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Complex Variables Functions. – *The stars at infinity in several complex variables*, by Anders Karlsson, communicated on 19 April 2024.

Abstract. – This text reviews certain notions in metric geometry that may have further applications to problems in complex geometry and holomorphic dynamics in several variables. The discussion contains a few new results and formulates a number of questions related to the asymptotic geometry and boundary estimates of bounded complex domains, boundary extensions of biholomorphisms, the dynamics of holomorphic self-maps, Teichmüller theory, and the existence of constant scalar curvature metrics on compact Kähler manifolds.

Keywords. – Kobayashi pseudo-metric, Denjoy–Wolff theorems, Teichmüller spaces, Kähler geometry, random iterations, metric geometry.

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1. From complex to metric

Pick's striking reformulation of the Schwarz lemma in [55] started a rich development of metric methods in complex analysis. The Schwarz–Pick lemma states that every holomorphic self-map of the unit disk D does not increase distances in the Poincaré metric. Modifying a definition of Carathèodory, Kobayashi defined in the 1960s a largest pseudo-distance k_Z on each complex space Z so that every holomorphic map is nonexpansive in these distances. A pseudo-distance is a metric except that there may exist distinct points x and y such that $d(x,y) = 0$. Indeed, $k_{\mathbb{C}} \equiv 0$. On the other hand, k_D is the Poincaré metric and one sees that Liouville's theorem that bounded entire functions are constant immediately follows from these assertions. This is not a shorter proof of this theorem but places it into a different framework that also explains Picard's little theorem. Complex spaces where the Kobayashi pseudo-distance is an actual distance are called *Kobayashi hyperbolic* and this is via Lang's conjectures from the 1970s connected to the finiteness of the number of rational solutions to diophantine equations [47, 48].

Metric methods are thus rather old in the subject of several complex variables, but as seen for example in [6, 16, 18, 19, 31, 49, 59], the metric perspective has developed strongly also in recent years. The present paper tries to outline a few further possible directions mostly to do with boundaries and intrinsic structures on them. One classic topic in complex analysis is the question of the extension of holomorphic maps to the boundary of the domain, a problem for which also the Bergman metric has been used. In Riemannian geometry, boundary maps were considered in the 1960s by Mostow for the proofs of his landmark rigidity theorems. At an early stage his ideas did not attract much interest from the Lie group community; instead encouragement came from Ahlfors, the famous complex analyst [54]. Nowadays, since the influence of Gromov having Mostow's and Margulis' work as a starting point, boundaries at infinity and extensions of maps between them are a staple of geometric group theory. In completing the circle as it were, these metric ideas have since been applied to the above-mentioned complex analytic question [8, 21].

In another direction, an important problem in complex geometry is to understand when a given compact Kähler manifold admits a constant scalar curvature Kähler metric in the same cohomology class. This study started in the 1950s by Calabi who for this purpose introduced a flow on a certain space of metrics on the underlying manifold. In response to a conjecture of Donaldson, Streets proposed to study a related weak Calabi flow on the metric completion of Calabi's space equipped with a certain L^2-type metric. This flow is nonexpansive in this metric and therefore metric versions of the Denjoy–Wolff theorem could be relevant for this set of problems. This is discussed more in Section 5 and we refer to [11, 20, 23] for references on this topic. We thus see an example of an entirely different use of metric methods, more in the spirit of Teichmüller theory, for problems in several variable complex geometry.

The present paper reviews the stars from [38]. Given a metric space X with a compactification $\overline{X}$, we associate an extra structure of the boundary $\partial X := \overline{X} \setminus X$. This boundary structure consists of subsets called *stars*, which are limits of generalized halfspaces, and is an isometry invariant in case the isometries act naturally on the boundary. Everything is defined in terms of distances, without any knowledge about the existence of geodesics. The interest of these notions comes from that

- the stars measure the failure of Gromov hyperbolicity (e.g. Proposition 3.3), and this is useful when extending the theory of Gromov hyperbolic spaces to more general metric spaces (e.g. [38, section 4]);
- the stars provide a way of describing the asymptotic geometry in any given compactification, and boundary estimates can sometimes be translated into qualitative information (e.g. Theorem 3.6, [38, Sections 8 and 9], [28, 52]);
- the stars restrict the limit sets of nonexpansive maps (Theorem 4.3);
- a contraction lemma dictates the dynamics of isometries (Lemma 2.4);
- there are conceivable versions of boundary extensions of isometric maps (cf. Question G).

As I have argued in a paper originally entitled *From linear to metric* [40], the methods discussed here are also useful in other mathematical and scientific subjects. And while the metric ideas in the present paper are rather unified and focused, their consequences belong to a diverse set of topics. Let me illustrate the latter by highlighting three such results. The final section establishes the following.

THEOREM (Theorem 6.1). *Let $R_n = f_1 \circ f_2 \circ \cdots \circ f_n$ be the composition of randomly selected holomorphic self-maps of a bounded domain X in $\mathbb{C}^N$. Let d denote the Kobayashi distance and assume X is a weak visibility domain in the sense of Bharali–Zimmer. Then, almost surely it holds that unless*

$$\frac{1}{n}d(x, R_n x) \to 0,$$

as $n \to \infty$, there is a random point $\xi \in \partial X$ independent of x such that

$$R_n x \to \xi,$$

as $n \to \infty$.

In Section 5 the following improvement of [58, Corollary 4.2] is obtained.

COROLLARY (Corollary 5.3). *Let (Y, ω) be a compact Kähler manifold that is geodesically unstable. Let Φ_t be the weak Calabi flow on the associated completed space $\mathcal{E}^2$ of Kähler metrics in the same class. Then $\Phi_t(x)$ lies on sublinear distance to a unique geodesic ray γ as $t \to \infty$.*

In particular, $\Phi_t(x)$ converges as $t \to \infty$ to the point defined by γ in the visual boundary, which is a significantly weaker statement. This type of statement was established and exploited in [24] for the purpose of partially confirming a conjecture of Tian.

Finally, recall the notion of extremal length initially studied by Grötzsch, Beurling and Ahlfors. Let M be a closed surface with complex structure x and α an isotopy class of a simple closed curve on M, and define

$$\mathrm{Ext}_x(\alpha) = \sup_{\rho \in [x]} \frac{\ell_\rho(\alpha)}{\mathrm{Area}(\rho)},$$

where the supremum is taken over all metrics in the conformal class of x. We have the following from Section 4.

COROLLARY (Corollary 4.2). *Let f be a holomorphic self-map of the Teichmüller space of M. Then there exists a simple closed curve β such that*

$$\lim_{n\to\infty} \mathrm{Ext}_{f^n(x)}(\beta)^{1/n} = \lim_{n\to\infty} \left(\sup_{\alpha} \frac{\mathrm{Ext}_{f^n(x)}(\alpha)}{\mathrm{Ext}_x(\alpha)} \right)^{1/n},$$

where the supremum is taken over all simple closed curves on M.

