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A cocyclic construction of S^1-equivariant homology and application to string topology

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Abstract. Given a space with a circle action, we study certain cocyclic chain complexes and prove a theorem relating cyclic homology to S^1-equivariant homology, in the spirit of celebrated work of Jones. As an application, we describe a chain level refinement of the gravity algebra structure on the (negative) S^1-equivariant homology of the free loop space of a closed oriented smooth manifold, based on work of Irie on chain level string topology and work of Ward on an S^1-equivariant version of operadic Deligne's conjecture.

1. Introduction

Let M be a closed oriented smooth manifold and $\mathcal{L}M = C^\infty(S^1, M)$ be the smooth free loop space of M. In a seminal paper [3] (and a sequel [4]), Chas–Sullivan discovered rich algebraic structures on the ordinary homology and S^1-equivariant homology of $\mathcal{L}M$, initiating the study of string topology. In particular, there is a Batalin–Vilkovisky (BV) algebra structure on (shifted) $H_*(\mathcal{L}M)$ ([3, Theorem 5.4]), which naturally induces a gravity algebra structure on (shifted) $H_*^{S^1}(\mathcal{L}M)$ ([3, Section 6], [4, page 18]).

The goal of this paper is to describe a chain level refinement of the string topology gravity algebra, and compare it with an algebraic counterpart related to the de Rham dg algebra $\Omega(M)$. Along the way we also obtain results on the relation between cyclic homology and S^1-equivariant homology, and an S^1-equivariant version of Deligne's conjecture.

In spirit, this paper may be compared with work of Westerland [33]. Westerland gave a homotopy theoretic generalization of the gravity operations on the (negative) S^1-equivariant homology of $\mathcal{L}M$, whereas we describe a chain level refinement.

Cyclic homology and S^1-equivariant homology. The close connection between cyclic homology (algebra) and S^1-equivariant homology (topology) was first systematically studied by Jones in [21]. One of the main theorems in that paper ([21, Theorem 3.3]) says that the singular chains $\{S_k(X)\}_{k\geq 0}$ of an S^1-space X can be made into a cyclic module, such that there are natural isomorphisms between three versions of cyclic homology (positive, periodic, negative) of $\{S_k(X)\}_{k\geq 0}$ and three versions of S^1-equivariant homology of X, in a way compatible with long exact sequences.

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The first result in this paper is a theorem "cyclic dual" to Jones' theorem. As far as the author knows, such a result did not appear in the literature.

Theorem 1.1 (See Theorem 3.1). *Let X be a topological space with an S^1-action. Then $\{S_*(X \times \Delta^k)\}_{k\geq 0}$ can be made into a cocyclic chain complex, such that there are natural isomorphisms between three versions of cyclic homology of $\{S_*(X \times \Delta^k)\}_{k\geq 0}$ and three versions of S^1-equivariant homology of X, in a way compatible with long exact sequences.*

Jones dealt with the cyclic set $\{\mathrm{Map}(\Delta^k, X)\}_k$ and the cyclic module $\{S_k(X)\}_k$, while we deal with the cocyclic space $\{X \times \Delta^k\}_k$ and the cocyclic complex $\{S_*(X \times \Delta^k)\}_k$. It is in this sense that these two theorems are "cyclic dual" to each other. In the special case that X is the free loop space of a topological space Y, Theorem 1.1 may also be viewed as "cyclic dual" to a result of Goodwillie ([18, Lemma V.1.4]). As does Jones' theorem, Theorem 1.1 has the advantage that it works for all S^1-spaces.

The cyclic structure on singular chains plays no role in Theorem 1.1; what matters is the cocyclic space. Indeed, the main motivation for the author to seek for a result like Theorem 1.1 is to study the S^1-equivariant homology of $\mathcal{L}M$, using a novel chain model of loop space homology defined via certain "de Rham chains", introduced by Irie [20].

Deligne's conjecture. What is called Deligne's conjecture asks whether there is an action of a certain chain model of the little disks operad on the Hochschild cochain complex of an associative algebra, inducing the Gerstenhaber algebra structure on Hochschild cohomology discovered by Gerstenhaber [12]. This conjecture, as well as some variations and generalizations, has been answered affirmatively by many authors, to whom we are apologetic not to list here. What is of most interest and importance to us is work of Ward [31].

Ward ([31, Theorem C]) gave a general solution to the question when a certain complex of cyclic (co)invariants admits an action of a chain model of the gravity operad, inducing the gravity algebra structure on cyclic cohomology. Recall that the gravity operad was introduced by Getzler [16] and is the S^1-equivariant homology of the little disks operad. So Ward's result can be viewed as an S^1-equivariant version of the operadic Deligne's conjecture ([31, Corollary 5.22]).

The second result in this paper is an extension, in a special case, of Ward's theorem. To state our result, let A be a dg algebra equipped with a symmetric, cyclic, bilinear form $\langle , \rangle : A \otimes A \to \mathbb{R}$ of degree $m \in \mathbb{Z}$ satisfying the Leibniz rule (see Example 5.9). Then $\langle , \rangle$ induces a dg A-bimodule map $\theta : A \to A^\vee[m]$, and hence a cochain map $\Theta : \mathrm{CH}(A, A) \to \mathrm{CH}(A, A^\vee[m])$ between Hochschild cochains. Let $\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$ be the subcomplex of cyclic invariants in $\mathrm{CH}(A, A^\vee[m])$. Let $\mathsf{M}_{\circlearrowleft}$ be the chain model of the gravity operad that Ward constructed (see also Example 5.3 (3)).

Theorem 1.2 (See Corollary 6.8). *Given $A, \langle , \rangle, \theta, \Theta$ as above, there is an action of $\mathsf{M}_{\circlearrowleft}$ on $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$, giving rise to a structure of a gravity algebra up to homotopy. If θ is a quasi-isomorphism and Θ restricts to a quasi-isomorphism $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])) \to$*

$\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$*, this descends to a gravity algebra structure on the cyclic cohomology of* A*, which is compatible with the BV algebra structure on Hochschild cohomology.*

Here the BV algebra structure on the Hochschild cohomology of A (when θ is a quasi-isomorphism) is well-known (e.g., Menichi [28, Theorem 18]), where the BV operator is given by Connes' operator (Example 2.6). By compatibility with a BV algebra structure we mean the content of Lemma 5.1. Note that Ward's original theorem only applies to the situation that θ is an isomorphism ([31, Corollary 6.2]).

Chain level structures in S^1-equivariant string topology. Let us say more about Irie's work [20]. Using his chain model and results of Ward ([31, Theorem A, Theorem B]), Irie obtained an operadic chain level refinement of the string topology BV algebra, and compared it with a solution to the ordinary Deligne's conjecture via a chain map defined by iterated integrals of differential forms.

The third result in this paper is a similar story in the S^1-equivariant context. Note that the string topology BV algebra induces gravity algebra structures on two versions (positive, i.e., ordinary, and negative) of S^1-equivariant homology of $\mathcal{L}M$ (Example 7.1).

Theorem 1.3 (See Theorem 7.6). *For any closed oriented C^∞-manifold M, there exists a chain complex $\widetilde{\mathcal{O}}_M^{\mathrm{cyc}}$ satisfying the following properties. Firstly, the homology of $\widetilde{\mathcal{O}}_M^{\mathrm{cyc}}$ is isomorphic to the negative S^1-equivariant homology of $\mathcal{L}M$, and $\widetilde{\mathcal{O}}_M^{\mathrm{cyc}}$ admits an action of $\mathsf{M}_{\circlearrowleft}$ (hence an up-to-homotopy gravity algebra structure) which lifts the gravity algebra structure mentioned above. Secondly, there is a morphism of $\mathsf{M}_{\circlearrowleft}$-algebras*

$$\widetilde{\mathcal{O}}_M^{\mathrm{cyc}} \to \Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M])) \tag{1.1}$$

which is induced by iterated integrals of differential forms, where the structure on the right-hand side follows from Theorem 1.2 and Θ comes from the Poincaré pairing. At homology level, the morphism (1.1) descends to a map (part of arrow 4 below) which fits into a commutative diagram of gravity algebra homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{1} & B \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} \\ C & \xrightarrow{4} & D. \end{array} \tag{1.2}$$

Here A is the S^1-equivariant homology of $\mathcal{L}M$, B is the negative cyclic cohomology of $\Omega(M)$, C is the negative S^1-equivariant homology of $\mathcal{L}M$, D is the cyclic cohomology of $\Omega(M)$. Arrows 1, 4 are defined by iterated integrals on free loop space, and arrow 2 (resp. 3) is the connecting map in the tautological long exact sequence for S^1-equivariant homology theories (resp. cyclic homology theories).

The crucial part of Theorem 1.3 is, of course, the chain level statement that fits well with structures on homology. The first part of Theorem 1.3 was conjectured by Ward in [31, Example 6.12], but the correct statement turns out to be more complicated, as we

actually lift gravity algebra structures on negative S^1-equivariant homology rather than S^1-equivariant homology, whereas they are naturally related by a morphism (arrow 2).

Other than the chain level statement, part of the results at homology level is known. For example, the fact that arrow 1 is a Lie algebra homomorphism appeared in work of Abbaspour–Tradler–Zeinalian as [1, Theorem 11]; the fact that (1.2) commutes was of importance to Cieliebak–Volkov [8] (the arrows are only treated as linear maps there).

In a forthcoming paper, the author is going to apply results in this paper to Lagrangian Floer theory, in view of cyclic symmetry therein (Fukaya [11]).

Outline

In Section 2, we review cyclic homology of mixed complexes. In Section 3, we prove Theorem 1.1. In Section 4, we review Irie's de Rham chain complex of differentiable spaces and apply Theorem 1.1 to it. In Section 5, we review basics of operads and algebraic structures. In Section 6, we prove Theorem 1.2. In Section 7, we prove Theorem 1.3.

Conventions

Vector spaces are over $\mathbb{R}$, algebras are associative and unital, graded objects are $\mathbb{Z}$-graded. Homological and cohomological gradings are mixed by the understanding $C_* = C^{-*}$, $C^* = C_{-*}$. As for sign rules, see Appendix A. For the sake of convenience, we may write $(-1)^{\varepsilon}$ for a sign that is apparent from the Koszul sign rule (Appendix A.1).

2. Preliminaries on cyclic homology

A convenient way to study different versions of cyclic homology is to work in the context of *mixed complexes*, which was introduced by Kassel [22]. By definition, a mixed cochain complex is a triple (C^*, b, B) consisting of a graded vector space C^* and linear maps $b : C^* \to C^{*+1}$, $B : C^* \to C^{*-1}$ such that

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

Let u be a formal variable of degree 2. Define graded $\mathbb{R}[u]$-modules $C[[u]]^*$, $C[[u, u^{-1}]^*$, $C[u^{-1}]^*$ by

$$C[[u]]^n := \Big\{ \sum_{i \geq 0} c_i u^i \mid c_i \in C^{n-2i} \Big\},$$
$$C[[u, u^{-1}]^n := \Big\{ \sum_{i \geq -k} c_i u^i \mid k \in \mathbb{Z}_{\geq 0},\ c_i \in C^{n-2i} \Big\},$$
$$C[u^{-1}]^n := \Big\{ \sum_{-k \leq i \leq 0} c_i u^i \mid k \in \mathbb{Z}_{\geq 0},\ c_i \in C^{n-2i} \Big\}.$$

Here the $\mathbb{R}[u]$-module structure on $C[u^{-1}]$ is induced by the identification $C[u^{-1}] = C[[u,u^{-1}]/uC[[u]]$. Then $b+uB$ is a differential on $C[[u]]^*$, $C[[u,u^{-1}]^*$, $C[u^{-1}]^*$, resulting in cohomology groups denoted by

$$\mathrm{HC}^*_{[[u]]}(C), \quad \mathrm{HC}^*_{[[u,u^{-1}]}(C), \quad \mathrm{HC}^*_{[u^{-1}]}(C).$$

These are three classical versions of *cyclic homology* of mixed complexes, called the negative, periodic and ordinary (positive) cyclic homology of (C^*, b, B), respectively. We prefer to distinguish them by suggestive symbols ($[[u]]$, $[[u,u^{-1}]$, $[u^{-1}]$) rather than names, as did in [8]. Here cohomological grading is used for cyclic homology since we deal with cochain complexes. If we move to homological grading $C_* := C^{-*}$ and replace u by v (a formal variable of degree -2), then the mixed chain complex (C_*, b, B) gives negative, periodic and ordinary (positive) cyclic homology theories

$$\mathrm{HC}^{[[v]]}_* = \mathrm{HC}^{-*}_{[[u]]}, \quad \mathrm{HC}^{[[v,v^{-1}]}_* = \mathrm{HC}^{-*}_{[[u,u^{-1}]}, \quad \mathrm{HC}^{[v^{-1}]}_* = \mathrm{HC}^{-*}_{[u^{-1}]}.$$

([8] also takes the Hom dual of C to define cyclic cohomology theories of (C,b,B), which we try to avoid in this article.)

For any mixed cochain complex (C^*, b, B), there is a *tautological exact sequence*

$$\cdots \to \mathrm{HC}^*_{[[u]]}(C) \xrightarrow{i_*} \mathrm{HC}^*_{[[u,u^{-1}]}(C) \xrightarrow{u\cdot p_*} \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} \mathrm{HC}^{*+1}_{[[u]]}(C) \to \cdots \tag{2.1}$$

which is induced by the short exact sequence

$$0 \to C[[u]] \xrightarrow{i} C[[u,u^{-1}] \xrightarrow{p} C[[u,u^{-1}]/C[[u]] \to 0$$

and the $(b+uB)$-cochain isomorphism

$$(C[[u,u^{-1}]/C[[u]])^* \xrightarrow[\cong]{\cdot u} C[u^{-1}]^{*+2}; \quad \sum_{-k\le i\le -1} c_i u^i \mapsto \sum_{-k\le i\le -1} c_i u^{i+1}.$$

The connecting map $B_{0*} : \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \to \mathrm{HC}^{*+1}_{[[u]]}(C)$ is induced by an anti-cochain map

$$B_0 : C[u^{-1}]^{*+2} \to C[[u]]^{*+1}; \quad \sum_{-k\le i\le 0} c_i u^i \mapsto B(c_0).$$

Similarly, from the short exact sequences

$$0 \to C[[u]]/uC[[u]] \xrightarrow{i} C[[u,u^{-1}]/uC[[u]] \xrightarrow{p} C[[u,u^{-1}]/C[[u]] \to 0,$$

$$0 \to uC[[u]] \xrightarrow{i^+} C[[u]] \xrightarrow{p_0} C[[u]]/uC[[u]] \to 0$$

one obtains the *Gysin–Connes exact sequences*

$$\cdots \to H^*(C,b) \xrightarrow{i_*} \mathrm{HC}^*_{[u^{-1}]}(C) \xrightarrow{u\cdot p_*} \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} H^{*+1}(C,b) \to \cdots, \tag{2.2a}$$

$$\cdots \to \mathrm{HC}^{*-2}_{[[u]]}(C) \xrightarrow{i^+_*\cdot u} \mathrm{HC}^*_{[[u]]}(C) \xrightarrow{p_{0*}} H^*(C,b) \xrightarrow{B_*} \mathrm{HC}^{*-1}_{[[u]]}(C) \to \cdots. \tag{2.2b}$$

The connecting maps $\mathrm{HC}^{*+2}_{[u^{-1}]}(C) \xrightarrow{B_{0*}} H^{*+1}(C,b)$ and $H^*(C,b) \xrightarrow{B_*} \mathrm{HC}^{*-1}_{[[u]]}(C)$ are induced by anti-cochain maps B_0 and B, respectively.

Lemma 2.1. *The map* $B_{0*}: \mathrm{HC}^{*+2}_{[u^{-1}]}(C) \to \mathrm{HC}^{*+1}_{[[u]]}(C)$ *in* (2.1) *and the exact sequences* (2.2) *fit into the following commutative diagram:*

$$\begin{array}{ccccccccc}
\cdots \to H^*(C,b) & \xrightarrow{i_*} & \mathrm{HC}^*_{[u^{-1}]}(C) & \xrightarrow{u\cdot p_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) & \xrightarrow{B_{0*}} & H^{*+1}(C,b) \to \cdots \\
\| \mathrm{id} & & \downarrow B_{0*} & & \downarrow B_{0*} & & \| \mathrm{id} \\
\cdots \to H^*(C,b) & \xrightarrow{B_*} & \mathrm{HC}^{*-1}_{[[u]]}(C) & \xrightarrow{i_*^+\cdot u} & \mathrm{HC}^{*+1}_{[[u]]}(C) & \xrightarrow{p_{0*}} & H^{*+1}(C,b) \to \cdots .
\end{array}$$

Proof. The left and the right squares commute since they commute at the level of cocycles. As for the middle square, let $c = \sum_{j=-k}^{0} c_j u^j \in Z^*(C[u^{-1}])$, then $B_0(u \cdot p(c)) = B(c_{-1})$ and $i^+(u \cdot B_0(c)) = B(c_0)u$. Since c is a cocycle,

$$0 = (b + uB)(c) = \sum_{j=-k}^{0} (b(c_j) + B(c_{j-1}))u^j \in C[u^{-1}].$$

In particular, $b(c_0) + B(c_{-1}) = 0$, so $B(c_0)u - B(c_{-1}) = (b + uB)(c_0)$ is exact. This proves $B_{0*} \circ (u \cdot p_*) = (i_*^+ \cdot u) \circ B_{0*}$. ∎

Definition 2.2. Let (C^*, b, B), (C''^*, b'', B'') be mixed cochain complexes.

(1) A series of linear maps $\{f_i : C^* \to (C'')^{*-2i}\}_{i\in\mathbb{Z}_{\geq 0}}$ is called an ∞-*morphism* from C^* to C''^* if $\sum_{i\geq 0} u^i f_i : (C[[u,u^{-1}]^*, b + uB) \to (C''[[u,u^{-1}]^*, b'' + uB'')$ is a cochain map, or equivalently, if $\{f_i\}_{i\geq 0}$ satisfies $b''f_0 = f_0 b$ and $B''f_{i-1} + b''f_i = f_{i-1}B + f_i b$ $(i \geq 1)$.

(2) An ∞-morphism $f = \{f_i\}_{i\geq 0} : C^* \to C''^*$ is called an ∞-*quasi-isomorphism* if $f_0 : (C^*, b) \to (C''^*, b'')$ is a cochain quasi-isomorphism.

(3) Given two ∞-morphisms $\{f_i\}_{i\geq 0}, \{g_i\}_{i\geq 0} : C^* \to C''^*$, a series of linear maps $\{h_i : C^* \to (C'')^{*-2i-1}\}_{i\in\mathbb{Z}_{\geq 0}}$ is called an ∞-*homotopy* between them if $h = \sum_{i\geq 0} u^i h_i : C[[u,u^{-1}]^* \to C''[[u,u^{-1}]^*$ is a $(b + uB, b'' + uB'')$-cochain homotopy between $\sum_{i\geq 0} u^i f_i$ and $\sum_{i\geq 0} u^i g_i$, or equivalently, if $\{h_i\}_{i\geq 0}$ satisfies $f_0 - g_0 = b''h_0 + h_0 b$ and $f_i - g_i = b''h_i + h_i b + B''h_{i-1} + h_{i-1}B$ $(i \geq 1)$.

A morphism between mixed complexes is an ∞-morphism $\{f_i\}_{i\geq 0}$ such that $f_i = 0$ for all $i > 0$, namely a single degree 0 linear map that commutes with both b and B. A quasi-isomorphism between mixed complexes is a morphism that is also a (b, b'')-quasi-isomorphism. A homotopy between two morphisms $f, g : (C^*, b, B) \to (C''^*, b'', B'')$ is an ∞-homotopy $\{h_i\}_{i\geq 0}$ such that $h_i = 0$ for all $i > 0$, namely a single degree -1 linear map h satisfying $f - g = b''h + hb$ and $B''h + hB = 0$.

The following important lemma goes back to [21, Lemma 2.1], and is a special case of [34, Lemma 2.3] which is stated for S^1-complexes (an ∞-version of mixed complexes). The proof is a spectral sequence argument using the u-adic filtration on $C[[u]]^*$ etc..

Lemma 2.3. *Let* $\{f_i\}_{i\geq 0} : (C^*, b, B) \to (C''^*, b'', B'')$ *be an* ∞*-quasi-isomorphism. Then* $\sum_{i\geq 0} u^i f_i$ *induces isomorphisms on* $\mathrm{HC}^*_{[[u]]}$, $\mathrm{HC}^*_{[[u,u^{-1}]}$ *and* $\mathrm{HC}^*_{[u^{-1}]}$. ∎

The following lemma illustrates the naturality of the tautological exact sequence and Connes–Gysin exact sequences for cyclic homology, with respect to ∞-morphisms between mixed complexes.

