Doc. Math. 29 (2024), 763–804
DOI 10.4171/DM/948

Motivic zeta functions of the Hilbert schemes of points on a surface

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Abstract. Let K be a discretely-valued field. Let $X \to \operatorname{Spec} K$ be a surface with trivial canonical bundle. In this paper we construct a weak Néron model of the schemes $\operatorname{Hilb}^n(X)$ over the ring of integers $R \subseteq K$. We exploit this construction in order to compute the motivic zeta function of $\operatorname{Hilb}^n(X)$ in terms of the motivic zeta functions of X and of its base-changes with respect to the totally ramified extensions of K. We determine the poles of $Z_{\operatorname{Hilb}^n(X)}$ and study its monodromy property, showing that if the monodromy conjecture holds for X then it holds for $\operatorname{Hilb}^n(X)$ too.

1. Introduction

The Motivic Zeta Function, introduced by Denef and Loeser, is a useful invariant attached to hypersurface singularities which encodes informations on other classical invariants of the singularity. This invariant is a formal series with coefficients in a Grothendieck ring of varieties which has been proved to be rational by Denef and Loeser in [5], in a sense that will become clear in Section 6. The most discussed open question concerning the motivic zeta function is the *Monodromy Conjecture* which claims the existence of a relationship between the poles of the function and the eigenvalues of the local monodromy operator associated to the hypersurface singularity. In this paper we focus on a natural counterpart, introduced in [18], of Denef's and Loeser's zeta function, whose argument consists of a punctured neighbourhood of the hypersurface singularity, together with a volume form on such neighbourhood, rather than involving the hypersurface itself.

Let K be a complete discretely valued field. Let R be its valuation ring and let k be its residue field, which we assume to be algebraically closed. Fix an algebraic closure $\bar{K}/K$; for a positive integer m coprime with the characteristic exponent of k, denote by $K(m)$ the unique extension $K \subseteq K(m) \subseteq \bar{K}$ such that $[K(m) : K] = m$ and let $R(m)$ be the integral closure of R in $K(m)$. Let $X \to \operatorname{Spec} K$ be a smooth algebraic variety. This datum is the arithmetic analog of the datum of a family of complex varieties over a small punctured disk, where one usually uses the fundamental group of the punctured disk to give a monodromy action on the cohomology. In our case the absolute Galois group of K, being canonically isomorphic to the étale fundamental group of $\operatorname{Spec} K$, replaces the usual fundamental group and acts on X, inducing an action on its cohomology: $H^*(X_{\bar{K}}, \mathbb{Q}_l)$, where

Mathematics Subject Classification 2020: 14G10 (primary); 14J42, 14C05 (secondary).
Keywords: motivic integration, motivic zeta functions, monodromy conjecture, Hilbert schemes.

l is any prime number coprime with the characteristic exponent of k. We will assume that the wild inertia subgroup acts trivially on the cohomology of X, so that the action of $\mathrm{Gal}(\bar{K}|K)$ can be identified with the action of its tame quotient, i.e., $\mathrm{Gal}(K^{\mathrm{tame}}|K)$, where K^{tame} is the union of all the tame extensions of K, which admits a topological generator σ. Hence we can study the monodromy action by focusing on the action of the linear operator

$$\sigma^*\colon H^*(X_{\bar{K}}, \mathbb{Q}_l) \to H^*(X_{\bar{K}}, \mathbb{Q}_l).$$

This operator is quasi-unipotent, meaning that there are $a, b \in \mathbb{N}^*$ such that

$$\big((\sigma^*)^a - \mathrm{id}\big)^b = 0,$$

hence its eigenvalues, often called *monodromy eigenvalues*, are roots of the unity.

Now assume X is a Calabi–Yau variety, i.e., a smooth proper algebraic variety whose canonical sheaf $\omega_X := \Omega^{\dim X}_{X/K}$ is a trivial line bundle and let ω be a volume form on X. Using this datum, we denote by $Z_{X,\omega}(T) \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$, the Motivic Zeta Function of Definition 6.1.6. This invariant is supposed to be closely related to the monodromy operator described above, for varieties satisfying the following property, introduced in [10, Definition 2.6]:

Definition (Monodromy property). Let X be a Calabi–Yau variety with a volume form ω. We say that X satisfies the monodromy property if there exists a finite subset $S \subseteq \mathbb{Z} \times \mathbb{N}^*$ such that

$$Z_{X,\omega}(T) \in \mathscr{M}_k^{\widehat{\mu}}\left[T, \frac{1}{1 - \mathbb{L}^a T^b} \colon (a, b) \in S\right],$$

and for all $(a, b) \in S$ we have that, for all the embeddings of $\mathbb{Q}_l$ in $\mathbb{C}$, $\exp(2\pi i a/b)$ is an eigenvalue of the monodromy operator.

The monodromy property has been proven for several classes of varieties: Halle and Nicaise proved it for Abelian varieties, [10], while Jaspers proved it in [14] when X is a K3 surface admitting a Crauder–Morrison model and Overkamp proved it for Kummer K3 surfaces in [20]. Yet we do not know whether all the K3 surfaces satisfy the Monodromy conjecture.

In this paper we study a slightly weaker version of this property for Hilbert schemes of points on surfaces; we report here a simplified version of our main result, Theorem 10.1.2; a more precise statement is provided in Section 10, after having introduced the notions involved:

Theorem. *Let X be a surface with trivial canonical bundle having the monodromy property.*

Then the property holds also for $\mathrm{Hilb}^n(X)$*, for every* $n \in \mathbb{N}^*$.

Sketch of the proof. We will outline here the main ideas involved in the proof. The computation of the motivic zeta function relies on the tools of motivic integration, thus on the construction of a weak Néron model of $\mathrm{Hilb}^n(X)$. After fixing an *sncd* model of X over the ring R (satisfying a technical assumption introduced in Section 9.1), call $\mathfrak{X}$ such a model, then for all integers m divisible enough and coprime with the characteristic exponent of k, it is possible to construct semistable models $\mathfrak{X}(m) \to \Delta(m) = \operatorname{Spec} R(m)$ of $X \times_K K(m)$. We give an explicit construction of such models, see Section 5, satisfying the following additional properties:

- The natural action of $\mu_m = \operatorname{Gal}(K(m)|K)$ on $X(m)$ extends to $\mathfrak{X}(m)$;
- For all points $x \in \mathfrak{X}(m)_{k,\mathrm{sm}}$, the stabilizer $\operatorname{Stab}_x \subseteq \mu_m$ acts trivially on the whole irreducible component $E \subseteq \mathfrak{X}(m)_k$ containing x.

The relative Hilbert functor provides a proper scheme $\operatorname{Hilb}^n(\mathfrak{X}(m)/\Delta(m)) \to \Delta(m)$ whose generic fibre is $\operatorname{Hilb}^n(X(m))$. This model inherits from $\mathfrak{X}(m)$ an action of μ_m, in particular it is a proper, equivariant model of $\operatorname{Hilb}^n(X(m))$ over $\Delta(m)$. If we remove the critical points of the map $\mathfrak{X}(m) \to \Delta(m)$, then we repeat the construction obtaining a smooth μ_m-equivariant subscheme $\operatorname{Hilb}^n(\mathfrak{X}(m)_{\mathrm{sm}}/\Delta(m)) \subseteq \operatorname{Hilb}^n(\mathfrak{X}(m)/\Delta(m))$. By the means of Weil restriction of scalars (see Section 4), we construct a smooth model of $\operatorname{Hilb}^n(X)$ over R, namely $(\operatorname{Res}_{\Delta(m)/\Delta} \operatorname{Hilb}^n(\mathfrak{X}(m)_{\mathrm{sm}}/\Delta(m)))^{\mu_m}$.

If m is sufficiently divisible (in Section 9 we construct a suitable value of m, denoted $\tilde{n}$), the model above satisfies the weak extension property with respect to $\operatorname{Spec} K \subseteq \operatorname{Spec} R$, hence it is a weak Néron model of $\operatorname{Hilb}^n(X)$. This model provides a direct way to compute the motivic integral $\int_{\operatorname{Hilb}^n(X)} \omega^{[n]}$, where $\omega^{[n]}$ is the volume form on $\operatorname{Hilb}^n(X)$ constructed in Section 9; choosing $m = m_1\tilde{n}$, for $m_1 \in \mathbb{N}^*$, allows us to compute the motivic integral of the Hilbert schemes $\operatorname{Hilb}^n(X(m_1)/K(m_1))$, therefore we get the formal series defining $Z_{\operatorname{Hilb}^n(X),\omega^{[n]}}$, since $\operatorname{Hilb}^n(X) \times_K K(m) = \operatorname{Hilb}^n(X(m))$. The value of the Zeta function is, indeed, written implicitly in Theorem 9.3.2 and explicitly in equation (9.2), i.e.,

$$Z_{\operatorname{Hilb}^n(X),\omega^{[n]}} = \sum_{\alpha \dashv n} \prod_{j=1}^{\infty} \left(\mathbb{L}^{(j-1)\alpha_j} \operatorname{Sym}^{\alpha_j}(Z_{X(j),\omega(j)})\right),$$

where $\alpha \dashv n$ denotes that α is a partition of n, in the sense of Definition 7.2.4.

Using this explicit formula and the results developed in Section 7.3, we find out that every pole of $Z_{\operatorname{Hilb}^n(X),\omega^{[n]}}$ is the sum of n poles of $Z_{X,\omega}$, possibly repeated, we do not know yet if the converse is true, i.e., whether every possible sum of n poles of $Z_{X,\omega}$ is a pole of $Z_{\operatorname{Hilb}^n(X),\omega^{[n]}}$. Putting the description of the poles of the Zeta Function together with the description of the cohomology of $\operatorname{Hilb}^n(X)$ in [8], one infers the claimed result about the monodromy property. ∎

A further byproduct of the computations we carried out in Section 7, we also achieved Proposition 10.1.5:

Proposition. *Let Y, Z be two Calabi–Yau varieties endowed with volume forms ω_1, ω_2 satisfying the monodromy property. Let ω be volume forms on $Y \times Z$ the volume form $\omega := \mathrm{pr}_Y^*\omega_1 \wedge \mathrm{pr}_Z^*\omega_2$. Then also $Y \times Z$, endowed with the volume form ω, satisfies the monodromy property.*

We conclude the paper by discussing the monodromy property in an explicit example, i.e., comparing the poles of the zeta functions of a quartic surface embedded in $\mathbb{P}^3_K$ and of its Hilbert scheme of degree 2.

Organisation of the paper

In Section 3, we introduce the Grothendieck rings of varieties and their equivariant versions, using them to define the main actors of the manuscript.

In Section 4, we recall the notion of a weak Néron model and show some techniques that we will use in order to construct them.

In Section 5, we adapt the theory of potential semistable reduction of families of surfaces in our case. We construct a specific equivariant semistable model satisfying a good property with respect to the Galois action of the extension.

In Section 6, we recall the notion of motivic integration and the definition of the Motivic Zeta Function. Then we explain the notion of rationality for power series with coefficients in $\mathscr{M}_k$, in $\mathscr{M}_k[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}]$ and in $\widehat{\mathscr{M}_k}$, giving also a definition of a pole of such functions. We conclude by stating the Monodromy Conjecture in two forms.

In Section 7, we discuss some facts concerning the poles of rational functions with coefficients in our motivic ring. These properties will be useful when applied to the formula that we will produce for the motivic zeta function of Hilbert schemes.

In Section 8, we summarize some facts about Hilbert schemes that will be involved in the computation of their motivic zeta functions.

In Section 9, we give a construction of the weak Néron models of Hilbert schemes of points on surfaces and use those models to compute their motivic zeta function.

We finally discuss the monodromy property in Section 10.

2. Notation and conventions

Throughout the manuscript, unless differently stated, R will be a complete DVR, K its fraction field and k its residue field, which we will assume to be algebraically closed. We fix an algebraic closure $\overline{K}$ of K, so that whenever we consider extensions of K and R we think of algebraic extensions of K in $\overline{K}$ and integral extensions of R in $\overline{K}$. When m is an integer coprime with the characteristic exponent of k, we denote by $K(m)$ the unique extension of K of degree m and by $R(m)$ the integral closure of R in $K(m)$; we denote by K^{tame} the tame closure of K, i.e.,

$$K^{\mathrm{tame}} := \bigcup_{\substack{m\geq 1\\ \operatorname{char} k\nmid m}} K(m).$$

We denote by $\Delta := \operatorname{Spec} R$, $\Delta(m) := \operatorname{Spec} R(m)$ and similarly $\Delta^* := \operatorname{Spec} K$, $\Delta^*(m) :=$ Spec $K(m)$, for m coprime with the characteristic exponent of k. By variety over a field F

we denote a reduced, separated scheme of finite type over F. The set of natural numbers $\mathbb{N}$ contains 0, while $\mathbb{N}^*$ denotes $\mathbb{N}\setminus\{0\}$.

3. Grothendieck ring of varieties

In this section we introduce the rings containing the coefficients of the formal series we will study later on.

3.1. A motivic ring

3.1.1. Fix a field k and consider the category of algebraic schemes of finite type Sch_k. Let $K_0(\mathrm{Sch}_k)$ be the abelian group whose generators are isomorphism classes of scheme of finite type over k and whose relations, called *scissor relations*, are generated by elements in the form

$$X - Y - (X\setminus Y),$$

whenever X is an algebraic variety and $Y \subseteq X$ is a closed subscheme. We denote by $[X]$ the class of $X \in \mathrm{Sch}_k$ in $K_0(\mathrm{Sch}_k)$.

