



Jinghao Huang · Fedor Sukochev

Hermitian operators and isometries on symmetric operator spaces

Received December 21, 2020; revised October 31, 2022

Abstract. Let \mathcal{M} be an atomless semifinite von Neumann algebra (or an atomic von Neumann algebra with all atoms having the same trace) acting on a (not necessarily separable) Hilbert space \mathcal{H} equipped with a semifinite faithful normal trace τ . Let $E(\mathcal{M}, \tau)$ be a symmetric operator space affiliated with \mathcal{M} , whose norm is order continuous and is not proportional to the Hilbertian norm $\|\cdot\|_2$ on $L_2(\mathcal{M}, \tau)$. We obtain a general description of all bounded hermitian operators on $E(\mathcal{M}, \tau)$. This is the first time that the description of hermitian operators on a symmetric operator space (even for a noncommutative L_p -space) is obtained in the setting of general (non-hyperfinite) von Neumann algebras. As an application, we resolve a long-standing open problem concerning the description of isometries raised in the 1980s, which generalizes and unifies numerous earlier results.

Keywords. Surjective isometry, hermitian operator, semifinite von Neumann algebra, symmetric operator space

1. Introduction

The main purpose of this paper is to answer the following long-standing open question concerning isometries on a symmetric operator space (see e.g. [4, 23, 48, 93]).

Question 1.1. *If $E(0, \infty)$ is a separable symmetric function space on $(0, \infty)$ and if (\mathcal{M}, τ) is a semifinite von Neumann algebra (on a separable Hilbert space) with a semifinite faithful normal trace τ , how can one describe the family of surjective isometries on the symmetric operator space $E(\mathcal{M}, \tau)$ associated with $E(0, \infty)$?*

This is one of the most fundamental questions in the theory of symmetric operator/function/sequence spaces, and it has attracted substantial interest.

The study of the above question has a very long history, initiated by Stefan Banach [10], who obtained the general form of isometries between L_p -spaces on a finite

Jinghao Huang: Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, P.R. China; jinghao.huang@hit.edu.cn

Fedor Sukochev: School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia; f.sukochev@unsw.edu.au

Mathematics Subject Classification (2020): Primary 46B04; Secondary 46L52, 47B15

measure space in the 1930s. This result was extended by Lamperti [61] to certain Orlicz function spaces over σ -finite measure spaces. Representations of isometries between more general complex symmetric function spaces were later obtained by Lumer and by Zaidenberg [39, 100, 101] (see Arazy’s paper [8] for the case of complex sequence spaces). More precisely, Zaidenberg showed that under mild conditions on the complex function spaces $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ over the atomless σ -finite measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, any surjective isometry T between two complex symmetric function spaces $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ must be of the elementary form

$$(Tf)(t) = h(t)(T_1 f)(t), \quad f \in E_1, \tag{1.1}$$

where T_1 is the operator induced by a regular set isomorphism [39, Definition 3.2.3] from Ω_1 onto Ω_2 and h is a measurable function on Ω_2 [39, Theorem 5.3.5] (see also [100, 101]). Let (Γ, Σ, μ) be a discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$. We denote by $\ell_p(\Gamma)$, $1 \leq p \leq \infty$, the L_p -space on (Γ, Σ, μ) [64, p. xi]. Whereas $\ell_p(\Gamma)$ is a well-studied object (see e.g. [45, 64, 85] and references therein) and the description of surjective isometries of $\ell_p(\Gamma)$ already follows from [86, 99], the case of arbitrary symmetric spaces $E(\Gamma)$ for uncountable Γ has remained untreated. The description of isometries of these classical Banach spaces is a simple corollary of our general result, Theorem 1.2 below. We also note that isometries on real symmetric function spaces and those on complex symmetric function spaces have substantial differences (see e.g. the works of Braverman and Semenov [18, 19], Jamison, Kamińska and Lin [49], and Kalton and Randrianantoanina [55, 56, 75, 76]). Throughout this paper, unless stated otherwise, we only consider complex Banach spaces and surjective linear isometries.

A noncommutative version of Banach’s description of isometries between L_p -spaces [10] was obtained by Kadison [53] in the 1950s, who showed that a surjective isometry between two von Neumann algebras can be written as a Jordan $*$ -isomorphism followed by multiplication by a unitary operator. After the non-commutative L_p -spaces were introduced by Dixmier [27] and Segal [83] in the 1950s, the study of L_p -isometries was conducted by Broise [20], Russo [79], Arazy [5], Tam [96], etc. A complete description (for the semifinite case) was obtained in 1981 by Yeadon [99], who proved that every isometry $T : L_p(\mathcal{M}_1, \tau_1) \xrightarrow{\text{into}} L_p(\mathcal{M}_2, \tau_2)$, $1 \leq p \neq 2 < \infty$, has the form

$$T(x) = uBJ(x), \quad x \in \mathcal{M}_1 \cap L_p(\mathcal{M}_1, \tau_2), \tag{1.2}$$

where u is a partial isometry in \mathcal{M}_2 , B is a positive self-adjoint operator affiliated with \mathcal{M}_2 and J is a Jordan $*$ -isomorphism from \mathcal{M}_1 onto a weakly closed $*$ -subalgebra of \mathcal{M}_2 (see [51, 52, 86, 97] for the case when $\mathcal{M}_1, \mathcal{M}_2$ are of type III).

Isometries of general symmetric operator spaces on semifinite von Neumann algebras have been widely studied since the notion of symmetric operator spaces was introduced in the 1970s (see e.g. [34, 35, 57, 72, 73, 92] and references therein). The question posed at the beginning of the paper indeed asks whether these isometries T have a natural description as in the cases of symmetric function spaces and noncommutative L_p -spaces (see (1.1) and (1.2)). One of the most important developments in this area is due to Sourour [90],

who described isometries on separable symmetric operator ideals, that is, when \mathcal{M} is the \ast -algebra $B(\mathcal{H})$ of all bounded linear operator on a separable Hilbert space \mathcal{H} . Adopting Sourour’s techniques, the second author [93] obtained the description of isometries on separable symmetric operator spaces affiliated with hyperfinite type II factors. However, the approach used in [90] strongly relies on the matrix representation of compact operators on a separable Hilbert space \mathcal{H} , which is not applicable to symmetric operator spaces affiliated with general semifinite von Neumann algebras. In the latter case, only partial results have been obtained. For example, the general form of isometries of Lorentz spaces on a finite von Neumann algebra was obtained in [23] (see also [69]). Under additional conditions on the isometries (e.g., disjointness-preserving, order-preserving, etc.), similar descriptions can be found in [1, 23, 25, 26, 40, 48, 58, 62, 68, 77, 91], which provide partial answers to the question posed at the outset of this paper.

The following theorem answers Question 1.1 in full generality.

Theorem 1.2. *Let \mathcal{M}_1 and \mathcal{M}_2 be atomless von Neumann algebras (or atomic von Neumann algebras whose atoms all have the same trace) equipped with semifinite faithful normal traces τ_1 and τ_2 , respectively. Let $E(\mathcal{M}_1, \tau_1)$ and $F(\mathcal{M}_2, \tau_2)$ be symmetric operator spaces whose norms are order continuous and are not proportional to $\|\cdot\|_2$. If $T : E(\mathcal{M}_1, \tau_1) \rightarrow F(\mathcal{M}_2, \tau_2)$ is a surjective isometry, then there exist nets $A_i \in F(\mathcal{M}_2, \tau_2)$, $i \in I$, disjointly supported from the right, and $B_i \in F(\mathcal{M}_2, \tau_2)$, $i \in I$, disjointly supported from the left, a surjective Jordan \ast -isomorphism $J : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ and a central projection $z \in \mathcal{M}_2$ such that*

$$T(x) = \|\cdot\|_F\text{-}\sum_{i \in I} J(x)A_i z + B_i J(x)(\mathbf{1} - z), \quad x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1,$$

where the series is taken as the limit of all finite partial sums. In particular, if \mathcal{M} is σ -finite, then the nets $\{A_i\}$ and $\{B_i\}$ are countable. If the trace τ is finite, then there exist elements $A, B \in F(\mathcal{M}_2, \tau_2)$ such that

$$T(x) = J(x)Az + BJ(x)(\mathbf{1} - z), \quad x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1.$$

This extends numerous earlier results on this topic (see e.g. [8, 23, 25, 26, 66–69, 79, 86, 90, 93, 97, 99]), and Theorem 1.2 yields the first description of surjective isometries on symmetric operator spaces associated with non-hyperfinite algebras. On the other hand, we show that if \mathcal{M} has atoms whose traces are different, then there exists a symmetric space $E(\mathcal{M}, \tau)$ (whose norm is not proportional to $\|\cdot\|_2$) and an isometry on $E(\mathcal{M}, \tau)$ which is not of the form (1.1) (see Example 6.2). This demonstrates that the assumption imposed on the von Neumann algebra is sharp.

Recall that the notion of hermitian operators on a Banach space was formulated by Lumer [66] in his seminal paper in the 1960s, for the purpose of extending Hilbert space type arguments to Banach spaces. This notion plays an important role in different fields such as operator theory on Banach spaces, matrix theory, optimal control theory and computer science (see e.g. [14, 39, 39, 40, 43, 66, 87, 98] and references therein).

The main method used in this paper for the description of isometries is to establish and employ the general description of hermitian operators on the symmetric operator spaces $E(\mathcal{M}, \tau)$. The following result is rather surprising as it shows that the stock of hermitian operators does not depend on the symmetric space $E(\mathcal{M}, \tau)$, and it is fully determined by the algebra \mathcal{M} .

Theorem 1.3. *Let $E(\mathcal{M}, \tau)$ be a symmetric space on an atomless semifinite von Neumann algebra (or an atomic von Neumann algebra with all atoms having the same trace) \mathcal{M} equipped with a semifinite faithful normal trace τ . Assume that $\|\cdot\|_E$ is order continuous and is not proportional to $\|\cdot\|_2$. Then a bounded operator T on $E(\mathcal{M}, \tau)$ is a hermitian operator on $E(\mathcal{M}, \tau)$ if and only if there exist self-adjoint operators a and b in \mathcal{M} such that*

$$Tx = ax + xb, \quad x \in E(\mathcal{M}, \tau).$$

In particular, T can be extended to a bounded hermitian operator on the von Neumann algebra \mathcal{M} .

This idea to employ hermitian operators to describe isometries lurks in the background of Lumer's description of isometries on Orlicz spaces [39], but the study of hermitian operators on noncommutative spaces is substantially more difficult than that of function spaces, and descriptions of hermitian operators are known only for very few operator spaces. For example, Sinclair [87] obtained the general form of hermitian operators on a C^* -algebra using earlier results on derivations on operator algebras; Sourour [90] obtained the general form of hermitian operators on separable operator ideals of $B(\mathcal{H})$ when \mathcal{H} is separable; and the case of symmetric operator spaces on hyperfinite type II factors was settled by the second author [93] by adopting Sourour's approach. For more general von Neumann algebras, the form of a hermitian operator on a symmetric space (even on noncommutative L_p -spaces, see [88, Theorem 4] and [89, Theorem 4.2] for partial results) has been unknown. Theorem 1.3 yields a complete description of hermitian operators on a symmetric operator space having order continuous norm by using a different approach than those in [90, 93].

The main ingredient of the proof of Theorem 1.3 is the following surprising observation: any bounded hermitian operator on $E(\mathcal{M}, \tau)$ can be "reduced" to a bounded hermitian operator on the so-called τ -compact ideal $C_0(\mathcal{M}, \tau)$ (which is a C^* -algebra), and therefore it can be written as the sum of left multiplication by a self-adjoint operator in \mathcal{M} and right multiplication by a self-adjoint operator in \mathcal{M} [87]. Having such a description at hand, we are able to describe isometries on symmetric operator spaces affiliated with \mathcal{M} and infer Theorem 1.2. However, the structure of a bounded hermitian operator on a von Neumann algebra is more complicated than for a factor, and this inference is far from straightforward. As pointed out in [25, p. 825], constructing a suitable Jordan $*$ -isomorphism from an isometry (even if this isometry is positive and finiteness-preserving [25]) is always 'problematic'. A simple adaptation of proofs in [90, 93] does not yield Theorem 1.2. Many new techniques are required in the proof of Theorem 1.2,

which are of interest in their own right and have potential usage in the future study of hermitian operators, Jordan $*$ -isomorphisms and isometries of vector-valued spaces.

Finally, as an application of Theorem 1.2, we consider a variant of Pełczyński’s problem on the uniqueness of symmetric structure of operator ideals for symmetric structure of $E(\mathcal{M}, \tau)$ affiliated with a II_1 factor, which establishes a noncommutative version of a result by Abramovich and Zaidenberg on uniqueness of symmetric structure of $L_p(0, 1)$ [2, Theorem 1] and its generalizations due to Zaidenberg [100] and Kalton and Randriantaoanina [56, 75].

2. Preliminaries

In this section, we recall the main notions of the theory of noncommutative integration, introduce some properties of generalised singular value functions and define noncommutative symmetric operator spaces. For details of von Neumann algebra theory, the reader is referred to e.g. [15, 28, 54, 94]. General facts concerning measurable operators may be found in [30, 71, 83] (see also the forthcoming book [32]). For the convenience of the reader, some of the basic definitions are recalled.

2.1. τ -Measurable operators and generalised singular values

In what follows, \mathcal{H} is a (not necessarily separable) Hilbert space and $(B(\mathcal{H}), \|\cdot\|_\infty)$ is the $*$ -algebra of all bounded linear operators on \mathcal{H} , and $\mathbf{1}$ is the identity operator on \mathcal{H} . Let \mathcal{M} be a von Neumann algebra on \mathcal{H} . Let $\mathcal{P}(\mathcal{M})$ be the set of all projections of \mathcal{M} . We denote by \mathcal{M}_p the reduced von Neumann algebra $p\mathcal{M}p$ generated by a projection $p \in \mathcal{P}(\mathcal{M})$.

A linear operator $x : \mathfrak{D}(x) \rightarrow \mathcal{H}$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of \mathcal{H} , is said to be *affiliated* with \mathcal{M} if $yx \subseteq xy$ for all $y \in \mathcal{M}'$, where \mathcal{M}' is the commutant of \mathcal{M} . A linear operator $x : \mathfrak{D}(x) \rightarrow \mathcal{H}$ is termed *measurable with respect to \mathcal{M}* if x is closed, densely defined, affiliated with \mathcal{M} , and there exists a sequence $\{p_n\}_{n=1}^\infty$ in the set $\mathcal{P}(\mathcal{M})$ of all projections of \mathcal{M} such that $p_n \uparrow \mathbf{1}$, $p_n(\mathcal{H}) \subseteq \mathfrak{D}(x)$ and $\mathbf{1} - p_n$ is a finite projection (with respect to \mathcal{M}) for all n . It should be noted that the condition $p_n(\mathcal{H}) \subseteq \mathfrak{D}(x)$ implies that $xp_n \in \mathcal{M}$. The collection of all operators measurable with respect to \mathcal{M} is denoted by $S(\mathcal{M})$; it is a unital $*$ -algebra with respect to strong sums and products (denoted simply by $x + y$ and xy for all $x, y \in S(\mathcal{M})$).

Let x be a self-adjoint operator affiliated with \mathcal{M} . We denote its spectral measure by $\{e^x\}$. It is well known that if x is a closed operator affiliated with \mathcal{M} with polar decomposition $x = u|x|$, then $u \in \mathcal{M}$ and $e \in \mathcal{M}$ for all projections $e \in \{e^{|x|}\}$. Moreover, $x \in S(\mathcal{M})$ if and only if x is closed, densely defined, affiliated with \mathcal{M} and $e^{|x|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that when \mathcal{M} is a von Neumann algebra of type III or a type I factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type II von Neumann algebras, this is no longer true. From now on, let \mathcal{M} be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ .

For any closed and densely defined linear operator $x : \mathfrak{D}(x) \rightarrow \mathcal{H}$, the *null projection* $n(x) = n(|x|)$ is the projection onto the kernel $\text{Ker}(x)$. The *left support* $l(x)$ is the projection onto the closure of the range $\text{Ran}(x)$, and the *right support* $r(x)$ of x is defined by $r(x) = \mathbf{1} - n(x)$.

An operator $x \in S(\mathcal{M})$ is called τ -*measurable* if there exists a sequence $\{p_n\}_{n=1}^\infty$ in $\mathcal{P}(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(\mathcal{H}) \subseteq \mathfrak{D}(x)$ and $\tau(\mathbf{1} - p_n) < \infty$ for all n . The collection of all τ -measurable operators is a unital $*$ -subalgebra of $S(\mathcal{M})$, denoted by $S(\mathcal{M}, \tau)$. It is well-known that a linear operator x belongs to $S(\mathcal{M}, \tau)$ if and only if $x \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator x affiliated with \mathcal{M} is τ -measurable (see [37]) if and only if

$$\tau(e^{|x|}(n, \infty)) \rightarrow 0, \quad n \rightarrow \infty.$$

Definition 2.1. Let a semifinite von Neumann algebra \mathcal{M} be equipped with a faithful normal semifinite trace τ and let $x \in S(\mathcal{M}, \tau)$. The *generalised singular value function* $\mu(x) : t \mapsto \mu(t; x)$, $t > 0$, of the operator x is defined by

$$\mu(t; x) = \inf\{\|xp\|_\infty : p \in \mathcal{P}(\mathcal{M}), \tau(\mathbf{1} - p) \leq t\}.$$

An equivalent definition in terms of the distribution function of the operator x is the following. For every self-adjoint operator $x \in S(\mathcal{M}, \tau)$, setting

$$d_x(t) = \tau(e^x(t, \infty)), \quad t > 0,$$

we have (see e.g. [37, 65])

$$\mu(t; x) = \inf\{s \geq 0 : d_{|x|}(s) \leq t\}.$$

Note that $d_x(\cdot)$ is a right-continuous function (see e.g. [32, 37]).

