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Existence of surfaces optimizing geometric and PDE shape functionals under reach constraint

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Abstract. This article deals with the existence of hypersurfaces minimizing general shape functionals under certain geometric constraints. We consider as admissible shapes orientable hypersurfaces satisfying a so-called *reach* condition, also known as the uniform ball property, which ensures $\mathcal{C}^{1,1}$ regularity of the hypersurface. In this paper, we revisit and generalize the results of Dalphin (2018 and 2020) and Guo and Yang (2013). We provide a simpler framework and more concise proofs of some of the results contained in these references and extend them to a new class of problems involving PDEs. Indeed, by using the signed distance, we avoid the intensive and technical use of local maps, as was the case in the above references. Our approach, originally developed to solve an existence problem in Privat, Robin, and Sigalotti's 2022 paper, can be easily extended to costs involving different mathematical objects associated with the domain, such as solutions of elliptic equations on the hypersurface.

1. Framework and main results

1.1. Introduction

In this paper, we are interested in the question of the existence of optimal sets for shape optimization problems involving surfaces. More precisely, we are interested in shape functionals written as

$$J(\Omega) = \int_{\partial\Omega} j(x, \nu_{\partial\Omega}(x), B_{\partial\Omega}(x)),\ d\mu_{\partial\Omega}(x),$$

where Ω denotes a smooth subset of $\mathbb{R}^d$, the word "smooth" is understood at this stage such that all the involved quantities make sense, ν denotes the outward-pointing normal vector to $\partial\Omega$, and $B_{\partial\Omega}$ is either a purely geometric quantity such as the mean curvature or the solution of a PDE on $\partial\Omega$ or on Ω.

We are then interested in the existence of solutions for the optimization problem

$$\boxed{\inf_{\Omega \text{ admissible}} J(\Omega)}.$$

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This kind of problem is very generic. What matters here is that the standard techniques, exposed and developed, for example, in [7,10], do not apply to $d-1$ objects and it is necessary to adopt a particular approach. The first question to ask is the choice of the set $\mathcal{O}_{\mathrm{ad}}$ of all admissible domains. Since the shape functionals we consider involve geometric quantities of the type "outward normal vector to the boundary" or "mean curvature", it is necessary that the manipulated surfaces are not too irregular. For this reason, we choose to impose a constraint that guarantees a uniform regularity, say $\mathcal{C}^{1,1}$, of the manipulated sets. This uniform regularity constraint is imposed by using the notion of "reach". Thus, the set $\mathcal{O}_{\mathrm{ad}}$ represents the set of surfaces having a reach uniformly bounded by below. The precise definition of this notion will be given in Section 1.3.

This kind of problem has been the subject of recent studies and results [4,5,9], which have provided positive answers to the existence issues. In their approach, the authors used an efficient, but nevertheless laborious, approach based on the parametrization of the manipulated surfaces, seen as regular manifolds, using local charts.

The objective of this paper is to promote a different approach, based on the extension of the functions defined on the manipulated surfaces to volume neighborhoods, the introduction of an extruded surface, and the rewriting of the surface integrals as volume integrals using ad hoc variable changes. This is a methodological paper, in which a proof method is presented that may work in many cases. The results contained in the article illustrate this point. We discuss possible generalizations of these results in the concluding section.

This method allows us to gain conciseness and provides much shorter and direct existence proofs than in the above references. The method also allows us to extend the field of investigation to new families of problems involving the solution of a PDE defined on a hypersurface. Nevertheless, some arguments used by the authors of [4,5,9] cannot be shortened by using our approach. We have therefore chosen to expound our method in a short article, in which we detail all the parts of the proof that can be condensed and we make the necessary reminders concerning the results that cannot be condensed.

The article is organized as follows: we introduce the definition of the reach of a surface as well as the class of admissible sets we will deal with in Section 1.3. The main results of this article, regarding several existence results for shape optimization problems involving surfaces, are provided in Section 1.4. The whole of Section 2 is devoted to the proofs of the main results. In these proofs, we detail the arguments based on our approach and which lead to simplified proofs of the results in [4,5,9]. In order to illustrate the potential of our approach, we also provide an existence result involving a general functional depending on the solution of a PDE on the sought manifold.

1.2. Notations

Let us recall some classical notations used throughout this paper.

- For the sake of notational simplicity, we will sometimes use the notation Γ (resp. Γ_n) to denote the hypersurfaces $\partial\Omega$ (resp. $\partial\Omega_n$).

- The Euclidean inner product (resp. norm) will be denoted $\langle\cdot,\cdot\rangle$ (resp. $\|\cdot\|$ or sometimes $|\cdot|$, when no confusion with other notations is possible).
- Given two positive integers $k \leqslant d$ and $\Omega \subset \mathbb{R}^d$, $\mathcal{H}^k(\Omega)$ denotes the k-dimensional Hausdorff measure of Ω.
- Given $\Omega \subset \mathbb{R}^d$, the distance (resp. signed distance) to Ω is defined for all $x \in \mathbb{R}^d$ by
$$d_\Omega(x) = \inf_{y\in\Omega} \|x-y\| \qquad (\text{resp. } b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^d\setminus\Omega}(x)).$$
- Given $\Omega \subset \mathbb{R}^d$ and $h > 0$, the tubular neighborhood $U_h(\Omega)$ is defined as
$$U_h(\Omega) = \{x \in \mathbb{R}^d \mid d_\Omega(x) \leqslant h\}.$$
- Given $\Omega \subset \mathbb{R}^d$, the reach of Ω is defined as
$$\operatorname{Reach}(\Omega) = \sup\{h > 0 \mid d_\Omega \text{ is differentiable in } U_h(\Omega) \setminus \Omega\}.$$
Recall that if $\partial\Omega$ is a non-empty compact $\mathcal{C}^{1,1}$-hypersurface of $\mathbb{R}^d$, then there exists $h > 0$ such that Ω satisfies a two-sided uniform ball condition, namely
$$\begin{aligned} \forall x \in \partial\Omega,\ \exists d_x \in \mathbb{R}^d \mid \|d_x\|_{\mathbb{R}^d} = 1,\ B_h(x - hd_x) \subset \Omega \text{ and}& \\ B_h(x + hd_x) \subset \mathbb{R}^d \setminus \Omega,& \end{aligned} \tag{$\mathcal{B}_h$}$$
where $B_h(x)$ stands for the open ball of radius h centered in x. See Figure 1. Furthermore, assuming $\mathcal{H}^d(\partial\Omega) = 0$, we have the simpler characterization
$$\operatorname{Reach}(\partial\Omega) = \sup\{h \mid \Omega \text{ satisfies } (\mathcal{B}_h)\}.$$
Conversely, if $\partial\Omega$ is non-empty and satisfies condition ($\mathcal{B}_h$), then its reach is larger than h and the Lebesgue measure of $\partial\Omega$ in $\mathbb{R}^d$ is equal to 0. Furthermore, $\partial\Omega$ is a $\mathcal{C}^{1,1}$ hypersurface of $\mathbb{R}^d$. We refer, for instance, to [4, Theorems 2.6 and 2.7].
- For a given oriented $\mathcal{C}^{1,1}$ hypersurface $\partial\Omega$, we denote by $\nabla_{\partial\Omega}$ or ∇_Γ the tangential gradient and by ∇ the full gradient in $\mathbb{R}^d$. When needed, each gradient will be assimilated to a line vector in $\mathbb{R}^d$.
- $\overline{\mathbb{N}}$ denotes $\mathbb{N} \cup \{+\infty\}$.
- $\mathcal{S}^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$.
- $M_d(\mathbb{R})$ denotes the linear space of $d \times d$ matrices with real entries, endowed with the Euclidean operator norm $\|\cdot\|$. Id denotes the identity matrix in $\mathbb{R}^d$.
- For a given $\mathcal{C}^{1,1}$ hypersurface $\partial\Omega$, we denote by $H_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ its mean curvature. We refer to Appendix A for proper definitions.
- Two oriented $\mathcal{C}^{1,1}$ hypersurfaces $\partial\Omega_1$ and $\partial\Omega_2$ are said to be isotopic if there exists a continuous function $H : \partial\Omega_1 \times [0,1] \to \mathbb{R}^d$ such that
  - $H(x,0) = x$ for all $x \in \partial\Omega_1$,
  - $H(\cdot,t)$ is a diffeomorphism onto its image for all $t \in [0,1]$,
  - $H(\cdot,1)$ is a diffeomorphism from $\partial\Omega_1$ to $\partial\Omega_2$.

Figure 1. In black, we have two hypersurfaces $\partial\Omega$. In red, we show the skeleton of $\partial\Omega$, which is the set where $d_{\partial\Omega}$ is not differentiable. Since the reach of $\partial\Omega$ is the distance to such a set, it provides bounds on the curvature radii (cf. Remark A.2), as illustrated by the left figure. Additionally, 'bottleneck folding' is not allowed, as illustrated by the right figure. Note that in these diagrams, it does not matter whether Ω is above or below in the left figure, or right or left in the right one.

1.3. Preliminaries on sets of uniformly positive reach

Given $r_0 > 0$ and a non-empty compact set $D \subset \mathbb{R}^d$, let us introduce the set $\mathcal{O}_{r_0}$ of admissible shapes whose reach is bounded by r_0, namely,

$$\mathcal{O}_{r_0} = \big\{\Omega \subset D \mid \Omega \text{ is open, Reach}(\partial\Omega) \geqslant r_0,\ \Omega \neq \emptyset, \text{ and } \mathcal{H}^d(\partial\Omega) = 0\big\}.$$

The elements of $\mathcal{O}_{r_0}$ are known to satisfy the following properties:

Lemma 1.1. *Let $\Omega \in \mathcal{O}_{r_0}$. Then,*

(1) *$\partial\Omega$ is a $\mathcal{C}^{1,1}$ $(d-1)$-submanifold. Conversely,*

$$\mathcal{O}_{r_0} = \big\{\Omega \subset D \mid \Omega \textit{ is open}, \text{Reach}(\partial\Omega) \geqslant r_0, \textit{ and } \partial\Omega \textit{ is a } (d-1)\textit{-submanifold}\big\}.$$

(2) *For $x \in \partial\Omega$, $\nabla b_\Omega(x)$ is the unit outward normal vector.*

(3) *For $h < r_0$, the vector field ∇b_Ω is $\frac{2}{r_0-h}$-Lipschitz continuous on the tubular neighborhood $U_h(\partial\Omega)$.*

(4) *The restriction of ∇b_Ω to $\partial\Omega$ is $\frac{1}{r_0}$-Lipschitz continuous.*

(5) *There exists a constant C depending only on d, r_0, and D such that $\mathcal{H}^{d-1}(\partial\Omega) \leqslant C$.*

Points 1 and 2 are proved in [7, Chapter 7, Theorem 8.2]. Points 3 and 4 are proved in [4, Theorems 2.7 and 2.8]. The proof of Point 5 is given in Section 2.1.1.