3.1.2. There is a unique ring structure on $K_0(\mathrm{Sch}_k)$ such that for all $X, Y \in \mathrm{Sch}_k$ one has $[X]\cdot[Y] = [X \times_k Y]$. With this ring structure, $K_0(\mathrm{Sch}_k)$ is called *the Grothendieck ring of varieties*. It is also characterized by the following universal property:

Universal property of $K_0(\mathrm{Sch}_k)$. Let R be a ring and let $\Psi\colon \mathrm{Sch}_k \to R$ a multiplicative and additive invariant constant on isomorphism classes, i.e., a function which associates to a variety X an element $\Psi(X) \in R$ such that $\Psi(X \times_k Y) = \Psi(X)\Psi(Y)$ and if $X = Y \cup Z$, then

$$\Psi(X) + \Psi(Y \cap Z) = \Psi(Y) + \Psi(Z).$$

Then there is a unique ring homomorphism $\varphi\colon K_0(\mathrm{Sch}_k) \to R$ such that

$$\forall X \in \mathrm{Sch}_k, \quad \varphi\big([X]\big) = \Psi(X).$$

3.2. Localised Grothendieck ring

3.2.1. A ring that is worth some consideration is obtained as a localisation of $K_0(\mathrm{Sch}_k)$.

Definition 3.2.2 (Localised Grothendieck ring of varieties). Let us denote by $\mathbb{L}$ the element $[\mathbb{A}^1_k] \in K_0(\mathrm{Sch}_k)$.

The localisation of $K_0(\mathrm{Sch}_k)$ with respect to $\mathbb{L}$,

$$\mathscr{M}_k := K_0(\mathrm{Sch}_k)[\mathbb{L}^{-1}],$$

is called *the localised Grothendieck ring of varieties*.

3.2.3. By combining the universal property of localisation and of $K_0(\mathrm{Sch}_k)$, one can define $\mathscr{M}_k$ as a universal ring for all the invariants $\mathrm{Sch}_k \to R$ which send $\mathbb{A}^1_k$ in R^*.

3.3. Completed Grothendieck ring

3.3.1. Consider the filtration $\mathcal{F}^n\mathscr{M}_k := \langle \mathbb{L}^r[X] \mid r \in \mathbb{Z}, \dim[X] + r \leq -n\rangle_{\mathbb{Z}}$.

Definition 3.3.2 (Completed Grothendieck ring of varieties). The completed Grothendieck ring of varieties $\widehat{\mathscr{M}_k}$ is the completion of $\mathscr{M}_k$ with respect to the filtration $\mathcal{F}^\bullet$.

3.4. Equivariant setting

3.4.1. All the three rings above have an equivariant version, i.e., can be constructed in the category Sch_k^G of algebraic varieties endowed with a good action of a finite group G.

Definition 3.4.2. A G-action on a scheme X is said to be *good* if every orbit is contained in an affine open subscheme of X. We say that X is a good G-variety (resp. G-scheme) if it is endowed with a good G-action.

3.4.3. Quasi-projective G-varieties are always good. Unless differently stated, our G-schemes are good.

3.4.4. As before, the *equivariant Grothendieck group of varieties* $K_0(\mathrm{Sch}_k^G)$ is the group generated on the isomorphism classes of G-schemes of finite type with relation of two kinds:

Scissor relations. Let X be a G-scheme and Y a G-invariant closed subscheme, then

$$X - Y - (X\backslash Y),$$

is 0 in $K_0(\mathrm{Sch}_k^G)$.

Trivializing relations. Let $S \in \mathrm{Sch}_k^G$ and let $V \to S$ be a G-equivariant affine bundle of rank d. Then

$$V - (S \times_k \mathbb{A}_k^d) \in \mathrm{Sch}_k^G$$

is set to 0, where G acts trivially on the second factor of $S \times_k \mathbb{A}_k^d$.

There is a unique ring structure on $K_0(\mathrm{Sch}_k^G)$ such that for every two $X, Y \in \mathrm{Sch}_k^G$, we have $[X] \cdot [Y] := [X \times_k Y]$, where the action of G on $X \times_k Y$ is the diagonal action.

3.4.5. A group homomorphism $G_2 \to G_1$ of finite groups, together with an action of G_1 on a scheme X, induces an action of G_2 on X via composition

$$G_2 \to G_1 \to \mathrm{Aut}(X),$$

which in turns induces a functor $\mathrm{Sch}_k^{G_1} \to \mathrm{Sch}_k^{G_2}$. Clearly this functor sends G_1-equivariant inclusions of schemes in G_2-equivariant inclusion of schemes, thus preserving scissor relations, and sends G_1-equivariant affine bundles to G_2-equivariant affine bundles, thus preserving also the trivializing relations; hence the functor descends to a group homomorphism $K_0(\mathrm{Sch}_k^{G_1}) \to K_0(\mathrm{Sch}_k^{G_2})$. This group homomorphism can be enhanced to a ring homomorphism since a diagonal action $G_1 \to \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ induces a diagonal action $G_2 \to \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$. It follows that any projective system of finite groups G_i induces a direct system of rings $K_0(\mathrm{Sch}_k^{G_i})$; therefore, given a profinite group

$$G := \varprojlim(G_i),$$

we can define the G-equivariant Grothendieck ring of varieties as

$$K_0(\mathrm{Sch}_k^G) := \varinjlim K_0(\mathrm{Sch}_k^{G_i}).$$

3.4.6. We use again the symbol $\mathbb{L} := [\mathbb{A}_k^1]$, where the group G acts trivially on the affine line; thus the trivializing relations tell us nothing more than:

$$[V] = \mathbb{L}^d[S],$$

whenever $V \to S$ is an equivariant affine bundle of rank d.

3.4.7. Similarly to what we did in the previous sections, we define the localisation $\mathscr{M}_k^G := K_0(\mathrm{Sch}_k^G)[\mathbb{L}^{-1}]$ and its completion $\widehat{\mathscr{M}_k^G}$ with respect to the filtration

$$\mathcal{F}^n\mathscr{M}_k^G := \langle \mathbb{L}^r[X] \mid r \in \mathbb{Z}, \dim[X] + r \leq -n \rangle_{\mathbb{Z}}.$$

4. Weak Néron models

4.1. Definition and basic constructions

4.1.1. In order to define the main objects involved in this manuscript we will need to introduce the notion of a weak Néron model and to develop some techniques involved in the construction of such models. We keep the usual conventions on R, K, k, etc.

4.1.2. The results we are going to state in this section hold in the context of algebraic spaces, but we will not work in such generality, thus we state them only in the context of schemes.

Definition 4.1.3. Let $X \to \mathrm{Spec}\, K$ be a smooth morphism of schemes. A *model* for X over Δ is a flat scheme $\mathfrak{X} \to \Delta$ together with an isomorphism $\mathfrak{X}_K \to X$.

Moreover we say that the model $\mathfrak{X}$ has a property **P** (e.g., smooth, proper) if the morphism $\mathfrak{X} \to \Delta$ has such property.

4.1.4. The notion we are mostly interested in is that of *Weak Néron Model*:

Definition 4.1.5. We say that a model $\mathfrak{X} \to \Delta$ of X has the weak extension property if there is a bijection $\mathrm{Hom}_R(\Delta, \mathfrak{X}) \xrightarrow{\sim} \mathrm{Hom}_K(\mathrm{Spec}\, K, X)$, $(f\colon \Delta \to \mathfrak{X}) \mapsto f|_{\Delta^*}$. In other words, if every K-point of X extends to a R-point of $\mathfrak{X}$.

A smooth model $\mathfrak{X} \to \Delta$ of $X \to \mathrm{Spec}\, K$ that satisfies the weak extension property is said to be a *weak Néron model* of X.

Example 4.1.6. Let $X \to \operatorname{Spec} K$ be a smooth and proper variety and let $\mathcal{X} \to \operatorname{Spec} R$ be a proper regular model of X, then the smooth locus of $\mathcal{X} \to \operatorname{Spec} R$ is a weak Néron model of X, because the sections $\operatorname{Spec} R \to \mathcal{X}$ do not meet the singular locus of $\mathcal{X}_0 := X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and finite étale covers of $\operatorname{Spec} R$ are just union of finite copies of $\operatorname{Spec} R$, since k is algebraically closed. See also [2, Section 3.1].

Since we will mostly work with *sncd* models, i.e., proper regular models whose central fibre $\mathcal{X}_0$ sits in $\mathcal{X}$ as a divisor with strict normal crossings, this construction will be often used throughout this manuscript.

4.2. Weil restriction of scalars

4.2.1. In this paragraph we study some generalities about the functor of the restriction of scalars. The main content of this paragraph is Proposition 4.2.6, which will allow us to construct weak Néron models of a finite, tamely ramified base-change of X.

Definition 4.2.2. Let $S' \to S$ be a morphism of schemes and $\mathcal{Y} \to S'$ be a scheme. The functor $\operatorname{Res}_{S'/S}(\mathcal{Y})\colon (\mathrm{Sch}_S)^{\mathrm{opp}} \to \mathrm{Sets}$ defined by $T \mapsto \mathcal{Y}(T \times_S S')$ is called the Weil restriction of scalars of $\mathcal{Y}$ along $S' \to S$.

When $\operatorname{Res}_{S'/S}(\mathcal{Y})$ is represented by a scheme, we say that the Weil restriction of $\mathcal{Y}$ along $S' \to S$ exists.

4.2.3. It follows from [2, Theorem 7.6] that if $S' \to S$ is a finite, flat and locally of finite presentation morphism and $\mathcal{Y} \to S'$ is quasi-projective, then the Weil restriction of $\mathcal{Y}$ along $S' \to S$ exists. This will always be the case, throughout this manuscript.

Remark 4.2.4. Consider arbitrary morphisms of schemes $S' \to S$ and $\mathcal{X} \to S$; then the universal property of fibered products implies the following:

$$\operatorname{Res}_{S'/S}(\mathcal{X} \times_S S') = \underline{\operatorname{Hom}}_S(S', \mathcal{X}),$$

where $\underline{\operatorname{Hom}}_S(S', \mathcal{X})$ is the sheaf $T \mapsto \operatorname{Hom}_T(S' \times_S T, \mathcal{X} \times_S T)$.

4.2.5. Let R be a complete DVR and let K be its fraction field and k be its residue field. We assume k is algebraically closed. Let $K \subseteq L$ be a finite tame extension with $\mathcal{G} := \operatorname{Gal}(L|K)$ and denote by R_L the integral closure of R in L. Let $X \to \operatorname{Spec} R_L$ be a $\mathcal{G}$-equivariant morphism. For an arbitrary scheme $T \to \operatorname{Spec} R$, the action of $\mathcal{G}$ on $\operatorname{Spec} R_L$ induces an action on T_{R_L}, thus, as constructed in [6, Construction 2.4], a right action on the Weil restriction, $\operatorname{Res}_{R_L/R}(\mathcal{X})$: more precisely, given $g \in \mathcal{G}$, it induces an automorphisms $\rho_{T_{R_L}}(g)\colon T_{R_L} \to T_{R_L}$ and an automorphism $\rho_X(g)\colon X \to X$; the action of $\mathcal{G}$ sends the point corresponding to the morphism $\psi\colon T_{R_L} \to \mathcal{X}$ to the composition $g(\psi) := \rho_X(g) \circ \psi \circ \rho_{T_{R_L}}(g)^{-1}$. The following proposition, already proved in [12, Theorem 3.1], provides a recipe that we will use for constructing weak Néron models of varieties.

Proposition 4.2.6. *Let $\mathfrak{X}' \to R_L$ be a Galois-equivariant weak Néron model for X_L, then*

$$\mathfrak{X} := (\operatorname{Res}_{R_L/R} \mathfrak{X}')^{\mathcal{G}}$$

is a weak Néron model for X.

4.2.7. The following lemma says that this construction is well behaved with respect to a tower of extensions:

Lemma 4.2.8. *Let $K \subseteq F \subseteq L$ be a tower of Galois extensions such that also $K \subseteq L$ is normal. Let $G := \mathrm{Gal}(L|K)$, $N := \mathrm{Gal}(L|F)$ and $G/N = H := \mathrm{Gal}(F|K)$. Denote by R_F and R_L the integral closures of R in F and L, respectively, and Δ_F, Δ_L their spectra. Let $\mathcal{F}\colon \mathrm{Sch}_{\Delta_L}^{\mathrm{opp}} \to \mathrm{Sets}$ be a functor endowed with an action of G compatible with its action on Δ_L.*

Then the following two functors are naturally isomorphic:

$$(\mathrm{Res}_{\Delta_L/\Delta}\,\mathcal{F})^G \cong \big(\mathrm{Res}_{\Delta_F/\Delta}\,(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})^N\big)^H.$$

Proof. The left-hand side is equal to

$$\big(\mathrm{Res}_{\Delta_F/\Delta}(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})\big)^G = \Big(\big(\mathrm{Res}_{\Delta_F/\Delta}(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})\big)^N\Big)^H,$$

where N acts as a subgroup of G and $H = G/N$ inherits the action of G on the N-invariant locus. Thus, we only need to show that there is an H-equivariant isomorphism of functors:

$$\mathrm{Res}_{\Delta_F/\Delta}\,(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})^N \cong \big(\mathrm{Res}_{\Delta_F/\Delta}(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})\big)^N.$$

Given a scheme morphism $T \to \Delta$, we have that

$$\begin{aligned}\mathrm{Res}_{\Delta_F/\Delta}\,(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})^N(T) &= (\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})^N(T \times_\Delta \Delta_F)\\ &= \big(\mathcal{F}(T \times_\Delta \Delta_L)\big)^N\\ &= \big(\mathrm{Res}_{\Delta_F/\Delta}(\mathrm{Res}_{\Delta_L/\Delta_F}\,\mathcal{F})(T)\big)^N,\end{aligned}$$

and we are done. ∎

Remark 4.2.9. The construction above can be made more explicit: in the following paragraphs we will describe the central fibre and the canonical divisor of $\mathfrak{X}$ in terms of $\mathfrak{X}'$.

