

Quantum fluctuations of many-body dynamics around the Gross–Pitaevskii equation

Cristina Caraci, Jakob Oldenburg, and Benjamin Schlein

Abstract. We consider the evolution of a gas of N bosons in the three-dimensional Gross–Pitaevskii regime (in which particles are initially trapped in a volume of order one and interact through a repulsive potential with scattering length of order $1/N$). We construct a quasi-free approximation of the many-body dynamics, whose distance to the solution of the Schrödinger equation converges to zero, as $N \rightarrow \infty$, in the $L^2(\mathbb{R}^{3N})$ -norm. To achieve this goal, we let the Bose–Einstein condensate evolve according to a time-dependent Gross–Pitaevskii equation. After factoring out the microscopic correlation structure, the evolution of the orthogonal excitations of the condensate is governed instead by a Bogoliubov dynamics, with a time-dependent generator quadratic in creation and annihilation operators. As an application, we show a central limit theorem for fluctuations of bounded observables around their expectation with respect to the Gross–Pitaevskii dynamics.

1. Introduction

In the Gross–Pitaevskii regime, we consider systems of N bosons confined by an external field in a volume of order one (after appropriate choice of the length unit) and interacting through a repulsive potential with small effective range of order $1/N$. The corresponding Hamilton operator is given by

$$H_N^{\text{trap}} = \sum_{j=1}^N [-\Delta_{x_j} + V_{\text{ext}}(x_j)] + \sum_{i < j}^N N^2 V(N(x_i - x_j)), \quad (1.1)$$

with $V_{\text{ext}}(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $V \geq 0$ compactly supported. According to bosonic statistics, H_N^{trap} acts as a self-adjoint operator on $L_s^2(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ consisting of wave functions that are symmetric with respect to permutations of the N particles. In [48, 53] it was proven that, to leading order in N , the ground state energy E_N^{trap} of (1.1) satisfies

$$\lim_{N \rightarrow \infty} \frac{E_N^{\text{trap}}}{N} = \min_{\varphi \in L^2(\mathbb{R}^3): \|\varphi\|_2=1} \mathcal{E}_{\text{GP}}(\varphi), \quad (1.2)$$

Mathematics Subject Classification 2020: 82C10 (primary); 81V73, 35Q40 (secondary).

Keywords: many-body quantum dynamics, Bogoliubov approximation, Gross–Pitaevskii equation.

with the Gross–Pitaevskii energy functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int [|\nabla\varphi|^2 + V_{\text{ext}}|\varphi|^2 + 4\pi\alpha|\varphi|^4] dx. \quad (1.3)$$

Here, $\alpha > 0$ denotes the scattering length of the interaction potential V , which is defined through the solution of the zero-energy scattering equation

$$\left[-\Delta + \frac{1}{2}V\right]f = 0,$$

with the boundary condition $f(x) \rightarrow 1$, as $|x| \rightarrow \infty$, by requiring that $f(x) = 1 - \alpha/|x|$ outside the support of V .

Let $\varphi_{\text{GP}} \in L^2(\mathbb{R}^3)$ denote the (unique, up to a phase) normalized minimizer of (1.3). It turns out that the ground state vector of (1.1) and, in fact, every sequence of approximate ground states, exhibit complete Bose–Einstein condensation in the one-particle state φ_{GP} . In other words, let us consider a normalized sequence $\psi_N \in L^2_s(\mathbb{R}^{3N})$ satisfying

$$\frac{1}{N} \langle \psi_N, H_N^{\text{trap}} \psi_N \rangle \rightarrow \mathcal{E}_{\text{GP}}(\varphi_{\text{GP}})$$

as $N \rightarrow \infty$ (i.e. ψ_N is a sequence of approximate ground states). Let γ_N denote the one-particle reduced density matrix associated with ψ_N , which is defined as the nonnegative operator on $L^2(\mathbb{R}^3)$ with the integral kernel

$$\gamma_N(x; y) = \int dx_2 \dots dx_N \bar{\psi}_N(x, x_2, \dots, x_N) \psi_N(y, x_2, \dots, x_N),$$

normalized so that $\text{tr } \gamma_N = 1$. Then, as first proven in [46, 47, 53],

$$\lim_{N \rightarrow \infty} \langle \varphi_{\text{GP}}, \gamma_N \varphi_{\text{GP}} \rangle = 1. \quad (1.4)$$

The convergence (1.4) implies that, in the states ψ_N , the fraction of particles orthogonal to φ_{GP} vanishes, in the limit $N \rightarrow \infty$.

Recently, the estimates (1.2), (1.4) have been improved in [18, 19, 52, 54] for integrable interaction potentials, through a rigorous version of Bogoliubov theory. In the translation-invariant setting, these improvements have previously been achieved in [8, 9]. This approach determines the ground state energy E_N^{trap} of (1.1), up to errors vanishing as $N \rightarrow \infty$. Moreover, it gives precise information on the low-energy excitation spectrum of (1.1) and on the depletion of the Bose–Einstein condensate (in particular, it shows that the rate of convergence in (1.4) is proportional to $1/N$) and it provides a norm approximation for the ground state vector. It is interesting to remark that, in the Gross–Pitaevskii regime, such precise estimates on the spectrum of the Hamiltonian cannot be obtained by restricting our attention to quasi-free states. Instead, it is important to take into account corrections that can be described by the action of a unitary operator on the Fock space of excitations of the condensate, given by the exponential of a cubic expression in creation

and annihilation operators. For related recent results concerning equilibrium properties of Bose gases in the (translation-invariant) Gross–Pitaevskii regime, beyond the Gross–Pitaevskii regime and in the thermodynamic limit, see [1, 3–5, 11, 12, 15, 21, 22, 31, 32, 40, 41].

Gross–Pitaevskii theory is not only useful to predict the ground state energy of Bose gases described by the Hamilton operator (1.1). It can also be used to approximate their time evolution. From the point of view of physics, it is relevant to study the dynamics of an equilibrium state of (1.1), after the external fields are switched off (so that the system is no longer at equilibrium). At (or very close to) zero temperature, it is therefore interesting to study the solution of the time-dependent Schrödinger equation

$$i \partial_t \psi_{N,t} = H_N \psi_{N,t} \quad (1.5)$$

with the translation-invariant Hamiltonian (obtained after turning off the trap)

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^2 V(N(x_i - x_j)), \quad (1.6)$$

for initial data $\psi_{N,0}$ approximating the ground state of (1.1). In [26–29], it was first proven that the time evolution $\psi_{N,t}$ of an initial data $\psi_{N,0}$ exhibiting Bose–Einstein condensate in a one-particle state $\varphi \in L^2(\mathbb{R}^3)$ still exhibits Bose–Einstein condensation, in a new one-particle state φ_t , given by the solution of the nonlinear time-dependent Gross–Pitaevskii equation

$$i \partial_t \varphi_t = -\Delta \varphi_t + 8\pi\alpha |\varphi_t|^2 \varphi_t \quad (1.7)$$

with the initial data $\varphi_{t=0} = \varphi$. More precisely, denoting by $\gamma_{N,t}$ the one-particle reduced density associated with the solution $\psi_{N,t} \in L^2_s(\mathbb{R}^{3N})$ of the Schrödinger equation (1.5), it turns out that

$$\lim_{N \rightarrow \infty} \langle \varphi_t, \gamma_{N,t} \varphi_t \rangle = 1 \quad (1.8)$$

for any fixed $t \in \mathbb{R}$, if (1.8) holds true at time $t = 0$. Analogous stability results were later established in [7, 55]. In [17], the convergence (1.8) is shown to hold with the optimal rate $1/N$, for every fixed $t \in \mathbb{R}$. It is easy to check that (1.8) implies the convergence $\gamma_{N,t} \rightarrow |\varphi_t\rangle\langle\varphi_t|$ in the trace-class topology (and also the convergence of the k -particle reduced density matrix associated with $\psi_{N,t}$ towards the product $|\varphi_t\rangle\langle\varphi_t|^{\otimes k}$, for every fixed $k \in \mathbb{N}$). However, (1.8) does not provide an approximation for the many-body wave function $\psi_{N,t}$ in the strong $L^2(\mathbb{R}^{3N})$ topology. To obtain a norm approximation, it is not enough to approximate the evolution of the condensate. It is instead crucial to take into account the evolution of its orthogonal excitations.

Norm approximations for many-body dynamics have been derived in the mean-field setting, where particles are initially trapped in a volume of order one and interact weakly through a potential whose range is comparable with the size of the trap (so that every particle interacts effectively with all other particles in the system). In this case, as shown

in [24, 35, 38, 39, 42, 44, 49, 58], the solution of the many-body Schrödinger equation can be approximated, after removing the condensate wave function (whose evolution is described here by the nonlinear Hartree equation; see also [2, 23, 30, 33, 34, 59]), by a unitary dynamics on the Fock space of excitations, with a generator quadratic in creation and annihilation operators, acting as a family of time-dependent Bogoliubov transformations (an approximation to arbitrary precision has recently been obtained in [14]). A similar norm approximation has been derived in [10, 16] for the many-body evolution generated by Hamilton operators having the form

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N} \sum_{i < j}^N N^{3\beta} V(N^\beta(x_i - x_j)), \quad (1.9)$$

with $\beta \in (0, 1)$, interpolating between the mean field and the Gross–Pitaevskii scaling (analogous results have also been obtained in [36, 37, 50] for $\beta < 1/3$, and in [43, 51] for $\beta < 1/2$; higher-order estimates have been derived, for sufficiently small $\beta > 0$, in [13]). To achieve this goal, it was important to combine the unitary dynamics with a quadratic generator, describing the evolution of excitations on macroscopic scales, with an additional Bogoliubov transformation generating the correct microscopic correlation structure.

In the present paper, we prove a norm approximation for the many-body dynamics generated by the Hamilton operator (1.6), in the Gross–Pitaevskii regime (i.e. with $\beta = 1$ in (1.9)). Compared with the techniques developed in [10, 16] for $\beta < 1$, there is an important difference in the construction of the approximating wave function. In fact, as already observed in [8, 18, 54] in the time-independent setting, for $\beta = 1$ microscopic correlations cannot be precisely modeled only through a Bogoliubov transformation; they require instead an additional unitary conjugation with a phase cubic in creation and annihilation operators. This makes our analysis significantly more involved. While the inclusion of the cubic phase is crucial to compare the generators of the full many-body evolution and of the quadratic dynamics (and thus to establish convergence for the corresponding evolutions), in the end it does not substantially change the $L^2(\mathbb{R}^{3N})$ -norm of the approximation and it can therefore be removed, providing a quasi-free norm approximation to the many-body evolution, similar to those obtained in [10, 16] for $\beta \in (0, 1)$.

2. Setting and main results

We aim at approximating the solution $\psi_{N,t}$ of the many-body Schrödinger equation (1.5), for a class of initial data exhibiting complete Bose–Einstein condensation in a normalized one-particle wave function $\varphi \in L^2(\mathbb{R}^3)$. In view of (1.4), from the point of view of physics it is interesting to choose φ as the minimizer of a Gross–Pitaevskii functional of the form (1.3). Here we will keep the choice of φ open, requiring only sufficient regularity.

First of all, we need to approximate the evolution of the condensate. While (1.7) provides a good approximation at the level of reduced density matrices, to derive a norm

approximation it is more convenient to consider a slightly modified, N -dependent, nonlinear equation. In the modified equation, the interaction potential appearing in the Hamilton operator (1.6) is corrected, to take into account correlations among particles. In order to describe correlations, we fix $\ell \in (0; 1/2)$ and we consider the ground state solution of the Neumann problem

$$\left[-\Delta + \frac{1}{2}V\right]f_\ell = \lambda_\ell f_\ell \quad (2.1)$$

on the ball $|x| \leq N\ell$ (we omit the N -dependence in the notation here for f_ℓ and for λ_ℓ ; notice that λ_ℓ scales as N^{-3}), with the normalization $f_\ell(x) = 1$ for $|x| = N\ell$. We extend f_ℓ to \mathbb{R}^3 , setting $f_\ell(x) = 1$ for all $|x| > N\ell$ and we also introduce the notation $w_\ell(x) = 1 - f_\ell(x)$. To describe correlations created by the rescaled interaction appearing in (1.1) and in (1.6), we will use the functions $f_{N,\ell}(x) = f_\ell(Nx)$, $w_{N,\ell}(x) = w_\ell(Nx) = 1 - f_{N,\ell}(x)$. By scaling, we observe that

$$\left[-\Delta + \frac{N^2}{2}V(N\cdot)\right]f_{N,\ell} = N^2\lambda_\ell f_{N,\ell} \quad (2.2)$$

on the ball $|x| \leq \ell$. Some important properties of λ_ℓ , f_ℓ , w_ℓ are collected in Lemma A.1 in Appendix A.

With $f_{N,\ell}$, we can now define the condensate wave function at time $t \in \mathbb{R}$ as the solution $\tilde{\varphi}_t$ of the modified Gross–Pitaevskii equation

$$i\partial_t \tilde{\varphi}_t = -\Delta \tilde{\varphi}_t + (N^3 V(N\cdot) f_\ell(N\cdot) * |\tilde{\varphi}_t|^2) \tilde{\varphi}_t, \quad (2.3)$$

with initial data $\tilde{\varphi}_{t=0} = \varphi$. As discussed in Lemma A.1, we have

$$\left| \int N^3 V(Nx) f_\ell(Nx) dx - 8\pi\alpha \right| = \left| \int V(x) f_\ell(x) dx - 8\pi\alpha \right| \leq \frac{C\alpha^2}{\ell N}.$$

It is therefore easy to check that, as $N \rightarrow \infty$, $\tilde{\varphi}_t$ converges to the solution of the limiting Gross–Pitaevskii equation (1.7) (with the same initial data). This convergence is part of the statement of Proposition A.2, in Appendix A, where we also collect some standard properties of the solutions of (1.7) and of (2.3) which will be used throughout the paper.

As explained in the introduction, to obtain a norm approximation for the time evolution it is not enough to approximate the evolution of the condensate; we also need to take into account its excitations. To this end, it is convenient to factor out the (time-dependent) condensate wave function introducing, for every $t \in \mathbb{R}$, the unitary map $U_{N,t} : L_s^2(\mathbb{R}^{3N}) \rightarrow \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$ into the Fock space of excitations

$$\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} = \bigoplus_{n=0}^N L_{\perp \tilde{\varphi}_t}^2(\mathbb{R}^3)^{\otimes n},$$

over the orthogonal complement $L_{\perp \tilde{\varphi}_t}^2(\mathbb{R}^3)$ of $\tilde{\varphi}_t$. The map $U_{N,t}$ is defined so that $U_{N,t} \psi_{N,t} = \{\alpha_{N,t}^{(0)}, \dots, \alpha_{N,t}^{(N)}\}$, corresponding to the unique decomposition

$$\psi_{N,t} = \sum_{n=0}^N \alpha_{N,t}^{(n)} \otimes_s \tilde{\varphi}_t^{\otimes (N-n)},$$

where $\alpha_{N,t}^{(n)} \in L^2_{\perp \tilde{\varphi}_t}(\mathbb{R}^3)^{\otimes_s n}$ is symmetric with respect to permutation and orthogonal to $\tilde{\varphi}_t$ in each coordinate. Denoting by $a(f)$, $a^*(g)$ the usual creation and annihilation operators, the action of $U_{N,t}$ is characterized (see [17, 45]) by the rules

$$\begin{aligned} U_{N,t} a^*(\tilde{\varphi}_t) a(\tilde{\varphi}_t) U_{N,t}^* &= N - \mathcal{N}, \\ U_{N,t} a^*(f) a(\tilde{\varphi}_t) U_{N,t}^* &= a^*(f) \sqrt{N - \mathcal{N}} =: \sqrt{N} b(f)^*, \\ U_{N,t} a^*(\tilde{\varphi}_t) a(g) U_{N,t}^* &= \sqrt{N - \mathcal{N}} a(g) =: \sqrt{N} b(g), \\ U_{N,t} a^*(f) a(g) U_{N,t}^* &= a^*(f) a(g), \end{aligned} \tag{2.4}$$

where $\mathcal{N} = \int a_x^* a_x dx$ is the number of particles operator, and $b^*(f)$, $b(g)$ are modified creation and annihilation operators satisfying the commutation relations

$$\begin{aligned} [b(f), b^*(g)] &= \left(1 - \frac{\mathcal{N}}{N}\right) \langle f, g \rangle - \frac{1}{N} a^*(f) a(g), \\ [b(f), b(g)] &= [b^*(f), b^*(g)] = 0, \end{aligned} \tag{2.5}$$

for all $f, g \in L^2_{\perp \tilde{\varphi}_t}(\mathbb{R}^3)$. Denoting by b_x , b_x^* , a_x , a_x^* the corresponding operator-valued distributions, we also find

$$\begin{aligned} [b_x, b_y^*] &= \delta(x - y) \left(1 - \frac{\mathcal{N}}{N}\right) - \frac{1}{N} a_y^* a_x, \\ [b_x, a_y^* a_z] &= \delta(x - y) b_z, \quad [b_x^*, a_y^* a_z] = -\delta(y - z) b_x^*, \end{aligned} \tag{2.6}$$

for all $x, y, z \in \mathbb{R}^3$.

After factoring out the condensate with the unitary operator $U_{N,t}$, we need to approximate the evolution of its orthogonal excitations in $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$. Here, we need to distinguish between microscopic excitations, varying on small length scales between $1/N$ and ℓ (which is chosen of order one) and macroscopic excitations, varying on scales of order one. Let us first worry about the microscopic excitations, characterizing all low-energy states. It is natural to include them on the initial data and to propagate them along the time evolution. These excitations only depend on time through the time dependence of the condensate wave function $\tilde{\varphi}_t$. We are going to describe them through a (generalized) Bogoliubov transformation.

We define the integral kernel

$$k_t(x, y) = -N w_\ell(N(x - y)) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y). \tag{2.7}$$

With Lemma A.1, we find $k_t \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ (with bounded norm, uniform in N). Hence, k_t defines a Hilbert–Schmidt operator on $L^2(\mathbb{R}^3)$, which we will again denote by k_t . To obtain a Bogoliubov transformation acting on the Hilbert space $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$, defined on the orthogonal complement of the condensate wave function $\tilde{\varphi}_t$, we set $\tilde{q}_t = 1 - |\tilde{\varphi}_t\rangle\langle \tilde{\varphi}_t|$ and

$$\eta_t = (\tilde{q}_t \otimes \tilde{q}_t) k_t. \tag{2.8}$$

We also denote $\mu_t = \eta_t - k_t$. With η_t , we define the antisymmetric operator

$$B_t = \frac{1}{2} \int [\eta_t(x; y) b_x^* b_y^* - \bar{\eta}_t(x; y) b_x b_y] dx dy, \quad (2.9)$$

and we consider the unitary operator e^{B_t} on $\mathcal{F}_{\perp \bar{\varphi}_t}^{\leq N}$. An important consequence of the bound $\|\eta_t\|_2 \leq \|k_t\|_2 \leq C$, uniformly in $N \in \mathbb{N}$ and $t \in \mathbb{R}$, is the fact that e^{B_t} does not substantially change the number of excitations. The proof of the following lemma can be found, for example, in [17, Lemma 3.1].

Lemma 2.1. *Let B_t be the anti-symmetric operator defined in (2.9). Then for every $n \in \mathbb{Z}$ there exists a constant $C > 0$ (depending only on $\|\eta_t\|$) such that*

$$e^{-B_t} (\mathcal{N} + 1)^n e^{B_t} \leq C (\mathcal{N} + 1)^n$$

as an operator inequality on $\mathcal{F}^{\leq N}$.

On states with few excitations (on which the commutation relations (2.5) are almost canonical), e^{B_t} approximately acts as a Bogoliubov transformation. In fact, we can write

$$e^{B_t} b(f) e^{-B_t} = b(\gamma_t(f)) + b^*(\sigma_t(\bar{f})) + d_t(f), \quad (2.10)$$

where we introduced the notation

$$\gamma_t = \cosh(\eta_t) = \sum_{j=0}^{\infty} \frac{(\eta_t \bar{\eta}_t)^j}{(2j)!}, \quad \sigma_t = \sinh(\eta_t) = \sum_{j=0}^{\infty} \frac{(\eta_t \bar{\eta}_t)^j \eta_t}{(2j+1)!}, \quad (2.11)$$

and where, from [9, Lemma 3.4], we have the bounds (denoting by d_x, d_x^* the operator-valued distributions associated with the operators d, d^*)

$$\|(\mathcal{N} + 1)^k d_t^\sharp(f) \xi\| \leq \frac{C}{N} \|f\| \|(\mathcal{N} + 1)^{k+3/2} \xi\|, \quad (2.12a)$$

$$\begin{aligned} \|(\mathcal{N} + 1)^{n/2} a_y d_x \xi\| &\leq \frac{C}{N} [(\|\eta_t(x; \cdot)\|_2 \|\eta_t(\cdot; y)\|_2 + |\eta_t(x; y)|) \\ &\quad \times \|(\mathcal{N} + 1)^{(n+2)/2} \xi\| \\ &\quad + \|\eta_t(\cdot; y)\|_2 \|a_x (\mathcal{N} + 1)^{(n+1)/2} \xi\| \\ &\quad + \|\eta_t(x; \cdot)\|_2 \|a_y (\mathcal{N} + 1)^{(n+3)/2} \xi\| \\ &\quad + \|a_x a_y (\mathcal{N} + 1)^{(n+2)/2} \xi\|], \end{aligned} \quad (2.12b)$$

$$\begin{aligned} \|(\mathcal{N} + 1)^{n/2} d_x d_y \xi\| &\leq \frac{C}{N^2} [(\|\eta_t(x; \cdot)\|_2 \|\eta_t(\cdot; y)\|_2 + |\eta_t(x; y)|) \\ &\quad \times \|(\mathcal{N} + 1)^{(n+4)/2} \xi\| \\ &\quad + \|\eta_t(\cdot; y)\|_2 \|a_x (\mathcal{N} + 1)^{(n+5)/2} \xi\| \\ &\quad + \|\eta_t(x; \cdot)\|_2 \|a_y (\mathcal{N} + 1)^{(n+5)/2} \xi\| \\ &\quad + \|a_x a_y (\mathcal{N} + 1)^{(n+4)/2} \xi\|], \end{aligned} \quad (2.12c)$$

for $\sharp = *, \cdot$, for all $f \in L^2(\mathbb{R}^3)$, $x, y \in \mathbb{R}^3$ and $n \in \mathbb{Z}$. Some bounds on the operators η_t , μ_t , γ_t , σ_t are collected in Lemma A.3, in Appendix A.

We are interested in the time evolution of initial data having the form

$$\psi_N = U_{N,0}^* e^{B_0} \xi_N, \quad (2.13)$$

under appropriate conditions on the excitation vector $\xi_N \in \mathcal{F}_{\perp\varphi}^{\leq N}$ (we will make assumptions on moments of the number of particles and kinetic energy operators, in the state ξ_N). This allows us to consider initial data which are expected to describe the ground state vector of a trapped Hamiltonian like (1.1) (see Remark 2.3 for more details).

We consider the many-body evolution $\psi_{N,t} = e^{-iH_N t} \psi_N$ of (2.13), generated by the translation-invariant Hamilton operator (1.6). At time $t \neq 0$, we expect $\psi_{N,t}$ to again have approximately the form (2.13), but now with φ replaced by the solution $\tilde{\varphi}_t$ of (2.3). For this reason, we introduce the excitation vector $\xi_{N,t} \in \mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ requiring that

$$e^{-iH_N t} \psi_N = U_{N,t}^* e^{B_t} \xi_{N,t}.$$

In other words, $\xi_{N,t} = \bar{U}_N(t; 0) \xi_N$, with the fluctuation dynamics

$$\bar{U}_N(t; s) = e^{-B_t} U_{N,t} e^{-iH_N(t-s)} U_{N,s}^* e^{B_s}. \quad (2.14)$$

While e^{B_t} takes care of the microscopic excitations of the condensate, to derive a norm approximation for $e^{-iH_N t} \psi_N$ we still need to take into account the evolution of the macroscopic excitations. To reach this goal, we introduce a unitary dynamics $\mathcal{U}_{2,N}(t; 0)$, whose generator is quadratic in creation and annihilation operators (a time-dependent Bogoliubov transformation), approximating the fluctuation dynamics $\bar{\mathcal{U}}_N(t; 0)$ and providing therefore an approximation of $\xi_{N,t} = \bar{U}_N(t; 0) \xi_N$. To this end, let us introduce the ‘‘projected’’ modified creation and annihilation operators

$$\tilde{b}_x^* = b^*(\tilde{q}_x) = b_x^* - \overline{\tilde{\varphi}_t(x)} b^*(\tilde{\varphi}_t), \quad \tilde{b}_x = b(\tilde{q}_x) = b_x - \tilde{\varphi}_t(x) b(\tilde{\varphi}_t), \quad (2.15)$$

where $q_x(y) = q_t(y, x) = \delta(x - y) - \tilde{\varphi}_t(x) \tilde{\varphi}_t(y)$ is the kernel of the projection orthogonal to the condensate wave function. Then we define the time-dependent self-adjoint operator

$$\begin{aligned} \mathcal{J}_{2,N}(t) &= \mathcal{J}_{2,N}^K(t) + \mathcal{J}_{2,N}^V(t) \\ &+ N^3 \lambda_\ell \int dx dy \chi_\ell(x - y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \tilde{b}_x^* \tilde{b}_y^* + \text{h.c.} \\ &+ \frac{1}{2} \int dx dy N w_{N,\ell}(x - y) [\Delta \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) + \tilde{\varphi}_t(x) \Delta \tilde{\varphi}_t(y)] \tilde{b}_x^* \tilde{b}_y^* + \text{h.c.} \\ &+ \int dx dy N \nabla w_{N,\ell}(x - y) [\nabla \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) - \tilde{\varphi}_t(x) \nabla \tilde{\varphi}_t(y)] \tilde{b}_x^* \tilde{b}_y^* + \text{h.c.} \\ &- \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) [\tilde{b}^*(\gamma_x^{(s)}) \tilde{b}^*(\gamma_y^{(s)}) + \tilde{b}^*(\gamma_x^{(s)}) \tilde{b}(\sigma_y^{(s)}) \\ &\quad + \tilde{b}(\sigma_x^{(s)}) \tilde{b}(\sigma_y^{(s)}) + \tilde{b}(\gamma_y^{(s)}) \tilde{b}(\sigma_x^{(s)})] + \text{h.c.} \\ &+ \int dx N^3 (V_N f_{N,\ell}) * |\tilde{\varphi}_t|^2(x) (a_x^* a_x - \tilde{b}_x^* \tilde{b}_x) \end{aligned}$$

$$\begin{aligned}
 &=: \mathcal{K} + \int dx N^3 (V_N f_{N,\ell}) * |\tilde{\varphi}_t|^2(x) a_x^* a_x \\
 &\quad + \int dx dy (G_t(x, y) \tilde{b}_x^* \tilde{b}_y + H_t(x, y) \tilde{b}_x^* \tilde{b}_y^* + \bar{H}_t(x, y) \tilde{b}_x \tilde{b}_y), \quad (2.16)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{J}_{2,N}^K(t) = \mathcal{K} + \int dx \left[\tilde{b}^*(-\Delta_x p_x) \tilde{b}_x + \frac{1}{2} \tilde{b}^*(\nabla_x p_x) \tilde{b}(\nabla_x p_x) + \tilde{b}_x^* \tilde{b}^*(-\Delta_x \mu_x) \right. \\
 \quad + \tilde{b}^*(-\Delta_x p_x) \tilde{b}^*(\eta_x) + \tilde{b}^*(p_x) \tilde{b}^*(-\Delta_x r_x) + \tilde{b}_x^* \tilde{b}^*(-\Delta_x r_x) \\
 \quad + \frac{1}{2} \tilde{b}^*(\nabla_x \eta_x) \tilde{b}(\nabla_x \eta_x) + \tilde{b}^*(\eta_x) \tilde{b}(-\Delta_x r_x) \\
 \quad \left. + \frac{1}{2} \tilde{b}^*(\nabla_x r_x) \tilde{b}(\nabla_x r_x) + \text{h.c.} \right] \quad (2.17)
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{J}_{2,N}^V(t) = \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\
 \quad \times [\tilde{b}^*(p_x) \tilde{b}_y^* + \tilde{b}^*(\gamma_x) \tilde{b}^*(p_y)] + \text{h.c.} \\
 + \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\
 \quad \times [\tilde{b}^*(\gamma_y) \tilde{b}(\sigma_x) + \tilde{b}^*(\gamma_x) \tilde{b}(\sigma_y) + \tilde{b}(\sigma_x) \tilde{b}(\sigma_y)] + \text{h.c.} \\
 + \int dx (N^3 (V_N f_{N,\ell}) * |\tilde{\varphi}_t|^2)(x) \\
 \quad \times (\tilde{b}^*(\gamma_x) \tilde{b}(\gamma_x) + \tilde{b}(\sigma_x) \tilde{b}(\gamma_x) + \tilde{b}^*(\gamma_x) \tilde{b}^*(\sigma_x) + \tilde{b}^*(\sigma_x) \tilde{b}(\sigma_x)) \\
 + \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \overline{\tilde{\varphi}_t(y)} \\
 \quad \times (\tilde{b}^*(\gamma_x) \tilde{b}(\gamma_y) + \tilde{b}(\sigma_x) \tilde{b}(\gamma_y) + \tilde{b}^*(\gamma_x) \tilde{b}^*(\sigma_y) + \tilde{b}^*(\sigma_y) \tilde{b}(\sigma_x)). \quad (2.18)
 \end{aligned}$$

Here we used the kernels $\eta_t, \mu_t, \gamma_t, \sigma_t$ introduced in (2.8), (2.11) and, additionally, we defined $p_t = \gamma_t - 1$ and $r_t = \sigma_t - \eta_t$. Moreover, for $s \in [0, 1]$, $\gamma_t^{(s)}$ and $\sigma_t^{(s)}$ are defined as γ_t, σ_t , but with η_t replaced by $s\eta_t$. Furthermore, we used the notation $f_x(y) = f_t(y, x)$ for kernels of operators acting on $L^2(\mathbb{R}^3)$ (here we drop the label t , to keep the notation as light as possible). Also, we set $V_N(x) = V(Nx)$ and we used the notation

$$\mathcal{K} = \int dx \nabla_x a_x^* \nabla_x a_x$$

for the kinetic energy operator. The self-adjoint operator $\mathcal{J}_{2,N}(t)$ generates a two-parameter family of unitary transformations $\mathcal{U}_{2,N}$, satisfying the equation

$$i \partial_t \mathcal{U}_{2,N}(t; s) = \mathcal{J}_{2,N}(t) \mathcal{U}_{2,N}(t; s) \quad (2.19)$$

with $\mathcal{U}_{2,N}(s; s) = 1$ for all $s \in \mathbb{R}$ (the well-posedness of (2.19) is part of the statement of the next theorem; it will be established in Proposition 4.4).