Lemma 2.4. *Let $f = \{f_i\}_{i\geq 0} : (C^*, b, B) \to (C'^*, b', B')$ be an ∞-morphism. Then $f = \sum_i u^i f_i$ induces a morphism between the exact sequence (2.1) for C and C', namely there is a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \mathrm{HC}^*_{[[u]]}(C) & \xrightarrow{i_*} & \mathrm{HC}^*_{[[u,u^{-1}]]}(C) & \xrightarrow{u\cdot p_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) & \xrightarrow{B_{0*}} & \mathrm{HC}^{*+1}_{[[u]]}(C)\cdots \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
\cdots \mathrm{HC}^*_{[[u]]}(C') & \xrightarrow{i'_*} & \mathrm{HC}^*_{[[u,u^{-1}]]}(C') & \xrightarrow{u\cdot p'_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C') & \xrightarrow{B'_{0*}} & \mathrm{HC}^{*+1}_{[[u]]}(C')\cdots .
\end{array}$$

Similarly, for the exact sequence (2.2a), there is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots H^*(C,b) & \xrightarrow{i_*} & \mathrm{HC}^*_{[u^{-1}]}(C) & \xrightarrow{u\cdot p_*} & \mathrm{HC}^{*+2}_{[u^{-1}]}(C) & \xrightarrow{B_{0*}} & H^{*+1}(C,b)\cdots \\
\downarrow f_{0*} & & \downarrow f_* & & \downarrow f_* & & \downarrow f_{0*} \\
\cdots H^*(C',b') & \xrightarrow{i'_*} & \mathrm{HC}^{*-1}_{[u^{-1}]}(C') & \xrightarrow{u\cdot p'_*} & \mathrm{HC}^{*+1}_{[u^{-1}]}(C') & \xrightarrow{B'_{0*}} & H^{*+1}(C',b')\cdots .
\end{array}$$

The case of the exact sequence (2.2b) is also similar.

Proof. We only write the proof for the first diagram since the others are similar. The left and the middle squares commute since they commute at the level of cocycles. Now let $c = \sum_{j=0}^{k} c_{-j}u^{-j} \in Z^{*+2}(C[u^{-1}])$. Then, $(b + uB)(c) = 0$ says $b(c_{-j}) + B(c_{-j-1}) = 0$ for all $j \in \{0,\dots,k\}$. Also recall the ∞-morphism f satisfies $B' f_{i-1} + b' f_i = f_{i-1}B + f_i b$. Using these relations, it is a straightforward computation to see

$$\sum_{i\geq 0} f_i(B(c_0))\cdot u^i - B'\Big(\sum_{0\leq j\leq k} f_j(c_{-j})\Big) = (b' + uB')\Big(\sum_{i\geq 0}\sum_{0\leq j\leq k} f_{i+j+1}(c_{-j})\cdot u^i\Big).$$

The left-hand side is $(f\circ B_0 - B'_0\circ f)(c)$, and the right-hand side is exact, so commutativity of the right square is proved. ∎

We now discuss some important examples of mixed (co)chain complexes and their cyclic homologies. Recall that a *cosimplicial object* in some category is a sequence of objects $C(k)$ $(k \in \mathbb{Z}_{\geq 0})$ together with morphisms

$$\delta_i : C(k-1) \to C(k) \quad (0 \leq i \leq k), \qquad \sigma_i : C(k+1) \to C(k) \quad (0 \leq i \leq k),$$

satisfying the following relations:

$$\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1} \quad (i < j);\\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1} \quad (i \leq j);
\end{aligned}$$

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & (i < j), \\ \mathrm{id} & (i = j,\ j+1), \\ \delta_{i-1}\sigma_j & (i > j+1). \end{cases}$$

A *cocyclic object* is a cosimplicial object $\{C(k)\}_k$ together with morphisms $\tau_k : C(k) \to C(k)$ satisfying the following relations:

$$\begin{aligned} &\tau_k^{k+1} = \mathrm{id}; \\ &\tau_k\delta_i = \delta_{i-1}\tau_{k-1} \quad (1 \le i \le k), \quad \tau_k\delta_0 = \delta_k; \\ &\tau_k\sigma_i = \sigma_{i-1}\tau_{k+1} \quad (1 \le i \le k), \quad \tau_k\sigma_0 = \sigma_k\tau_{k+1}^2. \end{aligned}$$

For example, let $\Delta^0 := \mathbb{R}^0$, $\Delta^k := \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid 0 \le t_1 \le \cdots \le t_k \le 1\}$ $(k > 0)$ be the standard simplices, then $\{\Delta^k\}_{k\in\mathbb{Z}_{\ge 0}}$ is a cocyclic set (topological space, etc.) with standard cocyclic maps $\delta_i : \Delta^{k-1} \to \Delta^k$, $\sigma_i : \Delta^{k+1} \to \Delta^k$, $\tau_k : \Delta^k \to \Delta^k$ defined by

$$\delta_i(t_1, \dots, t_{k-1}) := \begin{cases} (0, t_1, \dots, t_{k-1}) & (i = 0), \\ (t_1, \dots, t_i, t_i, \dots, t_{k-1}) & (1 \le i \le k-1), \\ (t_1, \dots, t_{k-1}, 1) & (i = k), \end{cases} \tag{2.3a}$$

$$\sigma_i(t_1, \dots, t_{k+1}) := (t_1, \dots, \widehat{t_{i+1}}, \dots, t_{k+1}) \quad (\text{miss } t_{i+1}) \quad (0 \le i \le k), \tag{2.3b}$$

$$\tau_k(t_1, \dots, t_k) := (t_2 - t_1, \dots, t_k - t_1, 1 - t_1). \tag{2.3c}$$

Remark 2.5. Equivalently, for $\widetilde{\Delta}^k := \{(s_0, s_1, \dots, s_k) \in [0,1]^{k+1} \mid s_0 + s_1 + \cdots + s_k = 1\}$ $(k \ge 0)$, $\tau_k : \widetilde{\Delta}^k \to \widetilde{\Delta}^k$ reads $\tau_k(s_0, s_1, \dots, s_k) = (s_1, \dots, s_k, s_0)$.

Example 2.6 (Cocyclic complex and Connes' version of cyclic cohomology). Consider the category of cochain complexes where the morphisms are degree 0 cochain maps. Let $((C(k)^*, d), \delta_i, \sigma_i, \tau_k)$ be a cocyclic cochain complex, then a mixed cochain complex (C, b, B) is obtained as follows. Let

$$\delta : C(k-1)^* \to C(k)^*; \qquad c_{k-1} \mapsto (-1)^{|c_{k-1}|+k-1} \sum_{0\le i\le k} (-1)^i \delta_i(c_{k-1}), \tag{2.4}$$

then $\delta^2 = 0$, $\delta d + d\delta = 0$. Let (C^*, b) be the *product* total complex of the double complex $(C(k)^l, d, \delta)_{l\in\mathbb{Z}}^{k\in\mathbb{Z}_{\ge 0}}$,

$$C^* := \prod_{l+k=*} C(k)^l = \prod_{k\ge 0} C(k)^{*-k}, \qquad b = d + \delta.$$

For later purpose we also introduce the *normalized* subcomplex (C^*_{nm}, b) of (C^*, b),

$$C^*_{\mathrm{nm}} := \prod_{k\ge 0} C_{\mathrm{nm}}(k)^{*-k}, \quad C_{\mathrm{nm}}(k) := \bigcap_{0\le i\le k-1} \mathrm{Ker}\big(\sigma_i : C(k) \to C(k-1)\big).$$

Note that the natural inclusion $\{C_{\mathrm{nm}}(k)\} \subset \{C(k)\}$ is not cosimplicial since δ_j does not restrict to $C_{\mathrm{nm}}(k)$. The natural inclusion $(C^*_{\mathrm{nm}}, b) \hookrightarrow (C^*, b)$ is a quasi-isomorphism (see [25, Proposition 1.6.5] or [20, Lemma 2.5]). Next, define the operator $B : C^* \to C^{*-1}$ by

$$B := Ns(1-\lambda) \quad (\textit{Connes' operator}),$$

where λ, N, s are given by (here $|c|$ is the degree of $c = (c_k)_{k\geq 0} \in \prod_{k\geq 0} C(k)$ in C^*)

$$\begin{aligned} \lambda|_{C(k)} &:= (-1)^k \tau_k, \\ N|_{C(k)} &:= 1 + \lambda + \cdots + \lambda^k, \\ s(c) &:= (-1)^{|c|-1}(\sigma_k \tau_{k+1}(c_{k+1}))_{k\geq 0}. \end{aligned}$$

Although C_{nm} is not closed under λ, N, it is closed under s, B. For $c_{k+1} \in C_{\mathrm{nm}}(k+1)^*$, there holds

$$s(\lambda(c_{k+1})) = (-1)^{|c_{k+1}|}\sigma_k \tau_{k+1}^2(c_{k+1}) = (-1)^{|c_{k+1}|}\tau_k \sigma_0(c_{k+1}) = 0,$$

so Connes' operator B has a simpler form on a normalized subcomplex,

$$B|_{C_{\mathrm{nm}}} = Ns.$$

To see (C^*, b, B) is a mixed complex, define

$$b' : C^* \to C^{*+1}, \quad c \mapsto b(c) - ((-1)^{|c_{k-1}|-1}\delta_k(c_{k-1}))_{k\geq 0}.$$

It is a routine calculation to see $(b')^2 = 0$, $N(1-\lambda) = (1-\lambda)N = 0$, $(1-\lambda)b = b'(1-\lambda)$, $bN = Nb'$ and $b's + sb' = 1$. It follows that $B^2 = Ns((1-\lambda)N)s(1-\lambda) = 0$ and $bB + Bb = Nb's(1-\lambda) + Nsb'(1-\lambda) = N(1-\lambda) = 0$, as desired. The identity $(1-\lambda)b = b'(1-\lambda)$ also implies that the space of *cyclic invariants*,

$$C_{\mathrm{cyc}} := \mathrm{Ker}(1-\lambda) \subset (C, b),$$

forms a subcomplex (we denote this inclusion by i_λ). This leads to *Connes' version of cyclic cohomology* of the cocyclic cochain complex,

$$\mathrm{HC}^*_\lambda(C) = \mathrm{HC}^*_\lambda(C(k), d, \delta_i, \sigma_i, \tau_k) := H^*(C_{\mathrm{cyc}}, b).$$

Since $B = Ns(1-\lambda)$ vanishes on C_{cyc}, (C^*_{cyc}, b) is also naturally a subcomplex of $(C[[u]]^*, b + uB)$. By an argument similar to [25, Theorems 2.1.5, 2.1.8] one sees that this inclusion $I_\lambda : (C^*_{\mathrm{cyc}}, b) \hookrightarrow (C[[u]]^*, b + uB)$ induces an isomorphism

$$I_{\lambda *} : \mathrm{HC}^*_\lambda(C) \cong \mathrm{HC}^*_{[[u]]}(C). \tag{2.5}$$

The short exact sequence $0 \to (C_{\mathrm{cyc}}, b) \xrightarrow{i_\lambda} (C, b) \xrightarrow{p_\lambda} (C/C_{\mathrm{cyc}}, b) \to 0$ induces *Connes' long exact sequence* (we follow the presentation of [23, Section 3.7])

$$\cdots \to \mathrm{HC}^*_\lambda(C) \xrightarrow{i_{\lambda *}} H^*(C, b) \xrightarrow{B_\lambda} \mathrm{HC}^{*-1}_\lambda(C) \xrightarrow{S_\lambda} \mathrm{HC}^{*+1}_\lambda(C) \to \cdots. \tag{2.6}$$

Here we have made use of an isomorphism $\mathrm{HC}_\lambda^{*-1}(C) \cong H^*(C/C_{\mathrm{cyc}}, b)$, which is a consequence of another short exact sequence

$$0 \to (C/C_{\mathrm{cyc}}, b) \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} (C_{\mathrm{cyc}}, b) \to 0 \tag{2.7}$$

and the fact that (C, b') is acyclic (since $b's + sb' = 1$). Lemma 2.9 below says (2.6) can be identified with (2.2b). Finally, we mention that $\mathrm{HC}^*_{[[u]]}(C_{\mathrm{nm}}) \cong \mathrm{HC}^*_{[[u]]}(C) \cong \mathrm{HC}^*_\lambda(C)$, where the first isomorphism follows from Lemma 2.3.

A subexample of Example 2.6 is as follows.

Example 2.7 (Cyclic cohomology of dg algebras). Let A^* be a dg algebra with unit 1_A. Then $\{\mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})\}_{k\geq 0}$ has the structure of a cocyclic cochain complex, where $\delta_i : \mathrm{Hom}^*(A^{\otimes k}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$, $\sigma_i : \mathrm{Hom}^*(A^{\otimes k+2}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$ and $\tau_k : \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R}) \to \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$ are

$$\begin{aligned}
\delta_i(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= \begin{cases} (-1)^\varepsilon \varphi(a_2 \otimes \cdots \otimes a_k \otimes a_{k+1}a_1) & (i = 0), \\ \varphi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) & (1 \leq i \leq k), \end{cases} \\
\sigma_i(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= \varphi(a_1 \otimes \cdots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \cdots \otimes a_{k+1}) \quad (0 \leq i \leq k), \\
\tau_k(\varphi)(a_1 \otimes \cdots \otimes a_{k+1}) &:= (-1)^\varepsilon \varphi(a_{k+1} \otimes a_1 \cdots \otimes a_k).
\end{aligned} \tag{2.8}$$

The associated mixed total complex is denoted by $\mathrm{CH}^*(A, A^\vee)$. For simplicity, denote cyclic homologies of $\mathrm{CH}^*(A, A^\vee)$ by $\mathrm{HC}^*_{[u^{-1}]}(A, A^\vee)$, $\mathrm{HC}^*_{[[u]]}(A, A^\vee) \cong \mathrm{HC}^*_\lambda(A, A^\vee)$, etc.. Classically, $\mathrm{HC}^*_\lambda(A, A^\vee)$ is called *(Connes') cyclic cohomology* of A.

Let us also recall that for any dg A-bimodule M^*, there is a structure of a cosimplicial complex on $\{\mathrm{Hom}^*(A^{\otimes k}, M)\}_{k\geq 0}$, where $\delta_i : \mathrm{Hom}^*(A^{\otimes k-1}, M) \to \mathrm{Hom}^*(A^{\otimes k}, M)$, $\sigma_i : \mathrm{Hom}^*(A^{\otimes k+1}, M) \to \mathrm{Hom}^*(A^{\otimes k}, M)$ are

$$\begin{aligned}
\delta_i(f)(a_1 \otimes \cdots \otimes a_k) &:= \begin{cases} (-1)^\varepsilon a_1 \cdot f(a_2 \otimes \cdots \otimes a_k) & (i = 0), \\ f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k) & (1 \leq i \leq k-1), \\ f(a_1 \otimes \cdots \otimes a_{k-1}) \cdot a_k & (i = k), \end{cases} \\
\sigma_i(f)(a_1 \otimes \cdots \otimes a_k) &:= f(a_1 \otimes \cdots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \cdots \otimes \cdots \otimes a_k) \quad (0 \leq i \leq k).
\end{aligned}$$

The associated total complex, denoted by $\mathrm{CH}^*(A, M)$, is called the *Hochschild cochain complex*, whose cohomology group, denoted by $\mathrm{HH}^*(A, M)$, is called the *Hochschild cohomology*. Taking $M^* = (A^\vee)^* = \mathrm{Hom}^*(A, \mathbb{R})$ with dg A-bimodule structure satisfying

$$(d\varphi)(a) + (-1)^{|\varphi|}\varphi(da) = 0, \quad \varphi(ab) = (-1)^{(|a|+|\varphi|)|b|}(b \cdot \varphi)(a) = (\varphi \cdot a)(b), \tag{2.9}$$

one sees that the cosimplicial structure on $\{\mathrm{Hom}^*(A^{\otimes k}, A^\vee)\}_k$ is the same as that on $\{\mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})\}$ described previously, in view of the natural isomorphism

$$\mathrm{Hom}^*(A^{\otimes k}, A^\vee) \cong \mathrm{Hom}^*(A^{\otimes k} \otimes A, \mathbb{R}) = \mathrm{Hom}^*(A^{\otimes k+1}, \mathbb{R})$$

from Hom-$\otimes$ adjunction. See Example 5.9 for further discussion.

Remark 2.8. We shall use the name "Connes' version of cyclic *co*homology" for "*co*cyclic complex", even if we work with chain complexes rather than cochain complexes. For a cocyclic chain complex $((C(k)_*, \partial), \delta_j, \sigma_i, \tau_k)$, Connes' version of cyclic cohomology is $\mathrm{HC}^\lambda_*(C) := H_*(C^{\mathrm{cyc}}, b)$ where $C^{\mathrm{cyc}}_* := \mathrm{Ker}(1-\lambda) \subset C_*$, and $\mathrm{HC}^\lambda_*(C)$ is isomorphic to $\mathrm{HC}^{[[v]]}_*$ of the mixed chain complex $(C_*, b, B) = (\prod_{k\geq 0} C(k)_{*+k}, \partial + \delta, Ns(1-\lambda))$.

Lemma 2.9. *In the situation of Example* 2.6*, the isomorphism* (2.5) *and the long exact sequences* (2.6) *and* (2.2b) *fit into the following commutative diagram:*

$$\begin{array}{ccccccccc}
\cdots \mathrm{HC}^*_\lambda(C) & \xrightarrow{i_{\lambda *}} & H^*(C,b) & \xrightarrow{B_\lambda} & \mathrm{HC}^{*-1}_\lambda(C) & \xrightarrow{-S_\lambda} & \mathrm{HC}^{*+1}_\lambda(C) \cdots \\
\cong \downarrow I_{\lambda *} & & \| \mathrm{id} & & \cong \downarrow I_{\lambda *} & & \cong \downarrow I_{\lambda *} \\
\cdots \mathrm{HC}^*_{[[u]]}(C) & \xrightarrow{p_{0*}} & H^*(C,b) & \xrightarrow{B_*} & \mathrm{HC}^{*-1}_{[[u]]}(C) & \xrightarrow{i^+_* \cdot u} & \mathrm{HC}^{*+1}_{[[u]]}(C) \cdots .
\end{array}$$

Proof. The left square commutes since it commutes at cochain level. To verify commutativity of the other two squares, we need explicit formulas of B_λ and S_λ. Since $(1-\lambda)b = b'(1-\lambda)$, there is a cochain isomorphism

$$(C/C_{\mathrm{cyc}}, b) \xrightarrow[\cong]{1-\lambda} (\mathrm{Im}(1-\lambda), b').$$

Also note that $C_{\mathrm{cyc}} = \mathrm{Im}N$ and $N|_{C_{\mathrm{cyc}}} = ((k+1)\mathrm{id}_{\mathrm{Ker}(1-(-1)^k\tau_k)})_{k\geq 0} : C_{\mathrm{cyc}} \to C_{\mathrm{cyc}}$ is a linear isomorphism. By definition, B_λ is the composition

$$H^*(C,b) \xrightarrow{p_{\lambda *}} H^*(C/C_{\mathrm{cyc}}, b) \xrightarrow[\cong]{1-\lambda} H^*(\mathrm{Im}(1-\lambda), b') \xrightarrow[\cong]{Q^{-1}_{\lambda *}} \mathrm{HC}^{*-1}_\lambda(C),$$

where by examining (2.7), $Q_{\lambda *} : \mathrm{HC}^{*-1}_\lambda(C) \xrightarrow{\cong} H^*(\mathrm{Im}(1-\lambda), b')$ is given on cocycles by

$$Q_\lambda : Z^{*-1}(C_{\mathrm{cyc}}, b) \to Z^*(\mathrm{Im}(1-\lambda), b'); \quad x \mapsto (b' \circ (N|_{C_{\mathrm{cyc}}})^{-1})(x).$$

Let us calculate that on $Z(C,b)$,

$$\begin{aligned} Q_\lambda B &= Q_\lambda Ns(1-\lambda) = b's(1-\lambda) = (1-sb')(1-\lambda) \\ &= 1-\lambda-s(1-\lambda)b = 1-\lambda. \end{aligned} \tag{2.10}$$

Thus $(B|_{C\to C_{\mathrm{cyc}}})_* = (Q_{\lambda *})^{-1} \circ (1-\lambda) \circ p_{\lambda *} = B_\lambda$, which says the middle square commutes. Similarly, S_λ is the composition

$$\mathrm{HC}^{*-1}_\lambda(C) \xrightarrow[\cong]{Q_{\lambda *}} H^*(\mathrm{Im}(1-\lambda), b') \xrightarrow[\cong]{(1-\lambda)^{-1}} H^*(C/C_{\mathrm{cyc}}, b) \xrightarrow{R_{\lambda *}} \mathrm{HC}^{*+1}_\lambda(C),$$

where $R_{\lambda *} : H^*(C/C_{\mathrm{cyc}}, b) \to \mathrm{HC}^{*+1}_\lambda(C)$ is induced by the map

$$R_\lambda : \{y \in C^* \mid b(y) \in C_{\mathrm{cyc}}\} \to Z^{*+1}(C_{\mathrm{cyc}}, b); \quad y \mapsto b(y).$$

(2.10) also holds on $Z(C/C_{\mathrm{cyc}}, b)$, and implies $(1-\lambda)^{-1}Q_{\lambda *} = (B|_{C/C_{\mathrm{cyc}}\to C_{\mathrm{cyc}}})^{-1}_*$.