4.3. Weil restriction and the central fibre

4.3.1. This subsection and the next one summarize some results contained in an unpublished manuscript of Lars Halle and Johannes Nicaise. The author is grateful to them for letting him use these results which are crucial for the computation of the motivic integral in Section 9.3. We keep the notation of the previous paragraph.

4.3.2. The inclusion $\mathfrak{X} \subseteq \mathrm{Res}_{R_L/R}\,\mathfrak{X}'$ corresponds, according to the definition of the restriction of scalars, to a map of R_L-schemes

$$h\colon \mathfrak{X} \times_R R_L \to \mathfrak{X}',$$

which gives, over the special fibres a morphism of k-schemes:

$$h_k\colon \mathfrak{X}_k \to \mathfrak{X}'_k.$$

4.3.3. On the other hand, we can characterize $\mathfrak{X}_k$ in a different way, using the *Greenberg schemes*. The following definition will cover the cases we will use:

Let $d = [L : K]$ and let $\mathfrak{m}$ be the maximal ideal of R_L; for $i \in \{0, \dots, d-1\}$, let

$$R_{L,i} := R_L/(\mathfrak{m}^{i+1}).$$

For a separated, smooth morphism $\mathcal{A} \to \operatorname{Spec} R_L$, consider the functor

$$\operatorname{Gr}_i(\mathcal{A}) := \operatorname{Res}_{R_{L,i}/k}(\mathcal{A} \times_{R_L} R_{L,i}),$$

which is representable by a separated, smooth scheme, as it follows from the proof of [2, Proposition 7.6]; this is also called the *level i Greenberg scheme of $\mathcal{A}$*.

Clearly, $\operatorname{Gr}_0(\mathcal{A}) = \mathcal{A}_k$, while $\operatorname{Gr}_{d-1}(\mathcal{A}) = \operatorname{Res}_{R_L/R}(\mathcal{A})_k$, since $d = [L : K]$ is also the ramification index of $\operatorname{Spec} R_L \to \operatorname{Spec} R$ at their closed points.

4.3.4. In our case we have that $\mathfrak{X}_k = (\operatorname{Gr}_{d-1}(\mathfrak{X}'))^{\mathcal{G}}$. Indeed, if T is a k-scheme, we have that

$$\mathfrak{X}_k(T) = \{f : T \times_R R_L \to \mathfrak{X}'\}^{\mathcal{G}} = \big(\operatorname{Gr}_{d-1}(\mathfrak{X}')(T)\big)^{\mathcal{G}}.$$

4.3.5. The natural truncation maps of Greenberg schemes define $\mathcal{G}$-equivariant affine bundles with affine $\mathcal{G}$-action (see [13, Proposition 3.12]), in particular $\operatorname{Gr}_{d-1}(\mathfrak{X}') \to \operatorname{Gr}_0(\mathfrak{X}') = \mathfrak{X}'_k$ is a composition of affine bundles. By taking the $\mathcal{G}$-invariant loci of this map, we get a description, at least locally, of $\mathfrak{X}_k$ as an affine bundle over $(\mathfrak{X}'_k)^{\mathcal{G}}$, in the sense that for each connected component $C \subseteq \mathfrak{X}_k$, there is a connected component $C' \subseteq (\mathfrak{X}'_k)^{\mathcal{G}}$ such that C is an $\mathbb{A}^r_k$-bundle over C', where $r = \dim(\mathfrak{X}'_k) - \dim C'$. In particular the following relation holds in $K_0(\mathrm{Var}_k)$:

$$[C] = \mathbb{L}^{\dim(\mathfrak{X}_k) - \dim C'}[C'].$$

4.4. Weil restriction and canonical divisor

4.4.1. Let us keep the notation introduced in the previous paragraph, but we also assume that X is a Calabi–Yau variety, i.e., it has trivial canonical bundle, and that a volume form $\omega \in \Omega^{\dim X}_{X/K}(X)$ is given. Let $\omega_L \in \Omega^{\dim X}_{X_L/L}(X_L)$ be the pull-back of ω under the base-change map. In this paragraph we will study the order of vanishing of ω on each component of $\mathfrak{X}_k$, which we will define as follows, adapting the definition given in [15, Section 4.1].

4.4.2. Let $p \in \mathfrak{X}_k$ be a closed point. Since R is an Henselian ring and since $\mathfrak{X} \to \operatorname{Spec} R$ is smooth, there is at least a section $\psi : \operatorname{Spec} R \to \mathfrak{X}$ such that $\psi(0) = p$. Consider the line bundle $\mathcal{L} := \psi^* \Omega^{\dim X}_{\mathfrak{X}/R}$ over $\operatorname{Spec} R$. There is $a \in \mathbb{Z}$ such that $\pi^a \omega$ extends to a global section $\omega' \in \Omega^{\dim X}_{\mathfrak{X}/R}(\mathfrak{X})$, where $\pi \in R$ is the uniformizer. So its pull-back $\psi^*(\omega')$ is a global section of $\mathcal{L}$. Let $M := \mathcal{L}/\psi^*\omega'\mathcal{O}_{\operatorname{Spec} R}$ be the quotient of $\mathcal{O}_{\operatorname{Spec} R}$-modules.

Definition 4.4.3. The order of ω at p is defined as:

$$\operatorname{ord}_p(\omega) := \inf\{b \in \mathbb{N} : \pi^b M = 0\} - a.$$

If $C \subseteq \mathfrak{X}_k$ is a connected component, then $\mathrm{ord}_p(\omega)$ does not depend on the coice of the closed point $p \in C$, so we define $\mathrm{ord}_C(\omega)$ as the order of ω at any of its closed point. If $\mathrm{ord}_C(\omega) > 0$ we say that C is a *zero* of ω, if $\mathrm{ord}_C(\omega) < 0$ we say that it is a *pole* of ω.

4.4.4. Let Z be a smooth scheme defined over a field F and let $V \to Z$ be a vector bundle over Z. Consider a cyclic group $G \cong \mu_d$ acting equivariantly on $V \to Z$ and let $z \in Z$ be a fixed point. There is a unique sequence of integers $(j_1, j_2, \dots, j_{\mathrm{rk}\, V})$ such that $0 \le j_1 \le j_2 \le \cdots \le j_{\mathrm{rk}\, V} \le d-1$ such that V_z has a basis $v_1, \dots, v_{\mathrm{rk}\, V}$ of eigenvectors such that $\zeta \star v_i = \zeta^{-j_i} \cdot v_i$ (where ζ is any generator of μ_d); the tuple $(j_i)_i$ is called the *tuple of exponents* of the G-action.

Definition 4.4.5. We define the *conductor* of the action of G in z as the sum:

$$c(V, z) := \sum_{i=1}^{\mathrm{rk}\, V} j_i.$$

If $C \subseteq Z^G$ is an irreducible subscheme, then for all $z, z' \in C$ one has that $c(V, z) = c(V, z')$, so we simply denote by $c(V, C)$ either of the conductors. Moreover we denote $c(Z, C)$ the conductor $c(T_Z, C)$.

Lemma 4.4.6. *Let C be a connected component of $\mathfrak{X}_k$ and let $C' = h(C)$, where h is the map defined in Section 4.3.2. Then:*

$$\mathrm{ord}_C(\omega) = \frac{\mathrm{ord}_{C'}(\omega_L) - c(\mathfrak{X}'_k, C')}{[L\colon K]}.$$

Proof. The map $\mathfrak{X} \times_R R_L \to \mathfrak{X}'$ induces a monomorphism

$$\alpha\colon (\psi')^* \Omega_{\mathfrak{X}'/R_L} \to \psi^* \Omega_{\mathfrak{X}/R} \otimes_R R_L$$

sending ω_L to $\omega \otimes 1$; in particular

$$\mathrm{length}\left(\alpha\left((\psi')^*\Omega_{\mathfrak{X}'/R_L}\right)/\langle \omega \otimes 1\rangle\right) = [L : K]\, \mathrm{length}\left((\psi')^*\Omega_{\mathfrak{X}'/R_L}/\langle \omega_L\rangle\right),$$

thus the statement will follow from the fact that $\operatorname{coker} \alpha = c(\mathfrak{X}'_k, C')$.

On the other hand, under the identification

$$T_{\mathfrak{X}/R} = \underline{\mathrm{Hom}}_R\left(R[\varepsilon]/(\varepsilon^2), \mathfrak{X}\right) = \underline{\mathrm{Hom}}_{R_L}\left(R_L[\varepsilon]/(\varepsilon^2), \mathfrak{X}'\right)^{\mathrm{Gal}(L|K)} = T_{\mathfrak{X}'/R_L}^{\mathrm{Gal}(L|K)},$$

the tangent map $T_h\colon T_{\mathfrak{X}/R \times_R R_L} \to T_{\mathfrak{X}'/R_L}$ induces a map

$$\beta\colon (\psi')^*(T_{\mathfrak{X}'/R_L})^{\mathrm{Gal}(L|K)} \otimes_R R_L \to (\psi')^* T_{\mathfrak{X}'/R_L}.$$

Fix a base of eigenvectors of $T_{\mathfrak{X}'_k}$; the upcoming Lemma 4.4.7, applied to the subspaces generated by each element of the base, implies that $\mathrm{coker}(\beta) = \oplus_{i=1}^d R_L/\mathfrak{m}_L^{j_i}$, hence

$$\mathrm{coker}(\alpha) = \bigwedge_{i=1}^{d} R_L/\mathfrak{m}_L^{j_i} = R_L/\mathfrak{m}_L^{c(\mathfrak{X}, C')}. \qquad \blacksquare$$

Lemma 4.4.7. *Let M a free R_L-module of rank 1. Assuming that* $\mathrm{Gal}(L|K)$ *acts R-linearly on M from the left. Let j be the exponent of the action induced on $M \otimes_{R_L} k$, as in Definition 4.4.5. Then the natural morphism $M^{\mathrm{Gal}(L|K)} \otimes_R R_L \to M$ has cokernel isomorphic to $R_L/\mathfrak{m}_L^j$.*

Proof. Let us fix an element $v \in M$ such that $0 \neq v \otimes 1 \in M \otimes_{R_L} k$; by our hypothesis we have that $(\zeta * v) \otimes 1 = \zeta^{-j} v \otimes 1$. Let π_L a uniformizer for R_L such that $\pi_L^d \in K$; if $0 \leq b \leq d-1$ the vectors $v_b := \pi_L^b v \otimes 1 \in M \otimes_R k$ form a base of the vector space $M \otimes_R k \cong M \otimes_{R_L} R_L/\mathfrak{m}_L^d$. Moreover that one is a base of G-eigenvectors, for $\zeta * v_b = \zeta^{b-j} v_b$.

By Henselianity we can lift the base $\{v_b\}$ to an R-base $\{w_b \colon 0 \leq b \leq d-1\}$ of M such that $\zeta * w_b = \zeta^{b-j} w_b$.

Now let $x = a_0 w_0 + \cdots + a_{d-1} w_{d-1} \in M$ be an arbitrary element. We have that $x \in M^G$ iff $x - \zeta * x = 0$, i.e., iff

$$\sum_{b=0}^{d-1} a_b (1 - \zeta^{b-j}) w_b = 0,$$

therefore, the R-module M^G is generated by w_j.

It follows that $M^G \otimes R_L$ is sent onto $\langle w_j \rangle \subseteq M$, which leads to our coveted statement. ∎

5. Equivariant semistable reduction

5.1. Preliminaries on logarithmic geometry

5.1.1. As seen in Example 4.1.6, every proper regular model of a smooth variety contains a weak Néron model, on the other hand it is not always possible to construct a semistable model over a DVR of all the varieties defined on its fraction field K. Given a variety X over K that admits a regular *sncd* model, under suitable assumptions there exists a finite extension F/K such that the basechange X_F admits a semistable model over the ring of integers in F. In this section we will show a way to produce such models over Galois extensions of a given field in such a way that the Galois group acts naturally on the model.

5.1.2. We use the standard language of monoids and logarithmic geometry, as in [7] or in [3] and we refer to [21] for the material concerning toric schemes. We only recall here the notions that will be used later in this section:

5.1.3. Let $V^\dagger = (V, \mathcal{M}_V)$ be a locally Noetherian fs logarithmic scheme and $x \in V$ be a point. We say that $V^\dagger$ is log regular at x if the following:

- The ring $\mathcal{O}_{V,x}/\mathcal{M}_{V,x}^+ \mathcal{O}_{V,x}$ is regular;
- $\dim \mathcal{O}_{V,x} = \dim \mathcal{O}_{V,x}/\mathcal{M}_{V,x}^+ \mathcal{O}_{V,x} + \dim \mathcal{M}_{V,x}$.

are satisfied, where $\mathcal{M}_{V,x}^{+}$ is the maximal ideal of the monoid $\mathcal{M}_{V,x}$, i.e., the subset of said monoid obtained by removing its invertible elements. A log structure $V^{\dagger}$ is logarithmically regular if it is log regular at each of its points.

