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Outer forms of type A_2 with infinite genus

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Abstract. Let G be an absolutely almost simple algebraic group over a field K. The genus $\mathbf{gen}_K(G)$ of G is the set of K-isomorphism classes of K-forms G' of G that have the same K-isomorphism classes of maximal K-tori as G. We construct an example of outer forms of type A_2 with infinite genus.

1. Introduction

Let K be a field and K^{sep} its separable closure. Two absolutely almost simple algebraic K-groups G_1 and G_2 are said to have the same K-isomorphism classes of maximal K-tori if every maximal K-torus of G_1 is K-isomorphic to some maximal K-torus of G_2, and vice versa. An algebraic K-group G' is called a K-form of an algebraic K-group G if G and G' are isomorphic over K^{sep}.

Definition 1.1 ([2, Def. 6.1]). Let G be an absolutely almost simple algebraic group over a field K. The genus $\mathbf{gen}_K(G)$ of G is the set of K-isomorphism classes of K-forms G' of G that have the same K-isomorphism classes of maximal K-tori as G.

The genus is trivial in some special cases and it is conjectured to be finite whenever the field K is finitely generated of "good" characteristic (see details in [6, §8]).

In a similar way one can define the genus of a division algebra.

Definition 1.2. The genus $\mathbf{gen}(\mathcal{D})$ of a finite-dimensional central division algebra $\mathcal{D}$ over a field K is defined as the set of classes $[\mathcal{D}'] \in \mathrm{Br}(K)$, where $\mathcal{D}'$ is a central division K-algebra having the same maximal subfields as $\mathcal{D}$.

If $\mathcal{D}$ is a finite-dimensional central division K-algebra, then it is well known that any maximal K-torus of the corresponding algebraic group $G = \mathrm{SL}_{1,\mathcal{D}}$ is of the form $\mathrm{R}_{E/K}(\mathbb{G}_m) \cap G$ (where $\mathrm{R}_{E/K}(\mathbb{G}_m)$ is the Weil restriction of the 1-dimensional split torus $\mathbb{G}_m$) for some maximal separable subfield E of $\mathcal{D}$. Thus the results on genus of division algebras from [4, 10] rephrased in the language of algebraic groups say that for any prime p, there exist fields (with infinite transcendence degree over the prime subfield) over which there are inner forms of type A_{p-1} with infinite genus. An example of groups

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of type G_2 with infinite genus is obtained in [1, Rem. 3.6 (b)]. In the present paper, we construct such an example for outer forms of type A_2.

Let F/K be a quadratic separable field extension and σ the non-trivial K-automorphism of F. An involution on an F-algebra $\mathcal{R}$ is called an F/K-involution if its restriction to F is σ. An isomorphism of F-algebras with involution

$$f : (\mathcal{R}, \tau) \to (\mathcal{R}', \tau')$$

is an F-algebra isomorphism $f : \mathcal{R} \to \mathcal{R}'$ such that $\tau' \circ f = f \circ \tau$. Let also E/F be a field extension such that E has an automorphism ψ of order 2 such that $\psi|F = \sigma$ (i.e., E has an F/K-involution). Then τ_E denotes the involution on $\mathcal{R} \otimes_F E$ defined by the formula $\tau_E(r \otimes e) := \tau(r) \otimes \psi(e)$, where $r \in \mathcal{R}$, $e \in E$. In particular, if L/F is a field extension linearly disjoint to E over F with an automorphism ϕ of order two extending σ, then ϕ_E is the automorphism of order two of the field $EL = L \otimes_F E$ which extends ϕ.

Let $\mathcal{A}$ be a central division F-algebra of degree n with an F/K-involution τ. Let L/F be a separable field extension of degree n, and let $\phi : L \to L$ be an automorphism of order two such that $\phi|F = \sigma$. An embedding of algebras with involution $(L, \phi) \hookrightarrow (\mathcal{A}, \tau)$ is by definition an injective F-homomorphism $f : L \to \mathcal{A}$ such that $\tau \circ f = f \circ \phi$. It is known that embeddings of maximal tori into the special unitary group $\mathrm{SU}(\mathcal{A}, \tau)$ can be described in terms of embeddings of fields with involution into the central simple algebra with involution $(\mathcal{A}, \tau)$ [5, Prop. 2.3].