Therefore for $x \in Z^{*-1}(C_{\mathrm{cyc}}, b)$,

$$S_\lambda([x]) = [b(y)] \in \mathrm{HC}_\lambda^{*+1}(C),$$

where y is any choice of elements in C^* satisfying $B(y) = x$ and $(1-\lambda)(b(y)) = 0$. For such x and y, $-b(y) - x \cdot u = (b + uB)(-y)$ is exact in $C^{*+1}[[u]]$, so $I_{\lambda *} \circ (-S_\lambda) = (i_*^+ \cdot u) \circ I_{\lambda *}$, i.e., the right square commutes. ∎

Example 2.10 (S^1-equivariant homology theories [21]). Let X be a topological S^1-space, namely a topological space with a continuous S^1-action $F^X : S^1 \times X \to X$. Let $(C_*, b) = (S_*(X), \partial)$ be the singular chain complex of X, and define the rotation operator $B = J : S_*(X) \to S_{*+1}(X)$ by

$$J(a) := F_*^X([S^1] \times a), \quad a \in S_*(X). \tag{2.11}$$

Here $[S^1] \in S_1(S^1)$ is the fundamental cycle of S^1, namely $[S^1] = \pi_{S^1}^{\Delta^1} : \Delta^1 = [0,1] \to \mathbb{R}/\mathbb{Z} = S^1$, and $\times$ is the simplicial cross product induced by standard decomposition of $\Delta^l \times \Delta^k$ into $(k+l)$-simplices (see [19, page 278]). Then $\partial J + J\partial = 0$ since

$$\partial([S^1] \times a) = \partial[S^1] \times a + (-1)^{\deg[S^1]}[S^1] \times \partial a = -[S^1] \times \partial a.$$

To see $J^2 = 0$, let us write down the cross product with $[S^1]$ explicitly. For $k \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, \ldots, k\}$, consider the embeddings $\iota_{k,j} : \Delta^{k+1} \to \Delta^1 \times \Delta^k$ defined by

$$\iota_{k,j}(t_1, \ldots, t_{k+1}) := (t_{j+1}, (t_1, \ldots, t_j, t_{j+2}, \ldots, t_{k+1})),$$

then for $(\sigma : \Delta^k \to X) \in S_k(X)$,

$$[S^1] \times \sigma = \sum_{0 \leq j \leq k} (-1)^j (\pi_{S^1}^{\Delta^1} \times \sigma) \circ \iota_{k,j} \in S_{k+1}(S^1 \times X).$$

Let $F^{S^1} : S^1 \times S^1 \to S^1$, $([t], [t']) \mapsto [t + t']$ be the rotation S^1-action on S^1, then

$$F_*^{S^1}([S^1] \times [S^1]) = F^{S^1} \circ (\pi_{S^1}^{\Delta^1} \times \pi_{S^1}^{\Delta^1}) \circ \iota_{1,0} - F^{S^1} \circ (\pi_{S^1}^{\Delta^1} \times \pi_{S^1}^{\Delta^1}) \circ \iota_{1,1} = 0.$$

From the commutative diagram

$$\begin{array}{ccc} S^1 \times S^1 \times X & \xrightarrow{F^{S^1} \times \mathrm{id}_X} & S^1 \times X \\ \downarrow{\scriptstyle \mathrm{id}_{S^1} \times F^X} & & \downarrow{\scriptstyle F^X} \\ S^1 \times X & \xrightarrow{F^X} & X, \end{array} \qquad \begin{array}{ccc} ([t], [t'], x) & \longmapsto & ([t + t'], x) \\ \downarrow & & \downarrow \\ ([t], F^X([t'], x)) & \longmapsto & F^X([t + t'], x) \end{array}$$

we conclude that for any $a \in S_*(X)$,

$$J^2(a) = F_*^X([S^1] \times F_*^X([S^1] \times a)) = F_*^X(F_*^{S^1}([S^1] \times [S^1]) \times a)) = 0.$$

For the mixed chain complex $(C_*, b, B) = (S_*(X), \partial, J)$, there is a natural isomorphism ([21, Lemma 5.1])

$$\mathrm{HC}_*^{[v^{-1}]}(S(X)) \cong H_*^{S^1}(X) := H_*(X \times_{S^1} ES^1),$$

namely $\mathrm{HC}_*^{[v^{-1}]}(S(X))$ is isomorphic to the S^1-equivariant homology of X, i.e., homology of the homotopy quotient (Borel construction). The other two cyclic homology groups of $(S_*(X), \partial, J)$ are called the *negative* and *periodic* S^1*-equivariant homology* of X, and are denoted by

$$G_*^{S^1}(X) := \mathrm{HC}_*^{[[v]]}(S(X)), \quad \widehat{H}_*^{S^1}(X) := \mathrm{HC}_*^{[[v,v^{-1}]}(S(X)),$$

respectively. The tautological exact sequence (2.1) translates into

$$\cdots \to G_*^{S^1}(X) \to \widehat{H}_*^{S^1}(X) \to H_{*-2}^{S^1}(X) \to G_{*-1}^{S^1}(X) \to \cdots, \tag{2.12}$$

and the Connes–Gysin exact sequences (2.2) translate into

$$\cdots \to H_*(X) \to H_*^{S^1}(X) \to H_{*-2}^{S^1}(X) \to H_{*-1}(X) \to \cdots, \tag{2.13a}$$

$$\cdots \to G_{*+2}^{S^1}(X) \to G_*^{S^1}(X) \to H_*(X) \to G_{*+1}^{S^1}(X) \to \cdots. \tag{2.13b}$$

We end this example by mentioning that (2.13a) coincides with the Gysin sequence associated to the S^1-fibration $X \times ES^1 \to X \times_{S^1} ES^1$.

Remark 2.11. There seems to be no interpretation of $G_*^{S^1}(X)$ and $\widehat{H}_*^{S^1}(X)$ as homology groups of some spaces naturally associated to X, but there are homotopy theoretic interpretations. For example, when X is a (finite) S^1-CW complex, [2, Lemma 4.4] says $G_*^{S^1}(X)$ is naturally isomorphic to the homotopy groups of the homotopy fixed point spectrum $(\mathbf{H} \wedge X_+)^{hS^1}$, where $\mathbf{H}$ is the Eilenberg–MacLane spectrum $\{K(\mathbb{Z}, n)\}$.

3. A cocyclic complex and an ∞-quasi-isomorphism

Let X be a topological space with S^1-action $F^X : S^1 \times X \to X$. There is a cocyclic structure on $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$, where $\delta_i^{X \times \Delta^k} := \mathrm{id}_X \times \delta_i^{\Delta^k}$, $\sigma_i^{X \times \Delta^k} := \mathrm{id}_X \times \sigma_i^{\Delta^k}$ ($\delta_i^{\Delta^k}$, $\sigma_i^{\Delta^k}$ are as in (2.3)), and $\tau_k^{X \times \Delta^k} : X \times \Delta^k \to X \times \Delta^k$ is defined by

$$\tau_k^{X \times \Delta^k}(x, t_1, \ldots, t_k) := (F^X([t_1], x), t_2 - t_1, \ldots, t_k - t_1, 1 - t_1).$$

Taking singular chains of the cocyclic space $\{X \times \Delta^k\}_{k \geq 0}$ yields a cocyclic chain complex $\{S_*(X \times \Delta^k)\}_{k \geq 0}$. Let us denote the associated mixed complex by

$$(S_*^{X\Delta}, b, B) := \Big(\prod_{k \geq 0} S_{*+k}(X \times \Delta^k), \partial + \delta, Ns(1 - \lambda) \Big).$$

The S^1-action on X extends to $X \times \Delta^k$ where the S^1-action on Δ^k is trivial, and then the rotation operator $J : S_*(X) \to S_{*+1}(X)$ defined in Example 2.10 extends component-by-component to $S_*^{X\Delta}$ by

$$J : S_*^{X\Delta} \to S_{*+1}^{X\Delta}, \quad (x_k)_{k\geq 0} \mapsto (J(x_k))_{k\geq 0}.$$

By Example 2.10, $J^2 = 0$ and $\partial J + J\partial = 0$. Since S^1 acts trivially on Δ^k, J commutes with δ_i, σ_i. It follows that $\delta J + J\delta = 0$ and $J(S^{X\Delta,\mathrm{nm}}) \subset S^{X\Delta,\mathrm{nm}}$, so $(S_*^{X\Delta}, b, J)$, $(S_*^{X\Delta,\mathrm{nm}}, b, J)$ are also mixed complexes. J also commutes with $\tau_k^{X\times\Delta^k}$ because of the commutative diagram

$$\begin{array}{ccc} S^1 \times X \times \Delta^k & \xrightarrow{\mathrm{id}_{S^1}\times\tau_k^{X\times\Delta^k}} & S^1 \times X \times \Delta^k \\ \downarrow{\scriptstyle F^{X\times\Delta^k}} & & \downarrow{\scriptstyle F^{X\times\Delta^k}} \\ X \times \Delta^k & \xrightarrow{\tau_k^{X\times\Delta^k}} & X \times \Delta^k, \end{array} \tag{3.1}$$

$$\begin{array}{ccc} ([t], x, t_1, \ldots, t_k) & \longmapsto & ([t], F^X([t_1], x), t_2 - t_1, \ldots, 1 - t_1) \\ \downarrow & & \downarrow \\ (F^X([t], x), t_1, \ldots, t_k) & \longmapsto & (F^X([t + t_1], x), t_2 - t_1, \ldots, 1 - t_1), \end{array}$$

so $JB + BJ = 0$. We will analyze the relationship between the mixed complexes

$$(S_*^{X\Delta}, b, B), \quad (S_*^{X\Delta}, b, J), \quad (S_*(X), \partial, J).$$

If there is no risk of confusion, we shall write $\delta_i^{X\times\Delta^k}$, $\sigma_i^{X\times\Delta^k}$, $\tau_k^{X\times\Delta^k}$ and the induced maps on singular chain complexes as $\delta_i, \sigma_i, \tau_k$ for short. Note that δ_i, σ_i do not involve S^1-action, so if we forget the S^1-action, there is still a total complex $(S_*^{X\Delta}, b = \partial + \delta)$ from the cosimplicial chain complex $\{S_*(X \times \Delta^k)\}_{k\in\mathbb{Z}_{\geq 0}}$.

Let us state the main theorem of this section.

Theorem 3.1. *Let X be a topological S^1-space. Then for both of the mixed complex structures (b, B) and (b, J) on $S_*^{X\Delta} = \prod_{k\geq 0} S_{*+k}(X \times \Delta^k)$, there are natural isomorphisms*

$$\begin{aligned} \mathrm{HC}_*^{[v^{-1}]}(S^{X\Delta}) &\cong H_*^{S^1}(X) \quad \text{as } \mathbb{R}[v^{-1}]\text{-modules}, \\ \mathrm{HC}_*^{[[v,v^{-1}]]}(S^{X\Delta}) &\cong \hat{H}_*^{S^1}(X) \quad \text{as } \mathbb{R}[[v, v^{-1}]]\text{-modules}, \\ \mathrm{HC}_*^{[[v]]}(S^{X\Delta}) &\cong G_*^{S^1}(X) \quad \text{as } \mathbb{R}[[v]]\text{-modules}. \end{aligned}$$

Furthermore, these isomorphisms throw the (tautological and Connes–Gysin) exact sequences (2.1), (2.2) for cyclic homology theories onto the (tautological and Gysin) exact sequences for S^1-equivariant homology theories.

Proof. The statement about isomorphisms is a consequence of Lemma 2.3, Corollary 3.5 and Proposition 3.7 below. The statement about long exact sequences is then a consequence of Lemma 2.4. ∎

Corollary 3.2. *For any topological S^1-space X, Connes' version of cyclic cohomology of the cocyclic chain complex $\{S_*(X \times \Delta^k)\}_{k\in\mathbb{Z}_{\geq 0}}$ is naturally isomorphic to the negative S^1-equivariant homology of X.* ∎

Lemma 3.3. *For any topological space X, the projection chain map*

$$\mathrm{pr}_0 : (S_*^{X\Delta}, b) \to (S_*(X), \partial); \quad (c_k)_{k\geq 0} \mapsto c_0$$

is a quasi-isomorphism.

Proof. Since pr_0 is surjective, it suffices to prove $\mathrm{Ker}(\mathrm{pr}_0)_* = \prod_{k\geq 1} S_{*+k}(X \times \Delta^k)$ is b-acyclic. Let us write $\widetilde{S}_* := \mathrm{Ker}(\mathrm{pr}_0)_*$ and consider the decreasing filtration $\mathcal{F}_p$ ($p \in \mathbb{Z}_{\geq 1}$) on $\widetilde{S}$ defined by $\mathcal{F}_p\widetilde{S}_* := \prod_{k\geq p} S_{*+k}(X \times \Delta^k)$. The E_1-page of the spectral sequence of this filtration is divided into columns indexed by $q \in \mathbb{Z}_{\geq 0}$, each of which looks like

$$0 \to H_q(X \times \Delta^1) \xrightarrow{\delta_*} H_q(X \times \Delta^2) \xrightarrow{\delta_*} H_q(X \times \Delta^3) \xrightarrow{\delta_*} \cdots . \tag{3.2}$$

For each $k \geq 1$, the map $p_k := \sigma_0\sigma_1 \cdots \sigma_{k-1} : X \times \Delta^k \to X \times \Delta^0 = X$, $(x, t_1, \ldots, t_k) \mapsto (x, 0) = x$ is a homotopy equivalence. Since $p_{k+1} = p_k\sigma_k$ and $\sigma_j\delta_i = \mathrm{id}$ $(i = j, j+1)$, we conclude that for any $k \geq 1$ and $0 \leq i \leq k$,

$$H_*(X \times \Delta^{k-1}) \xrightarrow{(\delta_i)_*} H_*(X \times \Delta^k)$$

is an isomorphism such that $(\delta_i)_* = (\sigma_{k-1})_*^{-1}$. Then since

$$\delta|_{S_q(\Delta^{k-1}\times X)} = (-1)^{q+k} \sum_{i=0}^{k} (-1)^i \delta_i,$$

(3.2) is nothing but

$$0 \to H_q(X \times \Delta^1) \xrightarrow[\cong]{(-1)^q(\sigma_1)_*^{-1}} H_q(X \times \Delta^2) \xrightarrow{0} H_q(X \times \Delta^3) \xrightarrow[\cong]{(-1)^q(\sigma_3)_*^{-1}} \cdots .$$

Thus all E_2-terms vanish. Finally, since the filtration $\mathcal{F}_p$ on $\widetilde{S}$ is complete (namely $\widetilde{S} = \varprojlim \widetilde{S}/\mathcal{F}_p\widetilde{S}$) and bounded above (since $\widetilde{S}_* = \mathcal{F}_1\widetilde{S}_*$), a standard convergence theorem [32, Theorem 5.5.10(2)] gives $H_*(\widetilde{S}, b) = 0$. ∎

Remark 3.4. The proof of Lemma 3.3 implies that more generally, for a cosimplicial complex $\{(C(k)_*, \partial)\}_{k\geq 0}$, if $\sigma_0\sigma_1 \cdots \sigma_{k-1} : C(k) \to C(0)$ is a quasi-isomorphism for each $k \geq 1$, then so is $\mathrm{pr}_0 : (\prod_{k\geq 0} C(k)_{*+k}, \partial + \delta) \to (C(0)_*, \partial)$ ([20, Lemma 8.3]).

Corollary 3.5. *For any topological S^1-space X, $\mathrm{pr}_0 : (S_*^{X\Delta}, b, J) \to (S_*(X), \partial, J)$ is a mixed complex quasi-isomorphism.* ∎

Note that $S_*(X) = S_*^{X\Delta}(0) = S_*^{X\Delta,\mathrm{nm}}(0)$ by a vacuum normalized condition. Since $(S_*^{X\Delta,\mathrm{nm}}, b) \hookrightarrow (S_*^{X\Delta}, b)$ is a quasi-isomorphism, Lemma 3.3 and Corollary 3.5 also hold

true if $S_*^{X\Delta}$ is replaced by $S_*^{X\Delta,\mathrm{nm}}$. In the following, we may use $S_*^{X\Delta,\mathrm{nm}}$ to simplify calculation involving Connes' operator B. One could also stick with $S_*^{X\Delta}$, though.

Recall the augmentation map $\varepsilon : S_0(X) \to \mathbb{R}$, $\sum \lambda_i \cdot (\Delta^0 \xrightarrow{u_i} X) \mapsto \sum \lambda_i$.

Lemma 3.6. *Consider the topological S^1-space S^1 with rotation action on itself.*

(1) *There exists a sequence of elements $\{\xi^n = (\xi_k^n)_{k\geq 0} \in S_{2n}^{S^1\Delta,\mathrm{nm}}\}_{n\in\mathbb{Z}_{\geq 0}}$ such that*

$$\varepsilon(\xi_0^0) = 1, \quad b(\xi^0) = 0, \quad b(\xi^n) = (J - B)(\xi^{n-1}) \quad (n \geq 1).$$

(2) *Suppose $\{\xi^n\}_{n\geq 0}$, $\{\xi'^n\}_{n\geq 0}$ both satisfy the conditions in (1). Then there exists a sequence of elements $\{\eta^n = (\eta_k^n)_{k\geq 0} \in S_{2n+1}^{S^1\Delta,\mathrm{nm}}\}_{n\in\mathbb{Z}_{\geq 0}}$ such that*

$$\xi^0 - \xi'^0 = b(\eta^0), \quad \xi^n - \xi'^n = b(\eta^n) - (J - B)(\eta^{n-1}) \quad (n \geq 1).$$

Proof. (1) Consider the isomorphisms $(\mathrm{pr}_0)_* : H_0(S^{S^1\Delta,\mathrm{nm}}, b) \xrightarrow{\cong} H_0(S^1)$ from Lemma 3.3 and $\varepsilon_* : H_0(S^1) \xrightarrow{\cong} \mathbb{R}$ induced by augmentation. Choose a 0-cycle ξ^0 in the homology class $(\varepsilon_* \circ (\mathrm{pr}_0)_*)^{-1}(1) \in H_0(S^{S^1\Delta,\mathrm{nm}}, b)$, then $\xi^0 = (\xi_k^0)_{k\geq 0}$ is as desired. Next, since $bB = -Bb$ and $bJ = -Jb$, $B(\xi^0)$ and $J(\xi^0)$ are 1-cycles. We claim that they are homologous. Since pr_0 is a quasi-isomorphism, it suffices to look at $\mathrm{pr}_0(B(\xi^0))$ and $\mathrm{pr}_0(J(\xi^0))$. By definition,

$$\mathrm{pr}_0(B(\xi^0)) = (Ns(\xi^0))_0 = \sigma_0\tau_1(\xi_1^0), \quad \mathrm{pr}_0(J(\xi^0)) = J(\xi_0^0) = F_*^{S^1}([S^1] \times \xi_0^0).$$

By construction, $\xi_0^0 \in S_0(S^1)$ is homologous to the map $\Delta^0 \ni 0 \mapsto [0] \in S^1$, and $\xi_1^0 \in S_1(S^1 \times \Delta^1)$ is homologous to the map $\Delta^1 \ni t \mapsto ([0], t) \in S^1 \times \Delta^1$. So $\mathrm{pr}_0(B(\xi^0))$, $\mathrm{pr}_0(J(\xi^0))$ are homologous to

$$\Delta^1 \to S^1 \times \Delta^1 \xrightarrow{\tau_1} S^1 \times \Delta^1 \xrightarrow{\sigma_0} S^1; \qquad t \mapsto ([0], t) \mapsto ([t], 1 - t) \mapsto [t],$$

$$\Delta^1 \to \Delta^1 \times \Delta^0 \to S^1 \times S^1 \xrightarrow{F^{S^1}} S^1; \qquad t \mapsto (t, 0) \mapsto ([t], [0]) \mapsto [t],$$

respectively. Namely they are both homologous to $[S^1]$. This proves the existence of $\xi^1 \in S_2^{S^1\Delta,\mathrm{nm}}$ satisfying $b(\xi^1) = (J - B)(\xi^0)$. Now suppose $\xi^0, \xi^1, \ldots, \xi^n$ $(n \geq 1)$ have been chosen as desired, to find ξ^{n+1}, simply notice that $(J - B)(\xi^n)$ is a $(2n + 1)$-cycle,

$$b((J - B)(\xi^n)) = -(J - B)(b(\xi^n)) = -(J - B)^2(\xi^{n-1}) = 0,$$

where $(J - B)^2 = 0$ since $J^2 = 0$, $B^2 = 0$ and $JB + BJ = 0$ (see (3.1)). Since

$$H_{2n+1}(S^{S^1\Delta,\mathrm{nm}}, b) \cong H_{2n+1}(S^1) = 0 \quad (n \geq 1),$$

$(J - B)(\xi^n)$ is exact, i.e., ξ^{n+1} exists.