In our first main theorem, we show that the Bogoliubov dynamics $\mathcal{U}_{2,N}$ can be used to describe the evolution of macroscopic excitations of the condensate, providing a norm approximation for the solution of the many-body Schrödinger equation.

Theorem 2.2. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, spherically symmetric and compactly supported. Let $\varphi \in H^6(\mathbb{R}^3)$. Let η_t be defined as in (2.8), with parameter $\ell > 0$ small enough. Then (2.19) defines a unique two-parameter strongly continuous unitary group $\mathcal{U}_{2,N}$, with $\mathcal{U}_{2,N}(t; s): \mathcal{F}_{\perp \tilde{\varphi}_s}^{\leq N} \rightarrow \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$. Moreover, let B_t be defined as in (2.9) and*

$$\begin{aligned} \kappa_N(t) = & \frac{N}{2} \langle \tilde{\varphi}_t, [N^3 V_N (1 - 2f_{N,\ell}) * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle - \frac{1}{2} \langle \tilde{\varphi}_t, (N^3 V_N f_{N,\ell} * |\tilde{\varphi}_t|^2) \tilde{\varphi}_t \rangle \\ & + \frac{1}{2} \int dx dy N^2 V_N(x-y) |\langle \gamma_y, \sigma_x \rangle|^2 \\ & + \int dx (N^3 V_N f_{N,\ell} * |\tilde{\varphi}_t|^2)(x) \langle \sigma_x, \sigma_x \rangle \\ & + \int dx dy N^3 V_N f_{N,\ell}(x-y) \tilde{\varphi}_t(x) \overline{\tilde{\varphi}_t(y)} \langle \sigma_x, \sigma_y \rangle \\ & + \frac{1}{2} \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \langle \sigma_x, \gamma_y \rangle + h.c. + \|\nabla_1 \sigma_t\|^2 \\ & - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) \langle \sigma_x^{(s)}, \gamma_y^{(s)} \rangle + h.c. \end{aligned} \quad (2.20)$$

Consider a sequence of normalized initial data $\psi_N \in L^2(\mathbb{R}^3)^{\otimes_s N}$, with excitation vectors

$$\xi_N = e^{-B_0} U_{N,0} \psi_N \quad (2.21)$$

satisfying

$$\langle \xi_N (\mathcal{K}^2 + \mathcal{N}^6) \xi_N \rangle \leq C, \quad (2.22)$$

uniformly in $N \in \mathbb{N}$. Then there exist constants $C, c > 0$ such that

$$\|e^{-iH_N t} \psi_N - e^{-i \int_0^t \kappa_N(s) ds} U_{N,t}^* e^{B_t} \mathcal{U}_{2,N}(t) \xi_N\| \leq C e^{ce^{c|t|}} N^{-1/8} \quad (2.23)$$

for any $t \in \mathbb{R}$, and $N \in \mathbb{N}$ large enough.

Remark 2.3. From (2.21), we have $\psi_N = U_{N,0}^* e^{B_0} \xi_N$. Thus, (2.23) is equivalent to

$$\|\bar{\mathcal{U}}_N(t; 0) \xi_N - e^{-i \int_0^t \kappa_N(s) ds} \mathcal{U}_{2,N}(t; 0) \xi_N\| \leq C e^{ce^{c|t|}} N^{-1/8} \quad (2.24)$$

with the fluctuation dynamics (2.14). In other words, the proof of Theorem 2.2 reduces to the comparison of the evolution $\bar{\mathcal{U}}_N$ with its quadratic approximation $\mathcal{U}_{2,N}$.

Remark 2.4. Observe that the choice $\xi_N = e^B \Omega$, with

$$B = \frac{1}{2} \int [\tau(x; y) b_x^* b_y^* - \bar{\tau}(x; y) b_x b_y] dx dy$$

and with a kernel $\tau \in (q_0 \times q_0)H^2(\mathbb{R}^3 \times \mathbb{R}^3)$ (with $q_0 = 1 - |\varphi\rangle\langle\varphi|$ projecting orthogonally to the condensate wave function) is compatible with the condition (2.22). This allows us to consider initial many-body wave functions of the form $\psi_N = U_{N,0}^* e^{B_0} e^B \Omega$ which are expected to approximate (in the $L^2(\mathbb{R}^{3N})$ norm which is preserved over time) the ground state vector of Hamilton operators of the form (1.1), describing trapped Bose gases in the Gross–Pitaevskii regime. This fact has been proven in [8] in the translation-invariant setting, for a gas confined on the unit torus. Notice that, in [8, equation (6.7)], the norm approximation of the ground state vector also includes the unitary operator e^A , with A cubic in creation and annihilation operators; in fact, this cubic phase can be removed, at the expense of another error of order smaller than $N^{-1/4}$ (arguing similarly to below, in the proof of Theorem 2.2, to show (4.13)). If the trapping potential is sufficiently smooth, it is also easy to verify the condition $\varphi \in H^6(\mathbb{R}^3)$, with φ minimizing the functional (1.3).

Remark 2.5. The double exponential in time on the right-hand side of (2.23) is due to the fact that we use estimates on high Sobolev norms for the solution of the modified Gross–Pitaevskii equation (2.3) and we apply a Grönwall argument for the propagation of many-body bounds. Assuming scattering for the solution of (2.3), the dependence on t could be substantially improved.

Remark 2.6. Theorem 2.2 confirms a conjecture formulated in [36, Section 10], for a different class of initial data (through the assumption (2.22), we impose a correlation structure on the initial wave function, generated by e^{B_0} , which is absent from the initial data discussed in [36]; it is not clear to us whether a norm approximation similar to (2.23) with an explicit rate can be obtained for “flat” data).

In (2.23), after factoring out the evolving Bose–Einstein condensate and the microscopic correlation structure, we approximate the evolution of the macroscopic correlations by the Bogoliubov dynamics $\mathcal{U}_{2,N}(t)$, which still depends on N . It is thus natural to ask whether $\mathcal{U}_{2,N}(t)$ approaches a limiting, N -independent, quadratic evolution $\mathcal{U}_{2,\infty}(t)$, as N tends to infinity. To answer this question, we start by defining the pointwise limit of $Nw_{N,\ell}(x)$, setting

$$w_{\infty,\ell}(x) = \begin{cases} \alpha \left[\frac{1}{|x|} - \frac{3}{2\ell} + \frac{x^2}{2\ell^3} \right], & |x| \leq \ell, \\ 0 & \text{otherwise,} \end{cases} \quad (2.25)$$

and the corresponding limiting integral kernel $k_{t,\infty} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$k_{t,\infty}(x; y) = -w_{\infty,\ell}(x - y)\varphi_t(x)\varphi_t(y), \quad (2.26)$$

where φ_t is the solution to the limiting (N -independent) Gross–Pitaevskii equation (1.7). Similarly to (2.8), we define $\eta_{t,\infty} = (q_t \otimes q_t)k_{t,\infty}$, projecting along $q_t = 1 - |\varphi_t\rangle\langle\varphi_t|$, orthogonally to φ_t and $\mu_{t,\infty} = \eta_{t,\infty} - k_{t,\infty}$. We also introduce the notation

$$\gamma_{t,\infty} = \cosh(\eta_{t,\infty}) = 1 + p_{t,\infty} \quad \text{and} \quad \sigma_{t,\infty} = \sinh(\eta_{t,\infty}) = \eta_{t,\infty} + r_{t,\infty}. \quad (2.27)$$

With this notation, we can introduce the generator of the limiting quadratic evolution, setting

$$\begin{aligned}
\mathcal{G}_{2,\infty}(t) &= \mathcal{G}_{2,\infty}^K(t) + \mathcal{G}_{2,\infty}^V(t) \\
&+ \frac{3\alpha}{\ell^3} \int dx dy \chi_\ell(x-y) \varphi_t(x) \varphi_t(y) \hat{a}_x^* \hat{a}_y^* + \text{h.c.} \\
&+ \frac{1}{2} \int dx dy w_{\infty,\ell}(x-y) [\Delta \varphi_t(x) \varphi_t(y) + \varphi_t(x) \Delta \varphi_t(y)] \hat{a}_x^* \hat{a}_y^* + \text{h.c.} \\
&+ \int dx dy \nabla w_{\infty,\ell}(x-y) [\nabla \varphi_t(x) \varphi_t(y) - \varphi_t(x) \nabla \varphi_t(y)] \hat{a}_x^* \hat{a}_y^* + \text{h.c.} \\
&- \int_0^1 ds \int dx dy \dot{\eta}_{t,\infty}(x,y) [\hat{a}^*(\gamma_{\infty,x}^{(s)}) \hat{a}^*(\gamma_{\infty,y}^{(s)}) + \hat{a}^*(\gamma_{\infty,x}^{(s)}) \hat{a}(\sigma_{\infty,y}^{(s)}) \\
&\quad + \hat{a}(\sigma_{\infty,x}^{(s)}) \hat{a}(\sigma_{\infty,y}^{(s)}) + \hat{a}^*(\gamma_{\infty,y}^{(s)}) \hat{a}(\sigma_{\infty,x}^{(s)})] \\
&\quad + \text{h.c.} \\
&+ 8\pi\alpha \int dx |\varphi_t(x)|^2 (a_x^* a_x - \hat{a}_x^* \hat{a}_x) \\
&=: \mathcal{K} + 8\pi\alpha \int dx |\varphi_t(x)|^2 a_x^* a_x \\
&+ \int dx dy (G_{t,\infty}(x,y) \hat{a}_x^* \hat{a}_y + H_{t,\infty}(x,y) \hat{a}_x^* \hat{a}_y^* \\
&\quad + \bar{H}_{t,\infty}(x,y) \hat{a}_x \hat{a}_y) \tag{2.28}
\end{aligned}$$

where \hat{a} , \hat{a}^* denote annihilation and creation operators, projected on $\mathcal{F}_{\perp\varphi_t}$ and where

$$\begin{aligned}
\mathcal{G}_{2,\infty}^K(t) &= \mathcal{K} + \int dx \left[\hat{a}^*(-\Delta_x p_{\infty,x}) \hat{a}_x + \frac{1}{2} \hat{a}^*(\nabla_x p_{\infty,x}) \hat{a}(\nabla_x p_{\infty,x}) + \right. \\
&\quad \hat{a}_x^* \hat{a}^*(-\Delta_x \mu_{\infty,x}) + \hat{a}^*(-\Delta_x p_{\infty,x}) \hat{a}^*(\eta_{\infty,x}) \\
&\quad + \hat{a}^*(p_{\infty,x}) \hat{a}^*(-\Delta_x r_{\infty,x}) + \hat{a}_x^* \hat{a}^*(-\Delta_x r_{\infty,x}) \\
&\quad + \frac{1}{2} \hat{a}^*(\nabla_x \eta_{\infty,x}) \hat{a}(\nabla_x \eta_{\infty,x}) + \hat{a}^*(\eta_{\infty,x}) \hat{a}(-\Delta_x r_{\infty,x}) \\
&\quad \left. + \frac{1}{2} \hat{a}^*(\nabla_x r_{\infty,x}) \hat{a}(\nabla_x r_{\infty,x}) + \text{h.c.} \right]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{2,\infty}^V(t) &= 4\pi\alpha \int dx \varphi_t(x)^2 [\hat{a}^*(p_{\infty,x}) \hat{a}_x^* + \hat{a}^*(\gamma_{\infty,x}) \hat{a}^*(p_y)] + \text{h.c.} \\
&+ 4\pi\alpha \int dx \varphi_t(x)^2 [2\hat{a}^*(\gamma_{\infty,x}) \hat{a}(\sigma_{\infty,x}) + \hat{a}(\sigma_{\infty,x}) \hat{a}(\sigma_{\infty,x})] + \text{h.c.} \\
&+ 16\pi\alpha \int dx |\varphi_t|^2(x) (\hat{a}^*(\gamma_{\infty,x}) \hat{a}(\gamma_{\infty,x}) + \hat{a}(\sigma_{\infty,x}) \hat{a}(\gamma_{\infty,x}) \\
&\quad + \hat{a}^*(\gamma_{\infty,x}) \hat{a}^*(\sigma_{\infty,x}) + \hat{a}^*(\sigma_{\infty,x}) \hat{a}(\sigma_{\infty,x})). \tag{2.29}
\end{aligned}$$

Here, $\gamma_{t,\infty}^{(s)}$, $\sigma_{t,\infty}^{(s)}$ are defined like $\gamma_{t,\infty}$, $\sigma_{t,\infty}$, but with $\eta_{t,\infty}$ replaced by $s\eta_{\infty,t}$. The two-parameter unitary evolution generated by $\mathcal{J}_{2,\infty}(t)$ is denoted by $\mathcal{U}_{2,\infty}(t; s)$. It satisfies the Schrödinger equation

$$i\partial_t \mathcal{U}_{2,\infty}(t, s) = \mathcal{J}_{2,\infty}(t) \mathcal{U}_{2,\infty}(t, s), \quad (2.30)$$

with initial condition $\mathcal{U}_{2,\infty}(s, s) = 1$ for all $s \in \mathbb{R}$ (the well-posedness of (2.30) is part of the statement of next theorem). Notice that $\mathcal{U}_{2,\infty}(t; s)$ maps the Fock space $\mathcal{F}_{\perp\varphi_s}$ into $\mathcal{F}_{\perp\varphi_t}$; here, there is no truncation on the number of particles.

Theorem 2.7. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, spherically symmetric and compactly supported. Let $\varphi \in H^6(\mathbb{R}^3)$ and $\eta_{t,\infty}$ be defined as after (2.26). Then there is a unique two-parameter strongly continuous unitary group $\mathcal{U}_{2,\infty}$ satisfying (2.30), with $\mathcal{U}_{2,\infty}(t; s): \mathcal{F}_{\perp\varphi_s} \rightarrow \mathcal{F}_{\perp\varphi_t}$. For normalized $\xi \in \mathcal{F}_{\perp\varphi}^{\leq N}$ satisfying*

$$\langle \xi, (\mathcal{K}^2 + \mathcal{N}^2)\xi \rangle < \infty, \quad (2.31)$$

we find $C, c > 0$ (only depending on the expectation (2.31)) such that

$$\|\mathcal{U}_{2,N}(t; 0)\xi - \mathcal{U}_{2,\infty}(t; 0)\xi\| \leq C e^{ce^{c|t|}} N^{-1/4} \quad (2.32)$$

for any $t \in \mathbb{R}$, and $N \in \mathbb{N}$ large enough. In (2.32), both $\mathcal{U}_{2,N}(t; 0)\xi$ and $\mathcal{U}_{2,\infty}(t; 0)\xi$ are thought of as vectors in the full Fock space \mathcal{F} , and $\|\cdot\|$ denotes the norm in this space.

Remark 2.8. Since here we compare two quadratic evolutions, condition (2.31) is milder than the corresponding assumption (2.22) in Theorem 2.2, where we compare the full many-body dynamics with a quadratic approximation.

Because its generator is quadratic in creation and annihilation operators, the evolution $\mathcal{U}_{2,\infty}$ acts as time-dependent Bogoliubov transformations. Thus, its action on annihilation and creation operators can be calculated explicitly.

Proposition 2.9. *Under the same assumptions as in Theorem 2.2, let $\mathcal{U}_{2,\infty}$ be the limiting quadratic evolution defined in equation (2.30). For every $t, s \in \mathbb{R}$ there exists a bounded linear map*

$$\Theta(t; s): L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

such that

$$\mathcal{U}_{2,\infty}^*(t; s)A(f, g)\mathcal{U}_{2,\infty}(t; s) = A(\Theta(t; s)(f, g))$$

for all $f, g \in L^2(\mathbb{R}^3)$. Here, $A(f, g) = a(f) + a^*(\bar{g})$. The map $\Theta(t; s)$ satisfies

$$\Theta(t; s)\mathcal{J} = \mathcal{J}\Theta(t; s), \quad S = \Theta(t; s)^*S\Theta(t; s),$$

where $\mathcal{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$ with J denoting complex conjugation on $L^2(\mathbb{R}^3)$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It can be written as

$$\Theta(t; s) = \begin{pmatrix} U(t, s) & JV(t, s)J \\ V(t, s) & JU(t, s)J \end{pmatrix} \quad (2.33)$$

for bounded linear maps $U(t, s), V(t, s): L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ satisfying

$$\begin{aligned} U^*(t, s)U(t, s) - V^*(t, s)V(t, s) &= 1, \\ U^*(t, s)JV(t, s)J &= V^*(t, s)JU(t, s)J. \end{aligned}$$

Remark 2.10. Differentiating the action of $\mathcal{U}_{2,\infty}$ on $A(f, g)$ yields the differential equation $-i\partial_t\Theta(t, s) = \Theta(t, s)\mathcal{A}(t)$ with

$$\mathcal{A}(t) = \begin{pmatrix} -\Delta + 8\pi\alpha|\varphi_t|^2 + q_t G_{t,\infty} q_t & -J(2q_t \overline{H_{t,\infty}} q_t)J \\ 2q_t \overline{H_{t,\infty}} q_t & -J(-\Delta + 8\pi\alpha|\varphi_t|^2 + q_t G_{t,\infty} q_t)J \end{pmatrix}.$$

This can be compared with [6, equation (2.22)] in the mean-field setting.

The proof follows from [6, Theorem 2.2] and [56, Proposition 1.3]; using the notation introduced in the last two lines of (2.28), the bounds $\|G_{t,\infty}\|_{\text{op}}, \|H_{t,\infty}\|_2 < \infty$, which are assumed in [56] follow from the analysis in the proof of Proposition 5.2.

Using the approximation in terms of the Bogoliubov dynamics $\mathcal{U}_{2,\infty}$, we can establish a central limit theorem for the evolution of initial data approximating ground states of the trapped Hamiltonian (1.1). Similar results have been obtained for the evolution generated by the Hamilton operator (1.9) in [6, 20] (if $\beta = 0$) and in [56] (for $0 < \beta < 1$). In the time-independent setting, an analogous central limit theorem has been established in [57], in the Gross–Pitaevskii regime.

Theorem 2.11. *Under the same assumptions as in Theorem 2.2, consider initial wave functions having the form*

$$\psi_N = U_{N,0}^* e^{B_0} e^{B} \Omega,$$

where B_0 is defined as in (2.9), and

$$B = \frac{1}{2} \int [\tau(x, y) b_x^* b_y^* - \bar{\tau}(x; y) b_x b_y] dx dy \quad (2.34)$$

with kernel $\tau \in (q_0 \otimes q_0)H^2(\mathbb{R}^3 \times \mathbb{R}^3)$ (where $q_0 = 1 - |\varphi\rangle\langle\varphi|$ projects orthogonally to the condensate wave function φ used in $U_{N,0}^*$). For a bounded operator O on $L^2(\mathbb{R}^3)$ we define

$$\mathcal{O}_{N,t} = \frac{1}{\sqrt{N}} \sum_{j=1}^N (O^{(j)} - \langle\varphi_t, O\varphi_t\rangle), \quad (2.35)$$

where $O^{(j)} = 1 \otimes \cdots \otimes O \otimes \cdots \otimes 1$ is the operator O acting only on the j th particle. We set, with U, V indicating the linear maps introduced in (2.33),

$$\begin{aligned} h_{\infty,t} &= \cosh(\eta_{t,\infty})q_{\varphi_t} O\varphi_t + \sinh(\eta_{t,\infty})\overline{q_{\varphi_t} O\varphi_t}, \\ n_t &= U(t, 0)h_{\infty,t} + \overline{V(t, 0)}\overline{h_{\infty,t}}, \\ f_t &= \cosh(\tau)n_t + \sinh(\tau)\overline{n_t}. \end{aligned}$$

Then there exist $c > 0$ and, for all $-\infty < a < b < \infty$, $C > 0$ such that, in the state $\psi_{N,t} = e^{-iH_{Nt}}\psi_N$,

$$|\mathbb{P}_{\psi_{N,t}}(\mathcal{O}_{N,t} \in [a, b]) - \mathbb{P}(\mathcal{G}_t \in [a, b])| \leq C e^{ce^{c|t|}} N^{-1/16} \quad (2.36)$$

for all N large enough. Here, \mathcal{G}_t is a centered Gaussian with variance $\|f_t\|^2$.

Remark 2.12. The bound (2.36) follows through standard arguments (see [20, Proof of Corollary 1.2]) from an estimate of the form

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} g(\mathcal{O}_{N,t}) - \frac{1}{\sqrt{2\pi}\|f_t\|} \int dx g(x) e^{-\frac{x^2}{2\|f_t\|^2}} \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2}|s|^3 \|O\|^3 + N^{-1}|s|^4 \|O\|^4) ds \end{aligned} \quad (2.37)$$

for the expectation of the random variable $g(\mathcal{O}_{N,t})$, valid for any $g \in L^1(\mathbb{R})$, with $(1 + s^4)\hat{g}(s) \in L^1(\mathbb{R})$. The proof of (2.37), which is based on the approximation (2.23) of the many body evolution, is given in Section 6. Similarly to [20, 56, 57], we could also extend (2.37) to a multivariate central limit theorem, proving that expectations of products of observables of the form (2.35) approach a Gaussian limit, as $N \rightarrow \infty$.

3. Fluctuation dynamics

While the wave function $e^{-i \int_0^t \kappa_N(s) ds} U_{N,t}^* e^{B_t} \mathcal{U}_{2,N}(t) \xi_N$ appearing in (2.23) provides a good norm approximation for the full evolution $e^{-iH_{Nt}} \psi_N = e^{-iH_{Nt}} U_{N,0}^* e^{B_0} \xi_N$, the difference between their energies does not converge to zero, as $N \rightarrow \infty$. For this reason, it seems difficult to show (2.23) directly. To circumvent this problem, we introduce an alternative approximation for the many-body evolution, this time having the correct energy. At the end, we will show that the two approximations are close in norm.

To define the new approximation of the many-body evolution, we will use a cubic phase. First, we define, as in [54], a cutoff $\Theta: \mathbb{N} \rightarrow \mathbb{R}$ in the number of particles, setting

$$\Theta(n) = \begin{cases} 1, & n \leq \frac{M}{2} + 10, \\ \frac{1}{2} \left(\frac{4n - 3M}{40 - M} + 1 \right), & \frac{M}{2} + 10 \leq n \leq M - 10, \\ 0, & n \geq M - 10, \end{cases}$$

for $M = N^\varepsilon \geq 50$ where $\varepsilon < 1$. Throughout this paper we always assume that N is sufficiently large so that this last condition holds true. Next we define the kernel

$$v_t(x, y) = -N w_m(N(x - y)) \tilde{\varphi}_t(y) \quad (3.1)$$

with $m = N^{-\alpha}$ and we introduce the antisymmetric operator

$$A_t = \frac{\Theta(N)}{\sqrt{N}} \int v_t(x, y) \tilde{b}_x^* \tilde{b}_y^* [\tilde{b}(\gamma_x) + \tilde{b}^*(\sigma_x)] dx dy - \text{h.c.}, \quad (3.2)$$

where we recall the definition (2.15) of the projected operators $\tilde{b}_x = b(\tilde{q}_x) = b_x - \tilde{\varphi}_t(x)b(\tilde{\varphi}_t)$. Notice that, in contrast with the quadratic kernel (2.7), we cut off (3.1) on a length scale $m = N^{-\alpha}$, vanishing as $N \rightarrow \infty$. This makes sure that, as discussed in Lemma A.5, $\|v\|_2 \leq C\sqrt{m} \leq CN^{-1/4}$ is small and therefore it allows us to compute the action of the phase e^{A_t} , which will be used in combination with the generalized Bogoliubov transformation e^{B_t} to generate the microscopic correlation structure, expanding the exponential to first and second order (all higher-order contributions will be negligible). In the following, we will choose $\alpha = \varepsilon = 1/2$.

An important observation is that conjugation with e^{A_t} does not substantially change the number of excitations and their energy. The proof of the next lemma is deferred to Section 7.

Lemma 3.1. *Let A_t be the anti-symmetric operator defined in (3.2). For every $k \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$e^{-A_t}(\mathcal{N} + 1)^k e^{A_t} \leq C e^{c|t|} (\mathcal{N} + 1)^k. \quad (3.3)$$

Let $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, with

$$\mathcal{K} = \int \nabla_x a_x^* \nabla_x a_x, \quad \mathcal{V}_N = \frac{1}{2} \int N^2 V(N(x-y)) a_x^* a_y^* a_x a_y. \quad (3.4)$$

Then, for every $k \in \mathbb{N}$, there is $C > 0$ such that

$$e^{-A_t}(\mathcal{H}_N + \mathcal{N} + 1)(\mathcal{N} + 1)^k e^{A_t} \leq C e^{c|t|} (\mathcal{H}_N + \mathcal{N} + 1)(\mathcal{N} + 1)^k \quad (3.5)$$

holds true as an operator inequality on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$.

Remark 3.2. In (3.5) it is crucial that the exponent k is the same on both sides of the inequality. In fact, this is the reason for the introduction of the cutoff $\Theta(\mathcal{N})$ in (3.2). For cubic phases without cutoff, such as the one used in the time-independent setting in [18], an estimate similar to (3.5) holds true, but only with an additional term $(\mathcal{N} + 1)^{k+2}$ on the right-hand side of the equation; see [18, Lemma 5.8].

With the cubic phase, we define a new fluctuation dynamics \mathcal{U}_N , setting

$$\mathcal{U}_N(t; s) = e^{-A_t} \bar{\mathcal{U}}_N(t; s) e^{A_s} = e^{-A_t} e^{-B_t} U_{N,t} e^{-iH_N(t-s)} U_{N,s}^* e^{B_s} e^{A_s}. \quad (3.6)$$

To prove Theorem 2.2, we will first show that the difference between $\bar{\mathcal{U}}_N(t; 0)\xi_N$ and $\mathcal{U}_N(t; 0)\xi_N$ is small in norm, in the limit $N \rightarrow \infty$. Afterwards, we will prove (2.24), but with $\bar{\mathcal{U}}_N$ replaced by \mathcal{U}_N . To this end, we will need some properties of the cubically renormalized fluctuation dynamics \mathcal{U}_N , which we establish in the rest of this section.

First of all, we need to control the growth of the number of particles and of the energy along the evolution \mathcal{U}_N . The proof of the next proposition is based on the estimates in [17, Proposition 6.1] for the dynamics $\bar{\mathcal{U}}_N$.

Proposition 3.3. *Under the same assumptions as in Theorem 2.2, let \mathcal{U}_N be defined as in (3.6). Then there exist $C, c > 0$ such that*

$$\langle \mathcal{U}_N(t; 0)\xi, (\mathcal{H}_N + \mathcal{N})\mathcal{U}_N(t; 0)\xi \rangle \leq C e^{c e^{c|t|}} \langle \xi, (\mathcal{H}_N + \mathcal{N} + 1)\xi \rangle,$$

for all $\xi \in \mathcal{F}_{\perp\varphi}^{\leq N}$ and all $t \in \mathbb{R}$.

