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Continuity of the temperature in a multi-phase transition problem. Part II

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Abstract. Local continuity is established for locally bounded, weak solutions to a doubly non-linear parabolic equation that models the temperature of a material undergoing a multi-phase transition. The enthalpy, as a maximal monotone graph of the temperature, is allowed to possess several jumps and/or infinite derivatives at the transition temperatures. The effect of the p-Laplacian-type diffusion is also considered. As an application, we demonstrate a continuity result for the saturation in the flow of two immiscible fluids through a porous medium, when irreducible saturation is present.

1. Introduction

Initiated in [11], we keep up the study of the continuity of the temperature of a material undergoing a multi-phase change. In this manuscript we consider the following non-linear parabolic partial differential equation:

$$\partial_t \widetilde{\beta}(u) - \operatorname{div}(|Du|^{p-2} Du) \ni 0 \quad \text{weakly in } E_T, \;\text{ for } p \geq 2. \tag{1.1}$$

Here E is an open set of $\mathbb{R}^N$ with $N \geq 1$ and $E_T := E \times (0, T]$ for some $T > 0$. The enthalpy $\widetilde{\beta}(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ defined by

$$\widetilde{\beta}(u) = \beta(u) + \sum_{i=0}^{\ell} \nu_i \mathcal{H}_{e_i}(u) \quad \text{for some } \ell \in \mathbb{N} \cup \{\infty\},\ e_i \in \mathbb{R},\ \text{and } \nu_i > 0. \tag{1.2}$$

We have assumed that $0 = e_o < e_1 < \cdots < e_\ell$ and used the notation

$$\mathcal{H}_{e_i}(u) = \begin{cases} 1 & u > e_i, \\ [0,1] & u = e_i, \\ 0 & u < e_i, \end{cases}$$

while $\beta(\cdot)$ is a continuous and piecewise C^1 function in $\mathbb{R}$ satisfying

$$\begin{cases} \beta' \geq \alpha_o & \text{for some constant } \alpha_o > 0, \\ \beta' < \infty & \text{except at } e_i \text{ for } i \in \{0, 1, \dots, \ell\}. \end{cases} \tag{1.3}$$

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Moreover, we stipulate that $\beta(\cdot)$ has the same graph near each e_i after translation, that is,

$$\begin{cases} \beta(u+e_i)=\beta(u)+\beta(e_i) \text{ for all } i\in\{0,1,\dots,\ell\}, \text{ if } |u|\le d \\ \qquad \text{for some } 0<d<\frac{1}{2}\min\big\{e_{i+1}-e_i:0\le i\le \ell-1\big\}. \end{cases} \tag{1.4}$$

Condition (1.4) is not restrictive but for ease of notation only, as we do not impose any growth condition on $\beta(\cdot)$ near e_i. Finally, we assume that for some $0<\bar{d}\le d$,

$$\beta \text{ is concave in } (0,\bar{d}) \text{ and is convex in } (-\bar{d},0), \tag{1.5}$$

$$\forall\, u\in(-\bar{d},\bar{d}), \quad \beta(u)=-\beta(-u). \tag{1.6}$$

The behavior of β in a neighborhood of the origin is depicted in Figure 1, whereas an example of graph of $\widetilde{\beta}(\cdot)$ is given in Figure 2.

The *main result* is that locally bounded, local weak solutions to (1.1) with $p\ge 2$ are locally continuous. Moreover, our estimates are structural and a modulus of continuity can be traced like in [11, Theorem 1.1], given explicit $\beta(\cdot)$. As an application of our argument, we establish a continuity result for the saturation in the flow of two immiscible fluids through a porous medium, when irreducible saturation is present (more about this can be found in Section 1.3).

1.1. Statement of the results

From here on, we will deal with the following more general parabolic partial differential equation modeled on (1.1):

$$\partial_t\widetilde{\beta}(u)-\operatorname{div}\mathbf{A}(x,t,u,Du)\ni 0 \quad \text{ weakly in } E_T. \tag{1.7}$$

Here $\widetilde{\beta}(\cdot)$ is defined in (1.2). The function $\mathbf{A}(x,t,u,\xi)\colon E_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ is assumed to be measurable with respect to $(x,t)\in E_T$ for all $(u,\xi)\in\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (u,ξ) for almost every $(x,t)\in E_T$. Moreover, we assume the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi\ge C_o|\xi|^p \\ |\mathbf{A}(x,t,u,\xi)|\le C_1|\xi|^{p-1} \end{cases} \quad \text{a.e. } (x,t)\in E_T,\ \forall\, u\in\mathbb{R},\ \forall\,\xi\in\mathbb{R}^N, \tag{1.8}$$

where C_o and C_1 are given positive constants, and we take $p\ge 2$.

In the remainder of the paper, the set $\{\alpha_o,\beta,\bar{d},d,\nu_i,p,N,C_o,C_1,\|u\|_{\infty,E_T}\}$ will be referred to as the *data*. A generic positive constant γ depending on the data will be used in the estimates.

The formal definition of local weak solution to (1.7) will be given in Section 1.4. Now we present the main theorem.

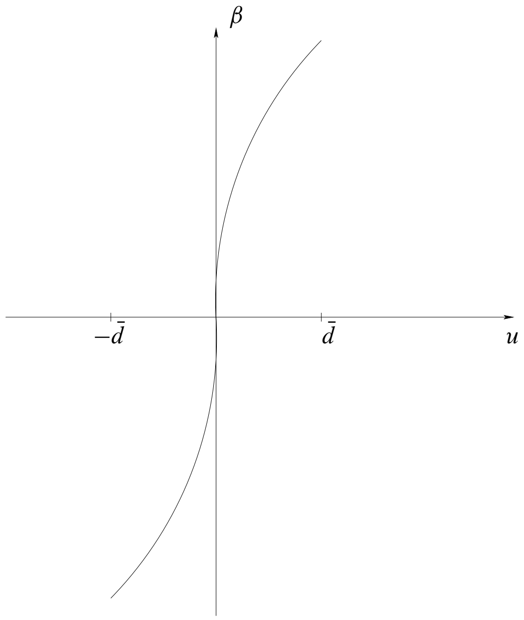

Figure 1. Graph of β at the origin

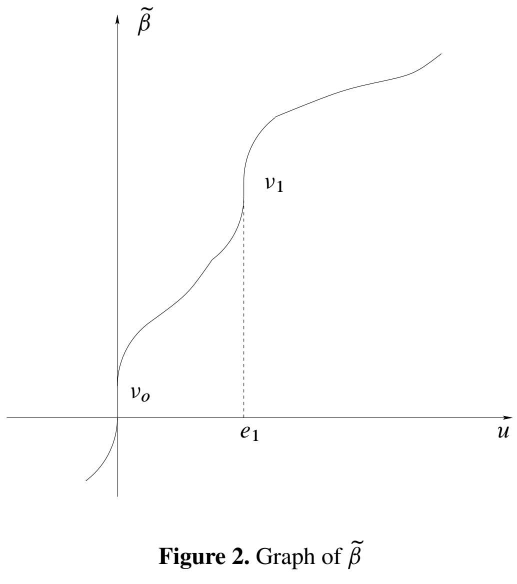

Figure 2. Graph of $\widetilde{\beta}$

Theorem 1.1. *Let u be a bounded weak solution to (1.7) in E_T, under the structure condition given by (1.8) for $p \geq 2$. The function u is continuous in any compact set $\mathcal{K} \subset E_T$. More precisely, for every pair of points $(x_1, t_1), (x_2, t_2) \in \mathcal{K}$, it holds that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \boldsymbol{\omega}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}}),$$

where the modulus $\boldsymbol{\omega}(\cdot)$ is determined by the data and the distance from $\mathcal{K}$ to the parabolic boundary of E_T.

Remark 1.1. The assumption about *bounded* solutions in Theorem 1.1 is quite standard when dealing with an equation like the one we consider here. For instance, given bounded boundary data, one should be able to prove a weak maximum principle (cf. [7, Chapter V, Theorem 3.3]) and apply Theorem 1.1 to construct locally continuous solutions. On the other hand, we do not know of results where *local* boundedness of u is proved when one has a graph $\widetilde{\beta}$ as in (1.2). However, it is conceivable, as seen, for example, in [7, Chapter V, Section 5] that *qualitative* information on the boundedness of u can be converted into *quantitative* one.

Remark 1.2. The perspective of our work is definitely *local* and *in the interior* of the domain E_T. Extending our results up to the boundary, both under Dirichlet and Neumann conditions, is a very interesting and open problem. Recent results in [17] suggest that a geometric density condition probably suffices to achieve global continuity for solutions to the Dirichlet problem, whereas at least a C^1 boundary should be required for the Neumann problem. However, at this stage, these are just speculations: extending our methods up to the boundary for a general $\widetilde{\beta}$ as in (1.2) poses quite a number of technical difficulties.

Remark 1.3. As we frequently point out in what follows in an explicit way, all the constants, parameters, and so on, depend on p, which is one of the data. We work in such a way that all estimates are stable as $p \downarrow 2$, that is, given any parameter γ, we always have $\lim_{p\downarrow 2} \gamma(p) = \gamma(2)$, where $\gamma(2)$ is a finite quantity. When studying the regularity of solutions to the parabolic p-laplacian, $p = 2$ represents a threshold value that separates the two quite different regimes which correspond to $p > 2$ and $1 < p < 2$, and the stability of the estimates as $p \downarrow 2$ (as it is the case here), or as $p \uparrow 2$ (which is beyond the framework of this manuscript) is a much sought-after condition; under this point of view, see, for example [7, Chapters III–IV].

Remark 1.4. The main argument can be adapted when lower-order terms are present. In fact, we will deal with some specific lower-order terms in Section 5, which bear particular physical meanings. For general lower-order terms, the modifications can be done as in [15, Chapter III]. We refrain from entering into details in this case.

Remark 1.5. Concerning the characterization of the modulus of continuity, there are interesting connections between solutions to the problem under consideration here and solutions to systems arising in the study of congested traffic dynamics (see [4]). Moreover, another comment is in order. The stability of all the estimates as $p \downarrow 2$ notwithstanding, the modulus of continuity does not improve when p tends to the limit value. Under this point of view, it is the same kind of situation that occurs when one compares the parabolic p-Laplacian with bounded and measurable coefficients that depend on (x, t) with linear parabolic equations with the same kind of coefficients: in both instances solutions are Hölder continuous (see, respectively, [7, Chapters III–IV] and [15, Chapter III]), and the only relevant fact is that for the Hölder continuity exponent α one has $\lim_{p\to 2} \alpha(p) = \alpha(2)$. The results of [12, Theorem 1.1] seem to suggest that the modulus of continuity depends on an interplay between p and N, but we refrain from going any further into details.

It could be remarked that the type of modulus of continuity one ends up with does not have a clear meaning in the application; indeed, in [11], for $\beta(u) = u$, the modulus of continuity established involves $\ln^{(6)}$, that is, logarithm composed with itself six times, and here things would not be very different. However, we think that this is not the case: as observed in [11, Corollary 1.1], once a modulus of continuity is obtained, we can localize and improve it.

Corollary 1.1 (Localization). *Under the hypotheses of Theorem 1.1, the modulus automatically improves to the one for the two-phase problem corresponding to a graph β that satisfies* (1.3)–(1.6).

Moreover, in our opinion there is also a more theoretical aspect that makes Theorem 1.1 interesting: under the purely qualitative assumptions given by (1.2)–(1.6) on the graph β, it is nevertheless possible to prove the continuity of the solution. Therefore, however singular, the diffusion process still ensures the regularity of u.

1.2. More general graphs

A priori, the number of jump points can be infinite, but we do not want them to cluster at any real point: this is the motivation of the assumption $d > 0$ in (1.4), where min might be inf if $\ell = +\infty$, and $\bar{d} > 0$ in (1.5)–(1.6). On the other hand, since we assume to work with bounded solutions u, that is, $-\|u\|_{\infty,E_T} \le u \le \|u\|_{\infty,E_T}$, even when an infinite number of jump points occurs in the graph $\widetilde{\beta}$, only finitely many of them actually have to be dealt with.

The above-defined $\widetilde{\beta}(\cdot)$ carries two types of singularities: vertical jumps brought by the Heaviside functions and infinite derivatives of $\beta(\cdot)$. Clearly, after proper rearrangement, they may or may not happen at a same temperature. Thus, taking into account the previous remark, $\widetilde{\beta}(\cdot)$ actually concerns a class of piecewise C^1 functions, where only finitely many jumps and finitely many infinite derivatives have to be dealt with.

It might seem restrictive to assume, as we do in (1.3)–(1.6), that infinite derivatives take place only at jump points. However, this is just for the sake of simplicity, and it does not imply any loss of generality. Indeed, as we discuss at the end of Section 1.3, nothing would be altered if there were no derivative blow-up at a jump point. Conversely, if we had such a blow-up at a point where no jump occurs, condition (2.5) in Lemma 2.4, which ensures that the effect due to the jump is negligible, would be automatically satisfied, and, once more, no change would take place. Under this point of view, see also the remarks before the statement of Lemma 2.4.

Condition (1.6) is not the most general one, and indeed it can be extended. From a technical point of view, the crucial task is preserving the validity of Lemma 2.4. This can be done, for instance, by assuming that the graph β is steeper on the left-hand side of the origin than on the right-hand one. Then, we could have

$$\begin{aligned} &\forall\, u \in (0, \bar{d}), \quad \beta(u) \le -\beta(-u); \\ &\forall\, u \in (0, \bar{d}), \quad \beta'(u) \le \beta'(-u); \\ &\text{for some suitable } 0 < \sigma_o < \min\{1, \bar{d}\}, \quad \int_0^{\sigma_o} \frac{\beta'(s)}{|\beta(-s)|}\, \mathrm{d}s = \infty. \end{aligned}$$

The case where the right-hand side is steeper than the left-hand one can be dealt with in an analogous way.

Similar considerations hold as far as (1.4) is concerned: it reduces the analysis of β to the study of its behavior in a neighborhood of the origin. However, one can dispense with such an assumption; if β had a different behavior at each discontinuity point e_i, then Lemma 2.4 would yield a different value of j_*, say $j_{*,i}$, and consequently of ξ_i, for each $i = 1, \dots, \ell$. Nevertheless, the arguments of Section 3 would remain the same, provided we choose

$$\xi = \min\{\xi_1, \xi_2, \dots, \xi_\ell\};$$

this is possible, since only a finite number of points e_i are considered, due the boundedness of u, as we have already discussed at the very beginning of this section.

1.3. Novelty and significance

As already mentioned at the beginning, this is the second part of an ongoing study about the local continuity for locally bounded, weak solutions to a doubly non-linear parabolic equation that models the temperature of a material undergoing a multi-phase transition, the so-called Stefan problem. This is a classical topic which has seen a huge amount of contributions since the pioneering work of Olga Oleĭnik in 1960 [18]: we refer to [11, Section 1.2] for a general introduction to the regularity of solutions to the problem, and the corresponding state of the art. The interested reader can see [9, 20, 24], and also [2], where the physically relevant investigation of the behavior of solutions to the Stefan problem, when a volumetric heat source is present, is considered. The understanding of the behavior of solutions reached so far notwithstanding, the general mathematical theory of weak solutions to multi-phase transitions is still fragmented, an overall comprehension is lacking, and there are yet a number of delicate and deep issues which are completely open, in particular, as far as quantitative moduli of continuity of solutions for general graphs β are concerned. Our study, which started in [11], builds on recent advances by the authors on the local regularity of solutions to the parabolic p-Laplacian; these progresses help in shedding light on some of the issues in the Stefan problem which still await full and satisfactory answers and foster concrete hopes of gaining a more thorough perspective.

In [11], the enthalpy, as a maximal monotone graph of the temperature, was allowed to possess several jumps at the transition temperatures, but otherwise was an absolutely continuous function β in $\mathbb{R}$, such that

$$0 < \alpha_o \le \beta' \le \alpha_1,$$

for two constants α_o and α_1.

In the present work we dispense with the bound above on β', and we allow β to have infinite derivatives at the transition temperatures. Besides the intrinsic mathematical interest of this kind of graphs, they are also significant from the point of view of applications; indeed, experimental measurements of enthalpy curves in the so-called *phase-change materials* (PCM) show graphs whose derivatives can blow up (see, for example, [19, Figures 3, 9, and 10]). Without entering into details here, it suffices to say that PCM are materials which release or absorb sufficient energy at phase transition to provide useful heat or cooling. Moreover, these graphs are usually obtained through measurements; despite their *qualitative* nature, we can conclude that u is continuous, as already remarked above.

There is another important instance of maximal monotone graphs with infinite derivatives, arising from real-world problems: in the so-called Buckley–Leverett model for the motion of two immiscible fluids in a porous medium (see [14, 16]), β presents two singularities, say at $u = 0$ and $u = 1$, where β can become vertical with an exponential speed, or even faster, and might also exhibit a jump in the case of irreducible saturation. The *Hölder* continuity of the saturation u was studied in [23], where a power-like behavior at $u = 0$ and $u = 1$ was considered. This result was extended in [8], where a weaker modulus of continuity was shown to hold, assuming no a priori knowledge about the singularity of β at both critical points; however, the presence of a jump could not be taken into account. This is the issue we consider in Section 5, where, as a straightforward application of the techniques developed in Sections 2–4, we prove that the saturation is continuous up to the irreducible value. Therefore, the continuity issue in this problem can be considered as definitely settled, and in our opinion this represents an interesting step forward with respect to the existing literature.

Besides the interest for applications, the continuity result is also very important from a mathematical perspective, as it shows the strong smoothing effect that the non-linearities, both of β and of $\mathbf{A}$, have.

Coming to the technical aspects, the main novelty is represented by Lemma 2.4; it is based on previous work developed in [8, Sections 4.3–4.5], but this is required to be properly adapted in order to take care of the more general context under consideration here.

There is another important technical feature that deserves proper comments; this also helps to understand why the approach we develop here works for $p \geq 2$, but breaks down when $1 < p < 2$. For simplicity, assume we have a single jump point at the origin. The "derivative" of $\widetilde{\beta}$ is infinite at such a point, and consequently, $\widetilde{\beta}$ is singular at $[u = 0]$. On the other hand, since $p > 2$, the p-laplacian is degenerate at $[Du = 0]$, and singular as $|Du| \to \infty$. Such a range for p is usually referred to as *degenerate regime*. In the classical De Giorgi approach to continuity of weak solutions, the essential point lies in showing that the oscillation of u reduces in a quantified way along a sequence of nested cylinders. Due to the two singularities of equations (1.7)–(1.8), here one needs to stretch or compress the cylinders according to the oscillation itself. Such an approach is called "intrinsic scaling" (see [7, Chapters III–IV]). Suppose $0 \leq u \leq 1$. If one works in the region of E_T where $u \approx 0$, the singularity of $\widetilde{\beta}$ prevails, and this calls for a stretching of the cylinder; on the other hand, in the region where $u \approx 1$, the singularity of $\widetilde{\beta}$ plays no role, and the dominant effect is due to the singularity of the p-laplacian. Luckily enough, in such a case $p > 2$ requires a stretching of the cylinder as well. Hence, in the degenerate regime, although independent, both singularities concur and a proper balancing is relatively easy. On the contrary, if we had $1 < p < 2$, the singularity coming from the p-laplacian would require a *compression* of the cylinder; therefore, balancing these two contrasting requirements is much more challenging and cannot be achieved with a straightforward adaptation of the techniques we employ here.

Moreover, an estimate of the modulus of continuity can be achieved once a specific expression of β is given; it suffices to trace all our computations, step by step, inserting its functional dependence.

Finally, even though β is assumed to be a continuous and piecewise C^1 function whose derivative β' blows up at e_i, if we had $\beta(u) \equiv u$, $\beta' \equiv 1$ and no blow-up occurred, the reasoning in Lemma 2.4 and Section 3 would not change, and the conclusions would remain the same. Therefore, the continuity result of [11, Theorem 1.1] can be retrieved as a particular case from the framework considered here.