(2) By construction, ξ^0 is homologous to ξ'^0, so η^0 exists. To inductively find η^n for $n \geq 1$, simply check that $\xi^n - \xi'^n + (J - B)(\eta^{n-1})$ is a $2n$-cycle, which is then exact since $H_{2n}(S^1) = 0$ $(n \geq 1)$. ∎

Proposition 3.7. *Let X be a topological S^1-space. Denote the transposition $X \times S^1 \to S^1 \times X$ by ν.*

(1) *Choose $\xi = \{\xi^n\}_{n\geq 0}$ as in Lemma 3.6 (1). Define a sequence of linear maps $f^\xi = \{f_n^\xi : S_*(X) \to S_{*+2n}^{X\Delta,\mathrm{nm}}\}_{n\geq 0}$ by*

$$f_n^\xi(a) := \big((F^{X\times\Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}))_*(a \times \xi_k^n)\big)_{k\geq 0}.$$

Then f^ξ is an ∞-quasi-isomorphism from $(S_*(X), \partial, J)$ to $(S_*^{X\Delta,\mathrm{nm}}, b, B)$.

(2) *For two choices ξ, ξ', the ∞-quasi-isomorphisms f^ξ, $f^{\xi'}$ are ∞-homotopic.*

Proof. (1) The verification of $f_0^\xi \circ \partial - b \circ f_0^\xi = 0$ is simpler than $f_n^\xi \circ \partial - b \circ f_n^\xi = B \circ f_{n-1}^\xi - f_{n-1}^\xi \circ J$ $(n \geq 1)$, so we omit it. Let us write $F_k := F^{X\times\Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}) : X \times S^1 \times \Delta^k \to X \times \Delta^k$. For $n \geq 1$,

$$\begin{aligned}\big((f_n^\xi \circ \partial - b \circ f_n^\xi)(a)\big)_k &= (F_k)_*(\partial a \times \xi_k^n - \partial(a \times \xi_k^n) - \delta(a \times \xi_{k-1}^n)) \\ &= (F_k)_*((-1)^{|a|}a \times (-\partial\xi_k^n - \delta\xi_{k-1}^n)) \\ &= (-1)^{|a|}(F_k)_*(a \times (B(\xi_{k+1}^{n-1}) - J(\xi_k^{n-1}))),\end{aligned}$$

where the last equality follows from $b(\xi^n) = (J - B)(\xi^{n-1})$. Now introduce maps

$$\begin{aligned} G_k : X \times S^1 \times \Delta^k &\to X \times \Delta^k, \\ (x, [t], t_1, \ldots, t_k) &\mapsto (F^X([t + t_1], x), t_2 - t_1, \ldots, 1 - t_1), \\ H_k : X \times S^1 \times S^1 \times \Delta^k &\to X \times \Delta^k, \\ (x, [t], [t'], t_1, \ldots, t_k) &\mapsto (F^X([t + t'], x), t_1, \ldots, t_k), \end{aligned}$$

then

$$\begin{aligned} G_k &= F_k \circ (\mathrm{id}_X \times \tau_k^{S^1\times\Delta^k}) = \tau_k^{X\times\Delta^k} \circ F_k, \\ H_k &= F_k \circ (\mathrm{id}_X \times F^{S^1\times\Delta^k}) = F_k \circ (F^X \times \mathrm{id}_{S^1\times\Delta^k}) \circ (\nu \times \mathrm{id}_{S^1\times\Delta^k}). \end{aligned}$$

It follows that

$$\begin{aligned} (-1)^{|a|}(F_k)_*(a \times B(\xi_{k+1}^{n-1})) &= B((F_k)_*(a \times \xi_{k+1}^{n-1})), \\ (-1)^{|a|}(F_k)_*(a \times J(\xi_k^{n-1})) &= (F_k)_*(J(a) \times \xi_k^{n-1}). \end{aligned}$$

This implies $(f_n^\xi \circ \partial - b \circ f_n^\xi)(a) = (B \circ f_{n-1}^\xi - f_{n-1}^\xi \circ J)(a)$, so f^ξ is an ∞-morphism. It remains to show that f_0^ξ is a quasi-isomorphism. Since ξ_0^0 is homologous to the 0-chain $\Delta^0 \to S^1$, $0 \mapsto [0]$, $\mathrm{pr}_0 \circ f_0^\xi$ is chain homotopic to $\mathrm{id}_{S_*(X)}$. Since pr_0 is a quasi-isomorphism, so is f_0^ξ.

(2) Choose η as in Lemma 3.6 (2). Define a sequence of linear maps $h^\eta = \{h_n^\eta : S_*(X) \to S_{*+2n+1}^{X\Delta,\mathrm{nm}}\}_{n\geq 0}$ by

$$h_n^\eta(a) := (-1)^{|a|}\big((F^{X\times\Delta^k} \circ (\nu \times \mathrm{id}_{\Delta^k}))_*(a \times \eta_k^n)\big)_{k\geq 0}.$$

Then similar calculation as before shows h^η is an ∞-homotopy between f^ξ and $f^{\xi'}$. ∎

4. The story of differentiable spaces

4.1. Differentiable spaces and de Rham chains

Materials in this subsection are collected from Irie [20]. The notion of differentiable spaces is a modification of that utilized by Chen [6], and the notion of de Rham chains is inspired by an idea of Fukaya [10].

Let $\mathcal{U} := \coprod_{n\geq m\geq 0} \mathcal{U}_{n,m}$ where $\mathcal{U}_{n,m}$ denotes the set of oriented m-dimensional C^∞-submanifolds of $\mathbb{R}^n$. Let X be a set. A *differentiable structure* $\mathcal{P}(X)$ on X is a family of maps $\{(U,\varphi)\}$ called *plots*, such that

- every plot is a map φ from some $U \in \mathcal{U}$ to X;
- if $\varphi : U \to X$ is a plot, $U' \in \mathcal{U}$ and $\theta : U' \to U$ is a submersion, then $\varphi \circ \theta : U' \to X$ is a plot.

A *differentiable space* is a pair of a set and a differentiable structure on it. A map $f : X \to Y$ between differentiable spaces is called *smooth*, if $(U, f \circ \varphi) \in \mathcal{P}(Y)$ for any $(U,\varphi) \in \mathcal{P}(X)$. A subset of a differentiable space and the product of a family of differentiable spaces admit naturally induced differentiable structures ([20, Example 4.2 (iii), (iv)]).

Remark 4.1. Differentiable structures are defined on *sets* rather than topological spaces. For later purpose, we say a differentiable structure and a topology on a set X are *compatible* if every plot is continuous.

Example 4.2. Here are some important examples of differentiable spaces.

(1) Let M be a C^∞-manifold. Consider two differentiable structures on it:
   (a) define $(U,\varphi) \in \mathcal{P}(M)$ if $\varphi : U \to M$ is a C^∞-map;
   (b) define $(U,\varphi) \in \mathcal{P}(M_{\mathrm{reg}})$ if $\varphi : U \to M$ is a (C^∞-)submersion.

   The set-theoretic identity map $\mathrm{id}_M : M_{\mathrm{reg}} \to M$ is smooth, but its inverse is not.

(2) Let $\mathcal{L}M := C^\infty(\mathbb{S}^1, M)$ be the smooth free loop space of M, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. There is a differentiable structure $\mathcal{P}(\mathcal{L}^M)$ on $\mathcal{L}M$ defined by: $(U,\varphi) \in \mathcal{P}(\mathcal{L})$ iff $U \times \mathbb{S}^1 \to M$, $(u,[t]) \mapsto \varphi(u)(t)$ is a C^∞-map.

(3) For each $k \in \mathbb{Z}_{\geq 0}$, the smooth free loop space of M with k inner marked points, denoted by $\mathcal{L}_{k+1}M$, is defined as

$$\{(\gamma, t_1, \ldots, t_k) \in \mathcal{L}M \times \Delta^k \mid \partial_t^m \gamma(0) = \partial_t^m \gamma(t_j) = 0\,(1 \leq \forall j \leq k,\ \forall m \geq 1)\}.$$

   It has induced differentiable structure $\mathcal{P}(\mathcal{L}^M_{k+1})$ as a subspace of $\mathcal{L}^M \times \Delta^k$, where Δ^k is viewed as a subspace of $\mathbb{R}^k$ with the differentiable structure in (1a).

(4) The smooth free Moore path space of M, denoted by ΠM, is defined as

$$\{(T,\gamma) \mid T \in \mathbb{R}_{\geq 0},\ \gamma \in C^\infty([0,T], M),\ \partial_t^m \gamma(0) = \partial_t^m \gamma(T) = 0\,(\forall m \geq 1)\}.$$

   Consider two differentiable structures $\mathcal{P}(\Pi^M), \mathcal{P}(\Pi^M_{\mathrm{reg}})$ on ΠM.

(a) Define $(U,\varphi)\in\mathcal{P}(\Pi^M)$ if $\varphi=(\varphi_T,\varphi_\gamma):U\to\Pi M$ satisfies the following conditions:

- $\varphi_T:U\to\mathbb{R}_{\geq 0}$ is a C^∞-map.
- The map

$$\widetilde{U}:=\{(u,t)\mid u\in U,\, t\in[0,\varphi_T(u)]\}\to M;\quad (u,t)\mapsto\varphi_\gamma(u)(t)$$

extends to a C^∞-map from an open neighborhood of $\widetilde{U}$ in $U\times\mathbb{R}$ to M.

(b) Define $(U,\varphi)\in\mathcal{P}(\Pi^M_{\mathrm{reg}})$ if: $(U,\varphi)\in\mathcal{P}(\Pi^M)$ and the map $U\to M$, $u\mapsto\varphi_\gamma(u)(t_0)$ is a submersion for $t_0=0,T$.

(5) For each $k\in\mathbb{Z}_{\geq 0}$, the smooth free Moore loop space of M with k inner marked points, denoted by $\mathcal{L}_{k+1}M$, is defined as

$$\begin{aligned}\{((T_0,\gamma_0),\dots,(T_k,\gamma_k))\in(\Pi M)^{k+1}\mid{}&\gamma_j(T_j)=\gamma_{j+1}(0)\,(0\leq j\leq k-1),\\ &\gamma_k(T_k)=\gamma_0(0)\}\\ =\{(T,\gamma,t_1,\dots,t_k)\in\Pi M\times\mathbb{R}^k\mid{}&0\leq t_1\leq\cdots\leq t_k\leq T,\,\gamma(0)=\gamma(T),\\ &\partial_t^m\gamma(t_j)=0\,(1\leq\forall j\leq k,\,\forall m\geq 1)\}.\end{aligned}$$

Apparently there are two ways to endow the set $\mathcal{L}_{k+1}M$ with differentiable structures, namely as a subset of $(\Pi M)^{k+1}$ or of $\Pi M\times\mathbb{R}^k$. It basically follows from [20, Lemma 7.2] that these two ways are equivalent. Let us denote by $\mathcal{L}^M_{k+1}$ (resp. $\mathcal{L}^M_{k+1,\mathrm{reg}}$) the differentiable space obtained from Π^M (resp. Π^M_{reg}).

Note that the inclusion of sets $\mathcal{L}_{k+1}M=\{T=1\}\subset\mathcal{L}_{k+1}M$, induced by the inclusion $\mathcal{L}_1M\times\Delta^k\subset\Pi M\times\mathbb{R}^k$, is also an inclusion of differentiable spaces $\mathcal{L}^M_{k+1}\hookrightarrow\mathcal{L}^M_{k+1}$.

The *de Rham chain complex* $(C^{\mathrm{dR}}_*(X),\partial)$ of a differentiable space X is defined as follows. For $n\in\mathbb{Z}$, let $\widetilde{C}^{\mathrm{dR}}_n(X):=\bigoplus_{(U,\varphi)\in\mathcal{P}(X)}\Omega_c^{\dim U-n}(U)$. For any $(U,\varphi)\in\mathcal{P}(X)$ and $\omega\in\Omega_c^{\dim U-n}(U)$, denote the image of ω under the natural inclusion $\Omega_c^{\dim U-n}(U)\hookrightarrow\widetilde{C}^{\mathrm{dR}}_n(X)$ by (U,φ,ω). Let $Z_n\subset\widetilde{C}^{\mathrm{dR}}_n(X)$ be the subspace spanned by all elements of the form $(U,\varphi,\pi_!\omega')-(U',\varphi\circ\pi,\omega')$, where $(U,\varphi)\in\mathcal{P}(X)$, $U'\in\mathcal{U}$, $\omega'\in\Omega_c^{\dim U'-n}(U')$, and $\pi:U'\to U$ is a submersion. Then define $C^{\mathrm{dR}}_n(X):=\widetilde{C}^{\mathrm{dR}}_n(X)/Z_n$. By abuse of notation we still denote the image of (U,φ,ω) under the quotient map $\widetilde{C}^{\mathrm{dR}}_n(X)\to C^{\mathrm{dR}}_n(X)$ by (U,φ,ω). Then $\partial:C^{\mathrm{dR}}_*(X)\to C^{\mathrm{dR}}_{*-1}(X)$ is defined by $\partial(U,\varphi,\omega):=(U,\varphi,d\omega)$. The homology of $(C^{\mathrm{dR}}_*(X),\partial)$ is denoted by $H^{\mathrm{dR}}_*(X)$.

Remark 4.3. For any oriented C^∞-manifold M, there exists $n\in\mathbb{Z}_{\geq 0}$ and an embedding $\iota:M\hookrightarrow\mathbb{R}^n$. Then $(\iota(M),\iota^{-1})\in\mathcal{P}(M_{\mathrm{reg}})\subset\mathcal{P}(M)$, and $(\iota(M),\iota^{-1},(\iota^{-1})^*\omega)\in C^{\mathrm{dR}}_*(M_{\mathrm{reg}})\subset C^{\mathrm{dR}}_*(M)$ for any $\omega\in\Omega_c(M)$. Such a de Rham chain is independent of choices of n and ι, and by abuse of notation we write it as (M,id_M,ω). If M is closed oriented, we call $(M,\mathrm{id}_M,1)$ the *fundamental de Rham cycle* of M (or M_{reg}).

Let X, Y be differentiable spaces. The *cross product* on de Rham chains is a chain map $C_k^{\mathrm{dR}}(X) \otimes C_l^{\mathrm{dR}}(Y) \to C_{k+l}^{\mathrm{dR}}(X \times Y)$, defined by

$$(U, \varphi, \omega) \times (V, \eta, \psi) := (-1)^{l \cdot \dim U} (U \times V, \varphi \times \psi, \omega \times \eta). \tag{4.1}$$

4.2. S^1-equivariant homology of differentiable S^1-spaces

Let X be a *differentiable S^1-space*, namely X is a differentiable space with a smooth map $F^X : S^1 \times X \to X$, where S^1 is endowed with the differentiable structure in Example 4.2 ((1a)). Let $(S^1, \mathrm{id}_{S^1}, 1) \in C_1^{\mathrm{dR}}(S^1)$ be the fundamental de Rham 1-cycle of S^1. Define

$$J : C_*^{\mathrm{dR}}(X) \to C_{*+1}^{\mathrm{dR}}(X); \quad a \mapsto F_*^X((S^1, \mathrm{id}_{S^1}, 1) \times a),$$

then J is clearly an anti-chain map. We claim $J^2 = 0$. Let $g : S^1 \times S^1 \to S^1$ be the smooth map $([t], [t']) \mapsto [t + t']$, then by the same arguments as in Example 2.10, to see $J^2 = 0$, it suffices to prove $g_*((S^1, \mathrm{id}_{S^1}, 1) \times (S^1, \mathrm{id}_{S^1}, 1)) = 0 \in C_2^{\mathrm{dR}}(S^1)$. This is easy, as we can see by the following:

$$\begin{aligned} g_*((S^1, \mathrm{id}_{S^1}, 1) \times (S^1, \mathrm{id}_{S^1}, 1)) &\overset{(4.1)}{=\!=} -g_*((S^1 \times S^1, \mathrm{id}_{S^1 \times S^1}, 1)) \\ &= -(S^1 \times S^1, \mathrm{id}_{S^1} \circ g, 1) = -(S^1, \mathrm{id}_{S^1}, g_!(1)) = 0. \end{aligned}$$

The middle equality on the second line holds since g is a submersion. Thus $(C_*^{\mathrm{dR}}(X), \partial, J)$ is a mixed chain complex. One can then define the positive (ordinary), periodic and negative "S^1-equivariant de Rham homology" of X as the $\mathrm{HC}_*^{[v^{-1}]}$, $\mathrm{HC}_*^{[[v]]}$ and $\mathrm{HC}_*^{[[v, v^{-1}]}$ versions of cyclic homology of $(C_*^{\mathrm{dR}}(X), \partial, J)$.

Consider Δ^k as a differentiable subspace of $\mathbb{R}^k$. Then the cocyclic maps $\delta_i, \sigma_i, \tau_k$ among $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$, defined by the same formulas as in Section 3, are smooth maps between differentiable spaces. So $\{X \times \Delta^k\}_{k \in \mathbb{Z}_{\geq 0}}$ is a cocyclic differentiable space and $\{C_*^{\mathrm{dR}}(X \times \Delta^k)\}_{k \in \mathbb{Z}_{\geq 0}}$ is a cocyclic chain complex, which gives rise to a mixed complex

$$(C_*^{\mathrm{dR}, X\Delta}, b, B) := \Big(\prod_{k \geq 0} C_{*+k}^{\mathrm{dR}}(X \times \Delta^k), \partial + \delta, Ns(1 - \lambda) \Big).$$

The smooth S^1-action $S^1 \times X \to X$ also extends trivially to $S^1 \times X \times \Delta^k \to X \times \Delta^k$ and gives a mixed complex $(C_*^{X\Delta}, b, J)$.

There is a counterpart of Theorem 3.1 for differentiable S^1-spaces, whose proof is also similar. We omit the details since we will not make essential use of it.

The *smooth singular chain complex* $(C_*^{\mathrm{sm}}(X), \partial)$ of a differentiable space X, introduced in [20, Section 4.7], is defined in a similar way as the singular chain complex of topological spaces, except that only "strongly smooth" maps $\Delta^k \to X$ are considered. The homology of $(C_*^{\mathrm{sm}}(X), \partial)$ is denoted by $H_*^{\mathrm{sm}}(X)$.

Smooth singular homology is related to singular homology and de Rham homology in the following way.

- Let X be a differentiable space with a fixed compatible topology (Remark 4.1). Then every strongly smooth map $\Delta^k \to X$ is continuous, hence there is a natural inclusion $(C_*^{\mathrm{sm}}(X), \partial) \hookrightarrow (S_*(X), \partial)$.
- C_*^{sm}, C_*^{dR} are functors from the category of differentiable spaces to the category of chain complexes. Given a cocycle $u = (u_k)_{k\geq 0} \in C_0^{\mathrm{dR},\mathrm{pt}\Delta} = \prod_{k\geq 0} C_k^{\mathrm{dR}}(\Delta^k)$ in the class $1 \in \mathbb{R} \cong H_0^{\mathrm{dR}}(\mathrm{pt}) \cong H_0^{\mathrm{dR},\mathrm{pt}}$, there is a natural transformation $\iota^u : C_*^{\mathrm{sm}} \to C_*^{\mathrm{dR}}$ defined by $\iota^u(X)_k : C_k^{\mathrm{sm}}(X) \to C_k^{\mathrm{dR}}(X)$, $\sigma \mapsto \sigma_*(u_k)$. The homotopy class of $\iota^u(X)$ does not depend on u (since $H_n^{\mathrm{dR},\mathrm{pt}} = 0$ when $n > 0$).

Assumption 4.4. X is a differentiable space with a fixed compatible topology, such that the chain maps discussed above induce isomorphisms $H_*(X) \xleftarrow{\cong} H_*^{\mathrm{sm}}(X) \xrightarrow{\cong} H_*^{\mathrm{dR}}(X)$.