Proof. From Lemma 3.1 (with $k = 0$), we find

$$\begin{aligned} & \langle \mathcal{U}_N(t; 0)\xi, (\mathcal{H}_N + \mathcal{N})\mathcal{U}_N(t; 0)\xi \rangle \\ &= \langle \bar{\mathcal{U}}_N(t; 0)e^{A_0}\xi, e^{A_t}(\mathcal{H}_N + \mathcal{N})e^{-A_t}\bar{\mathcal{U}}_N(t; 0)e^{A_0}\xi \rangle \\ &\leq C e^{c|t|} \langle \bar{\mathcal{U}}_N(t; 0)e^{A_0}\xi, (\mathcal{H}_N + \mathcal{N} + 1)\bar{\mathcal{U}}_N(t; 0)e^{A_0}\xi \rangle. \end{aligned}$$

From [17, Proposition 6.1 and the following remark] and applying Lemma 3.1 again, we conclude

$$\begin{aligned} \langle \mathcal{U}_N(t; 0)\xi, (\mathcal{H}_N + \mathcal{N})\mathcal{U}_N(t; 0)\xi \rangle &\leq C e^{c|t|} \langle e^{A_0}\xi, (\mathcal{H}_N + \mathcal{N} + 1)e^{A_0}\xi \rangle \\ &\leq C e^{c|t|} \langle \xi, (\mathcal{H}_N + \mathcal{N} + 1)\xi \rangle. \quad \blacksquare \end{aligned}$$

While controlling the growth of the expectation of \mathcal{N} , \mathcal{H}_N with respect to the fluctuation dynamics \mathcal{U}_N is enough to establish convergence of the reduced density, to obtain a norm approximation for the dynamics \mathcal{U}_N we need more precise information on its generator. To this end, we remark that (3.6) satisfies the Schrödinger-type equation

$$i \partial_t \mathcal{U}_N(t; s) = \mathcal{J}_N(t) \mathcal{U}_N(t; s),$$

with the time-dependent generator $\mathcal{J}_N(t)$ given by

$$\begin{aligned} \mathcal{J}_N(t) &= [i \partial_t e^{-A_t}] e^{A_t} \\ &+ e^{-A_t} [(i \partial_t e^{-B_t}) e^{B_t} + e^{-B_t} [(i \partial_t U_{N,t}) U_{N,t}^* + U_{N,t} H_N U_{N,t}^*] e^{B_t}] e^{A_t}. \end{aligned} \quad (3.7)$$

In the next proposition, we compute $\mathcal{J}_N(t)$ up to errors vanishing in the limit $N \rightarrow \infty$. The proof of this proposition is deferred to Section 8.

Proposition 3.4. *Under the same assumptions as in Theorem 2.2, let $\mathcal{J}_N(t)$ be the generator of the fluctuation dynamics \mathcal{U}_N , as defined in (3.7). Then we have, as a quadratic form on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$,*

$$\mathcal{J}_N(t) = \kappa_N(t) + \mathcal{J}_{2,N}(t) + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N}(t),$$

where $\kappa_N(t)$ and $\mathcal{J}_{2,N}(t)$ are defined as in (2.20), (2.16) and where the error term $\mathcal{E}_{\mathcal{J}_N}(t)$ satisfies

$$|\langle \xi_1, \mathcal{E}_{\mathcal{J}_N}(t)\xi_2 \rangle| \leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2}\xi_1\| \|(\mathcal{H}_N + \mathcal{N}^3 + 1)^{1/2}(\mathcal{N} + 1)\xi_2\|$$

for any $\xi_1, \xi_2 \in \mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$.

4. Quadratic evolution and the proof of Theorem 2.2

In this section we study the quadratic evolution defined in (2.19). First of all, we establish important properties of the time-dependent generator $\mathcal{J}_{2,N}$.

Proposition 4.1. *Let $\mathcal{J}_{2,N}$ be defined as in equation (2.16). Under the same assumptions as in Theorem 2.2, there exist constants $C, c > 0$ such that*

$$\pm(\mathcal{J}_{2,N}(t) - \mathcal{K}) \leq C e^{c|t|}(\mathcal{N} + 1), \quad (4.1)$$

$$(\mathcal{J}_{2,N}(t) - \mathcal{K})^2 \leq C e^{c|t|}(\mathcal{N} + 1)^2, \quad (4.2)$$

$$\pm \dot{\mathcal{J}}_{2,N}(t) \leq C e^{c|t|}(\mathcal{N} + 1), \quad (4.3)$$

$$\dot{\mathcal{J}}_{2,N}^2(t) \leq C e^{c|t|}(\mathcal{N} + 1)^2, \quad (4.4)$$

$$|\langle \xi_1, [\mathcal{N}, \mathcal{J}_{2,N}(t)] \xi_2 \rangle| \leq C e^{c|t|} \|(\mathcal{N} + 1)^{(1+j)/2} \xi_1\| \|(\mathcal{N} + 1)^{(1-j)/2} \xi_2\|, \quad (4.5)$$

for $j \in \mathbb{N}$ and $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$.

To show Proposition 4.1 (and some of the other bounds discussed in this section), the following lemma will be useful. Its proof is a straightforward adaptation of [10, Lemma 3.4, 3.6].

Lemma 4.2. *Let F be a bounded operator and J_1, J_2 two Hilbert–Schmidt operators on $L^2(\mathbb{R}^3)$. We also denote by F, J_1, J_2 the integral kernels of the three operators ($J_1, J_2 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, while F is in general a distribution). Let*

$$\begin{aligned} A_1 &= \int dx b^\#(J_{1,x}) b^\#(J_{2,x}), & A_2 &= \int dx b^\#(J_{1,x}) b_x, \\ A_3 &= \int dx dy F(x, y) b_x^* b_y \end{aligned} \quad (4.6)$$

where $\# = *, \cdot$. Then we have

$$\begin{aligned} |\langle \xi_1, A_1 \xi_2 \rangle| &\leq C \|J_1\|_2 \|(\mathcal{N} + 1)^{(1+p)/2} \xi_1\| \|(\mathcal{N} + 1)^{(1-p)/2} \xi_2\|, \\ |\langle \xi_1, A_2 \xi_2 \rangle| &\leq C \|J_1\|_2 \|J_2\|_2 \|(\mathcal{N} + 1)^{(1+p)/2} \xi_1\| \|(\mathcal{N} + 1)^{(1-p)/2} \xi_2\|, \\ |\langle \xi_1, A_3 \xi_2 \rangle| &\leq C \|F\|_{\text{op}} \|(\mathcal{N} + 1)^{(1+p)/2} \xi_1\| \|(\mathcal{N} + 1)^{(1-p)/2} \xi_2\|, \end{aligned} \quad (4.7)$$

for all $p \in \mathbb{Z}$ and $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$. Moreover,

$$\begin{aligned} A_1^* A_1 + A_1 A_1^* &\leq C \|J_1\|_2^2 (\mathcal{N} + 1)^2, \\ A_2^* A_2 + A_2 A_2^* &\leq C \|J_1\|_2^2 \|J_2\|_2^2 (\mathcal{N} + 1)^2, \\ A_3^2 &\leq C \|F\|_{\text{op}}^2 (\mathcal{N} + 1)^2. \end{aligned} \quad (4.8)$$

Remark 4.3. The bounds in Lemma 4.2 continue to hold true if we replace the operators b, b^* (or the corresponding operator-valued distributions) with the projected operators \tilde{b}, \tilde{b}^* , introduced in (2.15) and used in (2.16) to define the generator $\mathcal{J}_{2,N}(t)$. In fact, it is

easy to see that switching from b, b^* to \tilde{b}, \tilde{b}^* corresponds to multiplying the operators J_1, J_2, F by the orthogonal projection $\tilde{q}_t = 1 - |\tilde{\varphi}_t\rangle\langle\tilde{\varphi}_t|$ on the right and/or on the left; this does not increase the norms $\|J_1\|_2, \|J_2\|_2, \|F\|_{\text{op}}$ appearing on the right-hand side of (4.7), (4.8). Furthermore, (4.7), (4.8) (and their proofs) also hold true for operators $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ on the full Fock space \mathcal{F} (without truncation to $\mathcal{N} \leq N$), defined like A_1, A_2, A_3 , but with b, b^* replaced by the standard creation and annihilation operators a, a^* or by their projected versions \hat{a}, \hat{a}^* , used in the definition of the limiting generator (2.28).

Proof of Proposition 4.1. With the notation introduced in the last two lines on the right-hand side of (2.16), we claim that $\|G_t\|_{\text{op}}, \|H_t\|_2 \leq Ce^{c|t|}$. For most terms this follows easily from Lemma A.1, Proposition A.2 and Lemma A.3. In fact, all contributions (to G_t and to H_t) arising from the kinetic part (2.17) that involve derivatives of p, r or μ have L^2 -norm (and therefore also operator norm) less than $Ce^{c|t|}$. Also, the term

$$\begin{aligned} \int dx \tilde{b}^*(\nabla_x \eta_x) \tilde{b}(\nabla_x \eta_x) &= \int dz dw \left[\int dx \nabla_x \eta_t(z; x) \nabla_x \eta_t(x, w) \right] \tilde{b}_z^* \tilde{b}_w \\ &=: \int dz dw u_t(z, w) \tilde{b}_z^* \tilde{b}_w \end{aligned}$$

can be bounded with Lemma 4.2, since $\|u_t\|_{\text{op}} \leq \|u_t\|_2 \leq Ce^{c|t|}$ by Lemma A.3. To handle the contributions arising from the potential part (2.18), we observe that $N^3(V_N f_{N,\ell})(x-y) = N^3 V(N(x-y)) f_\ell(N(x-y))$ is the integral kernel of the differential operator $\hat{V} f_\ell(i\nabla/N)$ on $L^2(\mathbb{R}^3)$ (with operator norm bounded by $\hat{V}(0)$), that γ_t, p_t and σ_t are also bounded operators on $L^2(\mathbb{R}^3)$ and that all off-diagonal terms (contributing to H) involve at least one factor of p_t or σ_t (with bounded Hilbert–Schmidt norm, uniformly in N). Finally, let us consider the contributions from the other terms on the right-hand side of (2.16) (line 2 to line 7). Most of them can be handled as above, with the estimates from Lemma A.1, Proposition A.2 and Lemma A.3 (notice, in particular, that $N^3 \lambda_\ell \leq C$). Some more attention is needed for the term involving $\nabla w_{N,\ell}$. Using Lemma A.1, we bound

$$\begin{aligned} &|N \nabla w_{N,\ell}(x-y) [\nabla \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) - \tilde{\varphi}_t(x) \nabla \tilde{\varphi}_t(y)]| \\ &\leq \frac{C \chi(|x-y| \leq \ell)}{|x-y|^2} (|\nabla \tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y) - \tilde{\varphi}_t(x)| + |\tilde{\varphi}_t(x)| |\nabla \tilde{\varphi}_t(x) - \nabla \tilde{\varphi}_t(y)|) \\ &\leq \frac{C \|\tilde{\varphi}_t\|_{H^4} \chi(|x-y| \leq \ell)}{|x-y|} (|\nabla \tilde{\varphi}_t(x)| + |\tilde{\varphi}_t(x)|). \end{aligned} \quad (4.9)$$

Thus, the L^2 -norm of the left-hand side is bounded, uniformly in N . This concludes the proof of the bounds $\|G_t\|_{\text{op}}, \|H_t\|_2 \leq Ce^{c|t|}$. From Lemma 4.2 (and from Remark 4.3) we arrive at (4.1) and, using (4.8), to (4.2) (the second term on the right-hand side of (2.16), the one proportional to $a_x^* a_x$, can be handled in the same way). Since

$$[\mathcal{N}, \mathcal{J}_{2,N}(t)] = \int dx dy [2H_t(x, y) \tilde{b}_x^* \tilde{b}_y^* - 2\bar{H}_t(x, y) \tilde{b}_x \tilde{b}_y]$$

we also conclude (4.5).

As for (4.3) and (4.4), we observe that contributions to the time derivative $\dot{\mathcal{J}}_{2,N}(t)$ have the same form as contributions to $\mathcal{J}_{2,N}(t)$, either with a factor $\tilde{\varphi}_t$ replaced by $\dot{\tilde{\varphi}}_t$, or with one of the kernels $\eta_t, \gamma_t, \sigma_t, p_t, r_t$ replaced by $\dot{\eta}_t, \dot{\gamma}_t, \dot{\sigma}_t, \dot{p}_t, \dot{r}_t$, or with one of the operators \tilde{b}, \tilde{b}^* replaced by its time derivative (the projection depends on time). Using

$$\partial_t \tilde{b}_x^* = -\overline{\dot{\tilde{\varphi}}_t(x)} b^*(\tilde{\varphi}_t) - \overline{\tilde{\varphi}_t(x)} b^*(\dot{\tilde{\varphi}}_t),$$

a similar formula for $\partial_t \tilde{b}_x$ and the bounds in Lemma A.1, Proposition A.2 and Lemma A.3, we conclude that $\dot{\mathcal{J}}_{2,N}(t)$ can be written as the sum of terms of the form (4.6)¹ (with two projected operators \tilde{b}^\sharp or with one \tilde{b}^\sharp and one b^\sharp). For this reason, Lemma 4.2 also implies (4.3) and (4.4) (again, the term proportional to $a_x^* a_x$ on the right-hand side of (2.16) can be handled similarly). ■

With the help of Proposition 4.1, we obtain well-posedness of equation (2.19) (existence and uniqueness of the unitary quadratic evolution $\mathcal{U}_{2,N}$) and control on the growth of kinetic energy and the number of particles. Compared with the bounds obtained in Proposition 3.3 for the full fluctuation dynamics, here we can derive stronger estimates, controlling arbitrary moments of the number of particles operator \mathcal{N} and the second moment of the kinetic energy operator \mathcal{K} . These improvements (which we can only show for $\mathcal{U}_{2,N}$ and not for the full fluctuation dynamics \mathcal{U}_N) will play a crucial role in the proof of Theorem 2.2.

Proposition 4.4. *Under the same assumptions as in Theorem 2.2, let $\mathcal{J}_{2,N}$ be defined as in equation (2.16). Then $\mathcal{J}_{2,N}(t)$ generates a unique strongly continuous two-parameter group $\mathcal{U}_{2,N}$ of unitary transformations satisfying the Schrödinger equation (2.19). For every $t, s \in \mathbb{R}$, $\mathcal{U}_{2,N}(t; s): \mathcal{F}_{\perp \tilde{\varphi}_s}^{\leq N} \rightarrow \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$. Moreover, for every $k \in \mathbb{N}$ there exist $C, c > 0$ such that*

$$\begin{aligned} \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{N} + 1)^k \mathcal{U}_{2,N}(t; 0)\xi \rangle &\leq C e^{ce^{c|t|}} \langle \xi, (\mathcal{N} + 1)^k \xi \rangle, \\ \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{K}^2 \mathcal{U}_{2,N}(t; 0)\xi \rangle &\leq C e^{ce^{c|t|}} \langle \xi, (\mathcal{K}^2 + \mathcal{N}^2 + 1)\xi \rangle. \end{aligned} \quad (4.10)$$

Remark 4.5. From [10, Lemma 3.10] (extended trivially to the case $\beta = 1$), we have

$$\mathcal{V}_N \leq \frac{C}{N} (\mathcal{K}^2 + \mathcal{N}^2), \quad (4.11)$$

which also implies $\mathcal{H}_N \leq C(\mathcal{K}^2 + \mathcal{N}^2 + 1)$ and

$$(\mathcal{H}_N + \mathcal{N} + 1)(\mathcal{N} + 1) \leq C(\mathcal{K}^2 + (\mathcal{N} + 1)^2).$$

¹In fact, from (4.9) we find a contribution proportional to $\|\dot{\tilde{\varphi}}_t\|_{H^4}$; this is the only term where control of the H^6 -norm of $\tilde{\varphi}_t$ is needed.

Proof of Proposition 4.4. To prove the well-posedness, we proceed as in [44, Theorem 7]. Note that (4.1), (4.3) and (4.5) are precisely the inequalities shown in [44, Lemma 9] and needed to apply the abstract result in [44, Theorem 8].

To show that $\mathcal{U}_{2,N}(t; s)$ maps $\mathcal{F}_{\perp\tilde{\varphi}_s}^{\leq N}$ into $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$, let us consider $\xi \in \mathcal{F}_{\perp\tilde{\varphi}_s}^{\leq N}$, which implies that $a(\tilde{\varphi}_s)\xi = 0$. We compute

$$\begin{aligned} & \frac{d}{dt} \|a(\tilde{\varphi}_t)\mathcal{U}_{2,N}(t; s)\xi\|^2 \\ &= 2 \operatorname{Re} \left\langle a(\tilde{\varphi}_t)\mathcal{U}_{2,N}(t; s)\xi, \frac{d}{dt} (a(\tilde{\varphi}_t)\mathcal{U}_{2,N}(t; s)\xi) \right\rangle \\ &= 2 \operatorname{Im} \langle a(\tilde{\varphi}_t)\mathcal{U}_{2,N}(t; s)\xi, (-a(i\dot{\tilde{\varphi}}_t) + a(\tilde{\varphi}_t)\mathcal{J}_{2,N}(t))\mathcal{U}_{2,N}(t; s)\xi \rangle \\ &= -2 \operatorname{Im} \langle \mathcal{U}_{2,N}(t; s)\xi, a^*(\tilde{\varphi}_t)a(i\dot{\tilde{\varphi}}_t)\mathcal{U}_{2,N}(t; s)\xi \rangle \\ &\quad - i \langle \mathcal{U}_{2,N}(t; s)\xi, [a^*(\tilde{\varphi}_t)a(\tilde{\varphi}_t), \mathcal{J}_{2,N}(t)]\mathcal{U}_{2,N}(t; s)\xi \rangle = 0, \end{aligned}$$

where we used that $[a^*(\tilde{\varphi}_t)a(\tilde{\varphi}_t), \tilde{b}_x^*] = 0$ and

$$\left[a(\tilde{\varphi}_t), \mathcal{K} + \int dx N^3 (V_N f_{N,\ell}) * |\tilde{\varphi}_t|^2 a_x^* a_x \right] = a(i\dot{\tilde{\varphi}}_t).$$

Let us now show (4.10). From (4.5), we find

$$\begin{aligned} & \left| \frac{d}{dt} \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{N} + 1)^k \mathcal{U}_{2,N}(t; 0)\xi \rangle \right| \\ &= |\langle \mathcal{U}_{2,N}(t; 0)\xi, i[(\mathcal{N} + 1)^k, \mathcal{J}_{2,N}(t)]\mathcal{U}_{2,N}(t; 0)\xi \rangle| \\ &\leq C e^{c|t|} \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{N} + 1)^k \mathcal{U}_{2,N}(t; 0)\xi \rangle. \end{aligned}$$

The first estimate in (4.10) follows from Grönwall's lemma.

As for the second bound in (4.10), we first apply (4.2) to estimate

$$\begin{aligned} & \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{K}^2 \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\leq 2 \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\quad + 2 \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{J}_{2,N}(t) - \mathcal{K})^2 \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\leq 2 \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\quad + C e^{c|t|} \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{N} + 1)^2 \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\leq 2 \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle + C e^{c|t|} \langle \xi, (\mathcal{N} + 1)^2 \xi \rangle \quad (4.12) \end{aligned}$$

where, in the last inequality, we applied the first bound in (4.10). To control the first term on the right-hand side of the last equation, we observe that

$$\begin{aligned} & \frac{d}{dt} \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &= \langle \mathcal{U}_{2,N}(t; 0)\xi, [\mathcal{J}_{2,N}(t)\dot{\mathcal{J}}_{2,N}(t) + \dot{\mathcal{J}}_{2,N}(t)\mathcal{J}_{2,N}(t)]\mathcal{U}_{2,N}(t; 0)\xi \rangle \\ &\leq \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle + \langle \mathcal{U}_{2,N}(t; 0)\xi, \dot{\mathcal{J}}_{2,N}^2(t) \mathcal{U}_{2,N}(t; 0)\xi \rangle. \end{aligned}$$

With (4.4) and with the first bound in (4.10), we conclude that

$$\begin{aligned} & \frac{d}{dt} \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t)\mathcal{U}_{2,N}(t; 0)\xi \rangle \\ & \leq \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t)\mathcal{U}_{2,N}(t; 0)\xi \rangle + Ce^{c|t|} \langle \mathcal{U}_{2,N}(t; 0)\xi, (\mathcal{N} + 1)^2\mathcal{U}_{2,N}(t; 0)\xi \rangle \\ & \leq \langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t)\mathcal{U}_{2,N}(t; 0)\xi \rangle + Ce^{ce^{c|t|}} \langle \xi, (\mathcal{N} + 1)^2\xi \rangle. \end{aligned}$$

Applying Grönwall's lemma, we conclude that

$$\langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t)\mathcal{U}_{2,N}(t; 0)\xi \rangle \leq Ce^{ce^{c|t|}} \langle \xi, (\mathcal{J}_{2,N}^2(0) + \mathcal{N}^2 + 1)\xi \rangle.$$

With (4.2) (at time $t = 0$), we find therefore

$$\langle \mathcal{U}_{2,N}(t; 0)\xi, \mathcal{J}_{2,N}^2(t)\mathcal{U}_{2,N}(t; 0)\xi \rangle \leq Ce^{ce^{c|t|}} \langle \xi, (\mathcal{K}^2 + \mathcal{N}^2 + 1)\xi \rangle.$$

Inserting into (4.12), we obtain the desired bound. \blacksquare

Combining Propositions 3.3, 3.4 and 4.4, we can now proceed with the proof of our first main theorem.

Proof of Theorem 2.2. As observed in Remark 2.3, we have

$$\begin{aligned} & \|e^{-iH_N t} \psi_N - e^{-i \int_0^t \kappa_N(s) ds} U_{N,t}^* e^{B_t} \mathcal{U}_{2,N}(t; 0)\xi_N\| \\ & = \|\bar{\mathcal{U}}_N(t; 0)\xi_N - e^{-i \int_0^t \kappa_N(s) ds} \mathcal{U}_{2,N}(t; 0)\xi_N\|. \end{aligned}$$

Next we introduce cubic phases to pass from $\bar{\mathcal{U}}_N$ to the new fluctuation dynamics \mathcal{U}_N . To this end, we estimate

$$\begin{aligned} & \|e^{-iH_N t} \psi_N - e^{-i \int_0^t \kappa_N(s) ds} U_{N,t}^* e^{B_t} \mathcal{U}_{2,N}(t; 0)\xi_N\| \\ & \leq \|\bar{\mathcal{U}}_N(t; 0)e^{A_0}\xi_N - e^{-i \int_0^t \kappa_N(s) ds} e^{A_t} \mathcal{U}_{2,N}(t; 0)\xi_N\| \\ & \quad + \|e^{A_0}\xi_N - \xi_N\| + \|e^{A_t} \mathcal{U}_{2,N}(t; 0)\xi_N - \mathcal{U}_{2,N}(t; 0)\xi_N\|. \end{aligned}$$

Writing

$$\|e^{A_0}\xi_N - \xi_N\|^2 = 2 - 2\operatorname{Re} \langle \xi_N, e^{-A_0}\xi_N \rangle = \operatorname{Re} \int_0^1 ds \langle \xi_N, A_0 e^{-sA_0}\xi_N \rangle$$

and estimating (recalling the definition (3.2))

$$\begin{aligned} |\langle \xi_1, A_0 \xi_2 \rangle| & \leq CN^{-1/2} \int |v_0(x, y)| \|a_x a_y \xi_1\| [\|a_x \xi_2\| + \|(\mathcal{N} + 1)^{1/2} \xi_2\|] dx dy \\ & \quad + CN^{-1/2} \int |v_0(x, y)| \|a_x a_y (\mathcal{N} + 1)^{-1/2} \xi_2\| \\ & \quad \quad \times [\|a_x (\mathcal{N} + 1)^{1/2} \xi_1\| + \|(\mathcal{N} + 1) \xi_1\|] dx dy \\ & \leq CN^{-1/2} \left[\sup_x \|v_{0,x}\|_2 + \|v_0\|_2 \right] \|\mathcal{N} \xi_1\| \|(\mathcal{N} + 1)^{1/2} \xi_2\|, \end{aligned}$$

we obtain, using the estimates $\sup_x \|v_{0,x}\|_2, \|v_0\|_2 \leq C\sqrt{m} \leq CN^{-1/4}$ from Lemma A.5,

$$\|e^{A_0}\xi_N - \xi_N\|^2 \leq CN^{-3/4} \int_0^1 ds \|(\mathcal{N} + 1)\xi_N\| \|(\mathcal{N} + 1)^{1/2}e^{-sA_0}\xi_N\|.$$

With Lemma 3.1, we arrive at

$$\|e^{A_0}\xi_N - \xi_N\|^2 \leq CN^{-3/4} \|(\mathcal{N} + 1)\xi_N\| \|(\mathcal{N} + 1)^{1/2}\xi_N\|. \quad (4.13)$$

Similarly, also using Proposition 4.4, we find

$$\|e^{-A_0}\mathcal{U}_{2,N}(t;0)\xi_N - \mathcal{U}_{2,N}(t;0)\xi_N\|^2 \leq Ce^{ce^{|t|}} N^{-3/4} \|(\mathcal{N} + 1)\xi_N\| \|(\mathcal{N} + 1)^{1/2}\xi_N\|.$$

Hence, we conclude that

$$\begin{aligned} & \|e^{-iH_N t}\psi_N - e^{-i\int_0^t \kappa_N(s) ds} U_{N,t}^* e^{Bt} \mathcal{U}_{2,N}(t;0)\xi_N\| \\ & \leq \|\mathcal{U}_N(t;0)\xi_N - e^{-i\int_0^t \kappa_N(s) ds} \mathcal{U}_{2,N}(t;0)\xi_N\| + Ce^{ce^{|t|}} N^{-3/8}. \end{aligned} \quad (4.14)$$

We now compute

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{U}_N(t;0)\xi_N - e^{-i\int_0^t ds \kappa_N(s)} \mathcal{U}_{2,N}(t;0)\xi_N\|^2 \\ & = 2 \operatorname{Im} \langle \mathcal{U}_N(t)\xi_N, (\mathcal{J}_N(t) - \mathcal{J}_{2,N}(t) - \kappa_N(t)) e^{-i\int_0^t ds \kappa_N(s)} \mathcal{U}_{2,N}(t)\xi_N \rangle. \end{aligned}$$

With Proposition 3.4 and with the bound (4.11), we obtain

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{U}_N(t;0)\xi_N - e^{-i\int_0^t ds \kappa_N(s)} \mathcal{U}_{2,N}(t;0)\xi_N\|^2 \\ & \leq |\langle \mathcal{U}_N(t;0)\xi_N, \mathcal{E}_{\mathcal{J}_N}(t) \mathcal{U}_{2,N}(t;0)\xi_N \rangle| \\ & \quad + \|\mathcal{V}_N^{1/2} \mathcal{U}_N(t;0)\xi_N\| \|\mathcal{V}_N^{1/2} \mathcal{U}_{2,N}(t;0)\xi_N\| \\ & \leq |\langle \mathcal{U}_N(t;0)\xi_N, \mathcal{E}_{\mathcal{J}_N}(t) \mathcal{U}_{2,N}(t;0)\xi_N \rangle| \\ & \quad + \frac{1}{\sqrt{N}} \|\mathcal{V}_N^{1/2} \mathcal{U}_N(t;0)\xi_N\| \|(\mathcal{K} + \mathcal{N}) \mathcal{U}_{2,N}(t;0)\xi_N\|. \end{aligned}$$

From Proposition 3.4, we conclude that

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{U}_N(t;0)\xi_N - e^{-i\int_0^t \kappa_N(s) ds} \mathcal{U}_{2,N}(t;0)\xi_N\|^2 \\ & \leq \frac{Ce^{c|t|}}{N^{1/4}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \mathcal{U}_N(t;0)\xi_N\| \\ & \quad \times \|(\mathcal{H}_N + \mathcal{N}^3 + 1)^{1/2} (\mathcal{N} + 1) \mathcal{U}_{2,N}(t;0)\xi_N\| \\ & \quad + \frac{1}{\sqrt{N}} \|\mathcal{V}_N^{1/2} \mathcal{U}_N(t;0)\xi_N\| \|(\mathcal{K} + \mathcal{N}) \mathcal{U}_{2,N}(t;0)\xi_N\|. \end{aligned}$$

Using $\mathcal{V}_N \leq C \mathcal{K} \mathcal{N}$ (which follows from the Sobolev inequality, since $\|N^2 V(N \cdot)\|_{3/2} \leq \|V\|_{3/2} < \infty$), we arrive at

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{U}_N(t; 0)\xi_N - e^{-i \int_0^t ds \kappa_N(s)} \mathcal{U}_{2,N}(t; 0)\xi_N\|^2 \\ & \leq \frac{C e^{c|t|}}{N^{1/4}} \langle \mathcal{U}_N(t; 0)\xi_N, (\mathcal{H}_N + \mathcal{N} + 1)\mathcal{U}_N(t; 0)\xi_N \rangle \\ & \quad + \frac{C e^{c|t|}}{N^{1/4}} \langle \mathcal{U}_{2,N}(t; 0)\xi_N, (\mathcal{K}^2 + \mathcal{N}^6 + 1)\mathcal{U}_{2,N}(t; 0)\xi_N \rangle. \end{aligned}$$

With Propositions 3.3 and 4.4, we obtain

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{U}_N(t; 0)\xi_N - e^{-i \int_0^t ds \kappa_N(s)} \mathcal{U}_{2,N}(t; 0)\xi_N\|^2 \\ & \leq \frac{C e^{c|t|}}{N^{1/4}} \langle \xi_N, (\mathcal{K}^2 + \mathcal{N}^6 + 1)\xi_N \rangle. \end{aligned}$$

Integrating over t , using assumption (2.22) and combining with (4.14), we find (2.23). ■

5. Limiting quadratic evolution and proof of Theorem 2.7

In this section we show the well-posedness of the limiting Schrödinger equation (2.30), we control the growth of the number of particles with respect to the limiting quadratic evolution $\mathcal{U}_{2,\infty}$ and we show the convergence of $\mathcal{U}_{2,N}$ to $\mathcal{U}_{2,\infty}$ in the limit $N \rightarrow \infty$, as stated in Theorem 2.7.

