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On the realization of a class of $\mathrm{SL}(2, \mathbb{Z})$ representations

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Abstract. Let $p < q$ be odd primes and ρ_1 and ρ_2 be irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_p)$ and $\mathrm{SL}(2, \mathbb{Z}_q)$ of dimensions $\frac{p+1}{2}$ and $\frac{q+1}{2}$, respectively. We show that if $\rho_1 \oplus \rho_2$ can be realized as a modular representation associated with a modular fusion category $\mathcal{C}$, then $q - p = 4$. Moreover, if $\mathcal{C}$ contains a non-trivial étale algebra, then $\mathcal{C} \boxtimes \mathcal{C}(\mathbb{Z}_p, \eta) \cong \mathcal{Z}(\mathcal{A})$ as a braided fusion category, where $\mathcal{A}$ is a near-group fusion category of type $(\mathbb{Z}_p, p)$, which gives a partial answer to the conjecture of D. Evans and T. Gannon. We also show that there exists a non-trivial $\mathbb{Z}_2$-extension of $\mathcal{A}$ that contains simple objects of Frobenius–Perron dimension $\frac{\sqrt{p}+\sqrt{q}}{2}$.

1. Introduction

A braided spherical fusion category $\mathcal{C}$ is called modular if the S-matrix of $\mathcal{C}$ is non-degenerate (see Section 2). Modular fusion category connects with conformal field theory, quantum groups, representation theory, and mathematical physics, etc. [6, 9, 16, 17]. Combined with the T-matrix, which is defined by the ribbon structure θ of $\mathcal{C}$, these two matrices (S, T) are called the modular data of $\mathcal{C}$. The modular data enjoy many important algebraic and arithmetic properties. The modular data provides a projective congruence representation ρ of the modular group $\mathrm{SL}(2, \mathbb{Z})$ of level N [6, 9, 18], where $N = \mathrm{ord}(T)$. Moreover, ρ can be lifted to a linear congruence representation of $\mathrm{SL}(2, \mathbb{Z})$ of level n with $N \mid n \mid 12N$, that is, it factors through $\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}_n)$, and the linear representation satisfies the Galois symmetry [6].

Finite-dimensional representations of $\mathrm{SL}(2, \mathbb{Z}_n)$ are classified completely in [21, 22]. Thus, one could construct (or reconstruct) modular fusion categories from finite-dimensional congruence representations of $\mathrm{SL}(2, \mathbb{Z})$; see [18, 20, 30] for applications. In this paper, we are aimed to realize a class of finite-dimensional congruence representations of $\mathrm{SL}(2, \mathbb{Z})$ as a modular representation associated with a modular fusion category. Explicitly, let p be an odd prime, and let ρ be an irreducible $\frac{p+1}{2}$-dimensional representation of $\mathrm{SL}(2, \mathbb{Z}_p)$. It is well known that, up to isomorphism, there exist just two such representations [21]. However, neither of these two representations can be isomorphic to a modular representation associated with a modular fusion category [8]. Hence, we consider the following question.

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Question 1.1. Let $p < q$ be odd primes. Is there a modular fusion category $\mathcal{C}$ such that the associated modular representation $\rho_{\mathcal{C}} \cong \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are irreducible representations of dimension $\frac{p+1}{2}$ and $\frac{q+1}{2}$, respectively?

When $p = 3$ and $q = 7$, the answer is positive [18, Lemma 4.7]. We give a necessary condition on realizing the sum $\rho_1 \oplus \rho_2$ in Theorem 3.2, which states $q - p = 4$. Moreover, we show that if such a modular fusion category $\mathcal{C}$ does exist, then it is connected with a near-group fusion category $\mathcal{A}$ (see Section 3.2). We study the structure of $\mathcal{C}$ and the related near-group fusion category $\mathcal{A}$; and we also give a faithful $\mathbb{Z}_2$-extension of $\mathcal{A}$, which generalizes the fusion category $\mathcal{V}$ constructed by Ostrik in [4].

Since there exists a pointed modular fusion category $\mathcal{C}(\mathbb{Z}_p, \eta)$ of Frobenius–Perron dimension p such that $\mathcal{C} \boxtimes \mathcal{C}(\mathbb{Z}_p, \eta) \cong \mathcal{Z}(\mathcal{A})$ as a modular fusion category (Theorem 3.5), which then can be viewed as evidence that [12, Conjecture 2] might be true; and the modular data (of $\mathcal{C}$) obtained in this paper gives a partial solution to the modular data described with unknown parameters in [12, Proposition 7].

This paper is organized as follows: In Section 2, we recall some basic notions and notations of (modular) fusion categories, such as Frobenius–Perron dimension, global dimension, modular data, and the congruence representations of the modular group $\mathrm{SL}(2, \mathbb{Z})$. In Section 3, we consider the realization of a direct sum $\rho_1 \oplus \rho_2$ of two irreducible representations of dimensions $\frac{p+1}{2}$ and $\frac{q+1}{2}$, respectively. We show in Theorem 3.2 that if $\rho_1 \oplus \rho_2$ can be realized as a representation associated with a modular fusion category $\mathcal{C}$, then $q - p = 4$. Under the assumption that $\mathcal{C}$ contains a non-trivial connected étale algebra A, we prove that $\mathcal{C}_A^0$ is a pointed modular fusion category and $\mathcal{C}_A$ is a near-group fusion category of type $(\mathbb{Z}_p, p)$ in Theorem 3.5 and Theorem 3.8. At last, we construct a faithful $\mathbb{Z}_2$-extension $\mathcal{M}$ of $\mathcal{C}_A$, which contains simple objects of Frobenius–Perron dimension $\frac{\sqrt{p}+\sqrt{q}}{2}$, and we determine the fusion relations of $\mathcal{M}$ in Corollary 3.13.

2. Preliminaries

In this section, we recall some of the most used definitions and properties of modular fusion categories; we refer the reader to [7, 9–11, 17] for standard conclusions for fusion categories and braided fusion categories.

2.1. Fusion category

A $\mathbb{C}$-linear abelian category $\mathcal{C}$ over the complex number field $\mathbb{C}$ is called a fusion category if $\mathcal{C}$ is a finite semisimple tensor category [9]. In the following, we use $\mathcal{O}(\mathcal{C})$ and $\otimes$ to denote the set of isomorphism classes of simple objects of $\mathcal{C}$ and the tensor product on $\mathcal{C}$, respectively.

Let $\mathcal{C}$ be a fusion category. Its Grothendieck ring is then a fusion ring with $\mathbb{Z}_+$-basis $\mathcal{O}(\mathcal{C})$ and the multiplication is induced by the tensor product $\otimes$. There is a unique homomorphism $\mathrm{FPdim}(-)$, called the Frobenius–Perron homomorphism, from $\mathrm{Gr}(\mathcal{C})$ to $\mathbb{C}$ such that $\mathrm{FPdim}(X)$ is a positive algebraic integer for all non-zero objects X [9, 10]. The sum

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{X\in\mathcal{O}(\mathcal{C})} \mathrm{FPdim}(X)^2$$

is called the Frobenius–Perron dimension of $\mathcal{C}$.

A fusion category $\mathcal{C}$ is pivotal if it admits a pivotal structure j, which is a natural isomorphism from the identity functor id to the double dual functor $(-)^{**}$ [9]. Then there is a well-defined categorical trace $\mathrm{Tr}(-)$ for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, where X is an object of $\mathcal{C}$. Fix a pivotal structure j on $\mathcal{C}$, the categorical trace of id_X is called the categorical dimension of X and is denoted by $\dim(X)$, and the sum

$$\dim(\mathcal{C}) := \sum_{X\in\mathcal{O}(\mathcal{C})} \dim(X)\dim(X^*)$$

is called the global (or quantum) dimension of $\mathcal{C}$. Moreover, the categorical dimension induces a homomorphism from the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ to $\mathbb{C}$ [9, Proposition 4.7.12]. If $\dim(X) = \dim(X^*)$ for all objects X of $\mathcal{C}$, then $\mathcal{C}$ is called spherical.

Recall that a fusion ring R is categorifiable if there exists a fusion category $\mathcal{C}$ such that $\mathrm{Gr}(\mathcal{C}) = R$ as fusion ring [9, Definition 4.10.1], and $\mathcal{C}$ is called a categorification of R. For example, for any finite group G, the pointed fusion category Vec_G^{ω}, i.e., the category of G-graded finite-dimensional vector spaces over $\mathbb{C}$, is a categorification of the group ring $\mathbb{Z}[G]$, where $\omega \in Z^3(G, \mathbb{C}^*)$ is a normalized 3-cocycle on G and $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$.

2.2. Modular fusion category and modular representation

A braided fusion category $\mathcal{C}$ is a fusion category with a braiding c, which is a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ satisfying the hexagon equations [9]. In addition, if $\mathcal{C}$ is spherical, then $\mathcal{C}$ is called a pre-modular (or ribbon) fusion category and we use θ to denote the ribbon structure of $\mathcal{C}$.

Let $\mathcal{C}$ be a pre-modular fusion category. For any simple objects X, Y of $\mathcal{C}$, let $S_{X,Y} := \mathrm{Tr}(c_{Y,X}c_{X,Y})$, then

$$S = (S_{X,Y}), \quad T = (\delta_{X,Y}\theta_X)$$

is called the modular data of $\mathcal{C}$. If the S-matrix S is non-degenerate, then $\mathcal{C}$ is said to be a modular fusion category [7, 17]. For example, pointed modular fusion categories are in bijective correspondence with metric groups [7, Proposition 2.41]. We use $\mathcal{C}(G, \eta)$ to denote the modular fusion category determined by the metric group (G, η), where G is a finite abelian group and $\eta : G \to \mathbb{C}^*$ is a non-degenerate quadratic form, the modular data of $\mathcal{C}(G, \eta)$ is

$$S_{g,h} = \frac{\eta(gh)}{\eta(g)\eta(h)}, \theta_g = \eta(g), \forall g, h \in G.$$

The S-matrix of a modular fusion category $\mathcal{C}$ also satisfies the Verlinde formula [9], which states that for any objects $X, Y, Z \in \mathcal{O}(\mathcal{C})$,

$$N_{X,Y}^{Z} := \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)) = \frac{1}{\dim(\mathcal{C})} \sum_{W \in \mathcal{O}(\mathcal{C})} \frac{S_{X,W} S_{Y,W} S_{Z^*,W}}{\dim(W)}.$$

Recall that the modular group $\mathrm{SL}(2,\mathbb{Z})$ is generated by $\mathfrak{s} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathfrak{t} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with relations $\mathfrak{s}^4 = 1$ and $(\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2$. The modular data of a modular fusion category $\mathcal{C}$ determines a projective congruence representation ρ of the modular group $\mathrm{SL}(2,\mathbb{Z})$ of level $N = \mathrm{ord}(T)$ [2, 6, 9, 18], that is, $\ker(\rho)$ kills a congruence subgroup of level N, and

$$\rho : \mathfrak{s} \mapsto \frac{1}{\sqrt{\dim(\mathcal{C})}} S, \mathfrak{t} \mapsto T,$$

where $\sqrt{\dim(\mathcal{C})}$ is the positive square root of $\dim(\mathcal{C})$. Moreover, the projective representation ρ can be lifted to a linear congruence representation $\rho_{\mathcal{C}}$ of level n and $N \mid n$ by [6, Theorem II], where $n = \mathrm{ord}(\rho_{\mathcal{C}}(\mathfrak{t}))$. If $\mathrm{ord}(T)$ is odd, then there is a lifting ρ' of ρ such that $\mathrm{ord}(\rho'(\mathfrak{t})) = \mathrm{ord}(T)$ [6, Lemma 2.2].

