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Łojasiewicz inequalities for almost harmonic maps near simple bubble trees

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Abstract. We prove Łojasiewicz inequalities for the harmonic map energy for maps from surfaces of positive genus into general analytic target manifolds which are close to simple bubble trees and as a consequence obtain new results on the convergence of harmonic map flow and on the energy spectrum of harmonic maps with small energy. Our results and techniques are not restricted to particular targets or to integrable settings and we are able to lift general Łojasiewicz–Simon inequalities valid near harmonic maps $\hat{\omega}: S^2 \to N$ to the singular setting whenever the bubble $\hat{\omega}$ is attached at a point which is not a branch point.

1. Introduction

Let (Σ, g) be a closed orientable surface and let (N, g_N) be a closed Riemannian manifold of any dimension, which by Nash's embedding theorem can be assumed to be isometrically embedded $N \hookrightarrow \mathbb{R}^n$ in some Euclidean space. We recall that a map $u: \Sigma \to N$ is called a harmonic map if it is a critical point of the Dirichlet energy

$$E(u) := \frac{1}{2} \int_\Sigma |du|^2 \, dv_g.$$

Harmonic maps are characterised by $\tau_g(u) = 0$, where the tension of $u: \Sigma \to N \hookrightarrow \mathbb{R}^n$ can be described as $\tau_g(u) = P_u(\Delta_g u) = \Delta_g u + A(u)(\nabla u, \nabla u)$, Δ_g the Laplace–Beltrami operator of maps $u: (\Sigma, g) \to \mathbb{R}^n$ and $P_p: \mathbb{R}^n \to T_p N$ the orthogonal projection. Here and in the following, $A(p)(v, w) = -(dP_p)(v)(w)$, $v, w \in T_p N$, denotes the second fundamental form of $N \hookrightarrow \mathbb{R}^N$ and we write for short $A(u)(\nabla u, \nabla u) = g^{ij} A(u)(\partial_{x_i} u, \partial_{x_j} u)$.

In the study of harmonic maps from closed surfaces of positive genus, one is often confronted with the situation that the lowest possible energy level E_0 of homotopically non-trivial maps is not attained in the set of maps from the given surface Σ; instead, minimising sequences may undergo bubbling and converge to a limiting configuration which is a simple bubble tree consisting of a trivial base map and a single bubble $\hat{\omega}$ given by a non-trivial harmonic map $\hat{\omega}: S^2 \to N$. This singular behaviour means that the powerful techniques of Łojasiewicz inequalities as developed in the seminal work of

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Simon [18] do not apply, even in the simplest such situation of degree one maps from the torus to S^2. As a result, questions such as the discreteness of the energy spectrum near E_0 and the asymptotic behaviour of harmonic map flow for maps whose energy tends to E_0 are open in the setting of maps from higher genus surfaces.

The purpose of this paper is to address these and related questions, not only in the special situation of maps to the sphere mentioned above, but more generally for maps into closed analytic manifolds of arbitrary dimension which are close to simple bubble trees for which the underlying bubble is attached at a non-branched point.

To this end, we first recall that the results of [4, 11, 14, 19] imply that for any sequence of maps $u_n\colon \Sigma \to N$ with bounded energy and $\|\tau_g(u_n)\|_{L^2(\Sigma)} \to 0$, a subsequence converges strongly in $H^2_{\text{loc}}(\Sigma \setminus S)$ to a harmonic limit $u_\infty\colon \Sigma \to N$ away from a finite set of points S, where a finite number of bubbles form, and that in this convergence to a bubble tree there is no loss of energy and no formation of necks.

If no bubbles form and if N is analytic then we can apply the work of Simon [18], which establishes that there exist a neighbourhood of u_∞, a constant C and an exponent $\gamma_\infty \in (1,2]$ so that for maps $u\colon \Sigma \to N$ in this neighbourhood the Łojasiewicz estimate

$$|E(u) - E(u_\infty)| \leq C\|\tau_g(u)\|_{L^2(\Sigma)}^{\gamma_\infty} \tag{1.1}$$

holds true. While the method of Simon from [18] applies provided the maps are close to u_∞ in H^2, the above inequality is trivially satisfied for maps with bounded energy and large tension, so (1.1) holds whenever u is H^1-close to u_∞.

However, this result is not applicable for maps that undergo bubbling and the only setting in which this problem has been overcome is in the major works [20, 21] of Topping and [23] of Waldron on almost harmonic maps between spheres. These results are based on a delicate analysis of almost harmonic maps which exploits in particular that for maps between spheres the Dirichlet energy has a natural splitting into a holomorphic and an antiholomorphic part. This allowed Topping [21] to derive a Łojasiewicz estimate with optimal exponent

$$|E(u) - 4k\pi| \leq C\|\tau_{g_{S^2}}(u)\|_{L^2(S^2)}^2$$

for maps between spheres which are close to a large class of bubble trees and very recently for Waldron [23] to obtain Łojasiewicz estimates near general bubble trees.

Here we do not restrict our attention to a particular domain surface or a particular target, but instead restrict the limiting configuration to the simplest situation where strong convergence fails, i.e. where the maps converge to a simple bubble tree consisting of a constant base map and a single bubble. In this situation, the results of [4, 11, 14, 19] ensure that there exist a non-constant harmonic map $\hat{\omega}\colon S^2 \to N$, points $a_n \to a$ and bubble scales $\lambda_n \to \infty$ so that, after passing to a subsequence, $u_n \to \hat{\omega}(p^*)$ strongly in $H^2_{\text{loc}}(\Sigma \setminus \{a\})$ while on some fixed-sized ball $B_r(a)$, working in local isothermal coordinates $F_a\colon \Sigma \supset B_r(a) \to \mathbb{D}_{\tilde{r}} \subset \mathbb{R}^2$, we have

$$u_n \circ F_a^{-1} - \hat{\omega} \circ \pi_{\lambda_n}^{F_a(a_n)} \to 0 \quad \text{strongly in } H^1(\mathbb{D}_{\tilde{r}}) \cap L^\infty(\mathbb{D}_{\tilde{r}}). \tag{1.2}$$

Here, $\pi_\lambda^b := \pi(\lambda(x-b))$ for $\pi\colon \mathbb{R}^2 \to S^2 \setminus \{p^*\}$ the inverse of the stereographic projection from the north pole $p^* = (0,0,1)^\top$.

We note that despite this simple structure of the bubble tree, the result of Simon [18] is not applicable as we cannot view such maps as being H^1-close to a critical point $u_\infty\colon \Sigma \to N$ of the energy. The only exception to this is when the domain is a sphere, as in this case we can modify any such sequence by suitable Möbius transforms to obtain strong convergence to $\hat{\omega}$ on all of S^2. For the rest of the paper we will thus assume that Σ is a closed orientable surface of genus $\gamma \geq 1$. Since the energy is conformally invariant, we can assume that our domain is either a flat unit area torus or, for higher genus surfaces, that the metric g is hyperbolic, i.e. has (Gauss)-curvature -1.

While also in the present work one of the key steps will be to relate the rate at which the tension tends to zero with the rate at which the bubble concentrates, our method of proof will be very different to the ones in [21, 23]. In particular, we will not require any information on the behaviour of general almost harmonic maps beyond the well-known results on the bubble tree convergence recalled above. Instead, our analysis will follow the approach developed in the joint work [12] with Malchiodi and Sharp and we will derive our Łojasiewicz-estimate by comparing maps $u\colon \Sigma \to N$ which undergo bubbling to maps in a specific finite-dimensional set $\mathcal{Z}$ of what we call adapted bubbles. These adapted bubbles $z\colon \Sigma \to N$ provide models for maps converging to a simple bubble tree and are constructed so that the energy and its variations have the right properties on $\mathcal{Z}$. The key point of the method of proof is that a careful analysis of the energy and its variations on $\mathcal{Z}$ allows us to obtain Łojasiewicz estimates for much more general almost critical points, without ever having to analyse such general almost critical points. We also refer the reader to [12, Theorem 2.2] which establishes Łojasiewicz estimates near (non-compact) finite-dimensional manifolds of adapted critical points in the abstract setting of energies on Hilbert spaces, and to [16, 17] for recent applications of these ideas.

In the analysis of almost critical points of the H-surface energy in [12] the set of bubbles is explicitly known, indeed consists of rotations of the identity, and the bubbles are non-degenerate critical points, i.e. so that the second variation of the energy is definite in directions orthogonal to the action of Möbius transforms.

The present paper demonstrates that the ideas developed in [12] can be applied to far more general settings, where neither of these simplifications is present. On the one hand, we will not require any detailed information about the underlying bubbles $\hat{\omega}$. In particular, our proof does not rely on the explicit knowledge of the set of bubbles that for harmonic maps one would only have for special targets such as spheres. All we need to ask of the bubble is that it is attached to the base at a point that is not a branched point, i.e. that $d\hat{\omega}(p^*) \neq 0$, $p^* = \pi(\infty) = (0,0,1)^\top$.

Just as importantly, we will see that our method does not rely on the non-degeneracy of the underlying critical point that is present in [12] and we will be able to prove Łojasiewicz estimates even if $\hat{\omega}\colon S^2 \to N$ is a harmonic map which has non-integrable Jacobi fields. Indeed, we are able to lift the Łojasiewicz–Simon estimates [18]

$$|E(\widehat{\omega}) - E(\omega)| \le C \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^{\gamma_1} \tag{1.3}$$

and

$$\mathrm{dist}_{L^2}(\omega, \{\widetilde{\omega}: S^2 \to N \text{ harmonic}\}) \le C \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^{\gamma_2}, \tag{1.4}$$

from the regular setting of maps $\omega: S^2 \to N$ which are close to $\widehat{\omega}$ to obtain Łojasiewicz estimates with the same exponents $\gamma_1 \in (1,2]$ and $\gamma_2 \in (0,1]$ in the singular setting of maps from Σ which converge to simple bubble trees. To be more precise, we will prove the following theorem:

Theorem 1.1. *Let (Σ, g) be a closed oriented surface of positive genus and let (N, g_N) be a closed analytic manifold of any dimension. Let (u_n) be a sequence of maps with bounded energy which are almost harmonic in the sense that*

$$\mathcal{T}_n := \|\tau_g(u_n)\|_{L^2(\Sigma,g)} \to 0.$$

Suppose that u_n converges as described above to a bubble tree consisting of a constant base map $u_\infty: \Sigma \to N$ and a single bubble $\widehat{\omega}: S^2 \to N$ which is so that $d\widehat{\omega}(p^*) \neq 0$. Then, for sufficiently large n, we can bound the bubble scale λ_n in (1.2) by

$$\lambda_n^{-1} \le C \mathcal{T}_n |\log \mathcal{T}_n|^{\frac{1}{2}} \tag{1.5}$$

and the difference in energy by

$$|E(u_n, \Sigma) - E(\omega, S^2)| \le C \mathcal{T}_n^{\gamma_1} |\log \mathcal{T}_n|^{\frac{\gamma_1}{2}} \tag{1.6}$$

for the same exponent $\gamma_1 \in (1,2]$ for which (1.3) holds near $\widehat{\omega}: S^2 \to N$.

Furthermore, we can choose $\lambda_n \to \infty$, $a_n \to a$ and a sequence of harmonic maps $\omega_n: S^2 \to N$ which converge smoothly to ω_∞ so that

$$\|\nabla(u_n - \omega_n \circ \pi_{\lambda_n} \circ F_{a_n})\|_{L^2(B_{r_1}(a))} + \|\nabla u_n\|_{L^2(\Sigma \setminus B_{r_1}(a))} \le C \mathcal{T}_n^{\gamma_2} |\log \mathcal{T}_n|^{\frac{\gamma_2}{2}} \tag{1.7}$$

and

$$\|u_n - \omega_n(p^*)\|_{L^2(\Sigma,g)} \le C \mathcal{T}_n^{\gamma_2} |\log \mathcal{T}_n|^{\frac{\gamma_2}{2}} + C \mathcal{T}_n |\log \mathcal{T}_n| \tag{1.8}$$

and so that for every $r > 0$ there exists a constant C with

$$\|u_n \circ (\pi_{\lambda_n} \circ F_{a_n})^{-1} - \omega_n\|_{L^2(S^2 \setminus B_r(p^*))} \le C \mathcal{T}_n^{\gamma_2} |\log \mathcal{T}_n|^{\frac{\gamma_2}{2}}. \tag{1.9}$$

Here, $\gamma_2 \in (0,1]$ is the same exponent for which (1.4) holds, F_{a_n} are local isothermal coordinates centred at a_n as introduced in Remark 2.2 and $r_1 > 0$ is a fixed radius.

Remark 1.2. If all Jacobi fields along $\widehat{\omega}$ are integrable then we can drop the assumption that N is analytic and obtain the above result for $\gamma_1 = 2$ and $\gamma_2 = 1$. However, as observed by Lemaire–Wood [10], even energy minimisers can have non-integrable Jacobi fields. Conversely, the works of Gulliver–White [8] and Lemaire–Wood [9] establish that all Jacobi fields along harmonic spheres are integrable if the target is homotopic to S^2 or $\mathbb{C}P^2$.

Over the past decades, Łojasiewicz estimates have become a well-established tool in the analysis of variational problems in non-singular settings. However, to date there are few instances of Łojasiewicz estimates in settings with singularities or with a change of topology. In addition to [12,21] mentioned above, such results were obtained in the major papers of Colding–Minicozzi [2] and Chodosh–Schulze [1] on the uniqueness of blow-ups of mean curvature flow and by Glaudo–Figalli [6] and Deng–Sun–Wei [3] on critical points of the Sobolev inequality. One of the reasons that Łojasiewicz estimates have attracted a lot of interest is their versatility in applications both to variational problems and to the analysis of evolution equations. They can be used in particular to establish convergence of gradient flows, as well as to analyse the energy spectrum of critical points. As a consequence of Theorem 1.1 we will hence obtain new results both on the asymptotic behaviour of harmonic map flow

$$\partial_t u = -\nabla^{L^2} E(u) = \tau_g(u), \quad u(t=0) = u_0 \in H^1(\Sigma, N), \tag{1.10}$$

and on the energy spectrum of harmonic maps from higher genus surfaces into general analytic manifolds.

Simon's results [18] imply that the energy spectrum $\{E(u) : u\colon S^2 \to N \text{ harmonic}\}$ of harmonic maps from S^2 into any analytic manifold N is discrete below the level $2E_{S^2}$,

$$E_{S^2} := \min\big\{E(u) : u\colon S^2 \to (N, g_N) \text{ harmonic, non-constant}\big\},$$

since harmonic maps with energy $E(u_n) \to E_\infty < 2E_{S^2}$ can always be pulled back by suitable Möbius transforms to ensure that they subconverge strongly.

Conversely, for surfaces of positive genus, [18] only implies that the energy spectrum $\{E(u) : u\colon (\Sigma, g) \to (N, g_N) \text{ harmonic}\}$ is discrete below the energy level E_{S^2}. Theorem 1.1 now allows us to deduce the following result, which is, in particular, of interest for maps into three-manifolds, where the results [7] of Gulliver–Osserman–Royden ensure that area-minimising surfaces cannot have true branch points.

Corollary 1.3. *Let (N, g_N) be a closed analytic manifold of any dimension and let (Σ, g) be a closed surface of positive genus. Then the energy spectrum of harmonic maps from (Σ, g) to N below the level*

$$E^* := \min(2E_{S^2}, E_{(\Sigma,g)} + E_{S^2}, E^*_{S^2}) \tag{1.11}$$

is discrete, where

$$E_{(\Sigma,g)} := \inf\{E(u) : u\colon (\Sigma, g) \to N \textit{ harmonic, non-constant}\},$$
$$E^*_{S^2} := \inf\{E(\omega) : \omega\colon S^2 \to N \textit{ branched, harmonic, non-constant}\}.$$

To state our results on harmonic map flow, we first recall that the work of Struwe [19] establishes the existence of a global weak solution of (1.10) which has non-increasing energy and which is smooth away from finitely many times at which bubbling occurs.

While solutions of this flow always subconverge along a sequence of times $t_j \to \infty$ either to a harmonic map or to a bubble tree of harmonic maps, Topping [22] showed that one cannot expect that the whole flow converges as $t \to \infty$ for general smooth target manifolds. Conversely, it is conjectured that for analytic targets the flow must indeed converge. If no bubbling occurs at infinite time this already follows from the work of Simon [18], while for maps from S^2 to S^2 the Łojasiewicz inequalities of Topping [21] and Waldron [23] yield convergence results for the flow.

We can now establish convergence of harmonic map flow into any analytic target manifold (N, g_N) provided the initial energy is below the above-mentioned energy threshold. We stress that this constraint on the energy does allow for bubbling, though it restricts the potential limiting configurations to the simple bubble trees considered in Theorem 1.1.

Theorem 1.4. *Let (N, g) be a closed analytic manifold of any dimension and let $u_0 \in H^1(\Sigma, N)$ be any map with $E(u_0) \leq E^*$ for E^* defined in (1.11). Then the corresponding solution of harmonic map flow (1.10) either converges smoothly to a harmonic map $u_\infty\colon \Sigma \to N$ as $t \to \infty$ as described in [18] or it converges to a simple bubble tree in the following sense:*

There exist a point $a \in \Sigma$ and a harmonic sphere $\hat{\omega}\colon S^2 \to N$ so that the energy of $u(t)$ converges to $E_\infty = E(\hat{\omega})$ at a rate of

$$|E(u(t)) - E_\infty| \leq Ce^{-c_1\sqrt{t}}, \quad c_1 = c_1(N, (\Sigma, g), E_\infty) > 0 \tag{1.12}$$

if $\gamma_1 = 2$, respectively if $\gamma_1 \in (1, 2)$, at a rate of

$$|E(u(t)) - E_\infty| \leq Ct^{-\frac{\gamma_1}{2-\gamma_1}}(\log t)^{\frac{\gamma_1}{2-\gamma_1}}, \tag{1.13}$$

while for any $\alpha < \frac{\gamma_1 - 1}{\gamma_1}$ the maps converge in L^2 at a rate of

$$\|u(t) - \omega(p^*)\|_{L^2(\Sigma)} \leq C|E(u(t)) - E_\infty|^\alpha, \tag{1.14}$$

as well as in C^k on every compact subset K of $\Sigma \setminus \{p^*\}$ also at a rate of

$$\|u(t) - \omega(p^*)\|_{C^k(K)} \leq C|E(u(t)) - E_\infty|^\alpha. \tag{1.15}$$

Here, $\gamma_1 \in (1, 2]$ is so that (1.3) is valid with exponent γ_1 for all harmonic spheres with energy E_∞ and the constant C is allowed to depend on the setting, the specific solution and, in the cases of (1.14) and (1.15), additionally on $\alpha - \frac{\gamma_1 - 1}{\gamma_1} > 0$ and K.

Remark 1.5. As the set of harmonic spheres with energy $E_\infty < 2E_{S^2}$ is compact modulo Möbius transforms, there always exists an exponent $\gamma_1 \in (1, 2]$ so that (1.3) holds true for any harmonic sphere $\hat{\omega}$ with $E(\hat{\omega}) = E_\infty$. If all Jacobi fields along harmonic maps of energy E_∞ are integrable then we can drop the assumption that N is analytic and choose $\gamma_1 = 2$.

The first setting in which it was known that harmonic map flow must become singular, be it at finite or infinite time, is for degree ± 1 maps u_0 from the torus to the sphere, where the results of Eells–Wood [5] exclude the existence of a harmonic map that is homotopic to u_0. While $E^* = 8\pi$ for $N = S^2$, in this setting we obtain the following improvement of the above result:

Corollary 1.6. *Suppose that $u_0 \in H^1(T^2, S^2)$ has degree ± 1 and energy $E(u_0) \leq 12\pi$. Then the corresponding solution of harmonic map flow (1.10) either develops a bubble at a finite time T after which the flow converges exponentially to a constant or the flow converges to a simple bubble tree at a rate of $O(e^{-c\sqrt{t}})$ as described in Theorem 1.4.*

The paper is organised as follows: In Section 2 we explain the construction of the adapted bubbles $z: \Sigma \to N$ with which we will later compare more general almost harmonic maps $u: \Sigma \to N$ and state a version of our Łojasiewicz estimates for maps in a uniform $H^1 \cap L^\infty$-neighbourhood of the resulting finite-dimensional manifold $\mathcal{Z}$; compare Theorem 2.5. This theorem will then be proved in the subsequent Sections 3 and 4 and will in turn form the basis of the proof of Theorem 1.1 and of all other main results of the paper.

2. Definition of the adapted bubbles

Our set of adapted bubbles will be a finite-dimensional manifold of maps

$$\mathcal{Z} := \left\{ z_\lambda^{a,\omega} : \Sigma \to N,\ \lambda \geq \lambda_1,\ a \in \Sigma,\ \omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega}) \right\}$$

obtained by scaling maps $\omega: S^2 \to N$, which are elements of a suitable finite-dimensional manifold $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$, with a large factor λ, and then gluing them in a specific way to a point $a \in \Sigma$, as we describe in detail in the second part of this section.

A crucial point in the construction of this manifold $\mathcal{Z}$ is to ensure that the second variation of the energy is uniformly definite orthogonal to $\mathcal{Z}$. Therefore, the choice of the set of the underlying maps $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ from S^2 to N, which we use to define the elements of $\mathcal{Z}$, will crucially depend on the properties of the second variation of the energy at the limiting harmonic sphere $\hat{\omega}: S^2 \to N$.

We recall that $w \in \Gamma(\hat{\omega}^*TN)$ is called a Jacobi field along $\hat{\omega}$ if $d^2E(\hat{\omega})(w, v) = 0$ for all $v \in \Gamma(\hat{\omega}^*TN)$, or equivalently if w is a solution of

$$L_{\hat{\omega}}(w) := P_{\hat{\omega}}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tau(\pi_N(\hat{\omega} + \varepsilon w))\Big) = 0. \tag{2.1}$$

As the tension transforms according to $\tau(\omega \circ q) = \frac{1}{2}|\nabla q|^2 \tau(\omega) \circ q$ under conformal changes q, we know that any variation $M^{(\varepsilon)}$ of $\mathrm{Id}: S^2 \to S^2$ in the set of Möbius transforms $\text{Möb}(S^2)$ induces a Jacobi field $w = \frac{d}{d\varepsilon}|_{\varepsilon=0}(\hat{\omega} \circ M^{(\varepsilon)})$ along $\hat{\omega}$.

If the second variation of the energy is non-degenerate at $\hat{\omega}$ in the sense that all Jacobi fields are of this form, as was the case in [12], then we set

$$\mathcal{H}_1^{\sigma_1}(\hat{\omega}) := \left\{ \hat{\omega} \circ R : R \in \mathrm{SO}(3),\ |p^* - Rp^*| \leq \sigma_1 \right\}$$

for a sufficiently small number $\sigma_1 > 0$.