The structure of the paper is as follows: in Section 2 we collect all the preliminary tools; most of them are known and, therefore, we refer to elsewhere for their proofs, the only exception being Lemma 2.4, which is dealt with in Section 4. Section 3 is devoted to the proof of Theorem 1.1. Finally, Section 5 applies the main result to the flow of two immiscible fluids with irreducible saturation; the application is far from trivial, because it requires a careful analysis of particular lower-order terms. As clearly pointed out in [6], only the particular structure of the right-hand side combined with the incompressibility condition allows for the wanted regularity.

1.4. Definition of solution

A function

$$u \in L^\infty_{\text{loc}}(0, T; L^2_{\text{loc}}(E)) \cap L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(E))$$

is a local, weak sub(super)-solution to (1.7) with the structure conditions in (1.8), if for every compact set $K \subset E$ and every sub-interval $[t_1, t_2] \subset (0, T]$, there is a selection $v \subset \widetilde{\beta}(u)$, that is,

$$\big\{(z, v(z)) : z \in E_T\big\} \subset \big\{\big(z, \widetilde{\beta}[u(z)]\big) : z \in E_T\big\},$$

such that

$$\int_K v\zeta \,\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K [-v\partial_t\zeta + \mathbf{A}(x,t,u,Du)\cdot D\zeta]\mathrm{d}x\mathrm{d}t \le (\ge)\, 0$$

for all non-negative test functions

$$\zeta \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(K)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}_o(K)).$$

All the integrals are convergent as $v \in L^\infty_{\mathrm{loc}}(0,T; L^2_{\mathrm{loc}}(E))$.

A function that is both a local, weak sub-solution and a local, weak super-solution is termed a local, weak solution.

The use of test functions that involve u itself is standard in the regularity theory. Nevertheless, the above notion of solution, though standard in the existence theory, does not grant the admissibility of u a test function due to the lack of information in the time derivative and the jumps of $\widetilde{\beta}$. A common device to overcome this is to regularize (1.7). More precisely, for $\varepsilon \in (0, d)$, we let

$$\mathcal{H}^\varepsilon_{e_i}(u) = \begin{cases} 1 & u > e_i + \varepsilon, \\ \frac{1}{\varepsilon}(u - e_i) & e_i \le u \le e_i + \varepsilon, \\ 0 & u < e_i, \end{cases}$$

and define

$$\widetilde{\beta}_\varepsilon(u) \equiv \beta(u) + H_\varepsilon(u) := \beta(u) + \sum_{i=0}^{\ell} \nu_i \mathcal{H}^\varepsilon_{e_i}(u);$$

we now deal with

$$\partial_t[\beta(u) + H_\varepsilon(u)] - \operatorname{div}\mathbf{A}(x,t,u,Du) \le (\ge)\, 0 \quad \text{weakly in } E_T. \tag{1.9}$$

A function u is termed a local weak sub(super)-solution to (1.9) if

$$\begin{cases} \displaystyle\int_0^u [\beta'(s) + H'_\varepsilon(s)]s \,\mathrm{d}s \in C_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(E)), \\ u \in L^p_{\mathrm{loc}}(0,T; W^{1,p}_{\mathrm{loc}}(E)), \end{cases}$$

and for every compact subset K of E and every subinterval $[t_1, t_2]$ of $(0, T]$, we have

$$\int_K [\beta(u) + H_\varepsilon(u)]\zeta \,\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \big\{-[\beta(u) + H_\varepsilon(u)]\partial_t\zeta + \mathbf{A}(x,t,u,Du)\cdot D\zeta\big\}\mathrm{d}x\mathrm{d}t \le (\ge)\, 0$$

for all non-negative test functions

$$\zeta \in W^{1,2}_{\rm loc}(0,T;L^2(K)) \cap L^p_{\rm loc}(0,T;W^{1,p}_o(K)).$$

A function u that is both a local weak sub-solution and a local weak super-solution to (1.9) is termed a local weak solution.

Notice that the above notion of local weak solution to (1.9) still does not involve any time derivative of u. However, we may now use u as a test function, modulo a standard time mollification procedure (cf. [7, Chapter II]).

In this manuscript we assume that local solutions to (1.7) can be approximated by a sequence of solutions to (1.9) locally uniformly. In [10, Section 1] it is briefly explained why it is not restrictive to make such an assumption. The main goal is to establish a modulus of continuity for solutions to (1.9) uniform in ε, which then is inherited by solutions to (1.7) in the uniform convergence. Under this point of view, in [9, Section 5.1] there is an interesting discussion about how this way of proceeding might be used in order to prove existence of weak solutions to the equation.

2. Preliminary tools

2.1. Energy estimates

We denote by $K_R(x_o)$ the cube in $\mathbb{R}^N$ with center x_o and side length $2R$, whose faces are parallel with the coordinate planes. Moreover, for any $k \in \mathbb{R}$, we let

$$(u-k)_+ \equiv \max\{u-k,0\}, \quad (u-k)_- \equiv \max\{k-u,0\}.$$

The following energy estimate is standard (see, for example, [10, proof of estimate (2.5)]):

Proposition 2.1. *Let u be a local weak sub(super)-solution to (1.9) with (1.8) in E_T. There exists a constant $\gamma(C_o, C_1, p) > 0$ such that for all cylinders $Q_{R,S} = K_R(x_o) \times (t_o - S, t_o) \subset E_T$, every $k \in \mathbb{R}$, and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(x_o) \times (t_o - S, t_o)$, it holds that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o} \int_{K_R(x_o)\times\{t\}} \Big(\int_0^{(u-k)_\pm} \beta'(k\pm s)s\,\mathrm{d}s\Big)\zeta^p\,\mathrm{d}x \\
&\quad + \operatorname*{ess\,sup}_{t_o-S<t<t_o} \int_{K_R(x_o)\times\{t\}} \Big(\int_0^{(u-k)_\pm} H'_\varepsilon(k\pm s)s\,\mathrm{d}s\Big)\zeta^p\,\mathrm{d}x \\
&\quad + \iint_{Q_{R,S}} \zeta^p |D(u-k)_\pm|^p\,\mathrm{d}x\mathrm{d}t \\
&\le \gamma \iint_{Q_{R,S}} (u-k)_\pm^p |D\zeta|^p\,\mathrm{d}x\mathrm{d}t + \gamma \iint_{Q_{R,S}} \Big(\int_0^{(u-k)_\pm} \beta'(k\pm s)s\,\mathrm{d}s\Big)|\partial_t\zeta^p|\,\mathrm{d}x\mathrm{d}t \\
&\quad + \gamma \iint_{Q_{R,S}} \Big(\int_0^{(u-k)_\pm} H'_\varepsilon(k\pm s)s\,\mathrm{d}s\Big)|\partial_t\zeta^p|\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_{K_R(x_o)\times\{t_o-S\}}\Big(\int_0^{(u-k)_\pm}\beta'(k\pm s)s\,\mathrm{d}s\Big)\zeta^p\,\mathrm{d}x\\
&+\int_{K_R(x_o)\times\{t_o-S\}}\Big(\int_0^{(u-k)_\pm}H_\varepsilon'(k\pm s)s\,\mathrm{d}s\Big)\zeta^p\,\mathrm{d}x. \qquad (2.1)
\end{aligned}
$$

Now we discuss some simplification of the general energy estimate in (2.1). First of all, we deal with the terms containing H_ε. Since $H_\varepsilon'\ge 0$, we may discard the second term on the left-hand side. Meanwhile, since H_ε is a linear combination of Heaviside functions modulo ε, we have

$$
\int_0^{(u-k)_\pm}H_\varepsilon'(k\pm s)s\,\mathrm{d}s\le (u-k)_\pm\int_0^{(u-k)_\pm}H_\varepsilon'(k\pm s)\,\mathrm{d}s\le\Big(\sum_{i=0}^{\ell}\nu_i\Big)(u-k)_\pm,
$$

provided $\sum_{i=0}^{\ell}\nu_i$ is finite. Instead, if it is infinite, we let

$$
M:=\|u\|_{\infty,E_T},
$$

and estimate

$$
\int_0^{(u-k)_\pm}H_\varepsilon'(k\pm s)s\,\mathrm{d}s\le\sup_{|s|\le M}|H_\varepsilon(s)|(u-k)_\pm.
$$

Hence, in this case the subsequent estimates will depend also on M, but will be independent of ε.

Next we deal with the terms of β. By using the fact that $\beta'\ge\alpha_o$ in (1.3), we estimate

$$
\int_0^{(u-k)_\pm}\beta'(k\pm s)s\,\mathrm{d}s\ge\frac12\alpha_o(u-k)_\pm^2.
$$

On the other hand, we easily obtain

$$
\int_0^{(u-k)_\pm}\beta'(k\pm s)s\,\mathrm{d}s\le\sup_{|s|\le M}|\beta(s)|(u-k)_\pm.
$$

Taking into account these remarks, we reduce (2.1) to

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o}\frac12\alpha_o\int_{K_R(x_o)\times\{t\}}\zeta^p(u-k)_\pm^2\,\mathrm{d}x+\iint_{Q_{R,S}}\zeta^p|D(u-k)_\pm|^p\,\mathrm{d}x\mathrm{d}t\\
&\quad\le\gamma\iint_{Q_{R,S}}(u-k)_\pm^p|D\zeta|^p\,\mathrm{d}x\mathrm{d}t+\gamma\iint_{Q_{R,S}}(u-k)_\pm|\partial_t\zeta^p|\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\gamma\int_{K_R(x_o)\times\{t_o-S\}}\zeta^p(u-k)_\pm\,\mathrm{d}x,
\end{aligned}
$$

where the constant γ depends on the data and M.

If the cutoff function ζ is chosen to vanish at $t_o - S$, then we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o} \frac{1}{2}\alpha_o \int_{K_R(x_o)\times\{t\}} \zeta^p (u-k)_\pm^2 \,\mathrm{d}x + \iint_{Q_{R,S}} \zeta^p |D(u-k)_\pm|^p \,\mathrm{d}x\mathrm{d}t \\
&\quad \le \gamma \iint_{Q_{R,S}} \big[(u-k)_\pm^p |D\zeta|^p + (u-k)_\pm |\partial_t \zeta^p|\big] \,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{2.2}$$

which corresponds to [10, (2.5)].

On the other hand, if the cutoff function is chosen independent of the t variable, that is, $\zeta = \zeta(x)$, then we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o} \frac{1}{2}\alpha_o \int_{K_R(x_o)\times\{t\}} \zeta^p (u-k)_\pm^2 \,\mathrm{d}x + \iint_{Q_{R,S}} \zeta^p |D(u-k)_\pm|^p \,\mathrm{d}x\mathrm{d}t \\
&\quad \le \gamma \iint_{Q_{R,S}} (u-k)_\pm^p |D\zeta|^p \,\mathrm{d}x\mathrm{d}t + \gamma \int_{K_R(x_o)\times\{t_o-S\}} \zeta^p (u-k)_\pm \mathrm{d}x,
\end{aligned} \tag{2.3}$$

which corresponds to [10, (2.6)].

2.2. Logarithmic estimates

Letting k, u, and $Q_{R,S}$ be as in Proposition 2.1, we set

$$\mathcal{L} := \sup_{Q_{R,S}} (u-k)_\pm,$$

take $c \in (0, \mathcal{L})$, and introduce the following function in $Q_{R,S}$:

$$\Psi(x,t) \equiv \Psi(\mathcal{L}, (u-k)_\pm, c) := \ln_+ \Big(\frac{\mathcal{L}}{\mathcal{L} - (u-k)_\pm + c} \Big).$$

To simplify the notation, if we let $\Psi(s) = \Psi(\mathcal{L}, s, c)$, then

$$\Psi'(s) = \frac{1}{\mathcal{L} - s + c} \chi_{[s>c]}(s).$$

We may prove the following logarithmic energy estimate just like in [10, (2.7)] or in [11, Proposition 2.2]:

Proposition 2.2. *Let the hypotheses in Proposition 2.1 hold. There exists $\gamma > 1$ depending only on the data and on M, such that for any $\sigma \in (0,1)$,*

$$\begin{aligned}
\sup_{t_o-S\le t\le t_o} \int_{K_{\sigma R}(x_o)} \Psi^2(x,t)\,\mathrm{d}x &\le \frac{\gamma}{c} \int_{K_R(x_o)} \Psi(x, t_o - S)\,\mathrm{d}x \\
&\quad + \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_{R,S}} \Psi |\Psi'|^{2-p} \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

For the cylinder $\mathcal{Q} := K \times (T_1, T_2) \subset E_T$, we introduce the numbers $\mu^\pm$ and ω satisfying

$$\mu^+ \geq \operatorname*{ess\,sup}_{\mathcal{Q}} u, \quad \mu^- \leq \operatorname*{ess\,inf}_{\mathcal{Q}} u, \quad \omega \geq \mu^+ - \mu^-.$$

Then, we have the following consequence of the above logarithmic estimate, whose proof can be retrieved from [11, Lemma 2.4]; it indicates how the measure of a set near the supremum/infimum shrinks at each level of an arbitrarily long time interval, once initial pointwise information is assigned:

Lemma 2.1. *Let u be a local weak sub(super)-solution to (1.9) with (1.8) in E_T. For $\xi \in (0,1)$, set $\theta = (\xi\omega)^{2-p}$. Suppose that*

$$\pm(\mu^\pm - u(\cdot, t_1)) \geq \xi\omega \quad \text{a.e. in } K_\varrho(x_o).$$

Then, for any $\alpha \in (0,1)$ and $A \geq 1$, there exists $\bar{\xi} \in (0, \frac{1}{4}\xi)$ such that

$$|[\pm(\mu^\pm - u(\cdot, t)) \leq \bar{\xi}\omega] \cap K_{\frac{1}{2}\varrho}(x_o)| \leq \alpha |K_{\frac{1}{2}\varrho}| \quad \text{for all } t \in (t_1, t_1 + A\theta\varrho^p),$$

provided the cylinder $K_\varrho(x_o) \times (t_1, t_1 + A\theta\varrho^p)$ is included in $\mathcal{Q}$. Moreover, the dependence of $\bar{\xi}$ is given by

$$\bar{\xi} = \frac{1}{2}\xi \exp\Big\{ -\gamma(\text{data})\frac{A}{\alpha}\Big\}.$$

2.3. De Giorgi-type lemmas

For the cylinder $\mathcal{Q} \subset E_T$, we introduce the numbers $\mu^\pm$ and ω just like in Section 2.2.

Setting $(x_o, t_o) + Q_\varrho(\theta) = K_\varrho(x_o) \times (t_o - \theta\varrho^p, t_o)$, we now present the first De Giorgi-type lemma that can be shown by using the energy estimates in (2.2); for the detailed proof we refer to [17, Lemma 2.1]. If no confusion arises, we omit the vertex (x_o, t_o) for simplicity.

Lemma 2.2. *Let u be a local weak sub(super)-solution to (1.9) with (1.8) in E_T and let $\xi \in (0,1)$ and $\theta = (\xi\omega)^{2-p}$. There exists a constant $c_o \in (0,1)$ depending only on the data such that if*

$$|[\pm(\mu^\pm - u) \leq \xi\omega] \cap Q_\varrho(\theta)| \leq c_o(\xi\omega)^{\frac{N+p}{p}} |Q_\varrho(\theta)|,$$

then

$$\pm(\mu^\pm - u) \geq \frac{1}{2}\xi\omega \quad \text{a.e. in } Q_{\frac{1}{2}\varrho}(\theta),$$

provided $Q_\varrho(\theta)$ is included in $\mathcal{Q}$.

The next lemma is a variant of the previous one, involving quantitative initial data. For this purpose, we will use the forward cylinder at (x_o, t_1) with length $\theta > 0$

$$(x_o, t_1) + Q_\varrho^+(\theta) := K_\varrho(x_o) \times (t_1, t_1 + \theta\varrho^p).$$

The proof is based on (2.3) and can be retrieved from [11, Lemma 2.2].

Lemma 2.3. *Let u be a local weak sub(super)-solution to (1.9) with (1.8) in E_T. Assume that for some $\xi\in(0,1)$, it holds that*

$$\pm(\mu^\pm - u(\cdot,t_1)) \ge \xi\omega \quad \text{a.e. in } K_\varrho(x_o).$$

There exists a constant $\gamma_o\in(0,1)$ depending only on the data such that for any $\theta>0$, if

$$|[\pm(\mu^\pm - u)\le \xi\omega]\cap[(x_o,t_1)+Q_\varrho^+(\theta)]| \le \frac{\gamma_o(\xi\omega)^{2-p}}{\theta}|Q_\varrho^+(\theta)|,$$

then

$$\pm(\mu^\pm - u) \ge \frac{1}{2}\xi\omega \quad \text{a.e. in } K_{\frac12\varrho}(x_o)\times(t_1, t_1+\theta\varrho^p),$$

provided the cylinder $(x_o,t_1)+Q_\varrho^+(\theta)$ is included in $\mathcal{Q}$.

As we have seen, the previous two lemmas use the “simplified" version of energy estimates (2.2) and (2.3) only. In contrast, the next lemma examines the singularity of $\widetilde{\beta}$ at $[u=e_i]$ for any $i=0,1,\dots,\ell$ more carefully; due to the periodicity assumption, we are reduced to studying $[u=0]$. The lemma quantifies a measure condition to ensure that the singular effect due to the jump is negligible. As a result, the singularity due to $\beta'(0)=\infty$ prevails, which will be reflected by a time scaling different from the previous one. We will evoke an argument from [8, Sections 4.3–4.5] to deal with the situation.

The vertex (x_o,t_o) will be omitted from $Q_\varrho(A\omega^{2-p})$ for simplicity.

Lemma 2.4. *Let u be a local weak super-solution to (1.9) with (1.8) in E_T. Assume that for some α, $\eta\in(0,1)$ and $A>1$, it holds that*

$$|[u(\cdot,t)-\mu^- \ge \eta\omega]\cap K_\varrho| > \alpha|K_\varrho| \quad \text{for all } t\in(t_o - A\omega^{2-p}\varrho^p, t_o]. \tag{2.4}$$

There exists $\xi\in(0,\eta)$ determined by the data, α, η, M, and $\beta(\cdot)$ such that if

$$A \ge \frac{\beta(\frac18\xi\omega)}{\frac18\xi\omega}\,\xi^{2-p};$$

$|\mu^- - e_i|\le\frac14\xi\omega$ for some $i\in\{0,1,\dots,\ell\}$, $\frac54\xi\omega\in(0,\bar d)$; and it holds that

$$\iint_{Q_\varrho(\theta)}\Big(\int_u^k H_\varepsilon'(s)\chi_{[s<k]}\,\mathrm{d}s\Big)\mathrm{d}x\mathrm{d}t \le \beta(\xi\omega)\Big|\Big[u\le\mu^- + \frac12\xi\omega\Big]\cap Q_{\frac12\varrho}(\theta)\Big|, \tag{2.5}$$

where $k=\mu^-+\xi\omega$, and $\theta=\frac{\beta(\frac18\xi\omega)}{\frac18\xi\omega}(\xi\omega)^{2-p}$, then

$$u \ge \mu^- + \frac14\xi\omega \quad \text{a.e. in } Q_{\frac12\varrho}(\theta),$$

provided the cylinder $Q_\varrho(A\omega^{2-p})$ is included in $\mathcal{Q}$.

We will postpone the proof to Section 4.

Remark 2.1. A similar result to Lemma 2.4 also holds for sub-solutions. Since we do not use it, it is omitted.