Let ρ be an arbitrary irreducible finite-dimensional congruence representation of $\mathrm{SL}(2,\mathbb{Z})$ of level n, where n is a positive integer. Then it follows from the Chinese remainder theorem that ρ factors through the finite groups

$$\mathrm{SL}(2,\mathbb{Z}_n) \cong \mathrm{SL}(2,\mathbb{Z}_{p_1^{n_1}}) \times \cdots \times \mathrm{SL}(2,\mathbb{Z}_{p_r^{n_r}})$$

and $\rho \cong \bigotimes_{j=1}^{r} \rho_{p_j}$, where $n = \prod_{j=1}^{r} p_j^{n_j}$ and p_j are distinct primes, and ρ_{p_j} are finite-dimensional representations of subgroups $\mathrm{SL}(2,\mathbb{Z}_{p_j^{n_j}})$. Finite-dimensional irreducible representations of the group $\mathrm{SL}(2,\mathbb{Z}_{p^m})$ are completely classified and constructed explicitly in [21, 22].

Hence, one could try to reconstruct modular fusion categories from finite-dimensional congruence representations of $\mathrm{SL}(2,\mathbb{Z})$; see [2, 8, 18, 20, 30] and the references therein for details. For example, many important properties of modular representations are summarized and characterized in [18]; as an application, modular fusion categories with six simple objects (up to isomorphism) are classified by considering the type of the associated modular representation of $\mathcal{C}$ [18]. A representation ρ of $\mathrm{SL}(2,\mathbb{Z})$ is called realizable if there exists a modular fusion category $\mathcal{C}$ such that $\rho_{\mathcal{C}} \cong \rho$.

3. Realization and extension

In this section, we consider the realization of $\rho_1 \oplus \rho_2$ as a modular representation associated with a modular fusion category. Under the assumption that $\rho_1 \oplus \rho_2$ can be realized as a representation of a modular fusion category $\mathcal{C}$, we study the structure of $\mathcal{C}$ and show it is related to a certain near-group fusion category $\mathcal{A}$. At last, we construct a faithful $\mathbb{Z}_2$-extension of $\mathcal{A}$.

3.1. Realization

Let p be an odd prime. Let ρ be a $\frac{p+1}{2}$-dimensional irreducible representation of SL$(2,\mathbb{Z}_p)$. Then [8, (4.11)] says

$$\rho(\mathfrak{s}) = \beta_p \begin{pmatrix} 1 & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & & & \\ \vdots & & 2\cos\left(\frac{4\pi ajk}{p}\right) & \\ \sqrt{2} & & & \end{pmatrix} = \begin{pmatrix} \beta_p & B^T \\ B & D \end{pmatrix}, \rho(\mathfrak{t}) = \operatorname{diag}(1, T_1),$$

where $B^T := (\sqrt{2}\beta_p, \ldots, \sqrt{2}\beta_p)$ is a $\frac{p-1}{2}$-dimensional vector over $\mathbb{C}$, and

$$D := \left(2\beta_p \cos\left(\frac{4\pi ajk}{p}\right)\right) \quad \text{and} \quad T_1 := \operatorname{diag}\left(\zeta_p^a, \ldots, \zeta_p^{a\cdot\left(\frac{p-1}{2}\right)^2}\right)$$

are square matrices of order $\frac{p-1}{2}$, $1 \le j, k \le \frac{p-1}{2}$, $\beta_p := \left(\frac{a}{p}\right)\sqrt{\left(\frac{-1}{p}\right)\frac{1}{p}}$, where a is an integer coprime to p and $\left(\frac{a}{p}\right)$ is the classical Legendre symbol. Notice that ρ is non-degenerate, i.e., the eigenvalues of $\rho(\mathfrak{t})$ are multiplicity-free. Given an odd prime p, up to isomorphism, it is well known that there are exactly two such irreducible representations [21], depending on the value $\left(\frac{a}{p}\right)$.

It was proved in [8] that ρ cannot be realized by a rational conformal field theory (equivalently, it cannot be realized as a modular representation associated with a modular fusion category), as the corresponding fusion rings obtained from the Verlinde formula are not integer-valued fusion rings. However, it was also noted in [8] that one can obtain an integer-valued fusion ring from a direct sum of two such representations for different primes p, q such that $q - p = 4$.

Hence, one would like to answer the following question naturally.

Question 3.1. Let $p < q$ be odd primes. Furthermore, let ρ_1 and ρ_2 be irreducible representations of SL$(2,\mathbb{Z}_p)$ and SL$(2,\mathbb{Z}_q)$ such that $\dim(\rho_1) = \frac{p+1}{2}$ and $\dim(\rho_2) = \frac{q+1}{2}$, respectively. Is $\rho_1 \oplus \rho_2$ realizable?

When $p = 3$ and $q = 7$, the answer is positive; and $\mathcal{C}$ is a Galois conjugate of the modular fusion category $\mathcal{C}(\mathfrak{g}_2, 3)$ [18, Lemma 4.7]. We refer the reader to [1] for construction of the modular fusion category $\mathcal{C}(\mathfrak{g}, k)$, where $\mathfrak{g}$ is a simple Lie algebra. Notice that if $p = 1$ (of course, it is not a prime), and let ρ_0 be the trivial representation, then $\rho_0 \oplus \rho_2$ is realizable for all primes $q \ge 5$; moreover, the associated modular fusion category $\mathcal{C}$ is Grothendieck equivalent to $\mathcal{C}(\mathfrak{sl}_2, 2(q-1))_A^0$ [30, Theorem 3.12], where A is the non-trivial étale algebra of Rep$(\mathbb{Z}_2) \subseteq \mathcal{C}(\mathfrak{sl}_2, 2(q-1))$ and $\mathcal{C}(\mathfrak{sl}_2, 2(q-1))_A^0$ is the core of $\mathcal{C}(\mathfrak{sl}_2, 2(q-1))$; see [5, 7, 16] for details.

In the following theorem, we give a necessary condition to realize $\rho_1 \oplus \rho_2$ as modular representation associated with a modular fusion category.

Theorem 3.2. *If there is a modular fusion category $\mathcal{C}$ such that $\rho_{\mathcal{C}} \cong \rho_1 \oplus \rho_2$, then $q - p = 4$.*

Proof. It follows from [18, Theorem 3.23] that

$$\rho_{\mathcal{C}}(\mathfrak{s}) = V \begin{pmatrix} A & C_1^T & C_2^T \\ C_1 & D_1 & \mathbf{0} \\ C_2 & \mathbf{0} & D_2 \end{pmatrix} V, \ \rho_{\mathcal{C}}(\mathfrak{t}) = \begin{pmatrix} E_2 & & \\ & T_1 & \\ & & T_2 \end{pmatrix},$$

where V is a signed diagonal orthogonal matrix, $T_1 = \mathrm{diag}\Big(\zeta_p^{a_1}, \ldots, \zeta_p^{a_1 \cdot \left(\frac{p-1}{2}\right)^2}\Big)$ and $T_2 = \mathrm{diag}\Big(\zeta_q^{a_2}, \ldots, \zeta_q^{a_2 \cdot \left(\frac{q-1}{2}\right)^2}\Big)$, and

$$A = U \begin{pmatrix} \beta_p & \\ & \beta_q \end{pmatrix} U^T = \frac{1}{2} \begin{pmatrix} \beta_p + \beta_q & \nu(\beta_p - \beta_q) \\ \nu(\beta_p - \beta_q) & \beta_p + \beta_q \end{pmatrix}$$

with $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-\nu}{\sqrt{2}} \\ \frac{\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\nu^2 = 1$,

$$C_1 = (B_1, 0)U^T = \beta_p \begin{pmatrix} 1 & \nu \\ \vdots & \vdots \\ 1 & \nu \end{pmatrix}, C_2 = (0, B_2)U^T = \beta_q \begin{pmatrix} -\nu & 1 \\ \vdots & 1 \\ -\nu & 1 \end{pmatrix}.$$

Let $V = \mathrm{diag}\big(1, \epsilon_1, \ldots, \epsilon_{\frac{p+q}{2}}\big)$ where $\epsilon_j \in \{\pm 1\}$ for all $1 \leq j \leq \frac{p+q}{2}$; hence, we see

$$\rho_{\mathcal{C}}(\mathfrak{s}) = V \begin{pmatrix} A & C_1^T & C_2^T \\ C_1 & D_1 & \mathbf{0} \\ C_2 & \mathbf{0} & D_2 \end{pmatrix} V$$

$$= V \begin{pmatrix} \frac{1}{2}(\beta_p + \beta_q) & \frac{\nu}{2}(\beta_p - \beta_q) & \beta_p & \cdots & \beta_p & -\nu\beta_q & \cdots & -\nu\beta_q \\ \frac{\nu}{2}(\beta_p - \beta_q) & \frac{1}{2}(\beta_p + \beta_q) & \beta_p\nu & \cdots & \beta_p\nu & \beta_q & \cdots & \beta_q \\ \beta_p & \beta_p\nu & & & & & & \\ \vdots & \vdots & & & & & & \\ \beta_p & \beta_p\nu & & & D_1 & & \mathbf{0} & \\ -\nu\beta_q & \beta_q & & & & & & \\ \vdots & \vdots & & & \mathbf{0} & & D_2 & \\ -\nu\beta_q & \beta_q & & & & & & \end{pmatrix} V$$