5.1.4. A *(sharp) monoidal space* is a couple $(T, \mathcal{M}_T)$ consisting of a topological space T and a sheaf of (sharp) monoids over T. A morphism of monoidal spaces

$$(T', \mathcal{M}_{T'}) \to (T, \mathcal{M}_T)$$

is the datum of a continuous map $f: T' \to T$ and a map of sheaves of monoids

$$h: f^{-1}(\mathcal{M}_{T'}) \to \mathcal{M}_T.$$

Given an abstract monoid M, we can define its sharpification $M^{\sharp}$ by quotienting M by its submonoid of invertible elements $M^{\times}$, i.e., $M^{\sharp} := M \oplus_{M^{\times}} \{1\}$. Similarly, if $(T, \mathcal{M}_T)$ is a monoidal space, one can define its sharpification $(T, \mathcal{M}_T)^{\sharp} := (T, \mathcal{M}_T^{\sharp})$, where $\mathcal{M}_T^{\sharp}$ is the sheafification of $U \mapsto \mathcal{M}_T(U)^{\sharp}$.

5.1.5. Let P be a monoid, consider the set Spec P of its prime ideals, i.e., its submonoids whose complement in P is again a submonoid of P, endowed with the topology generated by $\{D(f): f \in P\}$, where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} P: f \notin \mathfrak{p}\}$. Let $\mathcal{M}_P$ be the sheaf of monoids such that $\mathcal{M}_P(D(f)) = P_f$. The space (Spec $P, \mathcal{M}_P$), or simply Spec P, is the *spectrum* of P. Moreover $(Spec P)^{\sharp}$ is called the *sharp spectrum* of P.

5.1.6. A *fan* is a sharp monoidal space $(F, \mathcal{M}_F)$, that can be covered by open subsets isomorphic to sharp spectra of monoids. A fan is *locally fine and saturated* (or locally fs) if it can be covered by spectra of fs monoids. The category of fans is a full subcategory of the category of sharp monoidal spaces, i.e., a morphism of fans is just a morphism of monoidal spaces between fans.

5.1.7. Let $(F, \mathcal{M}_F)$ be a fan. A subdivision is a morphism of fans $\varphi: (F', \mathcal{M}_{F'}) \to (F, \mathcal{M}_F)$ such that:

- For every $t \in F'$, the map induced on stalks $\mathcal{M}_{F,\varphi(t)}^{\mathrm{grp}} \to \mathcal{M}_{F,t}^{\mathrm{grp}}$ is surjective;
- The composition with φ induces a bijection

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F') \to \operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F).$$

5.1.8. Let $V^{\dagger} = (V, \mathcal{M}_V)$ be a log regular scheme. There is a fan associated with V whose underlying topological space is the topological subspace of V whose points are $F(V) := \{x \in V: \mathfrak{m}_x \text{ is generated by } \mathcal{M}_{V,x} \backslash \mathcal{O}_{V,x}^{\times}\}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{V,x}$, and whose structural sheaf $\mathcal{M}_{F(V)}$ is the restriction of $\mathcal{M}_V / \mathcal{O}_V^{\times}$.

5.2. The construction of a semistable model

5.2.1. Let $\Delta^{\dagger}$ be the log scheme supported on Δ with the divisorial log structure associated with $0 \in \Delta$. Similarly $\Delta^{\dagger}(m)$ shall denote the scheme $\Delta(m)$ introduced above endowed with the divisorial log structure for $0 \in \Delta(m)$. Let $S \to \Delta$ a finite type flat morphism of relative dimension 2 with smooth generic fibre and let $S^\dagger$ be the divisorial logarithmic structure associated with the central fibre S_k. Assume that

$$S_k = \sum_{i \in I} N_i E_i$$

is an snc divisor whose components, $\{E_i\}_{i \in I}$, have multiplicities, $\{N_i\}_{i \in I}$, coprime with the characteristic exponent of k, so that $S^\dagger \to \Delta^\dagger$ is logarithmically smooth. Let us fix a positive integer N such that the multiplicity of any irreducible component of S_k divides N and let m be an arbitrary positive integer; we further assume that N, m are coprime with the characteristic exponent of k. Let Γ be the fan associated with the logarithmic structure of $S^\dagger$.

5.2.2. The scheme S has a canonical stratification in locally closed subschemes described as follows:

- For $J \subseteq I$, denote $E_J := \bigcap_{i \in J} E_i$ endowed with its reduced scheme structure, in particular $E_\emptyset = S$;
- For $J \subseteq I$, denote $E_J^\circ := E_J \setminus \bigcup_{i \in I \setminus J} E_i$, in particular $E_\emptyset^\circ = S_K$.

The collection $\{E_J^\circ\}_{J \subseteq I}$ defines a partition of S with locally closed subschemes; if we omit $J = \emptyset$, they define a stratification of the central fibre S_k.

5.2.3. The fs-basechange $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger \to \Delta^\dagger$ which is a normal space, yet not necessarily regular, is canonically endowed with an equivariant μ_{mN}-action which induces an action on the fan of $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger$, which we call $\Gamma(mN)$. As described in [17, Section 2.3], its underlying scheme admits a stratification with locally closed subschemes $\{\widetilde{E}_J^\circ\}_{J \subseteq I}$ such that, for each $J \subseteq I$, $\widetilde{E}_J^\circ$ is the preimage of E_J° with respect to the map

$$S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger \to S^\dagger$$

and $\widetilde{E}_J^\circ \to E_J^\circ$ is a $\mathrm{Gal}(K(mN)|K)$-torsor with constant stabilizer.

5.2.4. In the literature it is well known how to perform and embedded resolution of a log regular scheme, for instance following the proof of [23, Theorem 4.8], it is possible to construct a subdivision of the polyhedral complex associated with $\Gamma(mN)$ that gives a μ_{mN}-equivariant logarithmic resolution of singularities of $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger$, which in turn gives an equivariant semistable model $S(mN) \to \Delta(mN)$ of $S_{K(mN)}$. Despite in [23] characteristic 0 is assumed, the semistable reduction works also in positive and mixed characteristic, as long as $S \to \Delta$ is logarithmically smooth.

Remark 5.2.5. The construction above works for every m because the relative dimension of $S \to \Delta$ is 2, see [23, Theorem 4.8, 2]; in higher dimension it could have been necessary to replace N with a sufficiently divisible number.

Remark 5.2.6. For our purposes we only need the existence of a toroidal equivariant resolution of $S^\dagger \times^{\mathrm{fs}}_{\Delta^\dagger} \Delta(mN)^\dagger$, not its specific properties.

5.2.7. We conclude the section with the following lemma which grasp the most important, at least for our purpose, property of the action of μ_{mN} on $S(mN)$; in fact this is the only reason why we did perform the construction in this way:

Definition 5.2.8. For each point $p \in S(mN)_k$ let $\mathrm{Stab}_p \subseteq \mu_{mN}$ be the subgroup consisting of the elements that fix p.

Lemma 5.2.9. Stab_{-} *is locally constant on* $S(mN)_{k,\mathrm{sm}}$. *In particular if a point* $p \in S(mN)_{k,\mathrm{sm}}$ *is fixed under the action of* μ_N, *then the whole connected component of* $S(mN)_{k,\mathrm{sm}}$ *containing* p *is fixed under such action.*

Proof. Let C be a connected component of $S(mN)_{k,\mathrm{sm}}$, which, by construction, is the preimage of some stratum $\widetilde{E}_J^{\circ}$ via the map

$$S(mN)^{\dagger} \to S^{\dagger} \times^{\mathrm{fs}}_{\Delta^{\dagger}} \Delta(mN)^{\dagger}.$$

As it follows from the proof of [4, Theorem 7.2.1], the map $C \to \widetilde{E}_J^{\circ}$ is a μ_{mN}-equivariant $\mathbb{G}^r_{m,k}$-torsor, for some $r \in \mathbb{N}$, with μ_{mN} acting trivially on the fibres $\mathbb{G}^r_{m,k}$. Therefore, for $\sigma \in \mu_{mN}$, its fixed locus $\mathrm{Fix}_C(\sigma)$ is a $\mathbb{G}^r_{m,k}$-invariant subscheme of C; moreover it is smooth over its fixed locus in $\widetilde{E}_J^{\circ}$, since mN is invertible in k.

As it follows from Section 5.2.3, the fixed locus of σ in $\widetilde{E}_J^{\circ}$ is either empty (in which case $\mathrm{Fix}_C(\sigma) = \emptyset$) or the whole stratum; since the only $\mathbb{G}^r_{m,k}$-invariant subschemes of C that are smooth over $\widetilde{E}_J^{\circ}$ are C itself and $\emptyset$, in both cases $\mathrm{Fix}_C(\sigma)$ is either C or empty. ■

6. The monodromy property

6.1. Motivic integration

6.1.1. Let us keep the usual conventions for R, K, k, etc. Let $Y \to \Delta^*(m)$ be a smooth Calabi–Yau variety and let ω be a volume form on Y. Fix a weak Néron model $\mathfrak{Y} \to \Delta(m)$ of Y. For a connected component $C \in \pi_0(\mathfrak{Y}_0)$ let $\mathrm{ord}_C(\omega)$ be the order of ω, considered as a meromorphic function, on the generic point of C.

6.1.2. It follows from a result of Loeser and Sebag (see [15, Proposition 4.3.1]), that the following definition does not depend on the choice of the weak Néron model of Y.

Definition 6.1.3. (Motivic integral) With the same notation introduced in this paragraph, we call motivic integral of the volume form ω on Y the element of $\mathscr{M}_k$ given by the following sum:

$$\int_Y \omega d\mu := \sum_{C \in \pi_0(\mathfrak{Y}_0)} [C]\mathbb{L}^{-\mathrm{ord}_C(\omega)}.$$

6.1.4. The motivic integral has an equivariant counterpart in $\mathscr{M}_k^G[\![T]\!]$, if X is a Calabi–Yau variety with a good action of a (pro)finite group G and ω is a G-equivariant volume form.

6.1.5. Now fix a Calabi–Yau variety $X \to \Delta^*$, together with a volume form $\omega \in \omega_X(X)$.

For every positive integer m, define $X(m) := X \times_{\Delta^*} \Delta^*(m)$, which is canonically endowed with a good action of $\mu_m \cong \mathrm{Gal}(K(m)|K)$. Since the map $\Delta^*(m) \to \Delta$ is an étale map, the basechange map $X(m) \to X$ is étale as well. The pull-back of ω through that map is thus a μ_m-equivariant volume form on $X(m)$, which we denote by $\omega(m)$. Using this construction, a formal series with coefficients in $\mathscr{M}_k^{\widehat{\mu}}$ is defined:

Definition 6.1.6 (Motivic Zeta Function). Keep the notation of this paragraph. The *Motivic Zeta Function* of X with respect to the volume form ω is the formal series

$$Z_{X,\omega}(T) := \Bigg(\sum_{\substack{m \geq 1 \\ \mathrm{char}\, k \nmid m}} \int_{X(m)} \omega(m) d\mu \Bigg) T^m \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!].$$

6.2. Rational functions in $\mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$

6.2.1. The notions discussed in this paragraph are introduced, for instance, in [19].

Definition 6.2.2. Let $F \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$, we say that F is rational if there is a finite set $S \subseteq \mathbb{N} \times \mathbb{N}^*$ such that $F \in \mathscr{M}_k^{\widehat{\mu}}[T, \frac{1}{1-\mathbb{L}^{-a}T^b} : (a,b) \in S]$.

In such a case, we say that F has a pole of order at most $n \in \mathbb{N}^*$ in $q \in \mathbb{Q}$ if there exist a finite set $S' \in \mathbb{N} \times \mathbb{N}^*$ such that $\frac{a}{b} = q \Rightarrow (a,b) \notin S'$ and a positive integer N such that also qN is an integer and

$$(1 - \mathbb{L}^{-qN} T^N)^n F \in \mathscr{M}_k^{\widehat{\mu}} \left[T, \frac{1}{1 - \mathbb{L}^a T^b} : (a,b) \in S' \right].$$

We say that F has a pole of order $n \geq 1$ in $q \in \mathbb{Q}$ if F has a pole of order at most n, but not a pole of order at most $n-1$.

Remark 6.2.3 (Historical note). Other types of zeta functions (such as Igusa's zeta function or its motivic analogue defined by Denef and Loeser) are defined as functions of complex variable $s \mapsto \zeta(s)$; these functions were paired with a formal series $Z(T)$ via the substitution of the formal variable $T = \mathbb{L}^{-s}$ (in the case of Denef's and Loeser's zeta function). With this substitution in mind, the poles of a rational function $F \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$ are those values of s where at least one denominator in every fractional expression of F would vanish.