Consider the algebra $\mathcal{M} = L^\infty(0, \infty)$ of all Lebesgue measurable essentially bounded functions on $(0, \infty)$. The algebra \mathcal{M} can be viewed as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0, \infty)$, with the trace given by integration with respect to Lebesgue measure m . It is easy to see that the algebra of all τ -measurable operators affiliated with \mathcal{M} can be identified with the subalgebra $S(0, \infty)$ of the algebra of Lebesgue measurable functions which consists of all functions f such that $m(\{|f| > s\})$ is finite for some $s > 0$. It should also be pointed out that the generalised singular value function $\mu(f)$ is precisely the decreasing rearrangement $\mu(f)$ of the function $|f|$ (see e.g. [11, 59]) defined by

$$\mu(t; f) = \inf\{s \geq 0 : m(\{|f| \geq s\}) \leq t\}.$$

For the convenience of the reader, we also recall the definition of the measure topology t_τ on the algebra $S(\mathcal{M}, \tau)$. For every $\varepsilon, \delta > 0$, we define the set

$$V(\varepsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \exists p \in \mathcal{P}(\mathcal{M}) \text{ such that } \|x(\mathbf{1} - p)\|_\infty \leq \varepsilon, \tau(p) \leq \delta\}.$$

The topology generated by the sets $V(\varepsilon, \delta)$, $\varepsilon, \delta > 0$, is called the *measure topology* t_τ on $S(\mathcal{M}, \tau)$ [32, 37, 71]. It is well known that the algebra $S(\mathcal{M}, \tau)$ equipped with the measure topology is a complete metrizable topological algebra [71]. We note that a sequence $\{x_n\}_{n=1}^\infty \subset S(\mathcal{M}, \tau)$ converges to zero in the measure topology t_τ if and only if $\tau(e^{|x_n|})(\varepsilon, \infty) \rightarrow 0$ as $n \rightarrow \infty$ for all $\varepsilon > 0$ [32].

The space $S_0(\mathcal{M}, \tau)$ of τ -compact operators is the space associated to the algebra of functions from $S(0, \infty)$ vanishing at infinity, that is,

$$S_0(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \mu(\infty; x) = 0\}.$$

The two-sided ideal $\mathcal{F}(\tau)$ in \mathcal{M} consisting of all elements of τ -finite range is defined by

$$\mathcal{F}(\tau) = \{x \in \mathcal{M} : \tau(r(x)) < \infty\} = \{x \in \mathcal{M} : \tau(s(x)) < \infty\}.$$

Note that $S_0(\mathcal{M}, \tau)$ is the closure of $\mathcal{F}(\tau)$ in the measure topology [30].

A further important vector space topology on $S(\mathcal{M}, \tau)$ is the *local measure topology* [30, 32]. A neighbourhood base for this topology is given by the sets $V(\varepsilon, \delta; p)$, $\varepsilon, \delta > 0$, $p \in \mathcal{P}(\mathcal{M}) \cap \mathcal{F}(\tau)$, where

$$V(\varepsilon, \delta; p) = \{x \in S(\mathcal{M}, \tau) : pxp \in V(\varepsilon, \delta)\}.$$

It is clear that the local measure topology is weaker than the measure topology [30, 32]. If $\{x_\alpha\} \subset S(\mathcal{M}, \tau)$ is a net and if $x_\alpha \rightarrow_\alpha x \in S(\mathcal{M}, \tau)$ in the local measure topology, then $x_\alpha y \rightarrow xy$ and $y x_\alpha \rightarrow yx$ in the local measure topology for all $y \in S(\mathcal{M}, \tau)$ [30, 32]. If $a_i \geq 0$ is an increasing net in $S(\mathcal{M}, \tau)$ and if $a \in S(\tau)$ is such that $a = \sup a_i$, then we write $0 \leq a_i \uparrow a$ [30, p. 212]. If $\{x_i\}$ is an increasing net in $S(\mathcal{M}, \tau)_+$ and $x \in S(\mathcal{M}, \tau)_+$ is such that $x_i \rightarrow x$ in the local measure topology, then $x_i \uparrow x$ (see e.g. [32, Chapter II, Proposition 7.6 (iii)]).

2.2. Symmetric spaces of τ -measurable operators

Let $E(0, \infty)$ be a Banach space of real-valued Lebesgue measurable functions on $(0, \infty)$ (with identification m -a.e.), equipped with a norm $\|\cdot\|_E$. The space $E(0, \infty)$ is said to be *absolutely solid* if $x \in E(0, \infty)$ and $|y| \leq |x|$, $y \in S(0, \infty)$ implies that $y \in E(0, \infty)$ and $\|y\|_E \leq \|x\|_E$. An absolutely solid space $E(0, \infty) \subseteq S(0, \infty)$ is said to be *symmetric* if for every $x \in E(0, \infty)$ and every $y \in S(0, \infty)$, the assumption $\mu(y) = \mu(x)$ implies that $y \in E(0, \infty)$ and $\|y\|_E = \|x\|_E$ [59]. Without loss of generality, throughout this paper, we always assume that $\|\chi_{(0,1)}\|_{E(0,\infty)} = 1$.

We now come to the definition of the main object of this paper.

Definition 2.2. Let \mathcal{M} be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ . Let \mathcal{E} be a linear subset in $S(\mathcal{M}, \tau)$ equipped with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that \mathcal{E} is a *symmetric space* if for every $x \in \mathcal{E}$, the conditions $y \in S(\mathcal{M}, \tau)$ and $\mu(y) \leq \mu(x)$ imply that $y \in \mathcal{E}$ and $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$.

Let $E(\mathcal{M}, \tau)$ be a symmetric space. It is well-known that any symmetrically normed space $E(\mathcal{M}, \tau)$ is a normed \mathcal{M} -bimodule (see e.g. [30, 32]). That is, for any symmetric operator space $E(\mathcal{M}, \tau)$, we have $\|axb\|_E \leq \|a\|_\infty \|b\|_\infty \|x\|_E, a, b \in \mathcal{M}, x \in E(\mathcal{M}, \tau)$. It is known that if $E(\mathcal{M}, \tau)$ has order continuous norm $\|\cdot\|_E$, i.e., $\|x_\alpha\|_E \downarrow 0$ whenever $0 \leq x_\alpha \downarrow 0 \subset E(\mathcal{M}, \tau)$, then $E(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$ [30, 32, 48].

The so-called Köthe dual is identified with an important part of the dual space. If $E(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau)$ is a symmetric space, then the Köthe dual $E(\mathcal{M}, \tau)^\times$ of E is defined by

$$E(\mathcal{M}, \tau)^\times = \left\{ x \in S(\mathcal{M}, \tau) : \sup_{\|y\|_E \leq 1, y \in E(\mathcal{M}, \tau)} \tau(|xy|) < \infty \right\}.$$

The Köthe dual $E(\mathcal{M}, \tau)^\times$ can be identified as a subspace of the Banach dual $E(\mathcal{M}, \tau)$ via the trace duality [30, p. 228]. Recall that $x \in L_1(\mathcal{M}, \tau) + \mathcal{M} := \{a \in S(\mathcal{M}, \tau) : \mu(a) \in L_1(0, \infty) + L_\infty(0, \infty)\}$ can be equipped with the norm $\|x\|_{L_1+L_\infty} = \int_0^1 \mu(s; x) ds$ and $x \in L_1(\mathcal{M}, \tau) \cap \mathcal{M} := \{a \in S(\mathcal{M}, \tau) : \mu(a) \in L_1(0, \infty) \cap L_\infty(0, \infty)\}$ can be equipped with the norm $\|x\|_{L_1 \cap L_\infty} := \max\{\|x\|_1, \|x\|_\infty\}$. In particular, $(L_1(\mathcal{M}, \tau) \cap \mathcal{M})^\times = L_1(\mathcal{M}, \tau) + \mathcal{M}$ and $L_1(\mathcal{M}, \tau) \cap \mathcal{M} = (L_1(\mathcal{M}, \tau) + \mathcal{M})^\times$ [30, Example 4].

The carrier projection $c_E \in \mathcal{M}$ of $E(\mathcal{M}, \tau)$ is defined by

$$c_E := \bigvee \{p : p \in \mathcal{P}(E)\}.$$

It is clear that c_E is in the centre $Z(\mathcal{M})$ of \mathcal{M} [30]. It is often assumed that the carrier projection c_E is equal to $\mathbf{1}$. Indeed, for any symmetric function space $E(0, \infty)$, the carrier projection of the corresponding operator space $E(\mathcal{M}, \tau)$ is always $\mathbf{1}$ (see e.g. [30, 57]). On the other hand, if \mathcal{M} is atomless or is atomic and all atoms have equal trace, then the carrier projection of any non-zero symmetric space $E(\mathcal{M}, \tau)$ is necessarily $\mathbf{1}$ [30, 32]. In this case, whenever $E(\mathcal{M}, \tau)$ has order continuous norm, then $E(\mathcal{M}, \tau)^\times$ is isometrically isomorphic to $E(\mathcal{M}, \tau)^*$ (see e.g. [35], [29, Proposition 6.4] or [30, Proposition 47 (v)]).

There exists a strong connection between symmetric function spaces and operator spaces [57] (see also [30, 65]). The operator space $E(\mathcal{M}, \tau)$ defined by

$$E(\mathcal{M}, \tau) := \{x \in S(\mathcal{M}, \tau) : \mu(x) \in E(0, \infty)\}, \quad \|x\|_{E(\mathcal{M}, \tau)} := \|\mu(x)\|_E,$$

is a complete symmetric space whenever $(E(0, \infty), \|\cdot\|_E)$ is a complete symmetric function space on $(0, \infty)$ [57]. In particular, for any symmetric function space $E(0, \infty)$, $F(\tau) \subset E(\mathcal{M}, \tau)$ [30, Lemma 18]. In the special case when $E(0, \infty) = L_p(0, \infty)$, $1 \leq p \leq \infty$, $E(\mathcal{M}, \tau)$ is the noncommutative L_p -spaces affiliated with \mathcal{M} and we denote the norm by $\|\cdot\|_p$. We note that if $E(0, \infty)$ is separable (i.e. has order continuous norm), then $E^\times(\mathcal{M}, \tau)$ is isometrically isomorphic to $E(\mathcal{M}, \tau)^*$ [30, p. 246]. Recall that every separable symmetric sequence/function space E is fully symmetric, that is, if $x \in E$ and $y \in \ell_\infty$ (resp. $y \in S(0, \infty)$) with

$$\int_0^s \mu(t; y) dt \leq \int_0^s \mu(t; x) dt, \quad s \geq 0,$$

(denoted by $y \ll x$), then $y \in E$ with $\|y\|_E \leq \|x\|_E$ (see e.g. [59, Chapter II, Theorem 4.10] or [32, Chapter IV, Theorem 5.7]).

3. Hermitian operators

Let X be a Banach space. Recall that a *semi-inner product* (abbreviated *s.i.p.*) on X is a mapping $\langle \cdot, \cdot \rangle$ of $X \times X$ into the field of complex numbers such that

- (1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for $x, y, z \in X$;
- (2) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $x, y \in X$ and $\lambda \in \mathbb{C}$;
- (3) $\langle x, x \rangle > 0$ for $0 \neq x \in X$;
- (4) $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$ for any $x, y \in X$.

When a s.i.p. is defined on X , we call X a *semi-inner-product space* (abbreviated *s.i.p.s.*). If X is a s.i.p.s., then $\langle x, x \rangle^{1/2}$ is a norm on X . On the other hand, every Banach space can be made into a s.i.p.s. (in general, in infinitely many ways) so that the s.i.p. is consistent with the norm, i.e., $\langle x, x \rangle^{1/2} = \|x\|$ for any $x \in X$ [39]. By virtue of the Hahn–Banach theorem, this can be accomplished by choosing one bounded linear functional f_x for each $x \in X$ such that $\|f_x\| = \|x\|$ and $f_x(x) = \|x\|^2$ (f_x is called a *support functional* of x), and then setting $\langle x, y \rangle = f_y(x)$ for arbitrary $x, y \in X$ [14, 39, 66]. Given a linear transformation T mapping a s.i.p.s. into itself, we denote by $W(T)$ the *numerical range* of T , that is, $\{\langle Tx, x \rangle : \langle x, x \rangle = 1, x \in X\}$. Let T be an operator on a Banach space $(X, \|\cdot\|)$. Although in principle there may be many different s.i.p. consistent with $\|\cdot\|$, nonetheless if the numerical range of T relative to one such s.i.p. is real, then the numerical range relative to any such s.i.p. is real (see e.g. [39, p. 107], [66, Section 6] and [14, p. 377]). If this is the case, T is said to be a *hermitian* operator on X .

From now on, unless stated otherwise, we always assume that \mathcal{M} is an atomless semifinite von Neumann algebra or an atomic semifinite von Neumann algebra with all atoms having the same trace (without loss of generality, we assume that $\tau(e) = 1$ for any atom $e \in \mathcal{M}$), and we assume that τ is a semifinite faithful normal trace on \mathcal{M} .

In particular, when \mathcal{M} is atomless (resp. atomic), the set

$$E(0, \tau(\mathbf{1})) := \{f \in S(0, \tau(\mathbf{1})) : \mu(f) = \mu(x) \text{ for some } x \in E(\mathcal{M}, \tau)\}$$

(resp.

$$\ell_E := \{f \in \ell_\infty : \mu(f) = \mu(x) \text{ for some } x \in E(\mathcal{M}, \tau)\})$$

is a symmetric function (resp. sequence) space [65, Theorem 2.5.3]. There exists a bijective correspondence between symmetric operator spaces and symmetric function/sequence spaces. Therefore, if $\|\cdot\|_E$ on $E(\mathcal{M}, \tau)$ is not proportional to $\|\cdot\|_2$ on $L_2(\mathcal{M}, \tau)$, then $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$ on $L_2(\mathcal{A}, \tau)$ for any maximal abelian von Neumann subalgebra \mathcal{A} of \mathcal{M} .

Sourour [90, Lemma 1] obtained Lemma 3.3 below in the setting of $B(\mathcal{H})$ by using a result due to Schneider and Turner (see e.g. [82, Lemma 3.1] and [39, Lemma 9.2.7]). Arazy gave a self-contained alternative proof in the setting of complex sequence spaces [8]. In the proof of the following lemma, we adapt Arazy’s proof. Due to the technical differences between the atomless case and atomic case, we provide a full proof below.

To prove Lemma 3.3 below, we need the following well-known proposition. For the sake of completeness, we provide a short proof below.

Proposition 3.1. *Let $p \in \mathcal{F}(\tau)$ be a projection and let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Then $\frac{\|p\|_E^2}{\tau(p)} p \in E(\mathcal{M}, \tau)^*$ is a support functional of $p \in E(\mathcal{M}, \tau)$, i.e.,*

$$\tau\left(p \cdot \frac{\|p\|_E^2}{\tau(p)} p\right) = \|p\|_E^2 = \|p\|_E \left\| \frac{\|p\|_E^2}{\tau(p)} p \right\|_{E^*}.$$

In particular, for any bounded hermitian operator T on $E(\mathcal{M}, \tau)$, we have

$$\tau(T(p)p) \in \mathbb{R}. \tag{3.1}$$

Proof. We only consider the case when \mathcal{M} is atomless; the atomic case uses the same argument (see also [8] or [39, Theorem 5.2.13]).

Note that

$$\|p\|_{E^*} = \|p\|_{E^{\times}} = \sup \left\{ \int_0^{\tau(p)} \mu(s; z) ds : z \in E(\mathcal{M}, \tau), \|z\|_E = 1 \right\}.$$

Since

$$\frac{\int_0^{\tau(p)} \mu(s; z) ds}{\tau(p)} \mu(p) = \frac{\int_0^{\tau(p)} \mu(s; z) ds}{\tau(p)} \chi_{(0, \tau(p))} \prec\prec \mu(z), \quad z \in E(\mathcal{M}, \tau), \|z\|_E = 1$$

(see e.g. [65, Section 3.6]), we obtain $\frac{\int_0^{\tau(p)} \mu(s; z) ds}{\tau(p)} \|p\|_E \leq \|z\|_E = 1$, and therefore

$$\|p\|_{E^*} \leq \frac{\tau(p)}{\|p\|_E}.$$

On the other hand, by [30, Remark 3] we have

$$\tau(p) \leq \|p\|_{E^*} \|p\|_E.$$

Hence, $\tau(p) = \|p\|_{E^*} \|p\|_E$, i.e. $\tau\left(p \cdot \frac{\|p\|_E^2}{\tau(p)} p\right) = \|p\|_E^2 = \|p\|_E \left\| \frac{\|p\|_E^2}{\tau(p)} p \right\|_{E^*}$. ■

Corollary 3.2. *Let $u \in \mathcal{F}(\tau)$ be a partial isometry and let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Then $\frac{\|u^*u\|_E^2}{\tau(u^*u)} u^* \in E(\mathcal{M}, \tau)^*$ is a support functional of $u \in E(\mathcal{M}, \tau)$. In particular, for any bounded hermitian operator T on $E(\mathcal{M}, \tau)$, we have*

$$\tau(T(u)u^*) \in \mathbb{R}.$$

Proof. Since $u \in \mathcal{F}(\tau)$, it follows that $r(u)$ and $l(u)$ are τ -finite projections. Hence, $r(u) \vee l(u)$ is also τ -finite. Therefore, there exists a unitary element v in $\mathcal{M}_{r(u) \vee l(u)}$ such that $v^*l(u)v = r(u)$ [95, Chapter XIV, Lemma 2.1].