Proposition 5.1. *Under the same assumptions as in Theorem 2.2, let $\mathcal{J}_{2,\infty}$ be defined as in (2.28). Then $\mathcal{J}_{2,\infty}(t)$ generates a unique strongly continuous two-parameter group $\mathcal{U}_{2,\infty}$ of unitary transformations satisfying (2.30). For every $t, s \in \mathbb{R}$, $\mathcal{U}_{2,\infty}(t; s): \mathcal{F}_{\perp\varphi_s} \rightarrow \mathcal{F}_{\perp\varphi_t}$, where φ_t denotes the solution of the limiting Gross–Pitaevskii equation (1.7). Moreover, for every $k \in \mathbb{N}$ there exist $C, c > 0$ such that*

$$\langle \mathcal{U}_{2,\infty}(t; 0)\xi, (\mathcal{N} + 1)^k \mathcal{U}_{2,\infty}(t; 0)\xi \rangle \leq C e^{c|t|} \langle \xi, (\mathcal{N} + 1)^k \xi \rangle \quad (5.1)$$

for all $\xi \in \mathcal{F}_{\perp\varphi}$.

Proof. The proof is very similar to the proof of Proposition 4.4. Combining Lemma 4.2 (and Remark 4.3) with the estimates in Lemma A.4, one can prove that the limiting generator $\mathcal{J}_{2,\infty}(t)$ satisfies bounds analogous to (4.1), (4.3), (4.5) in Proposition 4.1. The well-posedness of (2.30), the fact that $\mathcal{U}_{2,\infty}(t; s)$ maps $\mathcal{F}_{\perp\varphi_s}$ into $\mathcal{F}_{\perp\varphi_t}$ and the bound (5.1) can then be shown arguing exactly as in the proof of Proposition 4.4. ■

To prove the convergence of $\mathcal{U}_{2,N}$ towards $\mathcal{U}_{2,\infty}$, we bound the difference of the two generators. Since $\mathcal{J}_{2,N}(t)$ is only defined on the truncated Fock space $\mathcal{F}^{\leq N}$, our estimate is restricted to this space.

Proposition 5.2. *Under the assumptions of Theorem 2.2, we have, for every $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$ and for every $t \in \mathbb{R}$,*

$$\begin{aligned} & |\langle \xi_1, (\mathcal{J}_{2,N}(t; 0) - \mathcal{J}_{2,\infty}(t; 0))\xi_2 \rangle| \\ & \leq \frac{C e^{ce^{|t|}}}{\sqrt{N}} [\|(\mathcal{N} + 1)\xi_1\| \|(\mathcal{N} + 1)\xi_2\| + \|(\mathcal{N} + 1)^{1/2}\xi_1\| \|\mathcal{K}^{1/2}\xi_2\|]. \end{aligned}$$

Proof. From (2.16) and (2.28), we write

$$\begin{aligned} \mathcal{J}_{2,N}(t) &= \mathcal{K} + \int dx N^3 (V_N f_{N,\ell} * |\tilde{\varphi}_t|^2)(x) a_x^* a_x \\ & \quad + \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \tilde{b}_x^* \tilde{b}_y \\ & \quad + \int dx dy (G'_t(x, y) \tilde{b}_x^* \tilde{b}_y + H_t(x, y) \tilde{b}_x^* \tilde{b}_y^* + \bar{H}_t(x, y) \tilde{b}_x \tilde{b}_y), \\ \mathcal{J}_{2,\infty}(t) &= \mathcal{K} + 8\pi\alpha \int dx |\varphi_t(x)|^2 a_x^* a_x + 8\pi\alpha \int dx |\varphi_t(x)|^2 \hat{a}_x^* \hat{a}_x \\ & \quad + \int dx dy (G'_{t,\infty}(x, y) \hat{a}_x^* \hat{a}_y + H_{t,\infty}(x, y) \hat{a}_x^* \hat{a}_y^* + \bar{H}_{t,\infty}(x, y) \hat{a}_x \hat{a}_y). \end{aligned}$$

Observe that, with respect to (2.16), we extracted the operator on the second line from the $\tilde{b}^*(\gamma_x)\tilde{b}(\gamma_y)$ -contribution in the last summand in (2.18), denoting by G'_t the difference between G_t (as appearing in the last line of (2.16)) and this term (we isolate this term because it will require some additional care). In the expansion for $\mathcal{J}_{2,\infty}$, we extracted the corresponding term, proportional to $\hat{a}_x^* \hat{a}_x$, from $G_{t,\infty}$. Thus, we have

$$\mathcal{J}_{2,N}(t) - \mathcal{J}_{2,\infty}(t) = \text{I} + \text{II} + \text{III} + \text{IV},$$

where

$$\begin{aligned} \text{I} &= \int dx [N^3 (V_N f_{N,\ell} * |\tilde{\varphi}_t|^2)(x) - 8\pi\alpha |\varphi_t(x)|^2] a_x^* a_x, \\ \text{II} &= \int dx dy [\tilde{q}_t G'_t \tilde{q}_t - q_t G'_{t,\infty} q_t](x, y) a_x^* a_y - \int dx dy [\tilde{q}_t G'_t \tilde{q}_t](x, y) a_x^* \frac{\mathcal{N}}{N} a_y, \\ \text{III} &= \int dx dy [\tilde{q}_t P_t \tilde{q}_t - q_t P_{t,\infty} q_t](x, y) a_x^* a_y - \int dx dy [\tilde{q}_t P_t \tilde{q}_t](x, y) a_x^* \frac{\mathcal{N}}{N} a_y, \\ \text{IV} &= \int dx dy [(\tilde{q}_t \otimes \tilde{q}_t) H_t - (q_t \otimes q_t) H_{t,\infty}](x, y) a_x^* a_y^* \\ & \quad + \int dx dy [(\tilde{q}_t \otimes \tilde{q}_t) H_t](x, y) a_x^* a_y^* [\sqrt{1 - (\mathcal{N} + 1)/N} \sqrt{1 - \mathcal{N}/N} - 1] + \text{h.c.}, \end{aligned}$$

where we recall that $\tilde{q}_t = 1 - |\tilde{\varphi}_t\rangle\langle\tilde{\varphi}_t|$ and $q_t = 1 - |\varphi_t\rangle\langle\varphi_t|$ and where we introduced the notation $P_t, P_{t,\infty}$ for the operators with the integral kernels $P_t(x, y) = N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y)$, $P_{t,\infty}(x, y) = 8\pi\alpha |\varphi_t(x)|^2 \delta(x-y)$. By Lemma 4.2, we can bound

$$|\langle \xi_1, \text{I}\xi_2 \rangle| \leq C \|N^3 (V_N f_{N,\ell} * |\tilde{\varphi}_t|^2) - 8\pi\alpha |\varphi_t|^2\|_\infty \|(\mathcal{N} + 1)^{1/2}\xi_1\| \|(\mathcal{N} + 1)^{1/2}\xi_2\|.$$

For any $x \in \mathbb{R}^3$ we have

$$\begin{aligned} & |N^3(V_N f_{N,\ell} * |\tilde{\varphi}_t|^2)(x) - 8\pi\alpha|\varphi_t(x)|^2| \\ & \leq \int dz V(z) f_\ell(z) \left| |\tilde{\varphi}_t(x + z/N)|^2 - |\tilde{\varphi}_t(x)|^2 \right| \\ & \quad + \left| \int dz V(z) f_\ell(z) - 8\pi\alpha \right| |\tilde{\varphi}_t(x)|^2 + 8\pi\alpha \left| |\tilde{\varphi}_t(x)|^2 - |\varphi_t(x)|^2 \right|. \end{aligned}$$

Estimating

$$\left| |\tilde{\varphi}_t(x + z/N)|^2 - |\tilde{\varphi}_t(x)|^2 \right| = \left| \int_0^1 ds \frac{d}{ds} |\tilde{\varphi}_t(x + sz/N)|^2 \right| \leq 2|z| \|\nabla \tilde{\varphi}_t\|_\infty \|\tilde{\varphi}_t\|_\infty / N$$

and observing that $\left| |\tilde{\varphi}_t(x)|^2 - |\varphi_t(x)|^2 \right| \leq (\|\tilde{\varphi}_t\|_\infty + \|\varphi_t\|_\infty) \|\tilde{\varphi}_t - \varphi_t\|_\infty$, we conclude with Proposition A.2 and equation (A.2) from Lemma A.1 that

$$|\langle \xi_1, \mathbf{I} \xi_2 \rangle| \leq \frac{C e^{ce^{c|t|}}}{N} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{1/2} \xi_2\|.$$

Let us now consider the term II. Again with Lemma 4.2, we can estimate

$$\begin{aligned} |\langle \xi_1, \mathbf{II} \xi_2 \rangle| & \leq C \|\tilde{q}_t G'_t \tilde{q}_t - q_t G'_{t,\infty} q_t\|_{\text{op}} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{1/2} \xi_2\| \\ & \quad + \frac{C}{N} \|\tilde{q}_t G'_t \tilde{q}_t\|_{\text{op}} \|(\mathcal{N} + 1) \xi_1\| \|(\mathcal{N} + 1) \xi_2\|. \end{aligned}$$

Since $\|\tilde{q}_t - q_t\|_{\text{op}} \leq 2\|\varphi_t - \tilde{\varphi}_t\|_2$, we obtain, with Proposition A.2,

$$\|\tilde{q}_t G'_t \tilde{q}_t - q_t G'_{t,\infty} q_t\|_{\text{op}} \leq \frac{C e^{ce^{c|t|}}}{N} \|G'_t\|_{\text{op}} + \|G'_t - G'_{t,\infty}\|_{\text{op}}.$$

Going through the several contributions to G'_t , $G'_{t,\infty}$ in (2.16) and (2.28) (all diagonal terms) and applying the bounds in Proposition A.2 (iv) and Lemmas A.3 and A.4, we find that $\|G'_t\|_{\text{op}} \leq C e^{c|t|}$ (as already discussed in the proof of Proposition 4.4) and that

$$\|G'_t - G'_{t,\infty}\|_{\text{op}} \leq \frac{C e^{ce^{c|t|}}}{\sqrt{N}}. \quad (5.2)$$

In fact, to compare contributions from (2.18) with the corresponding contributions in (2.29), we often need to control the convergence of $N^3 V(N(x - y)) f_\ell(N(x - y))$ towards $8\pi\alpha\delta(x - y)$. Here, it is important to observe that, in all terms contributing to G'_t (which are compared with terms in $G'_{t,\infty}$), the factor $N^3 V(N(x - y)) f_\ell(N(x - y))$ appears in convolution with a kernel σ_x , σ_y , p_x , p_y (this is not the case for III; that is why this term has to be handled separately). To further illustrate this point, consider for example the difference $D_t - D_{t,\infty}$ contributing to $G'_t - G'_{t,\infty}$, with

$$\begin{aligned} D_t(y, z) & = \int dx N^3 V(N(x - y)) f_\ell(N(x - y)) \tilde{\varphi}_t^2(y) \sigma_t(x, z), \\ D_{t,\infty}(y, z) & = 8\pi\alpha \tilde{\varphi}_t^2(y) \sigma_t(y, z) \end{aligned}$$

(this difference arises from the term proportional to $\tilde{b}^*(\gamma_y)\tilde{b}(\sigma_x)$ in the second summand in (2.18)). With (A.2), we can bound

$$\begin{aligned}
 & |D_t(y, z) - D_{t,\infty}(y, z)| \\
 &= \left| \int dw V(w) f_\ell(w) \tilde{\varphi}_t^2(y) \sigma_t(y + w/N, z) - 8\pi\alpha \tilde{\varphi}_t^2(y) \sigma_t(y, z) \right| \\
 &\leq C e^{c|t|} \int dw V(w) f_\ell(w) |\sigma_t(y + w/N, z) - \sigma_t(y, z)| + \frac{C e^{c|t|}}{N} |\sigma_t(y, z)| \\
 &\leq \frac{C e^{c|t|}}{N} \int dw \int_0^1 ds V(w) f_\ell(w) |w| |\nabla_1 \sigma_t(y + sw/N, z)| \\
 &\quad + \frac{C e^{c|t|}}{N} |\sigma_t(y, z)|, \tag{5.3}
 \end{aligned}$$

which leads to

$$\|D_t - D_{t,\infty}\|_{\text{op}} \leq \|D_t - D_{t,\infty}\|_2 \leq \frac{C e^{c|t|}}{N} [\|\nabla_1 \sigma_t\|_2 + \|\sigma_t\|_2] \leq \frac{C e^{c|t|}}{\sqrt{N}}.$$

With (5.2), we obtain

$$|\langle \xi_1, \text{II} \xi_2 \rangle| \leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{N} + 1) \xi_1\| \|(\mathcal{N} + 1) \xi_2\|.$$

Similarly, we can also estimate

$$\begin{aligned}
 |\langle \xi_1, \text{III} \xi_2 \rangle| &\leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{N} + 1) \xi_1\| \|(\mathcal{N} + 1) \xi_2\| \\
 &\quad + \left| \int dz dx V(z) f_\ell(z) |\varphi_t(x)|^2 \langle a_x \xi_1, (a_x - a_{x+z/N}) \xi_2 \rangle \right|.
 \end{aligned}$$

To bound the last term, we proceed as in [10, Lemma 5.2]. We find

$$\begin{aligned}
 & \left| \int dz dx V(z) f_\ell(z) |\varphi_t(x)|^2 \langle a_x \xi_1, (a_x - a_{x+z/N}) \xi_2 \rangle \right| \\
 &= \left| \int dz dx \int_0^1 ds V(z) f_\ell(z) |\varphi_t(x)|^2 \frac{d}{ds} \langle a_x \xi_1, a_{x+sz/N} \xi_2 \rangle \right| \\
 &\leq \frac{1}{N} \int dz dx \int_0^1 ds V(z) f_\ell(z) |z| |\varphi_t(x)|^2 \|a_x \xi_1\| \|\nabla_x a_{x+sz/N} \xi_2\| \\
 &\leq \frac{C e^{c|t|}}{N} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|\mathcal{K}^{1/2} \xi_2\|.
 \end{aligned}$$

This is the only contribution where the kinetic energy is needed (exactly because, in contrast with contributions in G'_t , here the difference $N^3 V(N(x - y)) f_\ell(N(x - y)) - 8\pi\alpha \delta(x - y)$ acts directly on the operators $a_x^* a_y$, without convolution; therefore, some regularity of ξ_1, ξ_2 is needed).

Finally, to control the term IV, we bound

$$\begin{aligned} |\langle \xi_1, \text{IV} \xi_2 \rangle| &\leq C \|(\tilde{q}_t \otimes \tilde{q}_t) H_t - (q_t \otimes q_t) H_{t,\infty}\|_2 \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{1/2} \xi_2\| \\ &\quad + C \|(\tilde{q}_t \otimes \tilde{q}_t) H_t\|_2 \|(\mathcal{N} + 1) \xi_1\| \\ &\quad \times \left\| \left[\sqrt{1 - (\mathcal{N} + 1)/N} \sqrt{1 - \mathcal{N}/N} - 1 \right] \xi_2 \right\|. \end{aligned}$$

Using that $\|\tilde{q}_t - q_t\|_{\text{op}} \leq C e^{ce^{|t|}}/N$, that

$$\left\| \left[\sqrt{1 - (\mathcal{N} + 1)/N} \sqrt{1 - \mathcal{N}/N} - 1 \right] \xi_2 \right\| \leq \frac{C}{N} \|(\mathcal{N} + 1) \xi_2\|$$

and that, going through the several contributions to H_t , $H_{t,\infty}$ (the off-diagonal terms) in (2.16) and (2.28) and applying the bounds in Proposition A.2 (iv) and Lemmas A.3 and A.4, $\|H_t\|_2 \leq C e^{c|t|}$, $\|H_t - H_{t,\infty}\|_2 \leq C e^{ce^{|t|}}/\sqrt{N}$, we find that

$$|\langle \xi_1, \text{IV} \xi_2 \rangle| \leq \frac{C e^{ce^{|t|}}}{\sqrt{N}} \|(\mathcal{N} + 1) \xi_1\| \|(\mathcal{N} + 1) \xi_2\|$$

which concludes the proof of the proposition. \blacksquare

We can now proceed with the proof of Theorem 2.7.

Proof of Theorem 2.7. First of all, we observe that

$$\begin{aligned} \|\mathcal{U}_{2,N}(t; 0) \xi - \mathcal{U}_{2,\infty}(t; 0) \xi\|^2 &= 2 - 2 \operatorname{Re} \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathcal{U}_{2,N}(t; 0) \xi \rangle \\ &= 2 - 2 \operatorname{Re} \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0) \xi \rangle \end{aligned} \quad (5.4)$$

because $\mathcal{U}_{2,N}(t; 0) \xi = \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0) \xi$. We compute

$$\begin{aligned} &-i \frac{d}{dt} \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0) \xi \rangle \\ &= \langle \mathcal{U}_{2,\infty}(t; 0) \xi, [\mathcal{J}_{2,\infty}(t) \mathbb{1}(\mathcal{N} \leq N) - \mathbb{1}(\mathcal{N} \leq N) \mathcal{J}_{2,N}(t)] \mathcal{U}_{2,N}(t; 0) \xi \rangle. \end{aligned}$$

While we cannot move $\mathcal{J}_{2,N}(t)$ to the left of the projection $\mathbb{1}(\mathcal{N} \leq N)$, we can move $\mathcal{J}_{2,\infty}(t)$ to its right, generating a commutator. Thus

$$\begin{aligned} &-i \frac{d}{dt} \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0) \xi \rangle \\ &= \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathbb{1}(\mathcal{N} \leq N) (\mathcal{J}_{2,\infty}(t) - \mathcal{J}_{2,N}(t)) \mathcal{U}_{2,N}(t; 0) \xi \rangle \\ &\quad + \langle \mathcal{U}_{2,\infty}(t; 0) \xi, [\mathcal{J}_{2,\infty}(t), \mathbb{1}(\mathcal{N} \leq N)] \mathcal{U}_{2,N}(t; 0) \xi \rangle. \end{aligned}$$

With Proposition 5.2 and recalling the expression in the last two lines of (2.28) for the limiting generator $\mathcal{J}_{2,\infty}(t)$, we find

$$\begin{aligned} &\left| \frac{d}{dt} \langle \mathcal{U}_{2,\infty}(t; 0) \xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0) \xi \rangle \right| \\ &\leq \frac{C e^{ce^{|t|}}}{\sqrt{N}} \|(\mathcal{N} + 1) \mathcal{U}_{2,\infty}(t; 0) \xi\| \|(\mathcal{K} + \mathcal{N}^2 + 1)^{1/2} \mathcal{U}_{2,N}(t; 0) \xi\| \end{aligned}$$

$$\begin{aligned}
 & + \left| \int dx dy H_\infty(x, y) \langle \mathcal{U}_{2,\infty}(t; 0)\xi, \mathbb{1}(N-2 \leq \mathcal{N} \leq N) \hat{a}_x \hat{a}_y \mathcal{U}_{2,N}(t; 0)\xi \rangle \right| \\
 & + \left| \int dx dy H_\infty(x, y) \langle \mathcal{U}_{2,N}(t; 0)\xi, \hat{a}_x^* \hat{a}_y^* \mathbb{1}(N \leq \mathcal{N} \leq N+2) \mathcal{U}_{2,\infty}(t; 0)\xi \rangle \right|.
 \end{aligned}$$

In the last two terms, we estimate $\mathbb{1}(\mathcal{N} \geq N-2)$, $\mathbb{1}(\mathcal{N} \geq N) \leq C\mathcal{N}/N$ and we use Lemma 4.2 in combination with $\|H_\infty\|_2 \leq Ce^{c|t|}$. We obtain

$$\begin{aligned}
 & \left| \frac{d}{dt} \langle \mathcal{U}_{2,\infty}(t; 0)\xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0)\xi \rangle \right| \\
 & \leq \frac{Ce^{ce^{c|t|}}}{\sqrt{N}} \|(\mathcal{N}+1) \mathcal{U}_{2,\infty}(t; 0)\xi\| \|(\mathcal{K} + \mathcal{N}^2 + 1)^{1/2} \mathcal{U}_{2,N}(t; 0)\xi\|.
 \end{aligned}$$

From Propositions 4.4 and 5.1, we conclude that

$$\begin{aligned}
 & \left| \frac{d}{dt} \langle \mathcal{U}_{2,\infty}(t; 0)\xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0)\xi \rangle \right| \\
 & \leq \frac{Ce^{ce^{c|t|}}}{\sqrt{N}} \|(\mathcal{N}+1)\xi\| \|(\mathcal{K} + \mathcal{N}^2 + 1)^{1/2}\xi\|.
 \end{aligned}$$

Integrating over t and with assumption (2.31), we arrive at

$$\left| 1 - \langle \mathcal{U}_{2,\infty}(t; 0)\xi, \mathbb{1}(\mathcal{N} \leq N) \mathcal{U}_{2,N}(t; 0)\xi \rangle \right| \leq \frac{Ce^{ce^{c|t|}}}{\sqrt{N}}.$$

Inserting this on the right-hand side of (5.4) proves the desired estimate. \blacksquare

6. Central limit theorem: Proof of Theorem 2.11

Following Remark 2.12, in this section we aim at proving that

$$\begin{aligned}
 & \left| \mathbb{E}_{\psi_{N,t}} g(\mathcal{O}_{N,t}) - \frac{1}{\sqrt{2\pi}\|f_t\|} \int dx g(x) e^{-\frac{x^2}{2\|f_t\|^2}} \right| \\
 & \leq Ce^{ce^{c|t|}} \int ds |\hat{g}(s)| (N^{-1/8} + N^{-1/2}|s|^3 \|O\|^3 + N^{-1}s^4 \|O\|^4) \quad (6.1)
 \end{aligned}$$

for every $g \in L^1(\mathbb{R})$ with $\hat{g} \in L^1(\mathbb{R}, (1+s^4) ds)$.

For the initial wave function $\psi_N = U_{N,0}^* e^{B_0} e^B \Omega$ with B_0 defined as in (2.9) and with B given by (2.34), with $\tau \in (q_0 \otimes q_0) H^2(\mathbb{R}^3 \times \mathbb{R}^3)$, we find that (2.22) is satisfied, with $\xi_N = e^B \Omega$. Thus, Theorem 2.2 provides the norm approximation

$$\|e^{-iH_N t} \psi_N - U_{N,t}^* e^{Bt} \mathcal{U}_{2,N}(t) e^B \Omega\| \leq Ce^{ce^{c|t|}} N^{-1/8}.$$

Writing

$$\mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] = \langle \psi_{N,t}, g(\mathcal{O}_{N,t}) \psi_{N,t} \rangle = \int ds \hat{g}(s) \langle \psi_{N,t}, e^{is\mathcal{O}_{N,t}} \psi_{N,t} \rangle,$$

setting

$$\tilde{\mathcal{O}}_{N,t} = \frac{1}{\sqrt{N}} \sum_{p=1}^N (O^{(p)} - \langle \tilde{\varphi}_t, O \tilde{\varphi}_t \rangle)$$

and observing that, by Proposition A.2, $\|\mathcal{O}_{N,t} - \tilde{\mathcal{O}}_{N,t}\|_{\text{op}} \leq C e^{ce^{c|t|}} / \sqrt{N}$, we can therefore estimate

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle U_{N,t}^* e^{Bt} \mathcal{U}_{2,N}(t) e^B \Omega, e^{is\tilde{\mathcal{O}}_{N,t}} U_{N,t}^* e^{Bt} \mathcal{U}_{2,N}(t) e^B \Omega \rangle \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2} |s| \|O\|) ds. \end{aligned} \quad (6.2)$$

Next we conjugate the observable $e^{is\tilde{\mathcal{O}}_{N,t}}$ with the unitary operators defining the norm approximation. With the rules (2.4), we find

$$U_{N,t} \tilde{\mathcal{O}}_{N,t} U_{N,t}^* = \frac{1}{\sqrt{N}} d\Gamma(\tilde{q}_t O' \tilde{q}_t) + \phi(\tilde{q}_t O \tilde{\varphi}_t) \quad (6.3)$$

with $O' = O - \langle \tilde{\varphi}_t, O \tilde{\varphi}_t \rangle$. Here we set $\phi(f) = b^*(f) + b(f)$, while $d\Gamma(R)$ denotes the second quantization of the one-particle operator R and $\tilde{q}_t = 1 - |\tilde{\varphi}_t\rangle\langle\tilde{\varphi}_t|$.