6.2.4. The definition can be simplified, provided that we work on a ring $\mathscr{R}^{\widehat{\mu}}$ endowed with a map $\mathscr{M}_k^{\widehat{\mu}} \to \mathscr{R}^{\widehat{\mu}}$ such that the images of all the elements of the form $\mathbb{L}^r - 1$, with $r \in \mathbb{N} \backslash \{0\}$, are invertible. The minimal such choice for $\mathscr{R}^{\widehat{\mu}}$ is, clearly, the localisation of $\mathscr{M}_k^{\widehat{\mu}}$ with respect to that set of elements, i.e., $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^r - 1)^{-1} : r \in \mathbb{N}^*]$; this ring is also called the $\widehat{\mu}$-equivariant *Grothendieck ring of algebraic stacks*, for it can be obtained by repeating the construction we have seen in Section 3 starting with the category of algebraic stacks. Another natural choice for $\mathscr{R}^{\widehat{\mu}}$ is the completed Grothendieck ring of varieties: $\widehat{\mathscr{M}}_k^{\hat{\mu}}$, where the inverse of $1-\mathbb{L}^r = \mathbb{L}^r(1-\mathbb{L}^{-r})$ is

$$\mathbb{L}^{-r} + \mathbb{L}^{-2r} + \mathbb{L}^{-3r} + \cdots .$$

The following lemma clarifies why in this case it is easier to define a pole:

Lemma 6.2.5. *Let $\mathscr{R}^{\hat{\mu}}$ be a ring as above and $F \in \mathscr{R}^{\hat{\mu}}[\![T]\!]$ any rational function. Then exists a positive integer N and a finite set $S \subseteq \mathbb{Q}$ such that:*

$$F(T) = g(T) + \sum_{q\in S} \frac{f_q(T)}{(1-\mathbb{L}^{-qN}T^N)^{a_q}}, \tag{6.1}$$

for some polynomials $g, f_q \in \mathscr{R}^{\hat{\mu}}[T]$ and positive integers a_q.

Proof. We begin by noticing that, given $\mu > \nu$ positive integers, one has that

$$\frac{1}{(1-\mathbb{L}^{\mu}T^N)(1-\mathbb{L}^{\nu}T^N)} = \frac{1}{1-\mathbb{L}^{\mu-\nu}}\left(\frac{1}{1-\mathbb{L}^{\nu}T^N} - \frac{\mathbb{L}^{\mu-\nu}}{1-\mathbb{L}^{\mu}T^N}\right),$$

where $N \in \mathbb{N}$.

By repeatedly applying this step, one obtains the following identity

$$\frac{1}{(1-\mathbb{L}^{\mu}T^N)^a(1-\mathbb{L}^{\nu}T^N)^b} = \frac{r(T^N)}{(1-\mathbb{L}^{\mu}T^N)^a} + \frac{s(T^N)}{(1-\mathbb{L}^{\nu}T^N)^b},$$

for positive integers a, b and polynomials $r, s \in \mathbb{Z}[\mathbb{L}, \frac{1}{1-\mathbb{L}^{\mu-\nu}}, t]$ such that $\deg_t(r) < a$, $\deg_t(s) < b$. In particular any rational function in $\mathscr{R}^{\hat{\mu}}[\![T]\!]$ with two poles is the sum of two functions with a single pole. We conclude the proof by induction on the number of poles of F. ∎

Definition 6.2.6. If F is written as in (6.1) and $q \in S$, then we say that F has a pole of order at most a_q in q.

If, moreover, there is no integer $N' \in \mathbb{N}^*$ such that $f_q \in (1-\mathbb{L}^{-qN'}T^{N'})\mathscr{R}^{\hat{\mu}}[T]$, then F has a pole of order exactly a_q in q.

Remark 6.2.7. Let $F \in \mathscr{M}_k^{\hat{\mu}}[\![T]\!]$ and let $\widetilde{F} \in \mathscr{R}^{\hat{\mu}}[\![T]\!]$ be the image of F under the completion map (or localisation map).

Because of Lemma 6.2.5, any pole q of $\widetilde{F}$ of order $a \in \mathbb{N}^*$ is also a pole of F of order greater or equal than a.

6.3. Statement of the monodromy property

6.3.1. We conclude this section by explaining the main problem we are going to face. Let $\mathscr{R}$ be one of the three rings $\mathscr{M}_k, \mathscr{M}_k[(\mathbb{L}^r-1)^{-1} : 0 < r \in \mathbb{N}]$ or $\widehat{\mathscr{M}}_k$ and let $\mathscr{R}^{\hat{\mu}}$ be its $\hat{\mu}$-equivariant version. Let us fix a topological generator of $\mathrm{Gal}(K^{\mathrm{tame}}|K)$ as follows: for $m \in \mathbb{N}^*$ coprime with the carachteristic exponent of k, let us choose coherently uniformizers

$\varpi_m \in R(m)$ and primitive roots of the unity $\zeta_m \in K^{\mathrm{tame}}$, where by "coherently" here we mean that $\varpi_{ab}^b = \varpi_a$ and $\zeta_{ab}^b = \zeta_a$ for all $a, b \in \mathbb{N}^*$ coprime with the carachteristic exponent of k; then there exist a unique $\sigma \in \mathrm{Gal}(K^{\mathrm{tame}}|K)$ such that $\sigma(\varphi_m) = \zeta_m \varpi_m$ for all the relevant m. We give a statement of the monodromy property that depends on how the ring of coefficients for the zeta function is interpreted:

Definition 6.3.2 (Monodromy property in $\mathscr{R}^{\widehat{\mu}}$). Let $X \to \operatorname{Spec} K$ be a Calabi–Yau variety and let ω be a volume form on it. We say that X has the monodromy property, with respect to σ, if for all poles $q \in \mathbb{Q}$ of $Z_{X,\omega}(T) \in \mathscr{R}^{\widehat{\mu}}[\![T]\!]$ and for all the embeddings $\mathbb{Q}_l \hookrightarrow \mathbb{C}$, we have that $e^{2\pi i q}$ is an eigenvalue of σ^*.

Remark 6.3.3. In the subsequent discussion, we will omit to mention the relationship between the monodromy property and the choice of a particular topological generator of $\mathrm{Gal}(K^{\mathrm{tame}}|K)$; but the reader has to keep in mind that we fixed a topological generator σ and that the monodromy action and the monodromy property are defined with respect to such generator.

However, if X is defined over a "sufficiently basic" field, i.e., X is the basechange of a scheme defined over $\mathbb{Q}((t))$ or $\mathbb{F}_p((t))$ in the case of equal characteristic, or over $\mathbb{Q}_p$ in the case of mixed characteristic (where p is some prime number), then the lifts in $\mathrm{Gal}(\bar{K}|K)$ of two topological generators $\sigma, \tau \in \mathrm{Gal}(K^{\mathrm{tame}}|K)$ are conjugated with respect to an automorphism $g \in \mathrm{Aut}(\bar{K})$ that fixes $\mathbb{Q}((t))$, $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$, i.e., $\sigma = g^{-1}\tau g$; since X is the base-change of some scheme defined over one of the fields above, g induces an automorphism of $X_{\bar{K}}$, so that the two operators σ^* and τ^* are conjugated and have the same minimal polynomial, hence the same eigenvalues with the same multiplicities.

6.3.4. The monodromy property in the ring $\mathscr{M}_k^{\widehat{\mu}}$ implies the monodromy property in $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^r - 1)^{-1} : 0 < r \in \mathbb{N}]$ by Remark 6.2.7. In turns, the monodromy property in $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^r - 1)^{-1} : 0 < r \in \mathbb{N}]$ implies the version in $\widehat{\mathscr{M}}_k^{\widehat{\mu}}$.

7. Formal series in the motivic rings

In this section, the symbol $\mathscr{R}$ shall denote one of the rings $\mathscr{M}_k^{\widehat{\mu}}$, $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^r - 1)^{-1} : 0 < r \in \mathbb{N}]$ or $\widehat{\mathscr{M}_k}^{\widehat{\mu}}$, unless differently specified.

We will discuss some properties of power series with coefficients in $\mathscr{R}$, then we will describe some operations that shall be useful for computing the Motivic Zeta Function in our case.

7.1. Quotient by a group action

7.1.1. Let us fix a finite group G, let $N \trianglelefteq G$ and let $H = G/N$ be its quotient. Consider the equivariant versions of the ring $\mathscr{R}$, which we call $\mathscr{R}^G$ and $\mathscr{R}^H$, as in Section 3.4. For an arbitrary variety X endowed with an action of G, we consider the quotient X/N which is again an algebraic space endowed with an action of the group H, namely the quotient action.

7.1.2. The quotient X/N is not necessarily a scheme. In any case, as long as we consider good actions on X, there exists an open affine subscheme $\operatorname{Spec} A \subseteq X$ which is invariant under the action of G, so that $\operatorname{Spec} A^N \subseteq X/N$ is a scheme. We may, thus, repeat the argument for the closed G-invariant subscheme $X \setminus \operatorname{Spec} A$ and, by Noetherian induction, we stratify X as a union of G-schemes whose quotients with respect to the action of N are H-schemes; moreover a G-invariant stratification of each stratum induces an H-invariant stratification of its quotient. Thus there is a well defined map of groups $\mathscr{R}^G \to \mathscr{R}^H$ by $[X] \mapsto [X/N]$; this map does not preserve the products.

7.1.3. In the following definition, we extend the map above to a map

$$\mathscr{R}^G[\![T]\!] \to \mathscr{R}^H[\![T]\!]$$

and in the subsequent proposition we show that rationality of any power series is well behaved under this map.

Definition 7.1.4. If $F = \sum_n A_n T^n \in \mathscr{R}^G[\![T]\!]$, we define the *series of the quotients* with respect to N associated with F as

$$(F/N)(T) = \sum_n (A_n/N) T^n \in \mathscr{R}^H[\![T]\!].$$

Proposition 7.1.5. *For $f, g \in \mathscr{R}^G[T]$, with g being of the form $\prod_{j \in J}(1 - \mathbb{L}^{a_j} T^{b_j})$, let F be the rational function $F(T) := \frac{f(T)}{g(T)}$. Then $F/N = \frac{f/N}{g}$.*

In particular all the poles of F/N belong to the set of poles of F.

Proof. It suffices to show the statement in the case when $f = \alpha \in \mathscr{R}^G$ is a constant. Let

$$\frac{1}{g(T)} = \sum_n A_n T^n,$$

where $A_n \in \mathscr{R}$ has a trivial G-action. Then

$$\left(\frac{\alpha}{g(T)}\right) \Big/ N = \sum_n (\alpha A_n)/N T^n = \sum_n \alpha/N A_n T^n = \frac{\alpha/N}{g(T)}. \qquad \blacksquare$$

7.2. Power structures

7.2.1. We need to define a map on the Grothendieck rings which extends the symmetric product of a variety, allowing us to talk about the symmetric product of a "difference of varieties"; in order to do so, we need to use a power structure on $K_0(\mathrm{Var}_k)$, thus we begin by recalling what a power structure is, as introduced in [9].

Definition 7.2.2. Let A be a ring. A power structure on A is a map

$$\begin{gathered}\big(1 + tA[\![t]\!]\big) \times A \to 1 + tA[\![t]\!] \\ \big(F(t), X\big) \mapsto F(t)^X\end{gathered}$$

satisfying the following conditions for all $F, G \in 1 + tA[\![t]\!]$ and $X, Y \in A$:

- $F(t)^0 = 1$;

- $F(t)^1 = F(t)$;
- $(F(t)G(t))^X = F(t)^X \cdot G(t)^X$;
- $F(t)^{X+Y} = (F(t))^X (F(t))^Y$;
- $F(t)^{XY} = (F(t)^X)^Y$;
- $(1+t)^X \in 1 + Xt + t^2 A[\![t]\!]$;
- $F(t)^X|_{t \to t^n} = F(t^n)^X$.

In fact, the last two properties are not part of the original definition, but other authors include them in their definition.

7.2.3. In the rest of the section, for $F = \sum_n F_n T^n \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}[\![T]\!]$, with $F_0 = 1$, and for $X \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$ we denote by $F(T)^X$ the power structure introduced by Gusein-Zade, Luengo and Melle-Hernandez in [9], and that we describe as follows, taking also into account the actions of $\widehat{\mu}$:

Definition 7.2.4. Let $n \geq 0$ be an arbitrary natural number and $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}^*}$. We say that α is a *partition* of n if

$$\sum_{i>0} i\alpha_i = n.$$

In this case we write $\alpha \dashv n$ or $|\alpha| = n$. We define the length of a partition α as $\|\alpha\| := \sum_{i>0} \alpha_i$. We say that $\alpha \in \mathbb{N}^{\mathbb{N}^*}$ is a partition if it is a partition of n for some $n \in \mathbb{N}$.

Definition 7.2.5. Let $A(T) = 1 + \sum_{i>0} A_i T^i \in 1 + tK_0(\mathrm{Var}_k)^{\widehat{\mu}}[\![T]\!]$ and let M be a variety. We define $A(T)^{[M]}$ as

$$A(T)^{[M]} := 1 + \sum_{\alpha \text{ partition}} \pi_{G_\alpha}\Big(\Big[\prod_i M^{\alpha_i} \setminus \Delta\Big] \prod_i A_i^{\alpha_i}\Big) t^{|\alpha|},$$

where $G_\alpha = \prod_i \Sigma_{\alpha_i}$ acts simultaneously on $\prod_i M^{\alpha_i}$ and $\prod_i A_i^{\alpha_i}$ by permuting the factors, $\Delta \subseteq \prod_i M^{\alpha_i}$ is the large diagonal, i.e., the subscheme of points with at least two equal entries and $\pi_{G_\alpha} \colon \prod_i M^{\alpha_i} \to \mathrm{Sym}^{\|\alpha\|}(M)$ is the canonical projection.