We will endow the set $\mathcal{O}_{r_0}$ with a "sequential" topology, by introducing a notion of convergence in this set.

Definition 1.2 (R-convergence in $\mathcal{O}_{r_0}$). Given $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{r_0}^{\mathbb{N}}$, we say that $(\Omega_n)_{n\in\mathbb{N}}$ R-converges to $\Omega_\infty \in \mathcal{O}_{r_0}$ and we write $\Omega_n \xrightarrow{R} \Omega_\infty$ if

$$b_{\Omega_n} \to b_{\Omega_\infty} \quad \begin{cases} \text{in } \mathcal{C}(\overline{D}), \\ \text{in } \mathcal{C}^{1,\alpha}(U_r(\partial\Omega_\infty)),\ \forall r < r_0,\ \forall\alpha \in [0,1), \\ \text{weakly-}* \text{ in } W^{2,\infty}(U_r(\partial\Omega_\infty)),\ \forall r < r_0. \end{cases}$$

The next result justifies the interest of the class $\mathcal{O}_{r_0}$ endowed with the R-convergence for existence issues.

Proposition 1.3. *$\mathcal{O}_{r_0}$ is sequentially compact for the R-convergence.*

The proof of this proposition can be found in Appendix B. Let us end this section by providing several additional properties of the R-convergence.

Lemma 1.4. *If $\Omega_n \xrightarrow{R} \Omega_\infty$, then*

(1) *$\mathcal{H}^{d-1}(\partial\Omega_n)$ converges toward $\mathcal{H}^{d-1}(\partial\Omega_\infty)$ as $n \to +\infty$.*
(2) *$\mathcal{H}^{d}(\Omega_n)$ converges toward $\mathcal{H}^{d}(\Omega_\infty)$ as $n \to +\infty$.*
(3) *If all the $\partial\Omega_n$ belong to the same isotopic class, then $\partial\Omega_\infty$ also belongs to such a class.*

The proof of this lemma is given in Section 2.2.

Remark 1.5. According to Lemma 1.4, we obtain, for example, that for a given $\Omega_0 \in \mathcal{O}_{r_0}$ and $a \leqslant b$,

$$\left\{\Omega \in \mathcal{O}_{r_0} \mid a \leqslant \mathcal{H}^{d-1}(\partial\Omega) \leqslant b,\ \partial\Omega \text{ is isotopic to } \partial\Omega_0\right\}$$

is a sequentially compact set.

1.4. Main results

Let us introduce the general shape functional

$$F_1(\Omega) = \int_{\partial\Omega} j_1(x, \nu(x), H_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

where j_1 is continuous from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ to $\mathbb{R}$ and convex with respect to its last variable. We recall that ν and $H_{\partial\Omega}$ denote respectively the outward pointing normal vector and the mean curvature.

By Proposition 1.3, the set $\mathcal{O}_{r_0}$ is sequentially compact for the R-convergence. Therefore, in order to infer the existence of an optimal surface minimizing F_1 over $\mathcal{O}_{r_0}$ it is enough to prove the lower semicontinuity of functional F_1 (under suitable assumptions on the function j_1). This is the main purpose of the following result:

Theorem 1.6 ([4, Theorem 1.3]). *Let us assume that j_1 is continuous with respect to all variables and convex with respect to its last one. Then, F_1 is a lower semicontinuous shape functional for the R-convergence, that is, for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{r_0}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has*

$$\liminf_{n\to+\infty} F_1(\Omega_n) \geqslant F_1(\Omega_\infty).$$

As a consequence, the shape optimization problem

$$\inf_{\Omega\in\mathcal{O}_{r_0}} F_1(\Omega)$$

has a solution.

It is notable that, by applying Theorem 1.6 to both j_1 and $-j_1$, we get the following corollary:

Corollary 1.7. *If j_1 is continuous and linear in the last variable, then F_1 is a continuous shape functional for the R-convergence.*

Remark 1.8. In the case where $d = 3$, it is proved in [4, Theorem 1.3] that Theorem 1.6 holds if we replace the mean curvature by the Gaussian one in the definition of F_2. We do not provide a proof here, since most of the difficulties are related to the convergence of a product of weak-$*$-converging sequences and our approach does not change the proof in a significant way.

Let us now consider two classes of shape optimization problems involving either an elliptic PDE inside Ω or an elliptic PDE on the $\mathcal{C}^{1,1}$ hypersurface $\partial\Omega$.

Problems involving an elliptic PDE on a $\mathcal{C}^{1,1}$-hypersurface of $\mathbb{R}^d$. Given $f \in \mathcal{C}^0(D)$, we consider the problem of minimizing a shape functional depending on the solution $v_{\partial\Omega}$ of the equation

$$\Delta_\Gamma v_{\partial\Omega}(x) = f(x) \quad \text{in } \partial\Omega, \tag{1}$$

where $\Delta_{\partial\Omega}$ denotes the positive Laplace–Beltrami operator on $\partial\Omega$. Since we are not considering $\mathcal{C}^\infty$ manifolds but rather $\mathcal{C}^{1,1}$ ones, we need to explain how the PDE must be understood. We use here an energy formulation defining, for a closed and non-empty hypersurface $\partial\Omega$, the functional

$$\mathcal{E}_{\partial\Omega} : H^1_*(\partial\Omega) \ni u \mapsto \frac{1}{2}\int_{\partial\Omega} |\nabla_\Gamma u(x)|^2 d\mu_{\partial\Omega} + \int_{\partial\Omega} f(x)u(x)d\mu_{\partial\Omega},$$

where $H^1_*(\partial\Omega)$ denotes the Sobolev space of functions in $H^1(\partial\Omega)$ with zero mean on $\partial\Omega$. We hence define $v_{\partial\Omega}$ as the unique solution of the minimization problem

$$\min_{u\in H^1_*(\partial\Omega)} \mathcal{E}_{\partial\Omega}(u). \tag{2}$$

Lemma 1.9. *Let $\Omega \in \mathcal{O}_{r_0}$. Problem (2) has a unique solution $v_{\partial\Omega}$. Furthermore, if $\partial\Omega$ is $\mathcal{C}^2$ and if $f \in \mathcal{C}^0(D)$, then $v_{\partial\Omega}$ satisfies (1) almost everywhere in $\partial\Omega$.*

The proof of this result is postponed to Appendix C.

Let us introduce the shape functional

$$F_2(\Omega) = \int_{\partial\Omega} j_2(x, \nu(x), v_{\partial\Omega}(x), \nabla_\Gamma v_{\partial\Omega}(x))\, d\mu_{\partial\Omega}(x),$$

where $j_2 : \mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be continuous.

Theorem 1.10. *The shape functional F_2 is lower-semicontinuous for the R-convergence, that is, for every sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{r_0}^{\mathbb{N}}$ that R-converges toward Ω_∞, one has*

$$\liminf_{n\to+\infty} F_2(\Omega_n) \geqslant F_2(\Omega_\infty).$$

As a consequence, the shape optimization problem

$$\inf_{\Omega\in\mathcal{O}_{r_0}} F_2(\Omega)$$

has a solution.

Problems involving an elliptic PDE in a domain of $\mathbb{R}^d$. Finally, let us investigate the case of a shape criterion involving the solution of a PDE on a domain of $\mathbb{R}^d$. We consider hereafter a Poisson equation with non-homogeneous boundary condition, but we claim that all conclusions can be easily extended to a larger class of elliptic PDEs.

Let $h \in L^2(D)$, $g \in H^2(D)$, and define u_Ω as the solution of

$$\begin{cases} \Delta u_\Omega = h & \text{in } \Omega, \\ u_\Omega = g & \text{in } \partial\Omega. \end{cases} \tag{3}$$

Let us introduce the shape functional F_3 given by

$$F_3(\Omega) = \int_{\partial\Omega} j_3(x, \nu(x), u_\Omega(x), \nabla u_\Omega(x))\, d\mu_{\partial\Omega}(x),$$

where $j_3 : \mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous.

Theorem 1.11 ([5, Theorem 2.1]). *The shape functional F_3 is lower-semicontinuous for the R-convergence.*

It is notable that by adapting the proof of Theorem 1.10, it is possible to obtain a much shorter proof of this theorem. In order not to make this article unnecessarily heavy, we only give the main steps of the proof in Section 2.5. This example is mentioned both for the sake of completeness, in order to review the existing literature, and also to underline the potential of the approach introduced here, which allows us to find more direct proofs of all the known results and to extend them.

In addition, it is interesting to notice that our approach allows us to deal with problems involving PDEs both using weak formulations as in (3) and also those whose solutions are obtained using a minimization principle, as is the case in (1). The approach thus seems robust and we believe that it can be easily adapted to general families of problems (for example, to a general non-degenerate elliptic PDE).

2. Proofs

2.1. The extruded surface approach

One of the key ideas to prove sequential continuity of functionals involving an integral on the boundary is to approximate such an integral by an integral on a small tubular neighborhood (as done, e.g., in [6]).

Let us first illustrate the method by proving Point 5 of Lemma 1.1.

2.1.1. Proof of 1.1, Point 5. For $0 < h < r_0$, consider

$$T : (-h, h) \times \partial\Omega \to U_h(\partial\Omega),$$
$$(t, x) \mapsto x + t\nabla b_\Omega(x).$$

Since T is Lipschitz continuous, it is differentiable at almost every (t_0, x_0), with

$$d_{(t_0,x_0)}T(s, y) = y + s\nabla b_\Omega(x_0) + t_0 d_{x_0}\nabla b_\Omega(y), \quad \forall (s, y) \in \mathbb{R} \times T_{x_0}\partial\Omega. \tag{4}$$

Remark 2.1. Note that as $\nabla b_\Omega(x_0)$ is a normal unit vector to $\partial\Omega$ at x_0, we can identify the tangent hyperplane $T_{x_0}\partial\Omega$ with $\mathbb{R}^{d-1}$ endowed with a Euclidean structure inherited from that of $\mathbb{R}^d$. We will use this identification several times in this paper.