3. Proof of Theorem 1.1

Assume $(x_o, t_o) = (0, 0)$, introduce $Q_o = K_{8\varrho} \times (-(8\varrho)^{p-1}, 0]$, and set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_o} u, \quad \mu^- = \operatorname*{ess\,inf}_{Q_o} u, \quad \omega \ge \mu^+ - \mu^-.$$

Letting $\theta = (\frac{1}{4}\omega)^{2-p}$, for some $A(\omega) > 1$ to be determined in terms of the data and ω, we may assume that

$$Q_{8\varrho}(A\theta) \subset Q_o = K_{8\varrho} \times (-(8\varrho)^{p-1}, 0] \quad \text{such that} \quad \operatorname*{ess\,osc}_{Q_\varrho(A\theta)} u \le \omega; \tag{3.1}$$

the case when the set inclusion does not hold will be incorporated later.

3.1. Reduction of oscillation near the supremum

In this section, we work with u as a sub-solution near its supremum. Suppose for some $\bar{t} \in (-(A-1)\theta\varrho^p, 0]$

$$\Big|\Big[\mu^+ - u \le \frac{1}{4}\omega\Big] \cap [(0, \bar{t}) + Q_\varrho(\theta)]\Big| \le c_o\Big(\frac{1}{4}\omega\Big)^{\frac{N+p}{p}} |Q_\varrho(\theta)| =: \alpha |Q_\varrho(\theta)| \tag{3.2}$$

holds, where $c_o \in (0, 1)$ depends only on the data as determined in Lemma 2.2. An application of Lemma 2.2 (with $\xi = \frac{1}{4}$) then yields

$$\mu^+ - u \ge \frac{1}{8}\omega \quad \text{a.e. in } (0, \bar{t}) + Q_{\frac{1}{2}\varrho}(\theta). \tag{3.3}$$

In particular, estimate (3.3) holds at $t_1 := \bar{t} - \theta(\frac{1}{2}\varrho)^p$ and serves as the initial datum for an application of Lemmas 2.1 and 2.3. In fact, Lemma 2.3 determines some $\gamma_o \in (0, 1)$ satisfying that if for some $\eta \in (0, \frac{1}{8})$,

$$|[\mu^+ - u \le \eta\omega] \cap \widetilde{Q}| \le \frac{\gamma_o(\frac{1}{8}\omega)^{2-p}}{A\theta} |\widetilde{Q}| \quad \text{where } \widetilde{Q} := K_{\frac{1}{2}\varrho} \times (t_1, 0), \tag{3.4}$$

then

$$\mu^+ - u \ge \frac{1}{2}\eta\omega \quad \text{a.e. in } K_{\frac{1}{4}\varrho} \times (t_1, 0). \tag{3.5}$$

In the meantime, thanks to Lemma 2.1, (3.4) is fulfilled with the choice

$$\eta = \frac{1}{16} \exp\Big\{-\frac{\gamma A^2}{2^{p-2}\gamma_o}\Big\},$$

and hence, so is (3.5) in view of Lemma 2.3. As a result, we obtain a reduction of oscillation from estimate (3.5), no matter where the location of $\bar{t}$ is. More precisely,

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}(\theta)} u \le \Big(1 - \frac{1}{2}\eta\Big)\omega. \tag{3.6}$$

Note that $A(\omega)$ has not been fixed yet. It will be chosen at the final stage of the argument, and hence, simultaneously it will determine the value of η.

3.2. Reduction of oscillation near the infimum: Part I

Starting from this section, let us suppose contrary to (3.2) that, for any $\bar{t} \in (-(A-1)\theta\varrho^p, 0]$,

$$\Big|\Big[\mu^+ - u \le \frac{1}{4}\omega\Big] \cap [(0,\bar{t}) + Q_\varrho(\theta)]\Big| > \alpha |Q_\varrho(\theta)| \quad \text{for } \alpha = c_o\Big(\frac{1}{4}\omega\Big)^{\frac{N+p}{p}}.$$

Since $\mu^+ - \frac{1}{4}\omega \ge \mu^- + \frac{1}{4}\omega$ always holds, we may rephrase it as

$$\Big|\Big[u - \mu^- \ge \frac{1}{4}\omega\Big] \cap [(0,\bar{t}) + Q_\varrho(\theta)]\Big| > \alpha |Q_\varrho(\theta)|. \tag{3.7}$$

Suppose $\bar{t}$ is fixed for the moment. The measure information in (3.7) together with energy estimate (2.2) implies a local clustering phenomenon of u. This is the content of the next lemma, similar to [10, Proposition 5.1$^\pm$]; the proof can be reproduced as in [10, Sections 5–8] or [11, Lemma 3.1]. Throughout Sections 3.2–3.6, we will work with u as a super-solution near its infimum.

Lemma 3.1. *For every $\lambda \in (0,1)$ and $\eta \in (0,1)$, there exists a point $(x_*, t_*) \in (0,\bar{t}) + Q_\varrho(\theta)$, a number $\kappa \in (0,1)$, and a cylinder $(x_*, t_*) + Q_{\kappa\varrho}(\theta) \subset (0,\bar{t}) + Q_\varrho(\theta)$ such that*

$$\Big|\Big[u \le \mu^- + \frac{1}{4}\lambda\omega\Big] \cap [(x_*, t_*) + Q_{\kappa\varrho}(\theta)]\Big| \le \eta |Q_{\kappa\varrho}(\theta)|.$$

The constant κ is determined by the data, M, λ, η, α, and ω.

Although the location of $(x_*, t_*) + Q_{\kappa\varrho}(\theta) \subset (0,\bar{t}) + Q_\varrho(\theta)$ is only known qualitatively, we may use the quantified measure concentration to extract a pointwise estimate with the aid of Lemma 2.2, and then use the logarithmic energy estimate to propagate the measure information up to the level $\bar{t}$.

Indeed, if in Lemma 3.1 we choose $\lambda = \frac{1}{2}$ and $\eta = c_o(\frac{1}{4}\omega)^{\frac{N+p}{p}}$, where c_o is determined in Lemma 2.2, then Lemmas 2.2 and 3.1 would yield that

$$u \ge \mu^- + \frac{1}{16}\omega \quad \text{a.e. in } (x_*, t_*) + Q_{\frac{1}{2}\kappa\varrho}(\theta),$$

for some $(x_*, t_*) \in (0,\bar{t}) + Q_\varrho(\theta)$ and some $\kappa \in (0,1)$ depending on the data, M, and ω. In particular, we have

$$u\Big(\cdot, t_* - \theta\Big(\frac{1}{2}\kappa\varrho\Big)^p\Big) \ge \mu^- + \frac{1}{16}\omega \quad \text{a.e. in } K_{\frac{1}{2}\kappa\varrho}(x_*),$$

which serves as the initial datum to apply Lemma 2.1. In fact, setting $\alpha = \frac{1}{2}$ and $\xi = \frac{1}{16}$ in Lemma 2.1 and choosing $\widetilde{A}$ so large that

$$\Big(\frac{1}{4}\omega\Big)^{2-p}\varrho^p \le \widetilde{A}\Big(\frac{1}{16}\omega\Big)^{2-p}\Big(\frac{1}{2}\kappa\varrho\Big)^p, \quad \text{that is,} \quad \widetilde{A} \ge \frac{2^{4-p}}{\kappa^p},$$

it yields a number $\bar{\xi} \in (0, \frac{1}{4}\xi)$ such that

$$|[u(\cdot, \bar{t}) > \mu^- + \bar{\xi}\omega] \cap K_{\frac{1}{4}\kappa\varrho}(x_*)| > \frac{1}{2}|K_{\frac{1}{4}\kappa\varrho}|. \tag{3.8}$$

As in [11, Section 3.2, (3.17)], the dependence of $\bar{\xi}$ is traced by

$$\bar{\xi} = \frac{1}{32} \exp\Big\{-\frac{\gamma}{\omega^{p\bar{q}}}\Big\},$$

where

$$\bar{q} := \Big(\frac{N}{p} + 1\Big)\Big(3 + \frac{2}{p}\Big) + \frac{1}{p} + 1.$$

The measure information in (3.8) permits us to claim that

$$\begin{aligned}|[u(\cdot, \bar{t}) > \mu^- + \bar{\xi}\omega] \cap K_\varrho| &\geq |[u(\cdot, \bar{t}) > \mu^- + \bar{\xi}\omega] \cap K_{\frac{1}{4}\kappa\varrho}(x_*)| \\ &> \frac{1}{2}|K_{\frac{1}{4}\kappa\varrho}| = \frac{1}{2}\Big(\frac{1}{4}\kappa\Big)^N |K_\varrho| =: \bar{\alpha}|K_\varrho|.\end{aligned}$$

Thanks to the arbitrariness of $\bar{t}$, we have actually arrived at

$$|[u(\cdot, t) \geq \mu^- + \bar{\xi}\omega] \cap K_\varrho| > \bar{\alpha}|K_\varrho| \quad \text{for all } t \in (-(A-1)\theta\varrho^p, 0]. \tag{3.9}$$

Once more, as in [11, Section 3.2, (3.19)], the dependence of $\bar{\alpha}$ is traced by

$$\bar{\alpha} = \gamma(\text{data})\, \omega^{\bar{q}N}.$$

The analysis to be unfolded in the following sections relies on the measure information in (3.9). For simplicity, we will use (3.9) with $A - 1$ replaced by A; recall that A is a free, large number to be chosen.

3.3. Reduction of oscillation near the infimum: Part II

Let us first introduce the following intrinsic cylinders

$$\begin{cases} Q_\varrho(\widehat{\theta}) = K_\varrho \times (-\widehat{\theta}\varrho^p, 0), & \widehat{\theta} = \dfrac{\beta(\frac{1}{8}\xi\omega)}{\frac{1}{8}\xi\omega}(\xi\omega)^{2-p}, \\ Q_\varrho(\widetilde{\theta}) = K_\varrho \times (-\widetilde{\theta}\varrho^p, 0), & \widetilde{\theta} = (\delta\xi\omega)^{1-p}, \end{cases}$$

for some $\xi(\omega)$ and $\delta(\omega)$ in $(0, 1)$ to be determined later. We can always assume ξ and δ to be sufficiently small, so that on one hand $8\beta(\frac{1}{8}\xi\omega) \leq \delta^{1-p}$, which ensures $\widehat{\theta} \leq \widetilde{\theta}$, and on the other hand, $\widehat{\theta} \geq \theta$.

In addition, we may suppose that

$$8^p\widetilde{\theta} \leq A8^p\theta \tag{3.10}$$

for some $A(\omega)$ yet to be determined.

Throughout Sections 3.3–3.5, we always assume that $\frac{5}{4}\xi\omega \in (0, \bar{d})$ and

$$|\mu^- - e_i| \le \frac{1}{4}\delta\xi\omega \quad \text{for some } i \in \{0, 1, \dots, \ell\}, \tag{3.11}$$

for the same $\xi(\omega)$ and $\delta(\omega)$ in $(0, 1)$ introduced above, to be determined. When restriction (3.11) does not hold, the case will be examined in Section 3.6.

First of all, we turn our attention to Lemmas 2.2 and 2.4. In view of the measure information in (3.9), Lemma 2.4 is at our disposal, with α, η, and A replaced by $\bar{\alpha}$, $\bar{\xi}$, and $A/4^{2-p}$, respectively. Suppose ξ is determined in Lemma 2.4 in terms of the data and $\bar{\alpha}$, assume $\xi < \frac{1}{4}$ with no loss of generality, and let $\theta_* := (\xi\omega)^{2-p}$. If

$$\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\theta_*)\Big| \le c_o(\xi\omega)^{\frac{N+p}{p}} |Q_{\frac{1}{2}\varrho}(\theta_*)|$$

holds, then Lemma 2.2 yields that

$$u \ge \mu^- + \frac{1}{4}\xi\omega \quad \text{a.e. in } Q_{\frac{1}{4}\varrho}(\theta_*). \tag{3.12}$$

Analogously, if for $k = \mu^- + \xi\omega$,

$$\iint_{Q_\varrho(\widehat{\theta})} \int_u^k H_\varepsilon'(s)\chi_{[s<k]}\,\mathrm{d}s\mathrm{d}x\mathrm{d}t \le \beta(\xi\omega)\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\widehat{\theta})\Big|$$

holds, then, stipulating

$$A \ge \xi^{2-p}\frac{\beta(\frac{1}{8}\xi\omega)}{\frac{1}{8}\xi\omega}, \tag{3.13}$$

Lemma 2.4 yields that

$$u \ge \mu^- + \frac{1}{4}\xi\omega \quad \text{a.e. in } Q_{\frac{1}{2}\varrho}(\widehat{\theta}). \tag{3.14}$$

Consequently, either (3.12) or (3.14) yields a reduction of oscillation

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}(\theta_*)} u \le \Big(1 - \frac{1}{4}\xi\Big)\omega, \tag{3.15}$$

where we have taken into account that

$$\widehat{\theta} = \frac{\beta(\frac{1}{8}\xi\omega)}{\frac{1}{8}\xi\omega}(\xi\omega)^{2-p} > (\xi\omega)^{2-p} = \theta_*.$$

Remark 3.1. We may trace the dependence of ξ by

$$\xi = 2^{-mj_*}\bar{\xi} = \frac{1}{32}2^{-mj_*}\exp\Big\{-\frac{\gamma}{\omega^{p\bar{q}}}\Big\} \quad \text{with } m \in \mathbb{N},\ 2p \le m < 2p+1.$$

Here $j_* \equiv j_*(\omega)$ will be determined according to (4.6).

3.4. Reduction of oscillation near the infimum: Part III

This section starts dealing with the situation when the measure conditions in Section 3.3 are violated, that is, when the measure condition in Lemma 2.2 is violated, meaning that

$$\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\theta_*)\Big| > c_o(\xi\omega)^{\frac{N+p}{p}}|Q_{\frac{1}{2}\varrho}(\theta_*)|, \tag{3.16}$$

and when the condition in Lemma 2.4 is also violated: for $k = \mu^- + \xi\omega$, it holds that

$$\iint_{Q_\varrho(\widehat{\theta})} \int_u^k H'_\varepsilon(s)\chi_{[s<k]}\,\mathrm{d}s\mathrm{d}x\mathrm{d}t > \beta(\xi\omega)\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\widehat{\theta})\Big|. \tag{3.17}$$

Combining (3.16)–(3.17) and recalling $\widetilde{\theta} \ge \theta_*$, we obtain that, for all $r \in [2\varrho, 8\varrho]$,

$$\begin{aligned}
\iint_{Q_r(\widehat{\theta})} \int_u^k H'_\varepsilon(s)\chi_{[s<k]}\,\mathrm{d}s\mathrm{d}x\mathrm{d}t &\ge \iint_{Q_\varrho(\widehat{\theta})} \int_u^k H'_\varepsilon(s)\chi_{[s<k]}\,\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&\ge \beta(\xi\omega)\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\widehat{\theta})\Big| \\
&\ge \beta(\xi\omega)\Big|\Big[u \le \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\theta_*)\Big| \\
&\ge c_o(\xi\omega)^{\frac{N+p}{p}}\beta(\xi\omega)|Q_{\frac{1}{2}\varrho}(\theta_*)| \\
&\ge \frac{1}{8}c_o(\xi\omega)^{1+\frac{N+p}{p}}\frac{\beta(\xi\omega)}{\beta(\frac{1}{8}\xi\omega)}|Q_r(\widehat{\theta})| \\
&\ge \widetilde{\gamma}b(\xi\omega)|Q_r(\widehat{\theta})|,
\end{aligned} \tag{3.18}$$

where $\widetilde{\gamma} = c_o\frac{1}{8}16^{-N-p}$ and

$$b(\xi\omega) := (\xi\omega)^{1+\frac{N+p}{p}}.$$

Next, assuming δ has been chosen, we introduce a free parameter $\bar{\delta} \in (\delta, 2\delta)$ and set $\bar{\theta} := (\bar{\delta}\xi\omega)^{1-p}$. Recall also that

$$\theta = \Big(\frac{1}{4}\omega\Big)^{2-p}, \quad \widetilde{\theta} = (\delta\xi\omega)^{1-p}, \quad \widehat{\theta} = \frac{\beta(\frac{1}{8}\xi\omega)}{\frac{1}{8}\xi\omega}(\xi\omega)^{2-p},$$

and that we have assumed $\widetilde{\theta}(8\varrho)^p \le A\theta(8\varrho)^p \le (8\varrho)^{p-1}$ in (3.10). Hence, we have

$$Q_r(\widehat{\theta}) \subset Q_r(\bar{\theta}) \subset Q_r(\widetilde{\theta}) \subset Q_r(A\theta) \subset Q_o \quad \text{for any } r \in [2\varrho, 8\varrho].$$

According to (3.18), it is not hard to find some $t_* \in [-\widehat{\theta}r^p, 0]$ satisfying

$$\int_{K_r\times\{t_*\}} \int_u^k H'_\varepsilon(s)\chi_{[s<k]}\,\mathrm{d}s\mathrm{d}x \ge \widetilde{\gamma}b(\xi\omega)|K_r|. \tag{3.19}$$

Observe also that for any $t \in [-\bar{\theta} r^p, 0]$ and any $\bar{\delta} \in (\delta, 2\delta)$,

$$|K_r| \ge |[u \le \mu^- + \bar{\delta}\xi\omega] \cap K_r| \ge (\bar{\delta}\xi\omega)^{-2} \int_{K_r \times \{t\}} [u - (\mu^- + \bar{\delta}\xi\omega)]_-^2 \, \mathrm{d}x \tag{3.20}$$

holds. Denoting $\bar{k} = \mu^- + \bar{\delta}\xi\omega$ and enforcing that for some $i \in \{0, 1, \ldots, \ell\}$,

$$|\mu^- - e_i| \le \frac{1}{4}\delta\xi\omega \quad \text{and} \quad \varepsilon \le \frac{1}{4}\delta\xi\omega,$$

we use (3.19) and (3.20) to estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\bar{\theta} r^p < t < 0} \int_{K_r} \int_u^{\bar{k}} H'_\varepsilon(s)(s - \bar{k})_- \, \mathrm{d}s\mathrm{d}x \\
&\quad \ge (\bar{k} - e_i - \varepsilon) \operatorname*{ess\,sup}_{-\bar{\theta} r^p < t < 0} \int_{K_r} \int_u^{\bar{k}} H'_\varepsilon(s)\chi_{[s<\bar{k}]} \, \mathrm{d}s\mathrm{d}x \ge \widetilde{\gamma}(\bar{k} - e_i - \varepsilon) b(\xi\omega)|K_r| \\
&\quad \ge \frac{1}{2}\widetilde{\gamma} b(\xi\omega)(\delta\xi\omega)(\bar{\delta}\xi\omega)^{-2} \operatorname*{ess\,sup}_{-\bar{\theta} r^p < t < 0} \int_{K_r \times \{t\}} [u - (\mu^- + \bar{\delta}\xi\omega)]_-^2 \, \mathrm{d}x.
\end{aligned}$$

Here we require $\frac{5}{4}\xi\omega \in (0, \bar{d})$ because of Lemma 2.4, and in the first inequality above we have also assumed $\frac{9}{4}\xi\omega \le d$ by possibly further restricting the choice of ξ; hence, $(\mu^-, \bar{k}) \subset (e_i - \frac{1}{4}\delta\xi\omega, e_i + \frac{1}{4}\delta\xi\omega + 2\delta\xi\omega) \subset (e_i - d, e_i + d)$. The above analysis, together with (2.1), yields the following energy estimate:

Lemma 3.2. *Let u be a weak super-solution to (1.9) with (1.8) in E_T, under the measure information in (3.9). Let (3.16) and (3.17) hold true. Denoting*

$$b(\xi\omega) = (\xi\omega)^{1 + \frac{N+p}{p}},$$

and setting $k = \mu^- + \bar{\delta}\xi\omega$ with $\bar{\delta} \in (\delta, 2\delta)$, there exists a positive constant γ depending only on the data such that for all $\sigma \in (0, 1)$ and all $r \in [2\varrho, 8\varrho]$, we have

$$\begin{aligned}
&\delta\xi\omega(\bar{\delta}\xi\omega)^{-2} b(\xi\omega) \operatorname*{ess\,sup}_{-\bar{\theta}(\sigma r)^p < t < 0} \int_{K_{\sigma r} \times \{t\}} (u - k)_-^2 \, \mathrm{d}x + \iint_{Q_{\sigma r}(\bar{\theta})} |D(u - k)_-|^p \, \mathrm{d}x\mathrm{d}t \\
&\quad \le \frac{\gamma}{(1 - \sigma)^p r^p} \iint_{Q_r(\bar{\theta})} (u - k)_-^p \mathrm{d}x\mathrm{d}t + \frac{\gamma}{(1 - \sigma)\bar{\theta} r^p} \iint_{Q_r(\bar{\theta})} (u - k)_- \, \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

provided that for some $i \in \{0, 1, \ldots, \ell\}$,

$$|\mu^- - e_i| \le \frac{1}{4}\delta\xi\omega, \quad \frac{5}{4}\xi\omega \in (0, \bar{d}), \quad \textit{and} \quad \varepsilon \le \frac{1}{4}\delta\xi\omega.$$

The energy estimate in Lemma 3.2 yields the following De Giorgi-type lemma; notice that the time scaling used here is different from the one in Lemmas 2.2–2.4:

Lemma 3.3. *Suppose the hypotheses in Lemma 3.2 hold. Let* $\delta \in (0,1)$. *There exists a constant* $c_2 \in (0,1)$ *depending only on the data such that if*

$$|[u < \mu^- + 2\delta\xi\omega] \cap Q_{4\varrho}(\widetilde{\theta})| \le c_2 b(\xi\omega)|Q_{4\varrho}(\widetilde{\theta})|, \quad \text{where } \widetilde{\theta} = (\delta\xi\omega)^{1-p},$$

and $\frac{5}{4}\xi\omega \in (0, \bar{d})$, *then either* $|\mu^- - e_i| > \frac{1}{4}\delta\xi\omega$ *for all* $i \in \{0, 1, \dots, \ell\}$ *or*

$$u \ge \mu^- + \delta\xi\omega \quad \text{a.e. in } Q_{2\varrho}(\widetilde{\theta}),$$

provided $4^p\widetilde{\theta} \le A8^p\theta$.