$$
= \begin{pmatrix}
\frac{1}{2}(\beta_p+\beta_q) & \frac{\nu\epsilon_1}{2}(\beta_p-\beta_q) & \epsilon_2\beta_p & \cdots & \epsilon_{\frac{p+1}{2}}\beta_p & -\epsilon_{\frac{p+3}{2}}\nu\beta_q & \cdots & -\epsilon_{\frac{p+q}{2}}\nu\beta_q \\
\frac{\nu\epsilon_1}{2}(\beta_p-\beta_q) & \frac{1}{2}(\beta_p+\beta_q) & \epsilon_1\epsilon_2\beta_p\nu & \cdots & \epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu & \epsilon_1\epsilon_{\frac{p+3}{2}}\beta_q & \cdots & \epsilon_1\epsilon_{\frac{p+q}{2}}\beta_q \\
\epsilon_2\beta_p & \epsilon_1\epsilon_2\beta_p\nu & & & & & & \\
\vdots & \vdots & & & & & & \\
\epsilon_{\frac{p+1}{2}}\beta_p & \epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu & & & V_1D_1V_1 & & \mathbf{0} & \\
-\epsilon_{\frac{p+3}{2}}\nu\beta_q & \epsilon_1\epsilon_{\frac{p+3}{2}}\beta_q & & & & & & \\
\vdots & \vdots & & & \mathbf{0} & & V_2D_2V_2 & \\
-\epsilon_{\frac{p+q}{2}}\nu\beta_q & \epsilon_1\epsilon_{\frac{p+q}{2}}\beta_q & & & & & &
\end{pmatrix},
$$

where

$$
V = \begin{pmatrix} 1 & & & \\ & \epsilon_1 & & \\ & & V_1 & \\ & & & V_2 \end{pmatrix},\ V_1 = \begin{pmatrix} \epsilon_2 & & \\ & \ddots & \\ & & \epsilon_{\frac{p+1}{2}} \end{pmatrix},\ V_2 = \begin{pmatrix} \epsilon_{\frac{p+3}{2}} & & \\ & \ddots & \\ & & \epsilon_{\frac{p+q}{2}} \end{pmatrix}.
$$

Since the categorical dimensions of the simple objects are always non-zero, either the elements in the first or the second row are dimensions (multiplied with a non-zero scalar necessarily) of simple objects of $\mathcal{C}$, depending on which vector represents the unit object. We know $\beta_p = \frac{\mu_p}{\sqrt{p}}$ and $\beta_q = \frac{\mu_q}{\sqrt{q}}$, where $\mu_p = \left(\frac{a_1}{p}\right)\sqrt{\left(\frac{-1}{p}\right)}$ and $\mu_q = \left(\frac{a_2}{q}\right)\sqrt{\left(\frac{-1}{q}\right)}$ are 4th roots of unity. A classical theorem about Legendre symbols says $\left(\frac{a_1}{p}\right) \equiv a_1^{\frac{p-1}{2}} \bmod p$, so

$$
\mu_p = \left(\frac{a_1}{p}\right)\sqrt{\left(\frac{-1}{p}\right)} = \begin{cases} \left(\frac{a_1}{p}\right), & \text{if } p = 4k+1; \\ \left(\frac{a_1}{p}\right)\zeta_4, & \text{if } p = 4k+3. \end{cases},
$$

where ζ_4 is a 4th primitive root of unity. Notice that

$$
\begin{aligned}
|\beta_p+\beta_q|^2 &= \frac{(\mu_p\sqrt{q}+\mu_q\sqrt{p})(\overline{\mu}_p\sqrt{q}+\overline{\mu}_q\sqrt{p})}{pq} \\
&= \frac{(p+q)+(\overline{\mu}_p\mu_q+\overline{\mu}_q\mu_p)\sqrt{pq}}{pq}.
\end{aligned}
$$

We claim $\overline{\mu}_p\mu_q+\overline{\mu}_q\mu_p = 2\mathrm{Re}(\overline{\mu}_p\mu_q) = \pm 2$. In fact, $\overline{\mu}_p\mu_q+\overline{\mu}_q\mu_p \neq 0$, otherwise

$$
\dim(\mathcal{C}) = \frac{4}{|\beta_p+\beta_q|^2} = \frac{4pq}{p+q},
$$

then $p+q$ must contain a prime factor, which is coprime to pq. However, $\mathrm{ord}(\rho_{\mathcal{C}}(\mathfrak{t})) = pq$; it violates the Cauchy theorem of spherical fusion categories [2, Theorem 3.9]. Meanwhile, $\overline{\mu}_p\mu_q$ is a 4th root of unit, so $2\mathrm{Re}(\overline{\mu}_p\mu_q) = \pm 2$, as claimed. Therefore,

$$
\dim(\mathcal{C}) = \frac{4}{|\beta_p+\beta_q|^2} = pq\frac{\frac{p+q}{2} \pm \sqrt{pq}}{2},
$$

depending on the value of $\mathrm{Re}(\overline{\mu}_p\mu_q)$. Then

$$N(\dim(\mathcal{C})) = p^2q^2N\left(\frac{\frac{p+q}{2}\pm\sqrt{pq}}{2}\right) = p^2q^2\frac{(p-q)^2}{16},$$

where $N(\dim(\mathcal{C}))$ and $N\left(\frac{\frac{p+q}{2}\pm\sqrt{pq}}{2}\right)$ are the norms of $\dim(\mathcal{C})$ and $\frac{\frac{p+q}{2}\pm\sqrt{pq}}{2}$ over $\mathbb{Q}(\sqrt{pq})$, respectively. Again, the Cauchy theorem of spherical fusion categories [2, Theorem 3.9] implies that $\frac{\frac{p+q}{2}\pm\sqrt{pq}}{2}$ must be an algebraic unit in $\mathbb{Q}(\sqrt{pq})$, that is, $q-p=4$, as desired. ■

Below we calculate the dimensions of the simple objects of $\mathcal{C}$, denoted by $\varepsilon_{pq} := \frac{\sqrt{p}+\sqrt{q}}{2}$, then $\dim(\mathcal{C}) = pq\varepsilon_{pq}^{\pm 2}$. Since $q-p=4$, we have $\overline{\mu}_p\mu_q = \left(\frac{a_1}{p}\right)\left(\frac{a_2}{q}\right) = \pm 1$. That is, if a_1 and a_2 are both square residues or both non-square residues modulo p and q, respectively, then $\dim(\mathcal{C}) = pq\varepsilon_{pq}^{-2}$; otherwise, $\dim(\mathcal{C}) = pq\varepsilon_{pq}^{2}$.

We list the categorical dimensions in both cases explicitly. After identifying $\mathcal{O}(\mathcal{C})$ with the standard basis $\{e_1,\dots,e_{p+2}\}$ of the vector space $\mathbb{C}^{p+2}$, the S-matrix of $\mathcal{C}$ can be written as

$$S = \begin{pmatrix}
1 & \frac{\nu\epsilon_1(\beta_p-\beta_q)}{\beta_p+\beta_q} & \frac{2\epsilon_2\beta_p}{\beta_p+\beta_q} & \cdots & \frac{2\epsilon_{\frac{p+1}{2}}\beta_p}{\beta_p+\beta_q} & \frac{-2\epsilon_{\frac{p+3}{2}}\nu\beta_q}{\beta_p+\beta_q} & \cdots & \frac{-2\epsilon_{\frac{p+q}{2}}\nu\beta_q}{\beta_p+\beta_q} \\
\frac{\nu\epsilon_1(\beta_p-\beta_q)}{\beta_p+\beta_q} & 1 & \frac{2\epsilon_1\epsilon_2\beta_p\nu}{\beta_p+\beta_q} & \cdots & \frac{2\epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu}{\beta_p+\beta_q} & \frac{2\epsilon_1\epsilon_{\frac{p+3}{2}}\beta_q}{\beta_p+\beta_q} & \cdots & \frac{2\epsilon_1\epsilon_{\frac{p+q}{2}}\beta_q}{\beta_p+\beta_q} \\
\frac{2\epsilon_2\beta_p}{\beta_p+\beta_q} & \frac{2\epsilon_1\epsilon_2\beta_p\nu}{\beta_p+\beta_q} & & & & & & \\
\vdots & \vdots & & & & & & \\
\frac{2\epsilon_{\frac{p+1}{2}}\beta_p}{\beta_p+\beta_q} & \frac{2\epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu}{\beta_p+\beta_q} & & & \frac{2}{\beta_p+\beta_q}V_1D_1V_1 & & & \mathbf{0} \\
\frac{-2\epsilon_{\frac{p+3}{2}}\nu\beta_q}{\beta_p+\beta_q} & \frac{2\epsilon_1\epsilon_{\frac{p+3}{2}}\beta_q}{\beta_p+\beta_q} & & & & & & \\
\vdots & \vdots & & & \mathbf{0} & & & \frac{2}{\beta_p+\beta_q}V_2D_2V_2 \\
\frac{-2\epsilon_{\frac{p+q}{2}}\nu\beta_q}{\beta_p+\beta_q} & \frac{2\epsilon_1\epsilon_{\frac{p+q}{2}}\beta_q}{\beta_p+\beta_q} & & & & & &
\end{pmatrix}.$$

Case (1): $\overline{\mu}_p\mu_q = 1$. We can assume that a_1 and a_2 are both residues modulo p and q, respectively, the other case is same. Let $a_1 = a_2 = 1$. Then $\beta_p = \frac{1}{\sqrt{p}}$ and $\beta_q = \frac{1}{\sqrt{q}}$; if $p = 4k+1$, $\beta_p = \frac{\zeta_4}{\sqrt{p}}$ and $\beta_q = \frac{\zeta_4}{\sqrt{q}}$ if $p = 4k+3$, then $\dim(\mathcal{C}) = pq\varepsilon_{pq}^{-2}$. Let

$$d_1 := \sqrt{q}\varepsilon_{pq}^{-1} = \frac{\sqrt{q}(\sqrt{q}-\sqrt{p})}{2}, \qquad d_1' := d_1\varepsilon_{pq}^2 = \sqrt{q}\varepsilon_{pq} = \frac{\sqrt{q}(\sqrt{q}+\sqrt{p})}{2},$$

$$d_2 := \sqrt{p}\varepsilon_{pq}^{-1} = \frac{\sqrt{p}(\sqrt{q}-\sqrt{p})}{2}, \qquad d_2' := d_2\varepsilon_{pq}^2 = \sqrt{p}\varepsilon_{pq} = \frac{\sqrt{p}(\sqrt{q}+\sqrt{p})}{2}.$$

Then the first row of the S-matrix is

$$\left(1, \nu\epsilon_1\varepsilon_{pq}^{-2}, \epsilon_2 d_1, \ldots, \epsilon_{\frac{p+1}{2}} d_1, -\nu\epsilon_{\frac{p+3}{2}} d_2, \ldots, -\nu\epsilon_{\frac{p+q}{2}} d_2\right),$$

and the second row of the S-matrix is

$$\left(\nu\epsilon_1\varepsilon_{pq}^{-2}, 1, \epsilon_1\epsilon_2\nu d_1, \ldots, \epsilon_1\epsilon_{\frac{p+1}{2}}\nu d_1, \epsilon_1\epsilon_{\frac{p+3}{2}} d_2, \ldots, \epsilon_1\epsilon_{\frac{p+q}{2}} d_2\right).$$