In this paper, we construct a field E and a subfield $T \subset E$ such that $[E : T] = 2$ and there is an infinite set of (pairwise non-isomorphic) division E-algebras of degree 3 with E/T-involution such that a field extension L/E of degree 3 can be embedded as an algebra with involution into one algebra of this set if and only if it can be embedded as an algebra with involution into all other algebras from this set. Passing to the corresponding special unitary groups, we obtain an example of outer forms of type A_2 with infinite genus. The main result of the paper is the following theorem.

Theorem 1.3. *There exists a simple simply connected algebraic group G over a (certain) field T that is an outer form of type A_2 for which the genus* $\mathbf{gen}_T(G)$ *is infinite.*

Note that the field T constructed in the proof of this theorem is infinitely generated.

Below we use the following notation: $\mathrm{Alg}_3(F/K)$ is the set of isomorphism classes of central division F-algebras of degree 3 with F/K involution; $\mathrm{Ext}_3(F/K)$ is the set of isomorphism classes of field extensions of F of degree 3 with F/K-involution. The 3-torsion of the Brauer group $\mathrm{Br}(F)$ is denoted by ${}_3\mathrm{Br}(F)$. For a field extension E/F and a central simple F-algebra $\mathcal{A}$, $\mathcal{A}_E$ denotes the tensor product $\mathcal{A} \otimes_F E$ and $\mathrm{res}_{E/F} : \mathrm{Br}(F) \to \mathrm{Br}(E)$ denotes the restriction homomorphism. The restriction of $\mathrm{res}_{E/F}$ to the subgroup ${}_3\mathrm{Br}(F)$ will also be denoted by $\mathrm{res}_{E/F}$. For a central simple F-algebra $\mathcal{A}$, $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra and $\mathcal{A}^m$ denotes $\mathcal{A} \otimes_F \cdots \otimes_F \mathcal{A}$ (m times). For a quadratic form q over K and a field extension E/K, q_E denotes the quadratic form obtained by extension of scalars from K to E. Recall that a field extension E/F is called regular if E/F is separable and F is algebraically closed in E.

2. Preliminary results

We start with the following.

Lemma 2.1. *Let n be a positive integer, F a field of characteristic not dividing $2n$, F/K a quadratic field extension, σ the non-trivial K-automorphism of F, $\mathcal{A}$ a central simple F-algebra of degree n, and L/F a cyclic field extension of degree n. Then there exists a regular field extension M/F and a subfield $T \subset M$ such that $[M : T] = 2$ and*

(1) *$M = TF$ and the non-trivial T-automorphism of M extends σ;*

(2) *the composite ML splits $\mathcal{A}_M$;*

(3) *the homomorphism $\mathrm{res}_{M/F} : \mathrm{Br}(F) \to \mathrm{Br}(M)$ is injective.*

Proof. Let $F(x)$ be a purely transcendental extension of F of transcendence degree 1. Let also ϕ be a generator of the Galois group $\mathrm{Gal}(L(x)/F(x))$ and

$$\mathcal{C} := \mathcal{A}^{\mathrm{op}}_{F(x)} \otimes_{F(x)} \big(L(x)/F(x), \phi, x\big),$$

where $(L(x)/F(x), \phi, x)$ is a cyclic $F(x)$-algebra of degree n. Let also E be the function field of the Severi–Brauer variety of $\mathcal{C}$. Note that the kernel of the restriction homomorphism $\mathrm{res}_{E/F(x)} : \mathrm{Br}(F(x)) \to \mathrm{Br}(E)$ is generated by $[\mathcal{C}]$ (see, e.g., [8, Cor. 13.16]).

Let $\mathcal{B}$ be a central simple F-algebra of exponent bigger than 1. Assume that $\mathcal{B}$ is split by E, then $[\mathcal{B}_{F(x)}] = [\mathcal{C}^i]$ for some $1 \le i \le n$. If $i < n$, then the $F(x)$-algebra $\mathcal{C}^i$ ramifies at the discrete valuation (trivial on F) of $F(x)$ defined by the polynomial x, but $\mathcal{B}_{F(x)}$ is unramified at this valuation, hence $[\mathcal{B}_{F(x)}] \neq [\mathcal{C}^i]$. Since the exponent of $\mathcal{B}_{F(x)}$ is bigger than 1, then

$$[\mathcal{B}_{F(x)}] \neq [\mathcal{C}^n] = \big[F(x)\big].$$

Thus $\mathcal{B}_{F(x)}$ is not split by E, i.e., the homomorphism $\mathrm{res}_{E/F} : \mathrm{Br}(F) \to \mathrm{Br}(E)$ is injective.