Proof. For $n = 0, 1, \dots,$ we set

$$\begin{cases} k_n = \mu^- + \delta\xi\omega + \dfrac{\delta\xi\omega}{2^n}, \quad \tilde{k}_n = \dfrac{k_n + k_{n+1}}{2}, \\ \varrho_n = 2\varrho + \dfrac{\varrho}{2^{n-1}}, \quad \tilde{\varrho}_n = \dfrac{\varrho_n + \varrho_{n+1}}{2}, \\ K_n = K_{\varrho_n}, \quad \widetilde{K}_n = K_{\tilde{\varrho}_n}, \\ Q_n = Q_{\varrho_n}(\widetilde{\theta}), \quad \widetilde{Q}_n = Q_{\tilde{\varrho}_n}(\widetilde{\theta}). \end{cases}$$

We will use the energy estimate in Lemma 3.2 with the pair of cylinders $\widetilde{Q}_n \subset Q_n$. Note that the constant $\bar{\delta}$ in Lemma 3.2 is replaced by $(1 + 2^{-n})\delta$, as indicated in the definition of k_n. Enforcing $|\mu^- - e_i| \le \frac{1}{4}\delta\xi\omega$ and $\varepsilon \le \frac{1}{4}\delta\xi\omega$, the energy estimate in Lemma 3.2 yields that

$$\begin{aligned} (\delta\xi\omega)^{-1} b(\xi\omega) \operatorname*{ess\,sup}_{-\widetilde{\theta}\tilde{\varrho}_n^p < t < 0} \int_{\widetilde{K}_n \times \{t\}} (u - \tilde{k}_n)_-^2 \, \mathrm{d}x + \iint_{\widetilde{Q}_n} |D(u - \tilde{k}_n)_-|^p \, \mathrm{d}x\mathrm{d}t \\ \le \gamma \frac{2^{pn}}{\varrho^p} (\delta\xi\omega)^p |A_n|, \end{aligned}$$

where

$$A_n = [u < k_n] \cap Q_n.$$

Let $0 \le \phi \le 1$ be a cutoff function that vanishes on the parabolic boundary of $\widetilde{Q}_n$ and equals 1 in Q_{n+1}. An application of the Hölder inequality, the Sobolev imbedding [7, Chapter I, Proposition 3.1], and the above energy estimate gives that

$$\begin{aligned} \frac{\delta\xi\omega}{2^{n+3}} |A_{n+1}| &\le \iint_{\widetilde{Q}_n} \big(u - \tilde{k}_n\big)_- \phi \, \mathrm{d}x\mathrm{d}t \\ &\le \Big[\iint_{\widetilde{Q}_n} \big[\big(u - \tilde{k}_n\big)_- \phi\big]^{p\frac{N+2}{N}} \, \mathrm{d}x\mathrm{d}t \Big]^{\frac{N}{p(N+2)}} |A_n|^{1 - \frac{N}{p(N+2)}} \\ &\le \gamma \Big[\iint_{\widetilde{Q}_n} \big|D\big[(u - \tilde{k}_n)_- \phi\big]\big|^p \, \mathrm{d}x\mathrm{d}t \Big]^{\frac{N}{p(N+2)}} \\ &\qquad \times \Big[\operatorname*{ess\,sup}_{-\widetilde{\theta}\tilde{\varrho}_n^2 < t < 0} \int_{\widetilde{K}_n} \big(u - \tilde{k}_n\big)_-^2 \, \mathrm{d}x \Big]^{\frac{1}{N+2}} |A_n|^{1 - \frac{N}{p(N+2)}} \end{aligned}$$

$$\le \gamma[b(\xi\omega)]^{-\frac{1}{N+2}}(\delta\xi\omega)^{\frac{1}{N+2}}\Big[(\delta\xi\omega)^p\frac{2^{np}}{\varrho^p}\Big]^{\frac{N+p}{p(N+2)}}|A_n|^{1+\frac{1}{N+2}}.$$

In terms of $Y_n=|A_n|/|Q_n|$, the recurrence is rephrased as

$$Y_{n+1}\le\frac{\gamma C^n}{[b(\xi\omega)]^{\frac{1}{N+2}}}Y_n^{1+\frac{1}{N+2}},$$

for a constant γ depending only on the data and with $C=2^{\frac{2N+2+p}{N+2}}$. Hence, by [7, Chapter I, Lemma 4.1], there exists a positive constant c_2 depending only on the data such that $Y_n\to 0$ if we require that $Y_o\le c_2 b(\xi\omega)$. This concludes the proof. ∎

The next lemma concerns the smallness of the measure density of the set $[u\approx\mu^-]$. Its proof relies on (2.2) and the measure information in (3.9) will be employed.

Lemma 3.4. *Let u be a weak super-solution to (1.9) with (1.8) in E_T, under the measure information in (3.9). There exists a positive constant γ depending only on the data such that for any $s_*\in\mathbb{N}$, we have*

$$\Big|\Big[u\le\mu^-+\frac{\xi\omega}{2^{s_*}}\Big]\cap Q_{4\varrho}(\widetilde{\theta})\Big|\le\frac{\gamma}{\bar{\alpha}}s_*^{-\frac{p-1}{p}}|Q_{4\varrho}(\widetilde{\theta})|,\qquad \textit{where } \widetilde{\theta}=\Big(\frac{\xi\omega}{2^{s_*}}\Big)^{1-p},$$

provided $4^p\widetilde{\theta}\le A8^p\theta$.

Proof. The proof is identical to that of [11, Lemma 3.4]. ∎

3.5. Reduction of oscillation near the infimum: Part IV

Now we have all the prerequisites to reduce the oscillation under conditions (3.16) and (3.17). First of all, let ξ be determined by Lemma 2.4 in terms of the data and $\bar{\alpha}$ as in Section 3.3. Then, we choose the integer s_* large enough to satisfy

$$\frac{\gamma}{\bar{\alpha}s_*^{\frac{p-1}{p}}}\le c_2 b(\xi\omega),$$

where the constant c_2 and the quantity $b(\xi\omega)$ are defined in Lemma 3.3.

Next, we can fix $2\delta=2^{-s_*}$ in Lemma 3.3. Consequently, Lemma 3.3 can be applied, assuming that $|\mu^--e_i|\le\frac14\delta\xi\omega$ for some $i\in\{0,1,\dots,\ell\}$ and $\varepsilon\le\frac14\delta\xi\omega$, and we arrive at

$$u\ge\mu^-+\delta\xi\omega\qquad\text{a.e. in } Q_{2\varrho}(\widetilde{\theta}).$$

This would give us a reduction of oscillation

$$\operatorname*{ess\,osc}_{Q_{2\varrho}(\widetilde{\theta})}u\le(1-\delta\xi)\omega \tag{3.21}$$

with the above-defined δ and ξ. For the moment, as required in (3.10) and (3.13), the choice of A can be made from

$$A = \max\Big\{2^{p+4}\omega^{-1}(\delta\xi)^{1-p}, \xi^{2-p}\frac{\beta(\frac{1}{8}\xi\omega)}{\frac{1}{8}\xi\omega}\Big\} = 2^{p+4}\omega^{-1}(\delta\xi)^{1-p}. \tag{3.22}$$

To summarize the achievements in Sections 3.1–3.5, taking the reverse of (3.1), (3.6), and (3.15), as well as (3.21) all into account, if $|\mu^- - e_i| \le \frac{1}{4}\delta\xi\omega$ for some $i \in \{0, 1, \dots, \ell\}$ and $\varepsilon \le \frac{1}{4}\delta\xi\omega$ hold true, then for $\theta = (\frac{1}{4}\omega)^{2-p}$, taking into account that $\theta < \theta_*$, we have

$$\text{either} \quad \operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}(\theta)} u \le (1 - \bar{\eta}(\omega))\omega \quad \text{or} \quad \Big(\frac{1}{4}\omega\Big)^{p-2}[A(\omega)]^{-1} \le \varrho, \tag{3.23}$$

where

$$\bar{\eta} = \min\Big\{\frac{1}{2}\eta,\ \frac{1}{4}\xi,\ \delta\xi\Big\}.$$

Among them η is to be fixed, as the final choice of A is yet to be made.

3.6. Reduction of oscillation near the infimum: Part V

Let $\xi(\omega)$ and $\delta(\omega)$ be determined as in the previous sections. The analysis throughout Sections 3.3–3.5 has been founded on condition (3.11). We now examine the case when (3.11) does not hold, namely,

$$|\mu^- - e_i| > \frac{1}{4}\delta\xi\omega \quad \text{for all } i \in \{0, 1, \dots, \ell\}. \tag{3.24}$$

Notice that the analysis in Section 3.2 does not rely on condition (3.11), and thus, the measure information in (3.9) derived there is still at our disposal. In view of the determination of ξ starting from $\bar{\xi}$, we have that $\xi < \bar{\xi}$ (cf. Remark 3.1), and hence, (3.9) holds true with $\bar{\xi}$ replaced by $\delta\xi$.

Next, for $\tilde{\xi} \in (0, \frac{1}{8})$, we introduce the levels $k = \mu^- + \tilde{\xi}\delta\xi\omega$. According to (3.24) and assuming that $\varepsilon \le \frac{1}{4}\delta\xi\omega$, energy estimate (2.1)$_-$ written in $Q_\varrho(\vartheta) \subset Q_\varrho(A\theta)$ for some $0 < \vartheta < A\theta$ yields that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\vartheta\varrho^p < t < 0} \frac{1}{2}\alpha_o \int_{K_\varrho \times \{t\}} \zeta^p (u-k)_-^2 \,\mathrm{d}x + \iint_{Q_\varrho(\vartheta)} \zeta^p |D(u-k)_-|^p \,\mathrm{d}x\mathrm{d}t \\
&\quad \le \gamma \iint_{Q_\varrho(\vartheta)} \Big[(u-k)_-^p |D\zeta|^p + \beta'\Big(\frac{1}{8}\delta\xi\omega\Big)(u-k)_-^2 |\partial_t \zeta^p|\Big] \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Given this energy estimate and the measure information in (3.9), the theory of the parabolic p-Laplacian in [7, Chapter III] applies, bearing in mind that at this stage δ and ξ have been chosen and hence $\beta'(\frac{1}{8}\delta\xi\omega)$ is deemed a fixed quantity, although it could be quite large.

Lemma 3.5. *Let u be a weak super-solution to (1.9) with (1.8) in E_T. Suppose (3.9) and (3.24) hold true, and $\varepsilon \le \frac{1}{4}\delta\xi\omega$. There exists a positive constant $\widetilde{\xi}$ depending on the data, ω, and $\bar{\alpha}$, such that for $\vartheta = (\widetilde{\xi}\delta\xi\omega)^{2-p}$, we have*

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}(\vartheta)} u \le (1-\widetilde{\xi}\delta\xi)\omega,$$

provided $\vartheta \le A\theta$.

Remark 3.2. The dependence of $\widetilde{\xi}$ can be traced by

$$\widetilde{\xi} = \exp\Big\{-\gamma(\text{data})\bar{\alpha}^{-\frac{p}{p-1}}\Big[\beta'\Big(\frac{1}{8}\delta\xi\omega\Big)\Big]^{\frac{N+p+1}{p-1}}\Big\}.$$

Lemma 3.5 imposes a new condition on A in order to satisfy $\vartheta \le A\theta$, namely, $A \ge (4\widetilde{\xi}\delta\xi)^{2-p}$; taking into account the existing condition given by (3.22), the final choice of A is made by

$$A = \max\big\{(4\widetilde{\xi}\delta\xi)^{2-p},\ 2^{p+4}\omega^{-1}(\delta\xi)^{1-p}\big\}.$$

This choice of A also determines the value of η in (3.6).

Let us summarize what has been achieved in the previous sections. According to (3.23) and Lemma 3.5, we have

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}(\theta)} u \le (1-\widetilde{\eta}(\omega))\omega, \quad \text{or} \quad \omega^{p-2}[A(\omega)]^{-1} \le 4^{p-2}\varrho, \quad \text{or} \quad \delta(\omega)\xi(\omega)\omega \le 4\varepsilon,$$

where $\theta = (\frac{1}{4}\omega)^{2-p}$ and

$$\widetilde{\eta} = \min\big\{\bar{\eta},\ \widetilde{\xi}\delta\xi\big\}.$$

Moreover, the functions

$$(0,1) \ni \omega \mapsto \widetilde{\eta}(\omega),\ \delta(\omega),\ \xi(\omega),\ [A(\omega)]^{-1}$$

are increasing and satisfy

$$\widetilde{\eta},\ \delta,\ \xi,\ A^{-1} \to 0 \quad \text{as } \omega \to 0.$$

Remark 3.3. Starting from the previous conclusions, the final proof of continuity of u is given in a standard way, showing how the oscillation of u decreases in a controlled way along a sequence of nested cylinders. Moreover, once the functional dependence of β is given, an argument like the one in [11, Section 3.7] can be set up to quantify the modulus of continuity.

4. Proof of Lemma 2.4

We assume $(x_o, t_o) = (0, 0)$ for simplicity and use energy estimate (2.1) with $Q_{R,S} = Q_r(\theta)$ for $\frac{1}{2}\varrho \le r \le \varrho$, and with $k = \mu^- + \xi\omega$.

The length θ of the cylinder will be determined in what follows. For the moment we assume it as a given quantity.

We discard the second term on the left-hand side since $H'_\varepsilon \geq 0$, and we choose the test function such that $\zeta(\cdot, -\theta\varrho^p) = 0$. As for the first term on the left-hand side, we have

$$\int_u^k \beta'(s)(k-s)_- \, \mathrm{d}s \geq \min_{u \leq s \leq k} \beta'(s) \int_u^k (k-s)_- \, \mathrm{d}s = \frac{1}{2}\big(\min_{u \leq s \leq k} \beta'(s)\big)(u-k)_-^2,$$

and also for some $i \in \{0, 1, \ldots, \ell\}$,

$$e_i - \frac{1}{4}\xi\omega \leq \mu^- \leq e_i + \frac{1}{4}\xi\omega \quad \Longrightarrow \quad e_i + \frac{3}{4}\xi\omega \leq k \leq e_i + \frac{5}{4}\xi\omega.$$

Therefore, taking (1.4)–(1.5) into account, we estimate

$$\min_{u \leq s \leq k} \beta'(s) \geq \min\Big\{\beta'\Big(e_i + \frac{5}{4}\xi\omega\Big), \beta'\Big(e_i - \frac{1}{4}\xi\omega\Big)\Big\} = \min\Big\{\beta'\Big(\frac{5}{4}\xi\omega\Big), \beta'\Big(-\frac{1}{4}\xi\omega\Big)\Big\}.$$

If we denote

$$\beta'_*(\xi\omega) := \min\Big\{\beta'\Big(\frac{5}{4}\xi\omega\Big), \beta'\Big(-\frac{1}{4}\xi\omega\Big)\Big\},$$

then energy estimate (2.1) becomes

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta r^p < t < 0} \frac{1}{2}\beta'_*(\xi\omega) \int_{K_r \times \{t\}} (u-k)_-^2 \zeta^p \, \mathrm{d}x + \iint_{Q_r(\theta)} \zeta^p |D(u-k)_-|^p \, \mathrm{d}x\mathrm{d}t \\
&\quad \leq \gamma \iint_{Q_r(\theta)} (u-k)_-^p |D\zeta|^p \, \mathrm{d}x\mathrm{d}t + \gamma \iint_{Q_r(\theta)} \Big(\int_0^{(u-k)_-} \beta'(k-s)s \, \mathrm{d}s\Big) |\partial_t \zeta^p| \, \mathrm{d}x\mathrm{d}t \\
&\qquad + \gamma \iint_{Q_r(\theta)} \Big(\int_0^{(u-k)_-} H'_\varepsilon(k-s)s \, \mathrm{d}s\Big) |\partial_t \zeta^p| \, \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Next we treat the right-hand side of this energy estimate. Let us first estimate the last integral via the given measure-theoretical information in (2.5):

$$\begin{aligned}
&\iint_{Q_r(\theta)} \int_u^k H'_\varepsilon(s)(k-s)_+ \, \mathrm{d}s |\partial_t \zeta^p| \, \mathrm{d}x\mathrm{d}t \\
&\qquad \leq (k-\mu^-) \iint_{Q_r(\theta)} \int_u^k H'_\varepsilon(s) \chi_{[s<k]} \, \mathrm{d}s |\partial_t \zeta^p| \, \mathrm{d}x\mathrm{d}t \\
&\qquad \leq \xi\omega \iint_{Q_\varrho(\theta)} \int_u^k H'_\varepsilon(s) \chi_{[s<k]} \, \mathrm{d}s |\partial_t \zeta^p| \, \mathrm{d}x\mathrm{d}t \\
&\qquad \leq (\xi\omega)^2 \frac{\beta(\xi\omega)}{\xi\omega} \|\partial_t \zeta^p\|_\infty \Big| \Big[u \leq \mu^- + \frac{1}{2}\xi\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\theta) \Big| \\
&\qquad \leq (\xi\omega)^2 \frac{\beta(\xi\omega)}{\xi\omega} \|\partial_t \zeta^p\|_\infty \iint_{Q_r(\theta)} \chi_{[u<\mu^- + \xi\omega]} \, \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