If all Jacobi fields along $\hat{\omega}$ are integrable, but not necessarily induced by Möbius transforms, then we use that the set of harmonic maps near $\hat{\omega}$ is a manifold $\mathcal{H}(\hat{\omega})$ with $T_{\hat{\omega}}\mathcal{H}(\hat{\omega}) = \ker(L_{\hat{\omega}})$ on which the energy is constant; compare [18]. In this situation we can split

$$\ker(L_{\hat{\omega}}) = \mathcal{V}_0(\hat{\omega}) \oplus \mathcal{V}_{\text{Möb}}(\hat{\omega}), \quad \mathcal{V}_{\text{Möb}}(\hat{\omega}) := T_{\hat{\omega}}\{\hat{\omega} \circ M : M \in \text{Möb}(S^2)\}$$

L^2-orthogonally, fix a parametrisation $\Psi_1 : \ker(L_{\hat{\omega}}) \supset \mathcal{U} \to \mathcal{H}(\hat{\omega})$ with $\Psi_1(0) = \hat{\omega}$ and $d\Psi_1(0) = \text{Id}$ and consider the submanifold $\mathcal{H}_0(\hat{\omega}) = \Psi_1(\mathcal{U} \cap \mathcal{V}_0(\hat{\omega}))$ which, for $\mathcal{U}$ small, is transversal to the action of Möbius transforms. For suitably small $\sigma_1 > 0$ we then let $\mathcal{H}_0^{\sigma_1}(\hat{\omega}) := \{\Psi_1(w) : w \in \mathcal{V}_0(\hat{\omega}) \text{ with } \|w\|_{L^2} \leq \sigma_1\}$ and set

$$\mathcal{H}_1^{\sigma_1}(\hat{\omega}) := \{\omega \circ R : R \in \text{SO}(3), \omega \in \mathcal{H}_0^{\sigma_1}(\hat{\omega}) \text{ with } |p^* - Rp^*| \leq \sigma_1\}.$$

If we are instead dealing with a non-integrable setting, then we also need to consider maps that are obtained by adapting certain non-harmonic maps $\omega : S^2 \to S^2$ in order to obtain a set of adapted bubbles which is large enough to capture all non-definite directions of the second variation. In this situation we choose $\mathcal{H}_0(\hat{\omega})$ as a suitable submanifold of the manifold used by Simon in [18] and define $\mathcal{H}_0^{\sigma_1}(\hat{\omega})$ and $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ as described above. We discuss the precise definition of $\mathcal{H}_0(\hat{\omega})$ in this case in Appendix A and for now simply record that it has the following properties:

Lemma 2.1. *Let $\hat{\omega} : S^2 \to N$ be a harmonic map into an analytic target N, let $k \in \mathbb{N}$, $\beta > 0$ and let $\mathcal{H}_0(\hat{\omega})$ be the submanifold of $C^{k+2,\beta}(S^2, N)$ defined in Appendix A. Then*

$$\ker(L_{\hat{\omega}}) = T_{\hat{\omega}}\{\omega \circ M : \omega \in \mathcal{H}_0(\hat{\omega}),\ M \in \text{Möb}(S^2)\} \tag{2.2}$$

and there exists a constant C so that for any $\omega \in \mathcal{H}_0(\hat{\omega})$,

$$\|\tau_{g_{S^2}}(\omega)\|_{C^k(S^2)} \leq C\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} \tag{2.3}$$

and so that we can choose $\omega^{(\varepsilon)}$ in $\mathcal{H}_0(\hat{\omega})$ with $\omega^{(\varepsilon=0)} = \omega$, $\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^k(S^2)} \leq C$ and

$$\frac{d}{d\varepsilon} E(\omega^{(\varepsilon)}) \geq \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}. \tag{2.4}$$

Here and in the following, all derivatives with respect to ε will be evaluated at $\varepsilon = 0$.

Having thus chosen the set $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ of maps we want to scale and glue to a point $a \in \Sigma$, we now turn to the precise construction of the maps $z_\lambda^{a,\omega}$, for $a \in \Sigma$, $\omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ and sufficiently large λ. We note that the right definition of these maps $z_\lambda^{a,\omega}$ is crucial to ensure that the first and second variations of the energy on $\mathcal{Z}$ have the right properties for our method of proof to work; compare also [12, Theorem 2.2].

Let $\pi : \mathbb{R}^2 \to S^2 \setminus \{p^*\}$, $p^* := (0, 0, 1)^\intercal$, be the inverse stereographic projection

$$\pi(x) = \Big(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\Big)^\intercal$$

and set $\pi_\lambda(x) := \pi(\lambda x)$, $\lambda > 0$. We want to define our adapted bubbles

$$z_\lambda^{a,\omega} \colon \Sigma \to N \quad \text{for } \omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega}),\, \lambda \geq \lambda_1,\, a \in \Sigma,$$

λ_1, σ_1 chosen later, in a way that $z_\lambda^{a,\omega}(p) \approx \omega(\pi_\lambda(x))$ in the following local isothermal coordinates $x = F_a(p)$.

Remark 2.2. Given any $a \in \Sigma$ we let $F_a \colon B_\iota(a) \to \mathbb{D}_{r_0} = \{x \in \mathbb{R}^2 : |x| < r_0\}$, $\iota := \frac{1}{2}\operatorname{inj}(\Sigma, g)$, be as in [12, Remark 3.1]: If (Σ, g) is a flat unit area torus we set $r_0 = \iota$ and use Euclidean translations F_a to the origin on a fundamental domain as coordinates. Conversely, for higher genus surfaces we set $r_0 = \tanh(\iota/2)$ and choose an orientation-preserving isometric isomorphism F_a that maps $(B_\iota(a), g)$ to the disc $\mathbb{D}_{r_0}$ in the Poincaré hyperbolic disc $(\mathbb{D}_1, \frac{4}{(1-|x|^2)^2} g_E)$.

While in the higher genus case this only determines the maps F_a up to a rotation of the domain, the specific choice of F_a will not affect the definition of the set of adapted bubbles as rotations of the coordinates correspond to the action of the subgroup of SO(3) which fixes p^*. In the few places where a consistent choice of F_a for a in a neighbourhood of some a_0 is needed, we can fix a tiling of the Poincaré hyperbolic disc and use hyperbolic translations to the origin as explained in [12, Remark 3.4].

Since

$$\pi_\lambda(x) = p^* + \Big(\frac{2x}{\lambda |x|^2}, 0\Big)^{\mathsf{T}} + O(\lambda^{-2}) \quad \text{for } |x| \geq c > 0,$$

we can write

$$\begin{aligned} \tilde{\omega}_\lambda(x) &:= \omega(\pi_\lambda(x)) \\ &= \omega(p^*) + d\omega(p^*)\Big(\frac{2x}{\lambda |x|^2}, 0\Big)^{\mathsf{T}} + O(\lambda^{-2}) \quad \text{for } |x| \geq c > 0, \end{aligned} \tag{2.5}$$

where we note that this expansion is valid for the function $\tilde{\omega}_\lambda$, as well as its derivatives with respect to x. We will later on consider variations z_ε of adapted bubbles obtained by variations of either the bubble parameter λ_ε or of the underlying map $\omega^{(\varepsilon)} \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ and will always assume that these variations are chosen so that

$$|\partial_\varepsilon \lambda_\varepsilon| \leq C\lambda \quad \text{and} \quad \|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2(S^2)} \leq C \tag{2.6}$$

as this corresponds to variations of order 1 after rescaling. We note that for such variations, the expansion (2.5) also gives an expansion for $\partial_\varepsilon \tilde{\omega}_{\lambda_\varepsilon}^{(\varepsilon)}$ and its spatial derivatives with an error term of the same order $O(\lambda^{-2})$.

As in [12] we modify $\tilde{\omega}_\lambda$ with the help of the Green's function G, characterised by

$$-\Delta_p G(p, a) = 2\pi\delta_a - 2\pi(\operatorname{Area}(\Sigma, g))^{-1} \quad \text{on } \Sigma.$$

Letting G_a be the function that represents G in the above coordinates we recall that

$$G_a(x, y) := G(F_a^{-1}(x), F_a^{-1}(y)) = -\log|x - y| + J_a(x, y), \quad x, y \in \mathbb{D}_{r_0}$$

for a smooth harmonic function J_a which represents the regular part of Green's function; see [12] for more detail. In particular,

$$\nabla_y G_a(x,0) = \frac{x}{|x|^2} + \nabla_y J_a(x,0),$$

and we will use this to adapt the maps $\widetilde{\omega}_\lambda$ to give well-defined maps $v_\lambda^{a,\omega}\colon \Sigma \to \mathbb{R}^n$, which we will later project onto N to obtain our adapted bubbles $z_\lambda^{a,\omega}\colon \Sigma \to N \hookrightarrow \mathbb{R}^n$. To do this, we let $j_\lambda^{a,\omega}\colon \mathbb{D}_{r_0} \to \mathbb{R}^n$ be defined by

$$j_\lambda^{a,\omega}(x) := \frac{2}{\lambda}\, \mathrm{d}\omega(p^*)(\nabla_y J_a(x,0) - \nabla_y J_a(0,0),0)^\mathsf{T},$$

and fix a cut-off function $\phi \in C_c^\infty(\mathbb{D}_{r_0},[0,1])$ with $\phi \equiv 1$ on $\mathbb{D}_{\frac{r_0}{2}}$, $r_0 > 0$ as in Remark 2.2. The maps $v_\lambda^{a,\omega}\colon \Sigma \to \mathbb{R}^n$ are then defined as

$$v_\lambda^{a,\omega}(p) = \omega(p^*) + \frac{2}{\lambda}\, \mathrm{d}\omega(p^*)(\partial_{a^1} G(p,a) - \partial_{y_1} J_a(0,0), \partial_{a^2} G(p,a) - \partial_{y_2} J_a(0,0), 0)^\mathsf{T}$$

on $\Sigma \setminus B_\iota(a)$, where $\partial_{a^i} = (F_a^{-1})_* \partial_{y^i}$, while on $B_\iota(a)$ we set $v_\lambda^{a,\omega}(p) = \tilde{v}_\lambda^{a,\omega}(F_a(p))$ for $\tilde{v}_\lambda^{a,\omega}\colon \mathbb{D}_{r_0} \to \mathbb{R}^n$ given by

$$\tilde{v}_\lambda^{a,\omega} := \phi[\widetilde{\omega}_\lambda + j_\lambda^{a,\omega}] + (1-\phi)\Big[\omega(p^*) + \frac{2}{\lambda}\, \mathrm{d}\omega(p^*)(\nabla_y G_a(\cdot,0) - \nabla_y J_a(0,0),0)^\mathsf{T}\Big].$$

We note that for $N = S^2 \hookrightarrow \mathbb{R}^3$ and $\omega = \mathrm{Id}$ this definition of $v_\lambda^{a,\omega}$ essentially agrees with the choice of the adapted bubbles in the H-surface case in [12], except that here we need to ensure that $j_\lambda^{a,\omega}(0) = 0$ as our problem does not have the translation invariance present in [12].

On $B_\iota(a)$ we can use that the function $v_\lambda^{a,\omega}$ is represented in the above coordinates by

$$\tilde{v}_\lambda^{a,\omega} = \widetilde{\omega}_\lambda + j_\lambda^{a,\omega} + e_\lambda^{a,\omega} \tag{2.7}$$

for an error term $e_\lambda^{a,\omega}$ that is supported on $\mathbb{D}_{r_0} \setminus \mathbb{D}_{\frac{r_0}{2}}$ and there of order

$$\|e_\lambda^{a,\omega}\|_{C^2} + \|\partial_\varepsilon e_\lambda^{a,\omega}\|_{C^2} = O(\lambda^{-2}). \tag{2.8}$$

We will in particular use that since $j_\lambda^{a,\omega}(0) = 0$ we have

$$|\tilde{v}_\lambda^{a,\omega}(x) - \widetilde{\omega}_\lambda(x)| \le C\lambda^{-1}|x| + C\lambda^{-2} \quad \text{and} \quad |\nabla(\tilde{v}_\lambda^{a,\omega} - \widetilde{\omega}_\lambda)| \le C\lambda^{-1} \ \text{ on } \mathbb{D}_{r_0}, \tag{2.9}$$

and that analogous estimates also hold true for the derivatives of these quantities with respect to ε. Away from $B_{\tilde{\iota}}(a) := F_a^{-1}(\mathbb{D}_{r_0/2})$, we can instead use that

$$\|v_\lambda^{a,\omega} - \omega(p^*)\|_{C^2(\Sigma\setminus B_{\tilde{\iota}}(a))} + \|\partial_\varepsilon(v_\lambda^{a,\omega} - \omega^{(\varepsilon)}(p^*))\|_{C^2(\Sigma\setminus B_{\tilde{\iota}}(a))} \le C\lambda^{-1}. \tag{2.10}$$

We now let $\delta_N > 0$ be so that the nearest point projection π_N to N is well defined and smooth in a δ_N-tubular neighbourhood of $N \hookrightarrow \mathbb{R}^n$. Then, for sufficiently large $\lambda_1 \ge 2$, the above estimates imply that $\mathrm{dist}(v_\lambda^{a,\omega}(\cdot), N) \le C\lambda_1^{-1} < \delta_N$ on Σ, allowing us to project these maps to define our adapted bubbles $z_\lambda^{a,\omega}\colon \Sigma \to N$ by

$$z_\lambda^{a,\omega}(p) := \pi_N(v_\lambda^{a,\omega}), \quad \lambda \ge \lambda_1,\ a \in \Sigma,\ \omega \in \mathcal{H}_1^{\sigma_1}(\widehat{\omega}).$$

Remark 2.3. In the following, all results are to be understood as being true for the set of adapted bubbles $\mathcal{Z} = \mathcal{Z}_{\lambda_1}^{\sigma_1}$ for sufficiently large $\lambda_1 \geq 2$ and sufficiently small $\sigma_1 > 0$, both allowed to depend only on (Σ, g), N and $\widehat{\omega}$. At times we will furthermore need to consider a smaller subset of this set given by

$$\mathcal{Z}_{\bar{\lambda}}^{\bar{\sigma}} := \left\{ z_\lambda^{a,\omega} : \lambda \geq \bar{\lambda},\ a \in \Sigma \text{ and } \omega \in \mathcal{H}_1^{\bar{\sigma}}(S^2) \right\}$$

for suitable $\bar{\lambda} \geq \lambda_1$ and $0 < \bar{\sigma} \leq \sigma_1$. Furthermore, we use the convention that C denotes a constant, allowed to change from line to line, which only depends on $\widehat{\omega}$, (Σ, g) and N unless indicated otherwise, and we will use the shorthand $A \lesssim B$ to mean that $A \leq CB$ for such a constant C.

In the following it will be important that we do not work with respect to the standard inner product on $H^1(\Sigma, \mathbb{R}^n)$, but instead use an inner product that appropriately weighs the L^2-part of the norm in the bubble region.

Definition 2.4. Given any $z = z_\lambda^{a,\omega} \in \mathcal{Z}$ we consider the inner product

$$\langle v, w \rangle_z := \int_\Sigma \nabla v \nabla w + \rho_z^2 v w \, \mathrm{d}v_g, \quad v, w \in H^1(\Sigma, \mathbb{R}^n),$$

where the weight ρ_z is given by $\rho_z \equiv \frac{\lambda}{1+\lambda^2 r_0^2}$ on $\Sigma \setminus B_\iota(a)$, while

$$\rho_z(p) = \rho_\lambda(x) := \frac{1}{2\sqrt{2}} |\nabla \pi_\lambda(x)| = \frac{\lambda}{1 + \lambda^2 |x|^2} \quad \text{for } p = F_a^{-1}(x) \in B_\iota(a).$$

At times we will also want to use local versions of the above norm, so set

$$\|w\|_{z,\Omega}^2 := \|\nabla w\|_{L^2(\Omega)}^2 + \|\rho_z w\|_{L^2(\Omega)}^2, \quad \text{for } \Omega \subset \Sigma.$$

We note that the weight $\rho_z \colon \Sigma \to \mathbb{R}^+$ is continuous and that we can bound

$$|\nabla z| \leq C \rho_z \quad \text{and} \quad \|\nabla (P_z w)\|_z \leq C \|w\|_z \tag{2.11}$$

for any $w \in H^1(\Sigma, \mathbb{R}^n)$ and any $z \in \mathcal{Z}$. A short calculation (see Appendix B) gives

$$\left| \fint_\Sigma w \, \mathrm{d}v_g \right| \leq C_{(\Sigma,g)} (\log \lambda)^{\frac{1}{2}} \|w\|_z, \tag{2.12}$$

and thus allows us to bound

$$\|w\|_{L^p(\Sigma,g)} \leq C (\log \lambda)^{\frac{1}{2}} \|w\|_z, \quad p \in [1, \infty),\ C = C(p, (\Sigma, g)). \tag{2.13}$$

With these definitions in place we can finally formulate our main result in the form that we will prove in Sections 3 and 4 and that will subsequently form the basis of the proofs of all other main results.

Theorem 2.5. *Let (Σ, g) be any closed surface of positive genus, let N be any analytic closed manifold and let $\widehat{\omega}\colon S^2 \to N$ be any harmonic map with $d\widehat{\omega}(p^*) \neq 0$. Then there exist numbers $\varepsilon > 0$, $\bar{\lambda} \geq \lambda_1$, $\bar{\sigma} \in (0, \sigma_1)$ and $C < \infty$ so that for every $u \in H^1(\Sigma, N)$ for which there exists $\tilde{z} \in \mathcal{Z}_{\bar{\lambda}}^{\bar{\sigma}}$ with*

$$\|\nabla(u - \tilde{z})\|_{L^2(\Sigma,g)} + \|u - \tilde{z}\|_{L^\infty(\Sigma,g)} < \varepsilon,$$

$\mathcal{Z}_{\bar{\lambda}}^{\bar{\sigma}}$ as in Remark 2.3, we can bound

$$\operatorname{dist}(u, \mathcal{Z}) := \inf_{z \in \mathcal{Z}} \|u - z\|_z \leq C \|\tau_g(u)\|_{L^2(\Sigma,g)} (1 + |\log\|\tau_g(u)\|_{L^2(\Sigma,g)}|^{\frac{1}{2}}), \tag{2.14}$$

while the estimate

$$|E(\omega) - E(u)| \leq C \|\tau_g(u)\|_{L^2(\Sigma,g)}^{\gamma_1} (1 + |\log \|\tau_g(u)\|_{L^2(\Sigma,g)}|^{\frac{1}{2}})^{\gamma_1}, \tag{2.15}$$

holds true for the exponent $\gamma_1 \in (1, 2]$ for which (1.3) holds.

Furthermore, for each such u there exists $z \in \mathcal{Z}$ with $\|u - z\|_z = \operatorname{dist}(u, \mathcal{Z})$ and the bubble scale of any such $z = z_\lambda^{a,\omega}$ satisfies

$$\lambda^{-1} \leq C \|\tau_g(u)\|_{L^2(\Sigma,g)} \cdot (1 + |\log \|\tau_g(u)\|_{L^2(\Sigma,g)}|^{\frac{1}{2}}) \tag{2.16}$$

while the tension of the underlying map $\omega\colon S^2 \to N$ is controlled by

$$\|\tau_{g_{S^2}}(\omega)\|_{C^2(S^2)} \leq C \|\tau_g(u)\|_{L^2(\Sigma,g)} \cdot (1 + |\log \|\tau_g(u)\|_{L^2(\Sigma,g)}|^{\frac{1}{2}}). \tag{2.17}$$

3. Properties of the energy on the set of adapted bubbles

3.1. Basic properties of the second variation of the Dirichlet energy

We first recall the following standard expression for the second variation of the Dirichlet energy, for which we include a short proof for the convenience of the reader.

Lemma 3.1. *For any $u \in H^1(\Sigma, N)$ and $v, w \in \Gamma^{H^1 \cap L^\infty}(u^*TN)$ we can write the second variation of the Dirichlet energy $d^2E(u)(v, w) := \frac{d}{d\varepsilon}|_{\varepsilon=0} \frac{d}{d\delta}|_{\delta=0} E(\pi_N(u + \varepsilon v + \delta w))$ as*

$$d^2E(u)(v, w) = \int_\Sigma \nabla v \nabla w - A(u)(\nabla u, \nabla u) A(u)(v, w) \, \mathrm{d}v_g. \tag{3.1}$$

Proof. We write $u_\varepsilon := \pi_N(u + \varepsilon v) = u + \varepsilon v + O(\varepsilon^2)$ and use that the negative L^2-gradient of E is given by $\tau_g(u_\varepsilon) = \Delta_g u_\varepsilon + A(u_\varepsilon)(\nabla u_\varepsilon, \nabla u_\varepsilon) = P_{u_\varepsilon}(\Delta_g u_\varepsilon)$ to compute

$$\begin{aligned} d^2E(u)(v, w) &= -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int \tau_g(u_\varepsilon) w \, \mathrm{d}v_g = -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int \Delta_g u_\varepsilon P_{u_\varepsilon}(w) \, \mathrm{d}v_g \\ &= \int -\Delta_g v w \, \mathrm{d}v_g + \int \Delta_g u (-dP_u)(v)(w) \, \mathrm{d}v_g \\ &= \int \nabla v \nabla w \, \mathrm{d}v_g + \int \Delta_g u A(u)(v, w) \, \mathrm{d}v_g, \end{aligned}$$

which gives the claim since the normal component of $\Delta_g u$ is $-A(u)(\nabla u, \nabla u)$. ∎

We note that if $u \in W^{1,p}(\Sigma)$ for some $p > 2$, so in particular if $u = z \in \mathcal{Z}$, then d^2E has a unique extension to a continuous bilinear form on $\Gamma^{H^1}(u^*TN)$ and we will in the following consider d^2E on this space.

The above expression, combined with (2.11), immediately implies that d^2E is uniformly bounded on the (non-compact) set $\mathcal{Z}$ of adapted bubbles equipped with the weighted norms $\|\cdot\|_z$ in the sense that

$$|d^2E(z)(w,v)| \le C\|w\|_z\|v\|_z \quad \text{for every } z \in \mathcal{Z} \text{ and all } v,w \in \Gamma^{H^1}(z^*TN). \tag{3.2}$$

We also note that differences of second variation terms evaluated at different maps $\hat{u}, \tilde{u} \in H^2(\Sigma, N)$ and corresponding tangent vector fields $\tilde{v}_{1,2} \in \Gamma^{H^1}(\tilde{u}^*TN)$, $\hat{v}_{1,2} \in \Gamma^{H^1}(\hat{u}^*TN)$ can be bounded by

$$\begin{aligned}
&|d^2E(\tilde{u})(\tilde{v}_1,\tilde{v}_2) - d^2E(\hat{u})(\hat{v}_1,\hat{v}_2)| \\
&\quad\lesssim \int |\nabla(\hat{v}_1-\tilde{v}_1)||\nabla\hat{v}_2| + |\nabla\tilde{v}_1||\nabla(\hat{v}_2-\tilde{v}_2)| \\
&\qquad + \int (|\nabla\tilde{u}|^2 + |\nabla\hat{u}|^2)[|\hat{v}_1-\tilde{v}_1||\hat{v}_2| + |\hat{v}_2-\tilde{v}_2||\tilde{v}_1|] \\
&\qquad + \int |\hat{v}_1||\hat{v}_2|[|\hat{u}-\tilde{u}||\nabla\hat{u}|^2 + |\nabla(\hat{u}-\tilde{u})||\nabla\hat{u}| + |\nabla(\hat{u}-\tilde{u})|^2],
\end{aligned}$$

where all integrals are computed over (Σ, g).