As a result, we can identify $\mathbb{R} \times T_{x_0}\partial\Omega \ni (s, y) \mapsto y + s\nabla b_\Omega(x_0)$ with an orthogonal matrix. Moreover, up to the choice of a different orientation on $T_{x_0}\partial\Omega$, such a matrix belongs to the special orthogonal group $\mathrm{SO}(n)$. We use the same coordinate representation to identify $\mathbb{R} \times T_{x_0}\partial\Omega \ni (s, y) \mapsto d_{x_0}\nabla b_\Omega(y)$ with a $n \times n$ matrix. By uniform continuity of the determinant around $\mathrm{SO}(d)$, there exists $C_0 > 0$ such that, for every $M \in \mathrm{SO}(d)$ and every $l \in M_d(\mathbb{R})$ such that $\|l\| \leqslant C_0$,

$$\frac{1}{2} \leq \det(M + l) \leq \frac{3}{2}.$$

As ∇b_Ω is $\frac{2}{r_0}$-Lipschitz continuous on $\partial\Omega$, we have that for almost every $x_0 \in \partial\Omega$ and every $t_0 \in \mathbb{R}$, $\|t_0 d_{x_0}\nabla b_\Omega\| \leqslant \frac{2|t_0|}{r_0}$.

Let us fix $h < \min(r_0, r_0C_0/2)$ (independent of Ω), so that $\|t_0 d_{x_0}\nabla b_\Omega\| \leqslant C_0$ for almost every $x_0 \in \partial\Omega$ and every $t_0 \in (-h, h)$. By the change of variable formula, we then have

$$\mathcal{H}^{d-1}(\partial\Omega) = \int_{\partial\Omega} d\mu_{\partial\Omega} = \frac{1}{2h}\int_{U_h(\partial\Omega)} \det(d_{T^{-1}(y)}T)dy \leqslant \frac{3}{4h}\mathcal{H}^d(U_h(D)),$$

whence the conclusion.

2.1.2. Extruded surface and R-convergence. Let us now illustrate the power of this approach in the case of an R-converging sequence. Let $\Omega_n \xrightarrow{R} \Omega_\infty$. From now on, we use the notation $\Gamma_n := \partial\Omega_n$ for the hypersurfaces.

For $h < r_0$ and $n \in \overline{\mathbb{N}}$, let us define a parametrization of a neighborhood of Γ_n by

$$\begin{aligned} T_n : (-h,h) \times \Gamma_n &\to U_h(\Gamma_n), \\ (t,x) &\mapsto x + t\nabla b_{\Omega_n}(x). \end{aligned}$$

Lemma 2.2. *For every $\varepsilon > 0$, there exists $h > 0$ such that for all $n \in \overline{\mathbb{N}}$,*

$$1 - \varepsilon \leqslant \det(d_{(t_0,x_0)} T_n) \leqslant 1 + \varepsilon \quad \textit{for a.e. } (t_0, x_0) \in (-h,h) \times \Gamma_n.$$

Proof. We follow the same argument as in Section 2.1.1. More precisely, for a given $\varepsilon > 0$, there exists $C_0 > 0$ such that for every $M \in \mathrm{SO}(d)$ and every $l \in M_d(\mathbb{R})$ such that $\|l\| \leqslant C_0$,

$$1 - \varepsilon \leqslant \det(M + l) \leqslant 1 + \varepsilon.$$

Let us fix $h < \min(r_0, r_0 C_0/2)$ (independent of n). As ∇b_{Ω_n} is $\frac{2}{r_0}$-Lipschitz continuous on Γ_n, we get $\|t_0 d_{x_0} \nabla b_{\Omega_n}\| \leqslant C_0$ for almost every $x_0 \in \Gamma$ and every $t_0 \in (-h,h)$. Whence, using equation (4) with the previous estimate, we conclude the proof. ∎

Remark 2.3. In what follows, we will use the Bachmann–Landau notation $\mathrm{o}_{h\to 0}(1)$ for a function converging to 0 in L^∞ as h goes to 0 and for a given n, large enough. For example, Lemma 2.2 implies that

$$\det(dT_n) = 1 + \mathrm{o}_{h\to 0}(1) \quad \text{on } (-h,h) \times \Gamma_n,$$

which means $\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \exists h > 0, \forall n \in \overline{\mathbb{N}}, n \geqslant N_0$ implies

$$|\det(d_{(t_0,x_0)} T_n) - 1| \leqslant \varepsilon \quad \text{for a.e. } (t,x) \in (-h,h) \times \Gamma_n.$$

Let us now introduce the orthogonal projection p_n onto Γ_n, defined on $U_h(\Gamma_n)$ for every $h \in (0, r_0)$.

Lemma 2.4. *The following properties hold:*

1. *p_n coincides with the second component of $T_n^{-1} : U_h(\Gamma_n) \to (-h,h) \times \Gamma_n$.*
2. *For all $x \in U_h(\Gamma_n)$, $p_n(x) = x - b_{\Omega_n}(x)\nabla b_{\Omega_n}(x)$.*
3. *p_n converges toward p_∞ in $L^\infty(U_h(\Gamma_\infty))$.*

Proof. Properties (1) and (2) are obviously equivalent and are proved in [7, Chapter 7, Theorem 7.2]. Property (3) follows from the $\mathcal{C}^1$ convergence of b_{Ω_n} toward b_{Ω_∞}. ∎

We can now state the key equality to relate surface and volume integrals. Apply Lemma 2.2 with $\varepsilon \in (0,1)$ to select $h > 0$ such that $T_n : (-h,h) \times \Gamma_n \to U_h(\Gamma_n)$ is invertible for every $n \in \overline{\mathbb{N}}$[1].

[1] It is actually well known that the domain of invertibility of T_n contains $U_{r_0}(\Gamma_n)$.

Lemma 2.5. *For all $n \in \overline{\mathbb{N}}$, $f \in L^1(\Gamma_n)$, and $t \in (0,h)$, we have*

$$\int_{\Gamma_n} f(x)\, d\mu_{\Gamma_n}(x) = \frac{1}{2t}\int_{U_t(\Gamma_n)} f \circ p_n(y)\ \det(d_{T_n^{-1}(y)}T_n)\, dy.$$

Proof. Using the change of variable formula (also known as the area formula for Lipschitz continuous functions), one gets

$$\int_{-t}^{\top}\int_{\Gamma_n} f(x)\, d\mu_{\Gamma_n}(x)dt = \int_{U_t(\Gamma_n)} f \circ p_n(y) \det(d_{T_n^{-1}(y)}T_n)\, dy. \qquad \blacksquare$$

From now on, we will omit the $T_n^{-1}(y)$ inside the determinant to improve the readability.

Lemma 2.6. *For every $h \leqslant r_0/2$ and $0 < t < h$, there exists N_0 such that*

$$\forall n \geqslant N_0, \quad U_{h-t}(\Gamma_\infty) \subset U_h(\Gamma_n) \subset U_{h+t}(\Gamma_\infty).$$

Proof. By uniform convergence of b_{Ω_n} toward b_{Ω_∞}, we have that for n large enough

$$b_{\Omega_\infty}^{-1}((t-h, h-t)) \subset b_{\Omega_n}^{-1}((-h,h)) \subset b_{\Omega_\infty}^{-1}((-h-t, h+t)). \qquad \blacksquare$$

In order to perform changes of variable in surface integrals, it is convenient to use p_n directly as a way to map Γ_∞ onto Γ_n. To this aim, we define

$$\begin{aligned} \tau_n : \Gamma_\infty &\to \Gamma_n, \\ x &\mapsto p_n(x). \end{aligned}$$

Note that for n large enough, Lemma 2.6 ensures that τ_n is well defined. We also introduce $\mathrm{Jac}(\tau_n)$ to denote the Jacobian of τ_n. Then, we have the following lemma:

Lemma 2.7. *For n large enough, $\tau_n : \Gamma_\infty \to \Gamma_n$ is a diffeomorphism. In addition,*

$$\sup_{x\in\Gamma_\infty} |\mathrm{Jac}(\tau_n)(x) - 1| \overset{n\to\infty}{\longrightarrow} 0. \tag{5}$$

Proof. Let $x \in \Gamma_\infty$. We take $v \in T_x\Gamma_\infty$, and identify it with an element of the tangent hyperplane (see Remark 2.1). As v is tangent to Γ_∞ at x, we get

$$\langle v, \nabla b_{\Omega_\infty}(x)\rangle = 0.$$

Using Property 2 of Lemma 2.4, we get

$$d_x p_n(v) = v - \langle \nabla b_{\Omega_n}(x), v\rangle \nabla b_{\Omega_n}(x) - b_{\Omega_n}(x)\nabla^2 b_{\Omega_n}(x)v.$$

Let us now fix $h < \frac{r_0}{3}$. For n large enough, thanks to Lemma 2.6, we have $\Gamma_n \subset U_h(\Gamma_\infty)$. Thus,

$$\|d_x p_n(v) - v\| \leqslant \|\nabla b_{\Omega_n}(x) - \nabla b_{\Omega_\infty}(x)\|\|v\| + \|b_{\Omega_n}\|_{L^\infty(\Gamma_\infty)}\|\nabla^2 b_{\Omega_n}(x)\|\|v\|$$

$$\leqslant \|v\|\big(\|\nabla b_{\Omega_n} - \nabla b_{\Omega_\infty}\|_{L^\infty(U_{\frac{r_0}{3}}(\Gamma_\infty))} + \|b_{\Omega_n}\|_{L^\infty(\Gamma_\infty)}\|\nabla^2 b_{\Omega_n}(x)\|_{L^\infty(U_{\frac{r_0}{3}}(\Gamma_\infty))}\big).$$

We recall that both $\|\nabla b_{\Omega_n} - \nabla b_{\Omega_\infty}\|_{L^\infty(U_{\frac{r_0}{3}}(\Gamma_\infty))}$ and $\|b_{\Omega_n}\|_{L^\infty(\Gamma_\infty)}$ converge toward zero. In addition, the quantity $\|\nabla^2 b_{\Omega_n}(x)\|_{L^\infty(U_{\frac{r_0}{3}}(\Gamma_\infty))}$ is uniformly bounded. As a consequence,

$$\sup_{x\in\Gamma_\infty} \sup_{\substack{v\in T_x\Gamma_n \\ \|v\|=1}} \|d_x p_n(v) - v\| \overset{n\to\infty}{\longrightarrow} 0.$$

Using a similar argument to the one used in Lemma 2.2, we take the determinant and obtain (5).