Since E splits $\mathcal{C}$, then

$$[\mathcal{A}_E] = \big[\big(L(x)/F(x), \phi, x\big)_E\big] = \big[(EL/E, \phi', x)\big],$$

where ϕ' is the generator of $\mathrm{Gal}(EL/E)$. Thus EL splits $\mathcal{A}_E$.

Note that E/F is a regular extension of $F(x)$. For the following construction of the transfer of a regular field extension, we refer to [7, p. 220].

Let σ also denotes the $K(x)$-automorphism of $F(x)$ extending the automorphism σ of F. The automorphism σ of $F(x)$ can be extended to an isomorphism (which we also denote by σ) of E and another regular extension of $F(x)$ denoted by E_σ. Thus the following diagram commutes:

$$\begin{array}{ccccc} F & \hookrightarrow & F(x) & \hookrightarrow & E \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ F & \hookrightarrow & F(x) & \hookrightarrow & E_\sigma . \end{array} \tag{2.1}$$

Let $M = EE_\sigma$ be the free composite over F of E and E_σ. This free composite is F-isomorphic to the function field of the Severi–Brauer variety of the $E_\sigma(y)$-algebra

$$\mathcal{A}^{\mathrm{op}}_{E_\sigma(y)} \otimes_{E_\sigma(y)} \big(E_\sigma L(y)/E_\sigma(y), \psi', y\big),$$

where y is transcendental over E_σ (we replace x by y since the composite is free) and ψ' is the generator of the Galois group $\mathrm{Gal}(E_\sigma L(y)/E_\sigma(y))$. The field M is a regular extension of F. The isomorphisms

$$\sigma : E \to E_\sigma \quad \text{and} \quad \sigma^{-1} : E_\sigma \to E$$

have a unique extension to an automorphism $\overline{\sigma}$ of M of order two. Let $T := T_{F/K}(E)$ be the transfer of E with respect to the ground field descent $F \supset K$, i.e., T is the subfield of M of elements fixed under the action of $\overline{\sigma}$. Note that the composite TF coincides with M, $[M : T] = 2$, and $\overline{\sigma}$ extends σ.

The algebra $\mathcal{A}_M$ is split by ML since $\mathcal{A}_E$ is split by EL.

Finally, the diagram (2.1) induces the following commutative diagram for the corresponding Brauer groups:

$$\begin{array}{ccccccc}
\mathrm{Br}(F) & \xrightarrow{\mathrm{res}} & \mathrm{Br}\big(F(x)\big) & \xrightarrow{\mathrm{res}} & \mathrm{Br}(E) & & \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\mathrm{Br}(F) & \xrightarrow{\mathrm{res}} & \mathrm{Br}\big(F(x)\big) & \xrightarrow{\mathrm{res}} & \mathrm{Br}(E_\sigma) & \xrightarrow{\mathrm{res}} & \mathrm{Br}(M).
\end{array}$$

The injectivity of $\mathrm{res}_{E/F}$ implies the injectivity of $\mathrm{res}_{E_\sigma/F}$. Moreover, $\mathrm{res}_{M/E_\sigma}$ is injective by the same arguments as for $\mathrm{res}_{E/F}$, we just replace the ground field F by E_σ. Hence the homomorphism $\mathrm{res}_{M/F}$ is also injective. ∎

We also need the following.

Lemma 2.2. *Let F be a field of characteristic $\neq 2, 3$; F/K a quadratic field extension, σ the non-trivial K-automorphism of F, and L/F a field extension of degree 3 with an automorphism ϕ of order two such that $\phi|F = \sigma$. Then there exists a field extension $F(L)/F$ and a subfield $K(L) \subset F(L)$ such that $[F(L) : K(L)] = 2$ and*

(1) *$F(L) = K(L)F$ and the non-trivial $K(L)$-automorphism of $F(L)$, denoted by $\sigma_{F(L)}$, extends σ;*

(2) *$[F(L) : F] \leq 2$;*

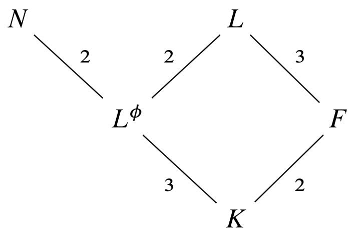
(3) *the composite $F(L)L$ is a cyclic extension of $F(L)$ of degree 3;*

(4) *the homomorphism $\mathrm{res}_{F(L)/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F(L))$ is injective.*

Proof. If the extension L/F is cyclic, then one can take $F(L) := F$, $K(L) := K$, and $\sigma_{F(L)} := \sigma$.