We will use this formula mainly for $\hat{u} = z_\lambda^{a,\omega} \in \mathcal{Z}$ and for maps $\tilde{u} = u_t = \pi_N(z + tw)$, $t \in [0,1]$, that interpolate between z and a map $u = z + w \in H^2(\Sigma, N)$ with $\|w\|_{L^\infty} < \delta_N$ and for vector fields obtained by projecting suitable $v_{1,2}\colon \Sigma \to \mathbb{R}^n$ onto the corresponding tangent spaces. In that situation the above formula, combined with (2.11), gives

$$\begin{aligned}
&|d^2E(u_t)(P_{u_t}v_1, P_{u_t}v_2) - d^2E(z)(P_z v_1, P_z v_2)| \\
&\quad\lesssim \int |w||\nabla v_1||\nabla v_2| + \int (|w|\rho_z + |\nabla w|)(|v_1||\nabla v_2| + |v_2||\nabla v_1|) \\
&\qquad + \int |v_1||v_2|(|w|\rho_z^2 + |\nabla w|\rho_z + |\nabla w|^2).
\end{aligned} \tag{3.3}$$

3.2. Uniform definiteness of the second variation orthogonal to $\mathcal{Z}$

One of the key features of our set of adapted bubbles is that the second variation of the energy is uniformly definite in directions orthogonal to $\mathcal{Z}$. Namely we prove the following lemma:

Lemma 3.2. *Let $\widehat{\omega}\colon S^2 \to N$ be any harmonic map and let $\mathcal{Z}$ be the set of adapted bubbles defined above. Then there exists $c_0 > 0$ so that for every $z \in \mathcal{Z}$ we can write the orthogonal complement $\mathcal{V}_z$ of $T_z\mathcal{Z}$ in $(\Gamma^{H^1}(z^*TN), \langle\cdot,\cdot\rangle_z)$ as an orthogonal sum $\mathcal{V}_z = \mathcal{V}_z^+ \oplus \mathcal{V}_z^-$ of spaces which are so that for $v^\pm \in \mathcal{V}_z^\pm$,*

$$d^2E(z)(v_+, v_-) = 0 \quad \text{and} \quad \pm d^2E(z)(v_\pm, v_\pm) \ge c_0\|v_\pm\|_z^2.$$

Here $\langle\cdot,\cdot\rangle_z$ and the associated norm are as in Definition 2.4.

We remark that an analogous property holds true for the manifold used in [18] in the proof of the classical Łojasiewicz–Simon inequality, and there directly follows as maps in this manifold are C^k close to $\hat{\omega}$ and as $T_{\hat{\omega}}\mathcal{H} = \ker(L_{\hat{\omega}})$.

Here we have to proceed with more care since our set $\mathcal{Z}$ is non-compact. As in the proof of the corresponding statement [12, Lemma 3.6] for the H-surface energy, we prove this result by establishing a uniform gap around 0 in the spectrum of the projected Jacobi operator, though here use energy considerations rather than Lorentz-space techniques as we do not have the explicit divergence structure present in [12].

Proof of Lemma 3.2. We note that for each fixed $z \in \mathcal{Z}$ the space $\Gamma^{H^1}(z^*TN)$ equipped with $\langle \cdot, \cdot \rangle_z$ is a Hilbert space, so Riesz's representation theorem allows us to consider the corresponding Jacobi operator $\tilde{L}_z \colon \Gamma^{H^1}(z^*TN) \to \Gamma^{H^1}(z^*TN)$ which is characterised by

$$d^2E(z)(v,w) = \langle \tilde{L}_z v, w\rangle_z \quad \text{for every } v, w \in \Gamma^{H^1}(z^*TN).$$

From the definition of the inner product and Lemma 3.1 it is easy to see that $\tilde{L}_z = \mathrm{Id} - K_z$ for $K_z \colon \Gamma^{H^1}(z^*TN) \to \Gamma^{H^1}(z^*TN)$ characterised by

$$\begin{aligned} P_z(-\Delta_g K_z(v)) + \rho_z^2 K_z(v) &= b_z(v) \\ &:= \rho_z^2 v + \sum \langle A(z)(\nabla z, \nabla z), \nu_z^j\rangle P_z(d\nu_z^j(v)), \end{aligned}$$

$\{\nu_p^j\}$ a local orthonormal frame of $T_p^\perp N$. We note that the right-hand side is bounded by $|b_z(v)| \leq C\rho_z^2|v|$, so as $\rho_z \in L^\infty(\Sigma)$ we have that K_z is a compact and self-adjoint operator on $(\Gamma^{H^1}(z^*TN), \langle \cdot, \cdot \rangle_z)$.

In order to construct the desired splitting of $\mathcal{V}_z$ we then consider the projected Jacobi operator $\hat{L}_z := P^{\mathcal{V}_z} \circ \tilde{L}_z|_{\mathcal{V}_z} = \mathrm{Id}_{\mathcal{V}_z} - \widehat{K}_z$, $P^{\mathcal{V}_z}$ the orthogonal projection onto $\mathcal{V}_z$, which can be equivalently characterised as the unique operator $\hat{L}_z \colon \mathcal{V}_z \to \mathcal{V}_z$ which is so that

$$d^2E(z)(v,w) = \langle \hat{L}_z v, w\rangle_z \quad \text{for every } v, w \in \mathcal{V}_z.$$

As $\widehat{K}_z = P^{\mathcal{V}_z} \circ K_z|_{\mathcal{V}_z}$ is also self-adjoint and compact, we know that the eigenvalues of $\hat{L}_z$ are real and tend to 1 and that there exists an orthonormal eigenbasis of $(\mathcal{V}_z, \langle \cdot, \cdot \rangle_z)$. It hence suffices to show that there exists $c_0 > 0$ so that, after increasing λ_1 and decreasing σ_1 if necessary, none of the operators $\hat{L}_z$, $z \in \mathcal{Z}$, has an eigenvalue in $[-c_0, c_0]$. This will imply that the lemma holds true for $\mathcal{V}_z^\pm$ chosen as the span of the eigenfunctions to positive, respectively negative, eigenvalues.

To prove this eigenvalue gap we argue by contradiction. Suppose there exist sequences of adapted bubbles $z_i = z_{\lambda_i}^{a_i,\omega_i} \in \mathcal{Z}$ with $\lambda_i \to \infty$ and $\omega_i \to \hat{\omega}$ and of elements $v_i \in \mathcal{V}_{z_i}$, normalised to $\|v_i\|_{z_i} = 1$, so that $\hat{L}_{z_i} v_i = \mu_i v_i$ for some $\mu_i \to 0$. We first claim that there exists a number $c_1 > 0$ so that for all sufficiently large i,

$$\int_{\mathbb{D}_{\lambda_i^{-1/3}}} \rho_{\lambda_i}^2 |\tilde{v}_i|^2 \, \mathrm{d}x \geq c_1, \quad \text{where } \tilde{v}_i := v_i \circ F_{a_i}^{-1}. \tag{3.4}$$

To see this we use that ρ_{z_i} is of order $O(\lambda_i^{-1})$ away from the ball $B_\iota(a_i)$, while $\rho_{\lambda_i} = \frac{\lambda_i}{1+\lambda_i^2|x|^2} \le \lambda_i^{-\frac{1}{3}}$ on $\mathbb{D}_{r_0} \setminus \mathbb{D}_{\lambda_i^{-1/3}}$. We can thus bound

$$\begin{aligned}\mu_i &= \langle \hat{L}_{z_i} v_i, v_i\rangle_{z_i} = \|v_i\|_{z_i}^2 - \langle K_{z_i}(v_i), v_i\rangle_{z_i} = 1 - \int_\Sigma b_{z_i}(v_i) v_i \,\mathrm{d}v_g \\ &\ge 1 - C\int_{\mathbb{D}_{\lambda_i^{-1/3}}} \rho_{\lambda_i}^2 |\tilde{v}_i|^2 \,\mathrm{d}x - C\lambda_i^{-\frac{2}{3}} \|v_i\|_{L^2(\Sigma)}^2 \\ &\ge 1 - C\lambda_i^{-\frac{2}{3}} \log(\lambda_i) - C\int_{\mathbb{D}_{\lambda_i^{-1/3}}} \rho_{\lambda_i}^2 |\tilde{v}_i|^2 \,\mathrm{d}x,\end{aligned}$$

where we use that the Poincaré hyperbolic metric is uniformly equivalent to the Euclidean metric on $\mathbb{D}_{r_0}$ in the penultimate step and (2.13) in the last step. As $\mu_i \to 0$ this yields the claimed lower bound (3.4) for all sufficiently large i.

We now proceed to construct a sequence of maps $w_i\colon S^2 \to \mathbb{R}^n$ that converges to a limit w_∞ which is a non-trivial Jacobi field at $\hat{\omega}$ but also orthogonal to

$$X_{\hat{\omega}} := T_{\hat{\omega}}\big\{\omega \circ M : \omega \in \mathcal{H}_0(\hat{\omega}),\ M \in \mathrm{M\ddot{o}b}(S^2)\big\}.$$

This leads to the desired contradiction since $\mathcal{H}_0(\hat{\omega})$ is chosen in a way that ensures that $X_{\hat{\omega}}$ agrees with the space $\ker(L_{\hat{\omega}}) = \ker(\hat{L}_{\hat{\omega}})$ of Jacobi fields at $\hat{\omega}$; compare Lemma 2.1.

To construct these maps w_i we first define

$$\widetilde{w}_i = \phi_i \tilde{v}_i(\lambda_i^{-1}\cdot) + (1-\phi_i)\bar{v}_i\colon \mathbb{R}^2 \to \mathbb{R}^n,$$

where we let $\bar{v}_i = ⨍_{A_i} \tilde{v}_i \,\mathrm{d}x$ be the mean value over the annulus $A_i = \mathbb{D}_{2\lambda_i^{-1/3}} \setminus \mathbb{D}_{\lambda_i^{-1/3}}$ and we set $\phi_i(x) = \phi(\lambda_i^{-\frac{2}{3}}|x|)$ for some fixed $\phi \in C_c^\infty([0,2),[0,1])$ with $\phi \equiv 1$ on $[0,1]$.

As $\widetilde{w}_i$ is constant near infinity, we have $w_i := \widetilde{w}_i \circ \pi^{-1} \in H^1(S^2,\mathbb{R}^n)$ and we can bound

$$\begin{aligned}\|w_i\|_{H^1(S^2)}^2 &= \int_{\mathbb{R}^2} |\nabla\widetilde{w}_i|^2 + |\partial_{x_1}\pi \wedge \partial_{x_2}\pi||\widetilde{w}_i|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} |\nabla\widetilde{w}_i|^2 + \frac{1}{2}|\nabla\pi|^2|\widetilde{w}_i|^2 \,\mathrm{d}x \\ &\lesssim \int_{\mathbb{D}_{2\lambda_i^{-1/3}}} |\nabla\tilde{v}_i|^2 + |\nabla\pi_{\lambda_i}|^2|\tilde{v}_i|^2 + ⨍_{A_i} |\tilde{v}_i - \bar{v}_i|^2 + |\bar{v}_i|^2 \|\nabla\pi\|_{L^2(\mathbb{R}^2\setminus\mathbb{D}_{\lambda_i^{2/3}})}^2 \\ &\lesssim \|v_i\|_{z_i}^2 + C\lambda_i^{-4/3}|\bar{v}_i|^2 \lesssim \|v_i\|_{z_i}^2,\end{aligned}$$

where the last step follows as $\rho_{\lambda_i} \ge c\lambda_i^{-\frac{1}{3}}$ on A_i and thus $\lambda_i^{-\frac{4}{3}}|\bar{v}_i|^2 \lesssim \lambda_i^{-\frac{2}{3}} \int_{A_i} |\tilde{v}_i|^2 \,\mathrm{d}x \lesssim \|v_i\|_{z_i}^2$.

After passing to a subsequence, the maps w_i thus converge to a limit w_∞ weakly in $H^1(S^2,\mathbb{R}^n)$, strongly in $L^2(S^2,\mathbb{R}^n)$ and almost everywhere. Away from the shrinking discs $\pi(\mathbb{R}^2 \setminus \mathbb{D}_{\lambda_i^{2/3}}) \subset S^2$, the maps $w_i = v_i \circ (\pi_{\lambda_i} \circ F_{a_i})^{-1}$ are tangential to N along

$$\hat{z}_i := z_i \circ (\pi_{\lambda_i} \circ F_{a_i})^{-1} = \omega_i + O(\lambda_i^{-1}) \to \hat{\omega}, \tag{3.5}$$

so the limit w_∞ must be tangential along $\hat{\omega}$, i.e. an element of $\Gamma^{H^1}(\hat{\omega}^*TN)$. Furthermore, w_∞ is non-trivial as $w_i \to w_\infty$ strongly in $L^2(S^2, \mathbb{R}^n)$ and as the L^2-norms of the maps w_i are bounded away from zero thanks to (3.4). We now want to prove that w_∞ is orthogonal to the space $X_{\hat{\omega}}$ with respect to the inner product

$$\langle v, w\rangle := \int_{S^2} \nabla w \nabla w + c_\gamma w v \, dv_{g_{S^2}}.$$

Remark 3.3. Here we set $c_\gamma = \frac{1}{4}$ if $\gamma = 1$ as this ensures that $(\pi_\lambda \circ F_a)^* g_{S^2} = c_\gamma \rho^2_{z_\lambda^{a,\omega}} g$ on $B_\iota(a)$, while we set $c_\gamma = 1$ if $\gamma \geq 2$ and use that in this case $((\pi_\lambda \circ F_a)^* g_{S^2})(p) = c_\gamma \rho^2_{z_\lambda^{a,\omega}}(1 + O(\text{dist}_g(p,a)^2))g(p)$ for $p \in B_\iota(a)$.

We can use the following lemma; see Appendix B for a sketch of the proof.

Lemma 3.4. *Let $z_i = z_{\lambda_i}^{a_i,\omega_i}$ be a sequence of adapted bubbles for which $\lambda_i \to \infty$ and $\omega_i \to \hat{\omega}$. Then there exist bases $\{e_j^i\}_{j=1}^K$ of $T_{z_i}\mathcal{Z}$ that are orthonormal with respect to $\langle\cdot,\cdot\rangle_{z_i}$ and that converge to an orthonormal basis $\{e_j^\infty\}_{j=1}^K$ of $(X_{\hat{\omega}}, \langle\cdot,\cdot\rangle)$ in the sense that $\hat{e}_j^i := e_j^i \circ (\pi_{\lambda_i} \circ F_{a_i})^{-1} \to e_j^\infty$ smoothly locally on $S^2 \setminus \{p^*\}$ while*

$$\lim_{\Lambda\to\infty} \limsup_{i\to\infty} \|e_j^i\|_{z_i, \Sigma \setminus B_{\Lambda\lambda_i^{-1}}(a_i)} = 0 \quad \textit{for } j = 1, \ldots, K. \tag{3.6}$$

As $w_i \rightharpoonup w_\infty$ in $H^1(S^2)$ we obtain that, for $j = 1, \ldots, K$,

$$\begin{aligned}
\langle w_\infty, e_j^\infty\rangle &= \lim_{\Lambda\to\infty} \int_{\pi(\mathbb{D}_\Lambda)} \nabla w_\infty \nabla e_j^\infty + c_\gamma w_\infty e_j^\infty \, dv_{g_{S^2}} \\
&= \lim_{\Lambda\to\infty} \lim_{i\to\infty} \int_{\pi(\mathbb{D}_\Lambda)} \nabla w_i \nabla \hat{e}_j^i + c_\gamma w_i \hat{e}_j^i \, dv_{g_{S^2}} \\
&= \lim_{\Lambda\to\infty} \lim_{i\to\infty} \int_{F_{a_i}^{-1}(\mathbb{D}_{\Lambda\lambda_i^{-1}})} \nabla v_i \nabla e_j^i + v_i e_j^i \rho_{z_i}^2 \, dv_g \\
&= -\lim_{\Lambda\to\infty} \lim_{i\to\infty} \int_{\Sigma \setminus F_{a_i}^{-1}(\mathbb{D}_{\Lambda\lambda_i^{-1}})} \nabla v_i \nabla e_j^i + v_i e_j^i \rho_{z_i}^2 \, dv_g = 0,
\end{aligned}$$

where we use Remark 3.3 in the third step, the orthogonality of v_i to $T_{z_i}\mathcal{Z}$ in the penultimate step and (3.6) as well as that $\|v_i\|_{z_i} = 1$ in the last step.

Having thus shown that $w_\infty \perp X(\hat{\omega}) = \ker(L_{\hat{\omega}})$, it now remains to show that

$$d^2E(\hat{\omega})(w_\infty, \eta) = 0 \quad \text{for all } \eta \in \Gamma^{H^1}(\hat{\omega}^*TN).$$

We note that this is trivially true if η itself is a Jacobi field and that it hence suffices to consider $\eta \in \Gamma^{H^1}(\hat{\omega}^*TN)$ with $\eta \perp \ker(\hat{L}_{\hat{\omega}})$.

Given such an η we set $\bar{\eta}_i := \fint_{A_{r_0}} \eta \circ \pi_{\lambda_i} \, dx$, $A_{r_0} := \mathbb{D}_{r_0} \setminus \mathbb{D}_{\frac{r_0}{2}}$, and define $\eta_i \in \Gamma^{H^1}(z_i^*TN)$ as $\eta_i = P_{z_i}(\bar{\eta}_i)$ on $\Sigma \setminus B_\iota(a_i)$, while for $p = F_{a_i}^{-1}(x) \in B_\iota(a_i)$,

$$\eta_i(p) := P_{z_i(p)}(\psi(x)\eta(\pi_{\lambda_i}(x)) + (1 - \psi(x))\bar{\eta}_i)$$

for a fixed cut-off $\psi \in C_c^\infty(\mathbb{D}_{r_0})$ with $\psi \equiv 1$ on $\mathbb{D}_{\frac{r_0}{2}}$.

As d^2E is conformally invariant and as $\lambda_i^{-1/3} \leq \frac{1}{2} r_0$ for sufficiently large i, we get

$$d^2E(z_i)(v_i, \eta_i) = \int_{\pi(\mathbb{D}_{\lambda_i^{2/3}})} \nabla(P_{\hat{z}_i}\eta)\nabla w_i + A(\hat{z}_i)(\nabla\hat{z}_i, \nabla\hat{z}_i)A(\hat{z}_i)(P_{\hat{z}_i}\eta, w_i)dv_{g_{S^2}} + \mathrm{err}_i, \tag{3.7}$$

for $\hat{z}_i$ defined by (3.5) and $|\mathrm{err}_i| \leq C\|v_i\|_{z_i}\|\eta_i\|_{z_i,\Sigma\setminus F_{a_i}^{-1}(\mathbb{D}_{\lambda_i^{-1/3}})}$. We recall that $\rho_{z_i} \lesssim \lambda_i^{-1}$ away from $B_{\tilde{\iota}}(a)$ and note that a short calculation, similar to the proof of (2.12) carried out in the appendix, gives $|\bar{\eta}_i| \lesssim (\log\lambda_i)^{\frac{1}{2}}\|\eta\|_{H^1(S^2)}$. As $\|v_i\|_{z_i} = 1$, this allows us to conclude that

$$\begin{aligned}|\mathrm{err}_i| &\lesssim \lambda_i^{-1}|\bar{\eta}_i| + \|\eta\circ\pi_{\lambda_i} - \bar{\eta}_i\|_{L^2(A_{r_0})} + \left\|\rho_{\lambda_i}|\eta\circ\pi_{\lambda_i}| + |\nabla(\eta\circ\pi_{\lambda_i})|\right\|_{L^2(\mathbb{D}_{r_0}\setminus\mathbb{D}_{\lambda_i^{-1/3}})}\\ &\lesssim \lambda_i^{-1}(\log\lambda_i)^{\frac{1}{2}}\|\eta\|_{H^1(S^2)} + \|\eta\|_{H^1(\pi(\mathbb{R}^2\setminus\mathbb{D}_{\lambda_i^{2/3}}))} \to 0.\end{aligned}$$

Combined with $w_i \rightharpoonup w_\infty$ in $H^1(S^2)$ and $\|\hat{z}_i - \hat{\omega}\|_{C^1(\pi(\mathbb{D}_{\lambda_i r_0}))} \to 0$, which follows as $\omega_i \to \hat{\omega}$ smoothly on S^2 and $\|\hat{z}_i - \omega_i\|_{C^1(\pi(\mathbb{D}_{\lambda_i r_0}))} \leq C\lambda_i^{-1} \to 0$, this shows that the right-hand side of (3.7) converges to $d^2E(\hat{\omega})(w_\infty, \eta)$. We thus conclude that

$$\begin{aligned}|d^2E(\hat{\omega})(w_\infty, \eta)| &= \lim_{i\to\infty}|d^2E(z_i)(v_i, \eta_i)| \leq |\langle \hat{L}_{z_i}v_i, P^{\mathcal{V}_i}\eta_i\rangle| + C\|v_i\|_{z_i}\|P^{T_{z_i}\mathcal{Z}}\eta_i\|_{z_i}\\ &\leq |\mu_i|\,\|\eta_i\|_{z_i} + C\|P^{T_{z_i}\mathcal{Z}}\eta_i\|_{z_i}.\end{aligned}$$

As $\|\eta_i\|_{z_i} \leq C\|\eta\|_{H^1(S^2)} + C\lambda_i^{-1}|\bar{\eta}_i|$ is uniformly bounded and as we have assumed that $\mu_i \to 0$, we know that the first term in this estimate tends to zero as $i \to \infty$. We can furthermore use Lemma 3.4 to see that for $j = 1, \ldots, K$,

$$\begin{aligned}\lim_{i\to\infty}\langle e_i^j, \eta_i\rangle_{z_i} &= \lim_{\Lambda\to\infty}\lim_{i\to\infty}\langle e_i^j, \eta_i\rangle_{z_i,\Sigma\setminus F_{a_i}^{-1}(\mathbb{D}_{\Lambda\lambda_i^{-1}})} + \lim_{\Lambda\to\infty}\lim_{i\to\infty}\langle \hat{e}_i^j, P_{\hat{z}_i}(\eta)\rangle_{\pi(\mathbb{D}_\Lambda)}\\ &= 0 + \lim_{\Lambda\to\infty}\langle e_\infty^j, \eta\rangle_{\pi(\mathbb{D}_\Lambda)} = \langle e_\infty^j, \eta\rangle = 0,\end{aligned}$$

where the last step follows as $\eta \perp \ker(L_{\hat{\omega}}) = X_{\hat{\omega}}$. Hence $\|P^{T_{z_i}\mathcal{Z}}\eta_i\|_{z_i} \to 0$ and we indeed obtain $d^2E(\hat{\omega})(w_\infty, \eta) = 0$. Thus $w_\infty \in \ker(L_{\hat{\omega}})$, contradicting the previously established fact that w_∞ is a non-trivial element of $(\ker(L_{\hat{\omega}}))^\perp$. ∎

3.3. Expansion of the energy on the set $\mathcal{Z}$ of adapted bubbles

The goal of this section is to identify variations in the space of adapted bubbles for which the leading-order term in the energy expansion appears with a known sign and scaling.