If the first rows are the categorical dimensions of the simple objects, that is, the first basis element e_1 is the unit object of $\mathcal{C}$, notice that

$$\dim(\mathcal{C}) < \sigma(\dim(\mathcal{C})) = pq\varepsilon_{pq}^2 \leq \mathrm{FPdim}(\mathcal{C}),$$

where $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{Q}(\sqrt{pq})/\mathbb{Q})$. Then the second row must be the Frobenius–Perron dimensions of the simple objects of $\mathcal{C}$ multiplied by the scalar $\nu\epsilon_1\varepsilon_{pq}^{-2}$. Since $\mathrm{FPdim}(X) > 0$, $X \in \mathcal{O}(\mathcal{C})$,

$$\nu\epsilon_1 = \nu\epsilon_{\frac{p+3}{2}} = \cdots = \nu\epsilon_{\frac{p+q}{2}} = 1, \epsilon_2 = \cdots = \epsilon_{\frac{p+1}{2}} = 1,$$

consequently, we obtain $\mathrm{FPdim}(\mathcal{C}) = pq\varepsilon_{pq}^2$ and

$$\mathrm{FPdim}(X) \in \left\{1, \varepsilon_{pq}^2, d_1', d_2'\right\}, \dim(X) \in \left\{1, \varepsilon_{pq}^{-2}, d_1, -d_2\right\}, \forall X \in \mathcal{O}(\mathcal{C}).$$

It is easy to see that the other formal codegrees of $\mathcal{C}$ are either $\frac{\dim(\mathcal{C})}{d_1^2} = p$ or $\frac{\dim(\mathcal{C})}{d_2^2} = q$, which cannot be the Frobenius–Perron dimension of $\mathcal{C}$ since $\mathcal{C}$ is not pointed; hence $\mathcal{C}$ is a Galois conjugate of a pseudo-unitary fusion category. Moreover, the modular data of $\mathcal{C}$ is

$$S = \begin{pmatrix} 1 & \varepsilon_{pq}^{-2} & d_1 & \cdots & d_1 & -d_2 & \cdots & -d_2 \\ \varepsilon_{pq}^{-2} & 1 & d_1 & \cdots & d_1 & d_2 & \cdots & d_2 \\ d_1 & d_1 & & & & & & \\ \vdots & \vdots & & 2d_1\cos\left(\frac{4\pi j_1 k_1}{p}\right) & & & \mathbf{0} & \\ d_1 & d_1 & & & & & & \\ -d_2 & d_2 & & & & & & \\ \vdots & \vdots & & \mathbf{0} & & & 2d_2\cos\left(\frac{4\pi j_2 k_2}{q}\right) & \\ -d_2 & d_2 & & & & & & \end{pmatrix},$$

$$T = \mathrm{diag}\left(1, 1, \zeta_p, \ldots, \zeta_p^{\left(\frac{p-1}{2}\right)^2}, \zeta_q, \ldots, \zeta_q^{\left(\frac{q-1}{2}\right)^2}\right),$$

where $1 \leq j_1, k_1 \leq \frac{p-1}{2}$ and $1 \leq j_2, k_2 \leq \frac{q-1}{2}$.

If the second row are the categorical dimensions of the simple objects, then e_2 is the unit object of $\mathcal{C}$ and the elements in the first row are the Frobenius–Perron dimensions of the simple objects multiplied by the scalar $\nu\epsilon_1\varepsilon_{pq}^{-2}$, similarly,

$$\nu\epsilon_1 = -\nu\epsilon_{\frac{p+3}{2}} = \cdots = -\nu\epsilon_{\frac{p+q}{2}} = 1, \epsilon_2 = \cdots = \epsilon_{\frac{p+1}{2}} = 1,$$

again we obtain

$$\text{FPdim}(X) \in \{1, \varepsilon_{pq}^2, d_1', d_2'\}, \dim(X) \in \{1, \varepsilon_{pq}^{-2}, d_1, -d_2\}, \forall X \in \mathcal{O}(\mathcal{C}).$$

Hence, $\text{FPdim}(\mathcal{C}) = pq\varepsilon_{pq}^2$. By using the same argument, we see that $\mathcal{C}$ is a Galois conjugate of a pseudo-unitary fusion category.

Case (2): $\overline{\mu}_p\mu_q = -1$. We can assume $a_1 = 1$ and a_2 is a non-square residue modulo q; the other case is the same. Then $\beta_p = \frac{1}{\sqrt{p}}$ and $\beta_q = \frac{-1}{\sqrt{q}}$ if $p = 4k+1$, $\beta_p = \frac{\zeta_4}{\sqrt{p}}$ and $\beta_q = \frac{-\zeta_4}{\sqrt{q}}$ if $p = 4k+3$; moreover, $\dim(\mathcal{C}) = pq\varepsilon_{pq}^2$. The first row of S is

$$\left(1, \nu\epsilon_1\varepsilon_{pq}^2, \epsilon_2 d_1', \ldots, \epsilon_{\frac{p+1}{2}} d_1', -\nu\epsilon_{\frac{p+3}{2}} d_2', \ldots, -\nu\epsilon_{\frac{p+q}{2}} d_2'\right),$$

and the second row of S is

$$\left(\nu\epsilon_1\varepsilon_{pq}^2, 1, \epsilon_1\epsilon_2\nu d_1', \ldots, \nu\epsilon_1\epsilon_{\frac{p+1}{2}} d_1', \epsilon_1\epsilon_{\frac{p+3}{2}} d_2', \ldots, \epsilon_1\epsilon_{\frac{p+q}{2}} d_2'\right).$$

Notice that $\dim(\mathcal{C}) = pq\varepsilon_{pq}^2$, $\dim(\mathcal{C})$ has a Galois conjugate $pq\varepsilon_{pq}^{-2} < \dim(\mathcal{C})$ and that the other formal codegrees of $\mathcal{C}$ are either p or q; hence $\text{FPdim}(X) = \dim(X)$ for all simple objects X of $\mathcal{C}$. Without loss of generality, we can take the elements in the first row to be the Frobenius–Perron dimensions of the simple objects of $\mathcal{C}$, then

$$-\nu\epsilon_{\frac{p+3}{2}} = \cdots = -\nu\epsilon_{\frac{p+q}{2}} = 1, \nu\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{\frac{p+1}{2}} = 1,$$

and $\text{FPdim}(X) \in \{1, \varepsilon_{pq}^2, d_1', d_2'\}$, $\forall X \in \mathcal{O}(\mathcal{C})$. In addition, up to isomorphism, we know that $\mathcal{C}$ contains $\frac{p-1}{2}$ simple objects of Frobenius–Perron dimension d_1' and $\frac{q-1}{2}$ simple objects of Frobenius–Perron dimension d_2', and a unique simple object X with $\text{FPdim}(X) = \varepsilon_{pq}^2$. Notice that the modular data of $\mathcal{C}$ is

$$S = \begin{pmatrix} 1 & \varepsilon_{pq}^2 & d_1' & \cdots & d_1' & d_2' & \cdots & d_2' \\ \varepsilon_{pq}^2 & 1 & d_1' & \cdots & d_1' & -d_2' & \cdots & -d_2' \\ d_1' & d_1' & & & & & & \\ \vdots & \vdots & & 2d_1'\cos\left(\frac{4\pi j_1 k_1}{p}\right) & & & \mathbf{0} & \\ d_1' & d_1' & & & & & & \\ d_2' & -d_2' & & & & & & \\ \vdots & \vdots & & \mathbf{0} & & & -2d_2'\cos\left(\frac{4\pi a_2 j_2 k_2}{q}\right) & \\ d_2' & -d_2' & & & & & & \end{pmatrix},$$

$$T = \text{diag}\left(1, 1, \zeta_p, \ldots, \zeta_p^{\left(\frac{p-1}{2}\right)^2}, \zeta_q^{a_2}, \ldots, \zeta_q^{a_2\left(\frac{q-1}{2}\right)^2}\right),$$

where $1 \le j_1, k_1 \le \frac{p-1}{2}$ and $1 \le j_2, k_2 \le \frac{q-1}{2}$.

Corollary 3.3. *Let $\mathcal{C}$ be a modular fusion category such that $\rho_{\mathcal{C}} \cong \rho_1 \oplus \rho_2$; then either $\mathcal{C}$ is a Galois conjugate of a pseudo-unitary fusion category or* $\dim(Y) = \text{FPdim}(Y)$ *for all simple objects Y of $\mathcal{C}$.*

Proposition 3.4. *Let $\mathcal{C}$ be a modular fusion category such that $\rho_\mathcal{C} \cong \rho_1 \oplus \rho_2$; then $\mathcal{C}$ must be a simple modular fusion category.*

Proof. On the contrary, assume that $\mathcal{C}$ contains a non-trivial fusion subcategory $\mathcal{D}$, which must be modular as $\mathcal{C}$ does not contain non-trivial simple objects of integer dimensions, hence $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{D}'_\mathcal{C}$ by [9, Theorem 8.21.4], where $\mathcal{D}'_\mathcal{C}$ is the centralizer of $\mathcal{D}$ in $\mathcal{C}$. In particular,

$$\mathrm{rank}(\mathcal{C}) = p + 3 = \mathrm{rank}(\mathcal{D})\mathrm{rank}(\mathcal{D}'_\mathcal{C}).$$

If $\dim(\mathcal{D})$ cannot be divided by p or q, then [26, Theorem 4.4] says that $\mathcal{D}$ is a non-trivial transitive subcategory in the sense of [20]. Assume $\mathrm{rank}(\mathcal{D}) = \frac{p-1}{2}$ with $p \geq 5$, so $\mathrm{rank}(\mathcal{D}'_\mathcal{C}) = 2 + \frac{8}{p-1}$, it is an integer if and only if $p = 5$; it is impossible as 9 is not a prime. Hence, both $\dim(\mathcal{D})$ and $\dim(\mathcal{D}'_\mathcal{C})$ are divided by some primes. Obviously, p or q cannot divide both $\dim(\mathcal{D})$ and $\dim(\mathcal{D}'_\mathcal{C})$, and we can assume $p \mid \dim(\mathcal{D})$ and $q \mid \dim(\mathcal{D}'_\mathcal{C})$; then $\dim(\mathcal{D}) = pu_1$ and $\dim(\mathcal{D}'_\mathcal{C}) = qu_2$, where u_j are non-trivial algebraic units. Therefore, $\mathrm{rank}(\mathcal{D}) = \frac{p+3}{2}$ and $\dim(\mathcal{D}'_\mathcal{C}) = \frac{q+3}{2}$ by [30, Theorem 3.13], which is a contradiction. ■

Let $\mathcal{C}$ be a braided fusion category. Recall that a commutative algebra A in $\mathcal{C}$ is said to be a connected étale algebra if the category $\mathcal{C}_A$ of right A-modules in $\mathcal{C}$ is semisimple and $\mathrm{Hom}_\mathcal{C}(I, A) = \mathbb{C}$ [5, Definition 3.1]. Let $(M, \mu_M) \in \mathcal{C}_A$, where $\mu_M : M \otimes A \to M$ is the right A-module morphism of M. Then M is a local (or dyslectic) module if $\mu_M = \mu_M \circ (c_{A,M} c_{M,A})$ [5,16], where c is the braiding of $\mathcal{C}$. The category of local modules over a connected étale algebra A is a braided fusion category, which will be denoted by $\mathcal{C}_A^0$ below.