2. Boundaries, halfspaces, and stars at infinity

Boundaries. Let (X,d) denote a metric space, for example, a complex domain with the Kobayashi metric assumed (Kobayashi) hyperbolic. Let $\overline{X}$ be a compactification of X with which we mean a compact Hausdorff space $\overline{X}$ and a topological embedding $i: X \to \overline{X}$ such that $\overline{X} = \overline{i(X)}$. Often we suppress the map i and consider X simply as a subset of $\overline{X}$. In the best case scenario the isometries of X extend naturally to homeomorphisms of $\overline{X}$; in this case we speak of an $\mathrm{Isom}(X)$-compactification. The corresponding *(ideal) boundary* is

$$\partial X := \overline{X} \setminus i(X).$$

There is also a weaker notion that only insists that i is continuous and injective, but that X is not necessarily homeomorphic to $i(X)$; we will refer to this as a *weak compactification*.

Example 2.1. In the case of bounded domains in complex vector spaces $\mathbb{C}^N$, we can consider the closure which often is called the *natural* boundary; here I will call it the *natural extrinsic* boundary. Complex automorphisms of the domain may not extend to the boundary; the question of when they do is a classical topic as mentioned in the introduction.

Example 2.2. For any metric space there is always an intrinsic method of a weak $\mathrm{Isom}(X)$-compactification, called *the horofunction bordification* or as I prefer, following Rieffel, *the metric compactification*, which is increasingly seen as being of fundamental importance. Given a base point x_0 of the metric space X, let

$$\Phi : X \to \mathbb{R}^X$$

be defined via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

With the topology of pointwise convergence, this is a continuous injective map and the closure $\overline{\Phi(X)}$ is compact and Hausdorff. The elements of $\overline{X}^h := \overline{\Phi(X)}$ are called *metric functionals*. In the case X is a proper geodesic space this construction gives a compactification in the stricter sense. If X is a proper metric space, the elements of ∂X are called *horofunctions*; the typical example of such comes from geodesic rays and are called *Busemann functions* (in the literature starting with Rieffel this word is also used for limits along almost geodesics).

A major question is to investigate the exact relation between these two examples: the metric compactification in the Kobayashi metric, which is the natural intrinsic compactification, and the natural extrinsic boundary of bounded complex domains. See for example [6]. Let me record it as follows.

QUESTION A. *Let X be a bounded complex domain with the Kobayashi metric. How and when can one relate the natural extrinsic boundary with the boundary from the metric compactification?*

Let (X, d) denote a metric space and $\overline{X}$ a compactification of X. For $x \in X$ and a subset $W \subset \overline{X}$ with $W \cap X \neq \emptyset$, we set

$$d(x, W) := \inf_{w \in W \cap X} d(x, w).$$

Halfspaces. Let $x_0 \in X$. The *(generalized) halfspace* defined by $W \subset \overline{X}$ with $W \cap X \neq \emptyset$ and real number C is

$$H(W, C) := H^{x_0}(W, C) := \{z : d(z, W) \leq d(z, x_0) + C\}.$$

We also use the notation $H(W) := H(W, 0)$. These notions have the advantage that they do not refer to geodesics which may or may not exist.

Stars at infinity. Let $\xi \in \partial X$ and denote by $\mathcal{V}_\xi$ the collection of open neighborhoods of ξ in $\overline{X}$. The *star of ξ based at x_0* is

$$S^{x_0}(\xi) = \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V)},$$

where the closure is taken in $\overline{X}$, and the *star of ξ* is

$$S(\xi) = \overline{\bigcup_{C \geq 0} \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V, C)}}.$$

It is immediate that $\xi \in S^{x_0}(\xi) \subset S(\xi)$. The second definition removes an a priori dependence on x_0. In all examples I can think of, the two definitions actually coincide, which in other words means that the first definition is independent of the base point chosen.

QUESTION B. *When is $S(\xi) = S^{x_0}(\xi)$?*

The motivation for calling these sets *stars* is that they are subsets at infinity of the space and they have a tendency to be star-shaped in appropriate senses or even to coincide with the notion of star in the theory of simplicial complexes. In addition they are closely related to visibility properties of the compactification (see Proposition 3.3) so one could appeal to the light emitting property of physical stars.

A *face* is a non-empty intersection of stars. The following notion will be used below, the *dual star of ξ*:

$$S^\vee(\xi) := \{\eta \in \partial X : \xi \in S(\eta)\}.$$

It was observed in [38] that in all the examples considered there, $S^\vee(\xi) = S(\xi)$, which could be called *star-reflexivity*, and raised the question whether or when it is the case. In an insightful paper by Jones and Kelsey [36] examples of homogeneous graphs, certain Diestel–Leader graphs, with their metric compactification were shown *not* to have this property. Understanding this phenomenon better has some additional interest in view of results like Theorem 4.3 or Proposition 3.2 below.

QUESTION C. *Which spaces and compactifications are star reflexive in the sense that $S^\vee(\xi) = S(\xi)$ for all $\xi \in \partial X$?*

An obvious case of $S^\vee(\xi) = S(\xi)$ is when $S(\xi) = \{\xi\}$. Such points are called *hyperbolic* because for example for any Gromov hyperbolic space with its standard Gromov boundary every point is hyperbolic. We call compactifications with this property *hyperbolic*. Another classical example of a hyperbolic compactification is the end compactification of any proper metric space. For bounded complex domains with Kobayashi metric, every natural extrinsic boundary point that is C^2-smooth and strictly pseudoconvex is hyperbolic as P. J. Thomas informed me; see also below and [38, Corollary 35].

QUESTION D. *For convex or pseudoconvex domains with the usual boundary, what are the Kobayashi stars?*

Hilbert's metric on convex domains has a very simple answer provided in [38, Proposition 32]: the stars and faces coincide with the usual homonymous notions for convex sets.

Teichmüller spaces of closed surfaces are of great importance and are biholomorphic to bounded pseudoconvex domains. They have a natural metric, the Teichmüller metric, which as Royden showed coincides with the Kobayashi metric. The recent papers by Duchin and Fisher [28] as well as by Liu and Shi [52] make substantial progress toward [38, Conjecture 46] determining the stars in the Teichmüller metric with the Thurston compactification which is defined in terms of topology and hyperbolic geometry. It has been observed for a long time that the complex analytic notions have a complicated relationship with the concepts from the approach of hyperbolic geometry, but here there is a hope of a clean tight connection. The very recent article [52] moreover considers statements analogous to the conjecture in [38] but for other metrics and compactifications of the Teichmüller spaces, and interestingly manages to prove these. See also [57] in this context.