In the integrable case, where all elements of $\mathcal{Z}$ are built out of harmonic maps $\omega\colon S^2 \to N$, we will only need to consider variations (z_ε) induced by a change of the bubble parameter. In the general case we will additionally need to consider (z_ε) induced by variations of the underlying maps $\omega^{(\varepsilon)} \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$. To treat both types of variations at the same time we first show the following lemma:

Lemma 3.5. *For any variation $z_\varepsilon = z_{\lambda_\varepsilon}^{a,\omega^{(\varepsilon)}}$ in $\mathcal{Z}$ for which (2.6) holds we have*

$$\frac{d}{d\varepsilon}E(z_\varepsilon) = \frac{d\lambda}{d\varepsilon}\cdot\int_{\mathbb{D}_{\frac{r_0}{2}}} j_\lambda^{a,\omega}\Delta\partial_\lambda\widetilde{\omega}_\lambda\,\mathrm{d}x + \frac{d}{d\varepsilon}E(\omega^{(\varepsilon)}) + \mathrm{err} \tag{3.8}$$

for an error term that is bounded by

$$|\mathrm{err}| \leq C\lambda^{-3} + C\lambda^{-2}\big[\|\partial_\varepsilon\omega^{(\varepsilon)}\|_{C^2(S^2)} + \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)}\big].$$

Proof. Let $z = z_\lambda^{a,\omega} \in \mathcal{Z}$ and let $z_\varepsilon = z_{\lambda_\varepsilon}^{a,\omega^{(\varepsilon)}}$ be a variation for which (2.6) holds. To lighten the notation we write for short $v = v_\lambda^{a,\omega}$, $j = j_\lambda^{a,\omega}$ and denote the corresponding variations by $\partial_\varepsilon v := \frac{d}{d\varepsilon}v_{\lambda_\varepsilon}^{a,\omega^{(\varepsilon)}}$, $\partial_\varepsilon j_\varepsilon := \frac{d}{d\varepsilon}j_{\lambda_\varepsilon}^{a,\omega^{(\varepsilon)}}$ and $\partial_\varepsilon\widetilde{\omega}_\lambda = \frac{d}{d\varepsilon}(\omega^{(\varepsilon)}\circ\pi_{\lambda_\varepsilon})$.

We first remark that away from the ball $B_\iota(a)$ we have $\Delta_g v = 0$ as the derivatives of the Green's function are harmonic functions. Combined with the estimate (2.8) on the error term in (2.7) and with $\tau_g(z) = P_z(\Delta_g z) = P_z(d\pi_N(v)(\Delta_g v) + d^2\pi_N(v)(\nabla v,\nabla v))$, we hence get

$$|\Delta_g v| + |\partial_\varepsilon\Delta_g v| + |\nabla v|^2 + |\partial_\varepsilon\nabla v|^2 + |\tau_g(z)| + |\partial_\varepsilon\tau_g(z)| \lesssim \lambda^{-2} \quad \text{on } \Sigma\setminus B_{\tilde{\iota}}(a). \tag{3.9}$$

We also note that (2.10) yields a bound of $|\partial_\varepsilon z| \lesssim \lambda^{-1} + \eta_0$ on this set, where here and in the following we write for short $\eta_0 := |\partial_\varepsilon\omega^{(\varepsilon)}(p^*)|$. We thus obtain

$$\begin{aligned}\frac{d}{d\varepsilon}E(z) &= -\int_\Sigma \partial_\varepsilon z\cdot\tau_g(z)\,\mathrm{d}v_g = -\int_{B_{\tilde{\iota}}(a)}\partial_\varepsilon z\cdot\Delta_g z\,\mathrm{d}v_g + O(\lambda^{-3} + \eta_0\lambda^{-2})\\ &= -\int_{\mathbb{D}_{\frac{r_0}{2}}}\partial_\varepsilon\tilde{z}\cdot\Delta\tilde{z}\,\mathrm{d}x + O(\lambda^{-3} + \eta_0\lambda^{-2})\end{aligned} \tag{3.10}$$

for $\tilde{z} = z\circ F_a^{-1}$. Here and in the following we can carry out all computations on $\mathbb{D}_{\frac{r_0}{2}}$ with respect to the Euclidean metric, as the above integral is conformally invariant. On this set we can write $\tilde{z} = \pi_N(\widetilde{\omega}_\lambda + j)$ as

$$\tilde{z} = \widetilde{\omega}_\lambda + P_{\widetilde{\omega}_\lambda}(j) + E = \widetilde{\omega}_\lambda + j - P_{\widetilde{\omega}_\lambda}^\perp(j) + E, \tag{3.11}$$

where the lower-order error term

$$E = \int_0^1 \frac{d}{dt}\pi_N(\widetilde{\omega}_\lambda + tj)\,\mathrm{d}t - P_{\widetilde{\omega}_\lambda}(j) = \int_0^1 (d\pi_N(\widetilde{\omega}_\lambda + tj) - d\pi_N(\widetilde{\omega}_\lambda))(j)\,\mathrm{d}t$$

satisfies the estimates

$$|E| + |\partial_\varepsilon E| \lesssim \lambda^{-2}|x|^2,\quad |\nabla E| + |\partial_\varepsilon\nabla E| \lesssim \lambda^{-2}|x|,\quad |\Delta E| + |\partial_\varepsilon\Delta E| \lesssim \lambda^{-2}. \tag{3.12}$$

Here and in the following we use that

$$|j| + |\partial_\varepsilon j| \lesssim \lambda^{-1}|x|,\quad |\partial_\varepsilon\nabla j| \lesssim \lambda^{-1},\quad |\partial_\varepsilon\nabla\pi_\lambda| \lesssim \rho_\lambda \quad \text{and} \quad \rho_\lambda|x| \leq 1, \tag{3.13}$$

while

$$|\partial_\varepsilon \widetilde{\omega}_\lambda| \lesssim \frac{(|\partial_\varepsilon \lambda| + \lambda)|x|}{1+\lambda^2|x|^2} + \eta_0 \lesssim (1+\lambda|x|)^{-1} + \eta_0 \quad \text{and} \quad |\partial_\varepsilon \nabla \widetilde{\omega}_\lambda| \lesssim \rho_z. \tag{3.14}$$

In the following it will also be useful to note that this implies that

$$|\partial_\varepsilon \tilde{z}| \lesssim (1+\lambda|x|)^{-1} + \eta_0, \tag{3.15}$$

that we can trivially bound

$$\begin{aligned}|\Delta \widetilde{\omega}_\lambda| + |\partial_\varepsilon \Delta \widetilde{\omega}_\lambda| &\lesssim [\|\omega\|_{C^2(S^2)} + \|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2(S^2)}][|\nabla \pi_\lambda|^2 + |\partial_\varepsilon \lambda||\nabla \pi_\lambda||\partial_\lambda \nabla \pi_\lambda|] \\ &\lesssim \rho_z^2, \end{aligned} \tag{3.16}$$

and that we have estimates of

$$\begin{aligned}&\left\|\rho_\lambda |x| + (1+\lambda|x|)^{-1}\right\|_{L^2(\mathbb{D}_{\frac{r_0}{2}})} \lesssim \lambda^{-1}(\log \lambda)^{\frac{1}{2}}, \\ &\left\|\rho_\lambda |x| + (1+\lambda|x|)^{-1}\right\|_{L^1(\mathbb{D}_{\frac{r_0}{2}})} \lesssim \lambda^{-1}.\end{aligned} \tag{3.17}$$

From (3.10), (3.11) and $\Delta_g j = 0$ we thus obtain

$$\begin{aligned}\frac{d}{d\varepsilon} E(z) = &-\int_{\mathbb{D}_{\frac{r_0}{2}}} \partial_\varepsilon \widetilde{\omega}_\lambda \Delta(\widetilde{\omega}_\lambda - P^\perp_{\widetilde{\omega}_\lambda} j) + \partial_\varepsilon (P_{\widetilde{\omega}_\lambda} j) \Delta \widetilde{\omega}_\lambda \\ &+ \mathrm{err}_1 + O(\lambda^{-3} + \eta_0 \lambda^{-2})\end{aligned} \tag{3.18}$$

for an error term that is bounded by

$$\begin{aligned}|\mathrm{err}_1| &\lesssim \int_{\mathbb{D}_{\frac{r_0}{2}}} |\partial_\varepsilon (P_{\widetilde{\omega}_\lambda} j)||\Delta P^\perp_{\widetilde{\omega}_\lambda} j| + |\partial_\varepsilon E|(|\Delta \widetilde{\omega}_\lambda| + |\Delta P^\perp_{\widetilde{\omega}_\lambda} j|) + |\Delta E||\partial_\varepsilon z_\varepsilon| \\ &\lesssim \lambda^{-2} \int_{\mathbb{D}_{\frac{r_0}{2}}} \rho_\lambda^2 |x|^2 + \rho_\lambda |x| + \lambda^{-2}\eta_0 + \lambda^{-2} \|(1+\lambda|x|)^{-1}\|_{L^1(\mathbb{D}_{\frac{r_0}{2}})} \\ &\lesssim \lambda^{-3} + \eta_0 \lambda^{-2}.\end{aligned}$$

We then note that

$$-\int_{\mathbb{D}_{\frac{r_0}{2}}} \partial_\varepsilon \widetilde{\omega}_\lambda \Delta \widetilde{\omega}_\lambda = -\int_{\mathbb{R}^2} \partial_\varepsilon \widetilde{\omega}_\lambda \tau(\widetilde{\omega}_\lambda) + \mathrm{err}_2 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(\omega^{(\varepsilon)}) + \mathrm{err}_2, \tag{3.19}$$

where $\mathrm{err}_2 = \int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{r_0}{2}}} \partial_\varepsilon \widetilde{\omega}_\lambda \tau(\widetilde{\omega}_\lambda)$ is also bounded by

$$|\mathrm{err}_2| \lesssim \|\partial_\varepsilon \widetilde{\omega}_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus \mathbb{D}_{\frac{r_0}{2}})} \|\tau_{g_{S^2}}(\omega)\|_{L^\infty(S^2)} \int_{\mathbb{R}^2 \setminus \mathbb{D}_{r_0/2}} |\nabla \pi_\lambda|^2 = O(\lambda^{-3} + \eta_0 \lambda^{-2}).$$

Here and in the following we use that π_λ is conformal and hence

$$\tau(\widetilde{\omega}_\lambda) = \frac{1}{2}|\nabla\pi_\lambda|^2 \cdot \tau_{g_{S^2}}(\omega) \circ \pi_\lambda. \tag{3.20}$$

Rewriting

$$\partial_\varepsilon(P_{\widetilde{\omega}_\lambda} j) \cdot \Delta\widetilde{\omega}_\lambda = \partial_\varepsilon(P_{\widetilde{\omega}_\lambda} j \cdot \Delta\widetilde{\omega}_\lambda) - P_{\widetilde{\omega}_\lambda} j \cdot \partial_\varepsilon \Delta\widetilde{\omega}_\lambda = \partial_\varepsilon(j \cdot \tau(\widetilde{\omega}_\lambda)) - P_{\widetilde{\omega}_\lambda} j \cdot \partial_\varepsilon \Delta\widetilde{\omega}_\lambda$$

and setting $\text{err}_3 := -\int_{\mathbb{D}_{\frac{r_0}{2}}} \partial_\varepsilon(j\tau(\widetilde{\omega}_\lambda))$, we hence obtain from (3.18) and (3.19) that

$$\begin{aligned}\frac{d}{d\varepsilon}E(z_\varepsilon) &= \frac{d}{d\varepsilon}E(\omega^{(\varepsilon)}) + \int_{\mathbb{D}_{\frac{r_0}{2}}} P_{\widetilde{\omega}_\lambda} j \cdot \partial_\varepsilon \Delta\widetilde{\omega}_\lambda + \Delta(P^\perp_{\widetilde{\omega}_\lambda} j) \cdot \partial_\varepsilon \widetilde{\omega}_\lambda \\ &\quad + \text{err}_3 + O(\lambda^{-3} + \eta_0 \lambda^{-2}) \\ &= \frac{d}{d\varepsilon}E(\omega^{(\varepsilon)}) + \int_{\mathbb{D}_{\frac{r_0}{2}}} j \cdot \partial_\varepsilon \Delta\widetilde{\omega}_\lambda + \text{err}_3 + \text{err}_4 + O(\lambda^{-3} + \eta_0\lambda^{-2}),\end{aligned}$$

where we integrate by parts in the second step. We can use (3.14) as well as that $j \in T_{\omega(p^*)}N$ to estimate the resulting the boundary term by

$$|\text{err}_4| \le \|\partial_\varepsilon \widetilde{\omega}_\lambda\|_{C^1(\partial\mathbb{D}_{\frac{r_0}{2}})} \|(P^\perp_{\widetilde{\omega}_\lambda} - P^\perp_{\omega(p^*)})(j)\|_{C^1(\partial\mathbb{D}_{\frac{r_0}{2}})} = O(\lambda^{-3} + \eta_0\lambda^{-2}),$$

while combining (3.20) with (3.13) and (3.17) allows us to estimate

$$\begin{aligned}|\text{err}_3| &\lesssim \lambda^{-1}\big[\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} + \|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2(S^2)}\big] \big\| |x| \rho_z^2 \big\|_{L^1(\mathbb{D}_{\frac{r_0}{2}})} \\ &\lesssim \lambda^{-2}(\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} + \|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2(S^2)}).\end{aligned}$$

We finally remark that $\int_{\mathbb{D}_{\frac{r_0}{2}}} j\partial_\varepsilon \Delta\widetilde{\omega}_\lambda = \frac{d\lambda}{d\varepsilon}\int_{\mathbb{D}_{\frac{r_0}{2}}} j\partial_\lambda \Delta\widetilde{\omega}_\lambda + \text{err}_5$ for

$$|\text{err}_5| \le \int_{\mathbb{D}_{\frac{r_0}{2}}} |j||\Delta((\partial_\varepsilon \omega^{(\varepsilon)}) \circ \pi_\lambda)| \lesssim \lambda^{-1}\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2} \big\| |x|\rho_z^2 \big\|_{L^1(\mathbb{D}_{\frac{r_0}{2}})} \lesssim \lambda^{-2}\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2}.$$

Altogether this yields the claim of Lemma 3.5. ∎

We now show that the integral $\int j\partial_\lambda \Delta\omega_\lambda$ appearing in (3.8) has a given sign and scaling in λ and indeed essentially only depends on $a \in \Sigma$, λ and $|d\omega(p^*)|$.

To state this in detail we first note that as $d\widehat{\omega}(p^*) \neq 0$ we can always assume that $\sigma_1 > 0$ is chosen small enough to ensure that

$$|d\omega(p^*)| \ge \frac{1}{2}|d\widehat{\omega}(p^*)| > 0 \quad \text{for all } \omega \in \mathcal{H}_1^{\sigma_1}(\widehat{\omega}).$$

Writing for short $\alpha_\omega := \frac{1}{\sqrt{2}}|d\omega(p^*)|_{g_{S^2}}$, we note that if ω is harmonic then the vectors $\{\alpha_\omega^{-1}\nabla_{e_1}\omega(p^*), \alpha_\omega^{-1}\nabla_{e_2}\omega(p^*)\}$ are orthonormal since harmonic maps from S^2 are weakly conformal. While this is not true for general elements $\omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ in the non-integrable case, the above will still hold up to a small error as elements of $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ are C^k close to the harmonic map $\hat{\omega}$.

So given any number $\eta_1 > 0$ we can assume that $\sigma_1 > 0$ is chosen small enough so that for any $\omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ there exists a matrix $S_\omega \in M(n)$ with

$$d\omega(p^*)(e_i) = \alpha_\omega S_\omega e_i \in \mathbb{R}^n, \ \ i = 1, 2, \quad \text{and} \quad |S_\omega^\mathsf{T} S_\omega - \mathrm{Id}| \le \eta_1. \tag{3.21}$$

Here we denote by $\{e_i\}$ the standard basis of both $\mathbb{R}^3$ and $\mathbb{R}^n$, as appropriate. We note that as S_ω will only be applied to elements of $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^n$ in the construction below the particular choice of S_ω in the other directions is irrelevant. We can then prove the following lemma:

Lemma 3.6. *For any $\omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$, $\lambda \ge \lambda_1$ and $a \in \Sigma$ we have*

$$\int_{\mathbb{D}_{\frac{r_0}{2}}} j_\lambda^{a,\omega} \Delta_g \partial_\lambda \tilde{\omega}_\lambda \, \mathrm{d}v_g = 4\pi |d\omega(p^*)|^2 \mathcal{J}(a) \lambda^{-3} + O(\lambda^{-4}) + O(|S_\omega^\mathsf{T} S_\omega - \mathrm{Id}| \lambda^{-3})$$

for S_ω as in (3.21) and $\mathcal{J}(a) := \lim_{x \to 0} (\partial_{y_1} \partial_{x_1} + \partial_{y_2} \partial_{x_2}) G_a(x, 0)$, where $G_a(x, y) = G(F_a^{-1}(x), F_a^{-1}(y))$ is the function that represents the Green's function in the coordinates F_a introduced in Remark 2.2.

Remark 3.7. We recall from [12] that the function $\mathcal{J}$, which depends only on the domain surface (Σ, g), is strictly negative on any surface of positive genus.

Proof of Lemma 3.6. Extending $\omega \colon S^2 \to N \hookrightarrow \mathbb{R}^n$ to a neighbourhood of S^2 by setting $\omega(x) = \omega(|x|^{-1} x)$, we can view $d\omega(p^*)$ as a map from $\mathbb{R}^3$ to $\mathbb{R}^n$ with $d\omega(p^*)(e_3) = 0$. Thus (3.21) allows us to write $d\omega(p^*)(\partial_\lambda \pi_\lambda) = \alpha_\omega S_\omega (\partial_\lambda \bar{\pi}_\lambda, 0_{n-2})^\mathsf{T}$, where $\bar{\pi}_\lambda = (\pi_\lambda^1, \pi_\lambda^2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by the first two components of the rescaled inverse stereographic projection π_λ. We can use this to estimate

$$\begin{aligned}
\|\partial_\lambda (\tilde{\omega}_\lambda - \alpha_\omega S_\omega (\bar{\pi}_\lambda, 0)^\mathsf{T})\|_{C^1(\partial \mathbb{D}_{\frac{r_0}{2}})} &= \|[d\omega(\pi_\lambda) - d\omega(p^*)](\partial_\lambda \pi_\lambda)\|_{C^1(\partial \mathbb{D}_{\frac{r_0}{2}})} \\
&\lesssim \|\omega\|_{C^2(S^2)} \|\pi_\lambda - p^*\|_{C^1(\partial \mathbb{D}_{\frac{r_0}{2}})} \|\partial_\lambda \pi_\lambda\|_{C^1(\partial \mathbb{D}_{\frac{r_0}{2}})} \\
&\lesssim \lambda^{-3}.
\end{aligned}$$

Integration by parts, also using $\|j_\lambda^{a,\omega}\|_{C^1(\mathbb{D}_{\frac{r_0}{2}})} = O(\lambda^{-1})$ and $\Delta_g j_\lambda^{a,\omega} = 0$, thus gives

$$\begin{aligned}
\int_{\mathbb{D}_{\frac{r_0}{2}}} j_\lambda^{a,\omega} \Delta \partial_\lambda \tilde{\omega}_\lambda &= \int_{\mathbb{D}_{\frac{r_0}{2}}} j_\lambda^{a,\omega} \Delta (\alpha_\omega S_\omega \partial_\lambda (\bar{\pi}_\lambda, 0)^\mathsf{T}) + O(\lambda^{-4}) \\
&= \alpha_\omega \int_{\mathbb{D}_{\frac{r_0}{2}}} (S_\omega^\mathsf{T} j_\lambda^{a,\omega}) \cdot (\Delta \partial_\lambda \bar{\pi}_\lambda, 0)^\mathsf{T} + O(\lambda^{-4}).
\end{aligned}$$

Writing for short $\hat{j}_a := 2\nabla_y J_a(\cdot,0) - 2\nabla_y J_a(0,0)\colon \mathbb{D}_r \to \mathbb{R}^2$, we now recall that $j_\lambda^{a,\omega}$ is given by $j_\lambda^{a,\omega} = \lambda^{-1} d\omega(p^*)(\hat{j}_a,0)^\top = \lambda^{-1}\alpha_\omega S_\omega(\hat{j}_a,0)^\top$. As $\lambda^{-1}\int|\hat{j}_a||\Delta\partial_\lambda\pi_\lambda| \lesssim \lambda^{-2}\int|x|\rho_\lambda^2 \lesssim \lambda^{-3}$ we thus obtain

$$\int_{\mathbb{D}_{\frac{r_0}{2}}} j_\lambda^{a,\omega}\Delta\partial_\lambda\tilde{\omega}_\lambda\,\mathrm{d}x = \alpha_\omega^2\lambda^{-1}\int_{\mathbb{D}_{\frac{r_0}{2}}}\hat{j}_a\Delta\partial_\lambda\bar{\pi}_\lambda\,\mathrm{d}x + O(\lambda^{-4} + |S_\omega^\top S_\omega - \mathrm{Id}|\lambda^{-3}). \quad (3.22)$$

Up to the factor α_ω^2 and the constant shift in $\hat{j}_a$, the leading-order term in (3.22) is exactly the same as the leading-order term obtained in the proof of [12, Lemma 3.7]. Following the argument there we can thus combine the Taylor expansion of $\hat{j}_a$ with the symmetries of $\Delta\bar{\pi}_\lambda = -|\nabla\pi_\lambda|^2\bar{\pi}_\lambda = \frac{-16\lambda x}{(1+\lambda^2|x|^2)^3}$ to compute

$$\begin{aligned}
&\alpha_\omega^2\lambda^{-1}\int_{\mathbb{D}_{\frac{r_0}{2}}}\hat{j}_a\Delta\partial_\lambda\bar{\pi}_\lambda\\
&\quad= -\alpha_\omega^2\lambda^{-1}\sum_{i=1,2}2\partial_{x_i}\partial_{y_i}J_a(0)\int_{\mathbb{D}_{\frac{r_0}{2}}}x_i\partial_\lambda\Big(\frac{16\lambda x_i}{(1+\lambda^2|x|^2)^3}\Big) + O(\lambda^{-4})\\
&\quad= \alpha_\omega^2\frac{8\pi}{\lambda^3}(\partial_{x_1}\partial_{y_1}J_a(0) + \partial_{x_2}\partial_{y_2}J_a(0)) + O(\lambda^{-4})\\
&\quad= \alpha_\omega^2\frac{8\pi}{\lambda^3}\mathcal{J}(a) + O(\lambda^{-4}).
\end{aligned}$$

Inserting this into (3.22) and using that $\alpha_\omega^2 = \frac{1}{2}|d\omega(p^*)|^2$ gives the claim of the lemma. ∎

We first use the above lemmas to control the variation of the energy induced by a change of the bubble parameter. To this end we note that given any $\eta_2 > 0$ we can choose $\sigma_1 > 0$ sufficiently small to ensure that

$$\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} \le \eta_2 \quad \text{for all } \omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$$

since $\hat{\omega}$ is harmonic. For suitable choices of σ_1 and λ_1 we can thus combine Lemmas 3.5 and 3.6 with Remark 3.7 to obtain the following corollary:

Corollary 3.8. *There exist constants $c_1 > 0$ and $C < \infty$ so that for any $z = z_\lambda^{a,\omega} \in \mathcal{Z}$,*

$$C\lambda^{-2} \ge -\lambda\frac{d}{d\lambda}E(z_\lambda^{a,\omega}) \ge c_1\lambda^{-2}.$$

Remark 3.9. As an immediate consequence we obtain

$$|E(z,\Sigma) - E(\omega,S^2)| \le C\lambda(z)^{-2} \quad \text{for any } z \in \mathcal{Z}.$$

In the non-integrable case we furthermore need to control the tension of the underlying map ω if $\omega \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ is not harmonic. To this end we let $\omega_0 \in \mathcal{H}_0^{\sigma_1}(\hat{\omega})$ and $R \in \mathrm{SO}(3)$ be so that $\omega = \omega_0 \circ R$ and set $\omega^{(\varepsilon)} = \omega_0^{(\varepsilon)} \circ R$ for $\omega_0^{(\varepsilon)}$ as in Lemma 2.1. As such a variation satisfies (2.6) and as (2.4) ensures that

$$\frac{d}{d\varepsilon} E(\omega^{(\varepsilon)}) = \frac{d}{d\varepsilon} E(\omega_0^{(\varepsilon)}) \geq \|\tau_{g_{S^2}}(\omega_0)\|_{L^2(S^2)} = \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)},$$

Lemma 3.5 immediately yields the following corollary:

Corollary 3.10. *There exists a constant $C < \infty$ so that for any $z = z_\lambda^{a,\omega} \in \mathcal{Z}$ for which ω is in the interior of $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$, there exists a variation $\omega^{(\varepsilon)}$ of ω in $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ satisfying (2.6) so that*

$$\frac{d}{d\varepsilon} E(z_\lambda^{a,\omega^{(\varepsilon)}}) \geq \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} - C\lambda^{-2}$$

for the corresponding variation of adapted bubbles (with fixed λ and a).