7.3. Symmetric powers

7.3.1. Keep the notation introduced in the previous paragraph. Let us begin with the following definition.

Definition 7.3.2 (Symmetric power in GRV). Let $\alpha \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$ and let $r \in \mathbb{N}$. The *r-th symmetric power of* α is the element $\mathrm{Sym}^r(\alpha) \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$ defined as

$$\mathrm{Sym}^r(\alpha) := [t^r](1-t)^{-\alpha},$$

where $[t^r]G(t)$ denotes the coefficient of t^r for a series G with coefficients in $K_0(\mathrm{Var}_k)^{\widehat{\mu}}$.

7.3.3. In particular, if $\alpha = [U]$, we have that

$$\mathrm{Sym}^r\big([U]\big) = \big[\mathrm{Sym}^r(U)\big];$$

in this sense $\mathrm{Sym}^\bullet$ extends the notion of symmetric power of an algebraic variety. In general, for $\alpha = [U] - [V]$ one gets an explicit formula by analyzing the coefficients of

$$(1-t)^{[V]-[U]} = (1-t)^{[V]} \cdot (1+t+t^2+\cdots)^{[U]}.$$

Example 7.3.4. In order to show how this computation can be handled, we compute explicitly the coefficient of t^2, that is, the expression for $\mathrm{Sym}^2([U]-[V])$. We know, from [9, Theorem 1], that

$$(1+t+t^2+\cdots)^{[U]} = 1 + [U]t + \left[\mathrm{Sym}^2(U)\right]t^2 + o(t^2),$$

thus we get

$$\begin{aligned}(1-t)^{[V]} &= \left(1 + [V]t + \left[\mathrm{Sym}^2(V)\right]t^2 + o(t^2)\right)^{-1} \\ &= 1 - [V]t + \left([V^2] - \left[\mathrm{Sym}^2(V)\right]\right)t^2 + o(t^2).\end{aligned}$$

It follows that

$$\mathrm{Sym}^2\left([U]-[V]\right) = \left[\mathrm{Sym}^2(U)\right] - \left[\mathrm{Sym}^2(V)\right] + [V]^2 - [U][V].$$

7.3.5. It is possible to extend the map

$$\mathrm{Sym}^r \colon K_0(\mathrm{Var}_k)^{\widehat{\mu}} \to K_0(\mathrm{Var}_k)^{\widehat{\mu}}$$

to a map $\mathrm{Sym}^r \colon \mathscr{M}_k^{\widehat{\mu}} \to \mathscr{M}_k^{\widehat{\mu}}$ by $\mathrm{Sym}^r(\mathbb{L}^{-s}\alpha) := \mathbb{L}^{-rs}\,\mathrm{Sym}^r(\alpha)$. In order to ensure that this map is well defined, we only need to show that, for $\alpha \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$ and for $s \in \mathbb{N}$, we have $\mathbb{L}^{-rs}\,\mathrm{Sym}^r(\alpha) = \mathbb{L}^{-r(s+1)}\,\mathrm{Sym}^r(\mathbb{L}\alpha)$.

Indeed, recalling that $\mathrm{Sym}^r(\mathbb{A}^n) \cong \mathbb{A}^{nr}$, we get

$$(1-t)^{\mathbb{L}\alpha} = \left((1-t)^{-\mathbb{L}}\right)^{-\alpha} = (1 + \mathbb{L}t + \mathbb{L}^2t^2 + \cdots)^{-\alpha} = (1-\mathbb{L}t)^{\alpha},$$

which implies that $\forall \alpha \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$, $\mathrm{Sym}^r(\mathbb{L}\alpha) = \mathbb{L}^r\,\mathrm{Sym}^r(\alpha)$.

7.3.6. The map Sym^r can also be defined at the level of $\widehat{\mathscr{M}_k}^{\widehat{\mu}}$; indeed, for $\alpha, \beta \in K_0(\mathrm{Var}_k)^{\widehat{\mu}}$, we have that:

$$\forall p \in \mathbb{N}, \quad (1-t)^{\alpha + \mathbb{L}^p\beta} - (1-t^{\alpha}) = (1-t)^{\alpha} \cdot \left((1-\mathbb{L}^p t)^{\beta} - 1\right) \in \mathbb{L}^p K_0(\mathrm{Var}_k)^{\widehat{\mu}}[\![t]\!],$$

thus $\mathrm{Sym}^r(\alpha + \mathbb{L}^p\beta) \equiv \mathrm{Sym}^r(\alpha) \pmod{\mathbb{L}^p}$.

7.3.7. We will also define a version of Sym^r defined over $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^n-1)^{-1} \colon n \in \mathbb{N}^*]$. Since Sym^1 is already defined as the identity map we can proceed inductively on r. Let us assume that all the maps Sym^i are defined for $1 \le i \le r-1$. Let us first check that for $\alpha \in \mathscr{M}_k^{\widehat{\mu}}$, the value of

$$\mathrm{Sym}^r\left(\frac{\alpha}{1}\right) := \frac{\mathrm{Sym}^r(\alpha)}{1} \in \mathscr{M}_k^{\widehat{\mu}}\left[(\mathbb{L}^n-1)^{-1} \colon n \in \mathbb{N}^*\right]$$

is well defined, i.e., that if $\frac{\alpha}{1} = \frac{\beta}{1}$, then

$$\frac{\mathrm{Sym}^r(\alpha)}{1} = \frac{\mathrm{Sym}^r(\beta)}{1}.$$

If $\gamma \in \mathscr{M}_k^{\hat{\mu}}$ is such that $(\mathbb{L}^n - 1)\gamma = 0$ for some $n \in \mathbb{N}$, then

$$\mathrm{Sym}^r(\mathbb{L}^n\gamma) = \mathrm{Sym}^r\left(\gamma + (\mathbb{L}^n - 1)\gamma\right) = \sum_{j=0}^{r} \mathrm{Sym}^j(\gamma)\,\mathrm{Sym}^{r-j}(0) = \mathrm{Sym}^r(\gamma),$$

thus $(\mathbb{L}^{nr} - 1)\,\mathrm{Sym}^r(\gamma) = 0$. By an inductive argument one proves that if $\frac{\gamma}{1} = 0$, then also $\frac{\mathrm{Sym}^r(\gamma)}{1} = 0$. It follows that, whenever $\frac{\alpha}{1} = \frac{\beta}{1}$, then

$$\frac{\mathrm{Sym}^r(\beta)}{1} = \frac{\mathrm{Sym}^r\left(\alpha + (\beta - \alpha)\right)}{1} = \sum_{j=1}^{r} \frac{\mathrm{Sym}^j(\alpha)}{1}\,\frac{\mathrm{Sym}^{r-j}(\beta - \alpha)}{1} = \frac{\mathrm{Sym}^r(\alpha)}{1}.$$

Then we define, recursively:

$$\mathrm{Sym}^r\left(\frac{\alpha}{\mathbb{L}^n - 1}\right) := (\mathbb{L}^{nr} - 1)^{-1} \sum_{i=1}^{r} \mathrm{Sym}^i(\alpha)\,\mathrm{Sym}^{r-i}\left(\frac{\alpha}{\mathbb{L}^n - 1}\right).$$

Example 7.3.8. We show how to compute this map in a specific case. For $\alpha = \frac{[U]}{1-\mathbb{L}^n}$, we have that $\mathrm{Sym}^2(\alpha) = \frac{[\mathrm{Sym}^2(U)]}{1-\mathbb{L}^{2n}} + \frac{\mathbb{L}^n \cdot [U]^2}{(1-\mathbb{L}^n)(1-\mathbb{L}^{2n})}$.

7.3.9. For a power series $F(T) = \sum A_n T^n \in \mathscr{R}^{\hat{\mu}}[\![T]\!]$, let us consider the power series obtained by plugging each coefficient of F into the above mentioned maps Sym^r:

$$\mathrm{Sym}^r(F)(T) := \sum \mathrm{Sym}^r(A_n) T^n.$$

These maps have very interesting properties when the function F is rational, indeed in this case we are able to control the poles of Sym^r thanks to the upcoming results:

Lemma 7.3.10. *Let*

$$F = \frac{\alpha T^h}{(1 - \mathbb{L}^{-qN} T^N)^e} \in \mathscr{R}^{\hat{\mu}}[\![T]\!],$$

where $q \in \mathbb{Q}$ is such that qN is an integer. Then, for $r > 0$, we have that $\mathrm{Sym}^r(F)$ *has at most one pole of order (at most) $r(e-1)+1$ in rq.*

Proof. Let $F = \sum_{m \geq 0} A_m T^{mN+h}$; then $A_m = \binom{m+e-1}{e-1}\alpha\mathbb{L}^{-mqN}$. It follows that

$$\mathrm{Sym}^r(A_m) = \sum_{\beta \dashv r} \binom{\binom{m+e-1}{e-1}}{\beta_1, \ldots, \beta_r} \alpha^{\beta_1} \cdot (\mathrm{Sym}^2\alpha)^{\beta_2} \cdots (\mathrm{Sym}^r\alpha)^{\beta_r} \mathbb{L}^{-rmqN}.$$

Once β is fixed,

$$\binom{\binom{m+e-1}{e-1}}{\beta_1, \ldots, \beta_r}$$

is either 0 for all $m \in \mathbb{Z}$ (this happens only if $e = 1$ and $\beta_1 + \cdots + \beta_r > 1$) or a polynomial in m of degree $(e-1)(\beta_1 + \cdots + \beta_r)$ with rational coefficients. Moreover it takes integral values whenever m is an integer.

It follows from the upcoming Lemma 7.3.11 that

$$\sum_{m \geq 0} \binom{\binom{m+e-1}{e-1}}{\beta_1, \ldots, \beta_r} \alpha^{\beta_1} \cdot (\mathrm{Sym}^2 \alpha)^{\beta_2} \cdots (\mathrm{Sym}^r \alpha)^{\beta_r} \mathbb{L}^{-rmqN} T^{mN+h}$$

is a suitable linear combination with integral coefficients of

$$\left\{ \frac{\alpha^{\beta_1} \cdot (\mathrm{Sym}^2 \alpha)^{\beta_2} \cdots (\mathrm{Sym}^r \alpha)^{\beta_r} T^h}{(1 - \mathbb{L}^{-rqN} T^N)^j} \right\}_{j=0}^{(e-1)(\beta_1 + \cdots + \beta_r)+1}.$$

Among all the partitions of r, the one which gives the highest possible order of the pole is the one maximizing $\beta_1 + \cdots + \beta_r$, namely $\beta = (r, 0, 0, \ldots)$, which gives a pole of order at most $r(e-1) + 1$. ∎

Lemma 7.3.11. *Consider the subring of* $\mathbb{Q}[x]$ *defined by*

$$A := \left\{ f \in \mathbb{Q}[x] \colon \forall m \in \mathbb{Z},\ f(m) \in \mathbb{Z} \right\}$$

and for $d \in \mathbb{N}$ *let* $A_d := \{ f \in A \colon \deg(f) \leq d \}$ *(setting by convention* $\deg 0 = -\infty$*). Then* A_d *is generated by* $\left\{ \binom{x+j}{j} \right\}_{j=0}^{d}$ *as a* $\mathbb{Z}$*-module.*

Proof. For all $j \in \mathbb{N}$, $\binom{x}{j}$ is a polynomial of degree j. For $m \in \mathbb{N}$, we have that

$$\binom{m}{j} \in \mathbb{N} \quad \text{and} \quad \binom{-m}{j} = \pm \binom{m+j-1}{j} \in \mathbb{Z},$$

thus $\binom{x}{j} \in A_d$ for $j \leq d$.

Moreover $\binom{m}{j} = 0$ for $0 \leq m < j$, while $\binom{j}{j} = 1$. It follows that given an arbitrary $f \in A_d$, there exist a unique linear combination

$$L(x) := \sum_{j=0}^{d} a_j \binom{x}{j},$$

with $a_j \in \mathbb{Z}$, such that for $m \in \{0, \ldots, d\}$ one has that

$$L(m) = f(m).$$

Since f and L are polynomials of degree at most d whose evaluations at $d+1$ points coincide, they are the same polynomial.

Thus $\{\binom{x}{j}\}_{j=0}^{d}$ generates A_d as a $\mathbb{Z}$-module, but $\{\binom{x+j}{j}\}_{j=0}^{d}$ is an equivalent $\mathbb{Z}$-base because the leading terms of $\binom{x}{j}$ and of $\binom{x+j}{j}$ coincide. ∎

7.3.12. For the rest of the section, we denote by $\mathscr{R}^{\widehat{\mu}}$ one of the two rings $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^n-1)^{-1}\colon n \in \mathbb{N}^*]$ or $\widehat{\mathscr{M}_k}^{\widehat{\mu}}$: the proofs we will present do not hold in general for functions with coefficients in $\mathscr{M}_k^{\widehat{\mu}}$.

Proposition 7.3.13. *For $i = 1, \dots, s$ let $F_i = \sum_{m\geq 0} A_m^{[i]} T^m \in \mathscr{R}^{\widehat{\mu}}[\![T]\!]$ be rational functions and let $\mathcal{Q}_i \subseteq \mathbb{Q}$ be the set of poles of F_i. Let $F = \sum_{m\geq 0} A_m^{[1]} \cdots A_m^{[s]} T^m \in \mathscr{R}^{\widehat{\mu}}[\![T]\!]$. Then F is also rational and its set of poles, $\mathcal{Q}$, is contained in $\mathcal{Q}_1 + \mathcal{Q}_2 + \cdots + \mathcal{Q}_s$.*

Moreover, for each $q \in \mathcal{Q}$, we have that

$$\operatorname{ord}_q(F) \leq \max\left\{1 - s + \sum_{i=1}^{s} \operatorname{ord}_{q_i}(F_i)\colon q_i \in \mathcal{Q}_i \text{ and } \sum q_i = q\right\}.$$

Remark 7.3.14. This statement holds, with the same proof, also if we consider functions $F_i \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$, provided that each of them is sum of functions with a single pole.