One can define the *star-distance* $d_\star$ as in [28, 38] to be the induced path distance on ∂X, in the extended sense that distances may be infinite, from defining

$$d_\star(\xi, \eta) = 0 \Longleftrightarrow \xi = \eta$$

and if $\xi \neq \eta$

$$d_\star(\xi, \eta) = 1 \iff \eta \in S(\xi) \text{ or } \xi \in S(\eta).$$

In the case of an Isom(X)-compactification the isometries obviously act by isometry also on $(\partial X, d_\star)$. It is a trivial concept in case X is Gromov hyperbolic with its standard boundary. This is related to Tits incidence geometry at infinity in nonpositive curvature and conjecturally [28, 38] the star distance restricted to the simple closed curves of the Thurston boundary isometric to the curve complex defined in pure topological terms; see [28] for more details and for a conjectural outline arriving at such a result.

Let me mention the following useful fact, the *sequence criterion* for star membership of Duchin–Fisher extending a lemma in [36].

LEMMA 2.3 ([28]). *Let (X, d) be a metric space and $\overline{X}$ a compactification of X. Assume that $\overline{X}$ is first countable. Then $\eta \in S(\xi)$ if and only if for every neighborhood U of η in $\overline{X}$, there are sequences $x_n \to \xi$, $y_n \to U$ and a constant $C \geq 0$ such that*

$$d(y_n, x_n) \leq d(y_n, x_0) + C.$$

In particular, if there are such sequences with $y_n \to \eta$, then $\eta \in S(\xi)$.

Isometries, when well-defined as maps of $\overline{X}$, preserve the star distance. This stands in contrast to an opposite phenomenon namely that topologically isometries tend to have strong contraction properties on $\overline{X}$ as expressed by Lemma 2.4 below. Particularly this is the case when there are many hyperbolic points but on the other hand it can reduce to no contraction for example in case of the Euclidean spaces with the usual visual boundaries (if one takes an ℓ^1 metric instead there is some contraction). The *north-south dynamics* is one of the most important features in the theory of word hyperbolic group and states that for any sequence of group elements g_n which converges to ξ^+ when $n \to \infty$ and g_n^{-1} to ξ^-, it holds that for any two neighborhoods V^+ of ξ^+ and V^- of ξ^- eventually everything outside U^- is mapped inside U^+ by g_n. This is generalized without any hyperbolicity assumption, and to any compactification, just adding an "H".

LEMMA 2.4 (The contraction lemma [38]). *Let (X, d) be a metric space and $\overline{X}$ a compactification of X. Let g_n be a sequence of isometries such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$ as n $n \to \infty$. Then for any neighborhoods V^+ of ξ^+ and V^- of ξ^-, there exists $N > 0$ such that*

$$g_n\big(X \setminus H(V^-)\big) \subset H(V^+)$$

for all $n \geq 1$.

In [38] a refinement in the case Isom(X)-compactifications is also formulated. In words, g_n eventually maps everything outside the star of ξ^- into any neighborhood of the star of ξ^+. Note that it is allowed that the two boundary points are the same, such as is the case for iterates of a parabolic isometry in hyperbolic geometry. The interplay between the invariance of the star distance and the contractive property of Lemma 2.4 can sometimes be used to rule out non-compact automorphism groups; see [38, Theorem 4] for an example. In a more classical direction it recovers Hopf's theorem on ends which states that any topological space X that is a regular covering space of a nice compact space must have either 0, 1, 2 or a continuum, of ends. In particular it applies to finitely generated groups and the ends of their Cayley graphs. This generalizes in view of Lemma 2.4 to any hyperbolic compactification with stars replacing ends.

QUESTION E. *Are there applications of the tension between the invariance of the star distance and the contraction lemma also for groups of biholomorphisms of certain complex domains?*

To exemplify this idea, we establish the following proposition.

PROPOSITION 2.5. *Let (X,d) be a proper metric space and $\overline{X}$ an* $\mathrm{Isom}(X)$*-compactification of X. Assume that a noncompact group of isometries of X fixes a finite set F of boundary points. Then F is contained in two stars.*

PROOF. By properness of X, compactness of $\overline{X}$, and the noncompactness of the isometry group, we can find isometries g_n such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$ as $n \to \infty$. Since F is finite, by passing to a finite index subgroup, which does not affect the noncompactness, we may assume that the group fixes the elements of F pointwise. Any point outside the two stars associated with $\xi^\pm$ must be contracted to these stars according to the contraction lemma. Such a point cannot be fixed; hence $F \subset S(\xi^+) \cup S(\xi^-)$. ∎

3. GEODESICS AND BOUNDARY ESTIMATES

Let us begin by the following simple observation.

PROPOSITION 3.1. *Let X be a proper metric space with compactification $\overline{X}$. To any geodesic ray γ there is an associated face of ∂X being the non-empty intersection of all the stars which contain limit points of $\gamma(t)$ as $t \to \infty$. In particular, all limit points of $\gamma(t)$ are contained in this face at infinity.*

PROOF. Since the space is proper, any geodesic ray only accumulates at the boundary. Take any two limit points $\gamma(n_i) \to \xi$ and $\gamma(k_j) \to \eta$. For any $n_i > k_j$ we have

$$d\big(\gamma(n_i), \gamma(k_j)\big) = n_i - k_j < d\big(\gamma(n_i), \gamma(0)\big) \le d\big(\gamma(n_i), x_0\big) + d\big(x_0, \gamma(0)\big).$$

This implies that $\gamma \in S(\eta)$ since $\gamma(n_i)$ stays closer to each neighborhood of η up to the constant $C = d(x_0, \gamma(0))$. Since the two limit points were arbitrary, this shows that any limit point belongs to the star of any other limit point. Thus this intersection of stars is non-empty and contains all limit points. ∎

Recall that any geodesic ray defines a Busemann function and thus converges to a boundary point in this compactification. In relation to Question A we have the following.

Question F. *When do Kobayashi geodesic rays converge to a boundary point in bounded complex domains?*

Partial results follow from works such as [6, 18] and earlier papers that identify the natural extrinsic boundary with the Gromov boundary since geodesic rays always converge in the latter boundary. Note that drawing from the analogy with Hilbert's metric on convex domains (discussed by Vesentini in [56]) the paper [29] suggests that it could be a more general phenomenon since it is shown there that Hilbert geodesic rays always converge even for general convex domains that often are not Gromov hyperbolic.