Proposition 4.5. *Let X be a set which satisfies Assumption 4.4 and admits an S^1-action that is both smooth (with respect to the differentiable structure) and continuous (with respect to topology). Then there are natural isomorphisms $\mathrm{HC}_*^{[v^{-1}]}(C^{\mathrm{dR},X\Delta}) \cong H_*^{S^1}(X)$, $\mathrm{HC}_*^{[[v,v^{-1}]}(C^{\mathrm{dR},X\Delta}) \cong \widehat{H}_*^{S^1}(X)$, as well as $\mathrm{HC}_*^{\lambda}(C^{\mathrm{dR},X\Delta}) \cong \mathrm{HC}_*^{[[v]]}(C^{\mathrm{dR},X\Delta}) \cong G_*^{S^1}(X)$, which are compatible with tautological and Connes–Gysin long exact sequences.*

Proof. Consider the mixed complex $(C_*^{\mathrm{sm},X\Delta}, b, B)$ associated to the cocyclic complex $\{C_{*+k}^{\mathrm{sm}}(X \times \Delta^k\}_k$. For any $k \geq 0$, $X \times \Delta^k$ is a differentiable and topological S^1-space satisfying Assumption 4.4. By construction, the inclusions $(C_*^{\mathrm{sm}}(X \times \Delta^k), \partial) \hookrightarrow (S_*(X \times \Delta^k), \partial)$ commute with the cocyclic maps $\delta_i, \sigma_i, \tau_k$, so there is an inclusion $(C_*^{\mathrm{sm},X\Delta}, b, B) \hookrightarrow (S_*^{X\Delta}, b, B)$. On the other hand, given a choice of u, the chain maps $\iota^u(X \times \Delta^k)_* : C_*^{\mathrm{sm}}(X \times \Delta^k) \to C_*^{\mathrm{dR}}(X \times \Delta^k)$ commute with cocyclic maps since ι^u is a natural transformation, so we obtain a morphism $(C_*^{\mathrm{sm},X\Delta}, b, B) \to (C_*^{\mathrm{dR},X\Delta}, b, B)$, whose homotopy class does not depend on u. Now consider the following commutative diagram of chain maps:

$$\begin{array}{ccccc}
(S_*^{X\Delta}, b) & \longleftarrow & (C_*^{\mathrm{sm},X\Delta}, b) & \longrightarrow & (C_*^{\mathrm{dR},X\Delta}, b) \\
\downarrow{\scriptstyle \mathrm{pr}_0} & & \downarrow{\scriptstyle \mathrm{pr}_0} & & \downarrow{\scriptstyle \mathrm{pr}_0} \\
(S_*(X), \partial) & \longleftarrow & (C_*^{\mathrm{sm}}(X), \partial) & \longrightarrow & (C_*^{\mathrm{dR}}(X), \partial).
\end{array}$$

By Lemma 3.3 and Remark 3.4, all vertical arrows are quasi-isomorphisms, and by assumption, the arrows in the second row are quasi-isomorphisms. Thus the arrows in the first row are quasi-isomorphisms. In this way we obtain quasi-isomorphisms of mixed complexes $(S_*^{X\Delta}, b, B) \leftarrow (C_*^{\mathrm{sm},X\Delta}, b, B) \to (C_*^{\mathrm{dR},X\Delta}, b, B)$, and get the desired isomorphisms by Lemma 2.3, Theorem 3.1 and Corollary 3.2. Compatibility with long exact sequences is a consequence of Lemma 2.4 and Lemma 2.9. ∎

Example 4.6. Let M be a closed oriented C^∞-manifold. It is proved in [20, Section 5, Section 6] that Assumption 4.4 is satisfied for M, M_{reg} (with manifold topology) and $\mathcal{L}^M$ (with Fréchet topology) in Example 4.2. Moreover, $\mathcal{L}^M$ is an S^1-space that Proposition 4.5 applies to.

4.3. Application to marked Moore loop spaces

Consider the various versions of smooth loop spaces in Example 4.2. The following lemma is proved in [20, Section 7].

Lemma 4.7. *For any closed oriented C^∞-manifold M and $k \in \mathbb{Z}_{\geq 0}$, the zig-zag of smooth maps between differentiable spaces*

$$\mathcal{L}^M_{k+1,\mathrm{reg}} \xrightarrow{\mathrm{id}_{\mathcal{L}_{k+1}M}} \mathcal{L}^M_{k+1} \xleftarrow{T=1} \mathcal{L}^M_{k+1} \hookrightarrow \mathcal{L}^M \times \Delta^k$$

induces a zig-zag of isomorphisms between de Rham homology groups

$$H^{\mathrm{dR}}_*(\mathcal{L}^M_{k+1,\mathrm{reg}}) \xrightarrow{\cong} H^{\mathrm{dR}}_*(\mathcal{L}^M_{k+1}) \xleftarrow{\cong} H^{\mathrm{dR}}_*(\mathcal{L}^M_{k+1}) \xrightarrow{\cong} H^{\mathrm{dR}}_*(\mathcal{L}^M \times \Delta^k).$$

The cocyclic structure on $\{\mathcal{L}^M \times \Delta^k\}_k$ restricts to $\{\mathcal{L}^M_{k+1}\}_k$. There is also a similar structure of a cocyclic set on $\{\mathcal{L}_{k+1}M\}_k$ as follows. Regarding $\mathcal{L}_{k+1}M \subset (\Pi M)^{k+1}$, $\delta_i : \mathcal{L}_k M \to \mathcal{L}_{k+1}M$, $\sigma_i : \mathcal{L}_{k+2}M \to \mathcal{L}_{k+1}M$, $\tau_k : \mathcal{L}_{k+1}M \to \mathcal{L}_{k+1}M$ are

$$\delta_i(T,\gamma,t_1,\dots,t_{k-1}) := \begin{cases} (T,\gamma,0,t_1,\dots,t_{k-1}) & (i=0), \\ (T,\gamma,t_1,\dots,t_i,t_i,\dots,t_{k-1}) & (1\leq i\leq k-1), \\ (T,\gamma,t_1,\dots,t_{k-1},T) & (i=k), \end{cases} \tag{4.2a}$$

$$\sigma_i(T,\gamma,t_1,\dots,t_{k+1}) := (T,\gamma,t_1,\dots,\widehat{t_{i+1}},\dots,t_{k+1}) \quad (0\leq i\leq k), \tag{4.2b}$$

$$\tau_k(T,\gamma,t_1,\dots,t_k) := (T,\gamma^{t_1},t_2-t_1,\dots,t_k-t_1,T-t_1), \tag{4.2c}$$

where $\gamma^{t_1}(t) := \gamma(t+t_1)$. These cocyclic maps are smooth for both $\{\mathcal{L}^M_{k+1}\}_k$ and $\{\mathcal{L}^M_{k+1,\mathrm{reg}}\}_k$. Note that if we view $\mathcal{L}_{k+1}M \subset (\Pi M)^{k+1}$, then (4.2c) can be written as $\tau_k((T_0,\gamma_0),\dots,(T_k,\gamma_k)) = ((T_1,\gamma_1),\dots,(T_k,\gamma_k),(T_0,\gamma_0))$.

Let us write $(C^{\mathcal{L}}_*, b, B) := (\prod_{k\geq 0} C^{\mathrm{dR}}_{*+k}(\mathcal{L}^M_{k+1,\mathrm{reg}}), \partial+\delta, Ns(1-\lambda))$ for the mixed total complex of the cocyclic chain complex $\{C^{\mathrm{dR}}_*(\mathcal{L}^M_{k+1,\mathrm{reg}})\}_k$.

Proposition 4.8. *For any closed oriented C^∞-manifold M, there are natural isomorphisms $\mathrm{HC}^{[v^{-1}]}_*(C^{\mathcal{L}}) \cong H^{S^1}_*(\mathcal{L}M)$, $\mathrm{HC}^{[[v,v^{-1}]]}_*(C^{\mathcal{L}}) \cong \hat{H}^{S^1}_*(\mathcal{L}M)$, and $\mathrm{HC}^{\lambda}_*(C^{\mathcal{L}}) \cong \mathrm{HC}^{[[v]]}_*(C^{\mathcal{L}}) \cong G^{S^1}_*(\mathcal{L}M)$, which are compatible with long exact sequences.*

Proof. The smooth maps $\mathcal{L}^M_{k+1,\mathrm{reg}} \xrightarrow{\mathrm{id}} \mathcal{L}^M_{k+1} \xleftarrow{T=1} \mathcal{L}^M_{k+1} \hookrightarrow \mathcal{L}^M \times \Delta^k$ commute with cocyclic maps, inducing a zig-zag of mixed complex morphisms between the mixed total complexes associated to the cocyclic de Rham chain complexes of these cocyclic differentiable spaces. By Lemma 4.7, this is a zig-zag of mixed complex quasi-isomorphisms. The rest is obvious in view of Lemma 2.3, Proposition 4.5 and Example 4.6. ∎

5. Preliminaries on operads and algebraic structures

Let $V = \{V_i\}_{i\in\mathbb{Z}}$ be a (homologically) graded vector space.

A Lie bracket of degree $n \in \mathbb{Z}$ is a Lie bracket on $V[-n]$, namely a bilinear map $[,] : V \otimes V \to V$ of degree n satisfying shifted skew-symmetry and Jacobi identity,

$$[a,b] = -(-1)^{(|a|-n)(|b|-n)}[b,a], \quad [a,[b,c]] = [[a,b],c] + (-1)^{(|a|-n)(|b|-n)}[b,[a,c]].$$

Note that in this definition, there is no need to apply sign change (A.1).

A structure of a *Gerstenhaber algebra* is a Lie bracket of degree 1 and a graded commutative (and associative, by default) product $\cdot$ satisfying the Poisson relation

$$[a, b\cdot c] = [a,b]\cdot c + (-1)^{(|a|+1)|b|} b\cdot[a,c].$$

A structure of a *Batalin–Vilkovisky (BV) algebra* is a graded commutative product $\cdot$ and a linear map $\Delta : V_* \to V_{*+1}$ (called the BV operator) such that $\Delta^2 = 0$, and

$$\begin{aligned}\Delta(a\cdot b\cdot c) = {}& \Delta(a\cdot b)\cdot c + (-1)^{|a|} a\cdot\Delta(b\cdot c) + (-1)^{(|a|+1)|b|} b\cdot\Delta(a\cdot c)\\ & - \Delta a\cdot b\cdot c - (-1)^{|a|} a\cdot\Delta b\cdot c - (-1)^{|a|+|b|} a\cdot b\cdot\Delta c. \end{aligned} \tag{5.1}$$

By induction, the defining relation (5.1) implies that for any $k \geq 2$,

$$\begin{aligned}\Delta(a_1\cdot a_2\cdot\cdots\cdot a_k) = {}& \sum_{1\leq i<j\leq k} (-1)^{\varepsilon(i,j)}\Delta(a_i\cdot a_j)\cdot a_1\cdot\cdots\widehat{a_i}\cdots\widehat{a_j}\cdots\cdot a_k\\ & - (k-2)\sum_{1\leq i\leq k}(-1)^{|a_1|+\cdots+|a_k|} a_1\cdot\cdots\cdot\Delta a_i\cdot\cdots\cdot a_k, \end{aligned} \tag{5.2}$$

where $\varepsilon(i,j)$ is from the Koszul sign rule. By [15, Proposition 1.2], a BV algebra is equivalently a Gerstenhaber algebra with a linear map $\Delta : V_* \to V_{*+1}$ such that $\Delta^2 = 0$ and

$$[a,b] = (-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta a\cdot b - a\cdot\Delta b. \tag{5.3}$$

Following Getzler [16], a structure of a *gravity algebra* is a sequence of graded symmetric linear maps $V^{\otimes k} \to V$ $(k \geq 2)$ of degree 1, $a_1\otimes\cdots\otimes a_k \mapsto \{a_1,\dots,a_k\}$ (which we call k-th bracket), satisfying the following generalized Jacobi relations:

$$\begin{aligned}&\sum_{1\leq i<j\leq k}(-1)^{\varepsilon(i,j)}\{\{a_i,a_j\},a_1,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_k,b_1,\dots,b_l\}\\ &\quad = \begin{cases}\{\{a_1,\dots,a_k\},b_1,\dots,b_l\} & (l>0),\\ 0 & (l=0).\end{cases}\end{aligned}$$

Note that the relation for $(k,l) = (3,0)$ implies that, with sign change (A.1), the second bracket becomes an honest Lie bracket on $V[-1]$.

The following lemma, which goes back to [3, Theorem 6.1], is well known to experts.

Lemma 5.1. *Let $(V_*, \cdot, \Delta)$ be a BV algebra, W_* be a graded vector space, with linear maps $\alpha : W_* \to V_*$, $\beta : V_* \to W_{*+1}$ such that $\Delta = \alpha \circ \beta$ and $\beta \circ \alpha = 0$. Then,*

(1) *W_* is a gravity algebra where the brackets $W^{\otimes k} \to W$ are*

$$x_1 \otimes \cdots \otimes x_k \mapsto \{x_1, \ldots, x_k\} := \beta(\alpha(x_1) \cdot \cdots \cdot \alpha(x_k)) \quad (k \geq 2). \tag{5.4}$$

(2) *Let $[,]$ be the Gerstenhaber bracket (5.3) on V_*. Then for any $x_1, x_2 \in W$,*

$$\alpha(\{x_1, x_2\}) = (-1)^{|x_1|}[\alpha(x_1), \alpha(x_2)].$$

Proof. To prove (1), first note that since $\cdot$ is graded commutative, $\{x_1, \ldots, x_k\}$ is graded symmetric in its variables. Next, the generalized Jacobi relations follow from a straightforward calculation based on (5.2) (see the proof of [7, Theorem 8.5]), and is omitted. The proof of (2) is trivial. ∎

A BV algebra homomorphism between two BV algebras is an algebra homomorphism that commutes with their BV operators. The case of gravity algebras is similar. The following lemma is obvious.

Lemma 5.2. *Suppose there is a commutative diagram of linear maps*

$$\begin{array}{ccccc} W_* & \xrightarrow{\alpha} & V_* & \xrightarrow{\beta} & W_{*+1} \\ \downarrow f & & \downarrow g & & \downarrow f \\ W'_* & \xrightarrow{\alpha'} & V'_* & \xrightarrow{\beta'} & W'_{*+1} \end{array}$$

such that $(V_*, W_*, \alpha, \beta)$ and $(V'_*, W'_*, \alpha', \beta')$ satisfy the assumptions in Lemma 5.1, and g is a BV algebra homomorphism. Then f is a gravity algebra homomorphism (for the induced structures on W, W'). ∎

Next we need to work in the language of operads. We collect some basics below, and refer the reader to [20, Section 2] or standard references [26, 27] for more details.

Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal category. A *nonsymmetric operad* (ns operad for short) $\mathcal{O}$ in $\mathcal{C}$ consists of the following data:

- an object $\mathcal{O}(k)$ in $\mathcal{C}$ for each $k \in \mathbb{Z}_{\geq 0}$;
- morphisms $\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(l) \to \mathcal{O}(k + l - 1)$ for each $1 \leq i \leq k$ and $l \geq 0$, called *partial compositions*, that are associative: for $x \in \mathcal{O}(k)$, $y \in \mathcal{O}(l)$, $z \in \mathcal{O}(m)$,

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z) \quad (1 \leq i \leq k,\ 1 \leq j \leq l,\ m \geq 0), \tag{5.5a}$$
$$(x \circ_i y) \circ_{l+j-1} z = (x \circ_j z) \circ_i y \quad (1 \leq i < j \leq k,\ l \geq 0,\ m \geq 0); \tag{5.5b}$$

- a morphism $1_{\mathcal{O}} : 1_{\mathcal{C}} \to \mathcal{O}(1)$, which is a two-sided unit for $\circ_i$.

An *operad* is an ns operad such that each $\mathcal{O}(k)$ admits a right action of the symmetric group $\mathfrak{S}_k$ ($\mathfrak{S}_0$ is the trivial group), in a way compatible with partial compositions.

An (ns) operad in the symmetric monoidal category of dg (resp. graded) vector spaces is called a (ns) *dg (resp. graded) operad*. A Koszul sign $(-1)^{|y||z|}$ should appear in (5.5b) in graded and dg cases. Taking homology yields a functor from the category of (ns) dg operads to the category of (ns) graded operads.

Example 5.3. Here are some examples of dg operads and graded operads.

(1) (Endomorphism operad $\mathcal{E}\mathrm{nd}_V$.) For any dg (resp. graded) vector space V_*, there is a dg (resp. graded) operad $\mathcal{E}\mathrm{nd}_V$ defined as follows. For each $k \geq 0$, $\mathcal{E}\mathrm{nd}_V(k)_* := \mathrm{Hom}_*(V^{\otimes k}, V)$, where $\mathrm{Hom}_*(V^{\otimes 0}, V) = \mathrm{Hom}_*(\mathbb{R}, V) = V_*$. For $1 \leq i \leq k$, $l \geq 0$, $f \in \mathrm{Hom}_*(V^{\otimes k}, V)$, $g \in \mathrm{Hom}_*(V^{\otimes l}, V)$, and $\sigma \in \mathfrak{S}_k$,

$$\begin{aligned}(f \circ_i g)(v_1 \otimes \cdots \otimes v_{k+l-1}) &:= (-1)^{\varepsilon} f(v_1 \otimes \cdots \otimes g(v_i \otimes \cdots) \otimes \cdots),\\ (f \cdot \sigma)(v_1 \otimes \cdots \otimes v_k) &:= (-1)^{\varepsilon} f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}),\\ 1_{\mathcal{E}\mathrm{nd}_V} &:= \mathrm{id}_V \in \mathrm{Hom}_0(V, V).\end{aligned}$$

Let $\mathcal{O}$ be a (ns) graded operad or dg operad. A structure of *algebra over* $\mathcal{O}$ on V, or say an action of $\mathcal{O}$ on V, means a morphism $\mathcal{O} \to \mathcal{E}\mathrm{nd}_V$ as (ns) operads.

(2) (Gerstenhaber operad $\mathcal{G}\mathrm{er}$, BV operad $\mathcal{BV}$, and gravity operad $\mathcal{G}\mathrm{rav}$.) These are graded operads that can be defined in terms of generators subject to the relations defining Gerstenhaber/BV/gravity algebras. A Gerstenhaber/BV/gravity algebra is exactly an algebra over $\mathcal{G}\mathrm{er}/\mathcal{BV}/\mathcal{G}\mathrm{rav}$.

(3) (Ward's construction [31].) There is a dg operad $\mathtt{M}_{\circlearrowleft}$ constructed from certain "labeled A_∞ trees", such that $H_*(\mathtt{M}_{\circlearrowleft}) \cong \mathcal{G}\mathrm{rav}$ as graded operads, and there are explicit homotopies measuring the failure of gravity relations on $\mathtt{M}_{\circlearrowleft}$ (while the Jacobi relation for the second bracket strictly holds). For this reason, an algebra over $\mathtt{M}_{\circlearrowleft}$ can be viewed as a gravity algebra up to homotopy. $\mathtt{M}_{\circlearrowleft}$ is closely related to the operad of "cyclic brace operations" (Section 6). There are other important properties of $\mathtt{M}_{\circlearrowleft}$ that we will use later (Proposition 5.6 (5)). Indeed, the notation of Ward [31] is $\mathtt{M}_{\circlearrowright}$, but we use $\mathtt{M}_{\circlearrowleft}$ for the reason of Remark 5.5.

Definition 5.4. ([20, Definition 2.6, 2.9].) Let $\mathcal{O}$ be an ns dg operad.

(1) A *cyclic structure* $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$ is a sequence of morphisms $\tau_k : \mathcal{O}(k) \to \mathcal{O}(k)$ $(k \geq 0)$ such that $\tau_k^{k+1} = \mathrm{id}_{\mathcal{O}(k)}$, $\tau_1(1_{\mathcal{O}}) = 1_{\mathcal{O}}$, and that for any $1 \leq i \leq k$, $l \geq 0$, $x \in \mathcal{O}(k)$, $y \in \mathcal{O}(l)$,

$$\tau_{k+l-1}(x \circ_i y) = \begin{cases} \tau_k x \circ_{i-1} y & (i \geq 2),\\ (-1)^{|x||y|} \tau_l y \circ_l \tau_k x & (i = 1,\ l \geq 1),\\ \tau_k^2 x \circ_k y & (i = 1,\ l = 0).\end{cases}$$

(2) A *multiplication* μ and a *unit* ε in $\mathcal{O}$ are elements $\mu \in \mathcal{O}(2)_0$, $\varepsilon \in \mathcal{O}(0)_0$ satisfying $\partial\mu = 0$, $\mu \circ_1 \mu = \mu \circ_2 \mu$, $\partial\varepsilon = 0$ and $\mu \circ_1 \varepsilon = \mu \circ_2 \varepsilon = 1_{\mathcal{O}}$.

Remark 5.5. An operad with a cyclic structure is called a *cyclic operad*. The cyclic relation in Definition 5.4 differs from some authors (in particular, Ward [31]) in the orientation of performing cyclic permutation, but they are equivalent. See, e.g., [28, Section 3].

Let $\mathcal{O} = (\mathcal{O}(k))_{k\geq 0}$ be an ns dg operad endowed with a multiplication μ and a unit ε. Then $\{(\mathcal{O}(k)_*, \partial)\}_{k\geq 0}$ is a cosimplicial chain complex where $\delta_i : \mathcal{O}(k-1)_* \to \mathcal{O}(k)_*$, $\sigma_i : \mathcal{O}(k+1)_* \to \mathcal{O}(k)_*$ $(0 \leq i \leq k)$ are

$$\delta_i(x) := \begin{cases} \mu \circ_2 x & (i = 0), \\ x \circ_i \mu & (1 \leq i \leq k-1), \\ \mu \circ_1 x & (i = k), \end{cases} \qquad \sigma_i(x) := x \circ_{i+1} \varepsilon. \tag{5.6}$$

Denote the associated total complex by $(\widetilde{\mathcal{O}}_*, b)$. If there is also a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$ such that μ is cyclically invariant, i.e., $\tau_2(\mu) = \mu$, then $\{(\mathcal{O}(k)_*, \partial), \delta_i, \sigma_i, \tau_k\}_{k\geq 0}$ is a cocyclic chain complex. Denote the associated mixed complex by $(\widetilde{\mathcal{O}}_*, b, B)$.