On the other hand, if we consider the second term on the right-hand side of the energy estimate,

$$
\begin{aligned}
&\iint_{Q_r(\theta)} \int_u^k \beta'(s)(k-s)_+ \,\mathrm{d}s |\partial_t \zeta^p| \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le (k-\mu^-) \iint_{Q_r(\theta)} \int_u^k \beta'(s)\chi_{[s<k]} \,\mathrm{d}s |\partial_t \zeta^p| \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le \xi\omega \|\partial_t \zeta^p\|_\infty \iint_{Q_r(\theta)} \int_u^k \beta'(s)\chi_{[s<k]} \,\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&\qquad \le \xi\omega \|\partial_t \zeta^p\|_\infty [\beta(\mu^- + \xi\omega) - \beta(\mu^-)] \iint_{Q_r(\theta)} \chi_{[u<\mu^-+\xi\omega]} \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Due to (1.4), we have

$$
\begin{aligned}
\beta(\mu^- + \xi\omega) - \beta(\mu^-) &\le \beta\Big(e_i + \frac{5}{4}\xi\omega\Big) - \beta\Big(e_i - \frac{1}{4}\xi\omega\Big) = \beta\Big(\frac{5}{4}\xi\omega\Big) - \beta\Big(-\frac{1}{4}\xi\omega\Big) \\
&\le \Big(\frac{5}{4}\xi\omega\Big)\Big[\frac{\beta(\frac{5}{4}\xi\omega)}{\frac{5}{4}\xi\omega} + \frac{\beta(-\frac{1}{4}\xi\omega)}{-\frac{1}{4}\xi\omega}\Big] \\
&\le \Big(\frac{5}{4}\xi\omega\Big) \max\Big\{\frac{\beta(\frac{5}{4}\xi\omega)}{\frac{5}{4}\xi\omega}, \frac{\beta(-\frac{1}{4}\xi\omega)}{-\frac{1}{4}\xi\omega}\Big\}.
\end{aligned}
$$

Hence, we arrive at

$$
\begin{aligned}
&\iint_{Q_r(\theta)} \int_u^k \beta'(s)(k-s)_+ \,\mathrm{d}s |\partial_t \zeta^p| \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le \gamma(\xi\omega)^2 \max\Big\{\frac{\beta(\frac{5}{4}\xi\omega)}{\frac{5}{4}\xi\omega}, \frac{\beta(-\frac{1}{4}\xi\omega)}{-\frac{1}{4}\xi\omega}\Big\} \|\partial_t \zeta^p\|_\infty \iint_{Q_r(\theta)} \chi_{[u<\mu^-+\xi\omega]} \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Consequently, under the assumption that $|\mu^- - e_i| \le \frac{1}{4}\xi\omega$ for some $i \in \{0, 1, \dots, \ell\}$, energy estimate (2.1) becomes

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta r^p<t<0} \frac{1}{2}\beta'_*(\xi\omega) \int_{K_r\times\{t\}} (u-k)_-^2 \zeta^p \,\mathrm{d}x + \iint_{Q_r(\theta)} \zeta^p |D(u-k)_-|^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le \gamma \iint_{Q_r(\theta)} (u-k)_-^p |D\zeta|^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad\quad + \gamma \|\partial_t \zeta^p\|_\infty (\xi\omega)^2 \theta_*(\xi\omega) \iint_{Q_r(\theta)} \chi_{[u<\mu^-+\xi\omega]} \,\mathrm{d}x\mathrm{d}t, \qquad (4.1)
\end{aligned}
$$

where we have set

$$
\theta_*(\xi\omega) := \max\Big\{\frac{\beta(\frac{5}{4}\xi\omega)}{\frac{5}{4}\xi\omega}, \frac{\beta(-\frac{1}{4}\xi\omega)}{-\frac{1}{4}\xi\omega}\Big\}, \quad \beta'_*(\xi\omega) := \min\Big\{\beta'\Big(\frac{5}{4}\xi\omega\Big), \beta'\Big(-\frac{1}{4}\xi\omega\Big)\Big\}.
$$

In the derivation of (4.1) we have used the *periodicity* of β expressed in (1.4), but otherwise, we have tried to keep its formulation as general as possible. Based on energy estimate (4.1), we can establish the following two claims:

Claim 1. *Letting* $\theta \equiv \theta(\xi\omega) := \theta_*(\frac{1}{2}\xi\omega)(\xi\omega)^{2-p}$, *there exists* $c_1 \in (0,1)$ *depending only on the data such that if*

$$|[u \le \mu^- + \xi\omega] \cap Q_\varrho(\theta)| \le c_1 \Big[\frac{\beta'_*(\xi\omega)}{\theta_*(\frac{1}{2}\xi\omega)}\Big] |Q_\varrho(\theta)|,$$

then

$$u \ge \mu^- + \frac{1}{2}\xi\omega \quad \textit{a.e. in } Q_{\frac{1}{2}\varrho}(\theta),$$

provided $|\mu^- - e_i| \le \frac{1}{4}\xi\omega$ *for some* $i \in \{0, 1, \dots, \ell\}$, *and* $\frac{5}{4}\xi\omega \in (0, \bar{d})$.

Proof. Recall that the wanted parameter ξ has not been determined yet. Assuming it has been fixed for the moment, for $n \in \mathbb{N}$, we introduce

$$\begin{cases} \varrho_n = \dfrac{\varrho}{2} + \dfrac{\varrho}{2^{n+1}}, \quad k_n = \mu^- + \xi_n\omega, \quad \xi_n = \dfrac{\xi}{2} + \dfrac{\xi}{2^{n+1}}, \\ K_n = K_{\varrho_n}, \quad Q_n = K_n \times (-\theta\varrho_n^p, 0). \end{cases}$$

The above energy estimate (see (4.1)) is used in Q_n instead of $Q_r(\theta)$, with k and ξ replaced by k_n and ξ_n, respectively. Moreover, by the definition of ξ_n, β'_*, θ_*, the property of β in (1.5), and the restriction $\frac{5}{4}\xi\omega \in (0, \bar{d})$, we estimate

$$\beta'_*(\xi_n\omega) \ge \beta'_*(\xi\omega), \quad \theta_*(\xi_n\omega) \le \theta_*\Big(\frac{1}{2}\xi\omega\Big).$$

To proceed, the quantity θ is chosen to be $\theta_*(\frac{1}{2}\xi\omega)(\xi\omega)^{2-p}$. As a result, energy estimate (4.1) may be written as

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\theta\varrho_n^p < t < 0} \frac{1}{2}\beta'_*(\xi\omega) \int_{K_n \times \{t\}} (u - k_n)_-^2 \zeta^p \,\mathrm{d}x + \iint_{Q_n} \zeta^p |D(u - k_n)_-|^p \,\mathrm{d}x\mathrm{d}t \\ &\quad \le \gamma \iint_{Q_n} (u - k_n)_-^p |D\zeta|^p \,\mathrm{d}x\mathrm{d}t + \gamma \|\partial_t \zeta^p\|_\infty (\xi\omega)^p \theta \iint_{Q_n} \chi_{[u<k_n]} \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Let $0 \le \zeta \le 1$ be a cutoff function that vanishes on the parabolic boundary of Q_n and equals the identity in Q_{n+1}. Moreover, suppose that its derivatives satisfy $|D\zeta| \le 2^n/\varrho$ and $|\partial_t \zeta| \le 2^{np}/\theta\varrho^p$. Then, the energy estimate becomes

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\theta\varrho_n^p < t < 0} \frac{1}{2}\beta'_*(\xi\omega) \int_{K_n \times \{t\}} (u - k_n)_-^2 \zeta^p \,\mathrm{d}x + \iint_{Q_n} \zeta^p |D(u - k_n)_-|^p \,\mathrm{d}x\mathrm{d}t \\ &\quad \le \gamma \frac{2^{np}}{\varrho^p} (\xi\omega)^p |A_n|, \end{aligned}$$

where $A_n = [u < k_n] \cap Q_n$.

An application of the Hölder inequality, the Sobolev imbedding [7, Chapter I, Proposition 3.1], and the above energy estimate gives that

$$
\begin{aligned}
\frac{\xi\omega}{2^{n+2}}|A_{n+1}| &\le \iint_{Q_n} (u-k_n)_-\zeta \,\mathrm{d}x\mathrm{d}t \\
&\le \Big[\iint_{Q_n} \big[(u-k_n)_-\zeta\big]^{p\frac{N+2}{N}} \,\mathrm{d}x\mathrm{d}t\Big]^{\frac{N}{p(N+2)}} |A_n|^{1-\frac{N}{p(N+2)}} \\
&\le \gamma\Big[\iint_{Q_n} \big|D\big[(u-k_n)_-\zeta\big]\big|^p \,\mathrm{d}x\mathrm{d}t\Big]^{\frac{N}{p(N+2)}} \\
&\qquad \times\Big[\operatorname*{ess\,sup}_{-\theta\varrho_n^p<t<0}\int_{K_n\times\{t\}} (u-k_n)_-^2\zeta^2 \,\mathrm{d}x\Big]^{\frac{1}{N+2}} |A_n|^{1-\frac{N}{p(N+2)}} \\
&\le \gamma[\beta_*'(\xi\omega)]^{-\frac{1}{N+2}}(\xi\omega)^{1+\frac{p-2}{N+2}}\frac{2^{n\frac{N+p}{N+2}}}{\varrho^{\frac{N+p}{N+2}}}|A_n|^{1+\frac{1}{N+2}}.
\end{aligned}
$$

In terms of $Y_n = |A_n|/|Q_n|$, the recurrence is rephrased as

$$
Y_{n+1} \le \gamma C^n \Big[\frac{\theta_*(\frac12\xi\omega)}{\beta_*'(\xi\omega)}\Big]^{\frac{1}{N+2}} Y_n^{1+\frac{1}{N+2}},
$$

where $C = 2^{\frac{2N+2+p}{N+2}}$ and γ is a constant depending only on the data. Hence, by [7, Chapter I, Lemma 4.1], there exists $c_1 \in (0,1)$ depending only on the data such that $Y_n \to 0$ if we require that $Y_o \le c_1\beta_*'(\xi\omega)/\theta_*(\frac12\xi\omega)$. This completes the proof. ∎

From the measure-theoretical information in (2.4), we obtain that

$$
|[u(\cdot,t)-\mu^- \ge \eta\omega]\cap K_{2\varrho}| > \alpha 2^{-N}|K_{2\varrho}| \quad \text{for all } t\in(t_o - A\omega^{2-p}\varrho^p, t_o]. \tag{4.2}
$$

The following claim hinges upon this measure information:

Claim 2. *For $j\in\mathbb{N}$, let $\theta_j := \theta_*(\frac12 2^{-j}\eta\omega)(2^{-j}\eta\omega)^{2-p}$. Assume the measure-theoretical information in (4.2) holds with $A\omega^{2-p} \ge \theta_{mj_*}$. There exists $\gamma>0$ depending only on the data such that for any $m,\ j_*\in\mathbb{N}$,*

$$
\Big|\Big[u \le \mu^- + \frac{\eta\omega}{2^{mj_*}}\Big]\cap Q_\varrho(\theta_{mj_*})\Big| \le \frac{\gamma^m 4^{m(N+p)}}{\alpha^m j_*^{m\frac{p-1}{p}}}|Q_\varrho(\theta_{mj_*})|,
$$

provided $|\mu^- - e_i| \le \frac14 2^{-mj_*}\eta\omega$ for some $i\in\{0,1,\dots,\ell\}$.

Proof. We employ energy estimate (4.1) in $Q_{2\varrho}(\theta_{mj_*})$ with the levels

$$
k_j := \mu^- + \frac{\eta\omega}{2^j}, \quad j = 0,1,\dots,j_*.
$$

According to the restriction $|\mu^- - e_i| \le \frac14 2^{-mj_*}\eta\omega$, we are allowed to employ energy estimate (4.1) with the above levels k_j. At this stage we are using neither (1.5) nor (1.6). To this end, the cutoff function ζ is chosen to vanish on the parabolic boundary of $Q_{2\varrho}(\theta_{mj_*})$ and equal the identity in $Q_\varrho(\theta_{mj_*})$, such that $|D\zeta| \le 1/\varrho$ and $|\partial_t\zeta| \le 1/(\theta_{mj_*}\varrho^2)$. Therefore, assuming m and j_* have been chosen and noticing that $\theta_j \le \theta_{mj_*}$, energy estimate (4.1) yields that

$$\iint_{Q_\varrho(\theta_{mj_*})} |D(u-k_j)_-|^p \,\mathrm{d}x\mathrm{d}t \le \frac{\gamma}{\varrho^p}\Big(\frac{\eta\omega}{2^j}\Big)^p\Big(1+\frac{\theta_j}{\theta_{mj_*}}\Big)|A_{j,2\varrho}| \le \frac{\gamma}{\varrho^p}\Big(\frac{\eta\omega}{2^j}\Big)^p|A_{j,2\varrho}|,$$

where $A_{j,2\varrho} := [u<k_j]\cap Q_{2\varrho}(\theta_{mj_*})$.

Next, we apply [7, Chapter I, Lemma 2.2] slicewise to $u(\cdot,t)$ for $t\in(-\theta_{mj_*}\varrho^p,0]$ over the cube K_ϱ, for levels $k_{j+1}<k_j$. Taking into account the measure-theoretical information in (4.2), this gives

$$\begin{aligned}
&(k_j-k_{j+1})|[u(\cdot,t)<k_{j+1}]\cap K_\varrho|\\
&\quad\le \frac{\gamma\varrho^{N+1}}{|[u(\cdot,t)>k_j]\cap K_\varrho|}\int_{[k_{j+1}<u(\cdot,t)<k_j]\cap K_\varrho}|Du(\cdot,t)|\,\mathrm{d}x\\
&\quad\le \frac{\gamma\varrho}{\alpha}\Big[\int_{[k_{j+1}<u(\cdot,t)<k_j]\cap K_\varrho}|Du(\cdot,t)|^p\,\mathrm{d}x\Big]^{\frac1p}|[k_{j+1}<u(\cdot,t)<k_j]\cap K_\varrho|^{1-\frac1p}\\
&\quad= \frac{\gamma\varrho}{\alpha}\Big[\int_{K_\varrho}|D(u-k_j)_-(\cdot,t)|^p\,\mathrm{d}x\Big]^{\frac1p}[|A_{j,\varrho}(t)|-|A_{j+1,\varrho}(t)|]^{1-\frac1p},
\end{aligned}$$

where $A_{j,\varrho}(t) := [u(\cdot,t)<k_j]\cap K_\varrho$. We now integrate the last inequality with respect to t over $(-\theta_{mj_*}\varrho^p,0]$ and apply Hölder's inequality in time. This procedure leads to

$$\begin{aligned}
\frac{\eta\omega}{2^{j+1}}|A_{j+1,\varrho}| &\le \frac{\gamma\varrho}{\alpha}\Big[\iint_{Q_\varrho(\theta_{mj_*})}|D(u-k_j)_-|^p\,\mathrm{d}x\mathrm{d}t\Big]^{\frac1p}[|A_{j,\varrho}|-|A_{j+1,\varrho}|]^{1-\frac1p}\\
&\le \frac{\gamma}{\alpha}\frac{\eta\omega}{2^j}|A_{o,2\varrho}|^{\frac1p}[|A_{j,\varrho}|-|A_{j+1,\varrho}|]^{1-\frac1p},
\end{aligned}$$

where $A_{j,\varrho} = [u<k_j]\cap Q_\varrho(\theta_{mj_*})$.

Now take the power $\frac{p}{p-1}$ on both sides of the above inequality to obtain

$$|A_{j+1,\varrho}|^{\frac{p}{p-1}} \le \Big(\frac{\gamma}{\alpha}\Big)^{\frac{p}{p-1}}|A_{o,2\varrho}|^{\frac{1}{p-1}}[|A_{j,\varrho}|-|A_{j+1,\varrho}|].$$

Add these inequalities from 0 to j_*-1 to obtain

$$j_*|A_{j_*,\varrho}|^{\frac{p}{p-1}} \le \Big(\frac{\gamma}{\alpha}\Big)^{\frac{p}{p-1}}|A_{o,2\varrho}|^{\frac{1}{p-1}}|A_{o,\varrho}|.$$

From this, we conclude

$$|A_{j_*,\varrho}| \le \frac{\gamma}{\alpha j_*^{\frac{p-1}{p}}}|A_{o,2\varrho}|^{\frac1p}|A_{o,\varrho}|^{\frac{p-1}{p}}.$$

Similarly, replacing ϱ by 2ϱ, we have

$$|A_{j_*,2\varrho}| \le \frac{\gamma}{\alpha j_*^{\frac{p-1}{p}}} |A_{o,4\varrho}|^{\frac{1}{p}} |A_{o,2\varrho}|^{\frac{p-1}{p}}.$$

The constant γ(data) appearing in the last two inequalities may be different, but we can take the larger one.

Suppose j_* has been chosen for the moment. We may repeat the above arguments for the same k_j but with $j = j_*, \dots, 2j_*$. We are allowed to employ energy estimate (4.1) with such k_j, due to $|\mu^- - e_i| \le \frac{1}{4} 2^{-2j_*} \eta\omega$. Hence, the energy estimate can be written in the same form with such choices of k_j, and the measure-theoretical condition in (4.2) permits us to apply [7, Chapter I, Lemma 2.2] just as above. Consequently, this will lead us to

$$|A_{2j_*,\varrho}| \le \frac{\gamma}{\alpha j_*^{\frac{p-1}{p}}} |A_{j_*,2\varrho}|^{\frac{1}{p}} |A_{j_*,\varrho}|^{\frac{p-1}{p}},$$
$$|A_{2j_*,2\varrho}| \le \frac{\gamma}{\alpha j_*^{\frac{p-1}{p}}} |A_{j_*,4\varrho}|^{\frac{1}{p}} |A_{j_*,2\varrho}|^{\frac{p-1}{p}}.$$

Here the constant γ is the same as above.