Proof. Let us assume for a moment that, $\forall i$, $F_i = \frac{\alpha_i T^{h_i}}{(1-\mathbb{L}^{-q_i N} T^N)^{e_i}}$, for some $\alpha_i \in \mathscr{R}$, $0 \leq h_i < N$ integers, $q_i \in \mathbb{Q}$, $0 < e_i \in \mathbb{N}$. In such case

$$A_{mN+h_i}^{[i]} = \binom{m + e_i - 1}{e_i - 1} \alpha_i \mathbb{L}^{-mq_i N}.$$

Thus $F = 0$ unless $h_i = h$ $\forall i$, while in this case we have that

$$\prod_{i=1}^{s} A_{mN+h}^{[i]} = \left(\prod_{i=1}^{s} \binom{m + e_i - 1}{e_i - 1} \alpha_i\right) \mathbb{L}^{-mqN},$$

where $q = q_1 + \cdots + q_s$. The degree of $\prod_{i=1}^{s} \binom{m+e_i-1}{e_i-1}$, seen as a polynomial in m, is $\sum e_i - s$, thus we get the desired result in this case.

For the general case it is enough to consider $F_1 = F_1' + F_1''$, where $F_1' = \sum B_m T^m$ and $F_1'' = \sum C_m T^m$; then, setting $F' := \sum B_m A_m^{[2]} \cdots A_m^{[s]} T^m$ and $F'' := \sum C_m A_m^{[2]} \cdots A_m^{[s]} T^m$, we have that

$$F = F' + F''$$

and if our statement holds for both F' and F'' then it holds also for F, in particular writing all the F_i as in equation (6.1), the proposition follows by an induction on the number of their summands. ∎

Lemma 7.3.15. *Let $F \in \mathscr{R}^{\widehat{\mu}}[\![T]\!]$ be a rational function whose set of poles is $\mathcal{Q} \subseteq \mathbb{Q}$, or let $F \in \mathscr{M}_k^{\widehat{\mu}}[\![T]\!]$ be the sum of functions with at most one pole. For all $r \in \mathbb{N}$, let $\Sigma^r \mathcal{Q}$ be the set of rational numbers that are sum of r elements of $\mathcal{Q}$. Then $\operatorname{Sym}^r F$ is also rational and its set of poles is contained in $\Sigma^r \mathcal{Q}$.*

Moreover, for each $q \in \mathcal{Q}$, we have that

$$\operatorname{ord}_q(\operatorname{Sym}^r F) \leq \max\left\{1 - r + \sum_{i=1}^{s} \operatorname{ord}_{q_i}(F)\colon q_i \in \mathcal{Q} \text{ and } \sum q_i = q\right\}.$$

Proof. If the Lemma holds for F and G, then by Proposition 7.3.13, it holds also for $F+G$, since $\mathrm{Sym}^r(F+G) = (\mathrm{Sym}^r F) + (\mathrm{Sym}^{r-1} F)G + \cdots + (\mathrm{Sym}^r G)$. Thus it is enough to write F as in equation (6.1) and notice that for each addendum the statement coincides with Lemma 7.3.10. ∎

8. Hilbert schemes

In this section we collect a few of the basic facts about Hilbert schemes that can be useful to understand the construction that shall appear in the incoming section.

8.1. The moduli problem

8.1.1. Hilbert schemes are the answer to one of the first moduli problems that mathematicians happen to face: classifying subschemes of a given variety.

Definition 8.1.2. Let $X \to S$ be a quasi-projective morphism of schemes.

The *Hilbert Functor* of S-subschemes of X is the functor:

$$\mathscr{H}(X/S)\colon \mathrm{Sch}_S \to \mathrm{Sets}$$

which associate to any S-scheme T the set

$$\mathscr{H}(X/S)(T) := \{V \subseteq X \times_S T \mid V \to T \text{ is proper and flat}\}$$

and associate to each morphism $T' \to T$ of S-schemes the map of sets:

$$\begin{aligned}&\mathscr{H}(X/S)(T) \to \mathscr{H}(X/S)(T')\\ &V \subseteq X \times_S T \mapsto V \times_T T' \subseteq (X \times_S T) \times_T T' \cong X \times_S T'.\end{aligned}$$

8.1.3. This functor is actually represented by a scheme $\mathrm{Hilb}(X/S)$ which is called the *Hilbert Scheme* of X over S. The scheme that can be constructed in this way is arguably unmanageable, for it has, typically, i.e., when $X \to S$ is not finite, an infinite amount of connected components whose dimension is not even bounded. It is often considered convenient to stratify this scheme as a disjoint union of open and closed subschemes each of them parametrizing subschemes of X with similar properties, i.e., classifying the subschemes of X according to an invariant.

Definition 8.1.4. Fix a relatively ample line bundle $\mathcal{L}$ over $f\colon X \to S$. Let $\mathcal{F} \in \mathrm{Coh}(X)$ an S-flat sheaf; then $f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})$ is a locally free $\mathcal{O}_S$-module, for $m \in \mathbb{Z}$. There is a polynomial $h_{X/S,\mathcal{F},\mathcal{L}} \in \mathbb{Q}[t]$ such that for $m >> 1$, $\mathrm{rk}(f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})) = h_{X/S,\mathcal{F},\mathcal{L}}(m)$, which is called *Hilbert polynomial* of $\mathcal{F}$.

8.1.5. Let $T \to S$ be a morphism of schemes and denote by $g\colon X_T := X \times_S T \to X$ the morphism induced by the base-change. The set of T-flat and proper (also over T)

subschemes of X_T is in natural bijection with the set of quotients of $\mathcal{O}_{X\times_S T}$ which are T-flat and have proper (over T) support, i.e.,

$$\begin{aligned}\mathscr{H}(X/S)(T) &= \{V \subseteq X \times_S T \mid V \to T \text{ is proper and flat}\}\\ &\cong \{\mathcal{O}_{X\times_S T} \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ is flat and proper over } T\}.\end{aligned}$$

Hence, given a polynomial $p \in \mathbb{Q}[t]$ and a relatively ample line bundle $\mathcal{L}$ over X, we can define a subfunctor of $\mathscr{H}(X/S)$ as:

$$\begin{aligned}&\mathscr{H}^p_{\mathcal{L}}(X/S)(T)\\ &\quad = \{\mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F} \mid T-\text{flat, with proper support and such that } h_{X_T/T,\mathcal{F},g^*\mathcal{L}} = p\},\end{aligned}$$

which is represented by an open and closed subscheme of $\mathrm{Hilb}(X/S)$ which we denote by $\mathrm{Hilb}^p_{\mathcal{L}}(X/S)$. We have, moreover, that

$$\mathrm{Hilb}(X/S) = \bigsqcup_{p\in\mathbb{Q}[t]} \mathrm{Hilb}^p_{\mathcal{L}}(X/S).$$

8.1.6. For the purposes of this manuscript, we shall only need to study the components corresponding to constant polynomials, which are called *Hilbert schemes of points*, since they parametrize 0-dimensional subschemes of X. Since the Hilbert polynomial of a finite subscheme of X is independent on the choice of the relatively ample line bundle $\mathcal{L}$, we can omit it from the notation and denote by $\mathrm{Hilb}^n(X/S)$ the Hilbert scheme corresponding to the constant polynomial n, which is often called *Hilbert scheme of n points on X*, with a slight abuse of language.

8.1.7. We shall need a lemma which describes how the Hilbert schemes behave under base-change.

Lemma 8.1.8. *Let $T \to S$ and $X \to S$ be morphisms of schemes, $\mathcal{L}$ an S-ample line bundle over X and let $p \in \mathbb{Q}[t]$. Let $f\colon X_T \to X$ the basechange morphism with respect to $T \to S$.*

Then $\mathrm{Hilb}^p_{f^*\mathcal{L}}(X_T/T) \cong \mathrm{Hilb}^p_{\mathcal{L}}(X/S) \times_S T$ *in* Sch_T.

In particular, if F is a field and $\mathrm{Spec}\, F \to S$ *is a point of S, the fibre over* $\mathrm{Spec}\, F$ *of* $\mathrm{Hilb}^p_{\mathcal{L}}(X/S)$ *coincide with the Hilbert scheme of the fibre:* $\mathrm{Hilb}^p_{\mathcal{L}|_{X_F}}(X_F/\,\mathrm{Spec}\, F)$.

Proof. Omitted. ∎

8.2. Properties of the Hilbert schemes of points

8.2.1. Let $X \to S$ be a flat morphism of schemes and let n be a positive integer. Consider the scheme $\mathrm{Sym}^n(X/S) := X^n/\Sigma_n$, where the symmetric group acts on the product by permuting the factors. There is a natural morphism

$$\mathrm{Hilb}^n(X/S) \to \mathrm{Sym}^n(X/S)$$

called Hilbert-to-Chow morphism, sending a subscheme of X to a weighted sum of finite S-subschemes of X.

Remark 8.2.2. If $X \to S$ is a smooth map of relative dimension 2, then $\mathrm{Hilb}^n(X/S)$ is a smooth scheme of relative dimension $2n$, in particular the Hilbert-to-Chow morphism is a resolution of singularities of $\mathrm{Sym}^n(X/S)$. This is no longer true if $\dim_S(X) > 2$ and $n > 2$, unless the couple $(n, \dim_S(X)) = (3, 3)$.

8.2.3. Our main motivation for studying the Hilbert schemes of points on a K3 surface is the following, see [1, p. 768, Proposition 6 and following].

Theorem 8.2.4. *If* $X \to \operatorname{Spec} \mathbb{C}$ *is a* K3 *surface, then* $\mathrm{Hilb}^n(X/\mathbb{C})$ *is an irreducible holomorphic symplectic variety of dimension* $2n$.

Remark 8.2.5. Also the case of an abelian surface A is interesting, because in that case the Hilbert scheme is anyway a smooth Calabi–Yau variety; moreover all IHS varieties of Kummer type can be constructed as deformations of subschemes of $\mathrm{Hilb}^n(A/K)$.

9. Hilbert schemes of points on a surface

9.1. Construction of a weak Néron model

9.1.1. Let $X \to \operatorname{Spec} K$ be a smooth surface with trivial canonical divisor and let $\omega \in \omega_{X/K}(X)$ be a volume form on it. Let $\mathfrak{X} \to \Delta$ be a regular model whose central fibre $\mathfrak{X}_k$ is a strict normal crossing divisor of $\mathfrak{X}$. Let us keep the notation of Section 5 concerning the field extensions over K and the corresponding base-changes. If $\operatorname{char} k = p > 0$, we add the further assumption that the central fibre $\mathfrak{X}_k$ has no components with multiplicity divisible by p.

9.1.2. The aim of this section is to provide a closed formula for the zeta function of $\mathrm{Hilb}^n(X)$; from the results developed in the previous sections, we will be able to expressit in terms of the zeta functions of $X(i)$ for $1 \leq i \leq n$.

9.1.3. Let a be the *lcm* of the multiplicities of the irreducible components of $\mathfrak{X}_k$. For $n \in \mathbb{N}$, with $n < p$ if $\operatorname{char} k = p > 0$, let $\tilde{n} := a \operatorname{lcm}(1, 2, \ldots, n)$ and let $K(\tilde{n})$ be the unique totally ramified extension of K whose degree is $\tilde{n}$, so that $\operatorname{Gal}(K(\tilde{n})/K) = \mu_{\tilde{n}}$.

9.1.4. For a positive integer m, coprime with the characteristic exponent of k, denote by $\mathfrak{X}(m\tilde{n})$ be the semistable model of $X(m\tilde{n})$ obtained from $\mathfrak{X}$ using the construction of Section 5.2. As usual we denote by $\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}$ the smooth locus of $\mathfrak{X}(m\tilde{n}) \to \Delta(m\tilde{n})$. Since $\mathrm{Hilb}^n(\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}/\Delta(m\tilde{n})) \to \Delta(m\tilde{n})$ is a smooth model of $\mathrm{Hilb}^n(X(m\tilde{n}))$, we have that

$$\mathfrak{X}^{[n]}(m) := \left(\mathrm{Res}_{\Delta(m\tilde{n})/\Delta(m)} \mathrm{Hilb}^n \left(\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}/\Delta(m\tilde{n})\right)\right)^{\mu_{\tilde{n}}} \to \Delta(m)$$

is a smooth model of $\mathrm{Hilb}^n(X(m))$.

Proposition 9.1.5. *Assume that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > n$, *then* $\mathfrak{X}^{[n]}(m) \to \Delta(m)$ *is a weak Néron model of* $\mathrm{Hilb}^n(X(m))$.

Proof. We assume that $a = 1$, i.e., that X has semistable reduction on K. The proof in the general case will descend from Proposition 4.2.6 and Lemma 4.2.8. We just need to show that every point $\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m)) \subseteq \mathfrak{X}^{[n]}(m)$ extends to a morphism $\Delta(m) \to \mathfrak{X}^{[n]}(m)$.