Theorem 3.5. *Let $\mathcal{C}$ be a modular fusion category such that $\rho_\mathcal{C} \cong \rho_1 \oplus \rho_2$. If $\mathcal{C}$ contains a non-trivial connected étale algebra A, then $\mathcal{C}_A^0$ is a pointed modular fusion category of dimension p. In particular, $\mathcal{C}$ cannot be braided equivalent to the Drinfeld center of a fusion category.*

Proof. As we noticed in Corollary 3.3, we have $\dim(Y) = \mathrm{FPdim}(Y)$ for all objects Y of $\mathcal{C}$ or $\mathcal{C}$ is a Galois conjugate of a pseudo-unitary fusion category. After replacing $\mathcal{C}$ by its Galois conjugate (if necessary), we know that the Frobenius–Perron dimensions of the objects coincide with the categorical dimensions of the objects.

Let A be a non-trivial connected étale algebra of $\mathcal{C}$. In a pseudo-unitary fusion category, we know that any connected étale algebra has trivial twist [25, Lemma 2.2.4]. Meanwhile, the modular fusion category $\mathcal{C}$ contains only two simple objects $\{I, X\}$ (up to isomorphism) with trivial twisting, and the categorical dimension of X is ε_{pq}^2. Therefore, $A = I \oplus nX$ for some $n \geq 1$. Since $\dim(\mathcal{C}) = pq\varepsilon_{pq}^2$ and $\dim(A) = 1 + n\varepsilon_{pq}^2$, so $\frac{\dim(\mathcal{C})}{\dim(A)^2}$ is an algebraic integer. Notice that

$$N\left(\frac{\dim(\mathcal{C})}{\dim(A)^2}\right) = \frac{p^2q^2}{\left(n^2 + 1 + n\frac{p+q}{2}\right)^2},$$

hence $1 + n^2 + n\frac{p+q}{2} = q$ as $\frac{p+q}{2} > p$. Then $n \leq 1$; otherwise $n\frac{p+q}{2} \geq q$; it is impossible.

Thus, $A = I \oplus X$, and [5, Remark 3.4] states that it is a $\mathcal{C}$-rigid algebra in the sense of [16]. Then it follows from [16, Theorem 4.5] that $\mathcal{C}_A^0$ is a modular fusion category and

$$\dim(\mathcal{C}_A^0) = \frac{\dim(\mathcal{C})}{\dim(A)^2} = \frac{pq\varepsilon_{pq}^2}{(1+\varepsilon_{pq}^2)^2} = p,$$

which must be pointed by [26, Theorem 5.12]. Moreover,

$$\mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\mathrm{rev}} \cong \mathcal{Z}(\mathcal{C}_A)$$

as modular fusion categories [5, Corollary 3.30], where $(\mathcal{C}_A^0)^{\mathrm{rev}} = \mathcal{C}_A^0$ as a fusion category but with reverse braiding [9]. Thus [5, Lemma 5.9] says that $\mathcal{C}$ is Witt equivalent to $\mathcal{C}(\mathbb{Z}_p, \eta)$, whose Witt equivalence class is non-trivial, so $\mathcal{C}$ cannot be braided tensor equivalent to the Drinfeld center of any spherical fusion category by [5, Proposition 5.8]. ∎

Remark 3.6. As we all know, there is a conformal embedding $G_{2,3} \subseteq E_{6,1}$ [5, Appendix], so the modular fusion category $\mathcal{C}(\mathfrak{g}_2, 3)$ contains a non-trivial étale algebra A such that there is a braided equivalence $\mathcal{C}(\mathfrak{g}_2, 3)_A^0 \cong \mathcal{C}(\mathfrak{e}_6, 1)$, which is braided equivalent to $\mathcal{C}(\mathbb{Z}_3, \eta)$ [4, Proposition A.4.1]. Note $\dim(A) = \frac{7+\sqrt{21}}{2} = 1 + \varepsilon_{21}^2$; hence $A = I \oplus X$ by Theorem 3.5.

However, when $p > 3$, we do not know currently whether there always exists an étale algebra structure on the object $I \oplus X$. We believe the answer is positive.

Remark 3.7. Let $\mathcal{I} : \mathcal{C}_A \to \mathcal{Z}(\mathcal{C}_A)$ be the right adjoint functor to the forgetful functor $F : \mathcal{Z}(\mathcal{C}_A) \to \mathcal{C}_A$. Then all simple direct summands of $\mathcal{I}(I)$ have trivial twists by [19, Theorem 4.1]. Let Z_j $(1 \le j \le \frac{p-1}{2})$ be the simple objects of $\mathcal{C}$ such that $\mathrm{FPdim}(Z_j) = \frac{\sqrt{q}(\sqrt{p}+\sqrt{q})}{2}$, then θ_{Z_j} are primitive p-th roots of unity. Let g be a generator of $\mathbb{Z}_p$. Then

$$\theta_{Z_j}^{-1} = \theta_{g^{k_j}} = \theta_{g^{-k_j}}$$

for a unique k_j with $1 \le k_j \le \frac{p-1}{2}$. Hence, up to isomorphism, $\mathcal{Z}(\mathcal{C}_A) = \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\mathrm{rev}}$ has exactly $p+1$ simple objects with trivial twists, which are

$$\left\{ I \boxtimes I, X \boxtimes I, Z_j \boxtimes g^{k_j}, Z_j \boxtimes g^{-k_j} \,\middle|\, 1 \le j \le \frac{p-1}{2} \right\}.$$

Indeed, in the next subsection, we will show that the Grothendieck ring $\mathrm{Gr}(\mathcal{C}_A)$ is commutative (see Theorem 3.8); therefore, $\mathcal{I}(I)$ must be multiplicity-free by [23, Corollary 2.16], and these objects are exactly the direct summands of $\mathcal{I}(I)$.

3.2. The structure of the fusion category $\mathcal{C}_A$

In this subsection, we show that the category $\mathcal{C}_A$ obtained in Theorem 3.5 is a near-group fusion category of type $(\mathbb{Z}_p, p)$.

Let G be a finite group, $\mathbb{Z}_+ := \mathbb{Z}_{\ge 0}$ and $n \in \mathbb{Z}_+$. Recall that a fusion ring R with $\mathbb{Z}_+$-basis $\{g \mid g \in G\} \cup \{X\}$ is called a near-group fusion ring of type (G, n) [28] if

$$gX = Xg = X, \ XX = \sum_{g \in G} g + nX.$$

When $n=0$, it is well known that R is categorifiable if and only if G is an abelian group, the corresponding fusion categories are called Tambara–Yamagami fusion categories, which are completely classified in [29]. We denote these fusion categories by $\mathcal{TY}(G,\tau,\mu)$, where τ is a non-degenerate bi-character on G and μ is a square root of $|G|^{-1}$.

Theorem 3.8. *$\mathcal{C}_A$ is a near-group fusion category of type $(\mathbb{Z}_p, p)$.*

Proof. As we have a braided tensor equivalence $\mathcal{C}\boxtimes(\mathcal{C}_A^0)^{\mathrm{rev}}\cong\mathcal{Z}(\mathcal{C}_A)$ by Theorem 3.5, then $\dim(\mathcal{C}_A)=p\sqrt{q}\varepsilon_{pq}=\frac{p(\sqrt{pq}+q)}{2}$, whose Galois conjugate is $\frac{p(-\sqrt{pq}+q)}{2}$. It was proved that fusion category $\mathcal{C}_A$ is faithfully graded by the following Galois group

$$\mathrm{Gal}(\mathbb{Q}(\mathrm{FPdim}(Y):Y\in\mathcal{O}(\mathcal{C}_A))/\mathbb{Q}(\mathrm{FPdim}(\mathcal{C}_A))),$$

which is an elementary abelian 2-group [13, Proposition 1.8], so the order of the Galois group is a factor of $\mathrm{FPdim}(\mathcal{C}_A)$ by [9, Theorem 3.5.2]. Since $2\nmid\mathrm{FPdim}(\mathcal{C}_A)$, we see

$$\mathbb{Q}(\mathrm{FPdim}(Y):Y\in\mathcal{O}(\mathcal{C}_A))=\mathbb{Q}(\sqrt{pq})=\mathbb{Q}(\varepsilon_{pq}^2).$$

Notice that [10, Proposition 8.15] says the ratio $\frac{\mathrm{FPdim}(\mathcal{C}_A)}{\mathrm{FPdim}((\mathcal{C}_A)_{\mathrm{int}})}$ is an algebraic integer, where $(\mathcal{C}_A)_{\mathrm{int}}$ is the maximal integral fusion subcategory of $\mathcal{C}_A$, so the only prime factor of $\mathrm{FPdim}((\mathcal{C}_A)_{\mathrm{int}})$ is p, as $\mathcal{C}_A^0$ is pointed by Theorem 3.5. Hence, $(\mathcal{C}_A)_{\mathrm{int}}=\mathcal{C}_A^0$.

Let Z be an arbitrary non-invertible simple object of $\mathcal{C}_A$ such that $\mathrm{FPdim}(Z)=\frac{a+b\sqrt{pq}}{2}$, which is an algebraic integer, where a and b are rational with $b\neq 0$. Then the minimal polynomial of $\mathrm{FPdim}(X)$ is

$$x^2-(\mathrm{FPdim}(Z)+\sigma(\mathrm{FPdim}(Z)))x+\mathrm{FPdim}(Z)\sigma(\mathrm{FPdim}(Z)),$$

where $\sigma(\sqrt{pq})=-\sqrt{pq}$. Note that $\mathrm{FPdim}(Z)+\sigma(\mathrm{FPdim}(Z))=a\in\mathbb{Q}$, so a is an integer. Furthermore, $m:=\mathrm{FPdim}(Z)\sigma(\mathrm{FPdim}(Z))=\frac{a^2-b^2pq}{4}$ is also an integer, then $b^2pq=a^2-4m\in\mathbb{Z}$. Assume $b=\frac{r}{s}$ where $(r,s)=1$, notice that $(pq,s)=1$; otherwise p or q is a factor of (r,s); it is a contradiction. So $b\in\mathbb{Z}$.