3.4. Estimates on the tension and the second variation on $\mathcal{Z}$

To prove our main result we furthermore need the following estimates on the scaling of the first and second variations of the energy at points of our adapted bubble set.

Lemma 3.11. *For any $z = z_\lambda^{a,\omega} \in \mathcal{Z}$ and $w \in \Gamma^{H^1}(z^*TN)$ with $\|w\|_z = 1$, we can bound*

$$|dE(z)(w)| \leq C\lambda^{-2}(\log\lambda)^{\frac{1}{2}} + C\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)},$$

while for all variations $z_\varepsilon = z_{\lambda_\varepsilon}^{a,\omega^{(\varepsilon)}}$ satisfying (2.6) we have

$$\begin{aligned} |d^2E(z)(\partial_\varepsilon z, w)| \leq {} & C\lambda^{-2}(\log\lambda)^{\frac{1}{2}} + C\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} + C\lambda^{-1}\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^2(S^2)} \\ & + C\|P_\omega(\partial_\varepsilon \tau_{g_{S^2}}(\omega^{(\varepsilon)}))\|_{L^2(S^2)}. \end{aligned} \tag{3.23}$$

Remark 3.12. For the variations $z_\varepsilon^{(1)} = z_{\lambda(1-\varepsilon)}^{a,\omega}$ considered in Corollary 3.8, this lemma yields a bound of

$$\|d^2E(z)(\partial_\varepsilon z_\varepsilon^{(1)}, \cdot)\| \leq C\lambda^{-2}(\log\lambda)^{\frac{1}{2}} + C\|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)},$$

where we compute the operator norm with respect to $\|\cdot\|_z$. For more general variations, the term $\|P_\omega(\partial_\varepsilon \tau_{g_{S^2}}(\omega^{(\varepsilon)}))\|_{L^2(S^2)} = \|L_\omega(\partial\omega^{(\varepsilon)})\|_{L^2(S^2)}$ can be of order one, but will be small since $T_{\hat{\omega}}\mathcal{H}_1^{\sigma_1}(\hat{\omega}) = \ker(L_{\hat{\omega}})$. For variations $z_\varepsilon^{(2)}$ as in Corollary 3.10 we will hence simply use that, after increasing λ_1 and decreasing $\sigma_1 > 0$ if necessary,

$$\|d^2E(z)(\partial_\varepsilon z^{(2)}, \cdot)\| \leq \eta_3$$

for a small constant $\eta_3 > 0$ that is chosen later on.

Proof of Lemma 3.11. The main step in the proof of the lemma is to derive suitable bounds on the tension $\tau_g(z)$ and its variation $P_z(\partial_\varepsilon \tau_g(z))$ on $B_{\tilde{\iota}}(a)$. To do this we can work in the usual isothermal coordinates in which z is represented by $\tilde{z}$ and estimate the tension of $\tilde{z}$ with respect to the Euclidean metric on $\mathbb{D}_{\frac{r_0}{2}}$. Writing $\tilde{z}$ as in (3.11) gives

$$\tau(\tilde{z}) = P_{\tilde{z}}(\Delta \tilde{z}) = T_1 + T_2 + \mathrm{err}_1 \quad \text{on } \mathbb{D}_{\frac{r_0}{2}}$$

for terms

$$T_1 := P_{\tilde{z}}(\Delta \widetilde{\omega}_\lambda) \quad \text{and} \quad T_2 := P_{\tilde{z}}(\Delta (P_{\widetilde{\omega}_\lambda} j))$$

that we analyse in detail below and an error term $\mathrm{err}_1 = P_{\tilde{z}}(\Delta E)$ for which (3.12) gives

$$|\mathrm{err}_1| + |\partial_\varepsilon \mathrm{err}_1| = O(\lambda^{-2}).$$

As we can write $T_1 = (P_{\widetilde{\omega}_\lambda} - P_{\tilde{z}})(A(\widetilde{\omega}_\lambda)(\nabla \widetilde{\omega}_\lambda, \nabla \widetilde{\omega}_\lambda)) + P_{\tilde{z}}(\tau(\widetilde{\omega}_\lambda))$ we can estimate

$$|T_1| \lesssim |\tilde{z} - \widetilde{\omega}_\lambda| \rho_\lambda^2 + |\tau(\widetilde{\omega}_\lambda)| \lesssim \lambda^{-1} |x| \rho_\lambda^2 + \|\tau_{g_{S^2}}(\omega)\|_{C^0(S^2)} \rho_\lambda^2;$$

compare (3.20). Furthermore, we can use (3.12)–(3.15) to bound

$$\begin{aligned} |P_{\tilde{z}} \partial_\varepsilon T_1| &\lesssim |\partial_\varepsilon (\tilde{z} - \widetilde{\omega}_\lambda)| \rho_\lambda^2 + |\tilde{z} - \widetilde{\omega}_\lambda| (|\partial_\varepsilon \widetilde{\omega}_\lambda| \rho_\lambda^2 + |\partial_\varepsilon \nabla \widetilde{\omega}_\lambda| \rho_\lambda) \\ &\quad + |\partial_\varepsilon \tilde{z}| |\tau(\widetilde{\omega}_\lambda)| + |P_{\tilde{z}}(\partial_\varepsilon \tau(\widetilde{\omega}_\lambda))| \\ &\lesssim \lambda^{-1} |x| \rho_\lambda^2 + \|\tau_{g_{S^2}}(\omega)\|_{C^0(S^2)} \rho_\lambda^2 + |P_{\tilde{z}}(\partial_\varepsilon \tau(\widetilde{\omega}_\lambda))|. \end{aligned}$$

To bound the last term we use that (3.20) and (3.13) give

$$\left| \partial_\varepsilon \tau(\widetilde{\omega}_\lambda) - \frac{1}{2} |\nabla \pi_\lambda|^2 (\partial_\varepsilon \tau_{g_{S^2}}(\omega)) \circ \pi_\lambda \right| \lesssim \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} \rho_\lambda^2,$$

thus allowing us to bound

$$\begin{aligned} |P_{\tilde{z}}(\partial_\varepsilon \tau(\widetilde{\omega}_\lambda))| &\le C \lambda^{-1} |x| |\partial_\varepsilon \tau(\widetilde{\omega}_\lambda)| + |P_{\widetilde{\omega}_\lambda}(\partial_\varepsilon \tau(\widetilde{\omega}_\lambda))| \\ &\lesssim \lambda^{-1} |x| \rho_\lambda^2 + \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} \rho_\lambda^2 \\ &\quad + |(P_\omega \partial_\varepsilon \tau_{g_{S^2}}(\omega)) \circ \pi_\lambda| \cdot |\nabla \pi_\lambda| \rho_\lambda. \end{aligned}$$

All in all we thus have an estimate of

$$|P_{\tilde{z}}(\partial_\varepsilon T_1)| \lesssim \lambda^{-1} |x| \rho_\lambda^2 + \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)} \rho_\lambda^2 + |(P_\omega \partial_\varepsilon \tau_{g_{S^2}}(\omega)) \circ \pi_\lambda| \cdot |\nabla \pi_\lambda| \rho_\lambda.$$

Since j is harmonic we have $T_2 = -P_{\tilde{z}}(\Delta(P^\perp_{\widetilde{\omega}_\lambda} j))$, so working with respect to a local orthonormal frame $\{\nu^k\}$ of $T^\perp N$ and summing over k we get

$$\begin{aligned} T_2 &= -\langle \nu^k_{\widetilde{\omega}_\lambda}, j \rangle P_{\tilde{z}}(\Delta \nu^k_{\widetilde{\omega}_\lambda}) - \Delta(\langle \nu^k_{\widetilde{\omega}_\lambda}, j \rangle)(P_{\tilde{z}} - P_{\widetilde{\omega}_\lambda})(\nu^k_{\widetilde{\omega}_\lambda}) \\ &\quad - 2\nabla(\langle \nu^k_{\widetilde{\omega}_\lambda}, j \rangle) P_{\tilde{z}}(\nabla \nu^k_{\widetilde{\omega}_\lambda}), \end{aligned} \tag{3.24}$$

allowing us to bound

$$|T_2| \lesssim |j| \rho_\lambda^2 + |\nabla j| |j| \rho_\lambda + \rho_\lambda |\langle \nu^k_{\widetilde{\omega}_\lambda}, \nabla j \rangle| \lesssim \lambda^{-1} |x| \rho_\lambda^2 + \lambda^{-1} \rho_\lambda (1 + \lambda |x|)^{-1}$$

since j maps into $T_{\omega(p^*)} N$ and $|\widetilde{\omega}_\lambda - \omega(p^*)| \lesssim (1 + \lambda |x|)^{-1}$.

Furthermore, differentiating (3.24) with respect to ε and using (3.13) gives

$$\begin{aligned}|\partial_\varepsilon T_2| &\lesssim [|j| + |\partial_\varepsilon j|]\rho_\lambda^2 + C[|\nabla j| + |\partial_\varepsilon \nabla j|]\rho_\lambda |j| \\ &\quad + |\langle \nu^k_{\widetilde{\omega}_\lambda}, \nabla j\rangle|\rho_\lambda + |\langle \nu^k_{\widetilde{\omega}_\lambda}, \partial_\varepsilon \nabla j\rangle|\rho_\lambda + |\partial_\varepsilon \widetilde{\omega}_\lambda||\nabla j|\rho_\lambda \\ &\lesssim \lambda^{-1}|x|\rho_\lambda^2 + \lambda^{-1}\rho_\lambda|\omega_\lambda - \omega(p^*)| + \lambda^{-1}(\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^1} + (1+|\lambda x|)^{-1})\rho_\lambda \\ &\lesssim \lambda^{-1}|x|\rho_\lambda^2 + \lambda^{-1}(\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^1} + (1+|\lambda x|)^{-1})\rho_\lambda,\end{aligned}$$

where the penultimate step uses (3.14), as well as that $\partial_\lambda \nabla j = \lambda^{-1}\nabla j \in T_{\omega(p^*)}N$ and thus $|P_{\omega(p^*)}(\partial_\varepsilon \nabla j)| \lesssim \lambda^{-1}\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^1}$.

All in all we thus find that on $\mathbb{D}_{\frac{r_0}{2}}$,

$$|\tau(\tilde{z})| \lesssim [\lambda^{-1}|x|\rho_\lambda + \lambda^{-1}(1+|\lambda x|)^{-1} + \|\tau_{g_{S^2}}(\omega)\|_{C^0(S^2)}\rho_\lambda]\rho_\lambda + \lambda^{-2}, \tag{3.25}$$

while

$$\begin{aligned}|P_{\tilde{z}}(\partial_\varepsilon \tau(\tilde{z}))| &\lesssim \lambda^{-1}[|x|\rho_\lambda + \|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^1(S^2)} + (1+|\lambda x|)^{-1}]\rho_\lambda + \lambda^{-2} \\ &\quad + \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)}\rho_\lambda^2 + |(P_\omega \partial_\varepsilon \tau_{g_{S^2}}(\omega))\circ \pi_\lambda||\nabla \pi_\lambda|\rho_\lambda,\end{aligned} \tag{3.26}$$

where we note that

$$\|(P_\omega \partial_\varepsilon \tau_{g_{S^2}}(\omega))\circ \pi_\lambda |\nabla \pi_\lambda|\,\|_{L^2(\mathbb{D}_{\frac{r_0}{2}})} = \sqrt{2}\|P_\omega(\partial_\varepsilon \tau_{g_{S^2}}(\omega))\|_{L^2(\pi_\lambda(\mathbb{D}_{\frac{r_0}{2}}))}.$$

As the energy is conformally invariant, $\|w\|_z = 1$ and as $\tau_g(z)$ and $\partial_\varepsilon \tau_g(z)$ are of order $O(\lambda^{-2})$ on $\Sigma \setminus B_{\tilde{\iota}}(a)$ (compare (3.9)), we hence get from (3.25), (2.13) and (3.17) that

$$\begin{aligned}|dE(z)(w)| = \left|\int_\Sigma \tau_g(z) w \,\mathrm{d}v_g\right| &\le C\lambda^{-2}\|w\|_{L^1(\Sigma)} + \int_{\mathbb{D}_{\frac{r_0}{2}}} |\tau(\tilde{z})||w\circ F_a^{-1}|\,\mathrm{d}x \\ &\lesssim \lambda^{-2}(\log\lambda)^{\frac12} + \lambda^{-1}\|\,|x|\rho_\lambda\|_{L^2(\mathbb{D}_{\frac{r_0}{2}})} + \lambda^{-1}\|(1+|\lambda x|)^{-1}\|_{L^2(\mathbb{D}_{\frac{r_0}{2}})} \\ &\quad + \|\tau_{g_{S^2}}(\omega)\|_{C^0(S^2)} \\ &\lesssim \lambda^{-2}(\log\lambda)^{\frac12} + \|\tau_{g_{S^2}}(\omega)\|_{C^0(S^2)},\end{aligned}$$

as claimed in the lemma. Finally, as w is tangential to N along z we have

$$\begin{aligned}d^2E(z)(\partial_\varepsilon z, w) &= -\int_\Sigma \partial_\varepsilon \tau_g(z)\cdot w\,\mathrm{d}v_g = -\int_\Sigma P_z(\partial_\varepsilon \tau_g(z))\cdot w\,\mathrm{d}v_g \\ &= -\int_{\mathbb{D}_{\frac{r_0}{2}}} P_{\tilde{z}}(\partial_\varepsilon \tau(\tilde{z}))\cdot w\circ F_a^{-1}\,\mathrm{d}x + O(\lambda^{-2}\|w\|_{L^1(\Sigma)})\end{aligned}$$

and inserting (3.26) and (3.17) immediately gives the second claim (3.23) of the lemma. ■

4. Proof of Theorem 2.5

We now turn to the proof of our first main result. To this end we first observe that two adapted bubbles with quite different scales λ_1, λ_2 respectively with quite different underlying maps ω_1 and ω_2 cannot be close. Namely, we have the following lemma for which we provide a short proof in the appendix.

Lemma 4.1. *Let $\widehat{\omega}$ be any harmonic sphere and let $\lambda_1 \geq 2$, $\sigma_1 > 0$ be any given numbers for which $\mathcal{Z} = \mathcal{Z}_{\lambda_1}^{\sigma_1}$ is well defined. Then there exist numbers $\varepsilon_3 > 0$ and $\lambda_2 \geq \lambda_1$ depending only on σ_1, $\widehat{\omega}$, (Σ, g) and N so that*

$$\|z_\lambda^{a,\omega} - z_{\tilde{\lambda}}^{\tilde{a},\tilde{\omega}}\|_{z_\lambda^{a,\omega}} \geq \varepsilon_3 \tag{4.1}$$

for all elements $z_\lambda^{a,\omega}, z_{\tilde{\lambda}}^{\tilde{a},\tilde{\omega}} \in \mathcal{Z}$ with $\lambda \geq \lambda_2$ for which either $\lambda\tilde{\lambda}^{-1} \notin [\frac{1}{2}, 2]$ or for which $\omega \in \mathcal{H}_1^{\frac{1}{3}\sigma_1}(\widehat{\omega})$ while $\tilde{\omega} \in \mathcal{H}_1^{\sigma_1}(\widehat{\omega}) \setminus \mathcal{H}_1^{\frac{2}{3}\sigma_1}(\widehat{\omega})$.

Furthermore, for any $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda_3 \geq \lambda_2$ so that for any $z_\lambda^{a,\omega}, z_{\tilde{\lambda}}^{\tilde{a},\tilde{\omega}} \in \mathcal{Z}$ with $\lambda \geq \lambda_3$ we have

$$\|z_\lambda^{a,\omega} - z_{\tilde{\lambda}}^{\tilde{a},\tilde{\omega}}\|_{L^\infty(\Sigma)} < \frac{1}{2}\varepsilon \quad \textit{whenever } \|z_\lambda^{a,\omega} - z_{\tilde{\lambda}}^{\tilde{a},\tilde{\omega}}\|_{z_\lambda^{a,\omega}} < \delta. \tag{4.2}$$

This lemma now allows us to prove that in the setting of Theorem 2.5 we have the following lemma:

Lemma 4.2. *Let $\widehat{\omega}$ be any harmonic sphere and let $\lambda_1 \geq 2$, $\sigma_1 > 0$ be any given numbers for which $\mathcal{Z} = \mathcal{Z}_{\lambda_1}^{\sigma_1}$ is well defined. Then for every $\varepsilon > 0$ there exist $\varepsilon_1 > 0$ and $\bar{\lambda} \geq \lambda_1$ so that for any $u \in H^1(\Sigma, N)$ for which*

$$\|u - z_0\|_{L^\infty(\Sigma)} + \|\nabla(u - z_0)\|_{L^2(\Sigma)} < \varepsilon_1 \quad \textit{for some } z_0 \in \mathcal{Z}_{\bar{\lambda}}^{\frac{1}{3}\sigma_1},$$

we have that the infimum $\mathrm{dist}(u, \mathcal{Z}) := \inf_{z \in \mathcal{Z}} \|u - z\|_z$ is attained on $\mathcal{Z}$ and for every minimiser z of this distance we have that $z \in \mathcal{Z} \setminus \partial Z$ and

$$\|u - z\|_{L^\infty(\Sigma)} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and let $u \in H^1(\Sigma, N)$ and $z_0 = z_{\lambda_0}^{a_0,\omega_0} \in \mathcal{Z}_{\bar{\lambda}}^{\frac{1}{3}\sigma_1}$ be so that the above assumptions are satisfied for numbers $\varepsilon_1 \in (0, \frac{1}{2}\varepsilon)$ and $\bar{\lambda} > \max(\lambda_3, 2\lambda_1)$ that are chosen below, λ_3 the constant from Lemma 4.1.

Since $\int \rho_z^2 \, dv_g$ is bounded uniformly on $\mathcal{Z}$, we have that $\|u - z_0\|_z \leq C\varepsilon_1$ for every $z \in \mathcal{Z}$. We can thus choose $\varepsilon_1 > 0$ small enough so that any $z = z_\lambda^{a,\omega} \in \mathcal{Z}$ with $\|z - u\|_z \leq \|z_0 - u\|_{z_0}$ must be so that

$$\|z_0 - z\|_z \leq \|z - u\|_z + \|z_0 - u\|_z \leq \|z_0 - u\|_{z_0} + \|z_0 - u\|_z \leq 2C\varepsilon_1 < \min(\delta, \varepsilon_3),$$

for $\delta, \varepsilon_3 > 0$ as in Lemma 4.1. As $\lambda(z_0) \geq \lambda_3 \geq \lambda_2$ we can apply this lemma to conclude that any such z is contained in the compact subset of adapted bubbles for which the parameters are constrained by $\lambda \in [\frac{1}{2}\lambda(z_0), 2\lambda(z_0)]$ and $\omega \in \mathcal{H}_1^{\frac{2}{3}\sigma_1}$. Hence $z \mapsto \|u - z\|_z$ achieves its minimum over $\mathcal{Z}$ on this compact subset and any minimiser in $\mathcal{Z}$ is contained in this subset of $\mathcal{Z} \setminus \partial Z$. Furthermore, the last part of Lemma 4.1 yields that any minimiser satisfies $\|u - z\|_{L^\infty} \leq \|z - z_0\|_{L^\infty} + \|u - z_0\|_{L^\infty} < \frac{\varepsilon}{2} + \varepsilon_1 \leq \varepsilon$. ■

As the norm $\|\cdot\|_z$ depends on z, we cannot expect that the difference $w = u - z$ between u and a minimiser z of $\tilde{z} \mapsto \|u - \tilde{z}\|_{\tilde{z}}$ is orthogonal to $T_z\mathcal{Z}$. However, as we will see in Lemma B.2 in the appendix, we can bound the variation of the weight ρ_z along any variation $z_\varepsilon \in \mathcal{Z}$ by

$$\|\partial_\varepsilon \rho_z\|_{L^2(\Sigma)} \leq C\|\partial_\varepsilon z\|_z. \tag{4.3}$$

This allows us to obtain that $w = u - z$ is almost orthogonal to $T_z\mathcal{Z}$ in the sense given in the following lemma:

Lemma 4.3. *Let $u \in H^1(\Sigma, N)$ and suppose that $z \in \mathcal{Z} \setminus \partial\mathcal{Z}$ minimises $\tilde{z} \mapsto \|u - \tilde{z}\|_{\tilde{z}}$ on $\mathcal{Z}$. Then $w := u - z$ satisfies*

$$\|P^{T_z\mathcal{Z}}(P_z w)\|_z \leq C\|w\|_{L^\infty(\Sigma)}\|w\|_z, \tag{4.4}$$

where P_z denotes the (pointwise) orthogonal projection from $\mathbb{R}^n$ to $T_{z(p)}N$, while $P^{T_z\mathcal{Z}}\colon \Gamma^{H^1}(z^*TN) \to T_z\mathcal{Z}$ is the $\langle\cdot,\cdot\rangle_z$-orthogonal projection.

Proof. Given any variation z_ε of a minimiser $z \in \mathcal{Z} \setminus \partial Z$ of $\tilde{z} \mapsto \|u - \tilde{z}\|_{\tilde{z}}$ we can combine the resulting constraint that $\frac{d}{d\varepsilon}|_{\varepsilon=0}\|u - z_\varepsilon\|_{z_\varepsilon}^2 = 0$ with (4.3) to conclude that at $\varepsilon = 0$,

$$\begin{aligned}\langle w, \partial_\varepsilon z\rangle_z &= \int_\Sigma \rho_z \partial_\varepsilon \rho_{z_\varepsilon}|w|^2\, \mathrm{d}v_g \\ &\leq C\|w\|_z\|w\|_{L^\infty(\Sigma)}\|\partial_\varepsilon \rho_{z_\varepsilon}\|_{L^2(\Sigma)} \\ &\leq C\|w\|_z\|w\|_{L^\infty(\Sigma)}\|\partial_\varepsilon z_\varepsilon\|_z.\end{aligned}$$

As $\|P_z w - w\|_z \leq C\|w\|_{L^\infty}\|w\|_z$ (compare (4.8) below), we thus obtain the bound

$$|\langle P_z w, v\rangle_z| \leq C\|w\|_{L^\infty}\|w\|_z\|v\|_z \quad \text{for every } v \in T_z\mathcal{Z},$$

which is equivalent to claim (4.4) of the lemma. ■

This almost orthogonality of w to $T_z\mathcal{Z}$ is sufficient to exploit the uniform definiteness of the second variation orthogonal to $\mathcal{Z}$. This is crucial to obtain the following initial estimate on w, which will play the role of [12, Lemma 2.7] in this new setting where we work with a family of distances induced by the norms $\|\cdot\|_z$ on the infinite-dimensional set of maps $H^1(\Sigma, N)$ rather than in a fixed Hilbert space.