Define $v' := u + v \cdot (r(u) \vee l(u) - r(u))$. Note that

$$\begin{aligned} (r(u) \vee l(u) - r(u))v^*u &= (r(u) \vee l(u) - r(u))v^*l(u)u \\ &= (r(u) \vee l(u) - r(u))r(u)v^*u = 0. \end{aligned}$$

We have

$$\begin{aligned} (v')^*v' &= (u + v \cdot (r(u) \vee l(u) - r(u)))^*(u + v \cdot (r(u) \vee l(u) - r(u))) \\ &= r(u) + (r(u) \vee l(u) - r(u)) = r(u) \vee l(u). \end{aligned}$$

Hence, v' is a unitary element in $\mathcal{M}_{r(u) \vee l(u)}$, and therefore

$$v'' := v' + (\mathbf{1} - r(u) \vee l(u))$$

is a unitary element in \mathcal{M} . It follows that $(v'')^*T(v''\cdot)$ on $E(\mathcal{M}, \tau)$ is also a bounded hermitian operator [39, p. 22]. Noting that $v''r(u) = u$, we obtain

$$\tau(T(u)u^*) = \tau((v'')^*T(v''r(u))r(u)) \stackrel{(3.1)}{\in} \mathbb{R}. \quad \blacksquare$$

Lemma 3.3. *Let $E(\mathcal{M}, \tau)$ be a symmetric space affiliated with \mathcal{M} , whose norm is order continuous and is not proportional to $\|\cdot\|_2$. Let $x_1, x_2 \in \mathcal{F}(\tau)$ be two partial isometries such that $l(x_1) \perp l(x_2)$ and $r(x_1) \perp r(x_2)$. Then, for any bounded hermitian operator $T : E(\mathcal{M}, \tau) \rightarrow E(\mathcal{M}, \tau)$, we have*

$$\tau(T(x_1)x_2^*) = 0.$$

Consequently, if $x_1 \in E(\mathcal{M}, \tau)$ and $x_2 \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}$ with $l(x_1) \perp l(x_2)$ and $r(x_1) \perp r(x_2)$, then

$$\tau(T(x_1)x_2^*) = 0.$$

Proof. We only handle the case of an atomless von Neumann algebra; the atomic case follows by a similar argument.

We first consider the case when x_1 and x_2 are two projections such that $x_1x_2 = 0$.

Since $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$, it follows that there exists a set $\{e_i\}_{1 \leq i \leq n}$ of pairwise orthogonal projections having the same trace such that $\|\cdot\|_E$ on $E(\mathcal{A})$ is not proportional to $\|\cdot\|_2$ on $L_2(\mathcal{A})$, where \mathcal{A} is the abelian weakly closed $*$ -algebra generated by $\{e_i\}_{1 \leq i \leq n}$. Let

$$t_{1,2} := \tau(T(e_1)e_2) \quad \text{and} \quad t_{2,1} := \tau(T(e_2)e_1).$$

By Proposition 3.1, we find that $\tau(T(e_1)e_1), \tau(T(e_2)e_2) \in \mathbb{R}$. We claim that

$$t_{i,j} = \overline{t_{j,i}} \tag{3.2}$$

when $i, j = 1, \dots, n$ and $i \neq j$. Define $x_\theta := e_1 + e^{i\theta}e_2, 0 \leq \theta \leq 2\pi$. In particular, $\tau(x_\theta x_\theta^*) = \tau(e_1 + e_2) \in \mathbb{R}$. By Corollary 3.2, we obtain

$$\tau(T(x_\theta)x_\theta^*) \in \mathbb{R},$$

i.e., $e^{i\theta}t_{i,j} + e^{-i\theta}t_{j,i} \in \mathbb{R}$ for all θ . Hence, (3.2) follows.

By [8, Lemma 4] (see also [4, 39]), there exist $1 < n < \infty$, $x = \sum_{k=1}^n x(k)e_k$, and $y = \sum_{k=1}^n y(k)e_k$ such that

- (1) $x(k), y(k) \geq 0$ for all k ;
- (2) $\|x\|_E = \|y\|_{E^*} = \tau(xy^*) = 1$;
- (3) x and y are linearly independent.

Let $\theta = (\theta_1, \dots, \theta_n)$, $0 \leq \theta_k \leq 2\pi$, and let

$$x_\theta = \sum_{k=1}^n e^{i\theta_k} x(k)e_k \quad \text{and} \quad y_\theta = \sum_{k=1}^n e^{i\theta_k} y(k)e_k.$$

By the assumption on x and y , we obtain

$$\|x_\theta\|_E = \|y_\theta\|_{E^*} = \tau(y_\theta^* x_\theta) \stackrel{(2)}{=} 1.$$

Therefore, by the definition of a hermitian operator, for any θ we have

$$\begin{aligned} 0 &= \text{Im} \tau(y_\theta^* T(x_\theta)) = \text{Im} \sum_{k,l=1}^n e^{i(\theta_k - \theta_l)} x(k)y(l)t_{l,k} \\ &\stackrel{3.2}{=} \frac{1}{2i} \sum_{k \neq l} x(k)y(l)(e^{i(\theta_k - \theta_l)} t_{l,k} - e^{i(\theta_l - \theta_k)} t_{k,l}) \\ &= \frac{1}{2i} \sum_{k \neq l} x(k)y(l)e^{i(\theta_k - \theta_l)} t_{l,k} - \frac{1}{2i} \sum_{k \neq l} x(k)y(l)e^{i(\theta_l - \theta_k)} t_{k,l} \\ &= \frac{1}{2i} \sum_{k \neq l} x(k)y(l)e^{i(\theta_k - \theta_l)} t_{l,k} - \frac{1}{2i} \sum_{k \neq l} x(l)y(k)e^{i(\theta_k - \theta_l)} t_{l,k} \\ &= \frac{1}{2i} \sum_{k \neq l} e^{i(\theta_k - \theta_l)} t_{l,k} (x(k)y(l) - x(l)y(k)). \end{aligned}$$

This implies that $t_{l,k}(x(k)y(l) - x(l)y(k)) = 0$ for all k and l .

Since x, y are linearly independent, it follows that there exist $k \neq l$ such that $x(k)y(l) - x(l)y(k) \neq 0$ and thus $t_{l,k} = 0$. Replacing in this argument x and y by $x_\pi = \sum x_k e_{\pi(k)}$ and $y_\pi = \sum y_k e_{\pi(k)}$, respectively, where π is an arbitrary permutation of $\{1, \dots, n\}$, we deduce that $t_{k,l} = 0$ for every $k \neq l$. Therefore,

$$\tau(T(p)q) = 0$$

for any projections p, q with $pq = 0$ and $\tau(p) = \tau(q) = \tau(e_1)$. Clearly, if the norm of the space $(E(\mathcal{A}), \|\cdot\|_{E(\mathcal{M}, \tau)})$ on the algebra generated by e_k , $1 \leq k \leq n$, is not proportional to $\|\cdot\|_2$, then the norm $\|\cdot\|_E$ on $E(\mathcal{A}')$ on the algebra generated by e'_k , $1 \leq k \leq 2n$, is not proportional to $\|\cdot\|_2$ either, where $\tau(e'_k) = \frac{1}{2}\tau(e_k)$. This implies that for any $k \in \mathbb{N}$, if p, q are τ -finite projections such that $pq = 0$ and $\tau(p) = \tau(q) = \frac{1}{2k}\tau(e_1)$, then

$$\tau(T(p)q) = 0. \tag{3.3}$$

The case when $\tau(p) \neq \tau(q)$ follows by standard approximation argument. Indeed, let p, q be τ -finite projections with $pq = 0$. For any $\varepsilon > 0$, there exist two sets $\{p_i\}_{1 \leq i \leq n}$

and $\{q_i\}_{1 \leq i \leq m}$ of τ -finite projections such that

$$\tau\left(p - \sum_{i=1}^n p_i\right), \tau\left(q - \sum_{i=1}^m q_i\right) \leq \varepsilon$$

and

$$\tau(p_i) = \tau(q_j) = \frac{1}{2^k} \tau(e_1), \quad 1 \leq i \leq n, 1 \leq j \leq m,$$

for some $k \in \mathbb{N}$. Note that

$$\begin{aligned} \tau(T(p)q) &= \tau\left(T(p)\left(q - \sum_{i=1}^m q_i\right)\right) + \tau\left(T\left(p - \sum_{i=1}^n p_i\right) \sum_{i=1}^m q_i\right) \\ &\quad + \tau\left(T\left(\sum_{i=1}^n p_i\right) \sum_{i=1}^m q_i\right) \\ &\stackrel{(3.3)}{=} \tau\left(T(p)\left(q - \sum_{i=1}^n p_i\right)\right) + \tau\left(T\left(p - \sum_{i=1}^n p_i\right) \sum_{i=1}^m q_i\right). \end{aligned}$$

Since

$$\begin{aligned} \left| \tau\left(T(p)\left(q - \sum_{i=1}^m q_i\right)\right) \right| &\leq \tau\left(\left|T(p)\left(q - \sum_{i=1}^m q_i\right)\right|\right) \\ &= \left\|T(p)\left(q - \sum_{i=1}^m q_i\right)\right\|_1 \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$ (see e.g. [31, Lemma 3.10] and [29]) and

$$\begin{aligned} \left| \tau\left(T\left(p - \sum_{i=1}^n p_i\right) \sum_{i=1}^m q_i\right) \right| &\leq \|T\| \left\|p - \sum_{i=1}^n p_i\right\|_E \left\|\sum_{i=1}^m q_i\right\|_{E^\times} \\ &\leq \|T\| \left\|p - \sum_{i=1}^n p_i\right\|_E \left\|\sum_{i=1}^m q_i\right\|_{E^\times} \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$ because the norm $\|\cdot\|_E$ is order continuous, it follows that (3.3) also holds.

The general case when x_1 and x_2 are partial isometries in $\mathcal{F}(\tau)$ such that $l(x_1) \perp l(x_2)$ and $r(x_1) \perp r(x_2)$ is reduced to the case just considered via the same argument as in Corollary 3.2. Thus, we obtain $\tau(T(x_1)x_2^*) = 0$, which completes the proof of the first assertion.

Now, we prove the second assertion. Since $E(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$, there exist two sequences $\{y_n\}$ and $\{y'_n\}$ in $\mathcal{F}(\tau)$ such that $0 \leq y_n \uparrow |x_1|$ and $0 \leq y'_n \uparrow |x_2|$ and y_n (resp. y'_n) are generated by spectral projections of $|x_1|$ (resp. $|x_2|$). Let $x_1 = u_1|x_1|$ and $x_2 = u_2|x_2|$ be the polar decompositions. By the first assertion of the lemma, $\tau(T(u_1y_n)(u_2y'_m)^*) = 0$ for all n and m . Since $u_1y_n \rightarrow_n x_1$ in $\|\cdot\|_E$, it follows that $T(u_1y_n) \rightarrow T(x_1)$ in $\|\cdot\|_E$, and so $T(u_1y_n) \rightarrow T(x_1)$ weakly. Hence, $\tau(T(x_1)(u_2y'_m)^*) = 0$ for each m .

Consider the special case when $x_2 \in \mathcal{F}(\tau)$. We may then assume in addition that $\|y'_m - x_2\|_\infty \rightarrow 0$. We obtain

$$\begin{aligned} |\tau(T(x_1)|x_2|u_2^*)| &= |\tau(T(x_1)r(x_2)(y'_m - |x_2|)u_2^*)| \\ &\leq \|T(x_1)r(x_2)\|_1 \|y'_m - |x_2|\|_\infty \rightarrow 0. \end{aligned} \tag{3.4}$$

For the general case, let $p := E^{|x_2|}(\delta, \infty)$, $\delta > 0$, be a τ -finite spectral projection of $|x_2|$ such that $\| |x_2|(\mathbf{1} - p) \|_{L_1 \cap L_\infty} \leq \varepsilon$. Hence,

$$\begin{aligned} |\tau(T(x_1)|x_2|u_2^*)| &\stackrel{(3.4)}{=} |\tau(T(x_1)(\mathbf{1} - p)|x_2|u_2^*)| \\ &\leq \|T(x_1)\|_{L_1 + L_\infty} \|(\mathbf{1} - p)|x_2|u_2^*\|_{L_1 \cap L_\infty} \\ &\leq \|T(x_1)\|_{L_1 + L_\infty} \cdot \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $|\tau(T(x_1)x_2^*)| = 0$. ■

The following result is a semifinite version of [90, Corollary 1].

Corollary 3.4. *Let $E(\mathcal{M}, \tau)$ be a symmetric space having order continuous norm and $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. For any $x \in E(\mathcal{M}, \tau)$, there exist $y, z \in E(\mathcal{M}, \tau)$ such that $T(x) = y + z$ and $r(y) \leq r(x)$ and $l(z) \leq l(x)$.*

Proof. Denote $A := T(x)$. Note that

$$A = l(x)Ar(x) + l(x)Ar(x)^\perp + l(x)^\perp Ar(x) + l(x)^\perp Ar(x)^\perp.$$

Assume that $l(x)^\perp Ar(x)^\perp \neq 0$. Let $p \in \mathcal{F}(\tau)$ be a τ -finite projection such that $z = l(x)^\perp Ar(x)^\perp p \neq 0$. Let $zp = u|zp|$ be the polar decomposition. Then

$$u^*l(x)^\perp Ar(x)^\perp p = u^*zp \geq 0,$$

i.e.,

$$\tau(T(x)r(x)^\perp pu^*l(x)^\perp) = \tau(u^*l(x)^\perp Ar(x)^\perp p) > 0.$$

Note that $l(l(x)^\perp upr(x)^\perp) \perp l(x)$ and $r(l(x)^\perp upr(x)^\perp) \perp r(x)$. By Lemma 3.3,

$$\tau(u^*l(x)^\perp Ar(x)^\perp p) = \tau(T(x)r(x)^\perp pu^*l(x)^\perp) = 0,$$

a contradiction. Taking $y = l(x)Ar(x) + l(x)^\perp Ar(x)$ and $z = l(x)Ar(x)^\perp$, we complete the proof (note that the choices of y and z are not necessarily unique). ■

The following lemma shows that any bounded hermitian operator T on $E(\mathcal{M}, \tau)$ (whose norm is not proportional to $\|\cdot\|_2$) maps the set of all τ -finite projections to a uniformly bounded set in \mathcal{M} , which should be compared with the estimates in [93, Remark 2.5].

Lemma 3.5. *Let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Assume that $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. Then $\|T(p)\|_\infty \leq 3\|T\|$ for any τ -finite projection $p \in \mathcal{P}(\mathcal{M})$.*

Proof. Let $p \in \mathcal{P}(\mathcal{M})$ be an arbitrary τ -finite projection. By Corollary 3.4,

$$T(p) = A_p p + p B_p \tag{3.5}$$

for some (not necessarily unique) $A_p, B_p \in E(\mathcal{M}, \tau)$ with $r(A_p) \leq p$ and $l(B_p) \leq p$. For two τ -finite projections $q_1, q_2 \in \mathcal{M}$ such that $q_1 q_2 = 0$ and $q_1 + q_2 = p$, we have

$$\begin{aligned} T(p) &= T(q_1 + q_2) = A_{q_1} q_1 + q_1 B_{q_1} + A_{q_2} q_2 + q_2 B_{q_2} \\ &= (A_{q_1} q_1 + A_{q_2} q_2) p + p (q_1 B_{q_1} + q_2 B_{q_2}). \end{aligned}$$

Therefore,

$$q_1 T(q_1) q_1 \stackrel{(3.5)}{=} q_1 A_{q_1} q_1 + q_1 B_{q_1} q_1 = q_1 T(p) q_1 \tag{3.6}$$

and

$$\begin{aligned} p^\perp A_p q_1 &\stackrel{(3.5)}{=} p^\perp T(p) q_1 = p^\perp (A_{q_1} q_1 + A_{q_2} q_2) p q_1 = p^\perp A_{q_1} q_1 \\ &= p^\perp (A_{q_1} q_1 + q_1 B_{q_1}) \stackrel{(3.5)}{=} p^\perp T(q_1). \end{aligned} \tag{3.7}$$

Consider the following decomposition (see Corollary 3.4 or (3.5)):

$$T(p) = p^\perp T(p) + T(p) p^\perp + p T(p) p = p^\perp A_p p + p B_p p^\perp + p T(p) p.$$

We claim that $\|p^\perp A_p\|_\infty \leq \|T\|_{E \rightarrow E}$. Assume otherwise. Then, taking $q = E^{|p^\perp A_p|}(\|T\|, \infty) \neq 0$, we obtain

$$\|T\| \|q\|_E < \|p^\perp A_p q\|_E \stackrel{(3.7)}{=} \|p^\perp T(q)\|_E \leq \|T(q)\|_E \leq \|T\| \|q\|_E,$$

a contradiction. Similarly, $\|B_p p^\perp\|_\infty \leq \|T\|$. Now, we aim to show that $\|p T(p) p\|_\infty \leq \|T\|$. Note that, by (3.5), we have

$$p T(p) p = p(A_p + B_p) p =: C.$$

We claim that C is self-adjoint. Indeed, assume that $p(A_p + B_p)p = a + ib$, where a, b are non-zero self-adjoint operators in $E(\mathcal{M}, \tau)$ and $r(a), r(b) \leq p$. Let $q := E^b(0, \infty) \leq p$ (or $q := E^b(-\infty, 0)$ if $E^b(0, \infty) = 0$) so that $q b q \neq 0$, which is a τ -finite projection. We obtain

$$\begin{aligned} \tau(T(q)q) &= \tau(qT(q)q) \stackrel{(3.6)}{=} \tau(qpT(p)pq) \stackrel{(3.5)}{=} \tau(qp(A_p + B_p)pq) \\ &= \tau(qaq + iqbq) \notin \mathbb{R}, \end{aligned}$$

which contradicts Proposition 3.1. Hence, $b = 0$. This implies that $p T(p) p$ is self-adjoint. Recall that for any $q \leq p$, we have

$$q T(p) q \stackrel{(3.6)}{=} q T(q) q.$$

Let $q := E^{|p T(p) p|}(\|T\| + \varepsilon, \infty)$, $\varepsilon > 0$. If $q \neq 0$, then

$$\begin{aligned} (\|T\| + \varepsilon) \|q\|_E &= \|(\|T\| + \varepsilon) q\|_E \leq \|p T(p) p q\|_E = \|q T(p) q\|_E = \|q T(q) p\|_E \\ &\leq \|T(q)\|_E \leq \|T\| \|q\|_E, \end{aligned}$$

a contradiction. Hence, $E^{|pT(p)p|}[\|T\| + \varepsilon, \infty) = 0$ for any $\varepsilon > 0$. This implies that $\|pT(p)p\|_\infty \leq \|T\|$.