When inserting into (6.2), the contribution of the first term on the right-hand side of (6.3) is small. Proceeding as in [57, Step 1 in Proof of Theorem 1.1], we arrive at

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle e^{Bt} \mathcal{U}_{2,N}(t) e^B \Omega, e^{is\phi(\tilde{q}_t O \tilde{\varphi}_t)} e^{Bt} \mathcal{U}_{2,N}(t) e^B \Omega \rangle \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2} |s|^3 \|O\|^3) ds. \end{aligned} \quad (6.4)$$

This bound relies on the control of the growth of the number of particles operator, which follows from Lemmas 2.1 and 3.1, Proposition 4.4 and from the estimate

$$\langle \xi, e^{-i\phi(f)} (\mathcal{N} + \alpha)^k e^{i\phi(f)} \xi \rangle \leq C \langle \xi, (\mathcal{N} + \alpha + \|f\|^2)^k \xi \rangle \quad (6.5)$$

for the action of the modified Weyl operator $e^{i\phi(f)}$. From (2.10) and (2.12), the action of e^{Bt} is given by

$$e^{-Bt} \phi(\tilde{q}_t O \tilde{\varphi}_t) e^{Bt} = \phi(h_t) + D,$$

with $h_t = \gamma_t(\tilde{q}_t O \tilde{\varphi}_t) + \sigma_t(\overline{\tilde{q}_t O \tilde{\varphi}_t})$ and with an error D satisfying

$$\|D\xi\| \leq \frac{C}{N} \|(\mathcal{N} + 1)^{3/2} \xi\|.$$

Proceeding as in [57, Step 2 in Proof of Theorem 1.1], from (6.4) we therefore arrive at

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle \mathcal{U}_{2,N}(t) e^B \Omega, e^{is\phi(h_t)} \mathcal{U}_{2,N}(t) e^B \Omega \rangle \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2} |s|^3 \|O\|^3 + N^{-1} |s|^4 \|O\|^4) ds. \end{aligned} \quad (6.6)$$

Before proceeding with the last two unitary conjugations, we replace the field $\phi(h_t) = b^*(h_t) + b(h_t)$ with $\phi_a(h_t) = a^*(h_t) + a(h_t)$. To this end, we observe that

$$|\langle \xi_1, (\phi(f) - \phi_a(f)) \xi_2 \rangle| \leq \frac{C \|f\|}{N} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\| \quad (6.7)$$

for all $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$ (the operators b, b^* are only defined on the truncated Fock space). Thus, for $\xi \in \mathcal{F}^{\leq N}$, we have, with the notation $\mathbb{1}^{\leq N} = \mathbb{1}(\mathcal{N} \leq N)$,

$$\begin{aligned} & |\langle \xi, e^{is\phi(h_t)} \xi \rangle - \langle \xi, e^{is\phi_a(h_t)} \xi \rangle| \\ &= \left| \int_0^s d\lambda \frac{d}{d\lambda} \langle \xi, e^{i\lambda\phi(h_t)} \mathbb{1}^{\leq N} e^{i(s-\lambda)\phi_a(h_t)} \xi \rangle \right| \\ &= \left| \int_0^s d\lambda \langle \xi, e^{i\lambda\phi(h_t)} \{(\phi(h_t) - \phi_a(h_t)) \mathbb{1}^{\leq N} + [\mathbb{1}^{\leq N}, \phi_a(h_t)]\} e^{i(s-\lambda)\phi_a(h_t)} \xi \rangle \right|. \end{aligned}$$

Writing

$$\begin{aligned} [\mathbb{1}^{\leq N}, a(h_t)] &= a(h_t) \mathbb{1}(N \leq \mathcal{N} \leq N + 1), \\ [\mathbb{1}^{\leq N}, a^*(h_t)] &= -\mathbb{1}(N \leq \mathcal{N} \leq N + 1) a^*(h_t), \end{aligned}$$

estimating $\mathbb{1}(N \leq \mathcal{N} \leq N + 1) \leq \mathcal{N}/N$ and applying (6.7) and (6.5) (and the analogous bound for the action of $e^{i\phi_a(f)}$), we conclude that

$$|\langle \xi, e^{is\phi(h_t)} \xi \rangle - \langle \xi, e^{is\phi_a(h_t)} \xi \rangle| \leq \frac{C |s| \|h_t\|}{N} \|(\mathcal{N} + s^2 \|h_t\|^2)^{1/2} \xi_1\| \|(\mathcal{N} + s^2 \|h_t\|^2) \xi_2\|.$$

Inserting into (6.6) we find, with Lemma 2.1 and Proposition 4.4,

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle \mathcal{U}_{2,N}(t) e^B \Omega, e^{is\phi_a(h_t)} \mathcal{U}_{2,N}(t) e^B \Omega \rangle \right| \\ & \leq C e^{ce^{|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2} |s|^3 \|O\|^3 + N^{-1} |s|^4 \|O\|^4) ds. \quad (6.8) \end{aligned}$$

Next we apply Theorem 2.7 to replace the quadratic evolution $\mathcal{U}_{2,N}$ with its limit $\mathcal{U}_{2,\infty}$. Moreover, we replace h_t with

$$h_{\infty,t} = \gamma_{t,\infty} (q_t O \varphi_t) + \sigma_{t,\infty} (\overline{q_t O \varphi_t}),$$

where $\gamma_{t,\infty}, \sigma_{t,\infty}$ are defined as in (2.27) and $q_t = 1 - |\varphi_t\rangle \langle \varphi_t|$. From Proposition A.2 and Lemma A.4, we find

$$\|h_t - h_{\infty,t}\|_2 \leq C e^{ce^{|t|}} \|O\| / \sqrt{N}.$$

From (6.8) we therefore obtain

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}} [g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle \mathcal{U}_{2,\infty}(t; 0) e^B \Omega, e^{is\phi_a(h_{\infty,t})} \mathcal{U}_{2,\infty}(t; 0) e^B \Omega \rangle \right| \\ & \leq C e^{ce^{|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2} |s|^3 \|O\|^3 + N^{-1} |s|^4 \|O\|^4) ds. \end{aligned}$$

The action of $\mathcal{U}_{2,\infty}(t; 0)$ on the operators a , a^* appearing in ϕ_a is explicit and described by Proposition 2.9. Setting $n_t = U(t; 0)h_{\infty,t} + \overline{V(t; 0)h_{\infty,t}}$, we find

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}}[g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) \langle e^B \Omega, e^{is\phi_a(n_t)} e^B \Omega \rangle \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2}|s|^3 \|O\|^3 + N^{-1}|s|^4 \|O\|^4) ds. \end{aligned} \quad (6.9)$$

Finally, we need to compute the action of e^B . To this end, we first replace the operator B in (2.34) with

$$B_a = \frac{1}{2} \int dx dy [\tau(x; y) a_x^* a_y^* - \text{h.c.}],$$

proceeding as we did above to replace $\phi(h_t)$ with $\phi_a(h_t)$ to show that $\|e^B \Omega - e^{B_a} \Omega\| \leq C/\sqrt{N}$. Then we use the explicit formula for the action of the Bogoliubov transformation e^{B_a} , which implies, setting $f_t = \cosh(\tau)n_t + \sinh(\tau)\bar{n}_t$, that

$$\begin{aligned} \langle e^{B_a} \Omega, e^{is\phi_a(g_t)} e^{B_a} \Omega \rangle &= \langle \Omega, e^{is\phi_a(f_t)} \Omega \rangle \\ &= \langle \Omega, e^{-s^2 \|f_t\|^2/2} e^{isa^*(f_t)} e^{isa(f_t)} \Omega \rangle = e^{-s^2 \|f_t\|^2/2}. \end{aligned}$$

From (6.9), we obtain

$$\begin{aligned} & \left| \mathbb{E}_{\psi_{N,t}}[g(\mathcal{O}_{N,t})] - \int ds \hat{g}(s) e^{-s^2 \|f_t\|^2/2} \right| \\ & \leq C e^{ce^{c|t|}} \int |\hat{g}(s)| (N^{-1/8} + N^{-1/2}|s|^3 \|O\|^3 + N^{-1}|s|^4 \|O\|^4) ds, \end{aligned}$$

which immediately implies (6.1). The statement of Theorem 2.11 now follows by standard arguments (see, for example, [20, Corollary 1.2]).

7. Control of the action of A_t and proof of Lemma 3.1

In this section, we consider the action of the cubic phase e^{A_t} on number and energy of excitations. To this end, we compute commutators of A_t with the Hamilton operator $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$.

Lemma 7.1. *Recall the definition of A_t in (3.2), with parameter $M = m^{-1} = N^{1/2}$, and recall the notation $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, with \mathcal{K} , \mathcal{V}_N the kinetic and potential energy operators on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$. On $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$, we have*

$$\begin{aligned} [\mathcal{H}_N, A_t] &= - \int dx dy N^{5/2} V_N(x-y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] \\ &+ \text{h.c.} + \mathfrak{E}_{[\mathcal{H}, A_t]}, \end{aligned} \quad (7.1)$$

where

$$\begin{aligned} & | \langle \xi_1, \mathfrak{E}_{[\mathcal{H}, A_t]} \xi_2 \rangle | \\ & \leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|. \end{aligned}$$

Furthermore,

$$\begin{aligned} & | \langle \xi_1, [\mathcal{H}_N, A_t] \xi_2 \rangle | \\ & \leq C e^{c|t|} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{n/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{-n/2} \xi_2\| \quad (7.2) \end{aligned}$$

for all $n \in \mathbb{N}$.

Remark 7.2. To apply Lemma 7.1 in the proof of Lemma 3.1, it is important that the total exponent of $\mathcal{H}_N + \mathcal{N}$ on the right-hand side of (7.2) is 1 (so that (3.5) follows by Grönwall’s lemma). To reach this goal, we inserted the cutoff $\Theta(\mathcal{N})$ in the definition (3.2) of A_t , similarly to [54] (in [18, Lemma 5.7], where the cubic phase does not have a cutoff, the exponent of \mathcal{N} increases).

Proof of Lemma 7.1. We proceed similarly to the proof of [18, Lemma 5.7]. We define A_t^1 as A_t in (3.2), but with \tilde{b}, \tilde{b}^* replaced by b, b^* . Using $[\mathcal{K}, a_x^*] = -\Delta_x a_x^*$ and that \mathcal{K} commutes with \mathcal{N} , we have

$$\begin{aligned} [\mathcal{K}, A_t] &= \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy (-\Delta_x - \Delta_y) v_t(x, y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] + \text{h.c.} \\ &\quad - \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy \nabla_x v_t(x, y) b_x^* b_y^* [2\nabla_x b_x + 2b(\nabla_x p_x) + b^*(\nabla_x \sigma_x)] + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* [b(-\Delta_x p_x) + b(\Delta p_x)] + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) \nabla_x b_x^* b_y^* b^*(\nabla_x \sigma_x) + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy dz v_t(x, y) b_x^* b_y^* \nabla_z b_z^* \nabla_z \sigma(z, x) + \text{h.c.} \\ &\quad + [\mathcal{K}, A_t - A_t^1] =: \sum_{i=1}^5 M_i + \text{h.c.} + [\mathcal{K}, A_t - A_t^1]. \end{aligned}$$

With Lemmas A.3 and A.5, we find

$$\begin{aligned} | \langle \xi_1, M_3 \xi_2 \rangle | &\leq C e^{c|t|} \sqrt{\frac{m}{N}} (\|\Delta_1 p\| + \|\Delta_2 p\|) \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\| \\ &\leq C e^{c|t|} N^{-3/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|. \end{aligned}$$

Writing $\nabla_x v_t(x, y) = -N \nabla_y w_{N,m}(x - y) \tilde{\varphi}_t(y)$ and integrating by parts, we can bound

$$\begin{aligned} | \langle \xi_1, M_2 \xi_2 \rangle | &\leq C e^{c|t|} \sqrt{\frac{m}{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \\ &\quad \times (\|\mathcal{K}^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\| + (\|\nabla_2 p\| + \|\nabla_2 \sigma\|) \|(\mathcal{N} + 1) \xi_2\|) \end{aligned}$$

$$\begin{aligned} &\leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \\ &\quad \times \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|. \end{aligned} \quad (7.3)$$

The term M_4 can be controlled analogously. In addition, M_5 satisfies the same bound, since $\|\nabla_1 \sigma\| \leq C e^{c|t|} \sqrt{N}$ by Lemma A.3. As for M_1 , the scattering equation (2.2) yields

$$\begin{aligned} (-\Delta_x - \Delta_y) v_t(x, y) &= -N^3 V_N(x - y) f_{N,m}(x - y) \tilde{\varphi}_t(y) \\ &\quad + N^3 \lambda_m f_{N,m}(x - y) \chi_m(x - y) \\ &\quad - 2N \nabla w_{N,m}(x - y) \nabla \tilde{\varphi}_t(y) + N w_{N,m}(x - y) \Delta \tilde{\varphi}_t(y). \end{aligned}$$

Let us denote the four contributions to M_1 corresponding to the four terms in the last identity by M_{11}, \dots, M_{14} . Then M_{14} is bounded similarly to M_3 , and M_{13} satisfies the same bound as M_2 after integration by parts (as usual, we use Proposition A.2 to bound norms of φ_t and of its derivatives). Since $N^3 \lambda_m \leq C m^{-3}$ by (A.1) and $0 \leq f_{N,m}(x - y) \leq 1$ we can bound

$$\begin{aligned} |(\xi_1, M_{12} \xi_2)| &\leq C e^{c|t|} \frac{1}{\sqrt{N} m^3} \left(\int dx dy \frac{m^2}{|x - y|^2} \|a_x a_y (\mathcal{N} + 1)^{-1/2} \xi_1\|^2 \right)^{1/2} \\ &\quad \times \left(\int dx dy \chi_m(x - y) \|(\mathcal{N} + 1)^{1/2} [b(\gamma_x) + b^*(\sigma_x)] \xi_2\|^2 \right)^{1/2} \\ &\leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|, \end{aligned}$$

where we used Hardy's inequality in the first integral and the choice $m = N^{-1/2}$. As for M_{11} , with $\Theta = 1 + (\Theta - 1)$ we write

$$\begin{aligned} M_{11} &= -\Theta(\mathcal{N}) N^{5/2} \int dx dy V_N(x - y) f_{N,m}(x - y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] \\ &= -N^{5/2} \int dx dy V_N(x - y) f_{N,m}(x - y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] + \mathcal{E}, \end{aligned} \quad (7.4)$$

where, using $\Theta(\mathcal{N}) - 1 \leq \mathbb{1}(\mathcal{N} \geq M/2)$ and Markov's inequality, we can estimate

$$\begin{aligned} |(\xi_1, \mathcal{E} \xi_2)| &\leq C e^{c|t|} \|\mathcal{V}_N^{1/2} \mathbb{1}(\mathcal{N} \geq M/2) \xi_1\| \|(\mathcal{N} + 1)^{1/2} \mathbb{1}(\mathcal{N} \geq M/2) \xi_2\| \\ &\leq \frac{C e^{c|t|}}{N^{1/4}} \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|. \end{aligned}$$

Finally, we have to control the term arising from the difference $A_t - A_t^1$. We observe that, on $\mathcal{F}^{\leq N} \times \mathcal{F}^{\leq N}$,

$$\begin{aligned} &b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] - \tilde{b}_x^* \tilde{b}_y^* [\tilde{b}(\gamma_x) + \tilde{b}^*(\sigma_x)] \\ &= b_x^* b_y^* \tilde{\varphi}_t(x) b(\tilde{\varphi}_t) + b^*(\tilde{\varphi}_t) \overline{\tilde{\varphi}_t(y)} b_x^* [\tilde{b}(\gamma_x) + \tilde{b}^*(\sigma_x)] \\ &\quad + b^*(\tilde{\varphi}_t) \overline{\tilde{\varphi}_t(x)} \tilde{b}_y^* [\tilde{b}(\gamma_x) + \tilde{b}^*(\sigma_x)]. \end{aligned} \quad (7.5)$$

Here we used the fact that, because of the projection in the kernel $\eta_t(x, y)$ and the definitions of p_t and σ_t in (2.11), we have $\langle \tilde{\varphi}_t, \sigma_x \rangle = 0 = \langle \tilde{\varphi}_t, p_x \rangle$. Note that as quadratic forms on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$, we have, for any operator D , with the shorthand notation $B_x = b(\gamma_x) + b^*(\sigma_x)$ and $\tilde{B}_x = \tilde{b}(\gamma_x) + \tilde{b}^*(\sigma_x)$,

$$\begin{aligned} [D, b_x^* b_y^* B_x - \tilde{b}_x^* \tilde{b}_y^* \tilde{B}_x] &= b_x^* b_y^* \tilde{\varphi}_t(x) [D, b(\tilde{\varphi}_t)] + [D, b^*(\tilde{\varphi}_t)] \overline{\tilde{\varphi}_t(y)} b_x^* \tilde{B}_x \\ &\quad + [D, b^*(\tilde{\varphi}_t)] \overline{\tilde{\varphi}_t(x)} \tilde{b}_y^* \tilde{B}_x. \end{aligned}$$

Applying this with $D = \mathcal{K}$, contributions arising from the commutator of this difference with \mathcal{K} can be controlled as before, using the bounds of Proposition A.2 for $\tilde{\varphi}_t$ and its derivatives. We conclude that

$$|\langle \xi_1, [\mathcal{K}, A_t - A_t^1] \xi_2 \rangle| \leq C e^{c|t|} N^{-3/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|.$$

Let us now consider $[\mathcal{V}_N, A_t]$. With $[\mathcal{V}_N, b_x^*] = \int ds N^2 V_N(x-s) b_x^* a_s^* a_s$ we can write

$$\begin{aligned} &[\mathcal{V}_N, A_t] \\ &= -\Theta(\mathcal{N}) N^{5/2} \int dx dy V_N(x-y) w_{N,m}(x-y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy ds v_t(x, y) N^2 V_N(x-s) b_x^* b_y^* a_s^* a_s [b(\gamma_x) + b^*(\sigma_x)] + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy ds v_t(x, y) N^2 V_N(y-s) b_x^* b_y^* a_s^* a_s [b(\gamma_x) + b^*(\sigma_x)] + \text{h.c.} \\ &\quad - \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy ds v_t(x, y) N^2 V_N(x-s) b_x^* b_y^* a_s^* a_s b_x + \text{h.c.} \\ &\quad - \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy dz ds v_t(x, y) N^2 V_N(z-s) \overline{p(z, x)} b_x^* b_y^* a_s^* a_s b_z + \text{h.c.} \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy dz ds v_t(x, y) N^2 V_N(z-s) \sigma(z, x) b_x^* b_y^* b_z^* a_s^* a_s + \text{h.c.} \\ &\quad + [\mathcal{V}_N, A_t - A_t^1] \\ &=: \sum_{i=1}^6 N_i + \text{h.c.} + [\mathcal{V}_N, A_t - A_t^1]. \end{aligned}$$

Proceeding as in (7.4) to replace the cutoff $\Theta(\mathcal{N})$ by 1, we find

$$N_1 = -N^{5/2} \int dx dy V_N(x-y) w_{N,m}(x-y) \tilde{\varphi}_t(y) b_x^* b_y^* B_x + \mathcal{E}', \quad (7.6)$$

where

$$|\langle \xi_1, \mathcal{E}' \xi_2 \rangle| \leq \frac{C e^{c|t|}}{N^{1/4}} \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|.$$

Noticing that $[a_s^* a_s, b^*(\sigma_x)] = \sigma_t(x; s) b_s^*$, and applying Lemma A.3 to prove that $|\sigma_t(x; s)| \leq CN$, we find

$$\begin{aligned} & | \langle \xi_1, N_2 \xi_2 \rangle | \\ & \leq C e^{c|t|} N^{-1/4} \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1 \| \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2 \|. \end{aligned} \quad (7.7)$$

The terms N_3, N_4 can be bounded similarly. As for N_5, N_6 , they can be estimated by Cauchy–Schwarz, using Lemma A.3 to show that $\sup_z \| p_z \|, \sup_z \| \sigma_z \| \leq C e^{c|t|}$. We find

$$\begin{aligned} & | \langle \xi_1, (N_5 + N_6) \xi_2 \rangle | \\ & \leq C e^{c|t|} \frac{\sqrt{mM}}{N} \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1 \| \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2 \| \\ & \leq C e^{c|t|} N^{-1} \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1 \| \| (\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2 \|. \end{aligned}$$

To bound $[\mathcal{V}_N, A_t - A_t^1]$ we argue as we did above to handle $[\mathcal{K}, A_t - A_t^1]$, using the identity (7.5). Combining all the estimates above, we obtain (7.1) (in particular, the large term on the right-hand side of (7.1) emerges summing the right-hand side of (7.4) and the right-hand side of (7.6)).

The second claim in the lemma follows analogously, noticing that powers of $(\mathcal{N} + 1)$ can be moved freely from one norm to the other, that in the bounds (7.3), (7.7) we can use the cutoff to remove the operator $(\mathcal{N} + 1)^{1/2}$ applied to ξ_2 , at the expense of an additional factor $M^{1/2} \leq CN^{1/4}$ and that also the main terms on the right-hand side of (7.4) and (7.6) can be controlled (with Cauchy–Schwarz) by the right-hand side of (7.2). ■

Now we are ready to show Lemma 3.1.

Proof of Lemma 3.1. The proof is similar to the proof of [18, Lemma 5.8]. To show (3.3), we observe that, since \mathcal{N} preserves the space $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$,

$$[\mathcal{N}, A_t] = [\mathcal{N}, A_t^1] = A_t^{1,\gamma} + 3A_t^{1,\sigma} + \text{h.c.}, \quad (7.8)$$

where, as above, A_t^1 is defined as A_t , but with \tilde{b}, \tilde{b}^* replaced by b, b^* and where

$$\begin{aligned} A_t^{1,\gamma} &= \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* b(\gamma_x), \\ A_t^{1,\sigma} &= \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* b^*(\sigma_x). \end{aligned}$$

From (7.8), it is easy to check that $\pm[\mathcal{N}, A_t] \leq C e^{c|t|} (\mathcal{N} + 1)$. With Grönwall’s lemma, we obtain (3.3) (first with $k = 1$ but then, using $[\mathcal{N}^k, A_t] = \sum_{j=1}^{k-1} \mathcal{N}^j [\mathcal{N}, A_t] \mathcal{N}^{k-j-1}$, also for arbitrary $k \in \mathbb{N}$).

Let us now consider (3.5). For $\xi \in \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$ and $s \in [0, 1]$ we define

$$f_\xi(s) = \langle \xi, e^{-sA_t} \mathcal{H}_N (\mathcal{N} + 1)^k e^{sA_t} \xi \rangle.$$

We have

$$f'_\xi(s) = \langle \xi, e^{-sA_t}([\mathcal{H}_N, A_t](\mathcal{N} + 1)^k + \mathcal{H}_N[(\mathcal{N} + 1)^k, A_t])e^{sA_t}\xi \rangle. \quad (7.9)$$

For $k = 0$, Lemma 7.1 implies, together with (3.3), that

$$|f'_\xi(s)| \leq C e^{c|t|} (f_\xi(s) + \langle \xi, (\mathcal{N} + 1)\xi \rangle). \quad (7.10)$$

The desired bound follows therefore from Grönwall's lemma.

For $k \geq 1$, the contribution of the term proportional to $[\mathcal{H}_N, A]$ is bounded similarly; using (7.2), with $\xi_1 = e^{sA_t}\xi$, $\xi_2 = (\mathcal{N} + 1)^k e^{sA_t}\xi$ and $n = k$, we find

$$\begin{aligned} \langle \xi, e^{-sA_t}[\mathcal{H}_N, A_t](\mathcal{N} + 1)^k e^{sA_t}\xi \rangle &\leq C e^{c|t|} \langle e^{sA_t}\xi, (\mathcal{H}_N + \mathcal{N} + 1)(\mathcal{N} + 1)^k e^{sA_t}\xi \rangle \\ &= C e^{c|t|} (f_\xi(s) + \langle \xi, (\mathcal{N} + 1)^{k+1}\xi \rangle). \end{aligned}$$

To handle the second contribution on the right-hand side of (7.9), we use (7.8). For $k = 1$, we write

$$\begin{aligned} \mathcal{H}_N[\mathcal{N}, A_t] &= [\mathcal{H}_N, A_t^{1,\gamma} + 3A_t^{1,\sigma}] + ((A_t^{1,\gamma} + 3A_t^{1,\sigma})\mathcal{H}_N + \text{h.c.}) \\ &\quad + \mathcal{H}_N[\mathcal{N}, A_t - A_t^1]. \end{aligned} \quad (7.11)$$

Contributions arising from the commutator $[\mathcal{H}_N, A_t^{1,\gamma} + 3A_t^{1,\sigma}]$ have already been estimated in the proof of Lemma 7.1. Thus

$$|\langle \xi, e^{-sA_t}[\mathcal{H}_N, A_t^{1,\gamma} + 3A_t^{1,\sigma}]e^{sA_t}\xi \rangle| \leq C e^{c|t|} f_\xi(s).$$

Contributions to the second term on the right-hand side of (7.11) have the form

$$\begin{aligned} A_t^{1,\gamma} \mathcal{K} &= \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* \int dz \nabla_z a_z^* \nabla_z a_z b(\gamma_x) \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* (-\Delta_x b_x + b(-\Delta p_x)), \end{aligned} \quad (7.12a)$$

$$A_t^{1,\sigma} \mathcal{K} = \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* b^*(\sigma_x) \int dz \nabla_z a_z^* (\mathcal{N} + 1)^{-1} (\mathcal{N} + 1) \nabla_z a_z, \quad (7.12b)$$

$$\begin{aligned} A_t^{1,\gamma} \mathcal{V}_N &= \frac{\Theta(\mathcal{N})}{2\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* \int dr dz N^2 V_N(r - z) a_r^* a_z^* a_r a_z b(\gamma_x) \\ &\quad + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* \int dr dz N^2 V_N(r - z) a_r^* a_r b_z \overline{\gamma(z, x)}, \end{aligned} \quad (7.12c)$$

$$A_t^{1,\sigma} \mathcal{V}_N = \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* b^*(\sigma_x) \int dr dz N^2 V_N(r - z) a_r^* a_z^* a_r a_z. \quad (7.12d)$$

The two commutators contributing to $A_t^{1,\nu} \mathcal{K}$, $A_t^{1,\nu} \mathcal{V}_N$ can be bounded similarly to the terms M_2 , M_3 and N_4 , N_5 , respectively, in the proof of Lemma 7.1. All other terms can be bounded directly with Cauchy–Schwarz; for instance

$$\begin{aligned} & \left| \left\langle e^{sA_t} \xi, \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* \int dz \nabla_z a_z^* \nabla_z a_z b(\gamma_x) e^{sA_t} \xi \right\rangle \right| \\ & \leq \frac{C e^{c|t|}}{\sqrt{N}} \left(\int dx dy dz \|a_x a_y \nabla_z a_z \Theta(\mathcal{N}) e^{sA_t} \xi\|^2 \right)^{1/2} \\ & \quad \times \left(\int dx dy dz N^2 w_{N,m}(x-y)^2 \|a(\gamma_x) \nabla_z a_z e^{sA_t} \xi\|^2 \right)^{1/2} \\ & \leq C e^{c|t|} \|(\mathcal{H}_N + 1)^{1/2} (\mathcal{N} + 1)^{1/2} e^{sA_t} \xi\|^2. \end{aligned}$$

The hermitian conjugates of the terms in (7.12) satisfy similar estimates. We handle the last term on the right-hand side of (7.11) analogously to the way we handled the terms proportional to $A_t - A_t^1$ in the proof of Lemma 7.1, using identity (7.5). Observe here that on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$ we have $\mathcal{H}_N b^*(\varphi_t) = [\mathcal{H}_N, b^*(\varphi_t)]$ allowing us to recover a commutator which has been estimated before. Thus we find that (7.10) also holds true for $k = 1$. For $k \geq 2$, we write

$$\mathcal{H}_N [(\mathcal{N} + 1)^k, A_t] = \sum_{j=1}^k (\mathcal{N} + 1)^j \mathcal{H}_N [\mathcal{N}, A_t] (\mathcal{N} + 1)^{k-j-1}$$

and we argue similarly to $k = 1$, after appropriately pulling factors of $(\mathcal{N} + 1)^{1/2}$ through the commutator $[\mathcal{N}, A_t]$. We obtain again estimate (7.10). The desired bounds follows by Grönwall’s lemma. \blacksquare

8. Generator of fluctuation dynamics: Proof of Proposition 3.4

In this section we study the properties of the generator $\mathcal{J}_N(t)$ of the full fluctuation dynamics (3.6). We start from expression (3.7). A first step in the proof of Proposition 3.4 consists of applying the rules (2.4) to compute the generator

$$\mathcal{L}_N(t) = (i \partial_t U_{N,t}) U_{N,t}^* + U_{N,t} H_N U_{N,t}^*.$$

We find $\mathcal{L}_N(t) = \sum_{j=0}^4 \mathcal{L}_{N,t}^{(j)}$ where as quadratic forms on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$

$$\begin{aligned} \mathcal{L}_{N,t}^{(0)} &= \frac{1}{2} \langle \tilde{\varphi}_t, [N^3 V_N (1 - 2f_{N,\ell}) * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle (N - \mathcal{N}) \\ & \quad - \frac{1}{2} \langle \tilde{\varphi}_t, [N^3 V_N * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle (\mathcal{N} + 1) \frac{(N - \mathcal{N})}{N}, \end{aligned} \quad (8.1a)$$

$$\mathcal{L}_{N,t}^{(1)} = \sqrt{N} b([N^3 V_N w_{N,\ell} * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t) - \frac{\mathcal{N} + 1}{\sqrt{N}} b([N^3 V_N * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t) + \text{h.c.}, \quad (8.1b)$$

$$\begin{aligned}
 \mathcal{L}_{N,t}^{(2)} = & \mathcal{K} + \int dx dy N^3 V_N(x-y) |\tilde{\varphi}_t(y)|^2 \left(b_x^* b_x - \frac{1}{N} a_x^* a_x \right) \\
 & + \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \bar{\tilde{\varphi}}_t(y) \left(b_x^* b_y - \frac{1}{N} a_x^* a_y \right) \\
 & + \frac{1}{2} \left[\int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) (b_x^* b_y^* + \text{h.c.}) \right], \tag{8.1c}
 \end{aligned}$$

$$\mathcal{L}_{N,t}^{(3)} = \int dx dy N^{5/2} V_N(x-y) \tilde{\varphi}_t(y) b_x^* a_y^* a_x + \text{h.c.}, \tag{8.1d}$$

$$\mathcal{L}_{N,t}^{(4)} = \mathcal{V}_N, \tag{8.1e}$$

where we recall that \mathcal{K} and \mathcal{V}_N are the kinetic and the potential energy operators, as defined on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ in (3.4).