I think this is related to questions of extending biholomorphisms $f : X \to Y$ to the boundaries. Since Kobayashi geodesic rays are mapped to Kobayashi geodesic rays, and if these, say emanating from x_0, are in bijective correspondence with boundary points, then there could be a hope for such an extension. But as the referee pointed out, a bijection of rays is not sufficient, as can be seen from examples in one complex variable using the Riemann mapping theorem and the Carathèodory prime ends theory. In Mostow's work in higher rank he obtained incidence preserving boundary maps. So even when there are no well-defined boundary maps one could formulate a vague, more general question in this direction.

Question G. *Are there results of the type that biholomorphims or proper holomorphic maps induce maps between the face lattices of the boundaries?*

For some trivial examples and for more discussion of this in the metric setting, see [38]. Obviously it will depend on the boundaries, and one optimistic possibility could be that if one takes the metric compactification (see Example 2.2 above) of the domain space, then it would map to the boundary (or its faces if the boundary is too large) of the range space. Some interesting results and insightful discussions of related type can be found in Bracci–Gaussier's papers [16, 17].

Recall that for $\xi \in \partial X$ in the metric compactification, h is the horofunction that ξ defines, and for each real number C there is an associated (closed) *horoball* defined as

$$\mathcal{H}_{\xi,C} = \{x \in X : h(x) \leq C\}.$$

(I emphasize that we are here primarily discussing horoballs in this sense, while there are also the more general notions of Abate's small and large horoballs defined in 1988 that have been of importance in complex geometry since then; see [1].) In view of the above discussion, the following question arises.

QUESTION H. *What are the relations between stars and the intersection of horoballs with the boundary?*

The following is a simple relation.

PROPOSITION 3.2. *Let ∂X be the metric boundary of a proper metric space. Let $\mathcal{H}_{\xi,C}$ be a horoball centered at $\xi \in \partial X$. Then*

$$\overline{\mathcal{H}_{\xi,C}} \cap \partial X \subset S^{\vee}(\xi).$$

PROOF. Let $x_n \to \xi$ in the metric compactification, which means that for the associated horofunction h,

$$h(y) = \lim_{n\to\infty} d(y, x_n) - d(x_0, x_n).$$

Suppose that a sequence y_k belongs to the fixed horoball $\mathcal{H}_{\xi,C}$, which means that for all k

$$C \geq h(y_k) = \lim_{n\to\infty} d(y_k, x_n) - d(x_0, x_n).$$

This implies that for any $C' > C$ and any k there is an N such that $d(x_n, y_k) \leq d(x_n, x_0) + C'$ for all $n > N$. From the definitions we then have that for any limit point η of the sequence y_k, it holds that $\xi \in S(\eta)$. ■

The *visibility property* of a compactification has its origin from Eberlein–O'Neill in nonpositive curvature and has recently entered into complex analysis in significant ways; see for example [13, 14, 19] for more discussion. One definition is as follows: for any two boundary points ξ and η there are disjoint closed neighborhoods V_ξ and V_η and a compact set K such that any geodesic segment connecting V_ξ and V_η must also meet K; alternatively formulated, there is a bound on the distance from x_0 to each such geodesics. Real hyperbolic spaces have this property while Euclidean spaces do not have it, in their standard visual (= metric) compactifications. In this context the following is immediate.

PROPOSITION 3.3. *Assume that X is a geodesic space which means that every two points can be connected by a geodesic segment. Suppose that for two distinct boundary points, there are disjoint neighborhoods of them such that all geodesics connecting these neighborhoods have bounded distance to x_0. Then the two stars are disjoint.*

PROOF. The assumption means that the distance between points near ξ and points near η is up to a bounded amount the sum of the respective distance to x_0. The conclusion now follows from the definition of stars: points near one of the boundary point will eventually all lie outside the halfspaces around the other. ■

If all stars are disjoint, then we must have that $S(\xi) = \{\xi\}$ for every $\xi \in \partial X$, and I call such compactifications hyperbolic as mentioned above.

COROLLARY 3.4. *If a compactification of a geodesic space has the visibility property, then it is a hyperbolic compactification.*

Since it seems not clear when Kobayashi domains are geodesic spaces, Bharali and Zimmer, see [13, 14], defined a weaker notion of visibility (see also these papers for a wealth of examples). Let X be a bounded domain in $\mathbb{C}^N$ with its associated Kobayashi distance d. Fix some $\kappa > 0$; by [14, Proposition 4.4] any two points in X can be joined by a $(1, \kappa)$-almost geodesic which means a path $\sigma : I \to X$ such that

$$|t - s| - \kappa \leq d\big(\sigma(t), \sigma(s)\big) \leq |t - s| + \kappa$$

for all $t, s \in I$. Let $\overline{X}$ be the closure X above referred to as the natural extrinsic compactification of X. We say that X is a *visibility domain* if for any two distinct boundary points ξ and η and neighborhoods V and W in $\overline{X}$ of these two points such that $\overline{V} \cap \overline{W} = \emptyset$, there exists a compact set K in X such that for any $x \in V \cap X$ and $y \in W \cap X$ and any $(1, \kappa)$-almost geodesic σ joining these two points, σ intersects K.

THEOREM 3.5. *Let X be a bounded domain in $\mathbb{C}^N$ and $\overline{X}$ its closure. Assume that it is a visibility domain for the Kobayashi distance. Then*

$$S(\xi) = \{\xi\}$$

for every $\xi \in \partial X = \overline{X} \setminus X$.

The proof is a minor adaptation of Proposition 3.3 in view of [14, Proposition 4.4]. See also the proof of Theorem 6.1 below.

QUESTION I. *Are there in some cases precise relations between visibility and boundary points being hyperbolic? Are hyperbolic compactifications a larger class than visibility compactifications?*

This is of interest in the Wolff–Denjoy context discussed below. Here is a way to get visibility and hyperbolic points from estimates for the Kobayashi distances, taken from [38].

THEOREM 3.6 ([38, Theorem 37]). *Let X be a bounded C^2-smooth domain in $\mathbb{C}^N$ which is complete in the Kobayashi metric. Assume that for the infinitesimal Kobayashi metric $K_X(z;v)$ there are some constants $\epsilon > 0$ and $c > 0$ such that*

$$K_X(z;v) \geq c\frac{\|v\|}{\delta(z,\partial X)^{\epsilon}}$$

for all $z \in X$ and $v \in \mathbb{C}^N$, where $\|\cdot\|$ and δ refer to the Euclidean norm and distance respectively. Then $S(\xi) = \{\xi\}$ for every $\xi \in \partial X$ and the compactification has the visibility property.

The estimate in the assumption of the theorem is established in [25] for smooth pseudoconvex bounded domains with boundary of finite type in the sense of D'Angelo. This has subsequently been extended in important ways, in [45, Lemma 5], the Goldilocks domain of Bharali–Zimmer in [14], and [13, Theorem 1.5].