As a result, we know that τ_n is a local diffeomorphism. It remains to prove that τ_n is injective. To this aim, we suppose that n is large enough to ensure that

$$\|\nabla b_{\Omega_n} - \nabla b_{\Omega_\infty}\|_{L^\infty(U_{\frac{r_0}{3}}(\Gamma_\infty))} < \frac{1}{2}.$$

Let $x, y \in \Gamma_\infty$ such that $p_n(x) = p_n(y)$. If $x \neq y$, this implies that there exists $t \in (-\frac{2r_0}{3}, \frac{2r_0}{3}) \setminus \{0\}$ such that

$$x = y + t\nabla b_{\Omega_n}(p_n(y)) = y + t\nabla b_{\Omega_n}(y).$$

As $\Omega_\infty \in \mathcal{O}_{r_0}$, it satisfies the r_0 uniform ball property (see $(\mathcal{B}_h)$). Thus, one has

$$B_{r_0}(y + r_0 \operatorname{sign} t \nabla b_{\Omega_\infty}(y)) \cap \Gamma_\infty = \emptyset.$$

But, we have

$$|x - y - r_0 \operatorname{sign} t\nabla b_{\Omega_\infty}(y)| = |t\nabla b_{\Omega_n}(y) - r_0 \operatorname{sign} t \nabla b_{\Omega_\infty}(y)| \leqslant \frac{t}{2} + |t - r_0 \operatorname{sign} t| < r_0.$$

This is a contradiction, and hence τ_n is injective, which implies that it is a diffeomorphism from Γ_∞ to Γ_n. ∎

2.2. Proof of Lemma 1.4

Suppose that $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{r_0}^{\mathbb{N}}$ R-converges toward $\Omega_\infty \in \mathcal{O}_{r_0}$.

2.2.1. Proof of Point 1. For $h < r_0$, using Lemma 2.5, we have

$$\mathcal{H}^{d-1}(\Gamma_n) = \int_{\Gamma_n} d\mu_{\Gamma_n}(x) = \frac{1}{2h}\int_{U_h(\Gamma_n)} \det(dT_n)\,dy.$$

Moreover, by Lemma 2.6,

$$\mathcal{H}^{d-1}(\Gamma_n) = \frac{1}{2h}\int_{U_{h-t}(\Gamma_\infty)} \det(dT_n)\,dy + \frac{1}{2h}\int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} \det(dT_n)\,dy$$

for $t \in (0, h)$ and n large enough. Let us compare the first term on the right-hand side with

$$\mathcal{H}^{d-1}(\Gamma_\infty) = \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} \det(dT_\infty)\, dy.$$

Using Lemma 2.2, $\det(dT_\infty) = \det(dT_n) + o_{h\to 0}(1)$ on $(-h, h) \times \Gamma_\infty$. In addition, $\frac{1}{2h} - \frac{1}{2(h-t)} = O(\frac{t}{h})$. Hence,

$$\frac{1}{2h} \int_{U_{h-t}(\Gamma_\infty)} \det(dT_n)\, dy = \mathcal{H}^{d-1}(\Gamma_\infty) + o_{h\to 0}(1) + O\Big(\frac{t}{h}\Big).$$

On the other hand, using again the relation $\det(dT_\infty) = \det(dT_n) + o_{h\to 0}(1)$,

$$\begin{aligned}
\frac{1}{2h} \int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} \det(dT_n)\, dy &\leqslant \frac{1}{2h} \int_{U_{h+t}(\Gamma_\infty)\setminus U_{h-t}(\Gamma_\infty)} \det(dT_n)\, dy \\
&= \frac{1}{2h}\Big(\int_{U_{h+t}(\Gamma_\infty)} \det(dT_n)\, dy - \int_{U_{h-t}(\Gamma_\infty)} \det(dT_n)\, dy\Big) \\
&= \frac{1}{2h}(2(h+t)\mathcal{H}^{d-1}(\Gamma_\infty) - 2(h-t)\mathcal{H}^{d-1}(\Gamma_\infty) + o_{h\to 0}(h)) \\
&= \frac{2t}{h}\mathcal{H}^{d-1}(\Gamma_\infty) + o_{h\to 0}(1) = O\Big(\frac{t}{h}\Big) + o_{h\to 0}(1).
\end{aligned}$$

By taking h arbitrarily small while $t = h^2$, we prove that $\mathcal{H}^{d-1}(\Gamma_n) \to \mathcal{H}^{d-1}(\Gamma_\infty)$.

2.2.2. Proof of Point 2. Using the uniform convergence of b_{Ω_n} to b_{Ω_∞}, we deduce that for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$b_{\Omega_\infty}^{-1}((-\infty, -\varepsilon]) \subset b_{\Omega_n}^{-1}((-\infty, 0)) \subset b_{\Omega_\infty}^{-1}((-\infty, \varepsilon)), \quad \forall n \geqslant N_0.$$

Hence, we get

$$\mathcal{H}^d(b_{\Omega_\infty} \leqslant -\varepsilon) \leqslant \mathcal{H}^d(\Omega_n) \leqslant \mathcal{H}^d(b_{\Omega_\infty} < \varepsilon).$$

By inner regularity of $\mathcal{H}^d$, $\mathcal{H}^d(b_{\Omega_\infty} \leqslant -\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} \mathcal{H}^d(b_{\Omega_\infty} < 0) = \mathcal{H}^d(\Omega_\infty)$. Similarly, by outer regularity $\mathcal{H}^d(b_{\Omega_\infty} < \varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} \mathcal{H}^d(b_{\Omega_\infty} \leq 0) = \mathcal{H}^d(\Omega_\infty)$, where we used that Ω_∞ belongs to $\mathcal{O}_{r_0}$.

2.2.3. Proof of Point 3. We want to prove that Γ_n is isotopic to Γ_∞ for n large enough. We consider

$$\begin{aligned}
\varphi_n(t, x) : [0, 1] \times \Gamma_\infty &\to \mathbb{R}^3, \\
(t, x) &\mapsto x + t(p_n(x) - x).
\end{aligned}$$

According to Lemma 2.7, $\varphi_n(1, \cdot) = \tau_n$ is a diffeomorphism from Γ_∞ onto Γ_n. In addition, following the proof of Lemma 2.7, we easily get that for $t \in (0, 1)$, $\varphi_n(t, \cdot)$ is a diffeomorphism onto its image.

2.3. Proof of Theorem 1.6

Suppose that $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{O}_{r_0}^{\mathbb{N}}$ R-converges toward $\Omega_\infty \in \mathcal{O}_{r_0}$. Let $0 < t < h$ small enough (to be fixed later) and n be large enough.

We recall that the unit normal vector to Γ_n is given by ∇b_{Ω_n} (see Lemma 2.4). Then, according to Lemma 2.5,

$$
\begin{aligned}
F_1(\Omega_n) &= \int_{\Gamma_n} j_1(x, \nabla b_{\Omega_n}(x), H_{\Gamma_n}(p_n(y))) d\mu_{\Gamma_n}(x) \\
&= \frac{1}{2h}\int_{U_h(\Gamma_n)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \det(d_{T_n^{-1}(y)} T_n)\, dy.
\end{aligned}
$$

Moreover, using Lemma 2.6,

$$
\begin{aligned}
F_1(\Omega_n) = {}& \frac{1}{2h}\int_{U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \det(dT_n)\, dy \\
&+ \frac{1}{2h}\int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \\
&\qquad \det(dT_n)\, dy. \qquad (6)
\end{aligned}
$$

The key idea is to prove that all arguments of j_1 in the first term converge toward their analogues for $n = \infty$ and to ensure that the second term is small for small t.

Let us start with comparing the first term in the right-hand side with $F_1(\Omega_\infty)$. Notice that

$$
\begin{aligned}
&\frac{1}{2h}\int_{U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \det(dT_n)\, dy \\
&\quad = \frac{1}{2(h-t)}\int_{U_{h-t}(\Gamma_\infty)} \frac{2(h-t)}{2h}\frac{\det(dT_n)}{\det(dT_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \\
&\qquad \det(dT_\infty)\, dy.
\end{aligned}
$$

By Lemma 2.2, we have

$$
\left\| \frac{2(h-t)}{2h}\frac{\det(dT_n)}{\det(dT_\infty)} - 1 \right\|_{L^\infty(U_h(\Gamma_\infty))} = o_{h\to 0}(1) + O\Big(\frac{t}{h}\Big).
$$

Let us now investigate the mean curvature term. Note that this term is slightly technical to handle for two reasons:

- the mean curvature H_{Γ_n} is defined as the trace of the shape operator, which is itself defined as the differential of the restriction to the hypersurface of ∇b_{Ω_n} (see Appendix A);
- the Hessian of b_{Ω_n} converges only in a weak sense.

We will use the following lemma, which is obtained thanks to the chain rule:

Lemma 2.8 ([6, Theorem 4.4]). *Let $h < r_0$ and $n \in \bar{\mathbb{N}}$. If $\nabla^2 b_{\Omega_n}(x)$ exists for $x \in U_h(\Gamma_\infty)$, then $\nabla^2 b_{\Omega_n}(p_n(x))$ exists and*

$$\nabla^2 b_{\Omega_n}(p_n(x)) = \nabla^2 b_{\Omega_n}(x)[\mathrm{Id} - b_{\Omega_n}(x)\nabla^2 b_{\Omega_n}(x)]^{-1}.$$

In addition, one has that $\nabla^2 b_{\Omega_n}(\tau_n^{-1}(p_n(x)))$ exists as well.

Notice that the last part of the statement is not explicitly contained in [6] but can be obtained by straightforwardly adapting the proof of its Theorem 4.4.