Next we have to consider the effect of the quadratic conjugation with the generalized Bogoliubov transformation e^{B_t} . We define the renormalized generator

$$\mathcal{G}_N(t) = (i\partial_t e^{-B_t}) e^{B_t} + e^{-B_t} \mathcal{L}_N(t) e^{B_t}. \tag{8.2}$$

In order to describe the operator $\mathcal{G}_N(t)$, we define

$$\begin{aligned}
 \kappa_{\mathcal{G}}(t) = & \frac{N}{2} \langle \tilde{\varphi}_t, [N^3 V_N(1 - 2f_{N,\ell}) * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle - \frac{1}{2} \langle \tilde{\varphi}_t, (N^3 V_N f_{N,\ell} * |\tilde{\varphi}_t|^2) \tilde{\varphi}_t \rangle \\
 & + \frac{1}{2} \int dx dy N^2 V_N(x-y) |\langle \gamma_y, \sigma_x \rangle|^2 \\
 & + \int dx (N^3 V_N * |\tilde{\varphi}_t|^2)(x) \langle \sigma_x, \sigma_x \rangle \\
 & + \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \bar{\tilde{\varphi}}_t(y) \langle \sigma_x, \sigma_y \rangle \\
 & + \frac{1}{2} \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \langle \sigma_x, \gamma_y \rangle + \text{h.c.} + \|\nabla_1 \sigma_t\|^2 \\
 & - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) \langle \sigma_x^{(s)}, \gamma_y^{(s)} \rangle + \text{h.c.} \tag{8.3}
 \end{aligned}$$

and the quadratic operator

$$\begin{aligned}
 \mathcal{G}_{2,N}(t) = & \mathcal{G}_{2,N}^K(t) + \mathcal{G}_{2,N}^V(t) \\
 & + N^3 \lambda_\ell \int dx dy \chi_\ell(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) b_x^* b_y^* + \text{h.c.} \\
 & + \frac{1}{2} \int dx dy N w_{N,\ell}(x-y) [\Delta \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) + \tilde{\varphi}_t(x) \Delta \tilde{\varphi}_t(y)] b_x^* b_y^* + \text{h.c.} \\
 & + \int dx dy N \nabla w_{N,\ell}(x-y) [\nabla \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) - \tilde{\varphi}_t(x) \nabla \tilde{\varphi}_t(y)] b_x^* b_y^* + \text{h.c.} \\
 & - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) [b^*(\gamma_x^{(s)}) b^*(\gamma_y^{(s)}) + b^*(\gamma_x^{(s)}) b(\sigma_y^{(s)}) \\
 & \quad + b(\sigma_x^{(s)}) b(\sigma_y^{(s)}) + b^*(\gamma_y^{(s)}) b(\sigma_x^{(s)})] + \text{h.c.}, \tag{8.4}
 \end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_{2,N}^K(t) = \mathcal{K} + \int dx & \left[b^*(-\Delta_x p_x) b_x + \frac{1}{2} b^*(\nabla_x p_x) b(\nabla_x p_x) + b_x^* b^*(-\Delta_x \mu_x) \right. \\
& + b^*(-\Delta_x p_x) b^*(\eta_x) + b^*(p_x) b^*(-\Delta_x r_x) + b_x^* b^*(-\Delta_x r_x) \\
& + \frac{1}{2} b^*(\nabla_x \eta_x) b(\nabla_x \eta_x) + b^*(\eta_x) b(-\Delta_x r_x) \\
& \left. + \frac{1}{2} b^*(\nabla_x r_x) b(\nabla_x r_x) + \text{h.c.} \right] \tag{8.5}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{2,N}^V(t) &= \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) [b^*(p_x) b_y^* + b^*(\gamma_x) b^*(p_y)] + \text{h.c.} \\
&+ \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell})(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\
&\quad \times [b^*(\gamma_y) b(\sigma_x) + b^*(\gamma_x) b(\sigma_y) + b(\sigma_x) b(\sigma_y)] + \text{h.c.} \\
&+ \int dx (N^3 V_N * |\tilde{\varphi}_t|^2)(x) \\
&\quad \times (b^*(\gamma_x) b(\gamma_x) + b(\sigma_x) b(\gamma_x) + b^*(\gamma_x) b^*(\sigma_x) + b^*(\sigma_x) b(\sigma_x)) \\
&+ \int dx dy N^3 V_N (x-y) \tilde{\varphi}_t(x) \overline{\tilde{\varphi}_t(y)} \\
&\quad \times (b^*(\gamma_x) b(\gamma_y) + b(\sigma_x) b(\gamma_y) + b^*(\gamma_x) b^*(\sigma_y) + b^*(\sigma_y) b(\sigma_x)). \tag{8.6}
\end{aligned}$$

The following proposition establishes, up to negligible errors, the form of $\mathcal{G}_N(t)$, in terms of (8.3), (8.5), (8.6).

Proposition 8.1. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, spherically symmetric and compactly supported. Let $\mathcal{G}_N(t)$ be defined as in equation (8.2). Assume ℓ in (2.7) is small enough but of order one in N . Let*

$$\mathcal{C}_{N,t} = \int dx dy N^{5/2} V_N (x-y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] + \text{h.c.} \tag{8.7}$$

Then on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$,

$$\mathcal{G}_N(t) = \kappa_{\mathcal{G}}(t) + \mathcal{G}_{2,N}(t) + \mathcal{C}_{N,t} + \mathcal{V}_N + \mathcal{E}_{\mathcal{G}_N}(t), \tag{8.8}$$

where the phase $\kappa_{\mathcal{G}}(t)$, $\mathcal{G}_{2,N}(t)$ and \mathcal{V}_N are given in equations (8.3), (8.4), (8.1) respectively, and the error term $\mathcal{E}_{\mathcal{G}_N}(t)$ satisfies

$$|\langle \xi_1, \mathcal{E}_{\mathcal{G}_N}(t) \xi_2 \rangle| \leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N}^3 + 1)^{1/2} (\mathcal{N} + 1) \xi_2\| \tag{8.9}$$

for any $\xi_1, \xi_2 \in \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$.

Proposition 8.1 is analogous to [18, Proposition 2.5 (b)] in the time-independent setting, where the solution of the Gross–Pitaevskii equation (2.3) (entering the definition of $\mathcal{L}_N(t)$ and $\mathcal{G}_N(t)$) is replaced by the minimizer of the Gross–Pitaevskii energy functional.² The main difference is the fact that Proposition 8.1 provides bounds for general matrix elements, while [18, Proposition 2.5 (b)] only deals with expectations. Since the two proofs are very similar, we only sketch the main steps in the proof of Proposition 8.1, referring to [18] for all details (we give some more details for the term $(i\partial_t e^{-B_t})e^{B_t}$, which is absent in [18]).

Sketch of proof of Proposition 8.1. We write $\mathcal{G}_N(t) = (i\partial_t e^{-B_t})e^{B_t} + \sum_{j=0}^4 \mathcal{G}_N^{(j)}(t)$, with $\mathcal{G}_N^{(j)}(t) = e^{-B_t} \mathcal{L}_N^{(j)}(t) e^{B_t}$, for $j = 0, \dots, 4$. We have

$$\begin{aligned} \mathcal{G}_{N,t}^{(0)} &= \frac{N}{2} \langle \tilde{\varphi}_t, [N^3 V_N (1 - 2f_{N,\ell}) * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle - \frac{1}{2} \langle \tilde{\varphi}_t, [N^3 V_N * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle \\ &\quad - \langle \tilde{\varphi}_t, [N^3 V_N w_{N,\ell} * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle \\ &\quad \times \left(\int dx [b^*(\gamma_x) b(\gamma_x) + b^*(\sigma_x) b(\sigma_x) + b^*(\gamma_x) b^*(\sigma_x) + b(\gamma_x) b(\sigma_x)] + \|\sigma_t\|^2 \right) \\ &\quad + \mathcal{E}_{N,t}^{(0)}, \end{aligned} \tag{8.10}$$

where

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(0)} \xi_2 \rangle| \leq \frac{C}{\sqrt{N}} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|$$

for all $\xi_1, \xi_2 \in \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$ and all $t \in \mathbb{R}$. Details can be found in [18, Corollary 4.3].

Proceeding as in [18, Lemma 4.4], we find

$$\mathcal{G}_{N,t}^{(1)} = \sqrt{N} [b(\gamma_{h_{N,t}}) + b^*(\sigma_{\bar{h}_{N,t}}) + \text{h.c.}] + \mathcal{E}_{N,t}^{(1)}, \tag{8.11}$$

with $h_{N,t} = (N^3 V_N w_{N,\ell} * |\tilde{\varphi}_t|^2) \tilde{\varphi}_t$ and

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(1)} \xi_2 \rangle| \leq \frac{C}{\sqrt{N}} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|.$$

We decompose $\mathcal{G}_N^{(2)}(t) = e^{-B_t} \mathcal{K} e^{B_t} + \mathcal{G}_N^{(2,V)}(t)$. Following [18, Lemma 4.6], we obtain

$$\begin{aligned} \mathcal{G}_{N,t}^{(2,V)} &= \int dx (N^3 V_N * |\tilde{\varphi}_t|^2)(x) \\ &\quad \times (b^*(\gamma_x) b(\gamma_x) + b(\sigma_x) b(\gamma_x) + b^*(\gamma_x) b^*(\sigma_x) + b^*(\sigma_x) b(\sigma_x) + \langle \sigma_x, \sigma_x \rangle) \end{aligned}$$

²Notice that the energy functional used in [18] was the limiting Gross–Pitaevskii functional, defined in terms of the scattering length α . In the present paper, on the other hand, we find it more convenient to work with the modified N -dependent Gross–Pitaevskii equation (2.3); as a consequence, some of the expressions emerging in the computation of $\mathcal{G}_N(t)$ are slightly different from the corresponding expressions in [18].

$$\begin{aligned}
& + \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\
& \quad \times (b^*(\gamma_x) b(\gamma_y) + b(\sigma_x) b(\gamma_y) + b^*(\gamma_x) b^*(\sigma_y) + b^*(\sigma_y) b(\sigma_x) + \langle \sigma_x, \sigma_y \rangle) \\
& + \frac{1}{2} \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\
& \quad \times [b^*(\gamma_y) b^*(\gamma_x) + b^*(\gamma_y) b(\sigma_x) + b^*(\gamma_x) b(\sigma_y) + b(\sigma_x) b(\sigma_y) \\
& \quad + \langle \sigma_x, \gamma_y \rangle (1 - \mathcal{N}/N) + d_x^*(b^*(\gamma_y) + b(\sigma_y)) \\
& \quad + (b^*(\gamma_x) + b(\sigma_x)) d_y^* + \text{h.c.}] + \mathcal{E}_{N,t}^{(2,V)}, \tag{8.12}
\end{aligned}$$

with

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(2,V)} \xi_2 \rangle| \leq \frac{C}{N^{1/2}} e^{c|t|} \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|,$$

where d_x, d_x^* are the operator-valued distributions associated with the fields d_t defined in (2.10). On the other hand, analogously to [18, Lemma 4.8], we conclude that

$$\begin{aligned}
\mathcal{G}_{N,t}^{(K)} & = \mathcal{K} + \int dx dy (-\Delta_x \eta_t)(x, y) b_x^* b_y^* + \int dx dy \overline{(-\Delta_x \eta_t)(x, y)} b_x b_y \\
& + \int dx [b^*(-\Delta_x p_x) b_x + b_x^* b(-\Delta_x p_x) + \nabla_x b^*(p_x) \nabla_x b(p_x) \\
& \quad + b^*(-\Delta_x p_x) b^*(\eta_x) + b^*(p_x) b^*(-\Delta_x r_x) + b_x^* b^*(-\Delta_x r_x) \\
& \quad + b(-\Delta_x r_x) b_x + b(-\Delta_x r_x) b(p_x) + b(\eta_x) b(-\Delta_x p_x) \\
& \quad + \nabla_x b^*(\eta_x) \nabla_x b(\eta_x) + b^*(\sigma_x) b(-\Delta_x r_x) + b^*(-\Delta_x r_x) b(\eta_x)] \\
& + \int dx [b(\nabla_x \eta_x) \nabla_x d_x + \nabla_x d_x^* b^*(\nabla_x \eta_x)] + \|\nabla_1 \sigma_t\|^2 (1 - \mathcal{N}/N) \\
& \quad + \frac{1}{N} \|\nabla_1 \eta_t\|^2 + \frac{\|\nabla_1 \eta_t\|^2}{N} \left(\int dx [b^*(\gamma_x) b(\gamma_x) + b^*(\sigma_x) b(\sigma_x) \right. \\
& \quad \quad \quad \left. + b^*(\gamma_x) b^*(\sigma_x) + b(\gamma_x) b(\sigma_x)] \right) \\
& \quad + \frac{\|\nabla_1 \eta_t\|^2}{N} \|\sigma_t\|^2 + \mathcal{E}_{N,t}^{(K)}, \tag{8.13}
\end{aligned}$$

with

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(K)} \xi_2 \rangle| \leq \frac{C}{N^{1/2}} e^{c|t|} \|(\mathcal{K} + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{K} + \mathcal{N}^3 + 1)^{1/2} (\mathcal{N} + 1) \xi_2\|.$$

As for the cubic term, we find

$$\begin{aligned}
\mathcal{G}_{N,t}^{(3)} & = \int dx dy N^{5/2} V_N(x-y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)] \\
& \quad - \sqrt{N} (b(\gamma(h_{N,t})) + b^*(\sigma(\overline{h_{N,t}}))) + \text{h.c.} + \mathcal{E}_{N,t}^{(3)}, \tag{8.14}
\end{aligned}$$

with

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(3)} \xi_2 \rangle| \leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1) \xi_2\|.$$

The proof of (8.14) is very similar to the proof of [18, Lemma 4.9].

Furthermore, following the strategy used in [18, Lemma 4.10], we obtain

$$\begin{aligned}
 \mathcal{E}_{N,t}^{(4)} &= \mathcal{V}_N + \frac{1}{2} \int dx dy N^2 V_N(x-y) |\langle \gamma_y, \sigma_x \rangle|^2 \left(1 + \frac{1}{N} - \frac{2\mathcal{N}}{N}\right) \\
 &\quad + \frac{1}{2} \int dx dy N^2 V_N(x-y) [b^*(\gamma_x) b^*(\gamma_y) + b^*(\gamma_x) b(\sigma_y) + b^*(\gamma_y) b(\sigma_x) \\
 &\quad \quad \quad + b(\sigma_x) b(\sigma_y)] k_t(x, y) + \text{h.c.} \\
 &\quad + \frac{1}{2} \int dx dy N^2 V_N(x-y) [d_x^*(b^*(\gamma_y) + b(\sigma_y)) \\
 &\quad \quad \quad + (b^*(\gamma_x) + b(\sigma_x)) d_y^*] k_t(x, y) + \text{h.c.} \\
 &\quad + \int dx dy N^3 (V_N w_{N,\ell}^2)(x-y) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 \\
 &\quad \times \left(\int du [b^*(\gamma_u) b(\gamma_u) + b^*(\sigma_u) b(\sigma_u) + b^*(\gamma_u) b^*(\sigma_u) + b(\gamma_u) b(\sigma_u)] + \|\sigma\|^2 \right) \\
 &\quad + \mathcal{E}_{N,t}^{(4)}, \tag{8.15}
 \end{aligned}$$

with

$$|\langle \xi_1, \mathcal{E}_{N,t}^{(4)} \xi_2 \rangle| \leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1) \xi_2\|.$$

Finally, we claim that

$$\begin{aligned}
 (i \partial_t e^{-B_t}) e^{B_t} &= - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) [b^*(\gamma_x^{(s)}) b^*(\gamma_y^{(s)}) + b^*(\gamma_x^{(s)}) b(\sigma_y^{(s)}) \\
 &\quad \quad \quad + b(\sigma_x^{(s)}) b(\sigma_y^{(s)}) + b^*(\gamma_y^{(s)}) b(\sigma_x^{(s)}) \\
 &\quad \quad \quad + \langle \sigma_x^{(s)}, \gamma_y^{(s)} \rangle] + \text{h.c.} + \mathcal{E}_{\partial_t}, \tag{8.16}
 \end{aligned}$$

where \mathcal{E}_{∂_t} satisfies

$$|\langle \xi_1, \mathcal{E}_{\partial_t} \xi_2 \rangle| \leq \frac{C}{N} e^{c|t|} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{3/2} \xi_2\|, \tag{8.17}$$

for any $\xi_1, \xi_2 \in \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$, $t \in \mathbb{R}$, and $N \in \mathbb{N}$ large enough.

To prove (8.16), we use (2.10) to expand

$$\begin{aligned}
 (i \partial_t e^{-B_t}) e^{B_t} &= \int_0^1 ds e^{-s B_t} (i \partial_t B_t) e^{s B_t} \\
 &= - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) [b^*(\gamma_x^{(s)}) b^*(\gamma_y^{(s)}) + b^*(\gamma_x^{(s)}) b(\sigma_y^{(s)}) \\
 &\quad \quad \quad + b(\sigma_x^{(s)}) b(\sigma_y^{(s)}) + b^*(\gamma_y^{(s)}) b(\sigma_x^{(s)}) \\
 &\quad \quad \quad + \langle \sigma_x^{(s)}, \gamma_y^{(s)} \rangle] + \text{h.c.} + \mathcal{E}_{\partial_t},
 \end{aligned}$$

where the superscript s denotes that we consider cosh and sinh of $s\eta_t$ instead of η_t . Here, \mathcal{E}_{∂_t} is given by

$$\begin{aligned} \mathcal{E}_{\partial_t} = & - \int_0^1 ds \int dx dy \dot{\eta}_t(x, y) \times [d_{x,s}^* (b^*(\gamma_y^{(s)}) + b(\sigma_y^{(s)})) + (b^*(\gamma_x^{(s)}) \\ & + b(\sigma_x^{(s)})) d_{y,s}^* + d_{x,s}^* d_{y,s}^* - \langle \sigma_x^{(s)}, \gamma_y^{(s)} \rangle \mathcal{N}/N] \\ & + \text{h.c.}, \end{aligned}$$

where $d_{x,s}$, $d_{x,s}^*$ are the operator-valued distributions associated with operators $d_t^{(s)}$, which are defined as d_t in (2.10), but with η_t replaced by $s\eta_t$. With (2.12) and with the bounds from Lemma A.3, we have

$$\begin{aligned} | \langle \xi_1, \mathcal{E}_{\partial_t} \xi_2 \rangle | & \leq \frac{C e^{c|t|}}{N} \int dx dy |\dot{\eta}_t(x, y)| \|(\mathcal{N} + 1)^{1/2} \xi_1\| \\ & \quad \times [(1 + |\eta_t(x, y)|) \|(\mathcal{N} + 1)^{3/2} \xi_2\| + \|a_x(\mathcal{N} + 1) \xi_2\| \\ & \quad + \|a_x a_y(\mathcal{N} + 1)^{1/2} \xi_2\|] \\ & \leq \frac{C e^{c|t|}}{N} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{3/2} \xi_2\|, \end{aligned}$$

which concludes the proof of (8.17).

Collecting the contributions on the right-hand sides of (8.12), (8.13) and (8.15) containing the operators d , d^* , we define

$$\begin{aligned} \mathcal{D}_{N,t} = & \int dx [b(\nabla_x \eta_x) \nabla_x d_x + \text{h.c.}] \\ & + \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell})(x - y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \\ & \quad \times [d_x^* (b^*(\gamma_y) + b(\sigma_y)) + (b(\gamma_x) + b^*(\sigma_x)) d_y^* + \text{h.c.}]. \end{aligned}$$

Proceeding as in [18, Section 4.5], we obtain

$$\begin{aligned} \mathcal{D}_{N,t} = & \frac{1}{2} \int dx dy N^3 (V_N f_{N,\ell} w_{N,\ell})(x - y) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 \\ & \quad \times \int dv [b^*(\sigma_v) b^*(\gamma_v) + b(\sigma_v) b(\gamma_v) + 2b^*(\sigma_v) b(\sigma_v)] \\ & + \frac{1}{2} \int dx dy N^3 (V_N w_{N,\ell} f_{N,\ell})(x - y) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 \|\sigma\|^2 + \mathcal{E}_{N,t}^{(D)}, \end{aligned} \quad (8.18)$$

with the error

$$| \langle \xi_1, \mathcal{E}_{N,t}^{(D)} \xi_2 \rangle | \leq \frac{C}{N} e^{c|t|} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{V}_N + \mathcal{N}^2 + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|.$$

To obtain the constant term (8.3), we first recombine the following terms, appearing in (8.10), (8.13) and (8.15):

$$\frac{1}{N} \|\nabla_1 \eta_t\|^2 + \frac{1}{2N} \int dx dy N^2 V_N(x - y) |\langle \gamma_y, \sigma_x \rangle|^2 - \frac{1}{2} \langle \tilde{\varphi}_t, [N^3 V_N * |\tilde{\varphi}_t|^2] \tilde{\varphi}_t \rangle.$$

Using the scattering equation (2.2) and the estimate

$$|\langle \sigma_x, \gamma_y \rangle - k_t(x, y)| \leq C |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|,$$

we end up with the second summand on the right-hand side of (8.3) (up to an error of order $1/N$). Moreover, observing that

$$\begin{aligned} & \left| \frac{1}{2} \int dx dy N^3 (V_N w_{N,\ell} f_{N,\ell})(x-y) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 + \frac{1}{N} \|\nabla_1 \eta_t\|^2 \right. \\ & \quad \left. + \int dx dy N^3 (V_N w_{N,\ell}^2)(x-y) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 - \langle \tilde{\varphi}_t, N^3 V_N w_{N,\ell} * |\tilde{\varphi}_t|^2 \tilde{\varphi}_t \rangle \right| \\ & \leq \frac{C}{N}, \end{aligned} \quad (8.19)$$

we conclude that the contributions proportional to $\|\sigma_t\|^2$ from equations (8.10), (8.13), (8.15) and (8.18) cancel (again, up to an error of order $1/N$); this leads to (8.3).

To derive (8.4), we use (8.19) again to show that the sum of all terms proportional to \mathcal{N} and all terms proportional to

$$\int du [b^*(\gamma_u)b(\gamma_u) + b^*(\sigma_u)b(\sigma_u) + b^*(\gamma_u)b^*(\sigma_u) + b(\gamma_u)b(\sigma_u)]$$

appearing on the right-hand side of (8.10), (8.12), (8.13), (8.15), (8.18) produces a small error, which can be absorbed into $\mathcal{E}_{\mathcal{H}_N}$ (in (8.18), we first decompose $2\sigma_t \bar{\sigma}_t = \gamma_t^2 + \sigma_t \bar{\sigma}_t - \mathbb{1}$). Furthermore, in the terms on the first line on the right-hand side of (8.13), we split $\eta_t = \mu_t + k_t$ and we combine the contribution associated with k_t with the contributions

$$\frac{1}{2} \int dx dy N^3 V_N(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) b_x^* b_y^*, \quad \frac{1}{2} \int dx dy N^2 V_N(x-y) k_t(x; y) b_x^* b_y^*$$

extracted from the third summands on the right-hand side of (8.12) and on the right-hand side of (8.15) (expanding $\gamma_t = \mathbb{1} + p_t$). Observing that

$$\begin{aligned} & N(\Delta_x + \Delta_y)w_{N,\ell}(x-y)\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) + N^3(V_N f_{N,\ell})(x-y)\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) \\ & = N[2\Delta w_{N,\ell}(x-y) + N^2(V_N f_{N,\ell})(x-y)]\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) \\ & \quad + Nw_{N,\ell}(x-y)[\Delta\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) + \tilde{\varphi}_t(x)\Delta\tilde{\varphi}_t(y)] \\ & \quad + 2N\nabla w_{N,\ell}(x-y)[\nabla\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) - \tilde{\varphi}_t(x)\nabla\tilde{\varphi}_t(y)] \end{aligned}$$

and that, from the scattering equation (2.2),

$$\begin{aligned} & N[2\Delta w_{N,\ell}(x-y) + N^2(V_N f_{N,\ell})(x-y)]\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) \\ & = 2N^3\lambda_\ell f_{N,\ell}\chi_\ell(x-y)\tilde{\varphi}_t(x)\tilde{\varphi}_t(y), \end{aligned}$$

we conclude that

$$\begin{aligned} & \int dx dy \left[\frac{1}{2}(-\Delta_x - \Delta_y)\eta_t(x; y) + \frac{1}{2}N^3 V_N(x-y)\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) \right. \\ & \quad \left. + \frac{1}{2}N^2 V_N(x-y)k_t(x; y) \right] b_x^* b_y^* \end{aligned}$$

$$\begin{aligned}
&= \int dx dy (-\Delta\mu_t)(x; y) b_x^* b_y^* \\
&\quad + N^3 \lambda_\ell \int dx dy \chi_\ell(x-y) f_{N,\ell}(x-y) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) b_x^* b_y^* \\
&\quad + \frac{1}{2} \int dx dy N w_{N,\ell}(x-y) [\Delta\tilde{\varphi}_t(x) \tilde{\varphi}_t(y) + \tilde{\varphi}_t(x) \Delta\tilde{\varphi}_t(y)] b_x^* b_y^* \\
&\quad + \int N \nabla w_{N,\ell}(x-y) [\nabla\tilde{\varphi}_t(x) \tilde{\varphi}_t(y) - \tilde{\varphi}_t(x) \nabla\tilde{\varphi}_t(y)] b_x^* b_y^*. \tag{8.20}
\end{aligned}$$

In the second term on the right-hand side we can write $f_{N,\ell} = 1 - w_{N,\ell}$ and check that the contribution of $w_{N,\ell}$ can be absorbed into the small error. Thus, combining (8.10), (8.11), (8.12), (8.13), (8.14), (8.15) and (8.16) with (8.18) and (8.20) and comparing with the definitions (8.3), (8.4), (8.7), we obtain the claim of Proposition 8.1. \blacksquare

With (8.2), we can rewrite the generator $\mathcal{J}_N(t)$ defined in (2.14) as

$$\mathcal{J}_N(t) = (i\partial_t e^{-A_t}) e^{A_t} + e^{-A_t} \mathcal{G}_N(t) e^{A_t}. \tag{8.21}$$

To conclude the proof of Proposition 3.4, we need therefore to control the action of A_t on the terms on the right-hand side of (8.8). We already have some information about the action of A_t on the Hamilton operator $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, thanks to Lemma 7.1. The action of A_t on the quadratic terms in $\mathcal{G}_{2,N}(t)$ (excluding the kinetic energy operator \mathcal{K}) is determined by the next lemma.

Lemma 8.2. *Let A_t be defined as in (3.2). Let $F: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{C}$. For any $\xi_1, \xi_2 \in \mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ we have*

$$\begin{aligned}
&\left| \int dr ds F(r, s) \langle \xi_1, [b_r^* b_s^*, A_t] \xi_2 \rangle \right|, \quad \left| \int dr ds \bar{F}(r, s) \langle \xi_1, [b_r b_s, A_t] \xi_2 \rangle \right| \\
&\leq C e^{c|t|} \|F\|_2 N^{-3/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|. \tag{8.22}
\end{aligned}$$

Moreover, assuming additionally that $F(r, s) = \overline{F(s, r)}$, we also have

$$\begin{aligned}
&\left| \int dr ds F(r, s) \langle \xi_1, [b_r^* b_s, A_t] \xi_2 \rangle \right| \\
&\leq C e^{c|t|} N^{-3/4} \min\left(\|F\|_2, \sup_s \int dr |F(r, s)|\right) \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|. \tag{8.23}
\end{aligned}$$

Lemma 8.2 is very similar to [18, Lemma 5.2]. The main differences are the presence of the cutoff $\Theta(\mathcal{N})$ in definition (3.2) of A_t (which plays no role in the proof) and the fact that bounds in [18, Lemma 5.2] only control commutators in expectation, while (8.22), (8.23) control all matrix elements. However, the proof only requires straightforward adaptations. We skip the details.

Finally, we need to control the action of A_t on the cubic operator $\mathcal{C}_{N,t}$ on the right-hand side of (8.8). This is the aim of the next lemma.