Lemma 4.4. *There exists $\varepsilon_2 > 0$ so that for any $u \in H^1(\Sigma, N)$ for which*

$$\|u - z\|_z = \inf_{\mathcal{Z}}\|u - \tilde{z}\|_{\tilde{z}} < \varepsilon_2 \quad \text{and} \quad \|z - u\|_{L^\infty(\Sigma)} < \varepsilon_2 \tag{4.5}$$

for some $z = z_\lambda^{a,\omega} \in \mathcal{Z} \setminus \partial\mathcal{Z}$, we can bound $w := u - z$ by

$$\|w\|_z^2 \leq C(dE(u) - dE(z))(\widetilde{w}_z) + C \log\lambda \|\tau_g(u)\|_{L^2(\Sigma,g)} \|w\|_z^2 \tag{4.6}$$

and therefore have

$$\|w\|_z \lesssim \|\tau_g(u)\|_{L^2(\Sigma,g)} (\log\lambda)^{\frac{1}{2}} + \lambda^{-2}(\log\lambda)^{\frac{1}{2}} + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} + (\log\lambda)^2 \|\tau_g(u)\|_{L^2(\Sigma,g)}^2.$$

Here, $\widetilde{w}_z = (P^{\mathcal{V}_z^+} - P^{\mathcal{V}_z^-})(P_z w)$ is defined using the $\langle\cdot,\cdot\rangle_z$-orthogonal projections from $\Gamma^{H^1}(z^*TN)$ to the subspaces $\mathcal{V}_z^\pm$ obtained in Lemma 3.1.

Both for this proof, and in later parts of the proof of Theorem 2.5, we consider the maps

$$u_t = \pi_N(z + tw) \quad \text{for } w = u - z \text{ and } t \in [0,1].$$

We note that these maps are well defined if $\varepsilon_2 < \delta_N$ and will be in $H^2(\Sigma, N)$ since any map $u \in H^1(\Sigma, N)$ with $\tau_g(u) \in L^2(\Sigma)$ is automatically in $H^2(\Sigma, N)$; see e.g. [13, 15]. We will also use that $|\nabla u_t| \leq |\nabla z| + |\nabla w|$ and thus that for $v \in H^1(\Sigma, \mathbb{R}^n)$,

$$\|v|\nabla u_t|\|_{L^2(\Sigma)}^2 + \|P_{u_t}(v)\|_z^2 \leq C\|v\|_z^2 + C\int |\nabla w|^2 |v|^2 \, dv_g.$$

We also remark that for any $s \in [0,1]$ we can write $\frac{d}{ds}u_s = d\pi_N(z + sw)(w) = P_{u_s}(w) + \mathrm{err}_s$ for an error term $\mathrm{err}_s \in \Gamma^{H^1}(u_s^*TN)$ that is bounded by

$$|\mathrm{err}_s| \leq C|w|^2 \quad \text{with } |\nabla \mathrm{err}_s| \leq C\rho_z |w|^2 + C|w||\nabla w|. \tag{4.7}$$

Integrating over $s \in [0,1]$ and using that also $|(P_{u_t} - P_{u_s})(w)| \leq C|w|^2$ we thus get

$$|w - P_{u_t} w| \leq C|w|^2 \quad \text{while } |\nabla(w - P_{u_t} w)| \leq C|w||\nabla w| + C|w|^2 \rho_z$$

for any $t \in [0,1]$ and we will in particular use that

$$\|w - P_{u_t} w\|_z \leq C\|w\|_{L^\infty} \|w\|_z. \tag{4.8}$$

Proof of Lemma 4.4. Let u and z be so that (4.5) is satisfied for a number $\varepsilon_2 \in (0, \delta_N)$ that is chosen below. We set $w_z := P_z(w)$, let $\widetilde{w}_z = (P^{\mathcal{V}_z^+} - P^{\mathcal{V}_z^-})(w_z)$ be as in the lemma and note that (4.8) implies that

$$\|w - w_z\|_z \leq C\varepsilon_2 \|w\|_z \quad \text{while } \|\widetilde{w}_z\|_z \leq \|w_z\|_z \leq C\|w\|_z.$$

We can hence combine Lemmas 3.2 and 4.3 with (3.2) to obtain

$$\begin{aligned} d^2E(z)(w_z, \widetilde{w}_z) &= d^2E(z)(P^{\mathcal{V}_z^+} w_z + P^{\mathcal{V}_z^-} w_z, P^{\mathcal{V}_z^+} w_z - P^{\mathcal{V}_z^-} w_z) \\ &\quad + d^2E(z)(P^{T_z\mathcal{Z}} w_z, \widetilde{w}_z) \\ &\geq c_0(\|P^{\mathcal{V}_z^+} w_z\|_z^2 + \|P^{\mathcal{V}_z^-} w_z\|_z^2) - C\|P^{T_z\mathcal{Z}} w_z\|_z \|\widetilde{w}_z\|_z \\ &\geq c_0(\|w_z\|_z^2 - \|P^{T_z\mathcal{Z}} w_z\|_z^2) - C\|w\|_{L^\infty} \|w\|_z^2 \\ &\geq (c_0(1 - C\varepsilon_2^2) - C\varepsilon_2)\|w\|_z^2 \geq \frac{c_0}{2}\|w\|_z^2, \end{aligned}$$

where $c_0 > 0$ is the constant obtained in Lemma 3.2 and where the last inequality holds after reducing $\varepsilon_2 > 0$ if necessary. As

$$\begin{aligned}(dE(u)-dE(z))(\widetilde{w}_z) &= \int_0^1 \frac{d}{dt}(dE(u_t)(\widetilde{w}_z))\,\mathrm{d}t = \int_0^1 \frac{d}{dt}(dE(u_t)(P_{u_t}\widetilde{w}_z))\,\mathrm{d}t\\ &= \int_0^1 d^2E(u_t)\Big(\frac{d}{dt}u_t, P_{u_t}\widetilde{w}_z\Big) + dE(u_t)\Big(\frac{d}{dt}P_{u_t}\widetilde{w}_z\Big)\,\mathrm{d}t\\ &= \int_0^1 d^2E(u_t)(P_{u_t}w + \mathrm{err}_t, P_{u_t}\widetilde{w}_z)\\ &\quad + \int_\Sigma \nabla u_t \nabla\Big(P_{u_t}\Big(\frac{d}{dt}P_{u_t}\widetilde{w}_z\Big)\Big)\,\mathrm{d}v_g\,\mathrm{d}t,\end{aligned}$$

we thus conclude that

$$\frac{c_0}{2}\|w\|_z^2 \le d^2E(z)(w_z,\widetilde{w}_z) \le (dE(u)-dE(z))(\widetilde{w}_z) + \sup_{[0,1]} T_1 + T_2 + T_3 \tag{4.9}$$

for

$$\begin{aligned}T_1 &:= |d^2E(u_t)(P_{u_t}w, P_{u_t}\widetilde{w}_z) - d^2E(z)(w_z,\widetilde{w}_z)|,\\ T_2 &:= \int |\nabla u_t|\Big|\nabla\Big(P_{u_t}\Big(\frac{d}{dt}P_{u_t}\widetilde{w}_z\Big)\Big)\Big|,\\ T_3 &:= |d^2E(u_t)(\mathrm{err}_t, P_{u_t}\widetilde{w}_z)|.\end{aligned}$$

To bound the first term we apply (3.3) for $v_1 = w$ and $v_2 = \widetilde{w}_z$ giving

$$\begin{aligned}T_1 &\le C\int |w||\nabla w||\nabla\widetilde{w}_z| + C\int (|w|\rho_z + |\nabla w|)(|w||\nabla\widetilde{w}_z| + |\nabla w||\widetilde{w}_z|)\\ &\quad + C\int |\widetilde{w}_z||w|(|w|\rho_z^2 + |\nabla w|\rho_z + |\nabla w|^2)\\ &\le C\|w\|_{L^\infty}\|w\|_z^2 + C\int |\widetilde{w}_z||\nabla w|^2\\ &\le \Big(C\varepsilon_2 + \frac{c_0}{8}\Big)\|w\|_z^2 + C\int |\widetilde{w}_z|^2|\nabla w|^2.\end{aligned} \tag{4.10}$$

As $\widetilde{w}_z$ is obtained using the non-local projections $P^{\mathcal{V}_z^\pm}$ we do not have a pointwise bound on $\widetilde{w}_z$. Instead we use that $|\Delta_g w| \le |\Delta_g u| + |\Delta_g z| \lesssim |\tau_g(u)| + \rho_z^2 + |\nabla w|^2$ (compare also (3.16)) to bound

$$\begin{aligned}I_1 := \int |\widetilde{w}_z|^2|\nabla w|^2 &= -\int w|\widetilde{w}_z|^2\Delta_g w - 2\int (w\nabla w)\cdot(\widetilde{w}_z\nabla\widetilde{w}_z)\\ &\le C\|w\|_{L^\infty}[\|\tau_g(u)\|_{L^2}\|\widetilde{w}_z\|_{L^4}^2 + \|w\|_z^2 + I_1].\end{aligned}$$

After possibly reducing $\varepsilon_2 > 0$ and applying (2.13) we thus get

$$I_1 \le C\varepsilon_2[\log\lambda\|\tau_g(u)\|_{L^2} + 1]\|w\|_z^2$$

and so obtain from (4.10) that

$$T_1 \le \Big(C\varepsilon_2 + \frac{c_0}{8}\Big)\|w\|_z^2 + C\varepsilon_2 \log\lambda \|\tau_g(u)\|_{L^2}\|w\|_z^2.$$

To bound T_2 we write, summing over repeated indices j,

$$\begin{aligned} P_{u_t}\Big(\frac{d}{dt}P_{u_t}\widetilde{w}_z\Big) &= P_{u_t}\Big(-\frac{d}{dt}(\langle \nu_{u_t}^j, \widetilde{w}_z\rangle \nu_{u_t}^j)\Big) = -\langle \nu_{u_t}^j, \widetilde{w}_z\rangle P_{u_t}\Big(\frac{d}{dt}\nu_{u_t}^j\Big) \\ &= -\langle \nu_{u_t}^j - \nu_z^j, \widetilde{w}_z\rangle P_{u_t}(d\nu_{u_t}^j(P_{u_t}w + \mathrm{err}_t)), \end{aligned}$$

where we used that $\widetilde{w}_z \in T_z N$ in the last step. We can thus estimate

$$\begin{aligned} T_2 &\lesssim \int |\nabla u_t|(\rho_z|w| + |\nabla w|)|w||\widetilde{w}_z| + |\nabla u_t||w|(|\nabla w||\widetilde{w}_z| + |\nabla \widetilde{w}_z||w|)\, dv_g \\ &\lesssim \|w\|_{L^\infty}\|w\|_z^2. \end{aligned}$$

Finally, we can use (3.1) and (4.7) to also bound

$$T_3 \lesssim \|\nabla \mathrm{err}_t\|_{L^2}\|\nabla\widetilde{w}_z\|_{L^2} + \int |\nabla u_t|^2|\mathrm{err}_t||\widetilde{w}_z| \lesssim \|w\|_{L^\infty}\|w\|_z^2.$$

All in all, we thus get

$$T_1 + T_2 + T_3 \le \Big(C\varepsilon_2 + \frac{c_0}{8}\Big)\|w\|_z^2 + C\varepsilon_2\log\lambda\|\tau_g(u)\|_{L^2}\|w\|_z^2.$$

Combined with (4.9) this gives the first claim (4.6) of the lemma provided $\varepsilon_2 > 0$ is chosen sufficiently small.

We can then combine (4.6) with the bound on $dE(z)$ obtained in Lemma 3.11 and with (2.13) to deduce that

$$\begin{aligned} \|w\|_z^2 \le C[\|\tau_g(u)\|_{L^2}(\log\lambda)^{\frac12} + \lambda^{-2}(\log\lambda)^{\frac12} + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}]\|\widetilde{w}_z\|_z \\ + C\|\tau_g(u)\|_{L^2}^2(\log\lambda)^2\|w\|_z + \|w\|_z^3. \end{aligned}$$

As we can assume that $\varepsilon_2 < \frac12$, we can absorb the last term into the right-hand side and use for a final time that $\|\widetilde{w}_z\|_z \le C\|w\|_z$ to obtain the second claim of the lemma. ■

We now want to derive suitable bounds on λ^{-1} and on $\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}$ in terms of the tension of u. To this end we will exploit the lower bounds on the variations of the energy in the specific directions of $T_z\mathcal{Z}$ obtained in Corollaries 3.8 and 3.10, as well as the bounds on the second variation from Lemma 3.11; see also Remark 3.12 .

These results tell us that for $z_\varepsilon^{(1)} := z_{(1-\varepsilon)\lambda}^{a,\omega}$,

$$\begin{aligned} dE(z)(\partial_\varepsilon z^{(1)}) &\ge c_1\lambda^{-2}, \\ \|d^2E(z)(\partial_\varepsilon z, \cdot)\| &\le C\lambda^{-2}(\log\lambda)^{\frac12} + C\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}, \end{aligned} \tag{4.11}$$

while for the variations $z_\varepsilon^{(2)} := z_\lambda^{a,\omega^{(\varepsilon)}}$ as considered in Corollary 3.10,

$$dE(z)(\partial_\varepsilon z^{(2)}) \geq \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} - C\lambda^{-2} \quad \text{and} \quad \|d^2E(z)(\partial_\varepsilon z, \cdot)\| \leq \eta_3 \tag{4.12}$$

for a number $\eta_3 > 0$ that we can still choose and for constants $c_1 > 0$ and $C < \infty$ that only depend on N, $\widehat{\omega}$ and (Σ, g).

Again writing $u_t = \pi_N(z + tw)$ and $w_z = P_z(w)$ for short, we have that for $i = 1, 2$,

$$dE(z)(\partial_\varepsilon z^{(i)}) = dE(u)(\partial_\varepsilon z^{(i)}) + d^2E(z)(\partial_\varepsilon z^{(i)}, w_z) - \int_0^1 T_4(t)\,\mathrm{d}t \tag{4.13}$$

for

$$\begin{aligned} T_4 &:= \frac{d}{dt}[dE(u_t)(\partial_\varepsilon z^{(i)})] - d^2E(z)(w_z, \partial_\varepsilon z^{(i)}) \\ &= dE(u_t)\Big(\frac{d}{dt}(P_{u_t}\partial_\varepsilon z^{(i)})\Big) + d^2E(u_t)\Big(\frac{d}{dt}u_t, P_{u_t}\partial_\varepsilon z^{(i)}\Big) - d^2E(z)(P_z w, \partial_\varepsilon z^{(i)}) \\ &= \int \nabla u_t \nabla\Big(P_{u_t}\Big(\frac{d}{dt}(P_{u_t}\partial_\varepsilon z^{(i)})\Big)\Big) + d^2E(u_t)(P_{u_t}w + \mathrm{err}_t, P_{u_t}\partial_\varepsilon z^{(i)}) \\ &\quad - d^2E(z)(P_z w, \partial_\varepsilon z^{(i)}). \end{aligned}$$

As $|\partial_\varepsilon z^{(i)}| \lesssim 1$ and $|\partial_\varepsilon \nabla z^{(i)}| \lesssim \rho_z$, we can use (3.3) to bound

$$|d^2E(u_t)(P_{u_t}w, P_{u_t}(\partial_\varepsilon z^{(i)})) - d^2E(z)(P_z w, \partial_\varepsilon z^{(i)})| \lesssim \|w\|_z^2,$$

while (3.1) and (4.7) ensure that $|d^2E(u_t)(\mathrm{err}_t, P_{u_t}(\partial_\varepsilon z^{(i)}))| \lesssim \|w\|_z^2$ also.

As $P_{u_t}(\frac{d}{dt}(P_{u_t}(\partial_\varepsilon z^{(i)}))) = -\sum_j \langle \nu_{u_t}^j - \nu_z, \partial_\varepsilon z^{(i)}\rangle P_{u_t}(d\nu_{u_t}(P_{u_t}w + \mathrm{err}_t))$, we can also bound

$$\begin{aligned} &\Big|\int \nabla u_t \nabla\Big(P_{u_t}\Big(\frac{d}{dt}P_{u_t}\partial_\varepsilon z^{(i)}\Big)\Big)\Big| \\ &\quad \lesssim \int [\rho_z + |\nabla w|][|w||\partial_\varepsilon z^{(i)}|(|\nabla w| + \rho_z|w|) + |w|^2|\partial_\varepsilon \nabla z^{(i)}|] \\ &\quad \lesssim \|w\|_z^2 \end{aligned}$$

and thus get $|T_4| \lesssim \|w\|_z^2$.

For the variation $z_\varepsilon^{(1)}$ which satisfies (4.11), we hence obtain from (4.13) that

$$\begin{aligned} c_1\lambda^{-2} &\leq dE(u)(\partial_\varepsilon z^{(1)}) + d^2E(z)(\partial_\varepsilon z^{(1)}, w_z) + C\|w\|_z^2 \\ &\leq \|\tau_g(u)\|_{L^2}\|\partial_\varepsilon z^{(1)}\|_{L^2} + C(\lambda^{-2}(\log\lambda)^{\frac{1}{2}} + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)})\|w\|_z + C\|w\|_z^2 \\ &\lesssim \lambda^{-1}(\log\lambda)^{\frac{1}{2}}\|\tau_g(u)\|_{L^2} + \lambda^{-4}(\log\lambda) + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^2 + \|w\|_z^2, \end{aligned}$$

where norms are computed over Σ unless stated otherwise and where we use that

$$\|\partial_\varepsilon z^{(1)}\|_{L^2} \lesssim \lambda^{-1} + \|(1 + \lambda|x|)^{-1}\|_{L^2(\mathbb{D}_{\frac{r_0}{2}})} \lesssim \lambda^{-1}(\log\lambda)^{\frac{1}{2}}.$$

Combined with Lemma 4.4, and after increasing λ_1 if necessary, we hence obtain

$$\lambda^{-2} \lesssim \lambda^{-1}(\log\lambda)^{\frac{1}{2}}\|\tau_g(u)\|_{L^2} + \log\lambda\|\tau_g(u)\|_{L^2}^2 + (\log\lambda)^4\|\tau_g(u)\|_{L^2}^4 \\ + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^2.$$

Thus, either $\lambda^{-1}(\log\lambda)^{-\frac{1}{2}} \le \|\tau_g(u)\|_{L^2}$ and so $\lambda^{-1} \le C\|\tau_g(u)\|_{L^2}(1 + |\log\|\tau_g(u)\|_{L^2}|^{\frac{1}{2}})$ or $\lambda^{-1} \le C\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}$, so in any case

$$\lambda^{-1} \lesssim \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} + \|\tau_g(u)\|_{L^2}(1 + |\log\|\tau_g(u)\|_{L^2}|^{\frac{1}{2}}). \tag{4.14}$$

On the other hand, applying (4.13) for the variation $z_\varepsilon^{(2)}$ which satisfies (4.12) as well as $\|\partial_\varepsilon z^{(2)}\|_{L^2} \le C$ and using Lemma 4.4 gives

$$\begin{aligned}\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} &\lesssim \lambda^{-2} + |dE(u)(\partial_\varepsilon z^{(2)})| + |d^2E(z)(\partial_\varepsilon z^{(2)}, w_z)| + \|w\|_z^2 \\ &\lesssim \lambda^{-2} + \|\tau_g(u)\|_{L^2}\|\partial_\varepsilon z^{(2)}\|_{L^2} + \eta_3\|w\|_z + \varepsilon_3\|w\|_z \\ &\le \lambda^{-2}(\log\lambda)^{\frac{1}{2}} + \|\tau_g(u)\|_{L^2}(\log\lambda)^{\frac{1}{2}} + (\log\lambda)^2\|\tau_g(u)\|_{L^2}^2 \\ &\quad + (\eta_3 + \varepsilon_3)\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}.\end{aligned}$$

As we can assume that ε_3 and η_3 are chosen small enough we thus conclude that

$$\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} \lesssim \lambda^{-2}(\log\lambda)^{\frac{1}{2}} + \|\tau_g(u)\|_{L^2}(\log\lambda)^{\frac{1}{2}} + \|\tau_g(u)\|_{L^2}^2(\log\lambda)^2. \tag{4.15}$$

We can thus eliminate $\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}$ from (4.14) and get

$$\lambda^{-1} \lesssim \lambda^{-2}\log\lambda + \|\tau_g(u)\|_{L^2}[1 + |\log\|\tau_g(u)\|_{L^2}|^{\frac{1}{2}} + (\log\lambda)^{\frac{1}{2}}] + \|\tau_g(u)\|_{L^2}^2(\log\lambda)^2.$$

For sufficiently large λ_1 we hence obtain our claimed bound (2.16) of

$$\lambda^{-1} \le C\|\tau_g(u)\|_{L^2(\Sigma)}[1 + |\log\|\tau_g(u)\|_{L^2(\Sigma)}|^{\frac{1}{2}}]. \tag{4.16}$$

Inserting this back into (4.15) and using (2.3) implies that also

$$\begin{aligned}\|\tau_{g_{S^2}}(\omega)\|_{C^k(S^2)} &\le C\|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)} \\ &\le C\|\tau_g(u)\|_{L^2(\Sigma)}[1 + |\log\|\tau_g(u)\|_{L^2(\Sigma)}|^{\frac{1}{2}}]\end{aligned} \tag{4.17}$$

as asserted in (2.17). From Lemma 4.4 we then obtain the claimed bound (2.15), i.e.

$$\operatorname{dist}(u, \mathcal{Z}) = \|w\|_z \le C\|\tau_g(u)\|_{L^2(\Sigma)}[1 + |\log\|\tau_g(u)\|_{L^2(\Sigma)}|^{\frac{1}{2}}]. \tag{4.18}$$

We now recall that $E(z_\lambda^{a,\omega}) - E(\omega) = O(\lambda^{-2})$ (compare Remark 3.9) and that $|E(\omega) - E(\hat{\omega})|$ is controlled by the classical Łojasiewicz–Simon inequality (1.3). Combined with the bound on $dE(z)$ from Lemma 3.11 this gives

$$\begin{aligned}&|E(u) - E(\hat{\omega})| \\ &\quad \le |E(u) - E(z)| + |E(\omega) - E(\hat{\omega})| + C\lambda^{-2}\end{aligned}$$

$$\lesssim |dE(z)(w)| + \int_0^1 |dE(z)(w) - dE(u_t)(w + \mathrm{err}_t)|\,\mathrm{d}t + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^{\gamma_1} + \lambda^{-2}$$
$$\lesssim \lambda^{-2}(\log\lambda)^{\frac{1}{2}}\|w\|_z + \|\tau_{g_{S^2}}(\omega)\|_{C^1(S^2)}\|w\|_z + \|w\|_z^2 + \|\tau_{g_{S^2}}(\omega)\|_{L^2(S^2)}^{\gamma_1} + \lambda^{-2}$$

since $|dE(z)(w) - dE(u_t)(w)| = |\int \nabla z \nabla(P_z w) - \nabla u_t \nabla(P_{u_t} w)\,\mathrm{d}v_g| \le C\|w\|_z^2$.