As $\nabla^2 b_{\Omega_n}$ is uniformly bounded on a neighborhood of Γ_∞ and that $b_{\Omega_n}(x) \leqslant h$ for $x \in U_h(\Gamma_n)$, there exists $C > 0$ such that

$$\operatorname*{ess\,sup}_{x \in U_h(\Gamma_n)} \|[\mathrm{Id} - b_{\Omega_n}(x)\nabla^2 b_{\Omega_n}(x)]^{-1} - \mathrm{Id}\| \leqslant Ch,$$

for small h. As a consequence, using Lemma A.3, one has

$$H_{\Gamma_n}(p_n(x)) = \operatorname{Tr}\nabla^2 b_{\Omega_n}(p_n(x)) = \operatorname{Tr}\nabla^2 b_{\Omega_n}(x) + \mathrm{O}(h).$$

Note also that $H_{\Gamma_n} \leqslant \frac{1}{r_0}$ on Γ_n. We can use the uniform continuity of j_1 on a compact set to ensure that for n large enough and $n = \infty$,

$$\begin{aligned}\frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} & j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y))) \det(dT_\infty)\,dy \\ &= \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), \operatorname{Tr}\nabla^2 b_{\Omega_n}(y)) \det(dT_\infty)\,dy \\ &\qquad + \mathrm{O}(h).\end{aligned} \tag{7}$$

The next step is to pass to the limit within the integral. Note that, by definition of R-convergence,

$$\begin{cases} p_n \overset{n\to\infty}{\longrightarrow} p_\infty & \text{strongly in } L^\infty(U_{\frac{r_0}{2}}(\Gamma_\infty)), \\ \nabla b_{\Omega_n} \circ p_n \overset{n\to\infty}{\longrightarrow} \nabla b_{\Omega_\infty} \circ p_\infty & \text{strongly in } L^\infty(U_{\frac{r_0}{2}}(\Gamma_\infty)), \\ \operatorname{Tr}\nabla^2 b_{\Omega_n} \overset{n\to\infty}{\longrightarrow} \operatorname{Tr}\nabla^2 b_{\Omega_\infty} & \text{weak-}* \text{ in } L^\infty(U_{\frac{r_0}{2}}(\Gamma_\infty)). \end{cases}$$

Thus, as j_1 is continuous from $\mathbb{R}^d \times \mathcal{S}^{d-1} \times \mathbb{R}$ to $\mathbb{R}$ and convex with respect to its last variable, it follows, for example, from [1, Theorem 1] that

$$L^\infty(U_{\frac{r_0}{2}}(\Gamma_\infty))^3 \ni (p, n, w) \mapsto \int_{U_{h-t}(\Gamma_\infty)} j_1(p(y), v(y), w(y)) \det(dT_\infty)\,dy$$

is sequentially lower-semicontinuous with respect to the strong convergence in (p, v) and weak-$*$ in w. Thus,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} & j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), \operatorname{Tr}\nabla^2 b_{\Omega_n}(y)) \det(dT_\infty)\,dy \\ &\geqslant \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} j_1(p_\infty(y), \nabla b_{\Omega_\infty}(p_\infty(y)), \operatorname{Tr}\nabla^2 b_{\Omega_\infty}(y)) \det(dT_\infty)\,dy\end{aligned}$$

$$= \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} j_1(p_\infty(y), \nabla b_{\Omega_\infty}(p_\infty(y)), \operatorname{Tr} \nabla^2 b_{\Omega_\infty}(p_\infty(y))) \det(dT_\infty)\, dy + \mathrm{O}(h)$$
$$= F_1(\Omega_\infty) + \mathrm{O}(h). \tag{8}$$

In order to conclude, let us check that the term in line (6) is small. Since j_1 is continuous on a compact set, it admits a minimum $m_0 \in \mathbb{R}$. Let $m_1 = \min(0, m_0) \leqslant 0$. Then,

$$\frac{1}{2h} \int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y)))\ \det(dT_n)\, dy$$
$$\geqslant \frac{1}{2h} \int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} m_1\ \det(dT_n)\, dy$$
$$\geqslant \frac{1}{2h} \int_{U_{h+t}(\Gamma_\infty)\setminus U_{h-t}(\Gamma_\infty)} m_1 \frac{\det(dT_n)}{\det(dT_\infty)} \det(dT_\infty)\, dy.$$

Using

$$\left\| \frac{\det(dT_n)}{\det(dT_\infty)} - 1 \right\|_{L^\infty(U_{2h}(\Gamma_\infty))} = \mathrm{o}_{h\to 0}(1)$$

and

$$\int_{U_{h\pm t}(\Gamma_\infty)} m_1\ \det(dT_\infty)\, dy = 2(h \pm t) m_1 \mathcal{H}^{d-1}(\Gamma_\infty),$$

we get

$$\frac{1}{2h} \int_{U_h(\Gamma_n)\setminus U_{h-t}(\Gamma_\infty)} j_1(p_n(y), \nabla b_{\Omega_n}(p_n(y)), H_{\Gamma_n}(p_n(y)))\ \det(dT_n)\, dy$$
$$\geqslant m_1 \mathcal{H}^{d-1}(\Gamma_\infty)\Big(\mathrm{o}_{h\to 0}(1) + \mathrm{O}\Big(\frac{t}{h}\Big)\Big). \tag{9}$$

Finally, combining equations (7)–(9), we obtain

$$\liminf_{n\to+\infty} F_1(\Omega_n) \geqslant (F_1(\Omega_\infty) + O(h))\Big(1 + \mathrm{O}\Big(\frac{t}{h}\Big)\Big) + m_1 \mathcal{H}^{d-1}(\Gamma_\infty)\Big(\mathrm{o}_{h\to 0}(1) + \mathrm{O}\Big(\frac{t}{h}\Big)\Big).$$

Hence, taking $h \to 0$ while ensuring $t = \mathrm{o}(h)$ gives

$$\liminf_{n\to+\infty} F_1(\Omega_n) \geqslant F_1(\Omega_\infty),$$

and finishes the proof.

2.4. Proof of Theorem 1.10

Let $(\Omega_n)_{n\in\mathbb{N}}$ denote a sequence that R-converges to Ω_∞, and let v_n denote the unique solution v_{Γ_n} to problem (2) for $\Omega = \Omega_n$. The difficult part here is that v_{Γ_n} is not defined on Γ_∞. Our main tool will be τ_n, the restriction to Γ_∞ of the orthogonal projection p_n on Γ_n. Those objects were introduced in Section 2.1.2 and we proved that τ_n is a diffeomorphism between Γ_∞ and Γ_n in Lemma 2.7.

We also have to be careful when we transport the tangential gradient of a function. In order to relate the tangential gradient and the ambient gradient, we establish the following pointwise estimate:

Lemma 2.9. *Let $n \in \mathbb{N}$ and $f_n \in H^1(\Gamma_n)$. Then, $f_n \circ \tau_n \in H^1(\Gamma_\infty)$ and, for almost every $x \in \Gamma_\infty$,*

$$\nabla_{\Gamma_\infty}(f_n \circ \tau_n)(x) = \nabla_{\Gamma_n} f_n(\tau_n(x))(\mathrm{Id} + C_n(x)), \tag{10}$$

where tangential gradients are understood as d-dimensional line vectors and

$$C_n(x) = (\nabla b_{\Omega_n}(x)^\top \nabla b_{\Omega_n}(x) - \mathrm{Id})\nabla b_{\Omega_\infty}(x)^\top \nabla b_{\Omega_\infty}(x) \\ + b_{\Omega_n}(x)\nabla^2 b_{\Omega_n}(x)(\nabla b_{\Omega_\infty}(x)^\top \nabla b_{\Omega_\infty}(x) - \mathrm{Id}).$$

In addition, C_n converges toward zero in the L^∞ norm, that is,

$$\operatorname{ess\,sup}_{x\in\Gamma_\infty} \|C_n(x)\| \overset{n\to\infty}{\longrightarrow} 0. \tag{11}$$

Proof. First notice that

$$\nabla_{\Gamma_\infty}(f_n \circ p_n)(x) = \nabla(f_n \circ p_n \circ p_\infty)(x)$$

for almost every $x \in \partial\Omega_\infty$, since the directional derivative of $f_n \circ p_n \circ p_\infty$ at the point x in the direction $\nabla b_{\Omega_\infty}(x)$ is zero. By Lemma 2.8, $\nabla^2 b_{\Omega_n}(x)$ is well defined for almost every x in Γ_∞. By Lemma 2.4 and the chain rule, we obtain almost everywhere on Γ_∞

$$\begin{aligned} &\nabla(f_n \circ p_n \circ p_\infty)(x) \\ &\quad = ((\nabla f_n) \circ p_n)(\mathrm{Id} - \nabla b_{\Omega_n}^\top \nabla b_{\Omega_n} - b_{\Omega_n}\nabla^2 b_{\Omega_n})(\mathrm{Id} - \nabla b_{\Omega_\infty}^\top \nabla b_{\Omega_\infty} - b_{\Omega_\infty}\nabla^2 b_{\Omega_\infty}) \\ &\quad = ((\nabla_{\Gamma_n} f_n) \circ \tau_n)(\mathrm{Id} - (\mathrm{Id} - \nabla b_{\Omega_n}^\top \nabla b_{\Omega_n})\nabla b_{\Omega_\infty}^\top \nabla b_{\Omega_\infty} \\ &\qquad - b_{\Omega_n}\nabla^2 b_{\Omega_n}(\mathrm{Id} - \nabla b_{\Omega_\infty}^\top \nabla b_{\Omega_\infty})), \end{aligned}$$

where we used that $\nabla f_n = \nabla_{\Gamma_n} f_n$, $p_n = \tau_n$, and $b_{\Omega_\infty} = 0$ on Γ_∞. This shows (10).

Let us now bound the L^∞ norm of C_n. There exists $C > 0$ such that, for every n satisfying $\Gamma_\infty \subset U_{\frac{r_0}{2}}(\Gamma_n)$,

$$\operatorname{ess\,sup}_{x\in\Gamma_\infty} \|\nabla^2 b_{\Omega_n}(x)(\nabla b_{\Omega_\infty}^\top(x)\nabla b_{\Omega_\infty}(x) - \mathrm{Id})\| \leqslant C.$$

In addition, $\|b_{\Omega_n}\|_{L^\infty(\Gamma_\infty)}$ converges to zero. Finally, using the uniform convergence of ∇b_{Ω_n} toward ∇b_{Ω_∞}, we get

$$\nabla b_{\Omega_n}^\top \nabla b_{\Omega_n} \nabla b_{\Omega_\infty}^\top \nabla b_{\Omega_\infty} \xrightarrow[n\to\infty]{L^\infty(\Gamma_\infty)} \nabla b_{\Omega_\infty}^\top (\nabla b_{\Omega_\infty} \nabla b_{\Omega_\infty}^\top)\nabla b_{\Omega_\infty} = \nabla b_{\Omega_\infty}^\top \nabla b_{\Omega_\infty}.$$

This concludes the proof of (11). ∎

From the solution v_n in $H^1_*(\Gamma_n)$, we introduce the function w_n defined on Γ_∞ by

$$w_n = v_n \circ \tau_n - \frac{1}{\mathcal{H}^{d-1}(\Gamma_\infty)} \int_{\Gamma_\infty} v_n \circ \tau_n \, d\mu_{\Gamma_\infty}.$$

Note that, defined as such, w_n belongs to $H^1_*(\Gamma_\infty)$.

