

So, one can think about an $\mathcal{A}_\infty$-module as a homotopy coherent module over $\mathcal{X}$.

Example A.28. We collect once more the standard examples.

- Let $\mathcal{M}$ be a graded module over a $\Bbbk$-linear category $\mathcal{X}$; then, it is an $\mathcal{A}_\infty$-module over $\mathcal{X}$ via
$$\mathrm{m}_i = \begin{cases} (\_) \circ (\_) & i = 2, \\ 0 & i \neq 2. \end{cases}$$
- Let $\mathcal{M}$ be a dg-module over a dg-algebra $\mathcal{X}$. Then, it defines an $\mathcal{A}_\infty$-module over $\mathcal{X}$ via
$$\mathrm{m}_i = \begin{cases} d_{\mathcal{M}} & i = 1, \\ (\_) \circ (\_) & i = 2, \\ 0 & i \notin \{1, 2\}. \end{cases}$$

Definition A.29 ([12, Section 4.2]). Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{A}_n$-modules over an $\mathcal{A}_n$-category $\mathcal{X}$ for $n \in \mathbb{N} \cup \{\infty\}$. A morphism of $\mathcal{A}_n$-modules consists of a set of morphisms:

$$f_i : \underbrace{\mathcal{M}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \cdots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{N}(a_i, a_0)$$

of degree $1 - i$ for $i \leq n$ such that we have for every $k < n$

$$\sum_{r+s+t} (-1)^{r+st} f_u \circ (\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}) = \sum_{n=r+s} (-1)^{(r-1)s} \mathrm{m}_{u'}(f_r \otimes \mathrm{Id}^s), \qquad (**_k)$$

where $u = r + s + t$ and $u' = 1 + s$.

Example A.30. We compute again $(**_k)$ for small k as follows.

$k = 1$. Similar to the cases above, $(**_1)$ boils down to

$$f_1 \circ \mathrm{m}_1 = \mathrm{m}_1 \circ f_1,$$

which means that f_1 defines a morphism of chain complexes.

$k = 2$. Here, we get

$$f_1 \circ \mathrm{m}_2 - f_2 \circ (\mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1) = \mathrm{m}_2 \circ (f_1 \otimes \mathrm{Id}_{\mathcal{X}}) + \mathrm{m}_1 \circ f_2.$$

This means that similar to the case of an $\mathcal{A}_n$-functor between $\mathcal{A}_n$-categories the equation $(**_2)$ encodes that f_1 is compatible with the action induced by m_2 up to a homotopy given by f_2.

These examples are another reason one can think about $\mathcal{A}_\infty$-structure as a notion for inductive homotopy coherent algebraic structures.

Definition A.31 ([12, Section 4.2]). An $\mathcal{A}_n$-morphism $f : \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism if it induces an isomorphism on cohomology

$$\mathrm{H}^*(f) : \mathrm{H}^*\mathcal{M} \xrightarrow{\sim} \mathrm{H}^*\mathcal{N}.$$

Definition A.32 ([12, Section 4.2]). Let $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M}' \to \mathcal{M}''$ be morphisms of $\mathcal{A}_\infty$-modules over a homologically unital $\mathcal{A}_\infty$-algebra $\mathcal{X}$. Then, the composition $f \circ g : \mathcal{M} \to \mathcal{M}''$ is given by

$$(f \circ g)_n = \sum_{n=r+s} (-1)^{(r-1)s} f_u(g_r \otimes \mathrm{Id}^{\otimes s}),$$

where we put $u = 1 - s$.

Definition A.33 ([12, Section 4.2]). Let $\mathcal{X}$ be a homologically unital $\mathcal{A}_\infty$-algebra; then, we define the category of $\mathcal{A}_\infty$-modules $\mathcal{C}_\infty(\mathcal{X})$ to be the category consisting of $\mathcal{A}_\infty$-modules and morphisms given by $\mathcal{A}_\infty$-morphisms.

Remark A.34. The identity of an object in $\mathcal{C}_\infty(\mathcal{X})$ is given by

$$\mathrm{Id} = (\mathrm{Id}, 0, \ldots).$$

Definition A.35 ([23, Section 1k]). Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{A}_i$-functor. Then, the functor

$$f_* : \mathcal{C}_\infty(\mathcal{Y}) \to \mathcal{C}_\infty(\mathcal{X})$$

is given on modules by

$$f_*\mathcal{M}(a) := \mathcal{M}(f(a))$$

for objects $a \in \mathrm{obj}(\mathcal{X})$. Higher compositions are given by

$$\mathrm{m}_k(m, x_{k-1}, \ldots, x_1) = \sum_{l<k} \sum_{s_1,\ldots,s_l} \mathrm{m}_l(m, f_{s_l}(x_{k-1}, \ldots, x_{k-s_l}), \ldots, f_{s_1}(a_{s_1}, \ldots, a_1)).$$

On morphisms, f^* is given by

$$f_*\varphi_k(m, x_{k-1}, \ldots, x_1) = \sum_{l<k} \sum_{s_1,\ldots,s_l} \varphi_l(m, f_{s_l}(x_{k-1}, \ldots, x_{k-s_l}), \ldots, f_{s_1}(a_{s_1}, \ldots, a_1)).$$

Remark A.36. We again choose the notation f_* over f^* as we will later model the category of sheaves on a projective scheme by modules over a $\Bbbk$-linear category, and under this construction, the functor f_* corresponds to the direct image, and so, the notation turns out to be more consistent and less confusing throughout this work.

Definition A.37 ([12, Section 4.2]). Let $\mathcal{X}$ be a homologically unital small $\mathcal{A}_\infty$-category. Then, we define the category

$$\mathcal{D}_\infty(\mathcal{X}) := \mathcal{C}_\infty(\mathcal{X})[\{\mathcal{A}_\infty - \text{quasi-isomorphism}\}^{-1}].$$

Remark A.38 ([12, Section 4.2]). More generally, one could consider $\mathcal{A}_\infty$-categories over commutative rings instead of a field $\Bbbk$. In this case, we would have to distinguish between the derived category of $\mathcal{A}_\infty$-modules, as we defined it, and the category of $\mathcal{A}_\infty$-modules up to homotopy. However, over a field, one can prove that actually every quasi-isomorphism of $\mathcal{A}_\infty$-modules is a homotopy equivalence and vice versa. In particular, in this case, the naively derived category arising by formally inverting quasi-isomorphisms and the category of $\mathcal{A}_\infty$-modules up to homotopy coincide.

The interpretation of $\mathcal{D}_\infty(\mathcal{X})$ as arising via $\mathcal{A}_\infty$-modules up to homotopy immediately gives that $\mathcal{D}_\infty(\mathcal{X})$ is well defined and there are no set-theoretic issues arising.

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