Combining the estimates $\|p^\perp A_p\|_\infty \leq \|T\|$, $\|B_p p^\perp\|_\infty \leq \|T\|$ and $\|pT(p)p\|_\infty \leq \|T\|$, we obtain

$$\|T(p)\|_\infty \leq 3\|T\|$$

for any τ -finite projection p . In particular, A_p and B_p can be taken with $\|A_p\|_\infty, \|B_p\|_\infty \leq 3\|T\|$ (see the proof of Corollary 3.4). ■

The following lemma is the key auxiliary tool in the proof of Proposition 3.8 below, which shows that any bounded hermitian operator T on $E(\mathcal{M}, \tau)$ is a bounded operator from $(\mathcal{F}(\tau), \|\cdot\|_\infty)$ into $(C_0(\mathcal{M}, \tau), \|\cdot\|_\infty)$. By applying the generalized Gleason theorem [44, Theorem 5.2.4] (see also [16, 41, 42, 70] for related results), we manage to prove all but one case in the following lemma. However, the case when a von Neumann algebra has an I_2 direct summand is exceptional, not covered by the generalized Gleason theorem. This special case is rather complicated and requires a careful study of the restriction of a hermitian operator to the type I_2 summand.

We denote by $C_0(\mathcal{M}, \tau)$ the closure in the norm $\|\cdot\|_\infty$ of the linear span of τ -finite projections in \mathcal{M} . Equivalently, $C_0(\mathcal{M}, \tau) = \{a \in S(\mathcal{M}, \tau) : \mu(a) \in L_\infty(0, \infty), \mu(\infty, a) = 0\}$ [65, Lemma 2.6.9].

Lemma 3.6. *Let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Assume that $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. Then T is a bounded operator from $(\mathcal{F}(\tau), \|\cdot\|_\infty)$ into $(C_0(\mathcal{M}, \tau), \|\cdot\|_\infty)$. In particular, T extends to a bounded operator from $C_0(\mathcal{M}, \tau)$ into $C_0(\mathcal{M}, \tau)$.*

Proof. There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where \mathcal{M}_1 has no type I_2 direct summand and \mathcal{M}_2 is either 0 or the type I_2 direct summand of \mathcal{M} (see e.g. [15, Chapter III.1.5.12]).

By Lemma 3.5, $\|T(p)\|_\infty \leq 3\|T\|$ for any τ -finite projection $p \in \mathcal{M}_1$. Moreover, $T(p) = A_p + B_p$ (see Corollary 3.4) for some $A_p, B_p \in E(\mathcal{M}, \tau)$ with $r(A_p) \leq p$ and $l(B_p) \leq p$. Since $p \leq \mathbf{1}_{\mathcal{M}_1}$, it follows that $A_p = A_p \mathbf{1}_{\mathcal{M}_1} \in \mathcal{M}_1$ and $B_p = \mathbf{1}_{\mathcal{M}_1} B_p \in \mathcal{M}_1$. On the other hand, $\tau(p) < \infty$ implies that $T(p) \in \mathcal{F}(\mathcal{M}_1) \subset C_0(\mathcal{M}_1, \tau) \subset C_0(\mathcal{M}, \tau)$. Let $\mathcal{P}_f(\mathcal{M}_1)$ be the set of all τ -finite projections in \mathcal{M}_1 . It follows from [44, Theorem 5.2.4] that for any τ -finite projection p , $T|_{\mathcal{P}_f(\mathcal{M}_1)}$ extends uniquely to a bounded linear operator R_p from the reduced algebra $p\mathcal{M}p$ into $C_0(\mathcal{M}, \tau)$. Moreover,

$$\|R_p|_{p\mathcal{M}_1p}\|_{\|\cdot\|_\infty \rightarrow \|\cdot\|_\infty} \leq 12\|T\|$$

(see the proof of [44, Theorem 5.2.4]). Moreover, by the uniqueness of the extension R_p [44, Theorem 5.2.4], if $p \geq q$, then R_p coincides with R_q on $q\mathcal{M}_1q$.

We claim that R_p coincides with T on $p\mathcal{M}_1p$. Indeed, let x be a positive operator in $p\mathcal{M}_1p$. Then there exists a sequence of positive operators x_n whose singular values are step functions such that $x_n \uparrow x$ and $\|x_n - x\|_\infty \rightarrow 0$. Since R_p coincides with T on

all projections $q \leq p$, it follows that $R_p(x_n) = T(x_n)$. Since $E(\mathcal{M}_1, \tau)$ has order continuous norm, it follows that $T(x_n) \rightarrow T(x)$ in $\|\cdot\|_E$, and hence in the measure topology [30, Proposition 20]. On the other hand, the $\|\cdot\|_\infty$ - $\|\cdot\|_\infty$ -boundedness of R_p implies that $R_p(x_n) \rightarrow R_p(x)$ in $\|\cdot\|_\infty$, and in the measure topology [30, Proposition 20]. Hence, $T(x) = R_p(x)$. Since p is an arbitrary τ -finite projection, it follows that T is a bounded linear operator from $(\mathcal{F}(\tau) \cap \mathcal{M}_1, \|\cdot\|_\infty)$ into $(C_0(\mathcal{M}_1, \tau), \|\cdot\|_\infty)$.

Now, we consider the case when \mathcal{M}_2 is a non-vanishing type I_2 von Neumann direct summand (if $\mathcal{M}_2 = 0$, then the lemma follows from the above result). It is known that \mathcal{M}_2 can be written as $\mathbb{M}_2 \otimes \mathcal{A}$, where \mathbb{M}_2 is the algebra of all $2 \otimes 2$ matrices and \mathcal{A} is a σ -finite commutative von Neumann algebra (see e.g. [54, 94] or [15, Chapter III.1.5.12]). For every element of the form $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$, where p is a projection in \mathcal{A} such that $\tau(\mathbf{1} \otimes p) < \infty$, we have (see Corollary 3.4)

$$T\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = A_p\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}B_p.$$

By Lemma 3.5, A_p and B_p are uniformly bounded. Assume that

$$A_p = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B_p = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Without loss of generality, we may assume in addition that a_2, a_4, b_3, b_4 are 0. Hence,

$$T\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ a_3 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (a_1 + b_1)p & b_2p \\ a_3p & 0 \end{pmatrix}.$$

Recall that $\|A_p\|_\infty, \|B_p\|_\infty \leq 3\|T\|$ (see the proof of Lemma 3.5). We infer that $a_1, a_3, b_1, b_2 \leq 3\|T\|$. Hence, for any $x \in \mathcal{A}$ whose singular value function is a step function, we have

$$T\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} h_1x & h_2x \\ h_3x & 0 \end{pmatrix},$$

where $\|h_1\|_\infty, \|h_2\|_\infty, \|h_3\|_\infty \leq 6\|T\|$. Therefore, $\|T\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\|_\infty \leq 12\|T\| \|x\|_\infty$. For any self-adjoint $x \in \mathcal{F}(\tau)$ there exists a sequence of self-adjoint elements $x_n \in \mathcal{A}$ such that $|x_n| \uparrow |x|, \|x_n - x\|_\infty \rightarrow 0$ and $\mu(x_n)$ are step functions. Since $E(\mathcal{M}, \tau)$ has order continuous norm, it follows

$$\left\| T\begin{pmatrix} x_n & 0 \\ 0 & 0 \end{pmatrix} - T\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\|_E \rightarrow 0,$$

and therefore $T\begin{pmatrix} x_n & 0 \\ 0 & 0 \end{pmatrix} \rightarrow T\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ in measure [30, Proposition 20]. On the other hand, $\|T\begin{pmatrix} x_n & 0 \\ 0 & 0 \end{pmatrix}\|_\infty \leq 12\|T\| \|x_n\|_\infty \leq 12\|T\| \|x\|_\infty$. Since the unit ball of \mathcal{M}_2 is closed in the measure topology [30, Theorem 32], we obtain

$$\left\| T\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty \leq 12\|T\| \|x\|_\infty$$

for any self-adjoint operator $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{F}(\tau) \cap \mathcal{M}_2$. Since every element in $\mathcal{F}(\tau) \cap \mathcal{M}_2$ is a linear combination of two self-adjoint elements, we obtain

$$\left\| T \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty \lesssim \|T\| \|x\|_\infty$$

for any $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{F}(\tau) \cap \mathcal{M}_2$. The same argument shows that $\left\| T \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \right\|_\infty \lesssim \|T\| \|x\|_\infty$. To estimate $\left\| T \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right\|_\infty$ and $\left\| T \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\|_\infty$, just note that

$$T'(\cdot) := \left(\begin{pmatrix} 0 & \mathbf{1}_{\mathcal{A}} \\ \mathbf{1}_{\mathcal{A}} & 0 \end{pmatrix} + \mathbf{1}_{\mathcal{M}_1} \right) T \left(\left(\begin{pmatrix} 0 & \mathbf{1}_{\mathcal{A}} \\ \mathbf{1}_{\mathcal{A}} & 0 \end{pmatrix} + \mathbf{1}_{\mathcal{M}_1} \right) \cdot \right)$$

is also a hermitian operator on $E(\mathcal{M}, \tau)$ and

$$T' \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = T \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

By taking linear combinations, we see that for any $x \in \mathcal{F}(\tau) \cap \mathcal{M}_2$,

$$\|T(x)\|_\infty \lesssim \|T\| \|x\|_\infty.$$

This completes the proof. ■

We prove below an analogue of Proposition 3.1 for the symmetric space $C_0(\mathcal{M}, \tau)$.

Proposition 3.7. *Let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Assume that $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. Then, for any operator $x \in C_0(\mathcal{M}, \tau)$ and a τ -finite projection $p \in \mathcal{P}(\mathcal{M})$ commuting with $|x|$, we have*

$$\langle Tx, pu^* \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} := \tau(T(x)pu^*) \in \mathbb{R},$$

where $x = u|x|$ is the polar decomposition.

Proof. We only consider the case when x is positive; the general case follows from the same argument by replacing Proposition 3.1 used below with Corollary 3.2.

Let $x_n := \sum_{1 \leq k \leq n} \alpha_k p_k \in \mathcal{F}(\tau)$ be such that $x_n \rightarrow x$ in $\|\cdot\|_\infty$, where p_k are τ -finite spectral projections of x_n which commute with p , and α_k are real numbers. For each p_k , we have

$$\begin{aligned} & \langle T p_k, p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \\ &= \langle T(pp_k), pp_k \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} + \langle T(pp_k), p - pp_k \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \\ & \quad + \langle T(p_k - pp_k), p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \\ & \stackrel{(3.3)}{=} \langle T(pp_k), pp_k \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \\ &= \tau(T(pp_k)pp_k) \stackrel{(3.1)}{\in} \mathbb{R}. \end{aligned}$$

Hence, $\langle Tx_n, p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \in \mathbb{R}$ for every n . Moreover,

$$\begin{aligned} & |\langle Tx, p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} - \langle Tx_n, p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)}| \\ & \leq \|T\|_{C_0(\mathcal{M}, \tau) \rightarrow C_0(\mathcal{M}, \tau)} \|x - x_n\|_\infty \|p\|_1 \rightarrow 0, \end{aligned}$$

which shows that $\langle Tx, p \rangle_{(C_0(\mathcal{M}, \tau), C_0(\mathcal{M}, \tau)^*)} \in \mathbb{R}$. ■

Proposition 3.8. *Let $E(\mathcal{M}, \tau)$ be an arbitrary symmetric operator space having order continuous norm. Assume that $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. Then T can be extended to a bounded operator on $C_0(\mathcal{M}, \tau)$ (still denoted by T) and, for any operator $x \in C_0(\mathcal{M}, \tau)$, there exists a support functional x' in $C_0(\mathcal{M}, \tau)^*$ of x such that $\langle Tx, x' \rangle \in \mathbb{R}$. In particular, T is a hermitian operator on $C_0(\mathcal{M}, \tau)$.*

Proof. By Lemma 3.6, T can be extended to a bounded operator on $C_0(\mathcal{M}, \tau)$.

Without loss of generality, we assume in addition that $\|x\|_\infty = 1$. Let $x = u|x|$ be the polar decomposition. Recall that $x \in C_0(\mathcal{M}, \tau)$. Hence, $\tau(E^{|x|}(1 - 1/n, 1]) < \infty$ for any $n > 0$. Recall that $(C_0(\mathcal{M}, \tau))^{\times} = L_1(\mathcal{M}, \tau)$ (see e.g. [84, Lemma 8] and [30, Theorem 53]). Define

$$x_n = \frac{E^{|x|}(1 - 1/n, 1]}{\tau(E^{|x|}(1 - 1/n, 1])} u^* \in L_1(\mathcal{M}, \tau) \subset C_0(\mathcal{M}, \tau)^*, \quad n \geq 1.$$

We have $\|x_n\|_{C_0(\mathcal{M}, \tau)^*} = \|x_n\|_1 = 1$ [30, p. 228]. Note that

$$1 - \frac{1}{n} \leq \tau(xx_n) = \frac{\tau(|x|E^{|x|}(1 - 1/n, 1])}{\tau(E^{|x|}(1 - 1/n, 1])} \leq 1. \tag{3.8}$$

By Alaoglu’s theorem [24, p. 130, Theorem 3.1], there exists a subnet $\{x_i\}$ of $\{x_n\}_n$ converging to some $x' \in C_0(\mathcal{M}, \tau)^*$ in the weak* topology of $C_0(\mathcal{M}, \tau)^*$ and $\|x'\|_{C_0(\mathcal{M}, \tau)^*} \leq 1$. On the other hand,

$$\|x'\|_{C_0(\mathcal{M}, \tau)^*} = \|x\|_\infty \|x'\|_{C_0(\mathcal{M}, \tau)^*} \geq x'(x) = \lim_i \tau(xx_i) \stackrel{(3.8)}{=} 1.$$

Hence, $\|x'\|_{C_0(\mathcal{M}, \tau)^*} = 1$. This implies that x' is a support functional of x . Therefore, by taking $p = E^{|x|}(1 - 1/n, 1]$ in Proposition 3.7, we obtain

$$\langle Tx, x' \rangle = w^* - \lim_i \langle Tx, x_i \rangle \in \mathbb{R}.$$

This completes the proof. ■

Recall that a *derivation* on an algebra \mathcal{A} is a linear operator satisfying the Leibniz rule. Although it is known that a derivation from $C_0(\mathcal{M}, \tau)$ into $C_0(\mathcal{M}, \tau)$ is not necessarily inner [12, 46] (see [80, Example 4.1.8] for examples of non-inner derivations on $K(\mathcal{H})$, the $*$ -algebra of all compact operators on a Hilbert space \mathcal{H}), it has been shown recently that every derivation δ from an arbitrary von Neumann subalgebra of \mathcal{M} into $C_0(\mathcal{M}, \tau)$ is inner, i.e., there exists an element $a \in C_0(\mathcal{M}, \tau)$ such that $\delta(\cdot) = [a, \cdot]$ [13, 46]. On the other hand, every derivation from $C_0(\mathcal{M}, \tau)$ into $C_0(\mathcal{M}, \tau)$ is *spatial*, i.e., it can be implemented by an element from \mathcal{M} (see e.g. [80, Theorem 2] and [9, Theorem 4.1]).

Lemma 3.9. *Every derivation δ from $C_0(\mathcal{M}, \tau)$ into $C_0(\mathcal{M}, \tau)$ is spatial. In particular, if δ is a $*$ -derivation, then the element implementing δ can be chosen to be self-adjoint.*

Proof. The first statement follows from [80, Theorem 2] (or [9, Theorem 4.1]) and the fact that $C_0(\mathcal{M}, \tau)$ is a C^* -algebra. For the second statement, see e.g. [46, Chapter 3.4, Remark 3.4.1]. ■

We now come to the main result of this section, which gives the full description of hermitian operators on a symmetric space $E(\mathcal{M}, \tau)$.

Theorem 3.10. *Let $E(\mathcal{M}, \tau)$ be a symmetric space affiliated with an atomless semifinite von Neumann algebra (or an atomic von Neumann algebra with all atoms having the same trace) \mathcal{M} equipped with a semifinite faithful normal trace τ . Assume that $\|\cdot\|_E$ is order continuous and is not proportional to $\|\cdot\|_2$. Then a bounded linear operator T on $E(\mathcal{M}, \tau)$ is hermitian if and only if there exist self-adjoint operators a and b in \mathcal{M} such that*

$$Tx = ax + xb, \quad x \in E(\mathcal{M}, \tau). \tag{3.9}$$

In particular, T can be extended to a bounded hermitian operator on the von Neumann algebra \mathcal{M} .

Proof. The ‘if’ part is obvious (see e.g. the argument in [90, p. 71] or [40, p. 167]).

By Corollary 3.8, T is a bounded hermitian operator on $C_0(\mathcal{M}, \tau)$. Recall that any hermitian operator T on a C^* -algebra \mathcal{A} is the sum of left multiplication by a self-adjoint operator in \mathcal{A} and a $*$ -derivation in \mathcal{A} (see e.g. [87, p. 213]). It follows from Lemma 3.9 that there exist self-adjoint elements $a, b \in \mathcal{M}$ such that

$$Tx = ax + xb, \quad x \in C_0(\mathcal{M}, \tau).$$

Noting that $\mathcal{F}(\tau) \subset C_0(\mathcal{M}, \tau)$, we obtain

$$Tx = ax + xb, \quad x \in \mathcal{F}(\tau).$$

Since $E(\mathcal{M}, \tau)$ has order continuous norm, it follows that $\mathcal{F}(\tau)$ is dense in $(E(\mathcal{M}, \tau), \|\cdot\|_E)$ (see e.g. [30, Proposition 46] or [48, Remark 2.9]). For any $x \in E(\mathcal{M}, \tau)$, there exists a sequence $\{y_n\} \subset \mathcal{F}(\tau)$ such that $\|y_n - x\|_E \rightarrow 0$. Hence,

$$Tx = \|\cdot\|_E\text{-}\lim_n T(y_n) = \|\cdot\|_E\text{-}\lim_n (ay_n + y_nb) = ax + xb, \quad x \in E(\mathcal{M}, \tau).$$

This completes the proof. ■

4. Isometries

The goal of this section is to answer the question posed in [23, 93] and stated at the outset of this paper. Throughout this section, unless stated otherwise, we always assume that \mathcal{M} is an atomless semifinite von Neumann algebra or an atomic semifinite von Neumann

algebra with all atoms having the same trace, and we assume that τ is a semifinite faithful normal trace on \mathcal{M} .