Assume that the extension L/F is not cyclic. Let N be the normal closure of the extension L^ϕ/K, where $L^\phi \subset L$ is the subfield of elements fixed by ϕ. Then $F \not\subset N$ and

NF is the normal closure of the extension L/F. Thus we have the following diagram of field extensions:



Let H be the Sylow 3-subgroup of the Galois group $\mathrm{Gal}(N/K)$. Then N^H, the fixed field of H, is an extension of K of degree 2 and N/N^H is a cyclic extension of degree 3. Hence NF is a cyclic extension of $N^H F$ of degree 3 and $[N^H F : F] = 2$. Let $K(L) := N^H$ and $F(L) := K(L)F$. Since $F \not\subset N$, then $[F(L) : K(L)] = 2$ and the field $F(L)$ has a $K(L)$-automorphism of order two extending σ. Note that $F(L)L = NF$, hence $F(L)L/F(L)$ is a cyclic extension of degree 3. Finally, since $[F(L) : F] = 2$, then the homomorphism $\mathrm{res}_{F(L)/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F(L))$ is injective. ■

Remark 2.3. In the notations of Lemma 2.2, for any field extension L' of F of degree 3, $F(L)$ and L' are linearly disjoint over F. Moreover, if L' has an automorphism ϕ' of order 2 extending σ, then the composite $F(L)L'$ has the automorphism $\phi'_{F(L)}$ which extends the automorphisms ϕ' and $\sigma_{F(L)}$.

Lemma 2.4. *Let F be a field of characteristic $\neq 2, 3$; F/K a quadratic field extension, σ the non-trivial K-automorphism of F, $\mathcal{A}$ a central simple F-algebra of degree 3 with an F/K-involution τ, and L/F a field extension of degree 3 with an automorphism ϕ of order two such that $\phi|F = \sigma$. Then there exists a field extension $F_{(K,L,\mathcal{A})}/F$ and a subfield $K_{(L,\mathcal{A})} \subset F_{(K,L,\mathcal{A})}$ such that $[F_{(K,L,\mathcal{A})} : K_{(L,\mathcal{A})}] = 2$ and*

1. *$F_{(K,L,\mathcal{A})} = K_{(L,\mathcal{A})}F$ and the non-trivial $K_{(L,\mathcal{A})}$-automorphism of $F_{(K,L,\mathcal{A})}$, denoted by $\sigma_{F_{(K,L,\mathcal{A})}}$, extends σ;*
2. *the homomorphism $\mathrm{res}_{F_{(K,L,\mathcal{A})}/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F_{(K,L,\mathcal{A})})$ is injective;*
3. *for any field extension L' of F of degree 3, $F_{(K,L,\mathcal{A})}$ and L' are linearly disjoint over F;*
4. *there is an embedding $(F_{(K,L,\mathcal{A})}L, \phi_{F_{(K,L,\mathcal{A})}}) \hookrightarrow (\mathcal{A}_{F_{(K,L,\mathcal{A})}}, \tau_{F_{(K,L,\mathcal{A})}})$ of algebras with involution.*

Proof. Let $F(L)$, $K(L)$ and $\sigma_{F(L)}$ be as in Lemma 2.2. Let also M and T be fields obtained by applying Lemma 2.1 for the quadratic field extension $F(L)/K(L)$, the $F(L)$-algebra $\mathcal{A}_{F(L)}$, the cyclic field extension $F(L)L/F(L)$ of degree 3. Then by Lemmas 2.1 and 2.2, the homomorphism $\mathrm{res}_{M/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(M)$ is injective; for any field extension L' of F of degree 3, M and L' are linearly disjoint over F and the composite ML splits $\mathcal{A}_M$. Thus there is an M-embedding $\varepsilon : ML \hookrightarrow \mathcal{A}_M$ of M-algebras.

Note that $\mathcal{A}_M$ has the M/T-involution τ_M which extends τ and ML has the automorphism ϕ_M of order two extending ϕ. By [5, Proposition 3.1], there exists an M/T-involution δ on $\mathcal{A}_M$ such that $\varepsilon : (ML, \phi_M) \hookrightarrow (\mathcal{A}_M, \delta)$ is an embedding of algebras with involution.