Then $\mathrm{FPdim}(Z)^2=\frac{\frac{a^2+b^2pq}{2}+ab\sqrt{pq}}{2}$, while

$$\mathrm{FPdim}(\mathcal{C}_A)=\frac{p(q+\sqrt{pq})}{2}=\sum_{Y\in\mathcal{O}(\mathcal{C}_A)}\mathrm{FPdim}(Y)^2\geq\mathrm{FPdim}(\mathcal{C}_A^0)+\mathrm{FPdim}(Z)^2,$$

so $\mathrm{FPdim}(Z)^2=\frac{\frac{a^2+b^2pq}{2}+ab\sqrt{pq}}{2}\leq\frac{p(q-2)+p\sqrt{pq}}{2}$, by comparing the rational and irrational parts, we obtain that $b^2\leq 1$; consequently $b=1$ ($b\neq -1$, otherwise $\mathrm{FPdim}(Z)$ has a Galois conjugate whose absolute value is strictly larger than $\mathrm{FPdim}(X)$, which is impossible [9, Theorem 3.2.1]). Therefore, up to isomorphism, $\mathcal{C}_A$ has exactly one non-invertible simple object Z. Since

$$\mathrm{FPdim}(\mathcal{C}_A)=p\sqrt{q}\varepsilon_{pq}=p+\left(\frac{p+\sqrt{pq}}{2}\right)^2,$$

$\mathrm{FPdim}(Z) = \frac{p+\sqrt{pq}}{2}$. By comparing the Frobenius–Perron dimensions of the simple objects, we see

$$Z \otimes Z = \bigoplus_{g \in \mathbb{Z}_p} g \oplus pZ,$$

i.e., $\mathcal{C}$ is a near-group fusion category of type $(\mathbb{Z}_p, p)$. ■

Remark 3.9. It is worth noting that the categorifications of near-group fusion rings were characterized with complicated linear and non-linear equations by using Cuntz algebra theory; see [15] and the references therein for details. Conclusions from [12, 15] suggest that there may exist an infinite family of near-group fusion categories of type $(G, |G|)$, where G is an abelian group. However, in order to show that such a near-group fusion category exists, one needs to solve these equations, which is a non-trivial task; see [15, Appendix A] for solutions for groups of small orders. With the help of computers, when $|G| \leq 13$, the answer is affirmative [12, Proposition 6], and recently this result is improved for cyclic groups of order less than 31 in [3].

Moreover, for an arbitrary abelian group G of odd order, let $\mathcal{A}$ be a near-group fusion category of type $(G, |G|)$; it was conjectured in [12, Conjecture 2] that

$$\mathcal{Z}(\mathcal{A}) \cong \mathcal{C} \boxtimes \mathcal{C}(G, \eta_1)$$

as a modular fusion category, we refer the reader to [12, Proposition 7] and [15, Theorem 6.8] for a detailed description of the modular data of $\mathcal{C}$.

Notice that $\mathcal{A}$ contains a unique non-trivial fusion subcategory $\mathrm{Vec}_{\mathbb{Z}_p}$, so $\mathcal{I}(I)$ contains a unique non-trivial étale subalgebra A such that $\mathcal{Z}(\mathcal{A})_A^0 \cong \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_p})$ as a braided fusion category and $\mathrm{FPdim}(A) = \frac{\dim(\mathcal{A})}{p} = \sqrt{q}\varepsilon_{pq}$ by [5, Theorem 4.10]. By comparing the Frobenius–Perron dimensions of the simple objects, we know $A = I \oplus X$, see Remark 3.6.

It was also conjectured in [12] that the modular data of $\mathcal{Z}(\mathcal{A})$ is determined by metric groups (G, η_1) and (H, η_2), where H is an abelian group of order $|G| + 4$. Indeed, if we require $\alpha = \beta = 1$, where α and β are the parameters in [12, Proposition 7], it is easy to see that the modular data $\mathcal{MD}_{G,H}(\eta_1, \eta_2)$ of [12] is exactly that of $\mathcal{C}$ in the pseudo-unitary situation. Hence, under the assumption that $\mathcal{C}$ contains a non-trivial étale algebra, Theorem 3.5 gives a partial positive answer to [12, Conjecture 2] and provides solutions to the conjectured modular data of $\mathcal{C}$, and our result suggests that the conjecture might be true.

Based on conclusions of the categorification of near-group fusion rings, we propose the following conjecture, and we believe there is an affirmative answer.

Conjecture 3.10. Let p, q, ρ_1, and ρ_2 be the notations as before. Then there exists a modular fusion category $\mathcal{C}$ such that $\rho_{\mathcal{C}} \cong \rho_1 \oplus \rho_2$ if and only if $q - p = 4$.

3.3. A faithful $\mathbb{Z}_2$-extension of $\mathcal{C}_A$

In this subsection, we provide a faithful $\mathbb{Z}_2$-extension $\mathcal{M}$ of the near-group fusion category $\mathcal{C}_A$. In particular, we prove that $\mathcal{M}$ contains simple objects of Frobenius–Perron dimension $\frac{\sqrt{p}+\sqrt{q}}{2}$. In the last part of this subsection, we construct a class of non-commutative fusion rings that are non-trivial $\mathbb{Z}_2$-extensions of near-group fusion rings of type $(\mathbb{Z}_n, n)$ for all $n \geq 1$.

For any odd prime p, note that there is a modular fusion category of Frobenius–Perron dimension $4p$, which is braided tensor equivalent to a $\mathbb{Z}_2$-equivariantization of a Tambara–Yamagami fusion category $\mathcal{TY}(\mathbb{Z}_p, \tau, \mu)$ [14, Proposition 5.1]. We refer the reader to [7,9] for the definition and properties of equivariantization and de-equivariantization of fusion categories by finite groups. Moreover, the modular data of $\mathcal{TY}(\mathbb{Z}_p, \tau, \mu)^{\mathbb{Z}_2}$ is given in [14, Example 5D] explicitly. In particular, $\mathcal{D}$ contains a Tannakian fusion subcategory $\mathrm{Rep}(\mathbb{Z}_2)$ and two simple objects of Frobenius–Perron dimension $\sqrt{p}$.

Let ρ' be a three-dimensional irreducible congruence representation of SL(2, $\mathbb{Z}$) of level 4 with

$$\rho'(\mathfrak{s}) = \mu_p \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{-1}{2} \end{pmatrix}, \rho'(\mathfrak{t}) = \mathrm{diag}(1, \xi_1, -\xi_1),$$

where $\beta_p = \mu_p \frac{1}{\sqrt{p}}$, ξ_1 is a square root of the central charge ξ (or $-\xi$) of $\mathcal{C}(\mathbb{Z}_p, \eta)$ [14].

Proposition 3.11. *Let $p \geq 3$ be an odd prime, and let ρ_1 be an irreducible representation of dimension $\frac{p+1}{2}$ of $SL(2, \mathbb{Z}_p)$. If $\rho_{\mathcal{C}} \cong \rho_1 \oplus \rho'$, then $\mathcal{C}$ is braided equivalent to a $\mathbb{Z}_2$-equivariantization of $\mathcal{TY}(\mathbb{Z}_p, \tau, \mu)$.*

Proof (sketched). Since ρ_1 and ρ' are non-degenerate, it follows from [18, Theorem 3.23] (see also Theorem 3.2) that

$$\rho_{\mathcal{C}}(\mathfrak{s}) = \begin{pmatrix} \frac{1}{2}\beta_p & \frac{\nu\epsilon_1}{2}\beta_p & \epsilon_2\beta_p & \cdots & \epsilon_{\frac{p+1}{2}}\beta_p & -\epsilon_{\frac{p+3}{2}}\frac{\nu\mu_p}{2} & -\epsilon_{\frac{p+5}{2}}\frac{\nu\mu_p}{2} \\ \frac{\nu\epsilon_1}{2}\beta_p & \frac{1}{2}\beta_p & \epsilon_1\epsilon_2\beta_p\nu & \cdots & \epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu & \epsilon_1\epsilon_{\frac{p+3}{2}}\frac{\nu\mu_p}{2} & \epsilon_1\epsilon_{\frac{p+5}{2}}\frac{\nu\mu_p}{2} \\ \epsilon_2\beta_p & \epsilon_1\epsilon_2\beta_p\nu & & & & & \\ \vdots & \vdots & & & & & \\ \epsilon_{\frac{p+1}{2}}\beta_p & \epsilon_1\epsilon_{\frac{p+1}{2}}\beta_p\nu & & V_1 D_1 V_1 & & \mathbf{0} & \\ \epsilon_{\frac{p+1}{2}}\nu\beta_p & \epsilon_1\epsilon_{\frac{p+1}{2}}\nu\beta_p & & & & & \\ -\epsilon_{\frac{p+3}{2}}\frac{\nu\mu_p}{2} & \epsilon_1\epsilon_{\frac{p+3}{2}}\frac{\nu\mu_p}{2} & & \mathbf{0} & & V_2 D_3 V_2 & \\ -\epsilon_{\frac{p+5}{2}}\frac{\nu\mu_p}{2} & \epsilon_1\epsilon_{\frac{p+5}{2}}\frac{\nu\mu_p}{2} & & & & & \end{pmatrix},$$

$$\rho_{\mathcal{C}}(\mathfrak{t}) = \mathrm{diag}\left(1, 1, \zeta_p^a, \ldots, \zeta_p^{a\left(\frac{p-1}{2}\right)^2}, \xi_1, -\xi_1\right),$$

where $\nu, \epsilon_1, \ldots, \epsilon_{\frac{p+5}{2}} \in \{\pm 1\}$, and

$$V_1 = \begin{pmatrix} \epsilon_2 & & \\ & \ddots & \\ & & \epsilon_{\frac{p+1}{2}} \end{pmatrix}, V_2 = \begin{pmatrix} \epsilon_{\frac{p+3}{2}} & \\ & \epsilon_{\frac{p+5}{2}} \end{pmatrix}, D_3 = \mu_p \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}.$$