To prove Theorem 4.4 below, we need the following auxiliary tool, which extends [90, Corollary 2] and [93, Corollary 3.2].

Corollary 4.1. *Let (\mathcal{M}, τ) be an atomless semifinite von Neumann algebra or an atomic von Neumann algebra whose atoms all have the same trace. Let $E(\mathcal{M}, \tau)$ be a symmetric operator space whose norm is order continuous and is not proportional to $\|\cdot\|_2$. Let T be a bounded hermitian operator on $E(\mathcal{M}, \tau)$. Then T^2 is also a hermitian operator on $E(\mathcal{M}, \tau)$ if and only if*

$$T(y) = ay + yb, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M},$$

for some self-adjoint operators $a \in \mathcal{M}_w$ and $b \in \mathcal{M}_{1-w}$, where $w \in \mathcal{P}(Z(\mathcal{M}))$.

Proof. (\Leftarrow) Note that $T^2(y) = a^2y + yb^2$ for all $y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}$. It follows from Theorem 3.10 that T^2 is a hermitian operator.

(\Rightarrow) Recall that, by Theorem 3.10, we have $T(y) = ay + yb$, $y \in \mathcal{M}$, for some self-adjoint elements $a, b \in \mathcal{M}$. Due to the assumption that T^2 is also hermitian, there exist self-adjoint operators $c, d \in \mathcal{M}$ such that

$$T^2(y) = cy + yd = a^2y + 2ayb + yb^2, \quad y \in \mathcal{M}.$$

The claim follows from Theorem A.6 in Appendix A. ■

Remark 4.2. Let T, a, b, w be defined as in Corollary 4.1. In particular, $l(a) \leq w$ and $l(b) \leq \mathbf{1} - w$. Define

$$\begin{aligned} z_a &:= \sup \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq w, pa \in Z(\mathcal{M})\}, \\ z_b &:= \sup \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq \mathbf{1} - w, pb \in Z(\mathcal{M})\}. \end{aligned}$$

For any $z_1, z_2 \in \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq w, pa \in Z(\mathcal{M})\}$ and $d \in \mathcal{M}$, we have

$$\begin{aligned} (z_1 \vee z_2)ad &= (z_1 + z_2 - z_1z_2)ad = z_1ad + z_2(\mathbf{1} - z_1)ad \\ &= z_1ad + z_2a(\mathbf{1} - z_1)d = dz_1a + d(\mathbf{1} - z_1)z_2a = d(z_1 \vee z_2)a. \end{aligned}$$

That is, $z_1 \vee z_2 \in \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq w, pa \in Z(\mathcal{M})\}$. Hence, $\{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq w, pa \in Z(\mathcal{M})\}$ is an increasing net with the partial order \leq of projections. Therefore, by Vigier’s theorem [65, Theorem 2.1.1], we infer that $z_a a \in Z(\mathcal{M})$. That is,

$$z_a \in \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq w, pa \in Z(\mathcal{M})\}.$$

Similarly,

$$z_b \in \{p \in \mathcal{P}(Z(\mathcal{M})) : p \leq \mathbf{1} - w, pb \in Z(\mathcal{M})\}.$$

We have

$$T(y) = a(w - z_a)y + yb((\mathbf{1} - w) - z_b) + (az_a + bz_b)y, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}. \quad (4.1)$$

In particular,

- (1) $w - z_a, (\mathbf{1} - w) - z_b, z_a + z_b$ are pairwise orthogonal projections in $Z(\mathcal{M})$;
- (2) if $p \in Z(\mathcal{M})$ is such that $p \leq w - z_a$ and $ap \in Z(\mathcal{M}, \tau)$ (or $p \leq (\mathbf{1} - w) - z_b$ and $bp \in Z(\mathcal{M})$), then $p = 0$;
- (3) $w - z_a$ (resp. $(\mathbf{1} - w) - z_b$) is the central support of $a(w - z_a)$ (resp. $b((\mathbf{1} - w) - z_b)$).

Remark 4.3. Let \mathcal{M} be a semifinite factor. It is an immediate consequence of Corollary 4.1 that if T a bounded hermitian operator on \mathcal{M} , then T^2 is hermitian if and only if T is left (or right) multiplication by a self-adjoint operator in \mathcal{M} (see also the proof of [90, Corollary 2] and [93, Corollary 3.2]).

Let \mathcal{M}_1 and \mathcal{M}_2 be von Neumann algebras. A complex-linear map $J : \mathcal{M}_1 \xrightarrow{\text{injective}} \mathcal{M}_2$ is called a *Jordan $*$ -isomorphism* if $J(x^*) = J(x)^*$ and $J(x^2) = J(x)^2, x \in \mathcal{M}_1$ (equivalently, $J(xy + yx) = J(x)J(y) + J(y)J(x), x, y \in \mathcal{M}_1$) (see e.g. [17, 86, 99]). A Jordan $*$ -isomorphism is called *normal* if it is completely additive (equivalently, ultraweakly continuous). Alternatively, we adopt the following equivalent definition: $J(x_\alpha) \uparrow J(x)$ whenever $x_\alpha \uparrow x \in \mathcal{M}_1^+$ (see e.g. [28, Chapter I.4.3]). If $J : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a surjective Jordan $*$ -isomorphism, then J is necessarily normal [78, Appendix A].

The following theorem is the main result of this section. Due to the complicated nature of hermitian operators on a von Neumann algebra distinct from a factor, the proof below is substantially more involved than those in [90, 93].

Theorem 4.4. *Let \mathcal{M}_1 and \mathcal{M}_2 be atomless von Neumann algebras (or atomic von Neumann algebras whose atoms have the same trace) equipped with semifinite faithful normal traces τ_1 and τ_2 , respectively. Let $E(\mathcal{M}_1, \tau_1)$ and $F(\mathcal{M}_2, \tau_2)$ be two symmetric operator spaces whose norms are order continuous and are not proportional to $\|\cdot\|_2$. If $T : E(\mathcal{M}_1, \tau_1) \rightarrow F(\mathcal{M}_2, \tau_2)$ is a surjective isometry, then there exist two nets $A_i \in F(\mathcal{M}_2, \tau_2), i \in I$, disjointly supported from the right, and $B_i \in F(\mathcal{M}_2, \tau_2), i \in I$, disjointly supported from the left, and a surjective Jordan $*$ -isomorphism $J : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ and a central projection $z \in \mathcal{M}_2$ such that*

$$T(x) = \|\cdot\|_F - \sum_{i \in I} J(x)A_i z + B_i J(x)(\mathbf{1} - z), \quad x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1,$$

where the series is taken as the limit of all finite partial sums.

Proof. The indices of von Neumann algebras \mathcal{M}_1 and \mathcal{M}_2 play no role in the proof below. So, to reduce the notation, we assume that $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$. We denote by L_a (resp. R_a) left (resp. right) multiplication by $a \in \mathcal{M}$, that is,

$$L_a(x) = ax$$

(resp. $R_a(x) = xa$) for all $x \in S(\mathcal{M}, \tau)$. For any self-adjoint operator $a \in \mathcal{M}$, TL_aT^{-1} and $TL_a^2T^{-1}$ are hermitian on $F(\mathcal{M}, \tau)$ (see e.g. [38, Lemma 2.3] or [50]).

We divide the proof into several steps.

Step 1. We aim to prove that there exists $z \in \mathcal{P}(Z(\mathcal{M}))$ (not depending on a below) such that

$$TL_aT^{-1} = L_{J_1(a)z} + R_{J_2(a)(1-z)}, \quad a = a^* \in \mathcal{M}, \tag{4.2}$$

where $J_1(a), J_2(a)$ are self-adjoint operators in \mathcal{M} . Let $z(f)$ be the central support of $f = f^* \in \mathcal{M}$. For any fixed $b = b^* \in \mathcal{M}$,

$$TL_bT^{-1}y \stackrel{(4.1)}{=} L_{J_1(b)y} + R_{J_2(b)y} + J_3(b)y, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}, \tag{4.3}$$

for some self-adjoint operators $J_1(b), J_2(b) \in \mathcal{M}$ and $J_3(b) = J_3(b)^* \in Z(\mathcal{M})$ such that

- (1) $z(J_1(b)), z(J_2(b))$ and $z(J_3(b))$ are pairwise orthogonal projections in $Z(\mathcal{M})$ (see Remark 4.2 (1, 3));
- (2) if $p \in Z(\mathcal{M})$ satisfies $p \leq z(J_1(b))$ and $J_1(b)p \in Z(\mathcal{M}, \tau)$ (or $p \leq z(J_2(b))$ and $J_2(b)p \in Z(\mathcal{M})$), then $p = 0$ (see Remark 4.2 (2)).

By (4.3), for any self-adjoint $a \in \mathcal{M}$, we have

$$TL_aT^{-1}y = L_{J_1(a)y} + R_{J_2(a)y} + J_3(a)y, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}, \tag{4.4}$$

and

$$TL_{a+b}T^{-1}y = L_{J_1(a+b)y} + R_{J_2(a+b)y} + J_3(a+b)y, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}. \tag{4.5}$$

Now, we consider the reduced algebra $\mathcal{M}_{z(J_1(b)) \wedge z(J_2(a))}$. For all

$$y \in (L_1 \cap L_\infty)(\mathcal{M}_{z(J_1(b)) \wedge z(J_2(a))}, \tau),$$

we have

$$\begin{aligned} L_{J_1(b)y} + R_{J_2(a)y} &= L_{J_1(b)y} + R_{J_2(b)y} + J_3(b)y + L_{J_1(a)y} + R_{J_2(a)y} + J_3(a)y \\ &\stackrel{(4.3),(4.4)}{=} TL_bT^{-1}y + TL_aT^{-1}y \\ &= TL_{b+a}T^{-1}y \\ &\stackrel{(4.5)}{=} L_{J_1(a+b)y} + R_{J_2(a+b)y} + J_3(a+b)y. \end{aligned}$$

By Theorem A.6, there exists a central projection

$$p \leq z(J_1(b)) \wedge z(J_2(a))$$

such that $J_1(b)p$ and $J_2(a)(z(J_1(b)) \wedge z(J_2(a)) - p)$ are in the centre $Z(\mathcal{M})$ of \mathcal{M} . However, by Remark 4.2 (2) (used twice),

$$p = 0 = z(J_1(b)) \wedge z(J_2(a)) - p.$$

That is,

$$z(J_1(b)) \wedge z(J_2(a)) = 0.$$

Note that a, b are arbitrary. Defining

$$z := \bigvee_{b=b^* \in \mathcal{M}} z(J_1(b)),$$

we obtain

$$\begin{aligned} TL_b T^{-1} y &\stackrel{(4.3)}{=} L_{J_1(b)} y + R_{J_2(b)} y + J_3(b) y \\ &= L_{J_1(b)+J_3(b)z} y + R_{J_2(b)+J_3(b)(1-z)} y, \quad y \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}. \end{aligned}$$

Replacing $J_1(b) + J_3(b)z$ (resp. $J_2(b) + J_3(b)(1 - z)$) with $J_1(b)$ (resp. $J_2(b)$), we obtain (4.2).

Step 2. Note that

$$L_{J_1(a)^2 z} + R_{J_2(a)^2(1-z)} \stackrel{(4.2)}{=} (TL_a T^{-1})^2 = TL_{a^2} T^{-1} \stackrel{(4.2)}{=} L_{J_1(a^2)z} + R_{J_2(a^2)(1-z)}$$

for every $a = a^* \in \mathcal{M}$. By a standard argument (see e.g. [93, p. 117]), we find that

$$J(\cdot) := J_1(\cdot)z + J_2(\cdot)(1 - z) \tag{4.6}$$

is an injective Jordan $*$ -isomorphism on \mathcal{M} . Let $0 \leq a_i \uparrow a \in \mathcal{M}$. Clearly, $J(a_i)z \uparrow \leq J(a)z$ (see e.g. [48, (12)] and [17, p. 211]). Since $a_i \uparrow a$, it follows that for any $x \in E(\mathcal{M}, \tau)$, $x^* a_i x \uparrow x^* a x$ [30, Proposition 1 (vi)] and $x^* a_i x \rightarrow x^* a x$ in the measure topology [30, Proposition 2 (iv)]. Since $t \mapsto t^{1/2}$ is an operator monotone function [33, Proposition 1.2], we deduce that $(x^*(a - a_i)x)^{1/2}$ is a decreasing net. Note that $(x^*(a - a_i)x)^{1/2} \rightarrow 0$ in measure [30, p. 213]. By [30, Proposition 2 (iii)], we see that $(x^*(a - a_i)x)^{1/2} \downarrow 0$, and therefore

$$E(\mathcal{M}, \tau) \ni |(a - a_i)^{1/2} x| = (x^*(a - a_i)x)^{1/2} \downarrow 0.$$

It follows from the order continuity of $\|\cdot\|_E$ that

$$\begin{aligned} \|J(a - a_i)z T(x)\|_F &\stackrel{(4.2)}{=} \|T((a - a_i)x)\|_F = \|(a - a_i)x\|_E \\ &\leq \|(a - a_i)^{1/2}\|_\infty \|(a - a_i)^{1/2} x\|_E \rightarrow 0 \end{aligned}$$

for all $x \in E(\mathcal{M}, \tau)$ such that $T(x)$ is a τ -finite projection in $\mathcal{F}(\tau)$ less than z . Therefore, $J(a_i)z \rightarrow J(a)z$ in the local measure topology [30, Proposition 20]. Hence, $J(a_i)z \uparrow J(a)z$ (see Section 2.1). The same argument shows that $J(a_i)(1 - z) \uparrow J(a)(1 - z)$. Therefore, $J(\mathcal{M})$ is weakly closed [48, Remark 2.16] and $J : \mathcal{M} \rightarrow J(\mathcal{M})$ is a surjective (normal) Jordan $*$ -isomorphism.

Step 3. We claim that J is surjective. Let $c = c^* \in \mathcal{M}$. Note that $T^{-1}(L_c z + R_{c(1-z)})T$ and its square $T^{-1}(L_{c^2 z} + R_{c^2(1-z)})T$ are hermitian operators. Hence, by Corollary 4.1, there exists a central projection $z' \in \mathcal{M}$ and $c' = (c')^* \in \mathcal{M}$ such that

$$T^{-1}(L_c z + R_{c(1-z)})T = L_{c' z'} + R_{c'(1-z')}. \tag{4.7}$$

Applying to (4.7) the argument used in Steps 1 and 2, we obtain a normal injective Jordan $*$ -isomorphism $J'' : \mathcal{M} \rightarrow \mathcal{M}$ such that $J''(c) = c'$. Moreover, for each $c \in J(\mathcal{M})$,

$$L_{J^{-1}(c)} \stackrel{(4.2)}{=} T^{-1}(L_{cz} + R_{c(1-z)})T \stackrel{(4.7)}{=} L_{c'z'} + R_{c'(1-z')}.$$

Hence, $L_{J^{-1}(c)-c'z'} = R_{c'(1-z')}$. In particular, $c'(1-z') \in Z(\mathcal{M})$. Hence, $L_{J^{-1}(c)} = L_{c'z'+c'(1-z')} = L_{c'} = L_{J''(c)}$. That is, for each $c \in J(\mathcal{M})$, we have $J''(c) = J^{-1}(c)$ (see also [93, Remark 3.3] for the case when \mathcal{M} is a factor). Hence, $J''(J(\mathcal{M})) = J^{-1}(J(\mathcal{M})) = \mathcal{M}$. By the injectivity of J'' , we conclude that $J(\mathcal{M}) = \mathcal{M}$. This proves the claim.

Step 4. Applying (4.2) to $T(x)$, we obtain

$$\begin{aligned} T(ax) &= J_1(a)zT(x) + T(x)(\mathbf{1} - z)J_2(a) \\ &= J_1(a)zT(x) + T(x)J_2(a)(\mathbf{1} - z) \\ &\stackrel{(4.6)}{=} J(a)zT(x) + T(x)J(a)(\mathbf{1} - z) \end{aligned}$$

for all $a = a^* \in \mathcal{M}$ and $x \in E(\mathcal{M}, \tau) \cap \mathcal{M}$. For any τ -finite projection e in \mathcal{M} , we have

$$T(xe) = J(x)T(e)z + T(e)J(x)(\mathbf{1} - z), \quad x \in \mathcal{M}.$$

Let $\{e_i\}_{i \in I}$ be a net of pairwise orthogonal τ -finite projections in \mathcal{M} such that $\sup_i e_i = \mathbf{1}$ [30, Corollary 8] and let $\{\lambda_\alpha\}$ be the collection of all finite subsets of I , partially ordered by inclusion. By the order continuity of $\|\cdot\|_E$, we have $\lim_\alpha \sum_{e \in \lambda_\alpha} xe_i = x$ [29, Theorem 6.13], and since T is an isometry, it follows that

$$\begin{aligned} T(x) &= \|\cdot\|_F\text{-}\lim_\alpha \sum_{e_i \in \lambda_\alpha} T(xe_i) = \|\cdot\|_F\text{-}\lim_\alpha \sum_{e_i \in \lambda_\alpha} J(x)T(e_i)z + T(e_i)J(x)(\mathbf{1} - z) \\ &= \|\cdot\|_F\text{-}\sum_{i \in I} J(x)T(e_i)z + T(e_i)J(x)(\mathbf{1} - z), \quad x \in E(\mathcal{M}, \tau) \cap \mathcal{M}. \end{aligned}$$

Note that

$$T(e_i) = J(e_i)T(e_i)z + T(e_i)J(e_i)(\mathbf{1} - z).$$

Recall that Jordan $*$ -isomorphisms preserve the disjointness for projections (see e.g. [48, Proposition 2.14]). Letting $B_i := T(e_i)z = J(e_i)T(e_i)z$ and $A_i := T(e_i)(\mathbf{1} - z) = T(e_i)J(e_i)(\mathbf{1} - z)$, we complete the proof. ■

5. Uniqueness of symmetric structure

Let C_E be the symmetric operator ideal in $B(\mathcal{H})$ generated by a symmetric sequence space E . We say that C_E has a unique symmetric structure if C_E isomorphic to some ideal C_F corresponding to a symmetric sequence space F implies that $E = F$ with equivalent norms. At the International Conference on Banach Space Theory and its Applications at Kent, Ohio (August 1979), Pełczyński posed the following question concerning

the symmetric structure of ideals of compact operators on the Hilbert space ℓ_2 (see also [7, Question (B)] and [6, Problem A]): *Does the ideal C_E of compact operators corresponding to an arbitrary separable symmetric sequence space E have a unique symmetric structure?* For readers who are interested in this topic, we refer to [6, 7, 47].