Let $\pi(\delta)$ and $\pi(\tau_M)$ be the 3-fold Pfister forms of involutions δ and τ_M respectively (see [3, §19.B]). Let $T(\pi(\delta))$ and $T(\pi(\tau_M))$ be the function fields of $\pi(\delta)$ and $\pi(\tau_M)$ respectively. Then the quadratic forms $\pi(\delta)_{T(\pi(\delta))}$ and $\pi(\tau_M)_{T(\pi(\tau_M))}$ are isotropic and hence hyperbolic since they are Pfister forms.

Let $K_{(L,\mathcal{A})}$ be the free composite over T of the fields $T(\pi(\delta))$ and $T(\pi(\tau_M))$. Let also $F_{(K,L,\mathcal{A})} := K_{(L,\mathcal{A})}F$. Since $F \not\subset K_{(L,\mathcal{A})}$, then $[F_{(K,L,\mathcal{A})} : K_{(L,\mathcal{A})}] = 2$ and $F_{(K,L,\mathcal{A})}$ has a $K_{(L,\mathcal{A})}$-automorphism $\sigma_{F_{(K,L,\mathcal{A})}}$ of order 2 extending σ. Note that the algebraic closure of F in $F_{(K,L,\mathcal{A})}$ is $F(L)$ and $[F(L) : F]$ is either 1 or 2. Therefore, $F_{(K,L,\mathcal{A})}$ and L' are linearly disjoint over F for any field extension L' of F of degree 3.

The extensions $T(\pi(\delta))/T$ and $T(\pi(\tau_M))/T$ are composition of a purely transcendental extension with a quadratic extension, hence the homomorphism

$$\operatorname{res}_{F_{(K,L,\mathcal{A})}/M} : {}_3\operatorname{Br}(M) \to {}_3\operatorname{Br}(F_{(K,L,\mathcal{A})})$$

is injective. Hence $\operatorname{res}_{F_{(K,L,\mathcal{A})}/F} : {}_3\operatorname{Br}(F) \to {}_3\operatorname{Br}(F_{(K,L,\mathcal{A})})$ is also injective.

The quadratic forms $\pi(\delta)_{K_{(L,\mathcal{A})}}$ and $\pi(\tau_M)_{K_{(L,\mathcal{A})}}$ are hyperbolic. Then by [3, Th. 19.6], the involutions $\delta_{F_{(K,L,\mathcal{A})}}$ and $\tau_{F_{(K,L,\mathcal{A})}}$ on $\mathcal{A}_{F_{(K,L,\mathcal{A})}}$ are conjugate. This means that there is an isomorphism $\xi : (\mathcal{A}_{F_{(K,L,\mathcal{A})}}, \delta_{F_{(K,L,\mathcal{A})}}) \to (\mathcal{A}_{F_{(K,L,\mathcal{A})}}, \tau_{F_{(K,L,\mathcal{A})}})$ of algebras with involution.

Moreover, the embedding $\varepsilon : (ML, \phi_M) \hookrightarrow (\mathcal{A}_M, \delta)$ of algebras with involution induces an embedding $(F_{(K,L,\mathcal{A})}L, \phi_{F_{(K,L,\mathcal{A})}}) \hookrightarrow (\mathcal{A}_{F_{(K,L,\mathcal{A})}}, \delta_{F_{(K,L,\mathcal{A})}})$ of algebras with involution. Indeed, $F_{(K,L,\mathcal{A})}L = ML \otimes_M F_{(K,L,\mathcal{A})}$. Let

$$\varepsilon_{F_{(K,L,\mathcal{A})}} : ML \otimes_M F_{(K,L,\mathcal{A})} \longrightarrow \mathcal{A}_{F_{(K,L,\mathcal{A})}}$$

be an $F_{(K,L,\mathcal{A})}$-embedding defined by the formula $\varepsilon_{F_{(K,L,\mathcal{A})}}(m \otimes a) := \varepsilon(m) \otimes a$, where $m \in ML$, $a \in F_{(K,L,\mathcal{A})}$. Then