4. WOLFF–DENJOY TYPE THEOREMS

An early application of the Schwarz–Pick lemma was found in 1926 seemingly as a conversation via *Comptes Rendus* of the French Academy of Sciences between Wolff and Denjoy. It states that any holomorphic self-map of the unit disk either has a fixed point, or its orbits converge to a single point in the boundary circle and every horodisk at that point is an invariant set. Extensions of this have generated a vast literature, starting with Valiron, Heins, H. Cartan, Hervé, Vesentini, Abate, Beardon, and many others; see [34] for references. Most extensions assume something like Gromov hyperbolicity or weaker property (like visibility or strict convexity). The stars will be used for a weaker conclusion but in a much more general setting.

I will mention two purely metric versions, one in terms of the metric compactification and the other one in terms of the stars at infinity for any given compactfication of interest.

Let X be a metric space and consider maps f between metric spaces that are *nonexpansive* in the sense that

$$d\big(f(x), f(y)\big) \leq d(x,y)$$

for all $x, y \in X$. Isometries are important examples and the composition of nonexpansive maps remains nonexpansive.

As was remarked in the very beginning Kobayashi provided a functor from complex spaces and holomorphic maps into psuedo-metric spaces and nonexpansive maps, thereby constituting a very significant class of examples.

One defines the *minimal displacement* $d(f) = \inf_x d(x, f(x))$ and the *translation number* $\tau(f) = \lim_{n\to\infty} d(x, f^n(x))/n$, which exists by a well-known subadditivity argument. These numbers are analogs of the operator norm and spectral radius, respectively; in particular note that $\tau(f) \leq d(f)$. They have been studied to some extent in the complex analytic literature, by Arosio, Bracci, Fiacchi, and Zimmer in particular; see [5–7].

The following, which I think of as a kind of weak spectral theorem in the metric category [40, 41], can be viewed as a partial extension of the theorem of Wolff and Denjoy.

THEOREM 4.1 ([37]). *Let $f : X \to X$ be a nonexpansive map of a metric space X. Then there is a metric functional h such that*

$$h(f^k x_0) \leq -\tau(f)k$$

for all $k \geq 1$, and

$$\lim_{n\to\infty} -\frac{1}{n}h(f^n x) = \tau(f).$$

The theorem implies as very special cases, with geometric input specific in each case, extensions of the Wolff–Denjoy theorem for holomorphic maps [37, 38], von Neumann's mean ergodic theorem [40, 41], and Thurston's spectral theorem for surface homeomorphisms [33, 39]. Moreover, it has been applied in non-linear analysis, see e.g. [50], and gave the classification of isometries of Gromov hyperbolic spaces even when non-locally compact and non-geodesic. It also provided new information for isometries of Riemannian manifolds. Its proof has subsequently been used several times in the setting of Denjoy–Wolff extensions in several complex variables; see [34]. Maybe it can also be useful for pseudo-holomorphic self-maps; see [15].

To illustrate how a metric generalization of the Wolff–Denjoy theorem can give back to complex geometry in a different way, we show the following. Extremal length was defined in the introduction.

COROLLARY 4.2. *Let f be a holomorphic self-map of the Teichmüller space of M. Then there exists a simple closed curve β such that*

$$\lim_{n\to\infty} \mathrm{Ext}_{f^n(x)}(\beta)^{1/n} = \lim_{n\to\infty} \left(\sup_{\alpha} \frac{\mathrm{Ext}_{f^n(x)}(\alpha)}{\mathrm{Ext}_x(\alpha)} \right)^{1/n},$$

where the supremum is taken over all simple closed curves on M.

PROOF. For more information and bibliographic details, see [39]. Holomorphic maps do not expand Teichmüller distance since it coincides with the Kobayashi metric d.

Kerckhoff showed the following formula (in particular implying that this expression is symmetric):

$$d(x,y) = \frac{1}{2}\log\sup_{\alpha}\frac{\mathrm{Ext}_x(\alpha)}{\mathrm{Ext}_y(\alpha)},$$

where the supremum is taken of isotopy classes of simple closed curves on M. Denote by τ the translation length $\tau(f)$ defined above. Liu and Su showed that the metric compactification coincides with the Gardiner–Masur compactification, and thanks also to Miyachi there is a description of the metric functionals. From Theorem 4.1 we then have

$$-\tau n \geq h\big(f^n(x)\big) = \log\sup_{\alpha}\frac{E_P(\alpha)}{\mathrm{Ext}_{f^n(x)}(\alpha)^{1/2}} - C \geq \log\frac{D}{\mathrm{Ext}_{f^n(x)}(\beta)^{1/2}} - C,$$

for some curve β (with $D = E_P(\beta) > 0$ which must exist) for all $n > 1$. This gives

$$\mathrm{Ext}_{f^n(x)}(\beta) \geq D^2 e^{2C} e^{2\tau n}.$$

The other inequality, for any $\epsilon > 0$ and all sufficiently large n, follows from the supremum in Kerckhoff's formula:

$$\mathrm{Ext}_{f^n(x)}(\beta) \leq C_1 e^{2(\tau+\epsilon)n}.$$

The result now follows. ■

Here is a result that applies to any compactification, in particular to holomorphic self-maps of bounded domains with the standard boundary as a subset of $\mathbb{C}^N$.

THEOREM 4.3 ([38, Theorem 11]). *Let $f : X \to X$ be a nonexpansive map of a proper metric space. Assume that $\overline{X}$ is a sequentially compact compactification of X. Then either the orbit is bounded or there is a boundary point $\xi \in \partial X$ such that for any $x \in X$, every limit point of $f^n(x)$ as $n \to \infty$ in ∂X is contained in $S^\vee(\xi)$.*

In the usual settings where one assumes something that implies $S^\vee(\xi) = \{\xi\}$, we of course may conclude that the orbits converge to this boundary point, as in the usual Denjoy–Wolff theorem. In view of [28], a corollary, which no doubt is only a partial result, can be formulated.

COROLLARY 4.4. *Let f be a holomorphic self-map of the Teichmüller space of a closed surface. If the orbit is unbounded and has an accumulation point ξ that is a uniquely ergodic foliation in the Thurston boundary, then $f^n(x) \to \xi$ as $n \to \infty$ and any x.*

A conceivable strengthening of the theorem could be that the limit set in the unbounded case has to be contained in a single face; compare with Proposition 3.1 for an analogy. In order to be less vague, while more risky, let us formulate the following conjecture.

Conjecture 4.5. *Let X be a bounded domain in $\mathbb{C}^N$ equipped with Kobayashi metric and assume it is a proper metric space. Let $\overline{X} \subset \mathbb{C}^N$ be the standard closure. Let $f : X \to X$ be holomorphic. Then either the orbits stay away from the boundary or there is a closed face $F \subset \partial X$ in the above metric sense such that for any $x \in X$ every accumulation point of $f^n(x)$ as $n \to \infty$ belongs to F.*

Theorem 4.3 provides a partial result. Note that one could also ask the same instead using the notion of face as the a priori non-metric sense of being an intersection of ∂X with a hyperplane. This relates to [1,3] which imply partial results on the conjecture in cases that the domain is convex or has a simple boundary.