We assume, in addition, that $\|e\|_E = 1$ for any atom $e \in \mathcal{M}$ if $\mathcal{M} = B(\mathcal{H})$ equipped with the standard trace, and $\|\mathbf{1}\|_E = 1$ if \mathcal{M} is a type II_1 factor equipped with the unique faithful normal tracial state. Here, we consider an analogue of Pełczyński’s problem in the sense of isometric isomorphisms. The above assumption implies that if $\|\cdot\|_E$ is proportional to $\|\cdot\|_2$, then $\|\cdot\|_E = \|\cdot\|_2$. Let $F(\mathcal{M}, \tau)$ be a symmetric operator space. If a symmetric operator space $E(\mathcal{M}, \tau)$ isometric to $F(\mathcal{M}, \tau)$ implies that $E(\mathcal{M}, \tau)$ coincides with $F(\mathcal{M}, \tau)$, then we say that $F(\mathcal{M}, \tau)$ has a *unique symmetric structure* up to isometry. The following corollary extends results in [2, 56, 74, 75, 100] to the noncommutative setting.

Corollary 5.1. *Let $\mathcal{M} = B(\mathcal{H})$ be equipped with the standard trace τ or \mathcal{M} be a II_1 factor equipped with the unique faithful normal tracial state τ . Let $E(\mathcal{M}, \tau)$ and $F(\mathcal{M}, \tau)$ be symmetric operator spaces whose norms are order continuous and are not proportional to the norm of $L_2(\mathcal{M}, \tau)$. Then T is a surjective isometry from $E(\mathcal{M}, \tau)$ to $F(\mathcal{M}, \tau)$ if and only if there exist a unitary element $u \in \mathcal{M}$ and a trace-preserving Jordan $*$ -isomorphism J such that*

$$T(x) = uJ(x), \quad x \in \mathcal{M}. \tag{5.1}$$

In particular, any symmetric space $E(\mathcal{M}, \tau)$ (including the case when $E(\mathcal{M}, \tau) = L_2(\mathcal{M}, \tau)$) has a unique symmetric structure up to isometry.

Proof. By Theorem 4.4, it suffices to show that the Jordan $*$ -isomorphism is trace-preserving. Indeed, when $\mathcal{M} = B(\mathcal{H})$, this follows from the fact that every $*$ -automorphism on $B(\mathcal{H})$ is inner [36, Corollary 5.42] (see also argument in [90, p. 75]). When \mathcal{M} is a II_1 factor, then the corollary follows from the same argument as in [93, pp. 118–119].

For the uniqueness of symmetric structure, we only need to show that if $L_2(\mathcal{M}, \tau)$ is isometric to $F(\mathcal{M}, \tau)$ (when $\mathcal{M} = B(\mathcal{H})$ or a II_1 factor), then $F(\mathcal{M}, \tau) = L_2(\mathcal{M}, \tau)$ (with the same norm). Indeed, all other cases follow from (5.1) immediately.

If there exists a surjective isometry $T : L_2(\mathcal{M}, \tau) \rightarrow F(\mathcal{M}, \tau)$, then $F(\mathcal{M}, \tau)^\times$ is isometric to $L_2(\mathcal{M}, \tau)$. That is, $F(\mathcal{M}, \tau)$ is isometric to $F(\mathcal{M}, \tau)^\times$. In particular, both $F(\mathcal{M}, \tau)$ and $F(\mathcal{M}, \tau)^\times$ have the Fatou property and order continuous norms [30, Theorem 45]. Hence, $F(\mathcal{M}, \tau)$ coincides with $F(\mathcal{M}, \tau)^{\times \times}$ with the same norm [30, Theorem 32]. If $\|\cdot\|_{F^\times}$ is proportional to $\|\cdot\|_2$, then the norm of the Köthe dual $F(\mathcal{M}, \tau)^{\times \times}$ is also proportional to $\|\cdot\|_2$. By the assumption that $\|e\|_E = 1$ for any atom $e \in \mathcal{M}$ if $\mathcal{M} = B(\mathcal{H})$ (or $\|\mathbf{1}\|_E = 1$ if \mathcal{M} is a type II_1 factor), we conclude that $F(\mathcal{M}, \tau)$ coincides with $L_2(\mathcal{M}, \tau)$ and $\|\cdot\|_F = \|\cdot\|_2$. Hence, we only need to consider the case when $\|\cdot\|_{F^\times}$ is not proportional to $\|\cdot\|_2$. Recalling that $F(\mathcal{M}, \tau)$ is isometric to $F(\mathcal{M}, \tau)^\times$, by (5.1), $F(\mathcal{M}, \tau)$ coincides with $F(\mathcal{M}, \tau)^\times$ with $\|\cdot\|_F = \|\cdot\|_{F^\times}$, and therefore, for any

$x \in F(\mathcal{M}, \tau)$, by the definition of Köthe dual, we have

$$\tau(xx^*) < \infty, \quad x \in F(\mathcal{M}, \tau),$$

i.e., $F(\mathcal{M}, \tau) = F(\mathcal{M}, \tau)^\times \subset L_2(\mathcal{M}, \tau)$. On the other hand, by the definition of the Köthe dual, $F(\mathcal{M}, \tau) \subset L_2(\mathcal{M}, \tau)$ implies that $F(\mathcal{M}, \tau)^\times \supset L_2(\mathcal{M}, \tau)$. Hence, $F(\mathcal{M}, \tau) = F(\mathcal{M}, \tau)^\times = L_2(\mathcal{M}, \tau)$ (as sets).

Since $\|x\|_2^2 = \tau(xx^*) \leq \|x\|_F \|x\|_{F^\times} = \|x\|_F^2$, we have $\|x\|_2 \leq \|x\|_F = \|x\|_{F^\times}$. Moreover,

$$\|x\|_{F^\times} = \sup_{\|y\|_F \leq 1} |\tau(xy)| \leq \sup_{\|y\|_2 \leq 1} |\tau(xy)| = \|x\|_2.$$

We see that $\|x\|_2 = \|x\|_F = \|x\|_{F^\times}$. This completes the proof. ■

6. Final remarks

Remark 6.1. Theorem 4.4 above covers all existing results of surjective isometries on (complex) symmetric operator/function/sequence spaces in the literature:

- (1) When τ is finite, we have $T(x) = T(z)J(x) + J(x)T(\mathbf{1} - z)$. This recovers and extends [23, Theorem 3.1], [25, Theorem 4.11] and [91, Theorem 6].
- (2) When $\mathcal{M} = L_\infty(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is some σ -finite atomless measure space, we have $T(x) = BJ_1(x)$, $x \in \mathcal{M}$, for some measurable function B on (Ω, Σ, μ) . In particular, Zaidenberg’s results are recovered (see (1.1) and [100, 101]).
- (3) When $\mathcal{M} = (\Gamma, \Sigma, \mu)$ is a discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$, we find that if T is an isometry on a symmetric space $E(\Gamma)$ whose norm is order continuous and is not proportional to $\|\cdot\|_2$, then

$$(Tx)(\gamma) = A(\gamma) \cdot x(\sigma(\gamma)), \quad x \in E(\Gamma), \gamma \in \Gamma,$$

where $A(\gamma)$ are unimodular scalars and σ is a permutation of Γ . Indeed, by Theorem 4.4, there exists a surjective Jordan $*$ -isomorphism J on $\ell_\infty(\Gamma)$. By the bijectivity and disjointness-preserving property of J , we infer that J maps atoms onto atoms in $\ell_\infty(\Gamma)$. Hence, J is generated by a permutation. This extends Arazy’s description of isometries on $E(\mathbb{N})$ [8, Theorem 1].

- (4) When \mathcal{H} is a (not necessarily separable) Hilbert space and $\mathcal{M} = B(\mathcal{H})$, Theorem 4.4 recovers and extends Sourour’s result [90, Theorem 2].
- (5) When \mathcal{M} is a hyperfinite type II-factor, Theorem 4.4 extends [93, Theorem 4.1], which was established under the assumption that \mathcal{H} is separable.
- (6) When $E = F = L_p$, Theorem 4.4 extends and complements results in e.g. [5, 58, 86, 96, 99].
- (7) When E and F are Lorentz spaces, [23, Theorem 6.1] (see also [21, 26, 60, 69]) is recovered.

- (8) When T is a positive isometry, several results are recovered and extended (see e.g. [1, Theorem 1], [23, Theorem 3.1], [91, Theorem 6], [48, Corollaries 5.4 and 5.5] and [25]).
- (9) When \mathcal{M} is an atomic von Neumann algebra, Theorem 4.4 extends the main result in [67], which was established under the assumption that \mathcal{M} is a σ -finite von Neumann algebra.

It is shown by Zaidenberg [101] (see also [39]) that under certain conditions, every surjective isometry between two complex symmetric function spaces on a σ -finite atomless measure space can be represented in the form of (1.1) (see [8] for the case of symmetric sequence spaces). In this respect, the condition imposed on the von Neumann algebras in Theorems 3.10 and 4.4 is very natural. One may expect that Theorems 3.10 and 4.4 (and [101, Theorem 1]) can be proved for more general von Neumann algebras, e.g., $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where \mathcal{M}_1 is an atomless von Neumann algebra (or an atomic von Neumann algebra with all atoms having the same trace but the trace of an atom in \mathcal{M}_1 is different from that in \mathcal{M}_2) and \mathcal{M}_2 is an atomic von Neumann algebra with all atoms having the same trace. See e.g. [39, Definition 5.3.1] or [11] for the definition of symmetric function spaces on general measure spaces. However, the following simple example shows that for an atomic von Neumann algebra (or measure space) whose atoms have different measures, isometries may have forms different from (1.1). This demonstrates that the condition imposed on the von Neumann algebras in Theorems 3.10 and 4.4 is sharp. For an arbitrary (not necessarily atomless or atomic with all atoms having the same trace) semifinite von Neumann algebra \mathcal{M} , it would be of interest to characterize those symmetric operator spaces $E(\mathcal{M}, \tau)$ such that all isometries on $E(\mathcal{M}, \tau)$ have elementary forms.

Example 6.2. Let (Ω, m) be a measure space consisting of two atoms e_1 and e_2 . Assume that $m(e_1) = 1$ and $m(e_2) = 2$. Then there exists a symmetric space $E(\Omega)$ whose norm is not proportional to the $L_2(\Omega)$ norm but which is isometric to the 2-dimensional Hilbert space. In particular, there exists a non-elementary isometry and a hermitian operator on $E(\Omega)$ which cannot be written as multiplication by an element in $L_\infty(\Omega)$.

Proof. A non-trivial projection in $E(\Omega)$ must be e_1, e_2 or $e_1 + e_2$, where e_1 (resp. e_2 and $e_1 + e_2$) denotes the indicator function on e_1 (resp. e_2 and $e_1 + e_2$).

For an element $ae_1 + be_2$, we define a norm by

$$\|ae_1 + be_2\|_E := \sqrt{|a|^2 + 3|b|^2}. \tag{6.1}$$

Indeed, this can be considered as the L_2 -norm on a measure space having an atom of measure 1 and the other of measure $\sqrt{3}$. We claim that $\|\cdot\|_E$ is symmetric. Indeed, assume that $x := a_1e_1 + a_2e_2 \geq 0$ and $y := b_1e_1 + b_2e_2 \geq 0$ and $\mu(x) \leq \mu(y)$.

If $b_1 \geq b_2$, then two cases are possible:

- If $a_1 \geq a_2$, then $b_1 \geq a_1$ and $b_2 \geq a_2$. In this case, $\|x\|_E \leq \|y\|_E$.
- If $a_1 \leq a_2$, then since $m(e_1) \leq m(e_2)$ and $b_1 \geq a_2$, it follows that $b_2 \geq a_2 \geq a_1$. Hence, $\|x\|_E \leq \|y\|_E$.

If $b_1 \leq b_2$, then two cases are possible:

- If $a_1 \geq a_2$, then $b_2 \geq a_1 \geq a_2$ and $b_1 \geq a_2$. Note that $\|x\|_E^2 = a_1^2 + 3a_2^2$ and $\|y\|_E^2 = b_1^2 + 3b_2^2$. We obtain

$$\|y\|_E^2 - \|x\|_E^2 = 3b_2^2 - a_1^2 + b_1^2 - 3a_2^2 \geq (3 - 1)b_2^2 - (3 - 1)a_2^2 \geq 0.$$

- If $a_1 \leq a_2$, then $b_2 \geq a_2$ and $b_1 \geq a_1$. Hence, $\|x\|_E \leq \|y\|_E$.

Consider the matrix

$$T := \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{i}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

that is, $T(e_1) = -\frac{i}{\sqrt{2}}e_1 + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{3}}e_2$, $T(e_2) = \sqrt{3}\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2$, $T(1) = (\frac{\sqrt{3}-i}{\sqrt{2}}, \frac{i+\sqrt{3}}{\sqrt{2}\sqrt{3}})$. For any $a, b \in \mathbb{C}$, we have

$$\begin{aligned} \|T(ae_1 + be_2)\|_E^2 &= \left\| \left(\frac{-ia}{\sqrt{2}} + \frac{\sqrt{3}b}{\sqrt{2}} \right) e_1 + \left(\frac{ia}{\sqrt{2}\sqrt{3}} + \frac{b}{\sqrt{2}} \right) e_2 \right\|_E^2 \\ &= \left| \frac{-ia}{\sqrt{2}} + \frac{\sqrt{3}b}{\sqrt{2}} \right|^2 + 3 \left| \frac{ia}{\sqrt{2}\sqrt{3}} + \frac{b}{\sqrt{2}} \right|^2. \end{aligned}$$

By the parallelogram law, we obtain

$$\|T(ae_1 + be_2)\|_E^2 = |a|^2 + 3|b|^2 \stackrel{(6.1)}{=} \|ae_1 + be_2\|_E^2.$$

This implies that T is an isometry on $E(\Omega)$.

Assume that T can be written in the elementary form, that is, there exists $B \in E(\Omega)$ and a Jordan isomorphism J on $L_\infty(\Omega)$ such that $T = BJ$. Since $J(e_1 + e_2) = e_1 + e_2$, it follows that either

$$J(e_1) = e_1, \quad J(e_2) = e_2$$

or

$$J(e_1) = e_2, \quad J(e_2) = e_1.$$

However, $T(e_1) = -\frac{i}{\sqrt{2}}e_1 + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{3}}e_2 \neq BJ(e_1)$ for any $B \in E(\Omega)$. Hence, T cannot be written in the elementary form.

Consider the hermitian operator T on $E(\Omega)$ defined by

$$T := \begin{pmatrix} 1 & i\sqrt{3} \\ -\frac{i}{\sqrt{3}} & 1 \end{pmatrix},$$

i.e., $T(e_1) = e_1 - \frac{i}{\sqrt{3}}e_2$ and $T(e_2) = i\sqrt{3}e_1 + e_2$. Assume that $T(x) = ax$ for some $a \in E(\Omega)$. Then $e_1 - \frac{i}{\sqrt{3}}e_2 = T(e_1) = ae_1 = \lambda e_1$ for some $\lambda \in \mathbb{C}$, a contradiction. ■

Remark 6.3. Recall that Zaidenberg’s description of isometries on complex symmetric function spaces only requires that a symmetric function space has a Fatou norm, which is a slightly weaker assumption than having an order continuous norm [39, Theorem 5.3.5].

Throughout this paper, we always consider symmetric spaces having order continuous norms. It will be interesting to verify Theorem 4.4 under a slightly weaker assumption that the symmetric operator spaces have Fatou norms only. This problem is yet unsolved. There are some partial answers in this direction obtained in [3, 4, 22, 63] in the setting of $B(\mathcal{H})$.

Remark 6.4. The structure of (real or complex) symmetric sequence space (under the assumption that the spaces in question have the Fatou property, which is a stronger assumption than having a Fatou norm) has been discussed by Braverman and Semenov [18, 19] and by Arazy [8]. Abramovich and Zaidenberg [2, Theorem 1] showed $L_p[0, 1]$, $1 \leq p < \infty$, has a unique symmetric structure up to isometry. The uniqueness of symmetric structure of separable complex symmetric function spaces on $[0, 1]$ was obtained by Zaidenberg [100]. By using a generalized Zaidenberg’s theorem [75, Theorem 1 and Proposition 3] (see also [56, Theorem 7.2]), the uniqueness of symmetric structure of separable real symmetric function spaces under some technical conditions was obtained by Randrianantoanina [75].

Remark 6.5. Note that when \mathcal{M} is a II_∞ factor, Corollary 5.1 may fail because the Jordan $*$ -isomorphism J on \mathcal{M} may not be trace-preserving. This is an oversight in the proof of [93, Theorem 4.1]. Indeed, letting $R_0 = \bigotimes_{1 \leq n < \infty} \mathbb{M}_2$ be the hyperfinite II_1 factor equipped with a faithful normal tracial state τ , we consider the hyperfinite II_∞ factor $\mathcal{M} = B(\mathcal{H}) \otimes R_0$ equipped with the trace $\text{Tr} \otimes \tau$. Let ϕ_1 be a $*$ -isomorphism from (R_0, τ) onto $(\bigotimes_{2 \leq n < \infty} \mathbb{M}_2, \tau)$, and ϕ_2 be a natural $*$ -isomorphism from $B(\mathcal{H}) \otimes \mathbf{1}_{R_0}$ onto $B(\mathcal{H}) \otimes \mathbb{M}_2 \otimes \mathbf{1}_{\bigotimes_{2 \leq n < \infty} \mathbb{M}_2}$. Clearly, $\phi_1 \otimes \phi_2$ is a $*$ -isomorphism on \mathcal{M} which does not preserve the trace $\text{Tr} \otimes \tau$. Indeed, ϕ_2 maps atoms in $B(\mathcal{H}) \otimes \mathbf{1}_{R_0}$ to atoms in $B(\mathcal{H}) \otimes \mathbb{M}_2 \otimes \mathbf{1}_{\bigotimes_{2 \leq n < \infty} \mathbb{M}_2}$. Let $p \in B(\mathcal{H})$ be an atom. In particular, $\text{Tr}(p) = (\text{Tr} \otimes \tau)(p \otimes \mathbf{1}_{R_0}) = 1$ and $(\text{Tr} \otimes \tau)(\phi_1(p) \otimes \mathbf{1}_{\bigotimes_{2 \leq n < \infty} \mathbb{M}_2}) = \frac{1}{2}$. This oversight in [93, Theorem 4.1] is rectified by Theorem 4.4 above. It is natural to compare this result with [39, Theorem 5.3.5] (see also (1.1) and [21, Main Theorem]), where the set-isomorphism may not necessarily preserve the measure.