$$\begin{aligned}
\varepsilon_{F_{(K,L,\mathcal{A})}}\big(\phi_{F_{(K,L,\mathcal{A})}}(m \otimes a)\big) &= \varepsilon_{F_{(K,L,\mathcal{A})}}\big(\phi_M(m) \otimes \sigma_{F_{(K,L,\mathcal{A})}}(a)\big) \\
&= \varepsilon\big(\phi_M(m)\big) \otimes \sigma_{F_{(K,L,\mathcal{A})}}(a) \\
&= \delta\big(\varepsilon(m)\big) \otimes \sigma_{F_{(K,L,\mathcal{A})}}(a) \\
&= \delta_{F_{(K,L,\mathcal{A})}}\big(\varepsilon(m) \otimes a\big) \\
&= \delta_{F_{(K,L,\mathcal{A})}}\big(\varepsilon_{F_{(K,L,\mathcal{A})}}(m \otimes a)\big).
\end{aligned}$$

Thus $\varepsilon_{F_{(K,L,\mathcal{A})}}$ is an embedding of algebras with involutions. Then $\xi \circ \varepsilon_{F_{(K,L,\mathcal{A})}}$ is an embedding $(F_{(K,L,\mathcal{A})}L, \phi_{F_{(K,L,\mathcal{A})}}) \hookrightarrow (\mathcal{A}_{F_{(K,L,\mathcal{A})}}, \tau_{F_{(K,L,\mathcal{A})}})$ of algebras with involution. ■

The following construction of the field $F_{(K,S,A)}$ is an adaptation of the construction from [10] for algebras with involutions. We give the details below for the reader's convenience.

Proposition 2.5. *Let F be a field of characteristic $\neq 2,3$; F/K a quadratic field extension, σ the non-trivial K-automorphism of F, $A \subset \mathrm{Alg}_3(F/K)$ and $S \subset \mathrm{Ext}_3(F/K)$. Then there exists a field extension $F_{(K,S,A)}/F$ and a subfield $K_{(S,\mathcal{A})} \subset F_{(K,S,A)}$ such that $[F_{(K,S,A)} : K_{(S,A)}] = 2$ and*

(1) *$F_{(K,S,A)} = K_{(S,A)}F$ and the non-trivial $K_{(S,A)}$-automorphism, denoted by $\sigma_{F_{(K,S,A)}}$, of $F_{(K,S,A)}$ extends σ;*
(2) *the homomorphism $\mathrm{res}_{F_{(K,S,A)}/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F_{(K,S,A)})$ is injective;*
(3) *for any field extension L' of F of degree 3, $F_{(K,L,\mathcal{A})}$ and L' are linearly disjoint over F;*
(4) *for any $\mathcal{A} \in A$ with an F/K-involution τ and $L \in S$ with a K-automorphism ϕ of order 2 extending σ, there is an embedding*

$$(F_{(K,S,A)}L, \phi_{F_{(K,S,A)}}) \hookrightarrow (\mathcal{A}_{F_{(K,S,A)}}, \tau_{F_{(K,S,A)}})$$

of algebras with involution.

Proof. Let $\mathcal{P} := \{(L,\mathcal{D}) \mid L \in S \text{ and } \mathcal{D} \in A\}$ be the set of pairs. Let also $<$ be a well-ordering on $\mathcal{P}$ and let $t_0 = (L_0, \mathcal{D}_0)$ denote its least element. Set $E_{t_0} := F_{(K,L_0,\mathcal{D}_0)}$ and $T_0 := K_{(L_0,\mathcal{D}_0)}$, where the fields $F_{(K,L_0,\mathcal{D}_0)}$ and $K_{(L_0,\mathcal{D}_0)}$ are constructed in Lemma 2.4. For $t = (L,\mathcal{D}) \in \mathcal{P}$, set

$$E^{<t} := \bigcup_{t'<t} E_{t'},\ T^{<t} := \bigcup_{t'<t} T_{t'},\ T_t := T^{<t}{}_{(E^{<t}L,\mathcal{D}_{E^{<t}})},\ E_t := E^{<t}{}_{(T^{<t},E^{<t}L,\mathcal{D}_{E^{<t}})},$$

where the fields E_t and T_t are obtained by applying Lemma 2.4 to the quadratic field extension $E^{<t}/T^{<t}$, the field extension $E^{<t}L/E^{<t}$ of degree 3, the automorphism $\phi_{E^{<t}}$ of $E^{<t}L$ extending the automorphism ϕ of L and the $E^{<t}$-algebra $\mathcal{D}_{E^{<t}}$. We also define $F_{(K,S,A)} := \bigcup_{t\in\mathcal{P}} E_t$ and $K_{(S,A)} := \bigcup_{t\in\mathcal{P}} T_t$.