Proposition 5.6. *Let $\mathcal{O} = (\mathcal{O}(k)_*, \partial)_{k\geq 0}$ be an ns dg operad. Define binary operations $\circ$ and $[,]$ on $\widetilde{\mathcal{O}}_* := \prod_{k\geq 0} \mathcal{O}(k)_{*+k}$ by the following: for $x = (x_k)_{k\geq 0}$, $y = (y_k)_{k\geq 0}$,*

$$(x \circ y)_k := \sum_{\substack{l+m=k+1 \\ 1\leq i\leq l}} (-1)^{(i-1)(m-1)+(l-1)(|y|+m)} x_l \circ_i y_m, \tag{5.7a}$$

$$[x, y] := x \circ y - (-1)^{(|x|-1)(|y|-1)} y \circ x. \tag{5.7b}$$

Then for $(\widetilde{\mathcal{O}}_*, \partial)$, statement (1a) below holds true.

If there is a multiplication μ and a unit ε on $\mathcal{O}$, define a binary operation $\cdot$ on $\widetilde{\mathcal{O}}_*$ by

$$(x \cdot y)_k := \sum_{l+m=k} (-1)^{l|y|} (\mu \circ_1 x_l) \circ_{l+1} y_m. \tag{5.8}$$

Then for $(\widetilde{\mathcal{O}}_*, b)$, statements (1b), (2a) below hold true.

If there is a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$, then for $\widetilde{\mathcal{O}}_*^{\mathrm{cyc}} = \mathrm{Ker}(1-\lambda) \subset \widetilde{\mathcal{O}}_*$, statement (3) below holds true.

If there is a multiplication μ, a unit ε and a cyclic structure $(\tau_k)_{k\geq 0}$ on $\mathcal{O}$ such that $\tau_2(\mu) = \mu$, then for $(\widetilde{\mathcal{O}}_*, b, B)$ and $\widetilde{\mathcal{O}}_*^{\mathrm{cyc}}$, the other statements below hold true.

(1) (a) *$(\widetilde{\mathcal{O}}_*, \partial, \circ)$ is a dg pre-Lie algebra (with shifted grading) such that $[,]$ is a Lie bracket of degree 1.*

(b) *$(\widetilde{\mathcal{O}}_*, b, \circ)$ is a dg pre-Lie algebra (with shifted grading) such that $[,]$ is a Lie bracket of degree 1, and $(\widetilde{\mathcal{O}}_*, b, \cdot)$ is a dg algebra.*

(2) (a) *$\cdot$ and $[,]$ induce a Gerstenhaber algebra structure on $H_*(\widetilde{\mathcal{O}}, b)$.*

(b) *$\cdot$ and Connes' operator B induce a BV algebra structure on $H_*(\widetilde{\mathcal{O}}, b)$ where the BV operator is $\Delta = B_*$.*

(c) *The above two structures on $H_*(\widetilde{\mathcal{O}}, b)$ are related by (5.3).*

(3) $\widetilde{\mathcal{O}}^{\mathrm{cyc}}_*$ *is closed under the operation* $[,]$. *The restriction of* $[,]$ *to* $\widetilde{\mathcal{O}}^{\mathrm{cyc}}_*$ *is called the cyclic bracket.*

(4) (a) *The BV algebra structure on* $H_*(\widetilde{\mathcal{O}}, b)$ *obtained in* (2b) *naturally induces gravity algebra structures on* $\mathrm{HC}^{\lambda}_*(\widetilde{\mathcal{O}})$, $\mathrm{HC}^{[[v]]}_*(\widetilde{\mathcal{O}})$ *and* $\mathrm{HC}^{[v^{-1}]}_*(\widetilde{\mathcal{O}})[-1]$.

(b) *The map* $B_{0*} : \mathrm{HC}^{[v^{-1}]}_*(\widetilde{\mathcal{O}})[-1] \to \mathrm{HC}^{[[v]]}_*(\widetilde{\mathcal{O}})$ *in* (2.1) *is a gravity algebra homomorphism. The map* $I_{\lambda *} : \mathrm{HC}^{\lambda}_*(\widetilde{\mathcal{O}}) \cong \mathrm{HC}^{[[v]]}_*(\widetilde{\mathcal{O}})$ *in* (2.5) *is a gravity algebra isomorphism.*

(c) *The Lie bracket on* $\mathrm{HC}^{\lambda}_*(\widetilde{\mathcal{O}})[-1]$ *induced from* (3) *coincides with the second bracket of its gravity algebra structure, up to sign change* (A.1).

(5) $\widetilde{\mathcal{O}}^{\mathrm{cyc}}_*$ *admits an action of the operad* $\mathsf{M}_{\circlearrowleft}$ *(see Example* 5.3*) which covers the cyclic bracket in* (3). *Via the isomorphism* $H_*(\mathsf{M}_{\circlearrowleft}) \cong \mathcal{G}\mathrm{rav}$, *this induces a gravity algebra structure on* $\mathrm{HC}^{\lambda}_*(\widetilde{\mathcal{O}})$ *which is the same as that in* (4a).

Proof. Statements (1), (2a) are exactly [20, Theorem 2.8 (i)–(iii)], which in turn follows from [31, Lemma 2.32]. Statements (2b), (2c) follow from [20, Theorem 2.10], which in turn is a consequence of [31, Theorem B]. Note that [20, Theorem 2.10] uses the normalized subcomplex $\widetilde{\mathcal{O}}^{\mathrm{nm}}_*$, but there is no difference on homology: as explained in [20, Section 2.5.4], the BV operator is just induced by Connes' operator $B = Ns$ on $\widetilde{\mathcal{O}}^{\mathrm{nm}}$.

Statement (3) is a straightforward consequence of [31, Corollary 3.3]. Alternatively, it is quite handy to use definition of cyclic structures to verify that if $\tau_{k_i} x_i = (-1)^{k_i} x_i$ ($x_i \in \mathcal{O}(k_i)$, $i = 1, 2$), then $\tau_{k_1+k_2-1}[x_1, x_2] = (-1)^{k_1+k_2-1}[x_1, x_2]$.

Statement (4a) is an application of Lemma 5.1 (1) to a part of the exact sequences (2.2), (2.6). Note that there is a transition between (co)homological gradings.

- For $\mathrm{HC}^{[[v]]}_*$, take $V_* = H_*(\widetilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{[[v]]}_*(\widetilde{\mathcal{O}})$, $\alpha = p_{0*}$, and $\beta = B_*$.
- For $\mathrm{HC}^{[v^{-1}]}_{*-1}$, take $V_* = H_*(\widetilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{[v^{-1}]}_*(\widetilde{\mathcal{O}})[-1] = \mathrm{HC}^{[v^{-1}]}_{*-1}(\widetilde{\mathcal{O}})$, $\alpha = B_{0*}$, and $\beta = i_*$.
- For HC^{λ}_*, take $V_* = H_*(\widetilde{\mathcal{O}}, b)$, $W_* = \mathrm{HC}^{\lambda}_*(\widetilde{\mathcal{O}})$, $\alpha = i_{\lambda *}$, and $\beta = B_\lambda$. Here the condition $\alpha \circ \beta = \Delta$ is satisfied because of Lemma 2.9.

Statement (4b) follows from Lemma 5.2, Lemma 2.1 and Lemma 2.9. Statement (4c) follows from Lemma 5.1 (2) and statement (2c).

Statement (5) is a consequence of [31, Theorem C], where the Maurer–Cartan element $\zeta = (\zeta_k)_{k\geq 2}$ is taken as $\zeta_2 = -\mu$ and $\zeta_k = 0$ ($k \neq 2$).

To see statement (5) covers statement (3), we need a concrete description of the action of $\mathsf{M}_{\circlearrowleft}$ on $\widetilde{\mathcal{O}}^{\mathrm{cyc}}_*$. For arity 2 it is the same as cyclic brace operations (see Example 6.3). ∎

Remark 5.7. The sign in (5.7a) comes from operadic suspension (see Appendix A.3). Indeed, $\widetilde{\mathcal{O}} = (\prod_{n\geq 0} \mathcal{s}\mathcal{O}(n))[-1]$.

Remark 5.8. Indeed, [20, Theorem 2.8 and 2.10] and [31, Theorem A and B] contain much stronger statements than Proposition 5.6 (1), (2) which we do not need (e.g., existence of an action of a chain model of the (framed) little 2-disks operad on $\widetilde{\mathcal{O}}$ or $\widetilde{\mathcal{O}}^{\mathrm{nm}}$).

Proposition 5.6 (1), (2) themselves were known much earlier, e.g., see [28, Section 1.2 and Theorem 1.3].

Example 5.9. Let $(A^*, d, \cdot)$ be a dg algebra with unit 1_A. Then $\mathcal{E}\mathrm{nd}_A$ admits a multiplication and a unit given by $\mu(a_1 \otimes a_2) := a_1 \cdot a_2$, $\varepsilon := 1_A$. Viewing A as a dg A-bimodule, the cosimplicial maps δ_i, σ_i in Example 2.7 are the same as (5.6) for $(\mathcal{E}\mathrm{nd}_A, \mu, \varepsilon)$ in (1). To discuss cyclic structures, suppose there is a graded symmetric bilinear form $\langle , \rangle : A \times A \to \mathbb{R}$ of degree $m \in \mathbb{Z}$, such that

$$d\langle a, b\rangle = \langle da, b\rangle + (-1)^{|a|}\langle a, db\rangle, \quad \langle ab, c\rangle = \langle a, bc\rangle \quad (\forall a, b, c \in A).$$

Namely, A is a dg version of a Frobenius algebra, but we do not require $\dim_{\mathbb{R}} A$ to be finite or $\langle , \rangle$ to be nondegenerate. Note that since $\langle , \rangle$ is symmetric, the relation $\langle ab, c\rangle = \langle a, bc\rangle$ is equivalent to $\langle , \rangle$ being *cyclic*, i.e.,

$$\langle ab, c\rangle = (-1)^{|a|(|b|+|c|)}\langle bc, a\rangle. \tag{5.9}$$

Now consider $A^\vee[m]$ with a dg A-bimodule structure characterized by (2.9) (the degree of $\varphi \in A^\vee[m]$ is now shifted). The degree 0 map

$$\theta : A \to A^\vee[m]; \quad \theta(a)(b) := \langle a, b\rangle \quad (\forall a, b \in A) \tag{5.10}$$

is a dg A-bimodule map, and $\mathrm{Hom}(-, \theta) : \mathrm{Hom}^*(A^{\otimes k}, A) \to \mathrm{Hom}^*(A^{\otimes k}, A^\vee[m])$ is a morphism of cosimplicial complexes. $\{\mathrm{Hom}^*(A^{\otimes k}, A^\vee[m]) = \mathrm{Hom}^{*+m}(A^{\otimes k+1}, \mathbb{R})\}_{k\geq 0}$ is moreover cocyclic with cyclic permutations $(\tau_k)_{k\geq 0}$ given in Example 2.7.

If θ happens to be an isomorphism, then $\{\mathrm{Hom}(-, \theta)^{-1} \circ \tau_k \circ \mathrm{Hom}(-, \theta)\}_{k\geq 0}$ endows $(\mathcal{E}\mathrm{nd}_A, \mu, \varepsilon)$ with a cyclic structure. All statements of Proposition 5.6 hold for $\mathcal{O} = \mathcal{E}\mathrm{nd}_A$.

If θ is a quasi-isomorphism, then $\widetilde{\mathcal{E}\mathrm{nd}_A} = \mathrm{CH}^*(A, A)$ and $\mathrm{CH}^*(A, A^\vee[m])$ are quasi-isomorphic through a natural map induced by θ. In this case, let us examine the statements (1)–(5) in Proposition 5.6 for $\mathcal{O} = \mathcal{E}\mathrm{nd}_A$.

(1)′ Statement (1) still holds honestly (it is irrelevant to θ).

(2)′ Statement (2a) holds honestly (it is irrelevant to θ). Statements (2b), (2c) "hold weakly" in the following sense: Connes' operator B on $\mathrm{CH}^*(A, A^\vee[m])$ induces a BV operator on $\mathrm{HH}^*(A, A) \cong \mathrm{HH}^*(A, A^\vee[m])$, making $\mathrm{HH}^*(A, A)$ into a BV algebra, which is compatible with its Gerstenhaber algebra structure. This is proved by Menichi [29, Theorem 18].

(3)′ Statement (3) "holds weakly" in the sense that the subspace of *weakly cyclic invariants* in $\mathrm{CH}^*(A, A)$, $\Theta^{-1}(\mathrm{CH}^*_{\mathrm{cyc}}(A, A^\vee[m]))$, is closed under the bracket (5.7b), and hence is a dg Lie subalgebra. Here $\Theta : \mathrm{CH}^*(A, A) \to \mathrm{CH}^*(A, A^\vee[m])$ is the cochain map induced by (5.10), and $\mathrm{CH}^*_{\mathrm{cyc}}(A, A^\vee[m]) := \mathrm{Ker}(1 - \lambda)$ is the subcomplex of cyclic invariants in $\mathrm{CH}^*(A, A^\vee[m])$, with respect to the cocyclic structure on $\{\mathrm{Hom}^{*+m}(A^{\otimes k+1}, \mathbb{R})\}_{k\geq 0}$. This result is rather simple and should be well known, e.g., it is stated without proof in [30, Lemma 4].

(4)′ Statement (4) "holds weakly" in the following sense: there are gravity algebra structures on $\mathrm{HC}^*_\lambda(A, A^\vee[m]) \cong \mathrm{HC}^*_{[[u]]}(A, A^\vee[m]) \cong \mathrm{HC}^*_{[[u]]}(A, A)$ and on $\mathrm{HC}^*_{[u^{-1}]}(A, A^\vee[m]) \cong \mathrm{HC}^*_{[u^{-1}]}(A, A)$, induced by the BV algebra structure on $\mathrm{HH}^*(A, A) \cong \mathrm{HH}^*(A, A^\vee[m])$ described in (3)′.

(5)′ Statement (5) "holds weakly" by Corollary 6.8, which largely generalizes (3)′.

Remark 5.10. Statements (3)′, (5)′ above hold true even if $\theta : A \to A^\vee[m]$ is not a quasi-isomorphism. If θ is a quasi-isomorphism, then so is $\Theta : \mathrm{CH}(A, A) \to \mathrm{CH}(A, A^\vee[m])$. If Θ also restricts to a quasi-isomorphism $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])) \to \mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$, then the structures in (3)′, (5)′ are compatible with those in (2)′, (4)′.

Remark 5.11. Statement (3)′ in Example 5.9 is irrelevant to the algebra structure on A. It holds true when A is just a graded vector space endowed with a symmetric bilinear form $\langle , \rangle : A \times A \to \mathbb{R}$ of degree m. In this case, we shall write $\Theta^{-1}(\prod_{k\geq 0} \mathrm{Hom}^{*+m}_{\mathrm{cyc}}(A^{k+1}, \mathbb{R}))$ in place of $\Theta^{-1}(\mathrm{CH}^*_{\mathrm{cyc}}(A, A^\vee[m]))$.

6. Cyclic brace operations

This section is devoted to the proof of Theorem 1.3. Recall $\widetilde{\mathcal{O}} := \prod_{k\geq 0} \mathcal{O}(k)$ if $\mathcal{O}$ is dg operad, and $\widetilde{\mathcal{O}}^{\mathrm{cyc}} := \mathrm{Ker}(1 - \lambda) \subset \widetilde{\mathcal{O}}$ if $\mathcal{O}$ is a dg cyclic operad .

Definition 6.1 (Brace operations via concrete formulas). Let $\mathcal{O}$ be a dg operad. For each $n \in \mathbb{Z}_{\geq 0}$, define an $(n + 1)$-ary operation on $\widetilde{\mathcal{O}}$ as follows. When $n = 0$, for $a \in \mathcal{O}(r)$, let $a\{\} := a$. When $n > 0$, for $a \in \mathcal{O}(r)$ and $b_j \in \mathcal{O}(t_j)$ $(1 \leq j \leq n)$, let

$$a\{b_1, \ldots, b_n\} := \sum_{i_1,\ldots,i_n} \pm(\cdots((a \circ_{i_1} b_1) \circ_{i_2} b_2) \cdots \circ_{i_n} b_n), \tag{6.1}$$

where the summation is taken over tuples $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 1}$ satisfying $i_{j+1} \geq i_j + t_j$ and $i_n \leq r - n + 1 + \sum_{l=1}^{n-1} t_l$. The sign $\pm$ is from iteration of (A.2).

Brace operations were first described by Getzler [14] in Hochschild context (generalizing the Gerstenhaber bracket [12] which corresponds to $n = 2$) and later by Gerstenhaber–Voronov [13] in operadic context. There is also an interpretation of brace operations via planar rooted trees, going back to the "minimal operad" of Kontsevich–Soibelman [24] (see also [9, Sections 7–9]), which allows for a variation in the cyclic invariant setting ([31]).

Let us fix terminologies about trees before moving to more details.

- A *tree without tails* is a contractible 1-dimensional finite CW complex. A 0-cell is called a *vertex*; the closure of a 1-cell is called an *edge* (identified with $[0, 1]$).
- A *tree with tails* is a tree without tails attached with copies of $[0, 1)$ called *tails* by gluing each $0 \in [0, 1)$ to some vertex.

The set of vertices, edges and tails in a tree T is denoted by V_T, E_T and L_T, respectively. The set of edges and tails at $v \in V_T$ is denoted by E_v and L_v, respectively. The *valence* of a vertex v is the number $|E_v \cup L_v|$. The *arity* of a vertex is its valence -1.

- An *oriented tree* is a tree with a choice of direction for each edge, from one vertex to the other. Such a choice of directions is called an orientation of the tree.
- A *rooted tree* is a tree with a choice of a distinguished tail called the *root*.

  Every rooted tree is naturally oriented by directions towards the root.
- A *planar tree* is a tree with a cyclic order on $E_v \cup L_v$ for each vertex v.

Every planar tree can be embedded into the plane in a way unique up to isotopy, so that at each vertex v, the cyclic order on $E_v \cup L_v$ is counterclockwise.

Every planar rooted tree T carries a natural *total order* on $E_T \cup L_T$, which can be obtained by moving counterclockwise along the boundary of a small tubular neighborhood of T in the plane. It starts from the root and is compatible with the cyclic order on $E_v \cup L_v$ for each $v \in V_T$, and also restricts to total orders on E_T, L_T and E_v, L_v for each $v \in V_T$.

- An *n-labeled tree* is a tree T with a bijection between $\{1, 2, \dots, n\}$ and V_T. If the number of vertices is not specified, it is just called a labeled tree.

The vertex with label i in an n-labeled tree T is denoted by $v_i(T)$, with arity $a_i(T)$. The notion of isomorphisms of trees (with various structures) is obvious. We shall view isomorphic trees as the same.

For $n \in \mathbb{Z}_{\geq 1}$, let $\mathsf{B}^s(n)$ be the set of n-labeled planar rooted trees without non-root tails, and let $\mathsf{B}(n)$ be the vector space spanned by $\mathsf{B}^s(n)$. Let $\overline{\mathsf{B}}^s(n)$ be the set of n-labeled planar rooted trees with tails, and let $\overline{\mathsf{B}}(n)$ be the vector space spanned by $\overline{\mathsf{B}}^s(n)$.

Given $T' \in \overline{\mathsf{B}}(n)$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$, define a set

$$\mathcal{T}(T', k) := \{T'' \in \overline{\mathsf{B}}^s(n) \mid T'' \text{ can be obtained by attaching tails to } T' \text{ so that } a_i(T'') = k_i \ (1 \leq \forall i \leq n)\}. \tag{6.2}$$

Definition 6.2 (Brace operations via trees). Let $\mathcal{O}$ be a dg operad. Define linear maps

$$\kappa_n : \mathsf{B}(n) \to \mathrm{Hom}(\widetilde{\mathcal{O}}^{\otimes n}, \widetilde{\mathcal{O}}), \quad \overline{\kappa}_n : \overline{\mathsf{B}}(n) \to \mathrm{Hom}(\widetilde{\mathcal{O}}^{\otimes n}, \widetilde{\mathcal{O}})$$

as follows: κ_n is the restriction of $\overline{\kappa}_n$. For $T' \in \overline{\mathsf{B}}^s(n)$ and $f_i \in \mathcal{O}(k_i)$ $(1 \leq i \leq n)$,

$$\overline{\kappa}_n(T')(f_1, f_2, \dots, f_n) := \sum_{T'' \in \mathcal{T}(T', k)} \pm \overline{\kappa}_n(T'')(f_1, f_2, \dots, f_n),$$

where by convention summation over the empty set is zero. If $a_i(T') = k_i$ $(1 \leq i \leq n)$, then $\overline{\kappa}_n(T')(f_1, \dots, f_n)$ is the operadic composition of $f_1, \dots, f_n$ in the obvious way described by T', where f_i is assigned to $v_i(T')$. The sign $\pm$ is from iteration of (A.2).