Step 1: convergence of $(w_n)_{n\in\mathbb{N}}$. Let us start by considering the sequence of energies $(\mathcal{E}_{\Gamma_n}(v_n))_{n\in\mathbb{N}}$. This sequence is bounded above by 0, since $\mathcal{E}_{\Gamma_n}(v_n) \leq \mathcal{E}_{\Gamma_n}(0) = 0$ for every n. By using the uniform Poincaré inequality stated in Proposition C.2 combined with the Cauchy–Schwarz inequality, we get that $(\int_{\Gamma_n} |v_n|^2 \, d\mu_{\Gamma_n}(x))_{n\in\mathbb{N}}$ is bounded. We now compute

$$
\begin{aligned}
\frac{1}{\mathcal{H}^{d-1}(\Gamma_\infty)} \int_{\Gamma_\infty} v_n \circ \tau_n \, d\mu_{\Gamma_\infty} &= \frac{1}{\mathcal{H}^{d-1}(\Gamma_\infty)} \int_{\Gamma_n} v_n \ \mathrm{Jac}(\tau_n) d\mu_{\Gamma_n} \\
&= \Big(\frac{\sqrt{\mathcal{H}^{d-1}(\Gamma_n)}}{\mathcal{H}^{d-1}(\Gamma_\infty)} \|v_n\|_{L^2(\Gamma_n)} \Big) \, \mathrm{o}_{n\to\infty}(1),
\end{aligned}
$$

where we used Lemma 2.7, the Cauchy–Schwarz inequality, and the fact that v_n has zero average on Γ_n. Hence, we infer that $w_n = v_n \circ \tau_n + \mathrm{o}_{n\to\infty}(1)$. In addition, by performing a change of variable and by using Lemmas 2.7 and 2.9, we get

$$
\begin{aligned}
&\int_{\Gamma_n} \Big(\frac{1}{2} |\nabla_{\Gamma_n} v_n(y)|^2 - f(y) v_n(y) \Big) \, d\mu_{\Gamma_n}(y) \\
&\quad = \int_{\Gamma_\infty} \Big(\frac{1}{2} |\nabla_{\Gamma_n} v_n(\tau_n(y))|^2 - f(\tau_n(y))(v_n \circ \tau_n)(y) \Big) \ \mathrm{Jac}(\tau_n)^{-1} d\mu_{\Gamma_\infty}(y) \\
&\quad = \int_{\Gamma_\infty} \Big(\frac{1}{2} |\nabla_{\Gamma_\infty} w_n(y)|^2 - f(\tau_n(y)) w_n(y) \Big) \, d\mu_{\Gamma_\infty}(y) + \mathrm{o}_{n\to\infty}(1),
\end{aligned}
$$

where we used that $\nabla_{\Gamma_\infty} w_n = \nabla_{\Gamma_\infty}(v_n \circ \tau_n)$, by definition of w_n. By using Proposition C.2 and again the Cauchy–Schwarz inequality, we successively infer that the sequences $(\int_{\Gamma_\infty} |w_n|^2 \, d\mu_{\Gamma_\infty}(x))_{n\in\mathbb{N}}$ and $(\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_n|^2 \, d\mu_{\Gamma_\infty}(x))_{n\in\mathbb{N}}$ are bounded. By using Theorem C.1, the sequence $(w_n)_{n\in\mathbb{N}}$ converges up to a subsequence toward $w_\infty \in H^1_*(\Gamma_\infty)$, weakly in $H^1(\Gamma_\infty)$, and strongly in $L^2(\Gamma_\infty)$. Up to extracting a subsequence, we get

$$
\begin{aligned}
\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_\infty(x)|^2 \, d\mu_{\Gamma_\infty} &\leqslant \liminf_{n\to+\infty} \int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_n(x)|^2 \, d\mu_{\Gamma_\infty}, \\
\lim_{n\to\infty} \int_{\Gamma_\infty} w_n(x) f(\tau_n(x)) \, d\mu_{\Gamma_\infty} &= \int_{\Gamma_\infty} w_\infty(x) f(x) \, d\mu_{\Gamma_\infty}.
\end{aligned}
$$

As a consequence,

$$
\mathcal{E}_{\Gamma_\infty}(w_\infty) \leqslant \liminf_{n\to+\infty} \mathcal{E}_{\Gamma_n}(v_n).
$$

Step 2: Minimality of w_∞. Let $u \in H^1_*(\Gamma_\infty)$ be given and define z_n in $H^1_*(\Gamma_n)$ by

$$
z_n = u \circ \tau_n^{-1} - \frac{1}{\mathcal{H}^{d-1}(\Gamma_n)} \int_{\Gamma_n} u \circ \tau_n^{-1} \, d\mu_{\Gamma_n}.
$$

Let $n \in \mathbb{N}$. By minimality, one has

$$
\mathcal{E}_{\Gamma_n}(v_n) \leqslant \mathcal{E}_{\Gamma_n}(z_n).
$$

By mimicking the arguments and computations of the first step, we easily get that

$$\mathcal{E}_{\Gamma_n}(z_n) = \mathcal{E}_{\Gamma_\infty}(u) + o_{n\to\infty}(1), \tag{12}$$

yielding at the end $\mathcal{E}_{\Gamma_\infty}(w_\infty) \leqslant \mathcal{E}_{\Gamma_\infty}(u)$. We infer that w_∞ is the unique solution to variational problem (2). Since the reasoning above holds for any closure point of $(w_n)_{n\in\mathbb{N}}$, it follows that the whole sequence $(w_n)_{n\in\mathbb{N}}$ converges toward w_∞, weakly in $H^1(\Gamma_\infty)$, and strongly in $L^2(\Gamma_\infty)$. Finally, using $u = w_\infty$ in (12), we obtain that

$$\mathcal{E}_{\Gamma_\infty}(w_\infty) = \liminf_{n\to+\infty} \mathcal{E}_{\Gamma_n}(v_n).$$

In particular, $(\|w_n\|^2_{H^1(\Gamma_\infty)})_{n\in\mathbb{N}}$ converges toward $\|w_\infty\|^2_{H^1(\Gamma_\infty)}$, which implies the strong convergence of w_n in $H^1(\Gamma_\infty)$.

Step 3: Lower-semicontinuity of F_2. Let us use the same notations as those used previously. Using a change of variable, we get

$$\begin{aligned} F_2(\Omega_n) &= \int_{\Gamma_n} j_2(x, \nabla b_{\Omega_n}(x), v_n(x), \nabla_{\Gamma_n} v_n(x))\, d\mu_{\Gamma_n}(x) \\ &= \int_{\Gamma_\infty} j_2(\tau_n(x), \nabla b_{\Omega_n}(\tau_n(x)), v_n(\tau_n(x)), \nabla_{\Gamma_n} v_n \circ \tau_n(x))\ \mathrm{Jac}(\tau_n)^{-1} d\mu_{\Gamma_\infty}(x). \end{aligned}$$

In addition, according to the results above and Lemma 2.4, the following convergences hold:

$$\begin{cases} \mathrm{Jac}(\tau_n)^{-1} \overset{n\to\infty}{\longrightarrow} 1 & \text{strongly in } L^\infty(\Gamma_\infty), \\ \tau_n \overset{n\to\infty}{\longrightarrow} \mathrm{Id}\,|_{\Gamma_\infty} & \text{strongly in } L^\infty(\Gamma_\infty), \\ \nabla b_{\Omega_n} \circ \tau_n \overset{n\to\infty}{\longrightarrow} \nabla b_{\Omega_\infty} & \text{strongly in } L^\infty(\Gamma_\infty), \\ v_n \circ \tau_n \overset{n\to\infty}{\longrightarrow} w_\infty & \text{strongly in } L^2(\Gamma_\infty), \\ \nabla_{\Gamma_n} v_n \circ \tau_n \overset{n\to\infty}{\longrightarrow} \nabla_{\Gamma_\infty} w_\infty & \text{strongly in } L^2(\Gamma_\infty), \end{cases}$$

where w_∞ is the unique solution to variational problem (2).

By applying [1, Theorem 1], one has

$$\liminf_{n\to+\infty} F_2(\Omega_n) \geqslant F_2(\Omega_\infty).$$

This is the desired conclusion.

2.5. Main steps in the proof of Theorem 1.11

First note that $u_\Omega - g$ solves (3) with source term $h - \Delta g$ and Dirichlet boundary condition. As a consequence, we can reduce our study to the case of homogeneous Dirichlet condition (i.e., $u_\Omega = 0$ on Γ).

The method relies on a uniform extension property proved by Chenais in [3] for surfaces satisfying an ε-cone condition, which is weaker than the uniform ball condition.

Lemma 2.10 ([3, Theorem II.1]). *There exists a positive constant C (depending only on r_0 and D) such that for every $\Omega \in \mathcal{O}_{r_0}$, there exists an extension operator $E_\Omega \in \mathcal{L}(H^2(\Omega), H^2(D))$ satisfying*

$$E_\Omega(u)|_\Omega = u, \quad \|E_\Omega\|_{\mathcal{L}(H^2(\Omega),H^2(D))} \leqslant C.$$

We will use this lemma to extend the solution of the PDEs to the whole box D. The next step is to find a uniform H^2 estimate of the solutions. In our case, such an estimate was proved by Dalphin, who extended a result for domains with $\mathcal{C}^2$ boundary obtained by Grisvard in [8].

Lemma 2.11 ([5, Proposition 3.1]). *There exists $C > 0$ (depending only on r_0 and D) such that for every $\Omega \in \mathcal{O}_{r_0}$ and $f \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\|f\|_{H^2(\Omega)} \leqslant C\|\Delta f\|_{L^2(\Omega)}.$$

As a consequence, we have a uniform $H^2(D)$ estimate on the extension of the solution u_Ω, namely,

$$\|E_\Omega(u_\Omega)\|_{H^2(D)} \leqslant C\|h\|_{L^2(D)}, \quad \forall \Omega \in \mathcal{O}_{r_0}. \tag{13}$$

Let us now consider $\Omega_n \xrightarrow{R} \Omega_\infty$. Using (13), we get that $(E_{\Omega_n}(u_{\Omega_n}))_{n\in\mathbb{N}}$ is uniformly bounded in $H^2(D)$. Up to extracting a subsequence, we can assume that

$$E_{\Omega_n}(u_{\Omega_n}) \overset{n\to\infty}{\longrightarrow} u^* \begin{cases} \text{weakly in } H^2(D), \\ \text{strongly in } H^1(D). \end{cases} \tag{14}$$

The next step is to prove that the restriction to Ω_∞ of u^* is u_{Ω_∞}.