Lemma 8.3. *Let A_t be defined as in (3.2), with parameters $M = m^{-1} = \sqrt{N}$ and $\mathcal{C}_{N,t}$ as in (8.7). Furthermore, let*

$$\begin{aligned} \Xi_1 = & -2 \int dx dy N^3 V_N(x-y) w_{N,m}(x-y) \overline{\tilde{\varphi}_t(x)} \tilde{\varphi}_t(y) \\ & \times (b^*(\gamma_y) b(\gamma_x) + b^*(\gamma_y) b^*(\sigma_x) + b(\gamma_x) b(\sigma_y) \\ & + b^*(\sigma_x) b(\sigma_y) + \langle \sigma_y, \sigma_x \rangle), \end{aligned} \quad (8.24a)$$

$$\begin{aligned} \Xi_2 = & -2 \int dx ((N^3 V_N w_{N,m}) * |\tilde{\varphi}_t|^2)(x) \\ & \times (b^*(\gamma_x) b(\gamma_x) + b^*(\gamma_x) b^*(\sigma_x) + b(\gamma_x) b(\sigma_x) \\ & + b^*(\sigma_x) b(\sigma_x) + \|\sigma_x\|^2). \end{aligned} \quad (8.24b)$$

Then, on $\mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N} \times \mathcal{F}_{\perp \tilde{\varphi}_t}^{\leq N}$, we have

$$[\mathcal{C}_{N,t}, A_t] = \Xi_1 + \Xi_2 + \mathcal{E}_{[\mathcal{C}_{N,t}, A_t]},$$

where $\mathcal{E}_{[\mathcal{C}_{N,t}, A_t]}$ is such that

$$\begin{aligned} & |\langle \xi_1, \mathcal{E}_{[\mathcal{C}_{N,t}, A_t]} \xi_2 \rangle| \\ & \leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|. \end{aligned} \quad (8.25)$$

Proof. We write $\mathcal{C}_{N,t} = \tilde{\mathcal{C}}_N + \tilde{\mathcal{C}}_N^*$, with

$$\tilde{\mathcal{C}}_N = N^{5/2} \int dx dy V_N(x-y) \tilde{\varphi}_t(y) b_x^* b_y^* [b(\gamma_x) + b^*(\sigma_x)].$$

We will also use the shorthand notation $B_x = b(\gamma_x) + b^*(\sigma_x)$. We have

$$\begin{aligned} [\mathcal{C}_{N,t}, A_t] = & \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) [\tilde{\mathcal{C}}_N, b_x^* b_y^* B_x] \\ & + \frac{\Theta(\mathcal{N})}{\sqrt{N}} \int dx dy v_t(x, y) [\tilde{\mathcal{C}}_N^*, b_x^* b_y^* B_x] \\ & + [\tilde{\mathcal{C}}_N + \tilde{\mathcal{C}}_N^*, \Theta(\mathcal{N})] \frac{1}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* B_x + \text{h.c.} \\ & + [\mathcal{C}_{N,t}, A_t - A_t^1]. \end{aligned} \quad (8.26)$$

With the notation $W = 1 - \mathcal{N}/N$ and $\tilde{W} = W - 1/N = 1 - (\mathcal{N} + 1)/N$ and with the commutation relations (2.6), we find

$$\begin{aligned} [b_z, b_x^* b_y^* B_x] = & W b_x^* \delta(z-y) B_x + W b_y^* \delta(z-x) B_x - \frac{2}{N} b_x^* a_y^* a_z B_x \\ & + b_x^* b_y^* W \sigma_t(z, x) - \frac{1}{N} b_x^* b_y^* a^*(\sigma_x) a_z, \end{aligned}$$

$$\begin{aligned}
[b_z^*, b_x^* b_y^* B_x] &= -b_x^* b_y^* W \overline{\gamma_t(z, x)} + \frac{1}{N} b_x^* b_y^* a_z^* a(\gamma_x), \\
[B_r, b_x^* b_y^* B_x] &= W b_y^* \overline{\gamma_t(x, r)} B_x + W b_x^* \overline{\gamma_t(y, r)} B_x - \frac{2}{N} b_y^* a_x^* a(\gamma_r) B_x \\
&\quad + b_x^* b_y^* W(\langle p_r, \sigma_x \rangle - \langle p_x, \sigma_r \rangle) \\
&\quad + \frac{1}{N} b_x^* b_y^* a^*(\sigma_r) a(\gamma_x) - \frac{1}{N} b_x^* b_y^* a^*(\sigma_x) a(\gamma_r), \\
[B_r^*, b_x^* b_y^* B_x] &= W \overline{\sigma_t(x, r)} b_y^* B_x + W \overline{\sigma_t(y, r)} b_x^* B_x \\
&\quad - \frac{2}{N} b_x^* a_y^* a(\sigma_r) B_x - b_x^* b_y^* W \delta(r - x) \\
&\quad - b_x^* b_y^* W(p_t(x, r) + \overline{p_t(r, x)} + \langle p_x, p_r \rangle - \langle \sigma_r, \sigma_x \rangle) \\
&\quad + \frac{1}{N} b_x^* b_y^* a^*(\gamma_r) a(\gamma_x) - \frac{1}{N} b_x^* b_y^* a^*(\sigma_x) a(\sigma_r).
\end{aligned}$$

A lengthy but straightforward computation leads to

$$\begin{aligned}
[b_r^* b_s^* B_r, b_x^* b_y^* B_x] &= -b_x^* b_y^* b_s^* \overline{p_t(r, x)} \widetilde{W} B_r - b_r^* b_x^* b_y^* \widetilde{W} \delta(s - x) B_r \\
&\quad - b_r^* b_x^* b_y^* \widetilde{W} \overline{p_t(s, x)} B_r + \frac{2}{N} b_r^* b_x^* b_y^* a_s^* a(\gamma_x) B_r \\
&\quad + b_r^* b_s^* b_y^* \widetilde{W} \overline{p_t(x, r)} B_x + b_r^* b_s^* b_x^* \widetilde{W} \delta(y - r) B_x \\
&\quad + b_r^* b_s^* b_x^* \widetilde{W} \overline{p_t(y, r)} B_x - \frac{2}{N} b_r^* b_s^* b_y^* a_x^* a(\gamma_r) B_x \\
&\quad + b_r^* b_s^* b_x^* b_y^* W(\langle p_r, \sigma_x \rangle - \langle p_x, \sigma_r \rangle) \\
&\quad + \frac{1}{N} b_r^* b_s^* b_x^* b_y^* a^*(\sigma_r) a(\gamma_x) - \frac{1}{N} b_r^* b_s^* b_x^* b_y^* a^*(\sigma_x) a(\gamma_r). \quad (8.27)
\end{aligned}$$

Let us label by R_1, \dots, R_{11} the contributions to $[\mathcal{C}_{N,t}, A_t]$ associated with the terms in (8.27). We claim that these terms can all be included in the error $\mathcal{E}_{[\mathcal{C}_{N,t}, A_t]}$. Let us consider first the terms with six creation and annihilation operators. As an example, using Proposition A.2, Lemma A.3 and the cutoff $\mathcal{N} \leq M$, we bound

$$\begin{aligned}
|\langle \xi_1, R_4 \xi_2 \rangle| &\leq \frac{C e^{c|t|}}{N^{3/2}} \|\mathcal{V}_N^{1/2} (\mathcal{N} + 1) \mathbb{1}^{\leq M} \xi_1\| \\
&\quad \times \left(\int dx dy dr ds N^3 V_N(r-s) N^2 w_{N,m}(x-y)^2 \|a(\gamma_x) B_r \mathbb{1}^{\leq M} \xi_2\|^2 \right)^{1/2} \\
&\leq C e^{c|t|} \sqrt{m \frac{M^2}{N^3}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|.
\end{aligned}$$

Here we used that $a(\gamma_x) b^*(\sigma_r) = (1 - \mathcal{N}/N)^{1/2} a^*(\sigma_x) a_y + (1 - \mathcal{N}/N) \sigma_t(x, r)$ and the fact that $\|\sigma_t\|_2 \leq C$. The other terms with six creation and annihilation operators, namely R_8, R_{10} and R_{11} , can be bounded in the same way. Next we consider terms with

a contraction, quartic in creation and annihilation operators. Let us start with

$$\begin{aligned}
 |\langle \xi_1, R_1 \xi_2 \rangle| &\leq \frac{C e^{c|t|}}{N} \|(\mathcal{N} + 1) \mathbb{1}^{\leq M} \xi_1\| \\
 &\quad \times \left(\int dx dy dr ds N^3 V_N(r-s) N^2 w_{N,m}(x-y)^2 |p_t(r,x)|^2 \right. \\
 &\quad \left. \times \|(\mathcal{N} + 1)^{1/2} B_r \xi_2\|^2 \right)^{1/2} \\
 &\leq C e^{c|t|} \sqrt{m \frac{M}{N^2}} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|,
 \end{aligned}$$

where we used Lemma A.3 to bound $\sup_r \|p_r\|_2 \leq C e^{c|t|}$. The term R_3 is estimated in exactly the same way. Furthermore,

$$\begin{aligned}
 |\langle \xi_1, R_5 \xi_2 \rangle| &\leq \frac{C e^{c|t|}}{N^{1/2}} \left(\int dx dy dr ds N^2 V_N(r-s) |p_t(r,x)|^2 \right. \\
 &\quad \left. \times \|(\mathcal{N} + 1)^{-1/2} a_y a_r a_s \mathbb{1}^{\leq M} \xi_1\|^2 \right)^{1/2} \\
 &\quad \times \left(\int dx dy dr ds N^3 V_N(r-s) |v_t(x,y)|^2 \|(\mathcal{N} + 1)^{1/2} B_r \xi_2\|^2 \right)^{1/2} \\
 &\leq C e^{c|t|} \sqrt{\frac{m}{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|,
 \end{aligned}$$

and R_7 can be bounded analogously. Also, R_9 satisfies the same estimate, since $|\langle p_r, \sigma_x \rangle| \leq \|p_r\| \|\sigma_x\|$. Finally, we control

$$\begin{aligned}
 |\langle \xi_1, R_2 \xi_2 \rangle| &\leq \frac{C e^{c|t|}}{N^{1/2}} \left(\int dx dy dr N^2 V_N(r-x) \|(\mathcal{N} + 1)^{-1/2} a_y a_r a_x \mathbb{1}^{\leq M} \xi_1\|^2 \right)^{1/2} \\
 &\quad \times \left(\int dx dy dr N^3 V_N(r-x) N^2 w_{N,m}(x-y)^2 \|(\mathcal{N} + 1)^{1/2} B_r \xi_2\|^2 \right)^{1/2} \\
 &\leq C e^{c|t|} \sqrt{\frac{m}{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|,
 \end{aligned}$$

and we observe that R_6 is essentially the same as R_2 , after renaming variables.

Let us now consider the second term on the right-hand side of (8.26). Here we have to compute the commutator $[B_r^* b_s b_r, b_x^* b_y^* B_x]$. The computations are more involved than in (8.27), because now there can be multiple contractions, leading to contributions that are quadratic or even constant in creation and annihilation operators. The main contributions are those where b_s and b_r are contracted with b_x^* , b_y^* . There are two such contributions. Assuming that b, b^* satisfy canonical commutation relations (it is easy to check that the corrections are negligible), they are given by

$$\begin{aligned}
 &- \Theta(\mathcal{N}) \int dx dy N^3 V_N(x-y) w_{N,m}(x-y) \overline{\tilde{\varphi}_t(x)} \tilde{\varphi}_t(y) B_y^* B_x \\
 &- \Theta(\mathcal{N}) \int dx (N^3 V_N w_{N,m} * |\tilde{\varphi}_t|^2)(x) B_x^* B_x + \text{h.c.} = \Xi_1 + \Xi_2 + \mathcal{E},
 \end{aligned}$$

with the error

$$|\langle \xi_1, \mathcal{E}\xi_2 \rangle| \leq C e^{c|t|} N^{-1/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)\xi_2\|$$

needed to remove the cutoff $\Theta(\mathcal{N})$ (arguing similarly to (7.4), with the choice $M = N^{1/2}$). Terms involving contractions between b_r , b_s and B_x (or between B_r^* and b_x^* , b_y^*) are smaller, because they produce factors of $\sigma_t(r, x)$ or $\sigma_t(s, x)$, which are in L^2 . As an example, consider the term $\delta(s - x)\sigma_t(r, x)B_r^*b_y^*$ (where we contracted b_s with b_x^* and b_r with B_x , ignoring again corrections to the canonical commutation relations). It produces a contribution S to (8.26), which can be bounded by

$$\begin{aligned} |\langle \xi_1, S\xi_2 \rangle| &\leq C e^{c|t|} \left(\int dx dy dr N^3 V_N(r - x) \|(\mathcal{N} + 1)^{-1/2} a_y B_r \xi_1\|^2 \right)^{1/2} \\ &\quad \times \left(\int dx dy dr N^3 V_N(r - x) |v_t(x, y)|^2 \right)^{1/2} \|(\mathcal{N} + 1)^{1/2} \xi_2\| \\ &\leq C e^{c|t|} N^{-1/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{1/2} \xi_2\|, \end{aligned}$$

where we used $|\sigma_t(r, x)| \leq CN|\tilde{\varphi}_t(r)\tilde{\varphi}_t(x)|$ from Lemma A.3 and $\|v_t\| \leq C\sqrt{m}$ from Lemma A.5 (and the choice $m = N^{-1/2}$). Terms arising from $[B_r^*b_s b_r, b_x^*b_y^*B_x]$ containing six or four creation and annihilation operators can be bounded as we did above with R_1, \dots, R_{11} .

The next term in (8.26) has the form

$$\begin{aligned} &[\tilde{\mathcal{C}}_N + \tilde{\mathcal{C}}_N^*, \Theta(\mathcal{N})] \frac{1}{\sqrt{N}} \int dx dy v_t(x, y) b_x^* b_y^* B_x + \text{h.c.} \\ &= \int dr ds N^2 V_N(r - s) \tilde{\varphi}_t(s) (\Theta(\mathcal{N} - 1) - \Theta(\mathcal{N})) b_r^* b_s^* b(\gamma_r) \\ &\quad \times \int dx dy v_t(x, y) b_x^* b_y^* B_x \\ &\quad + \int dr ds N^2 V_N(r - s) \tilde{\varphi}_t(s) (\Theta(\mathcal{N} - 3) - \Theta(\mathcal{N})) b_r^* b_s^* b^*(\sigma_x) \\ &\quad \times \int dx dy v_t(x, y) b_x^* b_y^* B_x \\ &\quad + \int dr ds N^2 V_N(r - s) \overline{\tilde{\varphi}_t(s)} (\Theta(\mathcal{N} + 1) - \Theta(\mathcal{N})) b^*(\gamma_r) b_s b_r \\ &\quad \times \int dx dy v_t(x, y) b_x^* b_y^* B_x \\ &\quad + \int dr ds N^2 V_N(r - s) \overline{\tilde{\varphi}_t(s)} (\Theta(\mathcal{N} + 3) - \Theta(\mathcal{N})) b(\sigma_r) b_s b_r \\ &\quad \times \int dx dy v_t(x, y) b_x^* b_y^* B_x + \text{h.c.} \\ &= \sum_{i=1}^4 T_i + \text{h.c.} \end{aligned} \tag{8.28}$$

The contribution T_2 is already normally ordered. It can simply be bounded by Cauchy–Schwarz. We find

$$\begin{aligned} |\langle \xi_1, T_2 \xi_2 \rangle| &\leq \int dx dy dr ds N^2 V_N(r-s) |\tilde{\varphi}_t(s)| |v_t(x, y)| \\ &\quad \times \|b_r b_s b(\sigma_x) b_x b_y (\mathcal{N} + 1)^{-3/2} \xi_1\| \|B_x(\mathcal{N} + 1)^{3/2} \mathbb{1}(\mathcal{N} \leq M) \xi_2\| \\ &\leq C e^{c|t|} \sqrt{\frac{m}{N}} \|\mathcal{V}_N^{1/2} \xi_1\| \|(\mathcal{N} + 1)^2 \mathbb{1}(\mathcal{N} \leq M) \xi_1\| \\ &\leq C e^{c|t|} N^{-1/4} \|\mathcal{V}_N^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|, \end{aligned}$$

where we used $\sup_x \|\sigma_x\|_2 \leq C e^{c|t|}$ from Lemma A.3 and $\sup_x \|v_x\|_2 \leq C \sqrt{m} e^{c|t|}$ from Lemma A.5 (with $m = N^{-1/2}$). All other normally ordered terms emerging from (8.28) can be treated analogously. On the other hand, terms involving commutators (produced through normal ordering) are closely related to the contributions discussed above from the first two terms on the right-hand side of (8.26). Due to the presence of the differences $\Theta(\mathcal{N} + 1) - \Theta(\mathcal{N})$ (or similar), the contributions where b_s, b_r are contracted with $b_x^* b_y^*$ (arising from T_3 and T_4) are also negligible here (since $\|(\Theta(\mathcal{N} + 1) - \Theta(\mathcal{N})) \xi\| \leq C/M \|\mathbb{1}(M/2 \leq \mathcal{N} \leq M) \xi\|$, we can gain a factor M^{-1} , arguing similarly to (7.4)).

Finally, we deal with the commutator $[\mathcal{C}_{N,t}, A_t - A_t^1]$ using the identity (7.5). The resulting terms can be treated analogously to the contributions to $[\mathcal{C}_{N,t}, A_t^1]$ (but these terms are less singular and thus simpler to handle). They all satisfy estimate (8.25). We skip the details. \blacksquare

We are now ready to proceed with the proof of Proposition 3.4.

Proof of Proposition 3.4. Recall from (8.21) that

$$\mathcal{J}_N(t) = (i \partial_t e^{-A_t}) e^{A_t} + e^{-A_t} \mathcal{G}_N(t) e^{A_t}, \quad (8.29)$$

where, by Proposition 8.1,

$$\mathcal{G}_N(t) = \kappa \mathcal{G}(t) + \mathcal{G}_{2,N}(t) + \mathcal{C}_{N,t} + \mathcal{V}_N + \mathcal{E}_{\mathcal{G}_N}(t),$$

with the error $\mathcal{E}_N(t)$ satisfying the bound (8.9). From Lemma 3.1, we find

$$\begin{aligned} &|\langle \xi_1, e^{-A_t} \mathcal{E}_{\mathcal{G}_N}(t) e^{A_t} \xi_2 \rangle| \\ &\leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N}^3 + 1)^{1/2} (\mathcal{N} + 1) \xi_2\|. \end{aligned}$$

With Lemmas 7.1, 8.3 and 8.2, we claim that

$$e^{-A_t} (\mathcal{G}_{2,N}(t) + \mathcal{C}_{N,t} + \mathcal{V}_N) e^{A_t} = \mathcal{G}_{2,N}(t) + \frac{1}{2} \Xi_1 + \frac{1}{2} \Xi_2 + \mathcal{V}_N + \mathcal{E}, \quad (8.30)$$

where Ξ_1, Ξ_2 are defined in equation (8.24) and

$$|\langle \xi_1, \mathcal{E} \xi_2 \rangle| \leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} (\mathcal{N} + 1)^{1/2} \xi_2\|.$$

To prove (8.30), we start by observing that

$$e^{-A_t}(\mathcal{G}_{2,N}(t) - \mathcal{K})e^{A_t} = (\mathcal{G}_{2,N}(t) - \mathcal{K}) + \int_0^1 ds e^{-sA_t}[(\mathcal{G}_{2,N}(t) - \mathcal{K}), A_t]e^{sA_t}.$$

Going through the terms in $\mathcal{G}_{2,N}(t) - \mathcal{K}$ in (8.4), we can check with Proposition A.2 and Lemma A.3 that they all have one of the forms considered in Lemma 8.2. It follows that

$$e^{-A_t}(\mathcal{G}_{2,N}(t) - \mathcal{K})e^{A_t} = (\mathcal{G}_{2,N}(t) - \mathcal{K}) + \mathcal{E}',$$

where, applying Lemma 3.1 also,

$$\begin{aligned} |(\xi_1, \mathcal{E}'\xi_2)| &= Ce^{c|t|}N^{-3/4} \int_0^1 ds \|(\mathcal{N} + 1)^{1/2}e^{sA_t}\xi_1\| \|(\mathcal{N} + 1)e^{sA_t}\xi_2\| \\ &\leq Ce^{c|t|}N^{-3/4} \|(\mathcal{N} + 1)^{1/2}\xi_1\| \|(\mathcal{N} + 1)\xi_2\|. \end{aligned}$$

On the other hand, with Duhamel's formula, we can write

$$\begin{aligned} e^{-A_t}\mathcal{H}_N e^{A_t} &= \mathcal{H}_N + \int_0^1 ds e^{-sA_t}[\mathcal{H}_N, A_t]e^{sA_t} \\ &= \mathcal{H}_N + \int_0^1 ds e^{-sA_t}(-\mathcal{C}_{N,t} + \mathcal{E}_{[\mathcal{H}_N, A_t]})e^{sA_t} \\ &= \mathcal{H}_N - \mathcal{C}_{N,t} + \int_0^1 ds e^{-sA_t}\mathcal{E}_{[\mathcal{H}_N, A_t]}e^{sA_t} \\ &\quad + \int_0^1 ds \int_0^s dr e^{-rA_t}(-\Xi_1 - \Xi_2 + \mathcal{E}_{[\mathcal{C}_{N,t}, A_t]})e^{rA_t} \\ &= \mathcal{H}_N - \mathcal{C}_{N,t} - \frac{1}{2}(\Xi_1 + \Xi_2) \\ &\quad + \int_0^1 ds e^{-sA_t}\mathcal{E}_{[\mathcal{H}_N, A_t]}e^{sA_t} + \int_0^1 ds \int_0^s dr e^{-rA_t}\mathcal{E}_{[\mathcal{C}_{N,t}, A_t]}e^{rA_t} \\ &\quad - \int_0^1 ds \int_0^s dr \int_0^r d\tau e^{-\tau A_t}[\Xi_1 + \Xi_2, A_t]e^{\tau A_t}. \end{aligned}$$

Similarly,

$$\begin{aligned} e^{-A_t}\mathcal{C}_{N,t}e^{A_t} &= \mathcal{C}_{N,t} + \int_0^1 ds e^{-sA_t}[\mathcal{C}_{N,t}, A_t]e^{sA_t} \\ &= \mathcal{C}_{N,t} + \int_0^1 ds e^{-sA_t}(\Xi_1 + \Xi_2 + \mathcal{E}_{[\mathcal{C}_{N,t}, A_t]})e^{sA_t} \\ &= \mathcal{C}_{N,t} + \Xi_1 + \Xi_2 \\ &\quad + \int_0^1 ds e^{-sA_t}\mathcal{E}_{[\mathcal{C}_{N,t}, A_t]}e^{sA_t} + \int_0^1 ds \int_0^s dr e^{-rA_t}[\Xi_1 + \Xi_2, A_t]e^{rA_t}. \end{aligned}$$

Applying Lemmas 7.1, 8.3 and 8.2 (noticing that the commutator of the quadratic operators Ξ_1, Ξ_2 with A_t is a sum of terms that can be bounded with (8.22), (8.23)) and propagating the estimates through the cubic phase with Lemma 3.1, we arrive at (8.30).

As for the first term on the right-hand side of (8.29), we observe that since $\sup_x \|\dot{v}_{t,x}\| \leq C e^{c|t|} N^{-1/4}$ from Lemma A.5,

$$\begin{aligned} |\langle \xi_1, (i \partial_t e^{-A_t}) e^{A_t} \xi_2 \rangle| &\leq \int_0^1 ds |\langle e^{sA_t} \xi_1, [i \partial_t A_t] e^{sA_t} \xi_2 \rangle| \\ &\leq C e^{c|t|} N^{-3/4} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1) \xi_2\|. \end{aligned}$$

We conclude that

$$\mathcal{J}_N(t) = \kappa_{\mathcal{G}}(t) + \mathcal{G}_{2,N}(t) + \frac{1}{2} \Xi_1 + \frac{1}{2} \Xi_2 + \mathcal{E}',$$

where

$$\begin{aligned} |\langle \xi_1, \mathcal{E}' \xi_2 \rangle| &\leq C e^{c|t|} N^{-1/4} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2} \xi_1\| \\ &\quad \times \|(\mathcal{H}_N + \mathcal{N}^3 + 1)^{1/2} (\mathcal{N} + 1) \xi_2\|. \end{aligned} \quad (8.31)$$

Next we observe that in the terms Ξ_1, Ξ_2 , as defined in (8.24), we can replace the parameter $m = N^{-1/2}$ with the fixed, N -independent, parameter $\ell \in (0; 1)$, at the expense of a small error. In fact, setting

$$\begin{aligned} \Xi'_1 &= -2 \int dx dy N^3 V_N(x-y) w_{N,\ell}(x-y) \overline{\tilde{\varphi}_t(x)} \tilde{\varphi}_t(y) \\ &\quad \times (b^*(\gamma_y) b(\gamma_x) + b^*(\gamma_y) b^*(\sigma_x) + b(\gamma_x) b(\sigma_y) \\ &\quad \quad + b^*(\sigma_x) b(\sigma_y) + \langle \sigma_y, \sigma_x \rangle), \end{aligned} \quad (8.32a)$$

$$\begin{aligned} \Xi'_2 &= -2 \int dx ((N^3 V_N w_{N,\ell}) * |\tilde{\varphi}_t|^2)(x) \\ &\quad \times (b^*(\gamma_x) b(\gamma_x) + b^*(\gamma_x) b^*(\sigma_x) + b(\gamma_x) b(\sigma_x) \\ &\quad \quad + b^*(\sigma_x) b(\sigma_x) + \|\sigma_x\|^2), \end{aligned} \quad (8.32b)$$

we find

$$|\langle \xi_1, (\Xi_j - \Xi'_j) \xi_2 \rangle| \leq \frac{C e^{c|t|}}{\sqrt{N}} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{K} + \mathcal{N} + 1)^{1/2} \xi_2\| \quad (8.33)$$

for $j = 1, 2$. To show (8.33), we argue as we do in the proof of Proposition 5.2 to control the terms I, II, III (in that proposition, we control convergence of $N^3 V_N f_{N,\ell}$ towards a δ -distribution, but the same argument implies convergence of $N^3 V_N f_{N,m}$ towards a δ -distribution and therefore allows us to control the difference $N^3 V_N (f_{N,\ell} - f_{N,m}) = N^3 V_N (w_{N,m} - w_{N,\ell})$; note that to handle $N^3 V_N f_{N,m}$ we need to use (A.2) with ℓ replaced by $m = N^{-1/2}$, which makes some of the estimates, like the one for the second term on the right-hand side of (5.3), worse).

Combining the operator $\mathcal{G}_{2,N}(t)$, as defined in (8.4), with the terms Ξ'_1, Ξ'_2 from (8.32) we obtain, as a quadratic form on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ (so that the “projected” operators \tilde{b}, \tilde{b}^* are the same as b, b^*), the operator $\mathcal{J}_{2,N}(t)$ in (2.16), up to a small error due to the term on the seventh line of (2.16), whose matrix elements can be bounded by

$$\begin{aligned} & \left| \int dx N^3 (V_N f_{N,\ell} * |\tilde{\varphi}_t|^2)(x) \langle \xi_1, (a_x^* a_x - b_x^* b_x) \xi_2 \rangle \right| \\ & \leq CN^{-1} \|(\mathcal{N} + 1)^{1/2} \xi_1\| \|(\mathcal{N} + 1)^{3/2} \xi_2\|. \end{aligned} \quad (8.34)$$

Remark that, despite its smallness on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$, we inserted this term into definition (2.16) to make sure that $\mathcal{U}_{2,N}(t; s)$ maps $\mathcal{F}_{\perp\tilde{\varphi}_s}^{\leq N}$ into $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$. Absorbing (8.34), together with (8.31), into the error term $\mathcal{E}_{\mathcal{J}_N}(t)$, we conclude the proof of Proposition 3.4. ■

A. Properties of $f_\ell, \varphi_t, \tilde{\varphi}_t, \eta_t, \eta_{\infty,t}, \nu_t$

In this appendix we collect some analytic properties of functions and kernels that are used throughout the paper to construct the approximation of the many-body dynamics.

In the first lemma, whose proof can be found in [9, 17, 25], we consider the ground state solution of the Neumann problem (2.1) on the ball $|x| \leq N\ell$, with the normalization $f_\ell(x) = 1$ for $|x| = N\ell$.

Lemma A.1. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, compactly supported and spherically symmetric. Fix $\ell > 0$ and let f_ℓ denote the solution of (2.1). For N large enough the following properties hold true:*

(i) *We have*

$$\lambda_\ell = \frac{3\alpha}{(\ell N)^3} (1 + \mathcal{O}(\alpha/\ell N)). \quad (A.1)$$

(ii) *We have $0 \leq f_\ell, w_\ell \leq 1$. Moreover, there exists a constant $C > 0$ such that*

$$\left| \int V(x) f_\ell(x) dx - 8\pi\alpha \right| \leq \frac{C\alpha^2}{\ell N} \quad (A.2)$$

for all $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$.

(iii) *There exists a constant $C > 0$ such that*

$$w_\ell(x) \leq \frac{C}{|x| + 1} \quad \text{and} \quad |\nabla w_\ell(x)| \leq \frac{C}{x^2 + 1} \quad (A.3)$$

for all $x \in \mathbb{R}^3$, $\ell \in (0; 1/2)$ and all $N \in \mathbb{N}$ large enough.

Next we consider solutions of the Gross–Pitaevskii equation (1.7) and of the modified N -dependent Gross–Pitaevskii equation (2.3). The proof of Proposition A.2 is essentially contained (up to straightforward changes) in [7, Theorem 3.1], [17, Proposition 4.2] and [10, Proposition B.1].