Combining the above estimates would yield that

$$|A_{2j_*,\varrho}| \le \frac{\gamma^2 4^{2(N+p)}}{\alpha^2 j_*^{2\frac{p-1}{p}}} |Q_\varrho(\theta_{mj_*})|.$$

The procedure can be iterated m times to yield that

$$|A_{mj_*,\varrho}| \le \frac{\gamma^m 4^{m(N+p)}}{\alpha^m j_*^{m\frac{p-1}{p}}} |Q_\varrho(\theta_{mj_*})|,$$

provided $|\mu^- - e_i| \le \frac{1}{4} 2^{-mj_*} \eta\omega$. ■

Up to now, we have not used (1.6) yet; we rely on it on the final part. Combining Claims 1 and 2, we can finish the proof of Lemma 2.4, provided we let

$$\xi := 2^{-mj_*} \eta,$$

and we choose m and j_* such that

$$0 < \frac{5}{4} 2^{-mj_*} \eta\omega < \bar{d}, \tag{4.3}$$
$$\frac{\gamma^m 4^{m(N+p)}}{\alpha^m j_*^{m\frac{p-1}{p}}} \le c_1 \frac{\beta'(2^{-mj_*}\eta\omega)}{\theta_*(\frac{1}{2} 2^{-mj_*}\eta\omega)}. \tag{4.4}$$

As for (4.3), it is certainly satisfied if we let

$$j_* > \frac{1}{m}\log_2\Big(\frac{5M}{4\bar{d}}\Big), \tag{4.5}$$

assuming for the moment that $m \in \mathbb{N}$ has already been chosen. Coming to (4.4), we have

$$\frac{\gamma^m 4^{m(N+p)}}{\alpha^m j_*^{m\frac{p-1}{p}}} \le c_1 \frac{\min\left\{\beta'(\frac{5}{4}2^{-mj_*}\eta\omega), \beta'(-\frac{1}{4}2^{-mj_*}\eta\omega)\right\}}{\max\left\{\frac{\beta(\frac{5}{8}2^{-mj_*}\eta\omega)}{\frac{5}{8}2^{-mj_*}\eta\omega}, \frac{\beta(-\frac{1}{8}2^{-mj_*}\eta\omega)}{-\frac{1}{8}2^{-mj_*}\eta\omega}\right\}}.$$

Due to (4.4), we can employ (1.5)–(1.6) to obtain

$$\max\left\{\frac{\beta(\frac{5}{8}2^{-mj_*}\eta\omega)}{\frac{5}{8}2^{-mj_*}\eta\omega}, \frac{\beta(-\frac{1}{8}2^{-mj_*}\eta\omega)}{-\frac{1}{8}2^{-mj_*}\eta\omega}\right\} = \frac{\beta(\frac{1}{8}2^{-mj_*}\eta\omega)}{\frac{1}{8}2^{-mj_*}\eta\omega},$$

$$\beta'\Big(\frac{5}{4}2^{-mj_*}\eta\omega\Big) \le \beta'\Big(\frac{1}{4}2^{-mj_*}\eta\omega\Big) = \beta'\Big(-\frac{1}{4}2^{-mj_*}\eta\omega\Big).$$

Therefore, (4.4) becomes

$$\frac{\gamma^m 4^{m(N+p)}}{\alpha^m j_*^{m\frac{p-1}{p}}} \le c_1 \frac{1}{8} 2^{-mj_*}\eta\omega\, \frac{\beta'(\frac{5}{4}2^{-mj_*}\eta\omega)}{\beta(\frac{1}{8}2^{-mj_*}\eta\omega)},$$

and if we take into account the monotonicity of β, we conclude it suffices to choose j_* such that

$$\frac{\gamma^{2p+1}4^{(2p+1)(N+p)}}{\alpha^{2p} j_*^{2(p-1)}} \le \frac{1}{10}c_1\Big(\frac{5}{4}2^{-2pj_*}\eta\omega\Big)\frac{\beta'(\frac{5}{4}2^{-2pj_*}\eta\omega)}{\beta(\frac{5}{4}2^{-2pj_*}\eta\omega)}, \tag{4.6}$$

where we let $m \in \mathbb{N}$ with $2p \le m < 2p+1$. We claim that there exists j_* such that (4.6) holds true. Indeed, if we let $s := \frac{5}{4}2^{-4j_*}\eta\omega$, then (4.6) can be rewritten as

$$\frac{C'}{\ln^{(2p-2)} s} \le C'' s \frac{\beta'(s)}{\beta(s)},$$

with s in a neighborhood of the origin, and C', C'' constants that depend on the data, α, η, and ω. If there did not exist a j_* satisfying (4.6), we would have

$$\frac{\beta'(\sigma)}{\beta(\sigma)} \le \frac{C'}{C''}\frac{1}{\sigma\ \ln^{(2p-2)}\sigma}, \qquad \forall\, \sigma \in (0, \sigma_o),$$

for some suitable $0 < \sigma_o < \min\{1, \bar{d}\}$; integrating both sides with respect to σ over $(0, \sigma_o)$, we would have a contradiction, since the integral on the right-hand side is finite, whereas the one on the left-hand side diverges. Hence, the required j_* does exist; if we denote it by $\tilde{j}_*$, and take into account (4.5) and the choice of m, we conclude that

$$j_* := \max\Big\{\tilde{j}_*, \frac{1}{4}\log_2\Big(\frac{5M}{4\bar{d}}\Big)\Big\},$$

whence ξ is determined.

5. Two immiscible fluids in porous media with irreducible saturation

5.1. Origin of the model

In nature, subsurface rocks were initially wet and the pores among them were saturated with water. It is important to understand how the oil in a reservoir eventually filled up these pores that were once occupied by the water. The displacement of the water by the oil is driven by the so-called *capillary pressure* that exists on the interface of the two immiscible fluids. The capillary pressure increases as the oil saturation increases, and meanwhile the water saturation is forced to decrease. Such a process continues until all water at the center of the pores is displaced, and the only water left is the layer adherent to the rock grains. In such a case, the remaining water becomes immobile, no matter how high the capillary pressure is exerted. This limiting saturation of water is called the *irreducible saturation* or *connate water saturation* (cf. [3, Section 9.24], [5, Section 2.24], [21, Section 3.4.2]).

5.2. Mathematical aspects

Let v_1 and v_2 stand for the saturations of the two fluids in a porous medium, say, water and oil in rock grains. Assuming Darcy's law and mass conservation, we could set up the system ([16], [3, Chapter 2], [5, Chapter 6], [21, Chapter 10])

$$\begin{cases} \partial_t v_i = \operatorname{div}(k_i(v_i)[Dp_i + e_i(v_i)]), \\ v_1 + v_2 = 1, \end{cases} \quad \text{in } E_T \equiv E \times (0, T]. \tag{5.1}$$

Here $k_i(v_i)$ for $i = 1, 2$ are the permeabilities and p_i are the hydrostatic pressures, whereas $e_i(v_i)$ represents the gravity forces.

If we set $v := v_1$, then the functions k_i and e_i can be recast into functions of v because of $(5.1)_2$. The difference $p_2 - p_1$ is the capillary pressure, which is a function of the saturation v and we denote by $p(v)$. The qualitative behaviors of k_i and p are shown in Figure 3. The irreducible saturation value of the first fluid (water) is denoted by d_o. The existence of solutions to proper initial-boundary value problems for (5.1) is established, for example, in the theorem of [1, Section 2.6] and in [13, Theorem 3].

Following the Kružkov–Sukorjanskiĭ transformation (cf. [14, Section 1] and [22, Appendix A]), we can transform system (5.1) into the following one:

$$\begin{cases} 0 \le v \le 1, \\ \partial_t v - \operatorname{div}(\mathcal{A}(v) Dv + \mathbf{B}(v)) = \mathbf{V} \cdot D\mathcal{C}(v), \\ \operatorname{div} \mathbf{V} = 0, \end{cases} \quad \text{in } E_T. \tag{5.2}$$

Here we have denoted

$$\mathbf{V} = \mathcal{K}(v)(Du + \mathbf{e}(v))$$

and the functions

$$[0, 1] \ni v \mapsto \mathcal{K}(v),\ \mathcal{A}(v),\ \mathbf{B}(v),\ \mathcal{C}(v),\ \mathbf{e}(v)$$

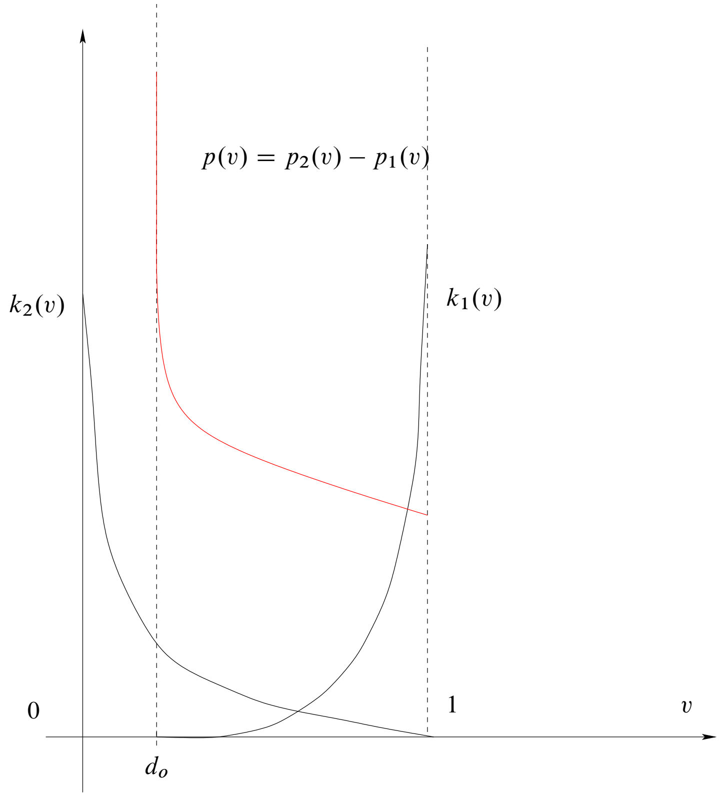

Figure 3. Permeabilities and Capillary Pressure

are continuous, and moreover, for some positive constants C_o and C_1,

$$\begin{cases} C_o \leq \mathcal{K}(v) \leq C_1, \\ \mathcal{A}(v) + |\mathbf{B}(v)| + |\mathcal{C}(v)| + |\mathbf{e}(v)| \leq C_1. \end{cases} \tag{5.3}$$

For an explicit relation between functions u, $\mathcal{K}$, $\mathcal{A}$, $\mathbf{B}$, $\mathcal{C}$ and quantities k_1, k_2, e_1, e_2, and p, we refer to [8, Section 1.1]. Here we only show the expression of $\mathcal{A}$ by

$$\mathcal{A}(v) = \frac{k_1(v)k_2(v)}{k_1(v) + k_2(v)} p'(v).$$

As pointed out in [1, Section 1], the variable u can be considered as a sort of *mean pressure*, and (5.2) can then be seen as an equation of continuity with pressure u and velocity v for an *idealized* incompressible fluid, which replaces the mixture of the two fluids.

According to Figure 3, it exhibits degeneracy near 0 and 1. In general, quantitative information on such degeneracy is unavailable, as the model is derived from hydrostatic experiments, dimensional analysis, and empirical arguments.

Therefore, on the degeneracy of $v \mapsto \mathcal{A}(v)$, we only assume qualitatively that

$$\begin{cases} \mathcal{A}(v) = 0 \quad \text{for } v \in [0, d_o] \cup \{1\}, \\ \mathcal{A}(v) > 0 \quad \text{for } v \in (d_o, 1), \\ \mathcal{A}(v) \text{ is increasing/decreasing in } [d_o, d_o + \delta]/[1-\delta, 1], \end{cases} \tag{5.4}$$

where d_o and δ are certain small positive constants.

The notion of a weak solution will be introduced in the next section. In what follows, the set $\{d_o, \delta, \mathcal{A}, N, C_o, C_1\}$ will be referred to as the *data*. We now state the main result of this section.

Theorem 5.1. *Let (u, v) be a local weak solution to (5.2) under conditions (5.3)–(5.4). Then, $\int_0^v \mathcal{A}(s)\,\mathrm{d}s$ is locally continuous in E_T. Moreover, the modulus of continuity over a compact set in E_T is determined by its distance to the parabolic boundary of E_T, $\mathcal{A}(\cdot)$, the local bound of u, and the data.*

Remark 5.1. Continuity cannot be claimed for v in general, as the ellipticity $\mathcal{A}(v)$ vanishes in the vicinity of 0. On the other hand, Theorem 5.1 implies that the composite function $F(v)$ is locally continuous in E_T for any continuous map $[0,1] \ni s \mapsto F(s)$ that vanishes in $[0, d_o]$. In particular, $(v - d_o)_+$ is locally continuous in E_T, and when $d_o = 0$, Theorem 5.1 recovers the continuity result in [8, Theorem 1.1].

5.3. Notion of solution

A local weak solution to (5.2) is a pair (u, v) in the functional spaces

$$\begin{cases} v \in C_{\mathrm{loc}}\big(0, T; L^2_{\mathrm{loc}}(E)\big), \quad u \in L^2_{\mathrm{loc}}\big(0, T; W^{1,2}_{\mathrm{loc}}(E)\big), \\ w := \displaystyle\int_0^v \mathcal{A}(s)\,\mathrm{d}s \in L^2_{\mathrm{loc}}\big(0, T; W^{1,2}_{\mathrm{loc}}(E)\big), \end{cases}$$

satisfying for any $(t_1, t_2) \subset (0, T]$,

$$\int_E v\zeta \,\mathrm{d}x \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_E [-v\partial_t\zeta + (Dw + \mathbf{B}(v) + \mathcal{C}(v)\mathbf{V}) \cdot D\zeta]\mathrm{d}x\mathrm{d}t = 0,$$

$$\int_{t_1}^{t_2}\int_E \mathcal{K}(v)(Du + \mathbf{e}(v)) \cdot D\varphi \,\mathrm{d}x\mathrm{d}t = 0,$$

for all test functions

$$\zeta \in W^{1,2}_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(E)) \cap L^2_{\mathrm{loc}}(0, T; W^{1,2}_{\mathrm{loc}}(E)), \quad \varphi \in L^2_{\mathrm{loc}}(0, T; W^{1,2}_{\mathrm{loc}}(E)).$$

Letting

$$\Phi(v) := w \equiv \int_0^v \mathcal{A}(s)\,\mathrm{d}s, \quad \widetilde{\beta} := \Phi^{-1},$$

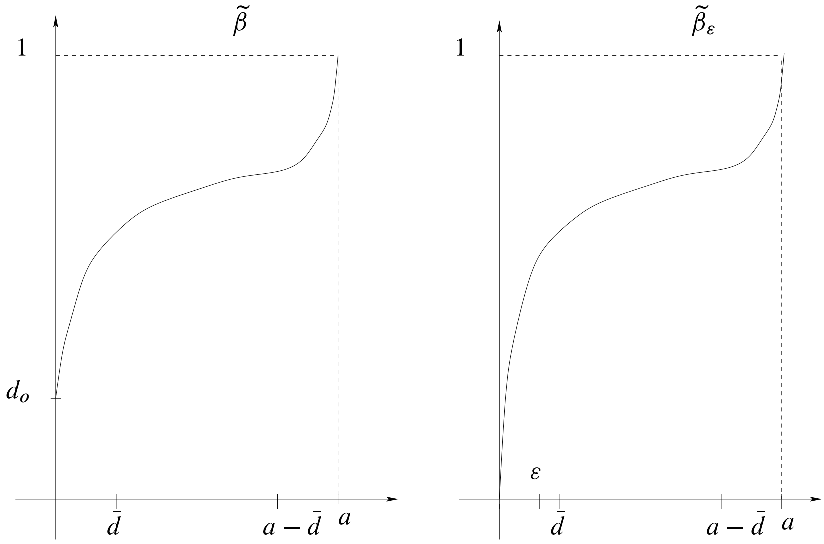

Figure 4. $\widetilde{\beta}$ and $\widetilde{\beta}_\varepsilon$

according to (5.2), formally we obtain

$$\partial_t \widetilde{\beta}(w) - \Delta w \ni \operatorname{div}(\mathbf{B}(v) + \mathscr{C}(v)\mathbf{V}). \tag{5.5}$$

Due to the conditions of $\mathcal{A}$ in (5.4), the graph of $\widetilde{\beta}$ verifies the properties

$$\begin{cases} \widetilde{\beta} = \beta + d_o \mathscr{H}_o : \ [0, a] \to [0, 1], \\ \beta \in C^1(0, a), \quad \beta' \geq \dfrac{1}{C_1}, \quad \beta'(0) = \beta'(a) = \infty, \\ \beta(0) = 0, \quad \beta(a) = 1 - d_o, \\ \beta \text{ is concave in } [0, \bar{d}], \text{ while convex in } [a - \bar{d}, a], \end{cases}$$

where $a = \Phi(1)$ and $\bar{d} > 0$ depends only on $\mathcal{A}$ and δ. The qualitative behavior of $\widetilde{\beta}(\cdot)$ is depicted in Figure 4.

As in Section 1.4, we regularize equation (5.5). For $\varepsilon \in (0, \frac{1}{2}a)$, we let $\mathscr{H}_o^\varepsilon$ be the mollification of $\mathscr{H}_o$ as in Section 1.4. Next we define a mollification of $\widetilde{\beta}$ by

$$\widetilde{\beta}_\varepsilon \equiv \beta + H_\varepsilon := \beta + d_o \mathscr{H}_o^\varepsilon,$$

and accordingly, introduce

$$\Phi_\varepsilon := \widetilde{\beta}_\varepsilon^{-1}, \quad \mathcal{A}_\varepsilon := \Phi_\varepsilon'.$$

As such, $\mathcal{A}_\varepsilon \to \mathcal{A}$ uniformly in $[0, 1]$. See Figure 5.

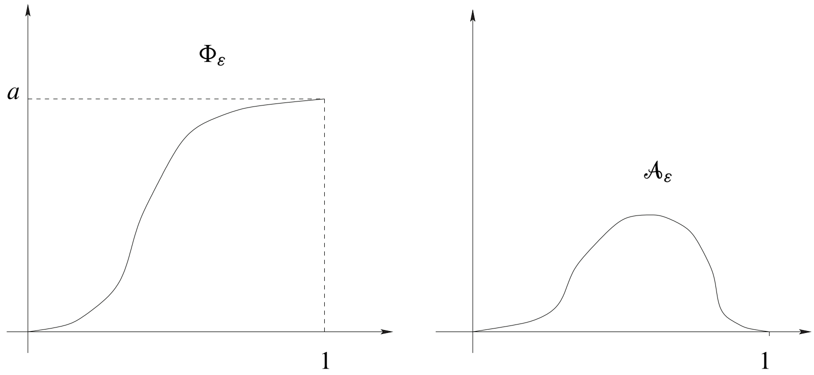

Figure 5. Φ_ε and $\mathcal{A}_\varepsilon = \Phi'_\varepsilon$

Let us still denote by (u, v) a local solution to (5.2) with $\mathcal{A}$ replaced by $\mathcal{A}_\varepsilon$. Then, $w := \Phi_\varepsilon(v)$ will satisfy

$$\partial_t \widetilde{\beta}_\varepsilon(w) - \Delta w = \operatorname{div}(\mathbf{B}(v) + \mathcal{C}(v)\mathbf{V}) \quad \text{weakly in } E_T. \tag{5.6}$$

The last equation of (5.2), that is, $\operatorname{div} \mathbf{V} = 0$, is used, whose dependence on ε is suppressed for simplicity. It plays the role of an incompressibility condition.

Our main assumption is that a local weak solution to (5.2) can be identified in a uniform convergence of solutions to the above regularized problem in (5.6). As such the proof of Theorem 5.1 consists in establishing a uniform estimate on the modulus of continuity for weak solutions to the regularized problem in (5.6).

In [1, Section 3] it was shown that the solutions constructed in E_T are local solutions to (5.2), so that local solutions to (5.2) do exist. On the other hand, we are not aware of an existence theory established under the stipulated approximation; this merits an independent study.

5.4. An auxiliary result

The last equation of (5.2) can be viewed as a family of elliptic equations parametrized by $t \in (0, T)$. Sufficient conditions can be given to ensure the local boundedness of u; a detailed discussion is given in the lemma in [1, Section 3.9]. Then, the conditions in (5.3) allow us to apply the standard elliptic theory and obtain the following (cf. [8, Proposition 2.1]):

Proposition 5.1. *Suppose $u \in L^\infty_{\mathrm{loc}}(E_T)$. Then, $u(\cdot, t) \in C^\epsilon_{\mathrm{loc}}(E)$, uniformly in t, on every interval $[\tau_1, \tau_2] \subset (0, T)$. More precisely, for every compact set $K \subset E$, there exist $\gamma > 1$ and $\epsilon \in (0, 1)$, depending only on $\{N, C_o, C_1\}$, $\|u\|_{\infty, K\times[\tau_1,\tau_2]}$, and the distance from*

$K \times [\tau_1, \tau_2]$ to the parabolic boundary of E_T, such that for all $x_1, x_2 \in K$,

$$|u(x_1,t) - u(x_2,t)| \le \gamma |x_1 - x_2|^{\epsilon} \quad \text{for all } t \in [\tau_1, \tau_2].$$

Moreover, there exists $\gamma > 1$ with the same dependence such that for every cylinder $Q_{R,S} \equiv K_R(x_o) \times (t_o - S, t_o) \subset K \times [\tau_1, \tau_2]$ and every $f \in L^2_{\mathrm{loc}}(0,T; W^{1,2}_{\mathrm{loc}}(E))$,

$$\iint_{Q_{R,S}} |Du|^2 f^2 \zeta^2 \,\mathrm{d}x\mathrm{d}t \le \gamma R^{2\epsilon} \Big(\iint_{Q_{R,S}} |Df|^2 \zeta^2 \,\mathrm{d}x\mathrm{d}t + \iint_{Q_{R,S}} f^2 |D\zeta|^2 \,\mathrm{d}x\mathrm{d}t \Big) + \iint_{Q_{R,S}} f^2 \zeta^2 \,\mathrm{d}x\mathrm{d}t,$$

where ζ is a non-negative cutoff function in $K_R(x_o)$ vanishing on $\partial K_R(x_o)$.