Here is another partial result.

Proposition 4.6. *Let X be a bounded domain in $\mathbb{C}^N$ equipped with Kobayashi metric. Let $\overline{X} \subset \mathbb{C}^N$ be the standard closure. Let $f : X \to X$ be holomorphic. Assume that $d(z, f^n(z)) \nearrow \infty$ monotonically for some $z \in X$. Then there is a closed face $F \subset \partial X$ in the above metric sense such that for any $x \in X$ every accumulation point of $f^n(x)$ as $n \to \infty$ belongs to F.*

Proof. Given two subsequences

$$f^{n_i}(z) \to \xi \in \partial X \quad \text{and} \quad f^{k_j}(z) \to \eta$$

as $i, j \to \infty$. For any $n_i > k_j$ we have

$$\begin{aligned} d\big(f^{n_i}(z), f^{k_j}(z)\big) &\le d\big(z, f^{n_i-k_j}(z)\big) < d\big(z, f^{n_i}(z)\big) \\ &\le d\big(f^{n_i}(z), x_0\big) + d(x_0, z). \end{aligned}$$

For any neighborhood V of η we can find a large enough j so that $f^{k_j}(z) \in V$, and the above inequality means that

$$f^{n_i}(z) \in H(V, C)$$

for all i large enough and where $C = d(x_0, z)$. Hence $\xi \in S(\eta)$. Since this was for two arbitrary sequences we must also have $\eta \in S(\xi)$ and can conclude that F being the intersection of all the stars of all accumulation points contains all accumulation points (even when changing z to x since the respective orbits stay on bounded distance and this does not influence the stars). ∎

The same argument would work if one merely knew that for some $a > 0$,

$$d\big(z, f^{an}(z)\big) \nearrow \infty.$$

This is presumably most often the case, but it may not be so easy to guarantee.

5. The Calabi flow

Given a Kähler manifold with a fixed Kähler class, a natural question is to determine whether there exists a canonical choice of Kähler metric in this class. One potential such choice, generalizing Riemann surface theory and Kähler–Einstein metrics, is to look for metrics of constant scalar curvature.

Let (Y, J, ω) be a compact connected Kähler manifold and consider the space $\mathcal{H}$ of smooth Kähler metrics in the cohomology class $[\omega]$, introduced by Calabi in the 1950s. This space can be equipped with a Riemannian metric (Mabuchi–Semmes–Donaldson) of Weil–Petersson or Ebin type with nonpositive sectional curvatures and such that the metric completion $\mathcal{E}^2$ is a CAT(0)-space admitting a concrete description in terms of plurisubharmonic functions (due to Darvas). The Calabi flow on $\mathcal{H}$ in the space of metrics does not expand distances as long as it exists. It is believed to exist for all times. Moreover it is expected that either the flow converges to a constant scalar curvature metric or it diverges and should asymptotically contain some information about the Kähler structure (made precise in a conjecture of Donaldson in terms of geodesic rays). Streets suggested to study a weak Calabi flow Φ_t, which is nonexpansive being the gradient flow of a convex function M, the K-energy of Mabuchi, and exists for all time and coincides with the Calabi flow when it exists. I refer to [11, 20, 58] for more information and appropriate references. The works of Mayer and Bacak in pure metric geometry play an important role here, and let me add another good reference [22] on gradient flows of convex functions on CAT(0)-spaces generally.

From a flow Φ_t we can define the time-one map $f(x) = \Phi_1(x)$, and we have that $f^n(x) = \Phi_n(x)$. From what has been said we have that f is a nonexpansive self-map of a complete CAT(0)-space.

Motivated by this it should be useful to see what can be said about the iteration of nonexpansive maps in the setting of CAT(0)-spaces (note by the way that since the Kobayashi metric is a supremum-type metric it is almost never CAT(0) apart from the well-known exceptional cases). Some results and arguments for locally compact spaces are contained in Beardon [9].

For general complete CAT(0)-spaces, in the case that the orbits of f are bounded, several authors have observed the existence of a fixed point, notably Kirk in [46]. I gave the following argument in my doctoral thesis.

Proposition 5.1. *Let f be a nonexpansive map of a complete* CAT(0)*-space. If the orbits are bounded, then f has a fixed point.*

Proof. It follows from CAT(0), in fact the uniform convexity, that bounded subset possesses a unique circumcenter, which is a point that is the center of a closed ball of minimal radius containing the set. The uniform convexity also leads to that the intersection of any nested sequence of decreasing closed convex sets is nonempty because the circumcenters form a Cauchy sequence and the limit is a point in the intersection.

Given an orbit $x_n := f_n^n(x)$ we can construct an invariant, bounded, closed convex set in the following way. For each $k > 1$ we let C_k be the intersection of the closed balls with centers at x_n and radius the diameter of the orbit. These are non-empty, bounded, and convex. Moreover $f : C_k \to C_{k+1}$, so the closure of the union of all C_k is the desired invariant set. Such sets are ordered by inclusion and every linearly ordered chain has a lower bound via the intersection; thus Zorn's lemma provides a minimal element. The circumcenter of this minimal element must be a fixed point of f. ∎

CAT(0)-spaces have the so-called *visual bordification* consisting of equivalence classes of geodesic rays (this is not related to the visibility property above). For proper spaces it coincides with the metric compactification.

THEOREM 5.2. *Let f be a nonexpansive map of a complete* CAT(0)*-space X. If the orbits are bounded, then f has a fixed point. If $d_f := \inf_x d(x, f(x)) > 0$, then there exists a unique geodesic ray γ from x such that*

$$\frac{1}{n} d\big(f^n(x), \gamma(d_f t)\big) \to 0,$$

as $n \to \infty$, and $b(f(x)) \leq b(x) - d_f$ for all x where b denotes the Busemann function associated with γ. In particular, $f(x)$ converges to the class of γ in the visual bordification. If $d_f = 0$, then there is a metric functional h such that $h(f(x)) \leq h(x)$ for all x.