Proposition A.2. *Let $V \in L^3(\mathbb{R}^3)$ be a nonnegative, spherically symmetric, compactly supported potential. Let $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\| = 1$. Recall the scattering solution (2.2) which enters the modified Gross–Pitaevskii equation (2.3); assume $\ell \in (0, 1/2)$.*

- (i) *For $\varphi \in H^1(\mathbb{R}^3)$, there exist unique global solutions $t \rightarrow \varphi_t$ and $t \rightarrow \tilde{\varphi}_t$ in $\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^3))$ of the Gross–Pitaevskii equation (1.7) and, respectively, of the modified Gross–Pitaevskii equation (2.3) with initial datum φ . We have $\|\varphi_t\| = \|\tilde{\varphi}_t\| = 1$ for all $t \in \mathbb{R}$. Furthermore, there exists a constant $C > 0$ such that*

$$\|\varphi_t\|_{H^1}, \|\tilde{\varphi}_t\|_{H^1} \leq C.$$

- (ii) *If $\varphi \in H^m(\mathbb{R}^3)$ for some $m \geq 2$, then $\varphi_t, \tilde{\varphi}_t \in H^m(\mathbb{R}^3)$ for every $t \in \mathbb{R}$. Moreover, there exist constants C depending on m and on $\|\varphi\|_{H^m}$, and $c > 0$ depending on m and on $\|\varphi\|_{H^1}$, such that for all $t \in \mathbb{R}$,*

$$\|\varphi_t\|_{H^m}, \|\tilde{\varphi}_t\|_{H^m} \leq C e^{c|t|}.$$

- (iii) *Suppose $\varphi \in H^4(\mathbb{R}^3)$. Then there exist constants $C > 0$ depending on $\|\varphi\|_{H^4}$, and $c > 0$ depending on $\|\varphi\|_{H^1}$, such that for all $t \in \mathbb{R}$*

$$\|\dot{\tilde{\varphi}}_t\|_{H^2}, \|\ddot{\tilde{\varphi}}_t\|_{H^2} \leq C e^{c|t|}.$$

Furthermore, if $\varphi \in H^6(\mathbb{R}^3)$ there exist constants $C > 0$ depending on $\|\varphi\|_{H^6}$, and $c > 0$ depending on $\|\varphi\|_{H^1}$, such that for all $t \in \mathbb{R}$

$$\|\dot{\varphi}_t\|_{H^4}, \|\dot{\tilde{\varphi}}_t\|_{H^4} \leq C e^{c|t|}.$$

- (iv) *Suppose $\varphi \in H^2(\mathbb{R}^3)$. Then there exist constants $C, c_1, c_2 > 0$ such that for all $t \in \mathbb{R}$,*

$$\|\varphi_t - \tilde{\varphi}_t\| \leq CN^{-1} \exp(c_1 \exp(c_2 |t|)). \quad (\text{A.4})$$

For $\varphi \in H^6(\mathbb{R}^3)$ there are constants $C, c > 0$ such that

$$\|\varphi_t - \tilde{\varphi}_t\|_{H^4} \leq CN^{-1} e^{ce^{c|t|}}$$

and

$$\|\dot{\varphi}_t - \dot{\tilde{\varphi}}_t\|_{H^2} \leq CN^{-1} e^{ce^{c|t|}}.$$

Now recall definition (2.8), depending on the parameters N, ℓ , of the kernel η_t appearing in the generalized Bogoliubov transformation e^{Bt} and the notation $\gamma_t = \cosh \eta_t$, $\sigma_t = \sinh \eta_t$. Furthermore, we set $p_t = \gamma_t - \mathbb{1}$, $r_t = \sigma_t - \eta_t$ and $\mu_t = \eta_t - k_t$ (recall (2.7)). Several bounds for the operators $\eta_t, \gamma_t, \sigma_t, p_t, r_t$ (for their integral kernels) and for their time derivatives are established in the next lemma, whose proof is a straightforward adaptation of [7, Lemmas 3.3 and 3.4], [17, Lemma 4.3], [10, Appendix C].

Lemma A.3. *Let $\tilde{\varphi}_t$ be the solution of (2.3) with initial datum $\varphi \in H^4(\mathbb{R})$. Let $w_\ell = 1 - f_\ell$ with f_ℓ the ground state solution of the Neumann problem (2.1) and let $\ell \in (0; 1/2)$.*

Let k_t, η_t, μ_t be defined as in (2.7), (2.8). Then there exist constants $C, c > 0$ depending only on $\|\varphi\|_{H^4}$ (or lower Sobolev norms of φ) and on V such that the following bounds hold uniformly in ℓ , for all $t \in \mathbb{R}$:

(i) We have

$$\|\eta_t\| \leq C\ell^{1/2}$$

and also

$$\|\nabla_j \eta_t\| \leq C\sqrt{N}, \quad \|\nabla_j \mu_t\| \leq C$$

for $j = 1, 2$. With $\nabla_1 \eta_t$ and $\nabla_2 \eta_t$ we indicate the kernels $\nabla_x \eta_t(x; y)$ and $\nabla_y \eta_t(x; y)$, and similar definitions hold for $\Delta_j \eta_t$, for $j = 1, 2$. Let σ_t, p_t, r_t be defined as in (2.11) and after (2.18), we obtain

$$\begin{aligned} \|\sigma_t\|, \|p_t\|, \|r_t\|, \|\nabla_j p_t\|, \|\nabla_j r_t\| &\leq C, \\ \|\Delta_j p_t\|, \|\Delta_j r_t\|, \|\Delta_j \mu_t\| &\leq C e^{c|t|}, \\ \|\nabla_j \sigma_t\| &\leq C e^{c|t|} \sqrt{N}. \end{aligned}$$

(ii) For a.e. $x, y \in \mathbb{R}^3$, $n \in \mathbb{N}$, $n \geq 2$, we have the pointwise bounds

$$\begin{aligned} |\eta_t(x; y)| &\leq \frac{C}{|x - y| + N^{-1}} |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|, \\ |\mu_t(x; y)|, |p_t(x; y)|, |r_t(x; y)| &\leq C |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|, \\ |\nabla_x \eta_t(x, y)| &\leq C(|\nabla \tilde{\varphi}_t(x)| + |\tilde{\varphi}_t(x)|) |\tilde{\varphi}_t(y)| \left(\frac{\chi(|x - y| \leq \ell)}{|x - y|^2} + 1 \right). \end{aligned}$$

(iii) Moreover, we have

$$\sup_x \|\eta_x\|^2, \sup_x \|k_{t,x}\|^2, \sup_x \|\mu_{t,x}\|^2 \leq C \|\tilde{\varphi}_t\|_{H^2}^2 \leq C e^{c|t|},$$

where we denote $\sup_x \|\eta_x\|^2 = \sup_x \int |\eta_t(x; y)|^2 dy$ and

$$\|\sigma_{t,x}\|, \|p_{t,x}\|, \|r_{t,x}\| \leq C e^{c|t|}.$$

(iv) For $j = 1, 2$ we have the following bounds for the time derivatives:

$$\|\partial_t \eta_t\|, \|\partial_t^2 \eta_t\| \leq C e^{c|t|},$$

and also

$$\|\partial_t \nabla_j \eta_t\| \leq C \sqrt{N} e^{c|t|}, \quad \|\partial_t \nabla_j \mu_t\| \leq C e^{c|t|}.$$

Furthermore,

$$\begin{aligned} \|\partial_t \sigma_t\|, \|\partial_t r_t\|, \|\partial_t p_t\|, \|\nabla_j \partial_t p_t\|, \|\nabla_j \partial_t r_t\|, \|\Delta_j \partial_t p_t\|, \|\Delta_j \partial_t r_t\|, \|\Delta_j \partial_t \mu_t\| \\ \leq C e^{c|t|}. \end{aligned}$$

(v) For a.e. $x, y \in \mathbb{R}^3$ we have the pointwise bounds

$$|\partial_t \eta_t(x; y)| \leq C \left[1 + \frac{1}{|x - y| + N^{-1}} \right] [|\dot{\tilde{\varphi}}_t(x)| |\tilde{\varphi}_t(y)| + |\tilde{\varphi}_t(x)| |\dot{\tilde{\varphi}}_t(y)| + |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|]$$

and

$$\begin{aligned} & |\partial_t \mu_t(x; y)|, |\partial_t r_t(x; y)|, |\partial_t p_t(x; y)| \\ & \leq C e^{c|t|} [|\dot{\tilde{\varphi}}_t(x)| |\tilde{\varphi}_t(y)| + |\tilde{\varphi}_t(x)| |\dot{\tilde{\varphi}}_t(y)| + |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|]. \end{aligned}$$

(vi) Finally, we have

$$\|\partial_t \eta_x\|, \|\partial_t k_{t,x}\|, \|\partial_t \mu_{t,x}\| \leq C e^{c|t|}$$

and

$$\|\partial_t \sigma_{t,x}\|, \|\partial_t p_{t,x}\|, \|\partial_t r_{t,x}\| \leq C e^{c|t|}.$$

While the kernels $\eta_t, \gamma_t, \sigma_t, p_t, r_t$ considered in the last lemma are used in the definition of the fluctuation dynamics \mathcal{U}_N and of its quadratic approximation $\mathcal{U}_{2,N}$, the limiting quadratic evolution $\mathcal{U}_{2,\infty}$ is defined in (2.30) in terms of limiting kernels $\eta_{\infty,t}, \gamma_{\infty,t}, \sigma_{\infty,t}, p_{\infty,t}, r_{\infty,t}$. To show the well-posedness of $\mathcal{U}_{2,\infty}$ and to compare it with $\mathcal{U}_{2,N}$, we need some bounds on these limiting objects.

Lemma A.4. *Let $w_{\infty,\ell}, \eta_{\infty,t}, \gamma_{\infty,t}, \sigma_{\infty,t}, p_{\infty,t}, r_{\infty,t}$ be defined as in (2.25)–(2.27).*

(i) *The limiting kernels satisfy*

$$\begin{aligned} & \|\eta_{t,\infty}\|, \|\sigma_{\infty}\|, \|p_{\infty}\|, \|r_{\infty}\| \leq C, \\ & \|\Delta_j p_{\infty}\|, \|\Delta_j r_{\infty}\|, \|\Delta_j \mu_{t,\infty}\| \leq C e^{c|t|}, \\ & \|\dot{\eta}_{t,\infty}\|, \|\dot{\sigma}_{\infty}\|, \|\dot{p}_{\infty}\|, \|\dot{r}_{\infty}\| \leq C e^{c|t|}, \\ & \|\Delta_j \dot{p}_{\infty}\|, \|\Delta_j \dot{r}_{\infty}\|, \|\Delta_j \dot{\mu}_{t,\infty}\| \leq C e^{c|t|}, \\ & \|\ddot{\eta}_{t,\infty}\| \leq C e^{c|t|}, \end{aligned}$$

for almost all $x \in \mathbb{R}^3$ and $j = 1, 2$.

(ii) *Let $R > 0$ be such that $V(x) = 0$ for $|x| > R$. Then we have*

$$|N w_{N,\ell}(x) - w_{\infty,\ell}(x)| \leq \begin{cases} \frac{C}{|x|}, & 0 \leq |x| \leq R/N, \\ \frac{C}{N|x|}, & R/N \leq |x| \leq \ell, \\ 0, & \ell \leq |x|, \end{cases} \quad (\text{A.5a})$$

$$|N \nabla w_{N,\ell}(x) - \nabla w_{\infty,\ell}(x)| \leq \begin{cases} \frac{C}{|x|^2}, & 0 \leq |x| \leq R/N, \\ \frac{C}{N|x|^2}, & R/N \leq |x| \leq \ell, \\ 0, & \ell \leq |x|. \end{cases} \quad (\text{A.5b})$$

(iii) We have

$$\begin{aligned} \|\eta_t - \eta_{t,\infty}\|, \|\dot{\eta}_t - \dot{\eta}_{t,\infty}\|, \|\sigma - \sigma_\infty\|, \|p - p_\infty\|, \|r - r_\infty\| &\leq \frac{C e^{c e^{c|t|}}}{\sqrt{N}}, \\ \|\Delta_2 p - \Delta_2 p_\infty\|, \|\Delta_2 r - \Delta_2 r_\infty\|, \|\Delta_2 \mu_t - \Delta_2 \mu_{t,\infty}\| &\leq \frac{C e^{c e^{c|t|}}}{\sqrt{N}}. \end{aligned}$$

Proof. The bounds in (i) follow from (2.25), which implies that $|w_{\infty,\ell}(x)| \leq C/|x|$, $|\nabla w_{\infty,\ell}(x)| \leq C/|x|^2$ and $\Delta w_{\infty,\ell}(x) = 4\pi\alpha\delta(x) + 3\alpha\chi(|x| \leq \ell)/\ell^3$.

The bounds (A.5) are trivial for $|x| > \ell$. In the region $|x| < R/N$, they follow by combining the estimates for $w_{\infty,\ell}$, $\nabla w_{\infty,\ell}$ with (A.3). In the region $R/N < |x| < \ell$, we apply the identity (see [8, 9])

$$w_{N,\ell}(x) = 1 - \frac{\sin(\sqrt{\lambda_{N,\ell}}(|x| - \ell))}{\sqrt{\lambda_{N,\ell}}|x|} - \frac{\ell}{|x|} \cos\left(\sqrt{\lambda_{N,\ell}}(|x| - \ell)\right)$$

and the corresponding expression for $\nabla w_{N,\ell}$. Using $\lambda_{N,\ell}\ell^2 \simeq 1/(N\ell) \ll 1$ and Taylor expanding sin and cos, we obtain (A.5).

The bounds in (iii) follow from (ii) and from (A.4) (because the limiting kernels $\eta_{t,\infty}$, $\dot{\eta}_{t,\infty}$, σ_∞ , p_∞ , r_∞ , $\mu_{t,\infty}$ are defined like η_t , $\dot{\eta}_t$, σ , p , r , μ_t , with $Nw_{N,\ell}$ and $\tilde{\varphi}_t$ replaced by $w_{\infty,\ell}$ and φ_t). We leave the details to the reader. ■

Finally, in the next lemma we collect some estimates for the kernel v_t introduced in (3.1), entering the definition of the cubic phase A_t in (3.2). The proof is a straightforward adaptation of the proof of Lemma A.3 above (with the parameter ℓ now replaced by m).

Lemma A.5. *Under the same conditions as in Lemma A.3, the kernel v_t defined as in equation (3.1) satisfies the following bounds:*

$$\|v_t\| \leq C\sqrt{m}, \quad \sup_x \|v_{t,x}\| \leq C e^{c|t|} \sqrt{m}, \quad \|v_{t,y}\| \leq C|\tilde{\varphi}_t(y)|\sqrt{m}$$

and the pointwise bound

$$|v_t(x, y)| \leq \frac{C|\tilde{\varphi}_t(y)|}{|x - y| + N^{-1}}.$$

Furthermore, the time derivative satisfies similar estimates:

$$\|\dot{v}_t\| \leq C e^{c|t|} \sqrt{m}, \quad \sup_x \|\dot{v}_{t,x}\| \leq C e^{c|t|} \sqrt{m}, \quad \|\dot{v}_{t,y}\| \leq C|\dot{\tilde{\varphi}}_t(y)|\sqrt{m}.$$

Funding. We gratefully acknowledge partial support from the Swiss National Science Foundation through the Grant ‘‘Dynamical and energetic properties of Bose–Einstein condensates’’, from the European Research Council through the ERC-AdG CLaQS and from the NCCR SwissMAP. C.C. warmly acknowledges the GNFM Gruppo Nazionale per la Fisica Matematica - INDAM.

References

- [1] A. Adhikari, C. Brennecke, and B. Schlein, [Bose–Einstein condensation beyond the Gross–Pitaevskii regime](#). *Ann. Henri Poincaré* **22** (2021), no. 4, 1163–1233 Zbl 1467.82006 MR 4229531
- [2] C. Bardos, F. Golse, and N. J. Mauser, [Weak coupling limit of the \$N\$ -particle Schrödinger equation](#). *Methods Appl. Anal.* **7** (2000), no. 2, 275–293 Zbl 1003.81027 MR 1869286
- [3] G. Basti, S. Cenatiempo, A. Giuliani, A. Olgiati, G. Pasqualetti, and B. Schlein, [Upper bound for the ground state energy of a dilute Bose gas of hard spheres](#). [v1] 2022, [v2] 2023, arXiv:2212.04431v2
- [4] G. Basti, S. Cenatiempo, A. Olgiati, G. Pasqualetti, and B. Schlein, [A second order upper bound for the ground state energy of a hard-sphere gas in the Gross–Pitaevskii regime](#). *Comm. Math. Phys.* **399** (2023), no. 1, 1–55 Zbl 07672991 MR 4567367
- [5] G. Basti, S. Cenatiempo, and B. Schlein, [A new second-order upper bound for the ground state energy of dilute Bose gases](#). *Forum Math. Sigma* **9** (2021), article no. e74 Zbl 1483.81165 MR 4342115
- [6] G. Ben Arous, K. Kirkpatrick, and B. Schlein, [A central limit theorem in many-body quantum dynamics](#). *Comm. Math. Phys.* **321** (2013), no. 2, 371–417 Zbl 1280.81157 MR 3063915
- [7] N. Benedikter, G. de Oliveira, and B. Schlein, [Quantitative derivation of the Gross–Pitaevskii equation](#). *Comm. Pure Appl. Math.* **68** (2015), no. 8, 1399–1482 Zbl 1320.35318 MR 3366749
- [8] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, [Bogoliubov theory in the Gross–Pitaevskii limit](#). *Acta Math.* **222** (2019), no. 2, 219–335 Zbl 1419.82064 MR 3974476
- [9] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, [Optimal rate for Bose–Einstein condensation in the Gross–Pitaevskii regime](#). *Comm. Math. Phys.* **376** (2020), no. 2, 1311–1395 Zbl 1439.82004 MR 4103969
- [10] C. Boccato, S. Cenatiempo, and B. Schlein, [Quantum many-body fluctuations around nonlinear Schrödinger dynamics](#). *Ann. Henri Poincaré* **18** (2017), no. 1, 113–191 Zbl 1358.81173 MR 3592092
- [11] C. Boccato, A. Deuchert, and D. Stocker, [Upper bound for the grand canonical free energy of the Bose gas in the Gross–Pitaevskii limit](#). *SIAM J. Math. Anal.* **56** (2024), no. 2, 2611–2660 Zbl 07849796 MR 4722361
- [12] C. Boccato and R. Seiringer, [The Bose gas in a box with Neumann boundary conditions](#). *Ann. Henri Poincaré* **24** (2023), no. 5, 1505–1560 Zbl 1522.81792 MR 4581120
- [13] L. Boßmann, N. Pavlović, P. Pickl, and A. Soffer, [Higher order corrections to the mean-field description of the dynamics of interacting Bosons](#). *J. Stat. Phys.* **178** (2020), no. 6, 1362–1396 Zbl 1439.82029 MR 4081233
- [14] L. Bossmann, S. Petrat, P. Pickl, and A. Soffer, [Beyond Bogoliubov dynamics](#). *Pure Appl. Anal.* **3** (2021), no. 4, 677–726 Zbl 1496.35332 MR 4384032
- [15] C. Brennecke, M. Caporaletti, and B. Schlein, [Excitation spectrum of Bose gases beyond the Gross–Pitaevskii regime](#). *Rev. Math. Phys.* **34** (2022), no. 9, article no. 2250027 Zbl 1513.81132 MR 4496997
- [16] C. Brennecke, P. T. Nam, M. Napiórkowski, and B. Schlein, [Fluctuations of \$N\$ -particle quantum dynamics around the nonlinear Schrödinger equation](#). *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 5, 1201–1235 Zbl 1419.81042 MR 3985542
- [17] C. Brennecke and B. Schlein, [Gross–Pitaevskii dynamics for Bose–Einstein condensates](#). *Anal. PDE* **12** (2019), no. 6, 1513–1596 Zbl 1414.35184 MR 3921312

- [18] C. Brennecke, B. Schlein, and S. Schraven, [Bogoliubov theory for trapped bosons in the Gross–Pitaevskii regime](#). *Ann. Henri Poincaré* **23** (2022), no. 5, 1583–1658 Zbl 1489.82046 MR 4415673
- [19] C. Brennecke, B. Schlein, and S. Schraven, [Bose–Einstein condensation with optimal rate for trapped bosons in the Gross–Pitaevskii regime](#). *Math. Phys. Anal. Geom.* **25** (2022), no. 2, article no. 12 Zbl 07528333 MR 4406847
- [20] S. Buchholz, C. Saffirio, and B. Schlein, [Multivariate central limit theorem in quantum dynamics](#). *J. Stat. Phys.* **154** (2014), no. 1-2, 113–152 Zbl 1291.82079 MR 3162535
- [21] C. Caraci, S. Cenatiempo, and B. Schlein, [Bose–Einstein condensation for two dimensional bosons in the Gross–Pitaevskii regime](#). *J. Stat. Phys.* **183** (2021), no. 3, article no. 39 Zbl 1508.81975 MR 4261710
- [22] C. Caraci, S. Cenatiempo, and B. Schlein, [The excitation spectrum of two-dimensional Bose gases in the Gross–Pitaevskii regime](#). *Ann. Henri Poincaré* **24** (2023), no. 8, 2877–2928 Zbl 07710871 MR 4612320
- [23] L. Chen, J. O. Lee, and B. Schlein, [Rate of convergence towards Hartree dynamics](#). *J. Stat. Phys.* **144** (2011), no. 4, 872–903 Zbl 1227.82046 MR 2826623
- [24] X. Chen, [Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions](#). *Arch. Ration. Mech. Anal.* **203** (2012), no. 2, 455–497 Zbl 1256.35099 MR 2885567
- [25] L. Erdős, B. Schlein, and H.-T. Yau, [Derivation of the Gross–Pitaevskii hierarchy for the dynamics of Bose–Einstein condensate](#). *Comm. Pure Appl. Math.* **59** (2006), no. 12, 1659–1741 Zbl 1122.82018 MR 2257859
- [26] L. Erdős, B. Schlein, and H.-T. Yau, [Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems](#). *Invent. Math.* **167** (2007), no. 3, 515–614 Zbl 1123.35066 MR 2276262
- [27] L. Erdős, B. Schlein, and H.-T. Yau, [Rigorous derivation of the Gross–Pitaevskii equation](#). *Phys. Rev. Lett.* **98** (2007), no. 4, article no. 040404
- [28] L. Erdős, B. Schlein, and H.-T. Yau, [Rigorous derivation of the Gross–Pitaevskii equation with a large interaction potential](#). *J. Amer. Math. Soc.* **22** (2009), no. 4, 1099–1156 Zbl 1207.82031 MR 2525781
- [29] L. Erdős, B. Schlein, and H.-T. Yau, [Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate](#). *Ann. of Math. (2)* **172** (2010), no. 1, 291–370 Zbl 1204.82028 MR 2680421
- [30] L. Erdős and H.-T. Yau, [Derivation of the nonlinear Schrödinger equation from a many body Coulomb system](#). *Adv. Theor. Math. Phys.* **5** (2001), no. 6, 1169–1205 Zbl 1014.81063 MR 1926667
- [31] S. Fournais and J. P. Solovej, [The energy of dilute Bose gases](#). *Ann. of Math. (2)* **192** (2020), no. 3, 893–976 Zbl 1455.81050 MR 4172623
- [32] S. Fournais and J. P. Solovej, [The energy of dilute Bose gases II: The general case](#). *Invent. Math.* **232** (2023), no. 2, 863–994 Zbl 07676261 MR 4574665
- [33] J. Fröhlich, A. Knowles, and A. Pizzo, [Atomism and quantization](#). *J. Phys. A* **40** (2007), no. 12, 3033–3045 Zbl 1115.81071 MR 2313859
- [34] J. Fröhlich, A. Knowles, and S. Schwarz, [On the mean-field limit of bosons with Coulomb two-body interaction](#). *Comm. Math. Phys.* **288** (2009), no. 3, 1023–1059 Zbl 1177.82016 MR 2504864
- [35] J. Ginibre and G. Velo, [The classical field limit of scattering theory for nonrelativistic many-boson systems. I](#). *Comm. Math. Phys.* **66** (1979), no. 1, 37–76 Zbl 0443.35067 MR 0530915

- [36] M. Grillakis and M. Machedon, [Beyond mean field: On the role of pair excitations in the evolution of condensates](#). *J. Fixed Point Theory Appl.* **14** (2013), no. 1, 91–111
Zbl 1304.81072 MR 3202027
- [37] M. Grillakis and M. Machedon, [Pair excitations and the mean field approximation of interacting bosons, I](#). *Comm. Math. Phys.* **324** (2013), no. 2, 601–636 Zbl 1277.82034 MR 3117522
- [38] M. Grillakis, M. Machedon, and D. Margetis, [Second-order corrections to mean field evolution of weakly interacting bosons. II](#). *Adv. Math.* **228** (2011), no. 3, 1788–1815 Zbl 1226.82033 MR 2824569
- [39] M. G. Grillakis, M. Machedon, and D. Margetis, [Second-order corrections to mean field evolution of weakly interacting bosons. I](#). *Comm. Math. Phys.* **294** (2010), no. 1, 273–301
Zbl 1208.82030 MR 2575484
- [40] F. Haberberger, C. Hainzl, P. T. Nam, R. Seiringer, and A. Triay, [The free energy of dilute Bose gases at low temperatures](#). [v1] 2023, [v2] 2024, arXiv:2304.02405v2
- [41] C. Hainzl, B. Schlein, and A. Triay, [Bogoliubov theory in the Gross–Pitaevskii limit: A simplified approach](#). *Forum Math. Sigma* **10** (2022), article no. e90 Zbl 1507.82066 MR 4498777
- [42] K. Hepp, [The classical limit for quantum mechanical correlation functions](#). *Comm. Math. Phys.* **35** (1974), 265–277 MR 0332046
- [43] E. Kuz, [Exact evolution versus mean field with second-order correction for bosons interacting via short-range two-body potential](#). *Differential Integral Equations* **30** (2017), no. 7-8, 587–630 Zbl 1413.81041 MR 3646465
- [44] M. Lewin, P. T. Nam, and B. Schlein, [Fluctuations around Hartree states in the mean-field regime](#). *Amer. J. Math.* **137** (2015), no. 6, 1613–1650 Zbl 1329.81430 MR 3432269
- [45] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, [Bogoliubov spectrum of interacting Bose gases](#). *Comm. Pure Appl. Math.* **68** (2015), no. 3, 413–471 Zbl 1318.82030 MR 3310520
- [46] E. H. Lieb and R. Seiringer, [Proof of Bose–Einstein condensation for dilute trapped gases](#). *Phys. Rev. Lett.* **88**, (2002), article no. 170409
- [47] E. H. Lieb and R. Seiringer, [Derivation of the Gross–Pitaevskii equation for rotating Bose gases](#). *Comm. Math. Phys.* **264** (2006), no. 2, 505–537 Zbl 1233.82004 MR 2215615
- [48] E. H. Lieb, R. Seiringer, and J. Yngvason, [Bosons in a trap: A rigorous derivation of the Gross–Pitaevskii energy functional](#). *Phys. Rev. A* **61**,(2000), article no. 043602
- [49] D. Mitrouskas, S. Petrat, and P. Pickl, [Bogoliubov corrections and trace norm convergence for the Hartree dynamics](#). *Rev. Math. Phys.* **31** (2019), no. 8, article no. 1950024
Zbl 1435.35321 MR 3999520
- [50] P. T. Nam and M. Napiórkowski, [Bogoliubov correction to the mean-field dynamics of interacting bosons](#). *Adv. Theor. Math. Phys.* **21** (2017), no. 3, 683–738 Zbl 1382.82032 MR 3695801
- [51] P. T. Nam and M. Napiórkowski, [A note on the validity of Bogoliubov correction to mean-field dynamics](#). *J. Math. Pures Appl. (9)* **108** (2017), no. 5, 662–688 Zbl 1376.35016 MR 3711470
- [52] P. T. Nam, M. Napiórkowski, J. Ricaud, and A. Triay, [Optimal rate of condensation for trapped bosons in the Gross–Pitaevskii regime](#). *Anal. PDE* **15** (2022), no. 6, 1585–1616
Zbl 1513.81135 MR 4507326
- [53] P. T. Nam, N. Rougerie, and R. Seiringer, [Ground states of large bosonic systems: The Gross–Pitaevskii limit revisited](#). *Anal. PDE* **9** (2016), no. 2, 459–485 Zbl 1342.35284 MR 3513141

- [54] P. T. Nam and A. Triay, [Bogoliubov excitation spectrum of trapped Bose gases in the Gross–Pitaevskii regime](#). *J. Math. Pures Appl. (9)* **176** (2023), 18–101 Zbl 1527.81119 MR 4612702
- [55] P. Pickl, [Derivation of the time dependent Gross–Pitaevskii equation with external fields](#). *Rev. Math. Phys.* **27** (2015), no. 1, article no. 1550003 Zbl 1320.35328 MR 3317556
- [56] S. Rademacher, [Central limit theorem for Bose gases interacting through singular potentials](#). *Lett. Math. Phys.* **110** (2020), no. 8, 2143–2174 Zbl 1446.35152 MR 4126876
- [57] S. Rademacher and B. Schlein, [Central limit theorem for Bose–Einstein condensates](#). *J. Math. Phys.* **60** (2019), no. 7, article no. 071902 Zbl 1421.82035 MR 3978931
- [58] I. Rodnianski and B. Schlein, [Quantum fluctuations and rate of convergence towards mean field dynamics](#). *Comm. Math. Phys.* **291** (2009), no. 1, 31–61 Zbl 1186.82051 MR 2530155
- [59] H. Spohn, [Kinetic equations from Hamiltonian dynamics: Markovian limits](#). *Rev. Modern Phys.* **52** (1980), no. 3, 569–615 MR 0578142

Received 21 September 2023; revised 15 April 2024; accepted 24 April 2024.

Cristina Caraci

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland;
cristina.caraci@math.uzh.ch

Jakob Oldenburg

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland;
jakob.oldenburg@math.uzh.ch

Benjamin Schlein

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland;
benjamin.schlein@math.uzh.ch