5.5. Energy estimates

Let ζ be a non-negative, piecewise smooth cutoff function in $Q_{R,S}$ vanishing on the set $\partial K_R(x_o) \times (t_o - S, t_o)$. Using $(w-k)_+\zeta^2$ for $k \in (0,1)$ as a test function in the weak formulation of (5.6) in $Q_{R,S}$, standard calculations permit us to produce energy estimates like in Section 2.1. As usual, we will denote by γ a generic positive constant depending on the data.

First of all, we obtain an energy estimate similar to the one in Proposition 2.1:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_o - S < t < t_o} \int_{K_R(x_o)\times\{t\}} \Big(\int_0^{(w-k)_\pm} \beta'(k \pm s) s \,\mathrm{d}s \Big) \zeta^2 \,\mathrm{d}x \\
&\quad + \operatorname*{ess\,sup}_{t_o - S < t < t_o} \int_{K_R(x_o)\times\{t\}} \Big(\int_0^{(w-k)_\pm} H'_\varepsilon(k \pm s) s \,\mathrm{d}s \Big) \zeta^2 \,\mathrm{d}x \\
&\quad + \iint_{Q_{R,S}} \zeta^2 |D(w-k)_\pm|^2 \,\mathrm{d}x\mathrm{d}t \\
&\le \gamma \iint_{Q_{R,S}} (w-k)_\pm^2 |D\zeta|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad + \gamma \iint_{Q_{R,S}} \Big(\int_0^{(w-k)_\pm} \beta'(k \pm s) s \,\mathrm{d}s \Big) |\partial_t \zeta^2| \,\mathrm{d}x\mathrm{d}t \\
&\quad + \gamma \iint_{Q_{R,S}} \Big(\int_0^{(w-k)_\pm} H'_\varepsilon(k \pm s) s \,\mathrm{d}s \Big) |\partial_t \zeta^2| \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{K_R(x_o)\times\{t_o - S\}} \Big(\int_0^{(w-k)_\pm} \beta'(k \pm s) s \,\mathrm{d}s \Big) \zeta^2 \,\mathrm{d}x \\
&\quad + \int_{K_R(x_o)\times\{t_o - S\}} \Big(\int_0^{(w-k)_\pm} H'_\varepsilon(k \pm s) s \,\mathrm{d}s \Big) \zeta^2 \,\mathrm{d}x \\
&\quad \mp \iint_{Q_{R,S}} (\mathbf{B}(v) + \mathcal{C}(v)\mathbf{V}) \cdot D[(w-k)_\pm \zeta^2] \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.7}$$

Apart from the last term, which is due to the right-hand side of (5.6) and we denote as I, all other terms can be estimated just like in Section 2.1. For simplicity of presentation, we will estimate I with $(w-k)_+$ only, as the other case is similar. We write it as $I = I_1 + I_2$, and first estimate I_1 by the boundedness of $\mathbf{B}$:

$$
\begin{aligned}
I_1 &:= -\iint_{Q_{R,S}} \mathbf{B}(v) D[(w-k)_+\zeta^2]\,\mathrm{d}x\mathrm{d}t \\
&\le C_1 \iint_{Q_{R,S}} [\zeta^2|D(w-k)_+| + 2\zeta(w-k)_+|D\zeta|]\,\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{4}\iint_{Q_{R,S}} \zeta^2|D(w-k)_+|^2\,\mathrm{d}x\mathrm{d}t + \iint_{Q_{S,R}} (w-k)_+^2|D\zeta|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad + 2C_1^2 \iint_{Q_{R,S}} \zeta^2\chi_{[w>k]}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

For I_2, we write it as

$$
\begin{aligned}
I_2 &= -\iint_{Q_{R,S}} \zeta^2\mathcal{C}(v)\mathbf{V}\cdot D(w-k)_+\,\mathrm{d}x\mathrm{d}t - 2\iint_{Q_{R,S}} \zeta\mathcal{C}(v)(w-k)_+ D\zeta\cdot\mathbf{V}\,\mathrm{d}x\mathrm{d}t \\
&= I_2^{(1)} + I_2^{(2)}.
\end{aligned}
$$

To proceed, we rewrite $I_2^{(1)}$ by using the fact that $\operatorname{div}\mathbf{V} = 0$ to obtain

$$
\begin{aligned}
I_2^{(1)} &= -\iint_{Q_{R,S}} \mathbf{V}\cdot D\Big(\int_k^w \mathcal{C}(\widetilde{\beta}_\varepsilon(s))\,\chi_{[s>k]}\,\mathrm{d}s\Big)\zeta^2\,\mathrm{d}x\mathrm{d}t \\
&= 2\iint_{Q_{R,S}} \zeta\Big(\int_k^w \mathcal{C}(\widetilde{\beta}_\varepsilon(s))\,\chi_{[s>k]}\,\mathrm{d}s\Big)\mathbf{V}\cdot D\zeta\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Consequently, we may estimate I_2 by using the conditions in (5.3) and Proposition 5.1 as

$$
\begin{aligned}
I_2 &\le 4C_1\iint_{Q_{R,S}} |\mathbf{V}|(w-k)_+\zeta|D\zeta|\,\mathrm{d}x\mathrm{d}t \\
&\le 4C_1^2\iint_{Q_{R,S}} [|Du|(w-k)_+\zeta|D\zeta| + C_1(w-k)_+\zeta|D\zeta|]\,\mathrm{d}x\mathrm{d}t \\
&\le 2C_1^2R^{-\epsilon}\iint_{Q_{R,S}} \zeta^2|Du|^2(w-k)_+^2\,\mathrm{d}x\mathrm{d}t + 2C_1^2\iint_{Q_{R,S}} (w-k)_+^2|D\zeta|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad + 2C_1^2\iint_{Q_{R,S}} \big(C_1^2\zeta^2 + R^\epsilon|D\zeta|^2\big)\chi_{[w>k]}\,\mathrm{d}x\mathrm{d}t \\
&\le \gamma C_1^2R^\epsilon\Big(\iint_{Q_{R,S}} \zeta^2|D(w-k)_+|^2\,\mathrm{d}x\mathrm{d}t + \iint_{Q_{R,S}} (w-k)_+^2|D\zeta|^2\,\mathrm{d}x\mathrm{d}t\Big) \\
&\quad + 2C_1^2\iint_{Q_{R,S}} (w-k)_+^2(R^{-\epsilon}\zeta^2 + |D\zeta|^2)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+ 2C_1^2 \iint_{Q_{R,S}} (C_1^2\zeta^2 + R^\epsilon |D\zeta|^2)\chi_{[w>k]}\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{1}{4}\iint_{Q_{R,S}} \zeta^2|D(w-k)_+|^2\,\mathrm{d}x\mathrm{d}t + 3C_1^2\iint_{Q_{R,S}} (w-k)_+^2(R^{-\epsilon}\zeta^2 + |D\zeta|^2)\,\mathrm{d}x\mathrm{d}t\\
&\quad + 2C_1^2 \iint_{Q_{R,S}} (C_1^2\zeta^2 + R^\epsilon |D\zeta|^2)\chi_{[w>k]}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In the last inequality we have imposed the condition that $\gamma C_1^2 R^\epsilon \le \min\{\frac{1}{4}, C_1^2\}$.

Collecting the above estimates about I, performing similar estimates about other terms of (5.7) as in Section 2.1, and choosing the cutoff function ζ to vanish at $t_o - S$, we obtain an analogue of (2.2):

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o} \frac{1}{C_1}\int_{K_R(x_o)\times\{t\}} \zeta^2(w-k)_\pm^2\,\mathrm{d}x + \frac{1}{2}\iint_{Q_{R,S}} \zeta^2|D(w-k)_\pm|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \gamma\iint_{Q_{R,S}} (w-k)_\pm^2(R^{-\epsilon}\zeta^2 + |D\zeta|^2)\,\mathrm{d}x\mathrm{d}t + \gamma\iint_{Q_{R,S}} (w-k)_\pm|\partial_t\zeta^2|\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \gamma\iint_{Q_{R,S}} (\zeta^2 + R^\epsilon|D\zeta|^2)\chi_{[(w-k)_\pm>0]}\,\mathrm{d}x\mathrm{d}t. \qquad (5.8)
\end{aligned}
$$

Taking ζ independent of t, an analogue of (2.3) is also in order:

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-S<t<t_o} \frac{1}{C_1}\int_{K_R(x_o)\times\{t\}} \zeta^2(w-k)_\pm^2\,\mathrm{d}x + \frac{1}{2}\iint_{Q_{R,S}} \zeta^2|D(w-k)_\pm|^2\,\mathrm{d}x\mathrm{d}t\\
&\le \gamma\iint_{Q_{R,S}} (w-k)_\pm^2(R^{-\epsilon}\zeta^2 + |D\zeta|^2)\,\mathrm{d}x\mathrm{d}t + \gamma\int_{K_R(x_o)\times\{t_o-S\}} \zeta^2(w-k)_\pm\mathrm{d}x\\
&+ \gamma\iint_{Q_{R,S}} (\zeta^2 + R^\epsilon|D\zeta|^2)\chi_{[(w-k)_\pm>0]}\,\mathrm{d}x\mathrm{d}t. \qquad (5.9)
\end{aligned}
$$

Remark 5.2. The main difference of energy estimates (5.8)–(5.9) from (2.2)–(2.3) lies in the last integral. However, if $(w-k)_\pm \le \xi\omega$ in $Q_{R,S}$ for some positive ξ and ω, then the last integral can be combined with the first integral on the right-hand side after enforcing $R^\epsilon \le (\xi\omega)^2$.

5.6. Logarithmic estimates

Letting k, w, and $Q_{R,S}$ be as in Section 5.5, we set

$$
\mathcal{L} := \sup_{Q_{R,S}} (w-k)_\pm,
$$

take $c \in (0, \mathcal{L})$, and introduce the following function in $Q_{R,S}$:

$$
\Psi(x,t) \equiv \Psi\big(\mathcal{L}, (w-k)_\pm, c\big) := \ln_+\Big(\frac{\mathcal{L}}{\mathcal{L} - (w-k)_\pm + c}\Big).
$$

As in Section 2.2, we let $\Psi(s)=\Psi(\mathcal{L},s,c)$, so that

$$\Psi'(s)=\frac{1}{\mathcal{L}-s+c}\chi_{[s>c]}(s),\quad \Psi''(s)=\Big(\frac{1}{\mathcal{L}-s+c}\Big)^2\chi_{[s>c]}(s)=[\Psi'(s)]^2.$$

We aim to derive an analogous logarithmic energy estimate for w just like in Proposition 2.2. For simplicity we will only work with $(w-k)_+$. As in the proof of [10, (2.7)] (see also [11, Proposition 2.2]), we use $(\Psi^2)'\zeta^2$ as a test function in the weak formulation of (5.6) in $Q_{R,S}$, where $\zeta=\zeta(x)$ is a non-negative cutoff function in $K_R(x_o)$ that equals the identity in $K_{\sigma R}(x_o)$ and satisfies $|D\zeta|\le 1/[(1-\sigma)R]$. The only care is needed for a term resulting from the right-hand side of (5.6), which we write as

$$I:=-\iint_{Q_{R,S}}(\mathbf{B}(v)+\mathscr{C}(v)\mathbf{V})\cdot D((\Psi^2)'\zeta^2)=I_1+I_2.$$

For I_1, we estimate by using $|\mathbf{B}|\le C_1$, $(\Psi^2)''=2(1+\Psi)(\Psi')^2$ and Young's inequality,

$$\begin{aligned}
I_1&=-\iint_{Q_{R,S}}\zeta^2(\Psi^2)''D(w-k)_+\cdot\mathbf{B}(v)\,\mathrm{d}x\mathrm{d}t\\
&\quad-2\iint_{Q_{R,S}}\zeta(\Psi^2)'\,\mathbf{B}(v)\cdot D\zeta\,\mathrm{d}x\mathrm{d}t\\
&\le 2C_1\iint_{Q_{R,S}}\zeta^2(1+\Psi)(\Psi')^2|D(w-k)_+|\,\mathrm{d}x\mathrm{d}t\\
&\quad+4C_1\iint_{Q_{R,S}}\zeta|D\zeta|\Psi\Psi'\,\mathrm{d}x\mathrm{d}t\\
&\le\frac12\iint_{Q_{R,S}}\zeta^2(1+\Psi)(\Psi')^2|D(w-k)_+|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad+2C_1^2\iint_{Q_{R,S}}\zeta^2(1+\Psi)(\Psi')^2\,\mathrm{d}x\mathrm{d}t\\
&\quad+4C_1\iint_{Q_{R,S}}\zeta|D\zeta|\Psi\Psi'\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

As far as the last term in the previous estimate is concerned, we have

$$\begin{aligned}
4C_1\iint_{Q_{R,S}}\zeta|D\zeta|\Psi\Psi'\,\mathrm{d}x\mathrm{d}t&\le\gamma\sup_{Q_{R,S}}(\Psi\Psi')^2\iint_{Q_{S,R}}\zeta^2\,\mathrm{d}x\mathrm{d}t+\iint_{Q_{S,R}}|D\zeta|^2\,\mathrm{d}x\mathrm{d}t\\
&\le\gamma\sup_{Q_{R,S}}(\Psi\Psi')^2|Q_{R,S}|+\frac{\gamma}{(1-\sigma)^2R^2}|Q_{R,S}|.
\end{aligned}$$

As for I_2, we write it as

$$\begin{aligned}
I_2&=-\iint_{Q_{R,S}}\mathscr{C}(v)\mathbf{V}\cdot D(\Psi^2)'\zeta^2\,\mathrm{d}x\mathrm{d}t-2\iint_{Q_{R,S}}\zeta\mathscr{C}(v)(\Psi^2)'D\zeta\cdot\mathbf{V}\,\mathrm{d}x\mathrm{d}t\\
&=I_2^{(1)}+I_2^{(2)}.
\end{aligned}$$

We then use the fact that div $\mathbf{V} = 0$ to estimate

$$\begin{aligned} I_2^{(1)} &= -\iint_{Q_{R,S}} \zeta^2 \mathcal{C}(v)(\Psi^2)'' D(w-k)_+ \cdot \mathbf{V}\,\mathrm{d}x\mathrm{d}t \\ &= -\iint_{Q_{R,S}} \zeta^2 D\Big(\int_k^w (\Psi^2(s))''\mathcal{C}(\widetilde{\beta}_\varepsilon(s))\mathrm{d}s\Big)\cdot \mathbf{V}\,\mathrm{d}x\mathrm{d}t \\ &= 2\iint_{Q_{R,S}} \zeta\Big(\int_k^w (\Psi^2(s))''\mathcal{C}(\widetilde{\beta}_\varepsilon(s))\mathrm{d}s\Big)\mathbf{V}\cdot D\zeta\,\mathrm{d}x\mathrm{d}t \\ &\le \gamma \iint_{Q_{R,S}} \int_k^w (\Psi^2(s))''\,\mathrm{d}s|\mathbf{V}||D\zeta|\zeta\,\mathrm{d}x\mathrm{d}t \\ &= \gamma \iint_{Q_{R,S}} (\Psi^2(w))'|\mathbf{V}||D\zeta|\zeta\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

As a result, we may estimate I_2 by using the definition of $\mathbf{V}$, the conditions in (5.3), and Proposition 5.1 as well as Young's inequality as

$$\begin{aligned} I_2 &\le \gamma \iint_{Q_{R,S}} (\Psi^2(w))'|\mathbf{V}||D\zeta|\zeta\,\mathrm{d}x\mathrm{d}t \\ &\le \gamma \sup_{Q_{R,S}} (\Psi\Psi')^2 \iint_{Q_{S,R}} |\mathbf{V}|^2\zeta^2\,\mathrm{d}x\mathrm{d}t + \iint_{Q_{S,R}} |D\zeta|^2\,\mathrm{d}x\mathrm{d}t \\ &\le \gamma \sup_{Q_{R,S}} (\Psi\Psi')^2 \frac{R^{2\epsilon}}{(1-\sigma)^2R^2}|Q_{R,S}| + \frac{\gamma}{(1-\sigma)^2R^2}|Q_{R,S}|. \end{aligned}$$

Upon using

$$\sup_{Q_{R,S}} \Psi(w) \le \ln\frac{\mathcal{L}}{c}, \qquad \sup_{Q_{R,S}} \Psi'(w) \le \frac{1}{c},$$

we arrive at the following logarithmic estimate:

Proposition 5.2. *There exists $\gamma > 1$ depending only on the data such that for any $\sigma \in (0,1)$,*

$$\begin{aligned} &\sup_{t_o-S\le t\le t_o} \int_{K_{\sigma R}(x_o)} \Psi^2(x,t)\,\mathrm{d}x \\ &\quad\le \frac{\gamma}{c}\int_{K_R(x_o)} \Psi(x,t_o-S)\,\mathrm{d}x + \frac{\gamma}{(1-\sigma)^2R^2}\iint_{Q_{R,S}} \Psi\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \frac{\gamma}{c^2}\Big(1+\ln\frac{\mathcal{L}}{c}\Big)^2\Big(1+\frac{R^{2\epsilon}}{(1-\sigma)^2R^2}\Big)|Q_{R,S}| + \frac{\gamma}{(1-\sigma)^2R^2}|Q_{R,S}|. \end{aligned}$$

Letting the quantities $\mu^\pm$ and ω be defined by the supremum/infimum and the oscillation of w over the cylinder $\mathcal{Q} = K \times (T_1, T_2)$ as in Section 2.2, employing Proposition 5.2, we have the following result parallel to Lemma 2.1; The change brought by the extra terms is an either-or statement:

Lemma 5.1. *Let w be a local weak sub(super)-solution to* (5.6) *in E_T. For $\xi \in (0,1)$, suppose that*

$$\pm(\mu^\pm - u(\cdot, t_1)) \geq \xi\omega \quad \text{a.e. in } K_\varrho(x_o).$$

Then, for any $\alpha \in (0,1)$ and $A \geq 1$, there exists $\bar{\xi} \in (0, \frac{1}{4}\xi)$ such that either

$$\omega \leq \Big(1 + \ln \frac{\xi}{\bar{\xi}}\Big)\frac{\varrho^\epsilon}{\bar{\xi}}$$

or

$$|[\pm(\mu^\pm - u(\cdot, t)) \leq \bar{\xi}\omega] \cap K_{\frac{1}{2}\varrho}(x_o)| \leq \alpha |K_{\frac{1}{2}\varrho}| \quad \text{for all } t \in (t_1, t_1 + A\varrho^2),$$

provided the cylinder $K_\varrho(x_o) \times (t_1, t_1 + A\varrho^2)$ is included in $\mathcal{Q}$. Moreover, the dependence of $\bar{\xi}$ is given by

$$\bar{\xi} = \frac{1}{2}\xi \exp\Big\{ -\gamma(\text{data})\frac{A}{\alpha}\Big\}.$$

5.7. De Giorgi-type lemmas

Let the quantities $\mu^\pm$ and ω be defined over $\mathcal{Q}$ as before. We now present some De Giorgi-type lemmas parallel to Lemmas 2.2–2.4. The first one hinges on the energy estimates in (5.8); the proof can be adapted from that of Lemma 2.2, recalling Remark 5.2.