To this aim, let us consider an arbitrary compact set K contained in the interior of Ω_∞ and a $\mathcal{C}^\infty$ function φ with compact support included in K. For n large enough, K is contained in the interior of Ω_n (see Lemma 2.6), and, therefore, one has $\varphi \in H_0^1(\Omega_n)$ for such integers n. Using the variational formulation of the PDE given by (3), we get

$$\int_D \langle \nabla E_{\Omega_n}(u_{\Omega_n}), \nabla\varphi\rangle - f\varphi = 0.$$

Using the density of $\mathcal{C}^\infty$ functions with compact support in $H_0^1(\Omega_\infty)$ and passing to the limit yields that $u^*|_{\Omega_\infty} = u_{\Omega_\infty}$.

Remark 2.12. In order to replace Dirichlet boundary conditions by Neumann's ones, one can follow similar steps as those leading to equation (14). Then, by considering the variational formulation with $\varphi \in \mathcal{C}^\infty(D)$ and passing to the limit in

$$\int_{\Gamma_n} g\partial_\nu\varphi \to \int_{\Gamma_\infty} g\partial_\nu\varphi$$

(a consequence of Corollary 1.7 if $g \in \mathcal{C}^0(D)$), one gets that $u^*|_{\Omega_\infty} = u_{\Omega_\infty}$.

The last step is to relate $F_3(\Omega_n)$ and $F_3(\Omega_\infty)$. Since the involved functions belong to Sobolev spaces and since one aims at comparing surface integrals with tubular ones, we need a suitable uniform trace result.

Lemma 2.13. *There exists C such that for every $h < \frac{r_0}{2}$, every $n \in \overline{\mathbb{N}}$, and every $f \in H^1(U_{\frac{r_0}{2}}(\Gamma_n))$,*

$$\|f - \tilde{f} \circ p_n\|_{L^2(U_h(\Gamma_n))} \leqslant Ch\|f\|_{H^1(U_{\frac{r_0}{2}}(\Gamma_n))},$$

where $\tilde{f}$ denotes the trace of f on Γ_n.

Proof. Let f be a smooth function. According to Lemma 2.4, every point $y \in U_h(\Gamma_n)$ can be written in a unique way as $y = x + t\nabla b_{\Omega_n}(x)$ with $x = p_n(y) \in \Gamma_n$ and $t \in (-h, h)$. Moreover, one has

$$|f(x + t\nabla b_{\Omega_n}(x)) - f(x)|^2 \leqslant C^2 \|\partial_{\nabla b_{\Omega_n}(x)} f(x + y\nabla b_{\Omega_n}(x))\|^2_{L^2_y(-\frac{r_0}{2}, \frac{r_0}{2})} |t|,$$

where $\partial_{\nabla b_{\Omega_n}}$ stands for the derivative in the direction $\nabla b_{\Omega_n}(x)$ and C is the norm of the continuous embedding of $H^1([-\frac{r_0}{2}, \frac{r_0}{2}])$ into the space $\mathcal{C}^{\frac{1}{2}}$ of $\frac{1}{2}$-Hölder continuous functions. Hence, using Lemma 2.2, we get

$$\begin{aligned}
\|f - f \circ p_n\|^2_{L^2(U_h(\Gamma_n))} &= \int_{-h}^{h} \int_{\Gamma_n} |f(x + t\nabla b_{\Omega_n}(x)) - f(x)|^2 \det(dT_n)\, dx dt \\
&\leqslant \int_{-h}^{h} \int_{\Gamma_n} C^2 \|\partial_{\nabla b_{\Omega_n}} f(x + y\nabla b_{\Omega_n}(x))\|^2_{L^2_y(-\frac{r_0}{2}, \frac{r_0}{2})} |t| \det(dT_n)\, dx dt \\
&\leqslant C^2 h^2 \int_{\Gamma_n} \|\partial_{\nabla b_{\Omega_n}} f(x + y\nabla b_{\Omega_n}(x))\|^2_{L^2_y(-\frac{r_0}{2}, \frac{r_0}{2})} (1 + \mathrm{o}_{h\to 0}(1))\, dx \\
&\leqslant C^2 h^2 \|f\|^2_{H^1(U_{\frac{r_0}{2}}(\Gamma_n))} (1 + \mathrm{o}_{h\to 0}(1)).
\end{aligned}$$

We conclude the proof thanks to the density of the smooth functions in H^1. ∎

Using that u_{Ω_n} is uniformly bounded in $H^2(D)$, let us apply Lemma 2.13 to u_{Ω_n} and ∇u_{Ω_n}. We obtain

$$\|u_{\Omega_n} - u_{\Omega_n} \circ p_n\|^2_{L^2(U_h(\Gamma_n))} + \|\nabla u_{\Omega_n} - (\nabla u_{\Omega_n}) \circ p_n\|^2_{L^2(U_h(\Gamma_n))} = \mathrm{o}_{h\to 0}(h).$$

The end of the proof is similar to the one of Theorem 1.6 and consists in using the extruded surface approach to prove

$$\begin{aligned}
\liminf_{n\to+\infty} F_3(\Omega_n) \geqslant (1 + \mathrm{o}_{h\to 0}(1)) \\
\times \liminf_{n\to+\infty} \frac{1}{2h} \int_{U_h(\Gamma_n)} j_3(x, \nabla b_{\Omega_n}(p_n(x)), E_{\Omega_n}(u_{\Omega_n})(x), \nabla E_{\Omega_n}(u_{\Omega_n})(x))\, dx
\end{aligned}$$

$$
\begin{aligned}
&\geqslant (1+o_{h\to 0}(1))\Big(1+\mathrm{O}\Big(\frac{t}{h}\Big)\Big)\\
&\qquad \times \liminf_{n\to+\infty} \frac{1}{2(h-t)} \int_{U_{h-t}(\Gamma_\infty)} j_3(x,\nabla b_{\Omega_\infty}(p_\infty(x)), u^*(x), \nabla u^*(x))\,dx\\
&\qquad + o_{h\to 0}(1) + \mathrm{O}\Big(\frac{t}{h}\Big)\\
&\geqslant F_3(\Omega_\infty) + o_{h\to 0}(1) + \mathrm{O}\Big(\frac{t}{h}\Big),
\end{aligned}
$$

which concludes the proof.

3. Conclusion

In this paper, we have introduced a new method to tackle the existence issue for shape optimization problems under uniform reach constraints on the considered shapes, of the type

$$
\inf_{\Omega\in\mathcal{O}_{r_0}} \int_{\partial\Omega} j(x,\nu_{\partial\Omega}(x), B_{\partial\Omega}(x))\,d\mu_{\partial\Omega}(x).
$$

While several references such as [4, 5, 9] have already addressed similar questions on the same type of problems, we believe that the approaches developed in this paper are on the one hand simpler, but also sufficiently robust to allow easy extension of the results to more general settings.

For example, we believe that minor adaptations of the developed proof techniques allow one to extend our results to the following cases without much effort:

- Under weaker regularity hypotheses, one could think of replacing the continuity assumption by lower semicontinuity on the integrand $j_{\{1,2,3\}}$. Another example would be to assume that f in equation (1) belongs to $H^{1/2}(D)$ instead of $\mathcal{C}(D)$. In fact, if $f\in H^{1/2}(D)$, then its restriction to $\partial\Omega$ is well defined and belongs to $L^2(\partial\Omega)$. The crucial aspect would then be to establish regularity of the integrand.
- More general PDEs could be considered (see Theorems 1.10 and 1.11). Extension to general elliptic equations associated with differential operators of the kind $\nabla_\Gamma\cdot(\sigma\nabla_\Gamma)$ satisfying a coercivity property should be straightforward. We also believe that our framework allows extensions to non-linear elliptic PDEs under reasonable assumptions.
- One could consider costs involving the solution of a minimization problem depending on Ω but not necessarily related to a PDE. Indeed, in the proof of Theorem 1.10, our study of the variational problem does not rely on the underlying PDE. We treated a case involving a convex minimization problem over the set of divergence-free vector fields on $\partial\Omega$ in [12].

All of those generalizations do not seem obvious when using other methods.

Let us conclude by discussing a more open-ended problem. A very interesting extension of this work would be to generalize our results to manifolds with boundary. For instance, we could examine open sets ω of $\partial\Omega$ (under the induced topology) such that $\partial\omega$ also satisfies a reach condition. In this scenario, ω becomes a $d-1$ submanifold with a boundary, prompting the question of whether adapted versions of Theorems 1.6, 1.10, and 1.11 are attainable (e.g., by imposing Dirichlet conditions on the submanifold's boundary).

A. Curvatures of a submanifold

Let us quickly review the definition of the mean curvature for an oriented $(d-1)$-submanifold of $\mathbb{R}^d$ with $\mathcal{C}^{1,1}$ regularity. To stick with our notation, we consider the submanifold to be the boundary of some $\Omega \in \mathcal{O}_{r_0}$.

Definition A.1. The Gauss map is the application which assigns to each $x \in \Gamma = \partial\Omega$ the direct unit normal vector to Γ at x. In our setting, it can be defined as

$$\begin{aligned} N\colon \Gamma &\to \mathcal{S}^{d-1}, \\ x &\mapsto \nabla b_\Omega(x). \end{aligned}$$

We can now define the following objects:

- The shape operator (or Weingarten map) is the differential of the Gauss map. For every $x \in \Gamma$, the tangent spaces $T_x\Gamma$ and $T_{N(x)}\mathcal{S}^{d-1}$ are equal as linear subspaces of $\mathbb{R}^d$, and the shape operator at x is self-adjoint where it is defined. See, for example, [11, Chapter 5] for a general introduction.
- The trace of the shape operator is called the mean curvature and is denoted H.[2]
- The determinant of the shape operator is called the Gauss curvature.

Remark A.2. The Gauss map is $\frac{1}{r_0}$-Lipschitz continuous (see Lemma 1.1), where r_0 is the reach of Γ. Thus, the shape operator is in L^∞ and, for almost every $x \in \Gamma$, all the eigenvalues $\kappa_1(x), \dots, \kappa_{d-1}(x)$ of the shape operator are bounded in modulus by $\frac{1}{r_0}$. This means, in particular, that the curvature radii are almost everywhere bigger than r_0.