Appendix A

In this appendix, we extend [93, Theorem 3.1] and [90, Lemma 2] to the setting of arbitrary von Neumann algebras. Our technique is different from that used in [90, 93]. We are grateful to Dmitriy Zanin for providing us with a correction of our initial argument and allowing us to use it in this paper.

Let \mathcal{M} be a von Neumann algebra. Let p be a projection in \mathcal{M} . We denote by $z(p)$ the central support of p .

Lemma A.1 ([81, Theorem 1.10.7]). *Let $p, q \in \mathcal{M}$ be projections such that*

$$pyq = 0, \quad \forall y \in \mathcal{M}.$$

Then $z(p)z(q) = 0$.

Lemma A.2. *Let $a, b, e, f \in \mathcal{M}$ be self-adjoint and such that*

$$ey + yf = ayb, \quad \forall y \in \mathcal{M}.$$

We have

(1) $[a, b] = 0$;

(2) $[b, [a, y]] = 0$ for every $y \in \mathcal{M}$.

Proof. Setting $y = \mathbf{1}$, we obtain $e + f = ab$. Taking adjoints, we obtain $e + f = ba$. Thus, $ab = ba$. This proves the first assertion.

Substituting $f = ab - e$, we obtain

$$[e, y] = [a, y]b, \quad \forall y \in \mathcal{M}.$$

Taking adjoints yields

$$[e, y] = b[a, y], \quad \forall y \in \mathcal{M}.$$

Comparing the right hand sides, we establish the second assertion. ■

Lemma A.3. *Let $a, b \in \mathcal{M}$ be commuting self-adjoint elements such that*

$$[b, [a, y]] = 0, \quad \forall y \in \mathcal{M}.$$

Then

$$[p, [q, y]] = 0, \quad \forall y \in \mathcal{M},$$

for all spectral projections p and q of a and b , respectively.

Proof. Note that for all $y \in \mathcal{M}$, we have

$$\begin{aligned} [b^n, [a, y]] &= [b, [b^{n-1}, [a, y]]] + [[b, [a, y]], b^{n-1}] = [b, [b^{n-1}, [a, y]]] \\ &= \cdots = [b, [b, \dots [b, [a, y]] \dots]] = 0. \end{aligned}$$

By linearity,

$$[P(b), [a, y]] = 0, \quad \forall y \in \mathcal{M},$$

for every polynomial P . Since the polynomials are norm-dense in the algebra of continuous functions, it follows that

$$[x, [a, y]] = 0, \quad \forall y \in \mathcal{M},$$

for every x in the C^* -algebra generated by b . By weak continuity of our equation,

$$[x, [a, y]] = 0, \quad \forall y \in \mathcal{M},$$

for every x in the von Neumann algebra generated by b . In particular,

$$[q, [a, y]] = 0, \quad \forall y \in \mathcal{M}.$$

Using the Jacobi identity

$$[q, [a, y]] + [a, [y, q]] + [y, [q, a]] = 0$$

and taking into account that $[q, a] = 0$, we have

$$[a, [q, y]] = 0, \quad \forall y \in \mathcal{M}.$$

Repeating the argument in the first paragraph of the proof, we complete the proof. ■

Lemma A.4. *If $p, q \in \mathcal{M}$ are commuting projections such that*

$$[p, [q, y]] = 0, \quad \forall y \in \mathcal{M}, \tag{A.1}$$

then

$$z(p)z(q)z(\mathbf{1} - p)z(\mathbf{1} - q) = 0.$$

Proof. Denote

$$z' := z(p)z(q)z(\mathbf{1} - p)z(\mathbf{1} - q).$$

Assume for contradiction that $z' \neq 0$. By (A.1), we have

$$[pz', [qz', y]] = 0, \quad \forall y \in \mathcal{M}_{z'},$$

and

$$z(pz')z(qz')z(z' - pz')z(z' - qz') \stackrel{[54, \text{Prop. 5.5.3}]}{=} z' \cdot z(p)z(q)z(\mathbf{1} - p)z(\mathbf{1} - q) = z'.$$

Hence, by passing to the reduced von Neumann algebra $\mathcal{M}_{z'}$, we may assume without loss of generality that $z' = \mathbf{1}$. In other words,

$$z(p) = z(q) = z(\mathbf{1} - p) = z(\mathbf{1} - q) = \mathbf{1}. \tag{A.2}$$

Obviously, assumption (A.1) is equivalent to

$$pqy + ypq = pyq + qyp. \tag{A.3}$$

Replacing y in (A.3) with $(\mathbf{1} - q)y(\mathbf{1} - p)$, we obtain

$$0 + 0 = p(\mathbf{1} - q)y(\mathbf{1} - p)q + 0, \quad \forall y \in \mathcal{M}.$$

By Lemma A.1, we have

$$z(p(\mathbf{1} - q)) \cdot z((\mathbf{1} - p)q) = 0. \tag{A.4}$$

Let $w_1 := z(p(\mathbf{1} - q))$ and $w_2 := z((\mathbf{1} - p)q)$. We have $w_1w_2 = 0$. Set $w_3 := \mathbf{1} - w_1 - w_2$. By (A.1), we have

$$\begin{aligned} [pw_1, [qw_1, y]] &= 0, \quad \forall y \in w_1\mathcal{M}, \\ [pw_2, [qw_2, y]] &= 0, \quad \forall y \in w_2\mathcal{M}, \\ [pw_3, [qw_3, y]] &= 0, \quad \forall y \in w_3\mathcal{M}. \end{aligned} \tag{A.5}$$

Step 1. We claim that

$$qw_1 \leq pw_1, \quad pw_2 \leq qw_2, \quad pw_3 = qw_3. \tag{A.6}$$

Note that

$$z((w_1 - pw_1) \cdot qw_1) \stackrel{[54, \text{Prop. 5.5.3}]}{=} w_1 \cdot z((\mathbf{1} - p)q) = w_1 \cdot w_2 \stackrel{(A.4)}{=} 0.$$

Hence,

$$(w_1 - pw_1) \cdot qw_1 = 0. \tag{A.7}$$

In other words, $qw_1 \leq pw_1$.

Similarly,

$$z(pw_2 \cdot (w_2 - qw_2)) \stackrel{[54, \text{Prop. 5.5.3}]}{=} z(p(\mathbf{1} - q)) \cdot w_2 = w_1 \cdot w_2 \stackrel{(A.4)}{=} 0.$$

Hence,

$$pw_2 \cdot (w_2 - qw_2) = 0.$$

In other words, $pw_2 \leq qw_2$.

Arguing similarly, we have $pw_3 \leq qw_3$ and $qw_3 \leq pw_3$. This completes the proof of (A.6).

Step 2. We claim that

$$w_1 = 0, \quad w_2 = 0, \quad w_3 = 0. \tag{A.8}$$

We only prove the first equality; the proofs of the other two are similar.

By (A.5), we have

$$[pw_1, [qw_1, (w_1 - pw_1)y]] = 0, \quad \forall y \in \mathcal{M}_{w_1}.$$

Since

$$(w_1 - pw_1) \cdot qw_1 \stackrel{(A.7)}{=} 0, \tag{A.9}$$

it follows that for all $y \in \mathcal{M}_{w_1}$,

$$\begin{aligned} 0 &= [pw_1, [qw_1, (w_1 - pw_1)y]] = [pw_1, qw_1(w_1 - pw_1)y - (w_1 - pw_1)yqw_1] \\ &\stackrel{(A.9)}{=} -[pw_1, (w_1 - pw_1)yqw_1]. \end{aligned}$$

Since $pw_1 \cdot (w_1 - pw_1) = 0$ and $qw_1 \stackrel{(A.6)}{\leq} pw_1$, it follows that

$$(w_1 - pw_1)yqw_1 = 0, \quad \forall y \in \mathcal{M}_{w_1}.$$

By Lemma A.1, we have

$$z(w_1 - pw_1) \cdot z(qw_1) = 0.$$

In other words,

$$w_1 \stackrel{(A.2)}{=} z(\mathbf{1} - p)z(q)w_1 \stackrel{[54, \text{Prop. 5.5.3}]}{=} z(w_1 - pw_1) \cdot z(qw_1) = 0.$$

This proves the first equality of (A.8).

Finally, $\mathbf{1} = w_1 + w_2 + w_3 = 0$, which is impossible. Hence, $z = 0$. This completes the proof. ■

Lemma A.5. *If $a, b \in \mathcal{M}$ are commuting elements such that*

$$[b, [a, y]] = 0, \quad \forall y \in \mathcal{M},$$

then there exists a central projection z such that both $a(\mathbf{1} - z)$ and bz are central.

Proof. Since a commutes with b , it follows from Lemmas A.3 and A.4 that

$$z(p)z(q)z(\mathbf{1} - p)z(\mathbf{1} - q) = 0$$

for any spectral projection p (resp. q) of a (resp. b).

For brevity, denote $z_q = z(q)z(\mathbf{1} - q)$. We have

$$z(pz_q) \cdot z(z_q - pz_q) \stackrel{[54, \text{Prop. 5.5.3}]}{=} z(p)z(\mathbf{1} - p)z_q = z(p)z(\mathbf{1} - p)z(q)z(\mathbf{1} - q) = 0.$$

Therefore,

$$\begin{aligned} 0 &= z(pz_q)z(z_q - pz_q) \geq z(pz_q)(z_q - pz_q) \geq (z_q - pz_q)z(pz_q)(z_q - pz_q) \\ &\geq (z_q - pz_q)pz_q(z_q - pz_q) = 0. \end{aligned}$$

This implies that

$$pz_q \geq z(pz_q) \geq pz_q,$$

that is,

$$pz_q = z(pz_q) \in Z(\mathcal{M}). \tag{A.10}$$

Thus, $pz(q)z(\mathbf{1} - q) = pz_q$ is a central projection. By the spectral theorem, we have $az(q)z(\mathbf{1} - q) \in Z(\mathcal{M})$.

Define $z' \in \mathcal{P}(Z(\mathcal{M}))$ by

$$\mathbf{1} - z' = \bigvee_q z(q)z(\mathbf{1} - q),$$

where the supremum is taken over all spectral projections q of b . We have $a(\mathbf{1} - z') = a \cdot \bigvee_q z(q)z(\mathbf{1} - q) \in Z(\mathcal{M})$. On the other hand (see e.g. [94, Chapter V, Proposition 1.1]),

$$z' = \bigwedge_q (\mathbf{1} - z(q)z(\mathbf{1} - q)).$$

In particular,

$$z' \leq \mathbf{1} - z(q)z(\mathbf{1} - q) \tag{A.11}$$

for every q . Thus,

$$z' - z(z'q)z(z' - z'q) \stackrel{[54, \text{Prop. 5.5.3}]}{=} z' \cdot (\mathbf{1} - z(q)z(\mathbf{1} - q)) \stackrel{(A.11)}{=} z'$$

for every q . That is,

$$z(z'q)z(z' - z'q) = 0$$

for every q . Hence, $z'q \in Z(\mathcal{M})$ for every spectral projection q of b (see e.g. the proof for (A.10)). By the spectral theorem, $bz' \in Z(\mathcal{M})$. ■

The following theorem is an immediate consequence of Lemmas A.2 and A.5. It should be compared with [93, Theorem 3.1] and [90, Lemma 2].

Theorem A.6. *Let $a, b, e, f \in \mathcal{M}$ be self-adjoint and such that*

$$ey + yf = ayb, \quad \forall y \in \mathcal{M}. \quad (\text{A.12})$$

Then there exists a central projection z such that $a(\mathbf{1} - z)$, $e(\mathbf{1} - z)$ and bz , fz are central.

Proof. By Lemmas A.2 and A.5, we deduce that there exists $z \in \mathcal{P}(Z(\mathcal{M}))$ such that $a(\mathbf{1} - z), bz \in Z(\mathcal{M})$. Replacing y with z in (A.12), we obtain

$$ez + fz = abz, \quad \forall y \in \mathcal{M}_z.$$

We have $fz = abz - ez$. Hence,

$$y f z \stackrel{(\text{A.12})}{=} (abz - ez)y = fzy, \quad \forall y \in \mathcal{M}_z.$$

This implies that $fz \in Z(\mathcal{M})$. The same argument shows that $e(\mathbf{1} - z) \in Z(\mathcal{M})$. ■

Acknowledgments. The authors would like to thank Professor Aleksey Ber for helpful discussions, and Professors Evgueni Semenov and Mikhail Zaidenberg for their interest and for pointing us to the existing literature and related problems in this field. We also thank Professor Dmitriy Zanin for pointing out a gap in our original proof of Corollary 4.1 in the earlier version of this paper and for providing Appendix A. We thank Thomas Scheckter for his careful reading of this paper.

Funding. Jinghao Huang was supported by the NNSF of China (grant number 12031004). Fedor Sukochev was supported by the Australian Research Council (FL170100052).

References

- [1] Abramovich, Y. A.: Isometries of normed lattices. *Optimizatsiya* **43** (60), 74–80 (1988) (in Russian) Zbl [0696.46005](#) MR [1007934](#)
- [2] Abramovich, Y. A., Zaidenberg, M.: A rearrangement invariant space isometric to L_p coincides with L_p . In: *Interaction between Functional Analysis, Harmonic Analysis, and Probability* (Columbia, MO, 1994), *Lecture Notes in Pure Appl. Math.* 175, Dekker, New York, 13–18 (1996) Zbl [0878.46019](#) MR [1358141](#)
- [3] Aminov, B., Chilin, V.: Isometries of perfect norm ideals of compact operators. *Studia Math.* **241**, 87–99 (2018) Zbl [1406.46002](#) MR [3732932](#)
- [4] Aminov, B. R., Chilin, V. I.: Isometries and Hermitian operators in complex symmetric sequence spaces. *Mat. Tr.* **20**, 21–42 (2017) (in Russian); English transl.: *Siberian Adv. Math.* **27**, 239–252 (2017) Zbl [1399.46011](#) MR [3676091](#)
- [5] Arazy, J.: The isometries of C_p . *Israel J. Math.* **22**, 247–256 (1975) Zbl [0319.47020](#) MR [512953](#)
- [6] Arazy, J.: Basic sequences, embeddings, and the uniqueness of the symmetric structure in unitary matrix spaces. *J. Funct. Anal.* **40**, 302–340 (1981) Zbl [0457.47035](#) MR [611587](#)
- [7] Arazy, J.: Isomorphisms of unitary matrix spaces. In: *Banach Space Theory and its Applications* (București, 1981), *Lecture Notes in Math.* 991, Springer, Berlin, 1–6 (1983) Zbl [0512.47036](#) MR [714168](#)
- [8] Arazy, J.: Isometries of complex symmetric sequence spaces. *Math. Z.* **188**, 427–431 (1985) Zbl [0538.47017](#) MR [771996](#)
- [9] Arveson, W.: On groups of automorphisms of operator algebras. *J. Funct. Anal.* **15**, 217–243 (1974) Zbl [0296.46064](#) MR [0348518](#)