By Lemma 2.4, $E_t = T_t F$ and $[E_t : T_t] = 2$ for any $t \in \mathcal{P}$. Then $F_{(K,S,A)} = K_{(S,A)}F$, $[F_{(K,S,A)} : K_{(S,A)}] = 2$ and the non-trivial $K_{(S,A)}$-automorphism of $F_{(K,S,A)}$ extends σ.

By Lemma 2.4 and transfinite induction, the homomorphism

$$\mathrm{res}_{F_{(K,S,A)}/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F_{(K,S,A)})$$

is injective and for any field extension L' of F of degree 3, $F_{(K,S,A)}$ and L' are linearly disjoint over F.

Finally, let $\mathcal{A} \in A$ with an F/K-involution τ, $L \in S$ with an automorphism ϕ of order 2 extending σ and $t = (L,\mathcal{A})$. By Lemma 2.4, there is an embedding $(E_t L, \phi_{E_t}) \hookrightarrow (\mathcal{A}_{E_t}, \tau_{E_t})$ of algebras with involution. Moreover, as in the proof of Lemma 2.4, this embedding induces the embedding

$$(F_{(K,S,A)}L, \phi_{F_{(K,S,A)}}) \hookrightarrow (\mathcal{A}_{F_{(K,S,A)}}, \tau_{F_{(K,S,A)}})$$

of algebras with involution. ∎

Theorem 2.6. *Let F be a field of characteristic $\neq 2, 3$, F/K a quadratic field extension, σ the non-trivial K-automorphism of F, $A \subset \mathrm{Alg}_3(F/K)$. Then there exists a field extension F_A/F and a subfield $K_A \subset F_A$ such that $[F_A : K_A] = 2$ and*

(1) *$F_A = K_A F$ and the non-trivial K_A-automorphism, denoted by σ_{F_A}, of F_A extends σ;*

(2) *the homomorphism $\mathrm{res}_{F_A/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F_A)$ is injective;*

(3) *for any central simple F-algebra $\mathcal{B}$ of degree 3 with an F/K-involution θ, the algebra $\mathcal{B}_{F_A}$ has an F_A/K_A-involution θ_{F_A} extending θ;*

(4) *if $L \in \mathrm{Ext}_3(F_A/K_A)$ with a K_A-automorphism ϕ of order 2 extending σ_{F_A}, then there is an embedding*
$$(L, \phi) \hookrightarrow (\mathcal{A}_{F_A}, \tau_{F_A})$$
of algebras with involution for any $\mathcal{A} \in A$ with an F/K-involution τ.

Proof. Let $F_0 := F$ and $K_0 := K$. We recursively define F_i and K_i, $i \in \mathbb{Z}_{>0}$, to be the fields $F_{i-1(K_{i-1},\mathrm{Ext}_3(F_{i-1}/K_{i-1}),\mathrm{res}_{F_{i-1}/F}(A))}$ and $K_{i-1(\mathrm{Ext}_3(F_{i-1}/K_{i-1}),\mathrm{res}_{F_{i-1}/F}(A))}$ constructed by applying Proposition 2.5 to the quadratic field extension F_{i-1}/K_{i-1}, the set
$$\mathrm{res}_{F_{i-1}/F}(A) \subset \mathrm{Alg}_3(F_{i-1}/K_{i-1})$$
and the set $\mathrm{Ext}_3(F_{i-1}/K_{i-1})$.

Let $F_A := \bigcup_{i \geq 0} F_i$ and $K_A := \bigcup_{i \geq 0} K_i$. Hence $F_A = K_A F$ and the non-trivial K_A-automorphism σ_{F_A} of F_A extends σ. Therefore, for any central simple F-algebra $\mathcal{B}$ of degree 3 with an F/K-involution θ, the F_A/K_A-involution θ_{F_A} extends θ.

By induction and Proposition 2.5, $\mathrm{res}_{F_A/F} : {}_3\mathrm{Br}(F) \to {}_3\mathrm{Br}(F_A)$ is injective.