PROOF. The bounded orbit case is treated in Proposition 5.1. It is known from [30] that for any nonexpansive self-map f of a CAT(0)-space, since it is a star-shaped hemi-metric space, the minimal displacement d_f equals the translation length τ_f. Therefore $\tau_f > 0$, and a special case of the main result in [44] then shows that there is a unique unit-speed geodesic ray γ from x such that

$$\frac{1}{n} d\big(f^n(x), \gamma(d_f n)\big) \to 0.$$

In particular, $f^n(x)$ converges to the visual boundary point $[\gamma]$ for any $x \in X$, which is strictly weaker. Let b be the Busemann function associated with the geodesic ray emanating from x_0 representing the class of γ. This is the unique Busemann function such that (cf. [43])

$$-\frac{1}{n} b(f^n x) \to d_f.$$

Theorem 4.1 now implies moreover that $b(f(x)) \leq b(x) - d_f$ which holds for all x in view of [30]. And in the remaining case that $d_f = 0$ and the infimum is not attained, Theorem 4.1 gives a metric functional h such that $h(f^n(x_0)) \leq 0$ for all $n > 0$. When the space is CAT(0) this can be improved; see [37] or [30], to give the remaining assertion for any x. ∎

Anyone from complex dynamics, or the attentive reader of the previous sections, will notice that this too is a Wolff–Denjoy type theorem in a purely metric setting, but now with implications for complex geometry. It is known, see [11], that either the trajectories of the weak Calabi flow converge to a constant scalar curvature metric, or they diverge, $d(x, \Phi_t(x)) \to \infty$ for any x. In the latter case one would like to know some directional behavior, for example the notion of weakly asymptotic geodesic ray introduced by Darvas–He and studied in [11], which is a notion much weaker (for example not necessarily unique) than the one in Theorem 5.2. In a special case there is more precise asymptotic convergence to a geodesic ray known [24, Theorem 6.3] used to prove the main result concerning uniqueness properties of constant scalar curvature metrics in that same paper. We will compare the above with Xia's paper [58]. Having as a starting point an inequality and conjecture of Donaldson in [27], Xia proved an analog of the conjecture when enlarging the space from $\mathcal{H}$ to $\mathcal{E}^2$; namely,

$$C := \inf_{x \in \mathcal{E}^2} \left|(\partial M)(x)\right| = \max \frac{-\mathbf{M}(\ell)}{\|\ell\|},$$

where the maximum on the right is taken over boundary points/geodesic rays in $\mathcal{E}^2$. The expression $|(\partial M)(x)|$ is the local upper gradient of the Mabuchi energy. If its infimum is strictly positive, (Y, ω) is called *geodesically unstable* (and in particular admitting no constant scalar curvature metric in its class). Recent results on geodesic stability and the existence of constant scalar curvature Kähler metrics can be found in [23].

Let me formulate the following that improves and reproves parts of [58, Corollary 1.2, Corollary 4.2].

Corollary 5.3. *Let (Y, ω) be a compact connected Kähler manifold that is geodesically unstable. Let $\Phi_t(x)$ be the weak Calabi flow on the associated space $\mathcal{E}^2$ starting from x. Then there exists a unique geodesic ray γ from x on sublinear distance to $\Phi_t(x_0)$; that is,*

$$\frac{1}{t} d\big(\Phi_t(x), \gamma(Cn)\big) \to 0,$$

where C is defined above. In particular $\Phi_t(x) \to [\gamma] \in \partial\mathcal{E}^2$ as $t \to \infty$, for any x.

Proof. The geodesic instability asserts that the infimum of the gradient is strictly positive, $C > 0$, which implies that the escape rate of the weak Calabi flow is linear; see [22]. The corollary is now a consequence of Theorem 5.2. ∎

There are many studies on the geodesics in spaces like $\mathcal{H}$ and $\mathcal{E}^p$; see [4, 12] for two recent papers in the complex setting and references therein. It might therefore be of interest that the argument I suggest here (from [44]) constructs the geodesic from the flow in a way that does not use any compactness argument. From my point of view, related to Theorem 4.1 above, it should also be fruitful to study the dual notion to geodesics, the metric functionals of $\mathcal{E}^1$ and $\mathcal{E}^2$.

Apart from the constant scalar curvature problem, automorphisms of the underlying Kähler manifold act by isometry on the Calabi space and also fall under Theorem 5.2.

Corollary 5.4. *Let f be a complex automorphism of a compact Kähler manifold. If*

$$d := \inf_x d\big(x, f(x)\big) > 0$$

for the action on $\mathcal{E}^2$, then there is a unique geodesic ray γ from x such that

$$\frac{1}{n} d\big(f^n(x), \gamma(dn)\big) \to 0,$$

and f fixes the corresponding boundary point in the visual bordification of M.

Proof. For CAT(0)-spaces any boundary limit point of the orbit is fixed by f as is well known, see for example [42], since two sequences of bounded distance from each other converge to the same equivalence class of geodesic ray when they converge. ■

Even in the case $d = 0$, it holds that f fixes a metric functional of $\mathcal{E}^p$ ([42]). A condition like $\inf_x d(x, f(x)) = 0$ where f is a diffeomorphism of an underlying compact manifold acting instead by isometry on a space of metrics was proposed by D'Ambra and Gromov in [26] as *(quasi-) unipotency* of f.

Question J. *How can we describe or understand the metric functionals for the Calabi–Mabuchi space and what do the relevant results discussed in this paper imply concretely for the Calabi flow, Donaldson's conjecture, and automorphisms of the Kähler manifold?*

A final remark is that in the standard visual bordification, the stars are identified in [38, Proposition 25] as

$$S(\xi) = S^{x_0}(\xi) = \big\{\eta : \angle(\eta, \xi) \leq \pi/2\big\} = S^{\vee}(\xi),$$

where $\angle$ denotes the Tits angle between two geodesic rays and is symmetric in its arguments.

Example. The stars for Euclidean spaces are half-spheres (the Tits angle coincides with the usual notion of angle) and for hyperbolic spaces they are points (since any two boundary points can be joined by a geodesic line giving the angle π).

6. Random iteration

The above discussion has involved iterations f^n of a holomorphic self-map f. In some contexts one meets the generalization to the composition of several different holomorphic maps. When looking at the asymptotics one can compose them in two ways, forward or backward. The latter, i.e.,

$$R_n = f_1 \circ f_2 \circ \cdots \circ f_n,$$

behaves best when studying individual orbits and appears for example in the theory of continued fractions. Indeed, a continued fraction expansion of a number is exactly such an expression where $f_i(z) = a_i/(b_i + z)$ are certain Möbius maps, and letting $n \to \infty$. Other examples considered by Ramanujan and Polya-Szegö are infinite radicals ($f_i(z) = \sqrt{a_i z + b_i}$) and iterated exponentials ($f_i(z) = a_i^z$); see [53] for more details. In these contexts one considers the limit of the corresponding $R_n(0)$ as $n \to \infty$. There is also a connection to Nevanlinna–Pick interpolation.

Some papers on this topic, see for example [2, 10, 35], take arbitrary sequences of maps (like in iterated function systems and the theory of fractals) and sometimes call them *random iteration*. Here we will only discuss R_n for actually randomly selected holomorphic maps f_i.