Lemma 5.2. *Let u be a local weak sub(super)-solution to* (5.6) *in E_T and let $\xi \in (0,1)$. There exists a constant $c_o \in (0,1)$ depending only on the data such that if*

$$|[\pm(\mu^\pm - u) \leq \xi\omega] \cap Q_\varrho| \leq c_o(\xi\omega)^{\frac{N+2}{2}}|Q_\varrho|, \tag{5.10}$$

then either

$$\xi\omega \leq \varrho^{\frac{\epsilon}{2}}$$

or

$$\pm(\mu^\pm - u) \geq \frac{1}{2}\xi\omega \quad \text{a.e. in } Q_{\frac{1}{2}\varrho},$$

provided $Q_\varrho = K_\varrho(x_o) \times (t_o - \varrho^2, t_o)$ is included in $\mathcal{Q}$.

A variant version involving quantitative initial data is formulated as Lemma 2.3. The proof is based on (5.9) and can be adapted from that of Lemma 2.3, recalling Remark 5.2.

Lemma 5.3. *Let w be a local weak sub(super)-solution to* (5.6) *in E_T. Assume that for some $\xi \in (0,1)$, it holds that*

$$\pm(\mu^\pm - w(\cdot, t_1)) \geq \xi\omega \quad \text{a.e. in } K_\varrho(x_o).$$

There exists a constant $\gamma_o \in (0,1)$ depending only on the data such that for any $\theta > 0$, if

$$|[\pm(\mu^\pm - w) \leq \xi\omega] \cap [(x_o, t_1) + Q_\varrho^+(\theta)]| \leq \frac{\gamma_o}{\theta}|Q_\varrho^+(\theta)|,$$

then either

$$\xi\omega \le \varrho^{\frac{\epsilon}{2}}$$

or

$$\pm(\mu^{\pm} - w) \ge \frac{1}{2}\xi\omega \quad \textit{a.e. in } K_{\frac{1}{2}\varrho}(x_o) \times (t_1, t_1 + \theta\varrho^2),$$

provided the cylinder $(x_o, t_1) + Q_\varrho^+(\theta) \equiv K_\varrho(x_o) \times (t_1, t_1 + \theta\varrho^2)$ *is included in* $\mathcal{Q}$.

The next lemma parallels Lemma 2.4. We only present a sketch of the proof, while keeping reference to the proof of Lemma 2.4 in Section 4 for more details. We omit (x_o, t_o) for simplicity.

Lemma 5.4. *Let* w *be a local weak super-solution to* (5.6) *in* E_T. *Assume that for some* α, $\eta \in (0, 1)$ *and* $A > 1$, *it holds that*

$$|[w(\cdot, t) - \mu^- \ge \eta\omega] \cap K_\varrho| > \alpha|K_\varrho| \quad \textit{for all } t \in (t_o - A\varrho^2, t_o].$$

There exists $\xi_o \in (0, \eta)$ *determined by the data,* α, η, *and* $\beta(\cdot)$ *such that if*

$$A \ge \theta_o = \frac{\beta(\frac{1}{8}\xi_o\omega)}{\frac{1}{8}\xi_o\omega}, \quad \mu^- \le \frac{1}{4}\xi_o\omega, \quad \frac{5}{4}\xi_o\omega \le \bar{d},$$

and it holds that

$$\iint_{Q_\varrho(\theta_o)} \Big(\int_w^k H'_\varepsilon(s)\chi_{[s<k]}\,\mathrm{d}s\Big)\mathrm{d}x\mathrm{d}t \le \beta(\xi_o\omega)\Big|\Big[w \le \mu^- + \frac{1}{2}\xi_o\omega\Big] \cap Q_{\frac{1}{2}\varrho}(\theta_o)\Big|, \tag{5.11}$$

where $k = \mu^- + \xi_o\omega$, *then either*

$$\xi_o\omega \le \varrho^{\frac{\epsilon}{2}}$$

or

$$w \ge \mu^- + \frac{1}{2}\xi_o\omega \quad \textit{a.e. in } Q_{\frac{1}{2}\varrho}(\theta_o),$$

provided the cylinder $Q_\varrho(A)$ *is included in* $\mathcal{Q}$.

Likewise, there exists $\xi_1 \in (0, \eta)$ *determined by the data,* α, η, *and* $\beta(\cdot)$ *such that if*

$$A \ge \theta_1 = \frac{\beta(a) - \beta(a - \frac{1}{8}\xi_1\omega)}{\frac{1}{8}\xi_1\omega}, \quad a - \mu^- \le \frac{1}{4}\xi_1\omega, \quad \frac{5}{4}\xi_1\omega \le \bar{d},$$

then either

$$\xi_1\omega \le \varrho^{\frac{\epsilon}{2}}$$

or

$$w \ge \mu^- + \frac{1}{2}\xi_1\omega \quad \textit{a.e. in } Q_{\frac{1}{2}\varrho}(\theta_1),$$

provided the cylinder $Q_\varrho(A)$ *is included in* $\mathcal{Q}$.

Proof. We first point out that the proof of Lemma 2.4 hinges upon energy estimate (4.1). Starting from (5.7) and employing (5.11) and $\mu^- \le \frac{1}{4}\xi_o\omega$, a straightforward adaption of the calculations in Section 4 will yield the following energy estimate analogous to (4.1) for $\frac{1}{2}\varrho \le r \le \varrho$, $\xi \ge \xi_o$, and $k = \mu^- + \xi\omega$:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta r^2<t<0} \frac{1}{2}\beta'\Big(\frac{5}{4}\xi\omega\Big)\int_{K_r\times\{t\}} (w-k)_-^2\zeta^2\,\mathrm{d}x + \iint_{Q_r(\theta)} \zeta^2|D(w-k)_-|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \gamma \iint_{Q_r(\theta)} (w-k)_-^2(r^{-\epsilon}\zeta^2+|D\zeta|^2)\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \gamma\|\partial_t\zeta^2\|_\infty(\xi\omega)^2\frac{\beta(\frac{5}{4}\xi\omega)}{\frac{5}{4}\xi\omega}\iint_{Q_r(\theta)} \chi_{[w<\mu^-+\xi\omega]}\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \gamma\iint_{Q_r(\theta)} (\zeta^2+r^\epsilon|D\zeta|^2)\chi_{[w<\mu^-+\xi\omega]}\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.12}$$

Now using (5.12), a version of Claims 1 and 2 can be reproduced. The last integral of (5.12) is absorbed by the first term on the right-hand side once imposing $\xi_o\omega \ge \varrho^{\frac{\epsilon}{2}}$. This procedure determines ξ_o just like in the proof of Lemma 2.4, which finishes the proof of the first part.

As for the second part, we let $\bar\beta(s) := \beta(a) - \beta(a-s)$. Then, starting from (5.7) and using $a - \mu^- \le \frac{1}{4}\xi_1\omega$, similar calculations will yield for $\frac{1}{2}\varrho \le r \le \varrho$, $\xi \ge \xi_1$, and $k = \mu^- + \xi\omega$ that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta r^2<t<0} \frac{1}{2}\bar\beta'\Big(\frac{1}{4}\xi\omega\Big)\int_{K_r\times\{t\}} (w-k)_-^2\zeta^2\,\mathrm{d}x + \iint_{Q_r(\theta)} \zeta^2|D(w-k)_-|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \gamma \iint_{Q_r(\theta)} (w-k)_-^2(r^{-\epsilon}\zeta^2+|D\zeta|^2)\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \gamma\|\partial_t\zeta^2\|_\infty(\xi\omega)^2\frac{\bar\beta(\frac{1}{4}\xi\omega)}{\frac{1}{4}\xi\omega}\iint_{Q_r(\theta)} \chi_{[w<\mu^-+\xi\omega]}\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \gamma\iint_{Q_r(\theta)} (\zeta^2+r^\epsilon|D\zeta|^2)\chi_{[w<\mu^-+\xi\omega]}\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.13}$$

Based on (5.13), the same procedure as before will determine ξ_1 and thus finish the proof of the second part. ∎

5.8. Proof of Theorem 5.1

The set-up is similar to that of Section 3. More specifically, we let $(x_o, t_o) = (0, 0)$, introduce $Q_o = K_{8\varrho} \times (-8\varrho, 0)$ for $\varrho \in (0, 1)$, and define $\mu^\pm$ and ω to be the supremum/infimum and the oscillation of w over Q_o, respectively. Moreover, we let $A(\omega) > 1$ be determined by the data and ω verify the intrinsic relation given by (3.1). We will follow the reasoning in Section 3 while keeping $p = 2$ and highlighting the main differences.

The argument starts with (3.2) for some $\bar{t} \in (-(A-1)\varrho^2, 0)$. Using Lemmas 5.1–5.3 in the place of Lemmas 2.1–2.3, we may reach a reduction of oscillation like in Section 3.1. More precisely, either

$$\omega \le (1 + |\ln \eta|)\frac{\varrho^{\frac{\epsilon}{2}}}{\eta} \quad \text{for } \eta = \frac{1}{16}\exp\{-\gamma A^2\}$$

or

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}} w \le \Big(1 - \frac{1}{2}\eta\Big)\omega. \tag{5.14}$$

The argument continues with (3.7) for any $\bar{t} \in (-(A-1)\varrho^2, 0)$. Fixing $\bar{t}$, we can perform analysis on the local clustering of u near its infimum just like Lemma 3.1, once we enforce proper alternative conditions whenever energy estimate (5.8) is employed in its proof (cf. Remark 5.2).

Lemma 5.5. *For every $\lambda \in (0,1)$ and $\eta \in (0,1)$, there exists a point $(x_*, t_*) \in (0, \bar{t}) + Q_\varrho$, a number $\kappa \in (0,1)$, and a cylinder $(x_*, t_*) + Q_{\kappa\varrho} \subset (0, \bar{t}) + Q_\varrho$ such that either*

$$\omega \le 4\varrho^{\frac{\epsilon}{2}}$$

or

$$\Big|\Big[w \le \mu^- + \frac{1}{4}\lambda\omega\Big] \cap [(x_*, t_*) + Q_{\kappa\varrho}]\Big| \le \eta |Q_{\kappa\varrho}|.$$

The constant κ is determined by the data, M, λ, η, α, and ω.

Using this Lemma 5.5, together with Lemmas 5.1–5.2, one can reason like in Section 3.2 and obtain an analogue of (3.9). More specifically, letting κ be determined as in Lemma 5.5 by $\lambda = \frac{1}{2}$ and $\eta = \alpha = c_o(\frac{1}{4}\omega)^{\frac{N+2}{2}}$, there exist

$$\bar{\alpha} = \frac{1}{2^{2N+1}}\kappa^N, \quad \bar{\xi} = \frac{1}{32}\exp\Big\{-\frac{\gamma}{\kappa^2}\Big\}$$

such that either

$$\omega \le (1 + |\ln \bar{\xi}|)\frac{\varrho^\epsilon}{\bar{\xi}}$$

or

$$|[w(\cdot, t) \ge \mu^- + \bar{\xi}\omega] \cap K_\varrho| > \bar{\alpha}|K_\varrho| \quad \text{for all } t \in (-A\varrho^2, 0]. \tag{5.15}$$

Let $\xi(\omega)$, $\delta(\omega) \in (0,1)$ to be determined and introduce the cylinder

$$Q_\varrho(\widetilde{\theta}) = K_\varrho \times (-\widetilde{\theta}\varrho^2, 0), \quad \widetilde{\theta} = (\delta\xi\omega)^{-1}$$

such that $8^2\widetilde{\theta} \le A$, where $A(\omega)$ is the number appearing in (5.15) and yet to be chosen.

Given the measure-theoretical information in (5.15), we first use Lemma 5.4 to determine ξ_o and ξ_1. Now one could reproduce the arguments in Sections 3.3–3.5, assuming either

$$\mu^- \le \frac{1}{4}\delta\xi_o\omega \quad \text{or} \quad a - \mu^- \le \frac{1}{4}\delta\xi_1\omega. \tag{5.16}$$

Indeed, let us suppose the second of (5.16) holds true. One can use the second part of Lemma 5.4 to obtain the reduction of oscillation

$$\operatorname*{ess\,osc}_{Q_{\frac14\varrho}} w \le \Big(1-\frac14\xi_1\Big)\omega,$$

after enforcing the conditions

$$\xi_1\omega \ge \varrho^{\frac{\epsilon}{2}},\quad A \ge \max\Big\{\frac{\beta(a)-\beta(a-\frac18\xi_1\omega)}{\frac18\xi_1\omega}, 8^2\widetilde{\theta}\Big\}.$$

Alternately, let us suppose the first of (5.16) holds true. One can use the first part of Lemma 5.4 to obtain the reduction of oscillation

$$\operatorname*{ess\,osc}_{Q_{\frac14\varrho}} w \le \Big(1-\frac14\xi_o\Big)\omega,$$

after enforcing the conditions

$$\xi_o\omega \ge \varrho^{\frac{\epsilon}{2}},\quad A \ge \max\Big\{\frac{\beta(\frac18\xi_o\omega)}{\frac18\xi_o\omega}, 8^2\widetilde{\theta}\Big\},$$

and *in addition* (5.11).

Fixing such ξ_o, if the measure condition in (5.10) with ξ replaced by ξ_o is satisfied, then Lemma 5.2 yields the same reduction of oscillation as above, once enforcing the inequality $\xi_o\omega \ge \varrho^{\frac{\epsilon}{2}}$.

Still assuming $(5.16)_1$ holds true, the case when both $(5.10)_{\xi=\xi_o}$ and (5.11) are violated can be treated like in Section 3.4. Consequently, the following energy estimate can be deduced from (5.7) (cf. Lemma 3.2):

Lemma 5.6. *Denoting $b(\xi_o\omega)=(\xi_o\omega)^{1+\frac{N+2}{2}}$, and setting $\bar\theta=(\bar\delta\xi_o\omega)^{-1}$ and $k=\mu^- + \bar\delta\xi_o\omega$ with $\bar\delta\in(\delta,2\delta)$, there exists a positive constant γ depending only on the data such that for all $\sigma\in(0,1)$ and all $r\in[2\varrho,8\varrho]$, we have*

$$\begin{aligned}
&\delta\xi_o\omega(\bar\delta\xi_o\omega)^{-2}b(\xi_o\omega)\operatorname*{ess\,sup}_{-\bar\theta(\sigma r)^2<t<0}\int_{K_{\sigma r}\times\{t\}}(w-k)_-^2\,\mathrm{d}x+\iint_{Q_{\sigma r}(\bar\theta)}|D(w-k)_-|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le\frac{\gamma}{(1-\sigma)^2r^2}\iint_{Q_r(\bar\theta)}(w-k)_-^2\mathrm{d}x\mathrm{d}t+\frac{\gamma}{(1-\sigma)\bar\theta r^2}\iint_{Q_r(\bar\theta)}(w-k)_-\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\frac{\gamma r^{\epsilon}}{(1-\sigma)^2r^2}\iint_{Q_r(\bar\theta)}\chi_{[w<k]}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

provided that

$$\mu^-\le\frac14\delta\xi_o\omega,\quad \frac54\xi_o\omega\le\bar d,\quad \textit{and}\quad \varepsilon\le\frac14\delta\xi_o\omega.$$

The energy estimate in Lemma 5.6 allows us to show the following De Giorgi-type lemma, whose proof is similar to that of Lemma 3.3:

Lemma 5.7. *Let $\delta \in (0,1)$. There exists a constant $c_1 \in (0,1)$ depending only on the data such that if*

$$|[w < \mu^- + 2\delta\xi_o\omega] \cap Q_{4\varrho}(\widetilde{\theta})| \leq c_1 b(\xi_o\omega)|Q_{4\varrho}(\widetilde{\theta})|, \quad \textit{where } \widetilde{\theta} = (\delta\xi_o\omega)^{-1},$$

then, enforcing $\mu^- \leq \frac{1}{4}\delta\xi_o\omega$, $\frac{5}{4}\xi_o\omega \leq \bar{d}$, and $\varepsilon \leq \frac{1}{4}\delta\xi_o\omega$, we have either

$$\delta\xi_o\omega \leq \varrho^{\frac{\epsilon}{2}}$$

or

$$w \geq \mu^- + \delta\xi_o\omega \quad \textit{a.e. in } Q_{2\varrho}(\widetilde{\theta}),$$

provided $4^2\widetilde{\theta} \leq A$.

We still need a version of Lemma 3.4, which is stated as follows; the proof hinges solely on energy estimate (5.8) and is analogous to that of Lemma 3.4:

Lemma 5.8. *Assume the measure information in (5.15) holds. There exists a positive constant γ depending only on the data such that for any $j_* \in \mathbb{N}$, we have either*

$$\xi_o\omega \leq \varrho^{\frac{\epsilon}{2}}$$

or

$$\left|\left[w \leq \mu^- + \frac{\xi_o\omega}{2^{j_*}}\right] \cap Q_{4\varrho}(\widetilde{\theta})\right| \leq \frac{\gamma}{\bar{\alpha}} j_*^{-\frac{1}{2}}|Q_{4\varrho}(\widetilde{\theta})|, \quad \textit{where } \widetilde{\theta} = \left(\frac{\xi_o\omega}{2^{j_*}}\right)^{-1},$$

provided $4^2\widetilde{\theta} \leq A\theta$.

Like in Section 3.5, we combine Lemmas 5.7 and 5.8 to determine j_* by

$$\frac{\gamma}{\bar{\alpha} j_*^{\frac{1}{2}}} \leq c_1 b(\xi_o\omega),$$

and determine δ by $2\delta = 2^{-j_*}$, and thus, $\widetilde{\theta}$ by $\widetilde{\theta} = (\delta\xi_o\omega)^{-1}$. Enforcing $\delta\xi_o\omega \geq \varrho^{\frac{\epsilon}{2}}$, Lemma 5.7 yields a reduction of oscillation

$$\operatorname*{ess\,osc}_{Q_{2\varrho}} w \leq (1 - \delta\xi_o)\omega.$$

This completes the argument when the first of (5.16) holds true.

The choice of A can be made out of

$$A(\omega) := \max\left\{\frac{\beta(\frac{1}{8}\xi_o\omega)}{\frac{1}{8}\xi_o\omega}, \frac{\beta(a) - \beta(a - \frac{1}{8}\xi_1\omega)}{\frac{1}{8}\xi_1\omega}, 8^2\widetilde{\theta}\right\} = 8^2\widetilde{\theta},$$

taking into account the value of $\widetilde{\theta}$ determined above. This also determines η in (5.14) by such A.

The remaining case is when (5.16) does not hold. This is done as in Lemma 3.5. Here the quantities $\{\xi_o, \xi_1, \delta\}$ being fixed and setting $\xi := \min\{\xi_o, \xi_1\}$, Lemma 3.5 can be reproduced after additionally enforcing $\delta\xi\omega \geq \varrho^{\frac{\epsilon}{2}}$.

Overall, we have achieved the following:

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{4}\varrho}} w \leq (1 - \tilde{\eta}(\omega))\omega, \quad \text{or} \quad [A(\omega)]^{-1} \leq \varrho, \quad \text{or} \quad \hat{\xi}\omega \leq \varrho^{\frac{\epsilon}{2}}, \quad \text{or} \quad \hat{\xi}\omega \leq \varepsilon,$$

where

$$\hat{\xi} := \min\left\{\frac{\bar{\xi}}{|\ln \bar{\xi}|}, \frac{\eta}{|\ln \eta|}, \delta\xi\right\}, \quad \tilde{\eta} := \min\left\{\frac{1}{2}\eta, \frac{1}{4}\xi, \delta\xi_o\right\}.$$

Moreover, the functions

$$(0,1) \ni \omega \mapsto \hat{\xi}(\omega),\ \eta(\omega),\ [A(\omega)]^{-1}$$

are increasing, determined only by the data, and satisfy

$$\hat{\xi},\ \eta,\ A^{-1} \to 0 \quad \text{as } \omega \to 0.$$

As a result, we can now set up an iteration scheme as discussed in Remark 3.3 and finish the proof.

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