Definition 6.1 and Definition 6.2 describe the same operations on $\widetilde{\mathcal{O}}$, as explained below. Consider $\beta_n \in \mathsf{B}^s(n+1)$ characterized by $E_{\beta_n} = \{e_1, \dots, e_n\}$, $V_{\beta_n} = \{v_1, \dots, v_{n+1}\}$

where v_i is labeled by i, $L_{\beta_n} = \{l_1\}$, $v_1 = l_1 \cap e_1 \cap \cdots \cap e_n$, $v_{i+1} \in e_i$ $(1 \le \forall i \le n)$, and the cyclic order on $E_{v_1} \cup L_{v_1}$ is $(l_1, e_1, \ldots, e_n)$. Then $\kappa_{n+1}(\beta_n) \in \mathrm{Hom}(\widetilde{\mathcal{O}}^{\otimes n+1}, \widetilde{\mathcal{O}})$ is exactly given by (6.1). Moreover, putting $\mathsf{B}(n)$ in degree $n-1$, $\mathsf{B} = \{\mathsf{B}(n)\}$ carries a *reduced* (meaning $\mathsf{B}(0) = 0$ and $\mathsf{B}(1) = \mathbb{R}$) graded operad structure (see [31, Definitions 2.11 and 2.13]) so that $\{\beta_n\}_{n\ge 0}$ generates B under operadic compositions and symmetric permutations, and

$$\kappa = \{\kappa_n\} : \mathsf{B} \to \mathcal{E}\mathrm{nd}_{\widetilde{\mathcal{O}}} \tag{6.3}$$

is a morphism of operads. B is called the *brace operad*, which tautologically controls brace operations on *brace algebras*, i.e., algebras over B. We have just seen that $\widetilde{\mathcal{O}}$ is naturally a brace algebra. For the purpose of this paper we omitted details of operadic compositions on B but explained how κ is defined.

In [31, Section 3.2], Ward introduced an operad $\mathsf{B}_{\circlearrowleft}$ which he called *cyclic brace operad*. Let $\mathsf{B}^s_{\circlearrowleft}(n)$ be the set of oriented n-labeled planar trees without tails. Then $\mathsf{B}_{\circlearrowleft}(n)$ is the graded vector space spanned by $\mathsf{B}^s_{\circlearrowleft}(n)$ modulo the relation that reversing direction on an edge produces a negative sign. If there is no risk of confusion, we will by abuse of notation not distinguish $T_{\circlearrowleft} \in \mathsf{B}^s_{\circlearrowleft}(n)$ from its image in $\mathsf{B}_{\circlearrowleft}(n)$. There is a morphism of operads $\rho : \mathsf{B}_{\circlearrowleft} \to \mathsf{B}$, which is induced by maps

$$\rho^s_n : \mathsf{B}^s_{\circlearrowleft}(n) \to \mathsf{B}(n), \quad T_{\circlearrowleft} \mapsto \sum_{T \in \mathcal{R}_1(T_{\circlearrowleft})} (-1)^{\varepsilon(T_{\circlearrowleft}, T)} T, \tag{6.4}$$

where $\mathcal{R}_1(T_{\circlearrowleft})$ is the set of labeled planar rooted trees that can be obtained by adding a root to the (non-rooted) tree underlying $T_{\circlearrowleft}$, and $\varepsilon(T_{\circlearrowleft}, T)$ is the number of edges in $E_{T_{\circlearrowleft}} = E_T$ whose direction from $T_{\circlearrowleft}$ does not agree with the direction from the rooted structure of T. Here and hereafter, in appropriate contexts we use f^s to denote a set-theoretic map which induces a linear map f.

A natural example of cyclic brace algebras, i.e., algebras over $\mathsf{B}_{\circlearrowleft}$, is as follows.

Example 6.3 ([31, Corollary 3.11]). Let $\widetilde{\mathcal{O}}$ be a dg cyclic operad. Consider $\widetilde{\mathcal{O}}^{\mathrm{cyc}} \subset \widetilde{\mathcal{O}}$ and $\kappa : \mathsf{B} \to \mathcal{E}\mathrm{nd}_{\widetilde{\mathcal{O}}}$ in (6.3). For $T \in \mathsf{B}^s_{\circlearrowleft}(n)$, $\rho(T) \in \mathsf{B}(n)$ and $(\kappa \circ \rho)(T) \in \mathrm{Hom}(\widetilde{\mathcal{O}}^{\otimes n}, \widetilde{\mathcal{O}})$. Restricting $(\kappa \circ \rho)(T)$ to $\widetilde{\mathcal{O}}^{\mathrm{cyc}}$ gives an element in $\mathrm{Hom}((\widetilde{\mathcal{O}}^{\mathrm{cyc}})^{\otimes n}, \widetilde{\mathcal{O}})$. Moreover, if $f_i \in \mathcal{O}(k_i)$ $(1 \le i \le n)$ are cyclic invariant, then so is $(\kappa \circ \rho)(T)(f_1, \ldots, f_n)$. (Such a claim appears in [31, Theorem 5.5] where it is referred to [31, Proposition 3.10], but there is no direct proof given in [31]. We will give a direct proof in a slightly different situation.) Hence $\kappa \circ \rho$ gives a morphism $\mathsf{B}_{\circlearrowleft} \to \mathcal{E}\mathrm{nd}_{\widetilde{\mathcal{O}}^{\mathrm{cyc}}}$.

Definition 6.4 (Cyclic brace operations). Let $\mathcal{O}$ be a dg cyclic operad. The cyclic brace operations on $\widetilde{\mathcal{O}}^{\mathrm{cyc}}$ are those characterized by the linear maps

$$\kappa_n \circ \rho_n : \mathsf{B}_{\circlearrowleft}(n) \to \mathrm{Hom}((\widetilde{\mathcal{O}}^{\mathrm{cyc}})^{\otimes n}, \widetilde{\mathcal{O}}^{\mathrm{cyc}})$$

discussed in Example 6.3.

Remark 6.5. It seems hard to write a direct formula for cyclic brace operations on $\widetilde{\mathcal{O}}^{\mathrm{cyc}}$ in terms of operadic compositions, in a way as explicit as (6.1).

Consider $\overline{\mathsf{B}}^s_{\circlearrowleft}(n) \supset \mathsf{B}^s_{\circlearrowleft}(n)$ and $\overline{\mathsf{B}}_{\circlearrowleft}(n) \supset \mathsf{B}_{\circlearrowleft}(n)$ by extending the definitions to labeled planar trees with tails. There is a forgetful map

$$w_n^s : \overline{\mathsf{B}}^s(n) \to \overline{\mathsf{B}}^s_{\circlearrowleft}(n) \setminus \mathsf{B}^s_{\circlearrowleft}(n)$$

forgetting the choice of root but keeping the orientation from rooted structure. Note that w_n^s induces $w_n : \overline{\mathsf{B}}(n) \to \overline{\mathsf{B}}_{\circlearrowleft}(n)/\mathsf{B}_{\circlearrowleft}(n)$. There is also a map

$$r_n^s : \overline{\mathsf{B}}^s_{\circlearrowleft}(n) \setminus \mathsf{B}^s_{\circlearrowleft}(n) \to \overline{\mathsf{B}}(n), \quad T'_{\circlearrowleft} \mapsto \sum_{T' \in \mathcal{R}_0(T'_{\circlearrowleft})} (-1)^{\varepsilon(T'_{\circlearrowleft}, T')} T',$$

where $\mathcal{R}_0(T'_{\circlearrowleft})$ is the set of n-labeled planar rooted trees obtained by choosing one of the tails in $T'_{\circlearrowleft}$ as the root, and $\varepsilon(T'_{\circlearrowleft}, T')$ is defined similar to $\varepsilon(T_{\circlearrowleft}, T)$ in (6.4).

It is clear that $(w_n \circ r_n)(T'_{\circlearrowleft}) = |\mathcal{R}_0(T'_{\circlearrowleft})| \cdot T'_{\circlearrowleft}$. To describe $r_n \circ w_n$, consider a map

$$t_n^s : \overline{\mathsf{B}}^s(n) \to \overline{\mathsf{B}}^s(n)$$

so that T' and $t_n^s(T')$ are the same after forgetting the root, and the root of $t_n^s(T')$ is the first non-root tail of T' (if there are no non-tail roots then $t_n^s(T') = T'$). Then for any $T' \in \mathcal{R}_0(T'_{\circlearrowleft})$, we have $\mathcal{R}_0(T'_{\circlearrowleft}) = \{T', t_n^s(T'), \dots, (t_n^s)^p(T')\}$, where $p = |\mathcal{R}_0(T'_{\circlearrowleft})| - 1$. It follows that

$$(r_n \circ w_n)(T') = \varepsilon(T'_{\circlearrowleft}, T') \cdot r_n(T'_{\circlearrowleft}) = \sum_{0 \le i \le |\mathcal{R}_0(T'_{\circlearrowleft})|-1} (-1)^{\varepsilon(T', t_n^i(T'))} \cdot t_n^i(T'). \quad (6.5)$$

Here $\varepsilon(T', t_n^i(T'))$ is the number of edges in $E_{T'} = E_{t_n^i(T')}$ whose direction towards the root of T' does not agree with the direction towards the root of $t_n^i(T')$.

Given $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\ge 0}$, define a map

$$\nu_k^s : \mathsf{B}^s_{\circlearrowleft}(n) \to \overline{\mathsf{B}}_{\circlearrowleft}(n), \quad T_{\circlearrowleft} \mapsto \sum_{T'_{\circlearrowleft} \in \mathcal{T}(T_{\circlearrowleft}, k)} T'_{\circlearrowleft},$$

where $\mathcal{T}(k, T_{\circlearrowleft}) \subset \overline{\mathsf{B}}^s_{\circlearrowleft}(n)$ is defined similar to $\mathcal{T}(k, T)$ in (6.2).

Lemma 6.6. *Let $\mathcal{O}$ be a dg operad. For any $T_{\circlearrowleft} \in \mathsf{B}^s_{\circlearrowleft}(n)$ and $f_i \in \mathcal{O}(k_i)$ $(1 \le i \le n)$, there holds*

$$(\kappa_n \circ \rho_n^s)(T_{\circlearrowleft})(f_1, \dots, f_n) = (\overline{\kappa}_n \circ r_n \circ \nu_k^s)(T_{\circlearrowleft})(f_1, \dots, f_n).$$

Proof. Consider the set of labeled planar rooted trees whose vertices have arities equal to $k = (k_1, \dots, k_n)$ in accordance with the labeling. Such a set can be represented as

$$\bigcup_{T \in \mathcal{R}_1(T_{\circlearrowleft})} \mathcal{T}(T, k) = \bigcup_{T'_{\circlearrowleft} \in \mathcal{T}(T_{\circlearrowleft}, k)} \mathcal{R}_0(T'_{\circlearrowleft}),$$

and the result follows. ∎

In the rest of this section, we take $\mathcal{O} = \mathcal{E}\mathrm{nd}_A$, where A is a dg algebra endowed with a symmetric, cyclic, bilinear form $\langle , \rangle$ of degree m. Recall from Example 5.9 that $\langle , \rangle$ induces $\theta : A \to A^\vee[m]$ and $\Theta : \mathrm{CH}(A, A) \to \mathrm{CH}(A, A^\vee[m])$. To deal with signs, we may work with $A[1]$ instead of A. As explained in Appendix A, the symmetric bilinear form $\langle , \rangle$ on A becomes anti-symmetric on $A[1]$, and the cyclic permutation τ_k on $\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R})$ reads as $\widetilde{\tau}_k = (-1)^k \tau_k = \lambda$ on $\mathrm{Hom}(A[1]^{\otimes k+1}, \mathbb{R})$. Since $\mathfrak{s}(\mathcal{E}\mathrm{nd}_A) \cong \mathcal{E}\mathrm{nd}_{A[1]}$, there is no need to take operadic suspension of $\mathtt{B}_{\circlearrowleft}$, and $\mathtt{B}_{\circlearrowleft}(n)$ stands in degree 0 when dealing with $A[1]$.

Since the pairing $\langle , \rangle$ is not necessarily nondegenerate, there is not always a cyclic structure on $\mathcal{E}\mathrm{nd}_A$ compatible with cyclic permutations on $\{\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R})\}$ via the map $\mathrm{Hom}(A^{\otimes k}, A) \to \mathrm{Hom}(A^{\otimes k}, A^\vee[m])$ induced by $\langle , \rangle$, so the discussion of Example 6.3 does not directly apply here. However, the following is true.

Proposition 6.7. *There is a natural action of* $\mathtt{B}_{\circlearrowleft}$ *on* $\Theta^{-1}(\prod_{k\geq 0} \mathrm{Hom}_{\mathrm{cyc}}^{*+m}(A^{\otimes k+1}, \mathbb{R}))$.

Proof. (This proposition is irrelevant to the multiplication on A; compare Remark 5.11.) Similar to Example 6.3, it suffices to show if $T_{\circlearrowleft} \in \mathtt{B}^s_{\circlearrowleft}(n)$ and $f_i \in \mathrm{Hom}(A^{\otimes k_i}, A)$ is weakly cyclic invariant in the sense that $\lambda(\theta \circ f_i) = \theta \circ f_i$, then $(\kappa \circ \rho)(T_{\circlearrowleft})(f_1, \ldots, f_n)$ is weakly cyclic invariant. This is immediate from Lemma 6.6 and Lemma 6.9 below. ∎

Corollary 6.8. *The cochain complex* $(\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])), d + \delta)$ *admits an action of* $\mathtt{M}_{\circlearrowleft}$. *Moreover, if* θ *is a quasi-isomorphism and* Θ *restricts to a quasi-isomorphism from* $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$ *to* $\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m])$, *this* $\mathtt{M}_{\circlearrowleft}$*-action lifts the gravity algebra structure on* $\mathrm{HC}^*_\lambda(A, A^\vee[m])$ *induced by the BV algebra structure on* $\mathrm{HH}^*(A, A^\vee[m])$ *(see Example 5.9 (4)′).*

Proof. As explained in the proof of [31, Theorem 5.5], to show $\mathtt{M}_{\circlearrowleft}$ acts on the space of weakly cyclic invariants, it suffices to consider cyclic brace operations, which is nothing but Proposition 6.7. In more details, Ward [31] defined a dg operad $\mathtt{M}$ using "A_∞-labeled planar rooted black/white trees" ($\mathtt{M}$ is isomorphic to the "minimal operad" of Kontsevich–Soibelman [24]), and $\mathtt{M}_{\circlearrowleft}$ is the non-rooted version of $\mathtt{M}$. $\mathtt{M}$ contains $\mathtt{B}$ as a graded suboperad, and acts on $\mathrm{CH}(A, A)$ extending brace operations. What $\mathtt{M}$ does more than $\mathtt{B}$ to $\mathrm{CH}(A, A)$ is generated by the operation $(f, g) \mapsto \mu_A\{f, g\}$ where $\mu_A \in \mathrm{Hom}(A^{\otimes 2}, A)$ is the multiplication. The action of $\mathtt{M}_{\circlearrowleft}$ on $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$ comes from an operad morphism $\mathtt{M}_{\circlearrowleft} \to \mathtt{M}$ which extends the morphism $\mathtt{B}_{\circlearrowleft} \to \mathtt{B}$ from (6.4). Therefore, that $\Theta^{-1}(\mathrm{CH}_{\mathrm{cyc}}(A, A^\vee[m]))$ is closed under the action of $\mathtt{M}_{\circlearrowleft}$ on $\mathrm{CH}(A, A)$ essentially follows from Proposition 6.7 and weakly cyclic invariance of μ_A, i.e., (5.9).

Now we explain why the $\mathtt{M}_{\circlearrowleft}$-action induces exactly the gravity algebra structure from the BV structure on homology under quasi-isomorphism assumptions; this is just by definition. The gravity algebra structure on $\mathrm{HC}^*_\lambda(A, A^\vee[m])$ follows from Lemma 5.1 and (2.6) with $V_* = \mathrm{HH}^{-*}(A, A) = \mathrm{HH}^{-*}(A, A^\vee[m])$, $W_* = \mathrm{HC}^{-*}_\lambda(A, A^\vee[m])$, $\alpha = i_{\lambda *}$, $\beta = B_\lambda$. The product $\cdot$ (5.8) on $\mathrm{CH}(A, A)$ is just the cup product $(f, g) \mapsto \mu_A\{f, g\}$, so the k-th gravity bracket (5.4) on $\mathrm{HC}_\lambda(A, A^\vee[m])$ is induced by the operation $(f_1, f_2, \ldots, f_k)$

$\mapsto \mu_A\{f_1, \mu_A\{f_2, \mu_A\{\cdots, \mu_A\{f_{k-1}, f_k\}\cdots\}\}\}$ at chain level, which is represented by a certain black/white tree with $k-1$ adjacent black vertices labeled by μ_A. Edges with both black vertices in such a tree should be contracted, creating a new tree with a single black vertex, see [31, Appendix A, (A.10)]. This gives exactly the trees representing the generators of $H_*(\mathtt{M}_{\circlearrowleft})$, see [31, Definition 5.12, Figure 2]. ∎

Lemma 6.9. *Let $T'_{\circlearrowleft} \in \overline{\mathtt{B}}_{\circlearrowleft}(n)$, $T' \in \mathcal{R}_0(T'_{\circlearrowleft})$, $f_i \in \mathrm{Hom}(A[1]^{\otimes k_i}, A[1])$ where $k_i = a_i(T')$ $(1 \le i \le n)$. Suppose every f_i is weakly cyclic invariant. Then*

$$\theta \circ ((\overline{\kappa}_n \circ r_n)(T'_{\circlearrowleft})(f_1, \ldots, f_n)) = \varepsilon(T'_{\circlearrowleft}, T') \cdot N(\theta \circ (\overline{\kappa}_n(T')(f_1, \ldots, f_n))).$$

Proof. In view of (6.5), it suffices to prove the following equality:

$$\theta \circ ((\overline{\kappa}_n \circ t_n)(T')(f_1, \ldots, f_n)) = \varepsilon(T', t_n(T')) \cdot \lambda(\theta \circ (\overline{\kappa}_n(T')(f_1, \ldots, f_n))). \quad (6.6)$$

If there is only one tail or only one vertex (equivalently, no edges) in T', then t_n acts trivially on $\overline{\mathtt{B}}(n)$, and (6.6) is obvious. Now suppose there are at least two tails and at least one edge in T'. Then there is a unique path in T' connecting the root l_1 to the first non-root tail l_2, consisting of successive edges with successive vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ $(k \ge 1)$, where v_{i_1}, v_{i_k} are vertices of l_1, l_2, respectively. Note that these $k-1$ successive edges are the only edges in T' whose directions towards l_1 and l_2 disagree, so

$$\varepsilon(T', t_n(T')) = k - 1.$$

If $k = 1$, (6.6) simply follows from cyclic invariance of $\theta \circ f_{i_1}$.