Inserting the bounds (4.16), (4.17) and (4.18) on λ, $\tau_{g_{S^2}}(\omega)$ and $\|w\|_z$ into this estimate yields finally the remaining claim (2.15) of Theorem 2.5.

5. Proofs of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. Let (u_n) be a sequence of almost harmonic maps which converges to a simple bubble tree as described in the introduction. We let λ_n, a_n be parameters so that (1.2) holds. From the definition of the adapted bubbles we hence obtain

$$\|u_n - z_{\lambda_n}^{a_n,\hat{\omega}}\|_{L^\infty(\Sigma,g)} + \|\nabla(u_n - z_{\lambda_n}^{a_n,\hat{\omega}})\|_{L^2(\Sigma,g)} \to 0 \quad \text{as } n \to \infty,$$

where we work on a fixed fundamental domain of Σ and use Euclidean, respectively hyperbolic, translations to the origin to get a consistent choice of coordinates F_{a_n} in the definition of the adapted bubbles. For sufficiently large n we can thus apply Theorem 2.5. This immediately yields the claim (1.6) on the energy $E(u_n)$. It also implies that the bubble scale $\tilde{\lambda}_n$ of elements $z_n = z_{\tilde{\lambda}_n}^{\tilde{a}_n,\tilde{\omega}_n} \in \mathcal{Z}$ which minimise $\tilde{z} \mapsto \|u_n - \tilde{z}\|_z$ is controlled by (2.16). As Lemma 4.1 implies that the originally chosen λ_n are so that $\lambda_n \in [\frac{1}{2}\tilde{\lambda}_n, 2\tilde{\lambda}_n]$ we also get the same bound on λ_n and for the rest of the proof we can assume that $\lambda_n = \tilde{\lambda}_n$ and $a_n = \tilde{a}_n$.

If $\widetilde{\omega}_n$ is harmonic, which will always be the case in the integrable setting, we can simply set $\omega_n = \widetilde{\omega}_n$. Otherwise we use that the classical Łojasiewicz–Simon inequality (1.4) implies that there exists a harmonic map $\omega_n \colon S^2 \to S^2$ which is C^k close to $\hat{\omega}$ so that

$$\|\omega_n - \widetilde{\omega}_n\|_{L^2(S^2)} \le C\|\tau(\widetilde{\omega}_n)\|_{L^2(S^2)}^{\gamma_2} \le C\mathcal{T}_n^{\gamma_2}|\log \mathcal{T}_n|^{\frac{\gamma_2}{2}}. \tag{5.1}$$

We note that the same type of estimate also holds for $\|\omega_n - \widetilde{\omega}_n\|_{C^1(S^2)}$ since both ω_n and $\widetilde{\omega}_n$ are elements of $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$. We can also use that

$$\|\widetilde{\omega}_n \circ \pi_{\lambda_n} \circ F_{a_n} - z_n\|_{C^1(B_\iota(a_n))} + \|z_n - \widetilde{\omega}_n(p^*)\|_{C^1(\Sigma \setminus B_\iota(a_n))} \lesssim \lambda_n^{-1};$$

compare (2.9) and (2.10). Combining this with (2.14) and the already-established bound (1.5) on the bubble scale, we get that, for $r_1 < \iota$ and sufficiently large n,

$$\|\nabla(u_n - \omega_n \circ \pi_{\lambda_n} \circ F_{a_n})\|_{L^2(B_{r_1}(a))} + \|\nabla u_n\|_{L^2(\Sigma \setminus B_{r_1}(a))}$$
$$\le C\|\omega_n - \widetilde{\omega}_n\|_{C^1(S^2)} + C\lambda_n^{-1} + \|u_n - z_n\|_{z_n} \le C\mathcal{T}_n^{\gamma_2}|\log \mathcal{T}_n|^{\frac{\gamma_2}{2}}$$

for the same exponent $\gamma_2 \in (0,1]$ for which (1.4) holds, as claimed in (1.7).

To establish the L^2-estimate (1.8) we note that

$$\|z_n - \widetilde{\omega}_n(p^*)\|_{L^2(\Sigma)} \lesssim \lambda_n^{-1} + \|(1+\lambda_n|x|)^{-1}\|_{L^2(\mathbb{D}_{r_0})} \lesssim \lambda_n^{-1}(\log \lambda_n)^{\frac{1}{2}} \lesssim \mathcal{T}_n(\log \mathcal{T}_n);$$

compare (3.17) and (1.5). Combined with (5.1) this gives

$$\|z_n - \omega_n(p^*)\|_{L^2(\Sigma)} \lesssim \mathcal{T}_n|\log \mathcal{T}_n| + \|\omega_n - \widetilde{\omega}_n\|_{C^0} \lesssim \mathcal{T}_n|\log \mathcal{T}_n| + \mathcal{T}_n^{\gamma_2}|\log \mathcal{T}_n|^{\frac{\gamma_2}{2}}.$$

Finally, (1.9) follows from (2.15) and (5.1) since we have a lower bound of $\rho_{\lambda_n} \geq c_\Lambda \lambda_n$, $c_\Lambda > 0$, on discs $\mathbb{D}_{\Lambda \tilde{\lambda}_n^{-1}}$. ∎

Proof of Corollary 1.3. Let N be an analytic manifold of any dimension, let (Σ, g) be a closed surface of genus at least 1 and suppose that there exists an accumulation point $\bar{E} < E^* = \min(E^*_{S^2}, 2E_{S^2}, E_{(\Sigma,g)} + E_{S^2})$ of the energy spectrum. Thus there are harmonic maps $u_i\colon \Sigma \to N$ with $E(u_i) \neq E(u_j)$ for $i \neq j$ and $E(u_i) \to \bar{E}$. We note that the maps u_i cannot subconverge smoothly to a harmonic map $u_\infty\colon \Sigma \to N$ as Simon's Łojasiewicz estimate ensures that all harmonic maps in a neighbourhood of u_∞ have the same energy. Thus the sequence must undergo bubbling: As each bubble requires energy of at least E_{S^2} and as $\bar{E} < 2E_{S^2}$ the corresponding bubble tree cannot contain multiple bubbles. As $\bar{E}$ is also less than $E_{S^2} + E_{(\Sigma,g)}$, the base map must furthermore be trivial. Finally, the assumption that $\bar{E} < E^*_{S^2}$ ensures that the bubble $\widehat{\omega}$ is not branched. We are hence in the setting of Theorem 1.1 and the resulting estimate (1.6) implies that $E(u_i) = E(\omega)$ for sufficiently large i leading to a contradiction. ∎

6. Convergence of harmonic map flow

Proof of Theorem 1.4. Let u be a solution of the harmonic map flow (1.10) as considered in Theorem 1.4. If there is any sequence $t_n \to \infty$ along which the flow converges strongly in H^1 to a (potentially trivial) harmonic map $u_\infty\colon \Sigma \to N$, then Simon's results from [18] imply that the flow converges indeed along all $t \to \infty$ to u_∞.

We can thus assume that for every sequence $t_n \to \infty$ with $\|\tau_g(u(t_n))\|_{L^2} \to 0$, a subsequence of $(u(t_n))$ converges to a non-trivial bubble tree. As the flow is not constant, and thus $E(u(t)) < E(u(0)) \leq E^*$, we can argue as in the proof of Corollary 1.3 to conclude that these bubble trees, which might depend on the chosen subsequence, are all simple and that the obtained bubbles ω are all unbranched and have energy $E(\omega) = E_\infty := \lim_{t\to\infty} E(t)$.

It is convenient to choose the rescalings in this convergence to a bubble tree to be around centres $a(t)$ and at scales $\lambda(t)$ which are chosen so that

$$E(u(t), F_{a(t)}^{-1}(\mathbb{D}_{\lambda(t)^{-1}})) = \sup_{a\in\Sigma} E\big(u(t), F_a^{-1}(\mathbb{D}_{\lambda(t)^{-1}})\big) = \frac{1}{2}E_{S^2},$$

as this ensures that the obtained bubbles are contained in a compact subset $K \subset H^2(\Sigma, N)$ of harmonic spheres: indeed, the upper bound of $\frac{1}{2}E_{S^2}$ on the energy of the maps

$u(t_n) \circ (\pi_{\lambda(t_n)} \circ F_{a(t_n)})^{-1}\colon S^2 \to N$ on balls with fixed radius gives such an upper bound for the bubbles also, which in turn makes it impossible for a sequence of such bubbles ω_n to undergo bubbling itself.

As Theorem 2.5 is applicable on a suitable $H^1 \cap L^\infty$-neighbourhood of each $\hat{\omega} \in K$, we can consider a finite cover of K by such neighbourhoods to deduce that there exist $\varepsilon, \bar{\lambda}, C > 0$ and $\gamma_1 > 1$ so that the Łojasiewicz estimate

$$|E(u) - E_\infty| \leq C \|\tau_g(u)\|_{L^2(\Sigma,g)}^{\gamma_1} (1 + |\log\|\tau_g(u)\|_{L^2(\Sigma,g)}|)^{\frac{\gamma_1}{2}} \tag{6.1}$$

holds true for every $u \in H^1(\Sigma, N)$ for which there exist $\hat{\omega} \in K$, $a \in \Sigma$ and $\lambda \geq \bar{\lambda}$ with

$$\|u - z_\lambda^{a,\hat{\omega}}\|_{H^1(\Sigma,g)} + \|u - z_\lambda^{a,\hat{\omega}}\|_{L^\infty(\Sigma,g)} < \varepsilon. \tag{6.2}$$

We note that there exist $\delta_0 > 0$ and $T \geq 0$ so that (6.2), and hence (6.1), hold true for all $u(t)$ with $t \geq T$ and $\|\tau_g(u(t))\|_{L^2} < \delta_0$; indeed, otherwise there would be $t_n \to \infty$ with $\|\tau(u(t_n))\|_{L^2} \to 0$ for which $(u(t_n))$ does not have a subsequence converging to a simple bubble tree.

As (6.1) is trivially true if $\|\tau_g(u(t))\|_{L^2} \geq \delta_0$ (after increasing C if necessary) and as we can assume that $E(T) - E_\infty \leq \frac{1}{2}$, we thus conclude that $E_d(t) := E(u(t)) - E_\infty$ satisfies

$$0 \leq E_d(t)^{\frac{2}{\gamma_1}} |\log E_d(t)|^{-1} \leq C_0 \|\tau_g(u(t))\|_{L^2(\Sigma)}^2 \tag{6.3}$$

for $t \geq T$ and some $C_0 > 0$, and thus

$$-\frac{d}{dt} E_d(t) = \|\tau_g(u(t))\|_{L^2(\Sigma)}^2 \geq C_0^{-1} E_d(t)^{\frac{2}{\gamma_1}} |\log E_d(t)|^{-1}. \tag{6.4}$$

We can now proceed as in [18] and [21] to establish the claimed convergence of the flow.

If $\gamma_1 = 2$ then (6.4) implies that

$$(\log E_d(t))^2 \geq 2C_0^{-1}(t - T) + (\log E_d(T))^2 \quad \text{for } t \geq T,$$

which allows us to conclude that $E(t) - E_\infty \leq Ce^{-c_1\sqrt{t}}$, as claimed in (1.12).

If $\gamma_1 \in (1, 2)$ then the above estimate implies that $\psi := E_d^{-\frac{2-\gamma_1}{\gamma_1}}$ satisfies

$$\frac{d}{dt}\psi = \frac{2-\gamma_1}{\gamma_1} E_d^{-\frac{2}{\gamma_1}} \|\tau_g(u(t))\|_{L^2(\Sigma)}^2 \geq \frac{2-\gamma_1}{C_0\gamma_1} |\log E_d|^{-1} \geq c(\log\psi)^{-1},$$

so we conclude that $\psi(t)(\log\psi(t) - 1) \geq c(t - T) - \psi(T)(\log\psi(T) - 1)$. The resulting bound of $\psi(t) \geq \tilde{c}t(\log t)^{-1}$ for some $\tilde{c} > 0$ then gives the claimed bound (1.13) on the decay of the energy.

Given any $0 < \alpha < \frac{\gamma_1 - 1}{\gamma_1}$ we now fix $\beta \in (\alpha, \frac{\gamma_1 - 1}{\gamma_1})$ and note that (6.3) gives

$$E_d(t)^{\beta-1} \|\tau_g(u)\|_{L^2} \geq E_d(t)^{-(\frac{\gamma_1-1}{\gamma_1} - \beta)} |\log E_d(t)|^{-\frac{1}{2}} \geq 1$$

for sufficiently large t. We hence obtain

$$-\frac{d}{dt}E_d(t)^\beta = \beta E_d{}^{\beta-1}\|\tau_g(u)\|^2_{L^2(\Sigma)} \geq \beta\|\tau_g(u)\|_{L^2(\Sigma)},$$

allowing us to conclude that for sufficiently large $t < \tilde{t}$,

$$\|u(t) - u(\tilde{t})\|_{L^2} \leq \int_t^{\tilde{t}} \|\tau_g(u(s))\|_{L^2}\, ds \leq C E_d(t)^\beta. \tag{6.5}$$

We now fix a sequence $t_n \to \infty$ for which $u(t_n)$ converges to a simple bubble tree and denote by $a \in \Sigma$ the point at which the corresponding bubble ω forms.

Applying the above estimate for $\tilde{t} = t_n$ and using that $\|u(t_n) - \omega(p^*)\|_{L^2} \to 0$ we get

$$\|u(t) - \omega(p^*)\|_{L^2} \leq C E_d(t)^\beta \tag{6.6}$$

for all sufficiently large t, which in particular implies (1.14).

To show that the point a where the bubble forms is independent of the chosen sequence and that the maps converge in C^k away from a, we can now follow the argument of [21] and combine (6.4) with estimates on the evolution of the energy on fixed-size balls as proven in [21, Lemma 3.3] and the C^k control on regions with low energy obtained in [19].

To be more precise, [19, Lemma 3.10′] (see also [21, Lemma 3.2]) ensures that there exists $\varepsilon_1 = \varepsilon_1(N) > 0$ so that for any $\Omega \subset \Sigma$, $r \in (0, \mathrm{inj}(\Sigma, g))$ and $k \in \mathbb{N}$ there exists a constant C so that following holds true: for any solution u of (1.10) which satisfies

$$\sup_{(x,t)\in\Omega\times[t_0,\infty)} E(u(t), B_r(x)) < \varepsilon_1$$

we can bound $\|u(t)\|_{C^k(\Omega_{\frac{r}{2}})} \leq C$ for $t \geq t_0 + 1$ and $\Omega_{\frac{r}{2}} := \{x \in \Sigma : \mathrm{dist}(x, \Omega) \leq \frac{r}{2}\}$.

Now let Ω be a fixed compact subset of $\Sigma \setminus \{a\}$. As $u(t_n) \to \omega(p^*)$ strongly in $H^1_{\mathrm{loc}}(\Sigma \setminus \{a\})$ along the particular sequence of times $t_n \to \infty$ chosen above, we can choose $r > 0$ so that $\sup_{x\in\Omega} E(u(t_n), B_{2r}(x)) \leq \frac{1}{2}\varepsilon_1$ for all n. Then [21, Lemma 3.3] and (6.5) allow us to bound

$$E(u(t), B_r(x)) \leq E(u(t_n), B_{2r}(x)) + Cr^{-1}\int_{t_n}^t \|\tau(u(s))\|_{L^2}\, ds \leq \frac{1}{2}\varepsilon_1 + C E_d(t)^\beta \leq \varepsilon_1$$

for all $x \in \Omega$ and $t \geq t_n$ for sufficiently large n.

Similarly to the argument in [21] we then combine the resulting uniform bounds on $\|u(t)\|_{C^l(\Omega_{\frac{r}{2}})}$, $l \in \mathbb{N}$, from [19] mentioned above, with the L^2-convergence of the flow using an interpolation argument: to this end we recall the standard interpolation inequality

$$\|f\|_{H^s(\Omega)} \leq C\|f\|^{\frac{m_2}{m_1+m_2}}_{H^{s-m_1}(\Omega_{\frac{r}{2}})}\|f\|^{\frac{m_1}{m_1+m_2}}_{H^{s+m_2}(\Omega_{\frac{r}{2}})}, \quad C = C(\Omega, r, m_{1,2}, s),$$

which holds for all $m_{1,2} \in \mathbb{N}$ with $m_1 \leq s$ and follows inductively from integration by parts. Given any $k \in \mathbb{N}$ and any $\delta > 0$ we can apply this inequality for $m_1 = s = k + 2$ and $m_2 = m_2(k, \delta)$ sufficiently large to conclude that, for $l = k + 2 + m_2$,

$$\|f\|_{H^{k+2}(\Omega)} \leq C\|f\|_{L^2(\Omega_{\frac{r}{2}})}^{1-\delta}\|f\|_{H^l(\Omega_{\frac{r}{2}})}^{\delta} \leq C\|f\|_{L^2(\Sigma)}^{1-\delta}\|f\|_{C^l(\Omega_{\frac{r}{2}})}^{\delta}.$$

Choosing $\delta > 0$ so that $(1-\delta)\beta \geq \alpha$ and combining this with (6.6) and the uniform C^l bounds on $u(t)$ on $\Omega_{\frac{r}{2}}$ hence gives the final claim of the theorem that

$$\begin{aligned}\|u(t) - \omega(p^*)\|_{C^k(\Omega)} &\leq C\|u(t) - \omega(p^*)\|_{H^{k+2}(\Omega)} \leq C\|u(t) - \omega(p^*)\|_{L^2(\Sigma)}^{1-\delta} \\ &\leq C E_d(t)^{(1-\delta)\beta} \leq C E_d(t)^{\alpha}.\end{aligned}$$

∎

Proof of Corollary 1.6. As [5] excludes the existence of a harmonic map of degree ± 1 from a torus (T^2, g) to S^2, any solution $u(t)$ of harmonic map flow as considered in the corollary will be non-constant, and so have $E(u(t)) < 12\pi$ for $t > 0$, and will need to become singular. Indeed, we claim that the constraint on the energy means that a single bubble must form, be it at finite or infinite time, and that this bubble must have the same degree as the original map. Indeed, the formation of either two bubbles with degree ± 1 or of a bubble of higher degree would only leave energy less than 4π, but at the same time would leave us with a limiting body map of a non-zero degree which is impossible. Similarly, the formation of more than two bubbles or of two bubbles which do not both have degree ± 1 would require initial energy greater than 12π and so is also excluded. Finally, if the degree of the bubble and the degree of the map did not agree then we would end up with a body map of degree k with $|k| \geq 2$, which would need energy at least 8π, leading again to a contradiction.

If the bubble forms at finite time then Simon's result [18] yields exponential convergence to a constant. Conversely, if the singularity forms at infinite time then the proof of Theorem 1.4 applies and yields convergence at a rate of $O(e^{-c\sqrt{t}})$ since all Jacobi fields along harmonic maps from S^2 to S^2 are integrable; see [8]. ∎

A. Definition of $\mathcal{H}_0(\hat{\omega})$ based on Simon's construction from [18]

Here we recall the key elements of Simon's argument that we need to define the manifold $\mathcal{H}_0(\hat{\omega})$ and to check that it has the properties stated in Lemma 2.1.

Let $\hat{\omega}\colon S^2 \to S^2$ be any harmonic sphere and let $L := L_{\hat{\omega}}$ be the Jacobi operator along $\hat{\omega}$ defined in (2.1), where here and in the following we work with the L^2 inner product and consider L as an operator on maps $w\colon S^2 \to \mathbb{R}^N$ which are tangential to N along $\hat{\omega}$.

The starting point of Simon's argument is that since $L := L_{\hat{\omega}}$ is a self-adjoint Fredholm operator, the linear equation $Lu = f$ has a unique solution $u \in \ker(L)^\perp$ if and only if $f \in \ker(L)^\perp$ and this solution furthermore satisfies $\|u\|_{C^{k+2,\beta}} \leq C_{k,\beta}\|f\|_{C^{k,\beta}}$ for any $k \in \mathbb{N}, \beta > 0$.

As explained in [18], these properties of L ensure that $\mathcal{N}: C^{k+2,\beta} \to C^{k,\beta}$ defined by

$$\mathcal{N}(w) := P^{\ker L}(w) + P_{\hat{\omega}}\big(\tau_{g_{S^2}}(\pi_N(\hat{\omega} + w))\big)$$

is so that the inverse function theorem yields a map $\Psi: \mathcal{U}_1 \subset C^{k,\beta} \to \mathcal{U}_2 \subset C^{k+2,\beta}$ between suitable neighbourhoods $\mathcal{U}_{1,2}$ of 0 in $\Gamma^{C^{k,\beta}}(\hat{\omega}^*TN)$ and $\Gamma^{C^{k+2,\beta}}(\hat{\omega}^*TN)$ so that $\|\Psi(f) - \Psi(g)\|_{C^{k+2,\beta}} \lesssim \|f - g\|_{C^{k,\beta}}$ and

$$\mathcal{N} \circ \Psi = \mathrm{Id}_{\mathcal{U}_1} \quad \text{and} \quad \Psi \circ \mathcal{N} = \mathrm{Id}_{\mathcal{U}_2}.$$

As $P^{\ker(L)^\perp}(\mathcal{N}(v)) = P^{\ker(L)^\perp}(P_{\hat{\omega}}(\tau_{g_{S^2}}(\pi_N(\hat{\omega} + v))))$ we have

$$P^{\ker(L)^\perp}\Big(P_{\hat{\omega}}\big(\tau_{g_{S^2}}\big(\pi_N(\hat{\omega} + \Psi(w))\big)\big)\Big) = P^{\ker(L)^\perp}(\mathcal{N} \circ \Psi(w)) = P^{\ker(L)^\perp} w.$$

This not only implies that the finite-dimensional manifold

$$\mathcal{H}(\hat{\omega}) := \big\{\pi_N(\hat{\omega} + \Psi(w)) : w \in \ker L \cap \mathcal{U}_1\big\}$$

contains all harmonic maps which are sufficiently close to $\hat{\omega}$, but also that

$$P_{\hat{\omega}}(\tau_{g_{S^2}}(\omega)) \in \ker(L) \quad \text{for every } \omega \in \mathcal{H}(\hat{\omega}).$$

From the equivalence of norms on the finite-dimensional space $\ker(L)$ and the fact that $\|\omega - \hat{\omega}\|_{C^{k+2,\beta}}$ will be small if we work on suitably small neighbourhoods $\mathcal{U}_{1,2}$, we hence get that, for $\omega \in \mathcal{H}(\hat{\omega})$,

$$\|\tau_{g_{S^2}}(\omega)\|_{C^k} \le C\|P_{\hat{\omega}}\tau_{g_{S^2}}(\omega)\|_{C^k} \le C\|P_{\hat{\omega}}\tau_{g_{S^2}}(\omega)\|_{L^2} \le C\|\tau_{g_{S^2}}(\omega)\|_{L^2}. \tag{A.1}$$

Now let $\mathcal{V}_0(\hat{\omega}) \subset \ker(L)$ be so that

$$\ker(L) = \mathcal{V}_0(\hat{\omega}) \oplus \mathcal{V}_{\text{Möb}}(\hat{\omega}) \quad \text{for } \mathcal{V}_{\text{Möb}}(\hat{\omega}) := T_{\hat{\omega}}\big\{\hat{\omega} \circ M : M \in \text{Möb}(S^2)\big\}$$

splits L^2-orthogonally and set

$$\mathcal{H}_0(\hat{\omega}) := \big\{\pi_N(\hat{\omega} + \Psi(w)) : w \in \mathcal{V}_0(\hat{\omega}) \cap \mathcal{U}_1\big\}.$$

This codimension 6 submanifold of $\mathcal{H}(\hat{\omega}) \subset C^{k+2,\beta}$ clearly satisfies (2.2), while (2.3) follows from (A.1).