We insist on the fact that N is defined only on Γ and thus the shape operator is not defined on $\mathbb{R}^d$ or any tubular neighborhood of Γ. Nevertheless, we have the following property:

[2] Note that in differential geometry it is common to define the mean curvature as the trace of the shape operator divided by $(d-1)$.

Lemma A.3. *The mean curvature coincides with the trace of* $\nabla^2 b_\Omega$ *on* Γ.

Proof. Let $x \in \Gamma$ and let $\mathcal{B}$ be an orthonormal basis of $T_x\Gamma$. Using the identification between $T_x\Gamma$ and the tangent hyperplane (see Remark 2.1), we obtain that $\{\nabla b_\Omega(x)\} \cup \mathcal{B}$ is an orthonormal basis of $\mathbb{R}^d$. The vector field ∇b_Ω is constant along the direction $\nabla b_\Omega(x)$ (see, e.g., [7, Theorem 7.8.5.ii]). As a consequence, the trace of $\nabla^2 b_\Omega$ and the mean curvature coincide. ∎

B. R-convergence: Proof of Proposition 1.3

The compactness property follows from two facts. First, the Arzelà–Ascoli theorem, combined with the fact that every function b_Ω, for $\Omega \in \mathcal{O}_{r_0}$, is 1-Lipschitz continuous. Second, the reach constraint which imposes a uniform bound on the second derivative of b_Ω. These two facts are used in [7] and [4] to get the sequential compactness results used below.

Let $(\Omega_n)_{n\in\mathbb{N}}$ denote a sequence in $\mathcal{O}_{r_0}$. By the compactness property of sets of uniformly positive reach proved in [7, Chapter 6], it follows that, up to a subsequence, b_{Ω_n} converges to b_{Ω_∞} for the C^0 topology on D. In [4], the convergence is shown to hold also for the strong $\mathcal{C}^{1,\alpha}$ topology (for $\alpha < 1$) and for the weak $W^{2,\infty}$ topology in an r-tubular neighborhood of $\partial\Omega_\infty$, with $r < r_0$.

As a consequence, $\operatorname{Reach}(\Gamma_\infty) \geqslant r_0$. In particular, according to Lemma 1.1, b_{Ω_∞} is $\mathcal{C}^{1,1}$ on $\overline{U_r(\Gamma_\infty)}$.

C. The Laplace–Beltrami equation on a manifold: Proof of Lemma 1.9

Let $(\partial\Omega, \mathfrak{g})$ denote a closed compact manifold. We explain hereafter how to understand the equation $\Delta_{\partial\Omega} v = h$ in $\partial\Omega$ in a weak sense, whenever $\Omega \in \mathcal{O}_{r_0}$. Indeed, under this assumption, $\partial\Omega$ is a $\mathcal{C}^{1,1}$ submanifold according to Lemma 1.1, not necessarily $\mathcal{C}^2$, which justifies why such an equation cannot be understood in a strong sense.

The key ingredient in what follows is the Rellich–Kondrachov lemma, stating the compactness of the embedding $H^1_*(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

Theorem C.1 (Rellich–Kondrachov theorem on surfaces). *Let* $\Omega \in \mathcal{O}_{r_0}$. *Let* $(u_n)_{n\in\mathbb{N}}$ *denote a sequence in* $H^1_*(\partial\Omega)$ *such that* $(\int_{\partial\Omega} |\nabla u_n(x)|^2 \, d\mu_{\partial\Omega})_{n\in\mathbb{N}}$ *is bounded. There exists* $u^* \in H^1_*(\partial\Omega)$ *such that, up to a subsequence,* $(u_n)_{n\in\mathbb{N}}$ *converges to* u^* *weakly in* $H^1_*(\partial\Omega)$ *and strongly in* $L^2(\partial\Omega)$.

Proof. According to [6, Theorem 4.5.ii], since $\partial\Omega$ is $\mathcal{C}^{1,1}$, the L^2 norm $\|\cdot\|_{L^2(\partial\Omega)}$ on the surface $\partial\Omega$ and the L^2 norm $L^2(\partial\Omega) \ni u \mapsto \|u \circ p_\Omega\|_{L^2(U_h(\partial\Omega))}$ on the thickened surface $U_h(\partial\Omega)$ are equivalent whenever $h > 0$ is small enough, where $p_\Omega(x)$ denotes the orthogonal projection of x onto $\partial\Omega$, that is, $p_\Omega(x) = x - b_\Omega(x)\nabla b_\Omega(x)$, and $U_h(\partial\Omega) = \{x \in \mathbb{R}^d \mid |b_\Omega(x)| < h \text{ and } p_\Omega(x) \in \partial\Omega\}$.

Similarly, according to [6, Theorem 4.7.v], since $\partial\Omega$ is $\mathcal{C}^{1,1}$, the norm $\|\cdot\|_{H^1_*(\partial\Omega)}$ defined as

$$\|u\|^2_{H^1_*(\partial\Omega)} = \int_{\partial\Omega} |\nabla_\Gamma u|^2 \, d\mu_{\partial\Omega},$$

and the norm $\|\cdot\|_{H^1_{U_h(\partial\Omega)}}$ given by

$$\|u\|_{H^1_{U_h(\partial\Omega)}} = \frac{1}{2h} \int_{U_h(\partial\Omega)} |\nabla_\Gamma u \circ p_\Omega|^2 \, d\mu_{\partial\Omega}$$

are equivalent whenever $h > 0$ is small enough. We conclude the proof by using the standard Rellich–Kondrakov theorem (see, e.g., [2, Section 9.3]) on the thickened surface $U_h(\partial\Omega)$. ∎

The following result is a Poincaré-type lemma, uniform with respect to the chosen surface in the set $\mathcal{O}_{r_0}$:

Proposition C.2 (Poincaré lemma on a surface). *Let $r_0 > 0$ and $\Omega \in \mathcal{O}_{r_0}$. There exists $C(r_0, D) > 0$ such that*

$$\forall u \in H^1_*(\Gamma), \quad \int_\Gamma |\nabla_\Gamma u(x)|^2 \, d\mu_\Gamma \geqslant C(r_0, D) \int_\Gamma |u(x)|^2 \, d\mu_\Gamma.$$

Proof. Let $(\Omega_n, v_n)_{n\in\mathbb{N}}$, with $v_n \in H^1_*(\Gamma_n)$, be a minimizing sequence for the problem

$$\inf_{\Omega\in\mathcal{O}_{r_0}} \inf_{u\in H^1_*(\Gamma)} \frac{\int_\Gamma |\nabla_\Gamma u(x)|^2 \, d\mu_\Gamma}{\int_\Gamma |u(x)|^2 \, d\mu_\Gamma}.$$

Let us argue by contradiction, assuming that

$$\int_{\Gamma_n} |\nabla_{\Gamma_n} v_n(x)|^2 \, d\mu_{\Gamma_n} \leqslant \frac{1}{n} \quad \text{and} \quad \int_{\Gamma_n} |v_n(x)|^2 \, d\mu_{\Gamma_n} = 1,$$

by homogeneity of the Rayleigh quotient. According to 1.3, we can assume without loss of generality that $(\Omega_n)_{n\in\mathbb{N}}$ R-converges toward $\Omega_\infty \in \mathcal{O}_{r_0}$.

Let p_n denote the orthogonal projection on Γ_n and let us introduce the function w_n defined in $U_h(\Gamma_n)$ for h as in Lemma 2.2 and n large enough by $w_n = v_n \circ p_n$. We follow exactly the same lines as in the first step of the proof of Theorem 1.10. A direct adaptation of the first step of the proof of Theorem 1.10 yields

$$\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_n(y)|^2 \, d\mu_{\Gamma_\infty}(y) = \int_{\Gamma_n} |\nabla_{\Gamma_n} v_n|^2 \, d\mu_{\Gamma_n}(x) + o(1).$$

We infer that

$$\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_n|^2 \, d\mu_{\Gamma_\infty}(x) \leqslant \frac{1}{n} + o(1).$$

By using Theorem C.1, we get that the sequence $(w_n)_{n\in\mathbb{N}}$ converges up to a subsequence toward $w^* \in H^1_*(\Gamma_\infty)$ weakly in $H^1(\Gamma_\infty)$ and strongly in $L^2(\Gamma_\infty)$. Up to extracting a subsequence, we get

$$\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w^*(x)|^2 \, d\mu_{\Gamma_\infty} \leqslant \liminf_{n\to+\infty} \int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} w_n(x)|^2 \, d\mu_{\Gamma_\infty} = 0,$$

$$\int_{\Gamma_\infty} |w^*(x)|^2 \, d\mu_{\Gamma_\infty} = 1,$$

$$\int_{\Gamma_\infty} w_n(x) \, d\mu_{\Gamma_\infty} = 0.$$

By using the first equality, we get that w^* is constant on Γ and we obtain a contradiction with the two last equalities above. ∎

Let us now prove Lemma 1.9. Let $(u_n)_{n\in\mathbb{N}}$ denote a minimizing sequence for problem (2). Since $(\mathcal{E}_\Gamma(u_n))_{n\in\mathbb{N}}$ is bounded, and since

$$\mathcal{E}_\Gamma(u_n) \geqslant C(d, r_0)\|u_n\|^2_{L^2(\Gamma)} - \|h\|_{L^2(\Gamma)}\|u_n\|_{L^2(\Gamma)}$$

according to Proposition C.2, we infer that $(\|u_n\|_{L^2(\Gamma)})_{n\in\mathbb{N}}$ is bounded. Since

$$\int_\Gamma |\nabla_\Gamma u_n(x)|^2 \, d\mu_\Gamma = \mathcal{E}_\Gamma(u_n) - \int_\Gamma u_n(x)h(x) \, d\mu_\Gamma \leqslant \|h\|_{L^2(\Gamma)}\|u_n\|_{L^2(\Gamma)} + \mathcal{E}_\Gamma(u_n),$$

we infer the existence of $u^* \in H^1_*(\Gamma)$ such that, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges weakly in $H^1_*(\Gamma)$ and strongly in $L^2(\Gamma)$. Up to extracting a subsequence, we get

$$\inf_{u\in H^1_*(\Gamma)} \mathcal{E}_\Gamma(u) = \liminf_{n\to+\infty} \mathcal{E}_\Gamma(u_n) \geqslant \int_\Gamma |\nabla_\Gamma u^*(x)|^2 \, d\mu_\Gamma + \int_\Gamma u^*(x)h(x) \, d\mu_\Gamma = \mathcal{E}_\Gamma(u^*)$$

and the existence follows. The uniqueness is standard and follows from the strong convexity of the functional $\mathcal{E}_\Gamma$.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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