In the same way as Theorem 3.2, if we identify the $\mathcal{O}(\mathcal{C})$ with the standard basis of the vector space, then

$$S = \begin{pmatrix} 1 & \nu\epsilon_1 & 2\epsilon_2 & \cdots & 2\epsilon_{\frac{p+1}{2}} & -\epsilon_{\frac{p+3}{2}}\nu\sqrt{p} & -\epsilon_{\frac{p+5}{2}}\nu\sqrt{p} \\ \nu\epsilon_1 & 1 & 2\epsilon_1\epsilon_2\nu & \cdots & 2\epsilon_1\epsilon_{\frac{p+1}{2}}\nu & \epsilon_1\epsilon_{\frac{p+3}{2}}\nu\sqrt{p} & \epsilon_1\epsilon_{\frac{p+5}{2}}\nu\sqrt{p} \\ 2\epsilon_2 & 2\epsilon_1\epsilon_2\nu & & & & & \\ \vdots & \vdots & & 4\epsilon_j\epsilon_k\cos\left(\frac{4\pi a jk}{p}\right) & & & \mathbf{0} \\ 2\epsilon_{\frac{p+1}{2}} & 2\epsilon_1\epsilon_{\frac{p+1}{2}}\nu & & & & & \\ -\epsilon_{\frac{p+3}{2}}\nu\sqrt{p} & \epsilon_1\epsilon_{\frac{p+3}{2}}\nu\sqrt{p} & & & & -\sqrt{p} & \epsilon_{\frac{p+3}{2}}\epsilon_{\frac{p+5}{2}}\sqrt{p} \\ -\epsilon_{\frac{p+5}{2}}\nu\sqrt{p} & \epsilon_1\epsilon_{\frac{p+5}{2}}\nu\sqrt{p} & & \mathbf{0} & & \epsilon_{\frac{p+3}{2}}\epsilon_{\frac{p+5}{2}}\sqrt{p} & -\sqrt{p} \end{pmatrix},$$

we know $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}) = 4p$, so $\mathcal{C}$ is pseudo-unitary. If the first row are Frobenius–Perron dimensions of the simple objects, then

$$\epsilon_{\frac{p+3}{2}} = \epsilon_{\frac{p+5}{2}} = -1, \nu = \epsilon_1 = \epsilon_2 = \cdots = \epsilon_{\frac{p+1}{2}} = 1;$$

if the second row are Frobenius–Perron dimensions of the simple objects, then

$$\nu = \epsilon_1 = \epsilon_2 = \cdots = \epsilon_{\frac{p+3}{2}} = \epsilon_{\frac{p+5}{2}} = 1.$$

From both cases, we know that $\mathcal{C}$ always contains a non-trivial Tannakian fusion subcategory $\mathrm{Rep}(\mathbb{Z}_2)$; hence its core $\mathcal{C}^0_{\mathbb{Z}_2}$ is a pointed modular fusion category of Frobenius–Perron dimension p [5, Corollary 3.32]. Since $\mathcal{C}$ is not integral, $\mathcal{C}_{\mathbb{Z}_2}$ must be a Tambara–Yamagami fusion category $\mathcal{TY}(\mathbb{Z}_p, \tau, \mu)$; hence $\mathcal{C} \cong \mathcal{TY}(\mathbb{Z}_p, \tau, \mu)^{\mathbb{Z}_2}$ [7, 14], as desired. ∎

We note that there exists a modular fusion category $\mathcal{C}$, which is also obtained from $\mathbb{Z}_2$-equivariantization of a Tambara–Yamagami fusion category of dimension $2p$, but $\rho_\mathcal{C} \ncong \rho_1 \oplus \rho'$; when $p = 5$, see [18, Theorem 4.15] for details.

Theorem 3.12. *There is a fusion category $\mathcal{M}$, which is a faithful $\mathbb{Z}_2$-extension of $\mathcal{C}_A$ and a non-degenerate fusion category $\mathcal{D}$ such that $\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{Z}(\mathcal{M})$; moreover, $\mathcal{M}$ contains exactly p simple objects of Frobenius–Perron dimension ε_{pq}.*

Proof. Indeed, let $\mathcal{D} = \mathcal{TY}(\mathbb{Z}_p, \tau, \mu)^{\mathbb{Z}_2}$ such that

$$\mathcal{D}^0_{\mathbb{Z}_2} \cong \mathcal{C}(\mathbb{Z}_p, \eta^{-1}) \cong \mathcal{C}(\mathbb{Z}_p, \eta)^{\mathrm{rev}},$$

where $\eta^{-1}(g) := \eta(g)^{-1}$ for all $g \in \mathbb{Z}_p$. Consequently, we have braided equivalences

$$(\mathcal{C} \boxtimes \mathcal{D})^0_{\mathbb{Z}_2} \cong \mathcal{C} \boxtimes \mathcal{D}^0_{\mathbb{Z}_2} \cong \mathcal{C} \boxtimes \mathcal{C}(\mathbb{Z}_p, \eta^{-1}) \cong \mathcal{Z}(\mathcal{C}_A),$$

by Theorem 3.5, so $\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{Z}(\mathcal{M})$ with fusion category $\mathcal{M}$ being a faithful $\mathbb{Z}_2$-extension of the near-group fusion category $\mathcal{C}_A$ by [11, Theorem 1.3].

Let $\mathcal{M} = \bigoplus_{h\in\mathbb{Z}_2} \mathcal{M}_h$ with $\mathcal{M}_e = \mathcal{C}_A$. Since $\mathcal{Z}(\mathcal{M})$ contains a simple object of Frobenius–Perron dimension $\sqrt{p}$, $\mathcal{M}$ contains an object M of Frobenius–Perron dimension $\sqrt{p}$. We claim that $M \in \mathcal{M}_h$. Indeed, assume $M = M_1 \oplus \mathcal{M}_2$ with $M_1 \in \mathcal{M}_e$ and $M_2 \in \mathcal{M}_h$, respectively, then $\mathrm{FPdim}(M_i) \in \mathbb{Q}(\sqrt{p})$ by [13, Lemma 1.1]. Meanwhile, $\mathrm{FPdim}(Z) \in \mathbb{Q}(\sqrt{pq})$ for all simple objects Z of $\mathcal{C}_A$, so $\mathrm{FPdim}(M_1)$ must be an integer and $\mathrm{FPdim}(M_2) = \sqrt{p} - \mathrm{FPdim}(M_1)$. If M_1 is a non-zero object, then $\mathrm{FPdim}(M_1) \geq 1$, which implies $\mathrm{FPdim}(M_2)$ admits a Galois conjugate whose absolute value is strictly larger than $\mathrm{FPdim}(M_2)$, it is impossible by [9, Theorem 3.2.1]. Hence, $M = M_2 \in \mathcal{M}_h$, as claimed.

Since $\mathcal{M}$ is $\mathbb{Z}_2$-graded, $M \otimes M \in \mathcal{M}_e$. Notice that $M \otimes M$ must be a direct sum of integral simple objects of $\mathcal{M}_e$, so $M \otimes M = \bigoplus_{g\in\mathbb{Z}_p} g$. Hence, M is simple and self-dual. Let $\mathcal{B}$ and $\mathcal{M}_{\mathrm{int}}$ be the maximal weakly integral and integral fusion subcategories of $\mathcal{M}$, respectively, then $\mathcal{B}$ is faithfully graded by an elementary abelian 2-group G with $\mathcal{M}_{\mathrm{int}}$ being the trivial component [9, Proposition 3.5.7]. Therefore, $\mathrm{FPdim}(\mathcal{B}) = p|G|$ by [9, Theorem 3.5.2], and $\mathrm{FPdim}(\mathcal{B})$ is a factor of $\mathrm{FPdim}(\mathcal{M})$ [10, Proposition 8.15], so $G = \mathbb{Z}_2$. In particular, $\mathcal{M}$ has a unique simple object M of Frobenius–Perron dimension $\sqrt{p}$.

Let $Y \in \mathcal{O}(\mathcal{M}_h)$ be an arbitrary simple object satisfying $Y \ncong M$, then $M \otimes Y \in \mathcal{M}_e$. Obviously, g cannot be a direct summand of $M \otimes Y$ for all invertible objects g of $\mathcal{M}_e$. Therefore, there exists a positive integer n_Y such that $M \otimes Y = n_Y X$, then

$$\mathrm{FPdim}(Y) = \frac{n_Y \mathrm{FPdim}(X)}{\mathrm{FPdim}(M)} = \frac{n_Y \sqrt{p}\varepsilon_{pq}}{\sqrt{p}} = n_Y \varepsilon_{pq}.$$

Notice that $Y \otimes Y^* \in \mathcal{M}_e$, so $Y \otimes Y^*$ is a direct sum of simple objects of $\mathcal{M}_e$. If

$$Y \otimes Y^* = \bigoplus_{g\in\mathbb{Z}_p} g \oplus m_Y X$$

for some positive integer m_Y, as $q = p + 4$, then

$$\begin{aligned}\mathrm{FPdim}(Y)^2 = n_Y^2 \varepsilon_{pq}^2 &= \frac{(p+2)n_Y^2 + n_Y^2\sqrt{pq}}{2} \\ &= p + m_Y\sqrt{p}\varepsilon_{pq} = \frac{(2+m_Y)p + m_Y\sqrt{pq}}{2}.\end{aligned}$$

By comparing the rational and irrational parts of the above equation, we obtain $n_Y^2 = m_Y$ and $p = n_Y^2$, which is absurd. Therefore, $Y \otimes Y^* = I \oplus m_Y X$; then the previous argument also implies $m_Y = n_Y = 1$. In particular, for any non-trivial invertible object g, we have $g \otimes Y \ncong Y$; hence the $\mathbb{Z}_2$-grading of $\mathcal{M}$ induces a transitive action of $\mathbb{Z}_p$ on $\mathcal{O}(\mathcal{M}_h)$. Up to isomorphism, $\mathcal{M}_h$ contains at least p non-isomorphic simple objects $\{Y_j\}_{j=1}^p$ of Frobenius–Perron dimension ε_{pq} and a unique simple object of Frobenius–Perron dimension $\sqrt{p}$. Then

$$\begin{aligned}\mathrm{FPdim}(\mathcal{M}_e) = \mathrm{FPdim}(\mathcal{M}_h) &\geq p\mathrm{FPdim}(Y_j)^2 + \mathrm{FPdim}(M)^2 \\ &= p\varepsilon_{pq}^2 + p = \mathrm{FPdim}(\mathcal{M}_e),\end{aligned}$$

thus $\mathcal{O}(\mathcal{M}_h) = \{M\} \cup \{Y_j \mid 1 \leq j \leq p\}$. ■

Corollary 3.13. *Let $\mathcal{M}$ be the $\mathbb{Z}_2$-extension of $\mathcal{C}_A$, and let Y be an arbitrary simple object of Frobenius–Perron dimension ε_{pq}. Then the fusion rules of $\mathcal{M}$ are given by the following relations*

$$X \otimes X = \bigoplus_{g \in \mathbb{Z}_p} g \oplus pX, g^i \otimes g^j = g^{i+j}, g \otimes X = X \otimes g = X,$$

$$M \otimes M = \bigoplus_{g \in \mathbb{Z}_p} g, g^j Y := g^j \otimes Y = Y \otimes g^{p-j}, M \otimes g^j Y = X = g^j Y \otimes M,$$

$$X \otimes M = M \otimes X = \bigoplus_{j=1}^{p} g^j Y, X \otimes g^j Y = g^j Y \otimes X = M \oplus \bigoplus_{j=1}^{p} g^j Y,$$

$$g^j Y \otimes g^k Y = g^{j+p-k} \oplus X.$$

In particular, non-invertible simple objects of $\mathcal{M}$ are self-dual.