Furthermore, for any $\omega = \pi_N(\hat{\omega} + \Psi(w_\omega)) \in \mathcal{H}_0(\hat{\omega})$ with $\tau_{g_{S^2}}(\omega) \neq 0$, we can split

$$P_{\hat{\omega}}(\tau_{g_{S^2}}(\omega)) = P^{\mathcal{V}_0(\hat{\omega})}(P_{\hat{\omega}}\tau_{g_{S^2}}(\omega)) + P^{\mathcal{V}_{\text{Möb}}(\hat{\omega})}(P_{\hat{\omega}}\tau_{g_{S^2}}(\omega)) \in \ker(L),$$

set $\mathcal{T} := \|\tau_{g_{S^2}}(\omega)\|_{L^2}$ and consider (for ε near 0)

$$\omega^{(\varepsilon)} := \pi_N\Big(\hat{\omega} + \Psi\Big(w_\omega - \frac{2\varepsilon}{\mathcal{T}} P^{\mathcal{V}_0(\hat{\omega})}(P_{\hat{\omega}}\tau_{g_{S^2}}(\omega))\Big)\Big) \in \mathcal{H}_0(\hat{\omega}).$$

The equivalence of norms on $\ker(L)$ implies that at $\varepsilon = 0$,

$$\begin{aligned}\|\partial_\varepsilon \omega^{(\varepsilon)}\|_{C^{k+2,\beta}} &\le C\mathcal{T}^{-1}\|P^{\mathcal{V}_0(\hat\omega)}(P_{\hat\omega}\tau_{g_{S^2}}(\omega))\|_{C^{k,\beta}} \\ &\le C\mathcal{T}^{-1}\|P^{\mathcal{V}_0(\hat\omega)}(P_{\hat\omega}\tau_{g_{S^2}}(\omega))\|_{L^2} \le C\end{aligned}$$

for a constant C that only depends on $\hat\omega$ and k.

The conformal invariance of the energy ensures that $\tau_{g_{S^2}}(\omega)$ is L^2-orthogonal to $\mathcal{V}_{\text{Möb}}(\omega) = T_\omega\{\omega \circ M,\ M \in \text{Möb}(S^2)\}$. As $d\Psi(0)|_{\ker(L)} = \text{Id}$ we can thus bound

$$\begin{aligned}&\Big\|\frac{2}{\mathcal{T}}\tau_{g_{S^2}}(\omega) + \partial_\varepsilon\omega^{(\varepsilon)}\Big\|_{L^2} \\ &\quad = \frac{2}{\mathcal{T}}\big\|P^{\mathcal{V}_0(\omega)}\tau_{g_{S^2}}(\omega) - d\pi_N(\omega)\big(d\Psi(w_\omega)\big(P^{\mathcal{V}_0(\hat\omega)}(P_{\hat\omega}\tau_{g_{S^2}}(\omega))\big)\big)\big\|_{L^2} \\ &\quad \le C(\|\hat\omega - \omega\|_{C^1} + \|w_\omega\|_{C^{k,\beta}}) \le 1,\end{aligned}$$

provided the neighbourhoods $\mathcal{U}_{1,2}$ are chosen sufficiently small.

As a consequence we obtain

$$\frac{d}{d\varepsilon}E(\omega^{(\varepsilon)}) = -\langle\tau_{g_{S^2}}(\omega), \partial_\varepsilon\omega^{(\varepsilon)}\rangle_{L^2} \ge \frac{2}{\mathcal{T}}\|\tau_{g_{S^2}}(\omega)\|_{L^2}^2 - \|\tau_{g_{S^2}}(\omega)\|_{L^2} = \|\tau_{g_{S^2}}(\omega)\|_{L^2},$$

which establishes the final property of the manifold $\mathcal{H}_0(\hat\omega)$ claimed in Lemma 2.1.

B. Proofs of technical lemmas

In this appendix we give the proofs of the auxiliary Lemma 3.4 used in the proof of Lemma 3.2, of the auxiliary Lemma 4.1 and of the estimate (4.3) used in the proof of Theorem 2.5 and the estimate (2.12) that we used throughout the paper.

To prove Lemma 4.1 we first show the analogous statement for the maps $\hat z_\lambda^{b,\omega} := \omega \circ M_\lambda^b : S^2 \to N$, where $M_\lambda^b(x) = \pi_\lambda(\pi^{-1}(\cdot) - b)$.

Lemma B.1. *Given any harmonic sphere $\hat\omega$ and any $\sigma_1 > 0$, there exists $\varepsilon_3 > 0$ so that*

$$\|\hat z_\lambda^{b,\omega} - \hat z_{\tilde\lambda}^{\tilde b,\tilde\omega}\|_{\lambda,b} \ge 2\varepsilon_3 \tag{B.1}$$

whenever $\omega, \tilde\omega \in \mathcal{H}_1^{\sigma_1}(\hat\omega)$ and $\lambda, \tilde\lambda > 0$ are either so that $\lambda^{-1}\tilde\lambda \notin [\frac{1}{2}, 2]$ or so that $\omega \in \mathcal{H}_1^{\frac{1}{3}\sigma_1}(\hat\omega)$ while $\tilde\omega \in \mathcal{H}_1^{\sigma_1}(\hat\omega) \setminus \mathcal{H}_1^{\frac{2}{3}\sigma_1}(\hat\omega)$.

Furthermore, given any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\|\hat z_\lambda^{b,\omega} - \hat z_{\tilde\lambda}^{\tilde b,\tilde\omega}\|_{L^\infty(S^2)} < \frac{1}{4}\varepsilon \quad \textit{whenever } \|\hat z_\lambda^{b,\omega} - \hat z_{\tilde\lambda}^{\tilde b,\tilde\omega}\|_{\lambda,b} < 2\delta. \tag{B.2}$$

Here we consider the norms on $H^1(S^2, \mathbb{R}^n)$ defined by

$$\|v\|_{\lambda,b} := \int_{S^2} |\nabla v|^2 + \frac{1}{2}c_\Sigma|\nabla M_\lambda^b|^2|v|^2 \, dv_{g_S^2}$$

for c_γ as in Remark 3.3 and use that these norms satisfy

$$\langle v \circ M_\lambda^b, \tilde v \circ M_\lambda^b\rangle_{\lambda,b} = \langle v, \tilde v\rangle_{1,0} \quad v, \tilde v \in H^1(S^2, \mathbb{R}^n). \tag{B.3}$$

Proof of Lemma B.1. Thanks to (B.3) it is enough to consider the case where $\lambda = 1$ and $b = 0$, so $\hat z_\lambda^{b,\omega} = \omega \in \mathcal{H}_1^{\sigma_1}(\hat\omega)$. We can then use that the closure of

$$F_1 := \{\hat z_{\tilde\lambda}^{\tilde b,\tilde\omega} : \tilde\omega \notin \mathcal{H}_1^{\frac{2}{3}\sigma_1}\}, \quad \text{respectively } F_2 := \{\hat z_{\tilde\lambda}^{\tilde b,\tilde\omega} : \tilde\lambda \notin [\tfrac12, 2]\}$$

in H^1 is disjoint from the compact sets $\mathcal{H}_1^{\frac13\sigma_1}$ respectively $\mathcal{H}_1^{\sigma_1}$. Thus the H^1-distance between F_1 and $\mathcal{H}_1^{\frac13\sigma_1}$ and between F_2 and $\mathcal{H}_1^{\sigma_1}$ is positive and this yields the first claim of the lemma.

As we can assume that $\delta < \varepsilon_3$ it then suffices to prove the second claim for maps $z_{\tilde\lambda}^{\tilde a,\tilde\lambda}$ with $\tilde\lambda \in [\frac12, 2]$. As such maps satisfy uniform C^2 bounds, we obtain (B.2) from Ehrling's lemma applied to $C^2(S^2, g_{S^2}) \subset\subset L^\infty(S^2, g_{S^2}) \hookrightarrow L^2(S^2, g_{S^2})$. ■

Proof of Lemma 4.1. As we may assume that $\varepsilon_3 < \min_{\omega\in\mathcal{H}_1^{\sigma_1}(\hat\omega)}\|\nabla\omega\|_{L^2(S^2)}$, as well as that λ_2 is sufficiently large, we have that (4.1) is trivially true if either $\tilde\lambda \notin [C^{-1}\lambda, C\lambda]$ or $d(a,\tilde a) \geq C\lambda^{-1}$ for a suitably large constant C.

In particular, we can assume that $d_\Sigma(a,\tilde a) \leq C\lambda^{-1} < \frac18\iota$, which ensures that the derivatives of the adapted bubbles $z_\lambda^{a,\omega}$, $z_{\tilde\lambda}^{\tilde a,\tilde\omega}$ respectively of the corresponding harmonic spheres $\hat z_\lambda^{0,\omega}$ and $\hat z_{\tilde\lambda}^{F_a(\tilde a),\omega}$ are of order $O(\lambda^{-1})$ outside $B_{\bar\iota}(a)$, respectively $\pi(\mathbb{D}_{r_0/2}) \subset S^2$. As the functions representing $z_\lambda^{a,\omega}$, $z_{\tilde\lambda}^{\tilde a,\tilde\omega}$ in the isothermal coordinates on $B_{\bar\iota}(a)$ agree up to H^1-errors of order $O(\lambda^{-1})$ with the functions representing $\hat z_\lambda^{0,\omega}$ and $\hat z_{\tilde\lambda}^{F_a(\tilde a),\omega}$ in stereographic coordinates, we thus have

$$\|\nabla z_\lambda^{a,\omega} - \nabla z_{\tilde\lambda}^{\tilde a,\tilde\omega}\|_{L^2(\Sigma)} = \|\nabla \hat z_\lambda^{0,\omega} - \nabla \hat z_{\tilde\lambda}^{F_a(\tilde a),\tilde\omega}\|_{L^2(S^2)} + O(\lambda^{-1}).$$

For the torus, we immediately get the same type of relationship for the weighted L^2-norms as well, while for higher genus surfaces we need to take into account an additional error term that results from the difference of the weights, which will be of order $O(\lambda^{-1}\log(\lambda)^{1/2})$ since $\int \rho_\lambda^2|x|^2\,dx = O(\lambda^{-2}\log(\lambda))$; compare Remark 3.3. In both cases we hence get

$$\|z_\lambda^{a,\omega} - z_{\tilde\lambda}^{\tilde a,\tilde\omega}\|_{z_\lambda^{a,\omega}} = \|\hat z_\lambda^{0,\omega} - \hat z_{\tilde\lambda}^{F_a(\tilde a),\tilde\omega}\|_{\lambda,0} + O(\lambda^{-1}\log(\lambda)^{\frac12})$$

and the first claim (4.1) of Lemma 4.1 follows from the corresponding estimates (B.1) of Lemma B.1.

Similarly, given $\varepsilon > 0$ and choosing $\delta > 0$ small enough so that (B.2) holds, it suffices to ensure that $\lambda_3(\log\lambda_3)^{-1/2} \geq C(\min(\varepsilon,\delta))^{-1}$ for a sufficiently large C to derive the required L^∞-bound (4.2) on the difference of the adapted bubbles from the corresponding property (B.2) of the bubbles stated in Lemma B.1. ■

To prove Lemma 3.4 as well as (4.3) we furthermore show the following lemma:

Lemma B.2. *There exists $C > 1$ so that for all smooth 1-parameter families $(b_\varepsilon) \subset \mathbb{R}^2$, $(a_\varepsilon) \subset \Sigma$, $(\lambda_\varepsilon) \subset [\lambda_1, \infty)$ and $(\omega^{(\varepsilon)}) \subset \mathcal{H}_1^{\sigma_1}(S^2)$ we have that $\hat{z}_\varepsilon := \hat{z}_{\lambda_\varepsilon}^{b_\varepsilon,\omega^{(\varepsilon)}}$ satisfies*

$$C^{-1}\|\partial_\varepsilon \hat{z}_\varepsilon\|_{\lambda_\varepsilon,b_\varepsilon} \leq \lambda_\varepsilon^{-1}|\partial_\varepsilon\lambda_\varepsilon| + \|\partial_\varepsilon\omega^{(\varepsilon)}\|_{C^2} + \lambda_\varepsilon|\partial_\varepsilon a_\varepsilon| \leq C\|\partial_\varepsilon\hat{z}_\varepsilon\|_{\lambda_\varepsilon,b_\varepsilon}, \tag{B.4}$$

while the adapted bubbles $z_\varepsilon = z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}} \in \mathcal{Z}$ satisfy

$$C^{-1}\|\partial_\varepsilon z_\varepsilon\|_{z_\varepsilon} \leq \lambda_\varepsilon^{-1}|\partial_\varepsilon\lambda_\varepsilon| + \|\partial_\varepsilon\omega^{(\varepsilon)}\|_{C^2} + \lambda_\varepsilon|\partial_\varepsilon a_\varepsilon| \leq C\|\partial_\varepsilon z_\varepsilon\|_{z_\varepsilon} \tag{B.5}$$

and

$$\|\partial_\varepsilon\rho_{z_\varepsilon}\|_{L^2(\Sigma)} \leq C\lambda_\varepsilon^{-1}|\partial_\varepsilon\lambda_\varepsilon| + C\lambda_\varepsilon|\partial_\varepsilon a_\varepsilon| \leq C\|\partial_\varepsilon z_\varepsilon\|_{z_\varepsilon}. \tag{B.6}$$

Proof. For variations of $\omega = \hat{z}_1^{0,\omega} \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$, the estimate (B.4) easily follows as $\mathcal{H}_1^{\sigma_1}(\hat{\omega})$ is a compact subset of a finite-dimensional submanifold of $C^k(S^2, N)$ which is transversal to the action of the Möbius transforms. We can then consider $\hat{z}_{\lambda_0^{-1}\lambda_\varepsilon}^{\lambda_0(b-b_0),\omega^{(\varepsilon)}}$ and use (B.3) to reduce the proof of (B.4) to this special case.

To derive (B.5) from (B.4) it then suffices to check that we only obtain error terms of lower order when we use the approximations $z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}} \approx 0$ on $\Sigma \setminus B_{\tilde{\iota}}(a_\varepsilon)$ and

$$z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}}(F_{a_0}^{-1}(x)) = \pi_N[\widetilde{\omega}_{\lambda_\varepsilon}^{(\varepsilon)}(F_{a_\varepsilon,a_0}(x)) + j_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}}(F_{a_\varepsilon,a_0}(x))] \approx \widetilde{\omega}_{\lambda_\varepsilon}^{(\varepsilon)}(x - b_\varepsilon) \quad \text{on } \mathbb{D}_{\frac{r_0}{2}},$$

where $F_{a_\varepsilon,a_0} := F_{a_\varepsilon} \circ F_{a_0}^{-1}$ and $b_\varepsilon := -F_{a_\varepsilon,a_0}(0) = -F_{a_\varepsilon}(a_0)$.

A short calculation shows that at $\varepsilon = 0$ we indeed have

$$\begin{aligned}
&\|\partial_\varepsilon\hat{z}_{\lambda_\varepsilon}^{b_\varepsilon,\omega^{(\varepsilon)}}\|_{\lambda_0,0,S^2\setminus\pi(\mathbb{D}_{\frac{r_0}{2}})} + \|\partial_\varepsilon[z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}} \circ F_{a_0}^{-1} - \widetilde{\omega}_{\lambda_\varepsilon}^{(\varepsilon)}(\cdot - b_\varepsilon)]\|_{\lambda_0,\mathbb{D}_{r_0}} \\
&\qquad + \|\partial_\varepsilon z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}}\|_{z,\Sigma\setminus F_{a_0}^{-1}(\mathbb{D}_{r_0})} \\
&\quad \leq C\lambda^{-1}(\lambda^{-1}|\partial_\varepsilon\lambda_\varepsilon| + \lambda|\partial_\varepsilon a_\varepsilon| + \|\partial_\varepsilon\omega_\varepsilon\|_{C^2(S^2)}),
\end{aligned}$$

so (B.5) follows from (B.4) after replacing C by $2C$ and after possibly increasing λ_1.

We finally recall that the weight does not depend on the underlying map $\omega^{(\varepsilon)} \in \mathcal{H}_1^{\sigma_1}(\hat{\omega})$ and is given by $\rho_{z_\lambda^{a,\omega}}(p) = \frac{\lambda}{1+\lambda^2|F_a(p)|^2}$ on $B_\iota(a)$, while $\rho_{z_\lambda^{a,\omega}} \equiv \frac{\lambda}{1+\lambda^2 r_0^2}$ elsewhere. The final claim (B.6) of the lemma thus follows from (B.5), as well as

$$\|\partial_\lambda\rho_{z_\lambda^{a,\omega}}\|_{L^2(\Sigma)}^2 = \int_{\mathbb{D}_{r_0}} \left|\partial_\lambda\Big(\frac{\lambda}{1+\lambda^2|x|^2}\Big)\right|^2 \mathrm{d}x + O(\lambda^{-4}) \leq C\lambda^{-2}$$

and

$$\begin{aligned}
\|\partial_\varepsilon\rho_{z_\lambda^{a_\varepsilon,\omega}}\|_{L^2(\Sigma)}^2 &= \int_{B_\iota(a)} \left|\partial_\varepsilon\Big(\frac{\lambda}{1+\lambda^2|F_{a_\varepsilon}(p)|^2}\Big)\right|^2 \mathrm{d}v_g \\
&\leq C\lambda^6\|\partial_\varepsilon F_{a_\varepsilon}\|_{L^\infty(B_\iota)}^2 \int_{B_\iota(a)} \frac{|F_a(p)|^2}{(1+\lambda^2|F_a(p)|^2)^4} \mathrm{d}v_g \leq C\lambda^2|\partial_\varepsilon a_\varepsilon|^2. \quad \blacksquare
\end{aligned}$$

To obtain Lemma 3.4 we first note that a short calculation shows that for variations with $\lambda^{-1}|\partial_\varepsilon\lambda| + \|\partial_\varepsilon\omega^{(\varepsilon)}\|_{C^2} + \lambda|\partial_\varepsilon a| = O(1)$ and for $\mu_\varepsilon = \lambda_0^{-1}\lambda_\varepsilon$, $b_\varepsilon = -F_{a_\varepsilon}(a_0)$ we have

$$\partial_\varepsilon z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}} \circ F_{a_0}^{-1} \circ \pi_{\lambda_0}^{-1} = \partial_\varepsilon \hat{z}_{\mu_\varepsilon}^{\lambda_0 b_\varepsilon,\omega^{(\varepsilon)}} + O(\lambda^{-1})$$

on the subsets $\pi_\lambda(\mathbb{D}_{\frac{r_0}{2}})$ which exhaust S^2 as $\lambda \to \infty$, while $\partial_\varepsilon(z_{\lambda_\varepsilon}^{a_\varepsilon,\omega^{(\varepsilon)}} - \omega^{(\varepsilon)}(p^*))$ is of order $O(\Lambda^{-1})$ in $H^1 \cap L^\infty(\Sigma \setminus B_{\Lambda\lambda^{-1}}(a_0))$.

Now let $\{e_j^\infty\}_{j=1}^K$ be an orthonormal basis of $X_{\hat{\omega}}$. For any fixed j we consider a variation $(b_\varepsilon, \mu_\varepsilon, \omega^{(\varepsilon)})$ of $(0, 1, \hat{\omega})$ so that $e_j^\infty = \partial_\varepsilon \hat{z}_{\lambda_\varepsilon}^{b_\varepsilon,\omega^{(\varepsilon)}}$ and, for i sufficiently large, corresponding variations a_i^ε with $\lambda_i \partial_\varepsilon b_\varepsilon = -\partial_\varepsilon F_{a_i^\varepsilon}(a_0)$, $\lambda_i^\varepsilon = \lambda_i \mu_\varepsilon$ and $\omega^{(\varepsilon,i)}$ in $\mathscr{H}_1^{\sigma_1}(\hat{\omega})$ so that $\partial_\varepsilon\omega^{(\varepsilon,i)} \to \partial_\varepsilon\omega$ in $C^1(S^2)$.

The above error estimates ensure that the resulting elements $\tilde{e}_j^i = \frac{d}{d\varepsilon}|_{\varepsilon=0} z_{\lambda_\varepsilon^i}^{a_\varepsilon^i,\omega^{(i,\varepsilon)}}$ of $T_{z_i}\mathcal{Z}$ converge to e_j^∞ in the sense described in the lemma. As we furthermore have that $\langle \tilde{e}_j^i, \tilde{e}_k^i \rangle_{z_i} = \delta_{jk} + o(1)$, we can hence obtain the desired orthonormal basis from Gram–Schmidt orthogonalisation.

For the sake of completeness we finally include the following proof:

Proof of (2.12). We can assume that $2\lambda^{-1} \leq r_0$ as the claim is trivially true for λ in a bounded range as $\rho_z \geq c\lambda^{-1}$, $c = c(\Sigma, g) > 0$.

As $\rho_\lambda^{-2} dv_g \geq c\lambda^2 dv_{g_E}$ on $\mathbb{D}_{2\lambda^{-1}}$, $c = \frac{1}{25}$, we can bound, writing for short $\widetilde{w} = w \circ F_a$,

$$\|w\|_z^2 \geq c\lambda^2 \int_{\mathbb{D}_{2\lambda^{-1}} \setminus \mathbb{D}_{\lambda^{-1}}} |\widetilde{w}|^2 \, \mathrm{d}x \geq c\lambda \int_{\lambda^{-1}}^{2\lambda^{-1}} \int_{S^1} |\widetilde{w}(re^{i\theta})|^2 \, \mathrm{d}\theta \, \mathrm{d}r.$$

We can thus choose $r \in [\lambda^{-1}, 2\lambda^{-1}]$ so that $|\fint_{S^1} \widetilde{w}(re^{i\theta})| \leq C\|w\|_z$ and hence bound

$$\begin{aligned} \left| \fint_{S^1} \widetilde{w}(r_0 e^{i\theta}) \, \mathrm{d}\theta \right| &\leq C\|w\|_z + \fint_{S^1} \int_r^{r_0} |\partial_s \widetilde{w}(se^{i\theta})| \, \mathrm{d}s \, \mathrm{d}\theta \\ &\leq C\|w\|_z + C[\log(r_0) - \log(r)]^{\frac{1}{2}} \|\nabla w\|_{L^2(\Sigma)} \\ &\leq C(\log \lambda)^{\frac{1}{2}} \|w\|_z. \end{aligned}$$

As a standard compactness argument gives

$$\left| \fint_{\partial B_\iota(a)} w \, \mathrm{d}S_g - \fint_\Sigma w \, \mathrm{d}v_g \right| \leq C\|\nabla w\|_{L^2(\Sigma,g)}$$

for some $C = C(\Sigma, g)$, we hence obtain claim (2.12) from the above bound on

$$\fint_{\partial B_\iota(a)} w \, \mathrm{d}S_g = \fint_{S^1} \widetilde{w}(r_0 e^{i\theta}) \, \mathrm{d}\theta. \qquad \blacksquare$$

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