Assume that $\mathcal{A} \in A$ with an F/K-involution τ and $L \in \mathrm{Ext}_3(F_A/K_A)$ with an automorphism ϕ of order two extending σ_{F_A}. Then there exists $i \geq 0$ and a field extension L' of F_i of degree 3 such that $L = F_A L'$ and $\phi_i := \phi|L'$ is a K_i-automorphism of order two. This means that $L' \in \mathrm{Ext}_3(F_i/K_i)$. By Proposition 2.5, there is an embedding
$$(F_{i+1}L', \phi_{i\,F_{i+1}}) \hookrightarrow (\mathcal{A}_{F_{i+1}}, \tau_{F_{i+1}})$$
of algebras with involution. As in the proof of Lemma 2.4, this embedding can be extended to an embedding $(L, \phi) \hookrightarrow (\mathcal{A}_{F_A}, \tau_{F_A})$ of algebras with involution. ∎

3. Construction

As a corollary to Theorem 2.6, we obtain the following.

Theorem 3.1. *There exists a field E and a subfield $T \subset E$ with $[E : T] = 2$ such that there is an infinite set B of pairwise non-isomorphic division E-algebras of degree 3 with E/T-involution and such that for any field extension L/E of degree 3 with an automorphism ϕ of order 2 extending the non-trivial T-automorphism of E, there is an embedding $(L, \phi) \hookrightarrow (\mathcal{A}, \tau)$ of algebras with involution for any $\mathcal{A} \in B$ with an E/T-involution τ.*

Proof. Let ξ_3 be a primitive 3th root of unity, $K = \mathbb{Q}(\xi_3)(x, y, z)$, the purely transcendental extension of the field $\mathbb{Q}(\xi_3)$ and $F = K(\sqrt{2})$. Then for $i > 0$, the symbol F-algebras $\mathcal{A}_i = (\frac{x+\sqrt{2}y^i}{x-\sqrt{2}y^i}, z)_3$ of degree 3 are pairwise non-isomorphic. Indeed, $\mathcal{A}_i$ ramifies at the discrete valuation (trivial on $\mathbb{Q}(\xi_3)(\sqrt{2})(y, z)$) of $F = \mathbb{Q}(\xi_3)(\sqrt{2})(y, z)(x)$ defined by the polynomial $x + \sqrt{2}y^i \in \mathbb{Q}(\xi_3)(\sqrt{2})(y, z)[x]$, but if $i \neq j$, then $\mathcal{A}_j$ is unramified at this valuation.

By the projection formula for the corestriction [9, Th. 3.2], we have that the corestriction to K of the F-algebra $\mathcal{A}_i$ is similar to the split K-algebra

$$\left(N_{F/K}\left(\frac{x+\sqrt{2}y^i}{x-\sqrt{2}y^i}\right), z\right)_3 = (1, z)_3.$$

Then by [3, Th. 3.1 (2)], the algebra $\mathcal{A}_i$ has an F/K-involution.

Now we apply Theorem 2.6 for the infinite set $A \subset \mathrm{Alg}_3(F/K)$ consisting of algebras $\mathcal{A}_i$, $i > 0$, and set $E := F_A$, $T := K_A$, $B := \mathrm{res}_{F_A/F}(A)$. ∎

Remark 3.2. The referee asked the following interesting question. The construction presented in the proof of the previous theorem yields algebras that are not isomorphic even as algebras without involution. One may wonder if one can construct (infinite) families of non-equivalent involutions supported on the *same* division algebra with isomorphic maximal *invariant* subfields.

Now we are in a position to prove the main Theorem 1.3.

Proof. We use notation from Theorem 3.1. Let $\mathcal{A}$ be an algebra with E/T-involution τ from the set B. Then the special unitary group $G = \mathrm{SU}(\mathcal{A}, \tau)$ is a simple simply connected outer form of type A_2 over T.

If $\mathcal{A}_1$ are $\mathcal{A}_2$ are different algebras with E/T-involutions τ_1 and τ_2 respectively from the set B, then the algebraic groups $\mathrm{SU}(\mathcal{A}_1, \tau_1)$ and $\mathrm{SU}(\mathcal{A}_2, \tau_2)$ are not isomorphic by [3, Th. 26.9]. Moreover, by [5, Prop. 2.3] the groups $\mathrm{SU}(\mathcal{A}_1, \tau_1)$ and $\mathrm{SU}(\mathcal{A}_2, \tau_2)$ have the same T-isomorphism classes of maximal T-tori.

Thus, the genus $\mathbf{gen}_T(G)$ is infinite. ∎

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