- [10] Banach, S.: Théorie des opérations linéaires. Warszawa (1932) JFM [58.0420.01](#)
- [11] Bennett, C., Sharpley, R.: Interpolation of Operators. Pure Appl. Math. 129, Academic Press, Boston, MA (1988) Zbl [0647.46057](#) MR [928802](#)
- [12] Ber, A., Huang, J., Levitina, G., Sukochev, F.: Derivations with values in ideals of semifinite von Neumann algebras. J. Funct. Anal. **272**, 4984–4997 (2017) Zbl [1372.46050](#) MR [3639520](#)
- [13] Ber, A., Huang, J., Levitina, G., Sukochev, F.: Derivations with values in the ideal of τ -compact operators affiliated with a semifinite von Neumann algebra. Comm. Math. Phys. **390**, 577–616 (2022) Zbl [1494.46067](#) MR [4384716](#)
- [14] Berkson, E.: Some types of Banach spaces, Hermitian operators, and Bade functionals. Trans. Amer. Math. Soc. **116**, 376–385 (1965) Zbl [0135.36502](#) MR [187100](#)
- [15] Blackadar, B.: Operator Algebras. Encyclopaedia Math. Sci. 122, Springer, Berlin (2006) Zbl [1092.46003](#) MR [2188261](#)
- [16] Botelho, F., Jamison, J., Molnár, L.: Surjective isometries on Grassmann spaces. J. Funct. Anal. **265**, 2226–2238 (2013) Zbl [1301.47100](#) MR [3091813](#)
- [17] Bratteli, O., Robinson, D. W.: Operator Algebras and Quantum Statistical Mechanics. Vol. 1. Texts Monogr. Phys., Springer, New York (1979) Zbl [0421.46048](#) MR [545651](#)
- [18] Braverman, M. Š., Semenov, E. M.: Isometries of symmetric spaces. Dokl. Akad. Nauk SSSR **217**, 257–259 (1974) (in Russian) Zbl [0338.46010](#) MR [0350376](#)
- [19] Braverman, M. Š., Semenov, E. M.: Isometries of symmetric spaces. Voronež. Gos. Univ. Trudy Naučn.-Issled. Inst. Mat. VGU **1975**, 7–18 (in Russian) MR [0493297](#)
- [20] Broise, M.: Sur les isomorphismes de certaines algèbres de von Neumann. Ann. Sci. École Norm. Sup. (3) **83**, 91–111 (1966) Zbl [0143.36104](#) MR [0213885](#)
- [21] Carothers, N. L., Haydon, R., Lin, P.-K.: On the isometries of the Lorentz function spaces. Israel J. Math. **84**, 265–287 (1993) Zbl [0799.46024](#) MR [1244671](#)
- [22] Chan, J.-T., Li, C.-K., Sze, N.-S.: Isometries for unitarily invariant norms. Linear Algebra Appl. **399**, 53–70 (2005) Zbl [1073.15022](#) MR [2152409](#)
- [23] Chilin, V. I., Medzhitov, A. M., Sukochev, P. A.: Isometries of noncommutative Lorentz spaces. Math. Z. **200**, 527–545 (1989) Zbl [0646.46060](#) MR [987585](#)
- [24] Conway, J. B.: A Course in Functional Analysis. 2nd ed., Grad. Texts in Math. 96, Springer, New York (1990) Zbl [0706.46003](#) MR [768926](#)
- [25] de Jager, P., Conradie, J.: Isometries between non-commutative symmetric spaces associated with semi-finite von Neumann algebras. Positivity **24**, 815–835 (2020) Zbl [07239377](#) MR [4129598](#)
- [26] de Jager, P., Conradie, J.: Extreme point methods in the study of isometries on certain non-commutative spaces. Glasgow Math. J. **64**, 462–483 (2022) Zbl [1496.46064](#) MR [4404108](#)
- [27] Dixmier, J.: Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert. Ann. of Math. (2) **51**, 387–408 (1950) Zbl [0036.35801](#) MR [33445](#)
- [28] Dixmier, J.: Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann), Gauthier-Villars, Paris (1969) Zbl [0175.43801](#) MR [0352996](#)
- [29] Dodds, P. G., de Pagter, B.: The non-commutative Yosida–Hewitt decomposition revisited. Trans. Amer. Math. Soc. **364**, 6425–6457 (2012) Zbl [1303.46054](#) MR [2958942](#)
- [30] Dodds, P. G., de Pagter, B.: Normed Köthe spaces: a non-commutative viewpoint. Indag. Math. (N.S.) **25**, 206–249 (2014) Zbl [1318.46041](#) MR [3151815](#)
- [31] Dodds, P. G., de Pagter, B., Sukochev, F.: Sets of uniformly absolutely continuous norm in symmetric spaces of measurable operators. Trans. Amer. Math. Soc. **368**, 4315–4355 (2016) Zbl [1362.46063](#) MR [3453373](#)
- [32] Dodds, P. G., de Pagter, B., Sukochev, F.: Theory of noncommutative integration. To appear

- [33] Dodds, P. G., Dodds, T. K.: On a submajorization inequality of T. Ando. In: *Operator Theory in Function Spaces and Banach Lattices*, Oper. Theory Adv. Appl. 75, Birkhäuser, Basel, 113–131 (1995) Zbl [0826.46054](#) MR [1322502](#)
- [34] Dodds, P. G., Dodds, T. K.-Y., de Pagter, B.: Noncommutative Banach function spaces. *Math. Z.* **201**, 583–597 (1989) Zbl [0653.46061](#) MR [1004176](#)
- [35] Dodds, P. G., Dodds, T. K.-Y., de Pagter, B.: Noncommutative Köthe duality. *Trans. Amer. Math. Soc.* **339**, 717–750 (1993) Zbl [0801.46074](#) MR [1113694](#)
- [36] Douglas, R. G.: *Banach Algebra Techniques in Operator Theory*. Pure Appl. Math. 49, Academic Press, New York (1972) Zbl [0247.47001](#) MR [0361893](#)
- [37] Fack, T., Kosaki, H.: Generalized s -numbers of τ -measurable operators. *Pacific J. Math.* **123**, 269–300 (1986) Zbl [0617.46063](#) MR [840845](#)
- [38] Fleming, R. J., Jamison, J. E.: Hermitian operators and isometries on sums of Banach spaces. *Proc. Edinburgh Math. Soc.* (2) **32**, 169–191 (1989) Zbl [0695.47019](#) MR [1001116](#)
- [39] Fleming, R. J., Jamison, J. E.: *Isometries on Banach Spaces: Function Spaces*. Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 129, Chapman & Hall/CRC, Boca Raton, FL (2003) Zbl [1011.46001](#) MR [1957004](#)
- [40] Fleming, R. J., Jamison, J. E.: *Isometries on Banach Spaces. Vol. 2*. Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 138, Chapman & Hall/CRC, Boca Raton, FL (2008) Zbl [1139.46001](#) MR [2361284](#)
- [41] Gehér, G. P., Šemrl, P.: Isometries of Grassmann spaces. *J. Funct. Anal.* **270**, 1585–1601 (2016) Zbl [1330.47045](#) MR [3447720](#)
- [42] Gehér, G. P., Šemrl, P.: Isometries of Grassmann spaces, II. *Adv. Math.* **332**, 287–310 (2018) Zbl [1486.47070](#) MR [3810254](#)
- [43] Giles, J. R.: Classes of semi-inner-product spaces. *Trans. Amer. Math. Soc.* **129**, 436–446 (1967) Zbl [0157.20103](#) MR [217574](#)
- [44] Hamhalter, J.: *Quantum Measure Theory*. Kluwer, Dordrecht (2003) Zbl [1038.81003](#) MR [2015280](#)
- [45] Hernández, F. L., Rodríguez-Salinas, B.: Lattice-embedding scales of L^p spaces into Orlicz spaces. *Israel J. Math.* **104**, 191–220 (1998) Zbl [0922.46025](#) MR [1622299](#)
- [46] Huang, J.: *Derivations with values into ideals of a semifinite von Neumann algebra*. Ph.D. thesis, Univ. of New South Wales (2019)
- [47] Huang, J., Sadovskaya, O., Sukochev, F.: On Arazy’s problem concerning isomorphic embeddings of ideals of compact operators. *Adv. Math.* **406**, art. 108530, 21 pp. (2022) Zbl [07567801](#) MR [4440075](#)
- [48] Huang, J., Sukochev, F., Zanin, D.: Logarithmic submajorisation and order-preserving linear isometries. *J. Funct. Anal.* **278**, art. 108352, 44 pp. (2020) Zbl [1444.46009](#) MR [4044743](#)
- [49] Jamison, J. E., Kamińska, A., Lin, P.-K.: Isometries of Musielak–Orlicz spaces. II. *Studia Math.* **104**, 75–89 (1993) Zbl [0815.46026](#) MR [1208040](#)
- [50] Jamison, J. E., Loomis, I.: Isometries of Orlicz spaces of vector valued functions. *Math. Z.* **193**, 363–371 (1986) Zbl [0605.46020](#) MR [862882](#)
- [51] Junge, M., Ruan, Z.-J., Sherman, D.: A classification for 2-isometries of noncommutative L_p -spaces. *Israel J. Math.* **150**, 285–314 (2005) Zbl [1354.46058](#) MR [2255812](#)
- [52] Junge, M., Sherman, D.: Noncommutative L^p modules. *J. Operator Theory* **53**, 3–34 (2005) Zbl [1078.46046](#) MR [2132686](#)
- [53] Kadison, R. V.: Isometries of operator algebras. *Ann. of Math.* (2) **54**, 325–338 (1951) Zbl [0045.06201](#) MR [43392](#)
- [54] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras. Vol. I*. Pure Appl. Math. 100, Academic Press, New York (1983) Zbl [0518.46046](#) MR [719020](#)
- [55] Kalton, N. J., Randrianantoanina, B.: Isometries on rearrangement-invariant spaces. *C. R. Acad. Sci. Paris Sér. I Math.* **316**, 351–355 (1993) Zbl [0811.46021](#) MR [1204303](#)

- [56] Kalton, N. J., Randrianantoanina, B.: Surjective isometries on rearrangement-invariant spaces. *Quart. J. Math. Oxford Ser. (2)* **45**, 301–327 (1994) Zbl [0846.46014](#) MR [1295579](#)
- [57] Kalton, N. J., Sukochev, F. A.: Symmetric norms and spaces of operators. *J. Reine Angew. Math.* **621**, 81–121 (2008) Zbl [1152.47014](#) MR [2431251](#)
- [58] Katavolos, A.: Are noncommutative L_p spaces really noncommutative? *Canad. J. Math.* **33**, 1319–1327 (1981) Zbl [0422.46053](#) MR [645227](#)
- [59] Kreĭn, S. G., Petunĭn, Y. Ĭ., Semĕnov, E. M.: *Interpolation of Linear Operators*. Transl. Math. Monogr. 54, Amer. Math. Soc., Providence, RI (1982) Zbl [0493.46058](#) MR [649411](#)
- [60] Krygin, A. V., Medzhitov, A. M.: Isometries of Lorentz spaces. In: *Mathematical Analysis, Algebra and Probability Theory*, Tashkent. Gos. Univ., Tashkent, 52–62 (1988) (in Russian) Zbl [0822.46029](#) MR [1095797](#)
- [61] Lamperti, J.: On the isometries of certain function-spaces. *Pacific J. Math.* **8**, 459–466 (1958) Zbl [0085.09702](#) MR [105017](#)
- [62] Le Merdy, C., Zadeh, S.: ℓ^1 -contractive maps on noncommutative L^p -spaces. *J. Operator Theory* **85**, 417–442 (2021) Zbl [07606468](#) MR [4249557](#)
- [63] Li, C.-K., Poon, Y.-T., Sze, N.-S.: Isometries for Ky Fan norms between matrix spaces. *Proc. Amer. Math. Soc.* **133**, 369–377 (2005) Zbl [1067.15027](#) MR [2093057](#)
- [64] Lindenstrauss, J., Tzafriri, L.: *Classical Banach Spaces. I*. *Ergeb. Math. Grenzgeb.* 92, Springer, Berlin (1977) Zbl [0362.46013](#) MR [0500056](#)
- [65] Lord, S., Sukochev, F., Zanin, D.: *Singular Traces*. *De Gruyter Stud. Math.* 46, De Gruyter, Berlin (2013) Zbl [1275.47002](#) MR [3099777](#)
- [66] Lumer, G.: Semi-inner-product spaces. *Trans. Amer. Math. Soc.* **100**, 29–43 (1961) Zbl [0102.32701](#) MR [133024](#)
- [67] Medzhitov, A.: Isometries of symmetric spaces on atomic von Neumann algebras. In: *Mathematical Analysis, Algebra and Geometry*, Tashkent State Univ., 52–54 (1989) (in Russian)
- [68] Medzhitov, A. M., Sukochev, F. A.: Positive isometries of noncommutative symmetric spaces. *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **1987**, no. 3, 20–25, 97 (in Russian) Zbl [0638.46045](#) MR [914648](#)
- [69] Medzhitov, A. M., Sukochev, F. A.: Isometries of noncommutative Lorentz spaces. *Dokl. Akad. Nauk UzSSR* **1988**, no. 4, 11–12 (in Russian) Zbl [0657.46044](#) MR [955368](#)
- [70] Mori, M.: Isometries between projection lattices of von Neumann algebras. *J. Funct. Anal.* **276**, 3511–3528 (2019) Zbl [1476.47026](#) MR [3944303](#)
- [71] Nelson, E.: Notes on non-commutative integration. *J. Funct. Anal.* **15**, 103–116 (1974) Zbl [0292.46030](#) MR [0355628](#)
- [72] Ovĉinnikov, V. I.: Symmetric spaces of measurable operators. *Dokl. Akad. Nauk SSSR* **191**, 769–771 (1970) (in Russian) Zbl [0215.49103](#) MR [0298435](#)
- [73] Ovĉinnikov, V. I.: Symmetric spaces of measurable operators. *Voronež. Gos. Univ. Trudy Nauĉn.-Issled. Inst. Mat. VGU* **1971**, 88–107 (in Russian) MR [0324436](#)
- [74] Potepun, L. I.: Hilbertian symmetric spaces. *Izv. Vysš. Uĉebn. Zaved. Mat.* **1974**, no. 1, 90–95 (in Russian) Zbl [0286.46026](#) MR [0361751](#)
- [75] Randrianantoanina, B.: Isometric classification of norms in rearrangement-invariant function spaces. *Comment. Math. Univ. Carolin.* **38**, 73–90 (1997) Zbl [0886.46007](#) MR [1455471](#)
- [76] Randrianantoanina, B.: Contractive projections and isometries in sequence spaces. *Rocky Mountain J. Math.* **28**, 323–340 (1998) Zbl [0928.46008](#) MR [1639806](#)
- [77] Randrianantoanina, B.: Injective isometries in Orlicz spaces. In: *Function Spaces* (Edwardsville, IL, 1998), *Contemp. Math.* 232, Amer. Math. Soc., Providence, RI, 269–287 (1999) Zbl [0935.46027](#) MR [1678340](#)
- [78] Rieckers, A., Roos, H.: Implementation of Jordan-isomorphisms for general von Neumann algebras. *Ann. Inst. H. Poincaré Phys. Théor.* **50**, 95–113 (1989) Zbl [0676.46046](#) MR [994044](#)

- [79] Russo, B.: Isometrics of L^p -spaces associated with finite von Neumann algebras. *Bull. Amer. Math. Soc.* **74**, 228–232 (1968) Zbl [0153.16604](#) MR [221294](#)
- [80] Sakai, S.: Derivations of W^* -algebras. *Ann. of Math. (2)* **83**, 273–279 (1966) Zbl [0139.30601](#) MR [193528](#)
- [81] Sakai, S.: C^* -Algebras and W^* -Algebras. *Ergeb. Math. Grenzgeb.* 60, Springer, New York (1971) Zbl [0219.46042](#) MR [0442701](#)
- [82] Schneider, H., Turner, R. E. L.: Matrices Hermitian for an absolute norm. *Linear Multilinear Algebra* **1**, 9–31 (1973) Zbl [0278.15012](#) MR [321955](#)
- [83] Segal, I. E.: A non-commutative extension of abstract integration. *Ann. of Math. (2)* **57**, 401–457 (1953) Zbl [0051.34201](#) MR [54864](#)
- [84] Semenov, E. M., Sukochev, F. A.: Sums and intersections of symmetric operator spaces. *J. Math. Anal. Appl.* **414**, 742–755 (2014) Zbl [1323.47081](#) MR [3167994](#)
- [85] Semenov, E. M., Sukochev, F. A., Usachev, A. S.: Geometry of Banach limits and their applications. *Uspekhi Mat. Nauk* **75**, no. 4, 153–194 (2020) (in Russian); English transl.: *Russian Math. Surveys* **75**, 725–763 (2020) Zbl [1464.46012](#) MR [4153700](#)
- [86] Sherman, D.: Noncommutative L^p structure encodes exactly Jordan structure. *J. Funct. Anal.* **221**, 150–166 (2005) Zbl [1084.46050](#) MR [2124900](#)
- [87] Sinclair, A. M.: Jordan homomorphisms and derivations on semisimple Banach algebras. *Proc. Amer. Math. Soc.* **24**, 209–214 (1970) Zbl [0175.44001](#) MR [250069](#)
- [88] Sourour, A. R.: On the isometries of $L^p(U, X)$. *Bull. Amer. Math. Soc.* **83**, 129–130 (1977) Zbl [0352.46025](#) MR [425604](#)
- [89] Sourour, A. R.: The isometries of $L^p(\Omega, X)$. *J. Funct. Anal.* **30**, 276–285 (1978) Zbl [0396.47020](#) MR [515230](#)
- [90] Sourour, A. R.: Isometries of norm ideals of compact operators. *J. Funct. Anal.* **43**, 69–77 (1981) Zbl [0474.47009](#) MR [639798](#)
- [91] Sukochev, F., Vekslar, A.: Positive linear isometries in symmetric operator spaces. *Integral Equations Operator Theory* **90**, art. 58, 15 pp. (2018) Zbl [1447.46014](#) MR [3836181](#)
- [92] Sukochev, F.: Symmetric Spaces of Measurable Operators on Finite von Neumann Algebras. Ph.D. thesis, Tashkent (1987) (in Russian)
- [93] Sukochev, F. A.: Isometries of symmetric operator spaces associated with AFD factors of type II and symmetric vector-valued spaces. *Integral Equations Operator Theory* **26**, 102–124 (1996) Zbl [0858.46047](#) MR [1405679](#)
- [94] Takesaki, M.: *Theory of Operator Algebras. I.* Springer, New York (1979) Zbl [0436.46043](#) MR [548728](#)
- [95] Takesaki, M.: *Theory of Operator Algebras. III.* Springer, New York (2003) Zbl [1059.46032](#) MR [1943007](#)
- [96] Tam, P. K.: Isometries of L_p -spaces associated with semifinite von Neumann algebras. *Trans. Amer. Math. Soc.* **254**, 339–354 (1979) Zbl [0408.46051](#) MR [539922](#)
- [97] Watanabe, K.: On isometries between noncommutative L^p -spaces associated with arbitrary von Neumann algebras. *J. Operator Theory* **28**, 267–279 (1992) Zbl [0836.46058](#) MR [1273046](#)
- [98] Xu, Y., Ye, Q.: Generalized Mercer kernels and reproducing kernel Banach spaces. *Mem. Amer. Math. Soc.* **258**, no. 1243, vi+122 pp. (2019) Zbl [1455.68009](#) MR [3923255](#)
- [99] Yeadon, F. J.: Isometries of noncommutative L^p -spaces. *Math. Proc. Cambridge Philos. Soc.* **90**, 41–50 (1981) Zbl [0483.46041](#) MR [611284](#)
- [100] Zaidenberg, M. G.: On the isometric classification of symmetric spaces. *Dokl. Akad. Nauk SSSR* **234**, 283–286 (1977) (in Russian); English transl.: *Soviet Math. Dokl.* **18**, 636–640 (1977) Zbl [0381.46016](#) MR [0442667](#)
- [101] Zaidenberg, M. G.: A representation of isometries on function spaces. *Mat. Fiz. Anal. Geom.* **4**, 339–347 (1997) Zbl [0905.47023](#) MR [1615808](#)