We formalize the setting as follows, more general than the usual *random* assumption of independently, identically distributed selected maps. Let (T, Ω, μ) be an ergodic measure preserving system with $\mu(\Omega) = 1$. Given a measurable map $f : \Omega \to G$ into a semigroup, we define the following *ergodic cocycle*: $R(n,\omega) = f(\omega)f(T\omega)\cdots f(T^{n-1}\omega)$ or in probabilistic notation leaving out the measure space: $R_n = f_1 f_2 \cdots f_n$. It is *integrable* if

$$\int_\Omega d\big(x, f(\omega)x\big) d\mu < \infty$$

for some $x \in X$. Then by a well-known consequence of Kingman's subadditive ergodic theorem, the limit has

$$\tau := \inf_n \frac{1}{n} \int d\big(x, R(n,\omega)x\big) d\mu = \lim_{n\to\infty} \frac{1}{n} d\big(x, R(n,\omega)x\big)$$

for almost every ω and by ergodicity τ is independent of ω.

Recall the notion of visibility domain from Section 3.

THEOREM 6.1. *Let $R_n = f_1 f_2 \cdots f_n$ be an integrable ergodic cocycle of holomorphic self-maps of a bounded domain X in $\mathbb{C}^N$ that is a visibility domain with the Kobayashi distance d. Then almost surely it holds that unless*

$$\frac{1}{n} d(x, R_n x) \to 0$$

as $n \to \infty$, there is a random point $\xi \in \partial X$ such that

$$R_n x \to \xi$$

$n \to \infty$, for any $x \in X$.

PROOF. Fix $x \in X$. Assume that for a.e. ω

$$\frac{1}{n} d\big(x, R(n,\omega)x\big) \to \tau > 0.$$

Take $0 < \epsilon < \tau$. By [44, Proposition 4.2], for a.e. ω, there is a sequence of $n_i \to \infty$ and K such that

$$d\big(x, R(n_i,\omega)x\big) - d\big(x, R(n_i - k, T^k\omega)x\big) \geq (\tau - \epsilon)k$$

for all $K < k < n_i$. Note that $d(R(n_i,\omega)x, R(k,\omega)x) \leq d(x, R(n_i - k, T^k\omega)x)$ by the nonexpansive property. This implies that

$$d\big(x, R(n_i,\omega)x\big) + d\big(x, R(k,\omega)x\big) - d\big(R(n_i,\omega)x, R(k,\omega)x\big) \geq (\tau - \epsilon)k$$

for all $K < k < n_i$ (for large k we could even insert a 2 on the right-hand side). Hence the left-hand side tends to infinity as $k < n_i \to \infty$. By compactness we may assume that $R(n_i,\omega)x \to \xi$ for some point $\xi = \xi(\omega) \in \partial X$.

Now suppose that for some subsequence k_j, $R(k_j,\omega)x \to \eta$ for some other boundary point η. Fix two disjoint closed neighborhoods V and W of ξ and η respectively. Consider all large enough j such that $R(k_j,\omega)x \in W$ and take a corresponding i_j such that $k_j < n_{i_j}$ and $R(n_{i_j},\omega)x \in V$, and take an almost geodesic σ_j joining these two orbit points. By the visibility assumption there is a C independent of j such that there exists t_j for which

$$d\big(x, \sigma_j(t_j)\big) < C,$$

for all large j. Since σ_j is an almost geodesic we have

$$d\big(R(k_j,\omega)x, \sigma_j(t_j)\big) + d\big(\sigma_j(t_j), R(n_{i_j},\omega)x\big) \leq d\big(R(n_{i_j},\omega)x, R(k_j,\omega)x\big) + 3\kappa.$$

Therefore by the triangle inequality

$$d\big(R(n_{i_j},\omega)x, R(k_j,\omega)x\big) > d\big(x, R(n_{i_j},\omega)x\big) + d\big(x, R(k_j,\omega)x\big) - 2C - 3\kappa.$$

So for an infinitude of n_i and ks we have

$$d\big(x, R(n_i,\omega)x\big) + d\big(x, R(k,\omega)x\big) - d\big(R(n_i,\omega)x, R(k,\omega)x\big) < 2C + 3\kappa,$$

but this contradicts the previous estimate. The conclusion follows. ∎

Without the visibility assumption, a more general result (in view of Theorem 3.5) holds corresponding to Theorem 4.3.

THEOREM 6.2 ([38, Theorem 18]). *Let (X, d) be a proper metric space, $\overline{X}$ a compactification, and $R_n = R(n, \omega)$ an integrable ergodic cocycle. Assume that $\tau > 0$. Then for a.e. ω there is a boundary point $\xi = \xi(\omega)$ such that*

$$R_n x \to S^\vee(\xi)$$

as $n \to \infty$.

The proof is similar to the previously stated theorem. It is not so easy in general to determine when the linear rate of escape τ is 0 or strictly positive. The difference in escape rate can be exemplified by Brownian motion in Euclidean spaces ($\tau = 0$ and no directional convergence) and hyperbolic spaces ($\tau > 0$ and asymptotic convergence). Another result that ultimately might prove to be yet more general is the following random version of Theorem 4.1 above.

THEOREM 6.3 (Ergodic theorem for noncommuting random operations, [32, 43]). *Let $R_n = R(n, \omega)$ be an integrable ergodic cocycle of nonexpansive maps of a metric space (X, d) assuming everything is measurable. Then there exists a.s. a metric functional $h = h^\omega$ of X such that*

$$\lim_{n\to\infty} -\frac{1}{n} h(R_n x) = \tau.$$

The proof in the isometry case uses the extension of the maps to the metric compactification, while the general case instead uses intricate subadditive ergodic theory. How to deduce results like Theorem 6.1 from this latter result is explained in the proof of [32, Corollary 5.2]. This reference also contains a result on the behavior of the Ahlfors–Beurling extremal length under the random iteration of holomorphic maps of Teichmüller spaces.

EXAMPLE. Let X be the Teichmüller space associated with a higher genus closed orientable surface. Let R_n be a non-degenerate random walk on the group of its complex automorphisms (which by a theorem of Royden coincides with the mapping class group in topology). The Teichmüller metric coincides with the Kobayashi metric (again by Royden). The metric compactification coincides with the Gardiner–Masur compactification of Teichmüller space [51]. The group in question acts properly on this space that has at most exponential growth and is a non-amenable group; it then follows from a theorem by Guivarch that the escape rate is $\tau > 0$. Therefore every random walk converges to a dual star at infinity of Teichmüller space. See [39] for more details and references. Note that [28] investigated the stars in the Thurston compactification, while for this other complex analytic boundary there has so far not appeared any study of its stars.

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"E quindi uscimmo a riveder le stelle."

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