If $k \ge 2$, for each $j \in \{1, \ldots, k-1\}$, denote the edge joining v_{i_j} to $v_{i_{j+1}}$ by $[v_{i_j}, v_{i_{j+1}}]$, which is identified with $[0, 1]$. By removing $\frac{1}{2} \in [v_{i_j}, v_{i_{j+1}}]$ for all $1 \le j \le k-1$, T' is cut into k pieces, where the j-th $(1 \le j \le k)$ piece T_j contains v_{i_j}. By regarding $[v_{i_j}, \frac{1}{2})$ and $(\frac{1}{2}, v_{i_{j+1}}]$ as tails, these pieces become labeled planar trees, where the planar structures are induced from T', and the vertex labeling is in the same order as T': say the vertices of T_j are labeled by $i_{j,1} < i_{j,2} < \cdots < i_{j,n_j}$ in T', then put their labels as $1, 2, \ldots, n_j$ in T_j. Let $T'_{j,-}$ (resp. $T'_{j,+}$) be the labeled planar rooted tree obtained by choosing the tail $(\frac{1}{2}, v_{i_j}]$ (resp. $[v_{i_j}, \frac{1}{2})$) in T_j as the root, where $(\frac{1}{2}, v_{i_1}]$ is indeed l_1 and $[v_{i_k}, \frac{1}{2})$ is indeed l_2. Suppose $(\frac{1}{2}, v_{i_j}]$ (resp. $[v_j, \frac{1}{2})$) is the p_j-th (resp. q_j-th) non-root tail in the total order on $L_{v_{i_j}}$ from the planar rooted structure of $T'_{j,+}$ (resp. $T'_{j,-}$). Since l_2 is the first non-root tail in T', we have $q_1 = \cdots = q_k = 1$, and

$$p_1 + \cdots + p_k - k + 1 = |L_{T'}| - 1 = k_1 + \cdots + k_n + 1 - n =: K.$$

Denote

$$F_j^+ := \overline{\kappa}_{n_j}(T'_{j,+})(f_{i_{j,1}}, \ldots, f_{i_{j,n_j}}), \quad F_j^- := \overline{\kappa}_{n_j}(T'_{j,-})(f_{i_{j,1}}, \ldots, f_{i_{j,n_j}}).$$

Then for any $x_1, \ldots, x_{K+1} \in A[1]$, there holds (Koszul sign $(-1)^\varepsilon$ is taken w.r.t. $A[1]$)

$$\begin{aligned}
&\langle(\overline{\kappa}_n \circ t_n)(T')(f_1, \ldots, f_n)(x_1 \otimes \cdots \otimes x_K), x_{K+1}\rangle \\
&\quad = (-1)^\varepsilon \langle F_k^+ \circ_{p_k} (F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+))(x_1 \otimes \cdots \otimes x_K), x_{K+1}\rangle \\
&\quad = (-1)^\varepsilon \langle F_k^-(\cdots \otimes x_{p_k - 1}), F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+)(x_{p_k} \otimes \cdots)\rangle \\
&\quad = -(-1)^\varepsilon \langle F_{k-1}^+ \circ_{p_{k-1}} (\cdots \circ_{p_2} F_1^+)(x_{p_k} \otimes \cdots), F_k^-(\cdots \otimes x_{p_k - 1})\rangle, \qquad (6.7)
\end{aligned}$$

where the second equality follows from cyclic invariance of $\theta \circ f_{i_k}$, and the third equality follows from (anti-)symmetry of $\langle , \rangle$. Iterating the above calculation by cyclic invariance of $\theta \circ f_{i_{k-1}}, \theta \circ f_{i_{k-2}}, \dots, \theta \circ f_{i_1}$ and (anti-)symmetry of $\langle , \rangle$, we see that (6.7) is equal to

$$\begin{aligned}
&(-1)^k(-1)^{\varepsilon}\langle x_{p_1+\cdots+p_k-k+1}, F_1^- \circ_{q_1} (F_2^- \circ_{q_2} (\cdots \circ_{q_{k-1}} F_k^-))(\cdots \otimes x_{p_1+\cdots+p_k-k})\rangle \\
&\quad = (-1)^{k-1}(-1)^{\varepsilon}\langle F_1^- \circ_1 (F_2^- \circ_1 (\cdots \circ_1 F_k^-))(x_{K+1} \otimes x_1 \otimes \cdots \otimes x_{K-1}), x_K\rangle \\
&\quad = (-1)^{\varepsilon(T',t_n(T'))}(-1)^{\varepsilon}\langle \overline{\kappa}_n(T')(f_1, \dots, f_n)(x_{K+1} \otimes x_1 \otimes \cdots), x_K\rangle.
\end{aligned}$$

Since $\widetilde{\tau}_K$ on $\mathrm{Hom}(A[1]^{\otimes K}, A[1]^{\vee}[m])$ corresponds to λ on $\mathrm{Hom}(A^{\otimes K}, A^{\vee}[m])$, this proves (6.6). The proof is now complete. ∎

Remark 6.10. It is easy to generalize Proposition 6.7 to A_∞ algebras with cyclic invariant symmetric bilinear forms (not necessarily nondegenerate), and the proof is similar.

7. Chain level structures in S^1-equivariant string topology

In this section, we describe chain level structures in S^1-equivariant string topology, based on the previous results. Let us first review the initial homology level structures discovered by Chas–Sullivan, and the chain level construction due to Irie.

Example 7.1 (String topology BV algebra and gravity algebra). Let M be a closed oriented manifold and $\mathcal{L}M$ be its free loop space. It was discovered by Chas–Sullivan in [3,4] that:

- There is a BV algebra structure $(\Delta, \cdot)$ on $\mathbb{H}_*(\mathcal{L}M) := H_{*+\dim M}(\mathcal{L}M)$. Here Δ is induced by the S^1-action of rotating loops (i.e., $\Delta = J_*$ where J is defined by (2.11)), $\cdot$ is induced by concatenation of crossing loops and is called the *loop product*. The associated Gerstenhaber bracket is called the *loop bracket*. We call this BV algebra the *string topology BV algebra*.
- There is a gravity algebra structure on $H^{S^1}_{*+\dim M-1}(\mathcal{L}M)$ (as an application of Lemma 5.1 to a part of the Gysin sequence (2.13a)), whose second bracket is the *string bracket* ([3, Theorem 6.1]) up to sign (A.1). We call this gravity algebra the *string topology gravity algebra*.

A similar application of Lemma 5.1 to a part of the Connes–Gysin sequence (2.2b) for the mixed complex $(S_*(\mathcal{L}M), \partial, J)$, together with Lemma 2.1 and Lemma 5.2, yields the following lemma.

Lemma 7.2. *For any closed oriented manifold M, there is a gravity algebra structure on $G^{S^1}_{*+\dim M}(\mathcal{L}M)$, such that the natural map $H^{S^1}_{*+\dim M-1}(\mathcal{L}M) \to G^{S^1}_{*+\dim M}(\mathcal{L}M)$ in (2.12) is a morphism of gravity algebras.* ∎

Example 7.3 (Irie's construction [20]). Given any closed oriented C^∞-manifold M, one can associate to M an ns cyclic dg operad $(\mathcal{O}_M, (\tau_k)_{k\geq 0}, \mu, \varepsilon)$ with a multiplication and a unit, defined by:

- For each $k \in \mathbb{Z}_{\geq 0}$, $(\mathcal{O}_M(k)_*, \partial) := (C^{\mathrm{dR}}_{*+\dim M}(\mathcal{L}^M_{k+1,\mathrm{reg}}), \partial)$.
- For each $k \in \mathbb{Z}_{\geq 1}, k' \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, \dots, k\}$, the partial composition $\circ_j : \mathcal{O}_M(k) \otimes \mathcal{O}_M(k') \to \mathcal{O}_M(k + k' - 1)$ is defined by
$$x \circ_j x' := (\mathrm{con}_j)_*(x \times_{\mathrm{ev}_j, \mathrm{ev}_0} x'),$$
where $\times_{\mathrm{ev}_j, \mathrm{ev}_0}$ is the fiber product of de Rham chains with respect to evaluation maps $\mathrm{ev}_j : \mathcal{L}^M_{k+1,\mathrm{reg}} \to M_{\mathrm{reg}}$ and $\mathrm{ev}_0 : \mathcal{L}^M_{k'+1,\mathrm{reg}} \to M_{\mathrm{reg}}$ (it is well defined because of a submersive condition), and $\mathrm{con}_j : \mathcal{L}_{k+1}M \times_{\mathrm{ev}_j, \mathrm{ev}_0} \mathcal{L}_{k'+1}M \to \mathcal{L}_{k+k'}M$ is the concatenation map defined by inserting the second loop into the first loop at the j-th marked point.
- For each $k \in \mathbb{Z}_{\geq 0}$, $\tau_k : \mathcal{O}_M(k)_* \to \mathcal{O}_M(k)_*$ is induced by (4.2c).
- $1_{\mathcal{O}_M} := (M, i_1, 1) \in \mathcal{O}_M(1)_0$, $\mu := (M, i_2, 1) \in \mathcal{O}_M(2)_0$, $\varepsilon := (M, i_0, 1) \in \mathcal{O}_M(0)_0$. Here for $k \geq 0$, $i_k : M \to \mathcal{L}_{k+1}M$ is the map $p \mapsto (0, \gamma_p, 0, \dots, 0)$, where γ_p is the constant loop of length 0 at $p \in M$.

By [20, Theorem 3.1 (ii)], there is an isomorphism $H_*(\widetilde{\mathcal{O}}_M, b) \cong \mathbb{H}_*(\mathcal{L}M)$ of BV algebras, where these BV algebra structures are from Proposition 5.6 and Example 7.1, respectively. (The crucial thing about $\widetilde{\mathcal{O}}_M$ is the chain level structure which refines the string topology BV algebra, but we do not need to use it.)

Let $(\Omega(M)^*, d, \wedge)$ be the dg algebra of differential forms on M. For each $k \geq 0$, there is a chain map $I_k : C^{\mathrm{dR}}_{*+\dim M}(\mathcal{L}^M_{k+1,\mathrm{reg}}) \to \mathrm{Hom}^{-*}(\Omega(M)^{\otimes k}, \Omega(M))$, called *iterated integral of differential forms*: for $\eta_1, \dots, \eta_k \in \Omega(M)$,
$$I_k(U, \varphi, \omega)(\eta_1 \otimes \cdots \otimes \eta_k) := (-1)^{\varepsilon_0}(\varphi_0)_!(\omega \wedge \varphi_1^*\eta_1 \wedge \cdots \wedge \varphi_k^*\eta_k),$$
where $\varepsilon_0 := (\dim U - \dim M)(|\eta_1| + \cdots + |\eta_k|)$ and $\varphi_j := \mathrm{ev}_j \circ \varphi$ $(0 \leq j \leq k)$. Moreover, $I = (I_k)_{k\geq 0} : \mathcal{O}_M \to \mathcal{E}\mathrm{nd}_{\Omega(M)}$ is a morphism of ns dg operads preserving multiplications and units ([20, Lemma 8.5]).

The paring $\langle \alpha, \beta \rangle := \int_M \alpha \wedge \beta$ is a graded symmetric bilinear form on $\Omega^*(M)$ of degree $m = -\dim M$, in line with Example 5.9. The induced dg $\Omega(M)$-bimodule map $\theta : \Omega^*(M) \to (\Omega(M)^\vee[-\dim M])^*$ is a quasi-isomorphism by Poincaré duality, hence induces a quasi-isomorphism
$$\Theta : \mathrm{CH}(\Omega(M), \Omega(M)) \xrightarrow{\simeq} \mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M]). \tag{7.1}$$

Lemma 7.4. *The composition*
$$\theta \circ I_k : \mathcal{O}_M(k)_* \to \mathrm{Hom}^{-*}(\Omega(M)^{\otimes k}, \Omega(M)^\vee[-\dim M]) \quad (k \geq 0) \tag{7.2}$$
is a morphism of cocyclic complexes.

Proof. The composition $\{\theta \circ I_k\}_k$ is one of cosimplicial maps, so it suffices to check $\tau_k \circ \theta \circ I_k = \theta \circ I_k \circ \tau_k$, which is a simple computation by definition. ∎

According to Lemma 7.4, $\Theta \circ I : \widetilde{\mathcal{O}}_M \to \mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M])$ preserves cyclic invariants. Moreover, the following is true.

Lemma 7.5. *The chain map*

$$I : (\widetilde{\mathcal{O}}_M^{\mathrm{cyc}})_* \to \Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M])) \tag{7.3}$$

is a morphism of $\mathsf{M}_{\circlearrowleft}$-algebras.

Proof. First, (7.3) is a morphism of $\mathsf{B}_{\circlearrowleft}$-algebras since $I : \mathcal{O}_M \to \mathcal{E}\mathrm{nd}_{\Omega(M)}$ is a morphism of ns operads, and the (cyclic) brace operations on the associated complexes are defined using operadic compositions. Then by the proof of Corollary 6.8, to show (7.3) is a morphism of $\mathsf{M}_{\circlearrowleft}$-algebras, it suffices to show $I_2(\mu) = \wedge$, where $\mu = (M, i_2, 1) \in \mathcal{O}_M(2)$ is the multiplication in $\mathcal{O}_M$. But this is obvious from definition. ∎

Theorem 7.6. *For any closed oriented C^∞-manifold M, the ns dg operad $\mathcal{O}_M$ with $(\tau_k)_{k\geq 0}$, μ, ε in Example 7.3 gives rise to the following data:*

(1) *A chain complex $\widetilde{\mathcal{O}}_M^{\mathrm{cyc}} := \mathrm{Ker}(1-\lambda) \subset \widetilde{\mathcal{O}}_M$ which is an algebra over $\mathsf{M}_{\circlearrowleft}$. In particular, $H_*(\widetilde{\mathcal{O}}_M^{\mathrm{cyc}})$ carries a gravity algebra structure.*

(2) *An isomorphism $H_*(\widetilde{\mathcal{O}}_M^{\mathrm{cyc}}) \cong G^{S^1}_{*+\dim M}(\mathcal{L}M)$ of gravity algebras, where the gravity algebra structure on $G^{S^1}_{*+\dim M}(\mathcal{L}M)$ is as in Lemma 7.2.*

(3) *A morphism $I : (\widetilde{\mathcal{O}}_M^{\mathrm{cyc}})_* \to \Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M), \Omega(M)^\vee[-\dim M]))$ of $\mathsf{M}_{\circlearrowleft}$-algebras, such that the induced map in homology fits into the following commutative diagram of gravity algebra homomorphisms:*

$$\begin{array}{ccc} H^{S^1}_{*+\dim M-1}(\mathcal{L}M) & \xrightarrow{1} & \mathrm{HC}^{-*+1}_{[u^{-1}]}(\mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M])) \\ \Big\downarrow 2 & & \Big\downarrow 3 \\ & & \mathrm{HC}^{-*}_{[[u]]}(\mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M])) \\ & & \Big\| (2.5) \\ G^{S^1}_{*+\dim M}(\mathcal{L}M) & \xrightarrow{4} & \mathrm{HC}^{-*}_{\lambda}(\mathrm{CH}(\Omega(M), \Omega(M)^\vee[-\dim M])). \end{array} \tag{7.4}$$

Here, arrows 1, 4 are induced by (7.2), arrow 2 is as in (2.12), arrow 3 is as in (2.1). The gravity algebra structures are those on the (negative) S^1-equivariant homology of $\mathcal{L}M$ (Example 7.1, Lemma 7.2) and on the (negative) cyclic cohomology of $\Omega(M)$ (Example 5.9) in view of the Poincaré duality.

Proof. Statement (1) follows from Proposition 5.6 (5). Statement (2) follows from Proposition 5.6 (4) and Proposition 4.8. As for statement (3), I is defined in Lemma 7.5; arrows 2, 3 are gravity algebra homomorphisms by Proposition 5.6 (4) and Example 5.9; arrows

1, 4 are gravity algebra homomorphisms by Lemma 7.4, Lemma 2.4 and Lemma 5.2; diagram (7.4) commutes by Lemma 2.4. Strictly speaking, since the grading of $\mathcal{O}_M(k)_*$ has been shifted by $\dim M$ from $C_*^{\mathrm{dR}}(\mathcal{L}^M_{k+1,\mathrm{reg}})$, there is a minor sign change for δ (2.4) in $\widetilde{\mathcal{O}}_M$ compared to $C^{\mathcal{L}}$ (the same thing happens in [20, Lemma 8.4]), and thus we should repeat the proof of Proposition 4.8 under new signs and use new isomorphisms to make the diagram commute without question of signs, but this is straightforward. ∎

Remark 7.7. Theorem 7.6 (1) is an easy combination of work of Irie and Ward, so it is not new. But it was not known before whether the chain level structures in statement (1) correctly fit with known homology level structures in string topology (it was not even known whether $H_*(\widetilde{\mathcal{O}}^{\mathrm{cyc}}_M)$ is isomorphic to the S^1-equivariant homology of $\mathcal{L}M$), so statement (2) is new. As for statement (3), some (perhaps not all) of the homology level statements are known, see the discussion after Theorem 1.3; the chain level statement is more crucial, and is new because the result that $\mathsf{M}_{\circlearrowleft}$ (nontrivially) acts on

$$\Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M),\Omega(M)^\vee[-\dim M]))$$

is new (Corollary 6.8).

Remark 7.8. It is known that if M is simply connected, then the iterated integral map $I : (\widetilde{\mathcal{O}}_M)_* \to \mathrm{CH}^{-*}(\Omega(M),\Omega(M))$ is a quasi-isomorphism (proved by K. T. Chen [5] and improved by Getzler–Jones–Petrack [17]). In this case, Lemma 2.3 implies that arrows 1, 4 in (7.4) are isomorphisms of gravity algebras.

Note that arrow 4 in (7.4) is not exactly induced by I, but is the composition $\Theta_* \circ I_*$. The author does not know an answer to the following question.

Conjecture 7.9. *For any closed oriented C^∞-manifold M, the quasi-isomorphism* (7.1) *restricts to a quasi-isomorphism on (weakly) cyclic invariants,*

$$\Theta^{-1}(\mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M),\Omega(M)^\vee[-\dim M])) \xrightarrow{\simeq} \mathrm{CH}^{-*}_{\mathrm{cyc}}(\Omega(M),\Omega(M)^\vee[-\dim M]).$$

A. Sign rules

A.1. Koszul sign rule

Compared to ungraded formulas, a sign $(-1)^{|a||b|}$ is produced in a graded setting whenever a symbol a travels across another symbol b. For example if A, B are graded vector spaces, the graded tensor product of graded linear maps $f : A \to B$ and $g : C \to D$, $f \otimes g : A \otimes C \to B \otimes D$, is defined by $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$.

A.2. Sign change rule for (de)suspension

Let $C = \{C^i\}_{i\in\mathbb{Z}}$ be a graded vector space. For any $n \in \mathbb{Z}$, define a shifted graded vector space $C[n] = \{C[n]^i\}_{i\in\mathbb{Z}}$ by $C[n]^i := C^{i+n}$. (In homological grading this turns into

$C[-n]_{-i} := C_{-i-n}$.) $C[-1]$ is often denoted by ΣC and is called the suspension of C. Let $s : C \to C[-1]$; $x \mapsto sx$ be the shifted identity map which is of degree 1. By the Koszul sign rule, for $x_1, \dots, x_k \in C$,

$$s^{\otimes k}(x_1 \otimes \cdots \otimes x_k) = (-1)^{\sum_{i=1}^{k}(k-i)|x_i|} sx_1 \otimes \cdots \otimes sx_k. \tag{A.1}$$

Here $|x_i|$ denotes the degree of x_i in C, and the sign $(-1)^{(k-i)|x_i|}$ comes from exchanging positions of $k-i$ copies of s with that of x_i. The sign change (A.1) identifies the graded exterior algebra of C with the graded symmetric algebra of $C[-1]$, as

$$(s \otimes s)((-1)^{|x_1||x_2|} x_2 \otimes x_1) = (-1)^{|x_1|}((-1)^{1+|sx_1||sx_2|} sx_2 \otimes sx_1).$$

The same rule applies to the sign change between A and $A[1]$, the desuspension of A.

A.3. Operadic suspension

Let $(\mathcal{O}, \circ_i)$ be a dg operad in cohomological grading. The operadic suspension of $\mathcal{O}$ is a dg operad $(\mathfrak{s}\mathcal{O})$ with partial compositions $\tilde{\circ}_i$ satisfying

$$\begin{aligned} \mathfrak{s}\mathcal{O}(n) &= \mathcal{O}(n)[1-n], \\ a \,\tilde{\circ}_i\, b &= (-1)^{(i-1)(m-1)+(n-1)|b;\mathcal{O}(m)|}, \end{aligned} \tag{A.2}$$

where $a \in \mathfrak{s}\mathcal{O}(n)$, $b \in \mathfrak{s}\mathcal{O}(m)$, $|b; \mathcal{O}(m)|$ is the degree of b in $\mathcal{O}(m)$. For an explanation of signs (which comes from the Koszul sign rule), see, e.g., [20, Section 2.5.4].

When $\mathcal{O} = \mathcal{E}\mathrm{nd}_A$ is the endomorphism operad of a dg algebra A, there is an isomorphism of dg operads $\mathfrak{s}\mathcal{O} = \mathfrak{s}(\mathcal{E}\mathrm{nd}_A) \cong \mathcal{E}\mathrm{nd}_{A[1]}$. Therefore, for signs related to $\mathfrak{s}(\mathcal{E}\mathrm{nd}_A)$, one may alternatively use the Koszul sign rule for $A[1]$ and perform (A.1) when necessary.

A.4. Cyclic permutation

If $(\mathcal{O}, \circ_i)$ is a cyclic dg operad, then $(\mathfrak{s}\mathcal{O}, \tilde{\circ}_i)$ also carries a cyclic structure where $\tilde{\tau}_k = (-1)^k \tau_k$ under the naive identification $\mathfrak{s}\mathcal{O}(k) = \mathcal{O}(k)$. On the other hand, let A be a dg algebra, consider the cocyclic complex $\{\mathrm{Hom}(A^{\otimes k+1}, \mathbb{R}), \tau_k\}$ and the operation $\tilde{\tau}_k$ on $\mathrm{Hom}(A[-1]^{\otimes k+1}, \mathbb{R})$ induced by τ_k under the linear isomorphism $s : A \to A[-1]$. Then the following equality says $\tilde{\tau}_k = (-1)^k \tau_k$ after sign change (A.1):

$$\tilde{\tau}_k \circ s^{\otimes k+1} = \tilde{\tau}_k \circ (s^{\otimes k} \otimes s) = (-1)^k (s \otimes s^{\otimes k}) \circ \tau_k,$$

where $s^{\otimes k}$ applies to $x_1 \otimes \cdots \otimes x_k \in A^{\otimes k}$ and s applies to $x_{k+1} \in A$. Therefore, when discussing cyclic homology theories of A under the naive identification $A = A[-1]$, the subspace of cyclic invariants in $C(k)$ is $\mathrm{Ker}(1-\lambda) = \mathrm{Ker}(1-\tilde{\tau}_k)$ and the operator N satisfies $N|_{C(k)} = \sum_{i=0}^{k} \lambda^i = \sum_{i=0}^{k} \tilde{\tau}_k^i$.

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