Proof. Let Y be a simple object of $\mathcal{M}$ of Frobenius–Perron dimension ε_{pq}. As $\mathcal{O}(\mathcal{M}_h)$ contains p simple objects of same Frobenius–Perron dimension, without loss of generality, we can choose Y to be self-dual, and it follows from Theorem 3.12 that $Y \otimes Y = I \oplus X$.

Let g be a non-invertible simple object. Then there exists a unique $1 \leq k \leq p-1$ such that $g \otimes Y \cong Y \otimes g^k$. Consequently,

$$\begin{aligned}\mathbb{C} = \mathrm{Hom}_{\mathcal{M}}(g \otimes Y, Y \otimes g^k) &\cong \mathrm{Hom}_{\mathcal{M}}(g, Y \otimes g^k \otimes Y) \\ &\cong \mathrm{Hom}_{\mathcal{M}}(g, Y \otimes Y \otimes g^{k^2}) = \mathrm{Hom}_{\mathcal{M}}(g, g^{k^2}),\end{aligned}$$

which means $k^2 \equiv 1 \bmod p$; then $k = 1, p-1$.

If $g \otimes Y = Y \otimes g$, then $g^j Y := g^j \otimes Y = Y \otimes g^j$ for all $1 \leq j \leq p$. As $X \otimes g^j = X$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}}(X \otimes g^j Y, g^k Y) &\cong \mathrm{Hom}_{\mathcal{M}}(X \otimes Y, g^k Y) \\ &\cong \mathrm{Hom}(X, g^k Y \otimes Y) \\ &\cong \mathrm{Hom}_{\mathcal{M}}(X, g^k \oplus X) = \mathbb{C}\end{aligned}$$

for all $1 \leq j, k \leq p$, we see $\bigoplus_{k=1}^p g^k Y \subseteq X \otimes g^j Y$. By computing the Frobenius–Perron dimension of $X \otimes g^j Y$ and its simple summands, we obtain

$$X \otimes g^j Y = M \oplus \bigoplus_{k=1}^{p} g^k Y,$$

which also implies the following relations

$$M \otimes X = \bigoplus_{j=1}^{p} g^j Y, \ M \otimes g^j Y = X.$$

Similarly, we have $M \otimes X = X \otimes X$ and $M \otimes g^j Y = g^j Y \otimes M$. Particularly, Gr$(\mathcal{M})$ is commutative. However, it follows from [18, Theorem 3.23 (iii)] and [14] that both $\mathcal{C}$ and $\mathcal{D}$ are self-dual modular fusion categories. Then the algebra homomorphism

$$\mathrm{Gr}(\mathcal{Z}(\mathcal{M})) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Gr}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective by [9, Lemma 9.3.10], so simple objects of $\mathcal{M}$ are self-dual, which is a contradiction. Hence, Gr$(\mathcal{M})$ cannot be commutative, so $g \otimes Y \cong Y \otimes g^{p-1}$; more generally, $g^j \otimes Y \cong Y \otimes g^{p-j}$ for all $1 \le j \le p-1$. Thus, for all $1 \le j, k \le p-1$, we obtain

$$(g^j \otimes Y) \otimes (g^k \otimes Y) = g^j \otimes g^{p-k} \otimes Y \otimes Y = g^{j+p-k} \oplus X.$$

In particular, $g^j Y$ is self-dual for all $1 \le j \le p$. Note that we still have

$$\mathrm{Hom}_{\mathcal{M}}(X \otimes g^j Y, g^k Y) \cong \mathrm{Hom}_{\mathcal{M}}(X \otimes Y, g^k Y) \cong \mathrm{Hom}_{\mathcal{M}}(X, g^{k+1} \oplus X) = \mathbb{C}$$

for all $1 \le j, k \le p$; then the fusion relations can be obtained in the same way. ■

Remark 3.14. When $p = 3$ and $q = 7$, the fusion category $\mathcal{M}$ is exactly the fusion category $\mathcal{V}$ constructed by Ostrik in [4, Proposition A.6.1].

It is easy to see that one can construct a fusion ring that is a $\mathbb{Z}_2$-extension of an arbitrary near-group fusion ring of type $(G, k|G|)$, where G is abelian and k is a non-negative integer. However, for some non-cyclic abelian groups G, the corresponding near-group fusion rings of type $(G, |G|)$ are not categorifiable; one can take $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [15, Proposition A.1] and [27, Theorem 1.1], for example, in these cases it is meaningless to consider the categorification of their extensions.

Hence, in the following definition, we only list the corresponding fusion ring, which contains a near-group fusion ring of type $(\mathbb{Z}_n, n)$.

Definition 3.15. Let R_0 be a near-group fusion ring of type $(\mathbb{Z}_n, n)$ determined by the cyclic group $\mathbb{Z}_n = \langle g \rangle$ and relations

$$g^j g^l = g^{j+l}, \quad g^j X = X g^j = X, \quad XX = \sum_{g \in \mathbb{Z}_n} g + nX.$$

Let $R \supseteq R_0$ be a fusion ring with $\mathbb{Z}_+$-basis $\{Y_j, g^j \mid 1 \le j \le n\} \cup \{M, X\}$ and the following fusion relations:

$$MM = \sum_{j=1}^{n} g^j, Y_j Y_l = g^{j+n-l} + X, g^i Y_j = Y_k = Y_j g^{n-i} (\text{where } i + j \equiv k \bmod n),$$

$$Y_j X = X Y_j = M + \sum_{l=1}^{n} Y_l, M Y_j = Y_j M = X, MX = XM = \sum_{j=1}^{n} Y_j.$$

A direct computation shows

$$\mathrm{FPdim}(X)=\frac{n+\sqrt{n^2+4n}}{2},\ \mathrm{FPdim}(M)=\sqrt{n},\ \mathrm{FPdim}(Y_j)=\frac{\sqrt{n}+\sqrt{n+4}}{2},$$

for all $1\leq j\leq n$. Then we obtain

$$\mathrm{FPdim}(R_0)=\frac{n^2+4n+n\sqrt{n^2+4n}}{2}, \mathrm{FPdim}(R)=n^2+4n+n\sqrt{n^2+4n}.$$

Hence [9, Proposition 3.5.3] says that R is a faithful $\mathbb{Z}_2$-extension of R_0. Also notice that R contains a fusion ring (generated by M) of Frobenius–Perron dimension $2n$, which is categorified as a Tambara–Yamagami fusion category $\mathcal{TY}(\mathbb{Z}_n,\tau,\mu)$.

In addition, we have the following proposition.

Proposition 3.16. *When $n\leq 3$, R is categorifiable. Moreover, there exists a braided fusion category $\mathcal{C}$ such that $Gr(\mathcal{C})=R$ if and only if $n=1$.*

Proof. If $n=1$, then $\mathrm{FPdim}(M)=1$, and it is easy to see that

$$\mathrm{Gr}(\mathcal{C}(\mathbb{Z}_2,\eta)\boxtimes\mathcal{C}(\mathfrak{sl}_2,3)_{\mathrm{ad}})=R,$$

where $\mathcal{C}(\mathfrak{sl}_2,3)_{\mathrm{ad}}$ is the adjoint fusion subcategory of $\mathcal{C}(\mathfrak{sl}_2,3)$ [1,9]. If $n=3$, then R is the Grothendieck ring of the fusion category $\mathcal{V}$ [4, Proposition A.6.1]. When $n\geq 3$, R is non-commutative, obviously it cannot be categorified as a braided fusion category.

If $n=2$, then $\mathrm{FPdim}(R)=12+4\sqrt{3}$. We claim that it can be categorified by $\mathcal{C}(\mathfrak{sl}_2,10)_A$, where A is a non-trivial connected étale algebra and $\mathrm{FPdim}(A)=3+\sqrt{3}$ by [16, Theorem 6.5]. Indeed, a direct computation shows that the Frobenius–Perron dimensions of the simple objects of $\mathcal{C}(\mathfrak{sl}_2,10)_A$ belong to $\{1,\sqrt{2},1+\sqrt{3},\sqrt{2+\sqrt{3}}\}$, and $\sqrt{2+\sqrt{3}}=\frac{\sqrt{2}+\sqrt{6}}{2}$. Since $\mathcal{C}(\mathfrak{sl}_2,10)_A$ contains a unique simple object X of Frobenius–Perron dimension $1+\sqrt{3}$ and two invertible objects I,g, we obtain

$$g\otimes X=X=X\otimes g,\quad X\otimes X=I\oplus g\oplus 2X,$$

i.e., X generates a near-group fusion category $\mathcal{A}$. Since $2\mathrm{FPdim}(\mathcal{A})=\mathrm{FPdim}(\mathcal{C}(\mathfrak{sl}_2,10)_A)$, $\mathcal{C}(\mathfrak{sl}_2,10)_A$ admits a faithful $\mathbb{Z}_2$-grading with the trivial component being $\mathcal{A}$ [9, Proposition 3.5.3], then the rest of the fusion relations follow from the principal diagram [16, Theorem 6.5].

However, when $n=2$, we claim that R cannot be categorified as a braided fusion category even if it is commutative. On the contrary, assume that there is a braided fusion category $\mathcal{B}$ such that $\mathrm{Gr}(\mathcal{B})=R$. Since $\mathcal{C}$ always contains an Ising category $\mathcal{I}$ as a fusion subcategory, which is modular by [7, Corollary B.12], $\mathcal{B}\cong\mathcal{I}\boxtimes\mathcal{D}$ as a braided fusion category [7, Theorem 3.13], where $\mathcal{D}$ is a braided fusion subcategory of $\mathcal{B}$ such that $\dim(\mathcal{D})=3+\sqrt{3}$ by [7, Theorem 3.14]. So there exists a Galois conjugate of $\mathcal{D}$ whose global dimension is $3-\sqrt{3}$, which contradicts the conclusion of [24, Theorem 1.1.2]. ■

We end this section by proposing the following question.

Question 3.17. Assume that there is a near-group fusion category $\mathcal{A}$ such that $\mathrm{Gr}(\mathcal{A}) = R_0$. Is R categorifiable when $n \geq 4$?

Indeed, R is categorifiable when n is odd and $\mathcal{A}$ exists and $\mathcal{Z}(\mathcal{A}) \cong \mathcal{C}(\mathbb{Z}_n, \eta) \boxtimes \mathcal{C}$ by the construction of $\mathcal{M}$ in Theorem 3.12.

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