Consider a point $\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m))$ and let $Z \subseteq X(m\tilde{n})$ the ($\mu_{\tilde{n}}$-invariant) subscheme representing such point. Either the closure of Z in $\mathfrak{X}(m\tilde{n})$ is contained in $\mathfrak{X}(m\tilde{n})_{\mathrm{sm}}$ or at least one point $P \in \operatorname{supp} Z$ specializes to the singular locus $\mathfrak{X}(m\tilde{n})_{k,\mathrm{sing}}$. In the first case $\overline{Z}$ represents a morphism $\Delta(m) \to \mathfrak{X}^{[n]}(m)$ extending the given

$$\operatorname{Spec} K(m) \to \operatorname{Hilb}^n(X(m)).$$

If $P \in \operatorname{supp} Z$ specializes to $\mathfrak{X}(m\tilde{n})_{k,\mathrm{sing}}$, then its residue field $k(P)$ contains strictly $K(m\tilde{n})$, for $\mathfrak{X}(m\tilde{n})$ is a regular model of $X(m\tilde{n})$.

Let $Q \in X(m)$ be the image of P under the map $\operatorname{Spec} k(P) \hookrightarrow X(m\tilde{n}) \to X(m)$. The degree $[k(Q) : K(m)]$ cannot divide $\tilde{n}$, otherwise $k(P)$ would be $K(m\tilde{n})$, thus

$$[k(Q) \colon K(m)] \geq n + 1.$$

Since Z is $\mu_{\tilde{n}}$-invariant, then it contains the whole orbit of P which is the reduced scheme associated with the preimage of Q under the map

$$\pi \colon X(m\tilde{n}) \to X(m\tilde{n})/\mu_{\tilde{n}} \cong X(m).$$

We have that $\pi^{-1}(Q) = \operatorname{Spec}(k(Q) \otimes_{K(m)} K(m\tilde{n}))$, which is a reduced $K(m\tilde{n})$-algebra of dimension $[k(Q) : K(m)] > n$.

This contradicts the fact that Z is a subscheme of length n, thus this case cannot occur and we are done. ∎

9.1.6. Our goal is to study the motivic integral of $\operatorname{Hilb}^n(X(m))$ using the models $\mathfrak{X}^{[n]}(m)$ constructed above. In order to simplify notation, we perform the following computations only for $m = 1$, working with $\mathfrak{X}^{[n]} = \mathfrak{X}^{[n]}(1)$; similar arguments apply when $m > 1$. In fact, when we are not concerned about the action of groups, this is equivalent to renaming $K(mm')$ as $K(m')$, for all m' coprime with the characteristic exponent of k (and similarly for $R(-)$, $X(-)$, etc.); on the other hand, under this change of notation, we assume that K admits a subfield F such that K/F is a Galois extension of degree m and that $\mathfrak{X} \to \Delta$ is an equivariant model with respect to the action of $\operatorname{Gal}(K|F)$, this will allow us to perform computations in the equivariant Grothendieck rings.

9.1.7. Even though $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{\mathrm{sm}}/\Delta(\tilde{n}))$ is not a weak Néron model of $X(\tilde{n})$, it is a smooth model and the results of Section 4.3 apply. It follows that the connected components of $\mathfrak{X}_k^{[n]}$ are in bijection with the connected components of $\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mu_{\tilde{n}}}$, moreover if $C \subseteq \mathfrak{X}_k^{[n]}$ is sent to C' via this bijection, then

$$[C] = \mathbb{L}^{2n-\dim C'}[C'].$$

9.1.8. For any $d \in \mathbb{N}$ dividing $\tilde{n}$ we denote by $Y(d) \subseteq \mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$ the subscheme consisting of the points whose stabilizer is exactly $\mathrm{Gal}(K(\tilde{n})|K(d))$. Then $Y(d)$ is $\mathrm{Gal}(K(\tilde{n})|F)$-invariant and inherits an action of $\mathrm{Gal}(K(d)|F)$, since $\mathrm{Gal}(K(\tilde{n})|F)$ is an abelian group and since points in the same orbit have the same stabilizer; furthermore, the action of $\mathrm{Gal}(K(d)|F)$ on $Y(d)$ naturally induces an action of

$$\mathrm{Gal}(K|F) \cong \mathrm{Gal}\left(K(d)|F\right)/\,\mathrm{Gal}\left(K(d)|K\right)$$

on the quotient $Y(d)/\,\mathrm{Gal}(K(d)|K)$. Because of Lemma 5.2.9, $Y(d)$ is an open and closed subscheme of $\mathfrak{X}_{k,\mathrm{sm}}$ and we have that:

$$\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}^{\mathrm{Gal}(K(\tilde{n})|K(d))} = \bigsqcup_{d'|d} Y(d').$$

Remark 9.1.9. Since $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}^{\mathrm{Gal}(K(\tilde{n})|K(d))}$ is a scheme of pure dimension 2, it can be identified with the central fibre of $\mathrm{Res}_{\Delta(\tilde{n})/\Delta(d)}\left(\mathfrak{X}(\tilde{n})_{\mathrm{sm}}\right)^{\mathrm{Gal}(K(\tilde{n})|K(d))}$, which is a weak Néron model of $X(d)$, via the map h_k described in Section 4.3.2. Let

$$C \subseteq \mathrm{Res}_{\Delta(\tilde{n})/\Delta(d)}\left(\mathfrak{X}(\tilde{n})_{\mathrm{sm}}\right)^{\mathrm{Gal}(K(\tilde{n})|K(d))}$$

be a connected component. It follows from the identity in Lemma 4.4.6 and from the fact that $\mathrm{Gal}(K(\tilde{n})|K(d))$ acts trivially on $T_{h_k(C)}$ that

$$\mathrm{ord}_{h_k(C)}\left(\omega(\tilde{n})\right) = \frac{\tilde{n}}{d}\,\mathrm{ord}_C\left(\omega(d)\right).$$

9.1.10. The following statement gives a $\mathrm{Gal}(K|F)$-equivariant decomposition of the central fibre as a union of closed and open subschemes; it is an *ad hoc* partition that will be very useful for our computation:

Proposition 9.1.11. $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mathrm{Gal}(K(\tilde{n})|K)}$ *admits the following* $\mathrm{Gal}(K|F)$*-equivariant decomposition as disconnected union of subschemes:*

$$\mathrm{Hilb}^n\left(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}\right)^{\mathrm{Gal}(K(\tilde{n})|K)} \cong \bigsqcup_{\alpha\dashv n}\prod_{j=1}^{n}\mathrm{Hilb}^{\alpha_j}\left(Y(j)/\,\mathrm{Gal}\big(K(j)|K\big)\right).$$

Moreover, for a fixed partition $\alpha \dashv n$, the isomorphism above restricts to an isomorphism between $\prod_{j=1}^{n}\mathrm{Hilb}^{\alpha_j}(Y(j)/\,\mathrm{Gal}(K(j)|K))$ and a finite union of connected components of $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mathrm{Gal}(K(\tilde{n})|K)}$.

Proof. Fix a k-scheme S. For any closed subscheme $Z \subseteq \mathfrak{X}(\tilde{n})_S$ endowed with a finite map $Z \to S$, set $Z_j := Z \cap Y(j)_S$. If Z is stable under the action of $\mathrm{Gal}(K(\tilde{n})|K)$, then every Z_j, which is the intersection of two stable schemes, is stable as well; moreover the induced action of $\mathrm{Gal}(K(j)|K) = \mathrm{Gal}(K(\tilde{n})|K)/\,\mathrm{Gal}(K(\tilde{n})|K(j))$ on $Y(j)_S$ is free, thus the induced maps $\pi_j\colon Y(j)_S \to Y(j)_S/\,\mathrm{Gal}(K(j)|K)$ and $Z_j \to \pi_j(Z_j)$ are étale of degree j.

In this way, from any invariant finite S-subscheme of $\mathfrak{X}(\tilde{n})_S$ we construct a finite subscheme in each $Y(j)_S/\operatorname{Gal}(K(j)|K)$; on the other hand given a sequence of finite S-subschemes of $Y(j)_S/\operatorname{Gal}(K(j)|K)$ of length α_j we get a unique $\operatorname{Gal}(K(\tilde{n})|K)$-stable S-subscheme of $\mathfrak{X}(\tilde{n})_S$ whose length is $\sum_j j\alpha_j$.

Moreover, for $g \in \operatorname{Gal}(K(\tilde{n})|F)$, the subscheme $gZ \in \mathfrak{X}(\tilde{n})_S$ only depends on the coset $g\operatorname{Gal}(K(\tilde{n})|K)$ (as Z is $\operatorname{Gal}(K(\tilde{n})|K)$-invariant) and not on g itself; as $Y(j)$ is $\operatorname{Gal}(K(\tilde{n})|F)$-invariant for all j, the schemes $gZ \cap Y(j)_S$ and $g(Z \cap Y(j)_S)$ coincide. Hence, the correspondence between S-subschemes of $\mathfrak{X}(\tilde{n})_S$ and S-points of

$$\bigsqcup_{\alpha \dashv n} \prod_{j=1}^{n} \operatorname{Hilb}^{\alpha_j}\big(Y(j)/\operatorname{Gal}\big(K(j)|K\big)\big)$$

is preserved under the action of $\operatorname{Gal}(K|F)$.

The last statement follows directly from the fact that the $Y(j)$-s are themselves open and closed subschemes of $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$. ■

9.1.12. Thus, recalling that $\operatorname{Hilb}^{\alpha_j}(Y(j)/\operatorname{Gal}(K(j)|K))$ is pure of dimension $2\alpha_j$, we conclude that

$$\mathfrak{X}_k^{[n]} = \bigsqcup_{\alpha \dashv n} \mathfrak{X}_{k,\alpha}^{[n]},$$

where $\mathfrak{X}_{k,\alpha}^{[n]}$ is an affine bundle of rank $\sum_j 2(j-1)\alpha_j$ on

$$\operatorname{Hilb}^{\alpha_1}\big(Y(1)\big) \times \cdots \times \operatorname{Hilb}^{\alpha_n}\big(Y(n)/\operatorname{Gal}\big(K(n)|K\big)\big).$$

We thus have the following:

Corollary 9.1.13. *The following equation holds in the equivariant Grothendieck ring of varieties:*

$$\big[\mathfrak{X}_k^{[n]}\big] = \sum_{\alpha \dashv n} \prod_{j=1}^{n} \mathbb{L}^{2(j-1)\alpha_j}\big[\operatorname{Hilb}^{\alpha_j}\big(Y(j)/\operatorname{Gal}\big(K(j)|K\big)\big)\big].$$

9.2. The volume form on $\mathfrak{X}^{[n]}$

9.2.1. There is a volume form, $\omega^{[n]}$, on $\operatorname{Hilb}^n(X)$ that naturally arises from the given $\omega \in \omega_{X/K}$. In this paragraph we will recall its construction and compute its zeroes and poles on $\mathfrak{X}^{[n]}$.

9.2.2. Let $\mathrm{pr}_i\colon X^n \to X$, for $i \in \{1,\dots,n\}$, denote the projections on the factors. Then $\mathrm{pr}_1^*\omega \wedge \cdots \wedge \mathrm{pr}_n^*\omega$ is a global section of $\omega_{X^n/K}$ which, being invariant under the permutation of coordinates, descends to a global section of $\omega_{\operatorname{Sym}^n X/K}$, which we denote by φ. Let finally $\omega^{[n]}$ be the pull-back of φ through the Hilbert–Chow morphism, thus

$$\omega^{[n]} \in H^0\big(\operatorname{Hilb}^n(X), \omega_{\operatorname{Hilb}^n(X)}\big)$$

is a volume form on $\operatorname{Hilb}^n(X)$.

9.2.3. Now we will compute the zeroes and poles of $\omega^{[n]}$ seen as a rational section of $\omega_{\mathfrak{X}^{[n]}/\Delta}$. Since it is a volume form on the generic fibre of $\mathfrak{X}^{[n]}$, its zeroes or poles are all irreducible components of the central fibre. Let us fix a connected component $C \subseteq \mathfrak{X}_k^{[n]}$ and let us denote by C' the connected component of $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mathrm{Gal}(K(\tilde{n})|K)}$ such that $C \to C'$ is the affine bundle described in Section 4.3.5. In the following lemma we compute the conductor of the action of $\mathrm{Gal}(K(\tilde{n})|K)$ at points of C' in terms of the partition of n corresponding to the stratum of $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})/\Delta(\tilde{n}))^{\mathrm{Gal}(K(\tilde{n})|K)}$ containing C'.

Lemma 9.2.4. *Consider the decomposition of* $\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})^{\mathrm{Gal}(K(\tilde{n})|K)}$ *of Proposition 9.1.11 and fix a point* $[Z]$ *lying inside the stratum corresponding to* $\alpha \dashv n$. *Then:*

- *The conductor of the action of* $\mu_{\tilde{n}}$ *at* $[Z]$ *is*

$$c\big(\mathrm{Hilb}^n\big(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}\big), [Z]\big) = \tilde{n}\sum_{j=1}^{n}(j-1)\alpha_j.$$

- *If we denote by* $[Z_j]$ *the point of* $\mathrm{Hilb}^{\alpha_j}(Y(j)/\,\mathrm{Gal}(K(j)|K))$ *corresponding to*

$$Z_j/\,\mathrm{Gal}\,\big(K(j)|K\big),$$

then one has that:

$$\mathrm{ord}_{[Z]}\big(\omega^{[n]}(\tilde{n})\big) = \tilde{n}\sum_{j=1}^{n}\mathrm{ord}_{[Z_j/\,\mathrm{Gal}(K(j)|K)]}\big(\omega^{[\alpha_j]}(j)\big),$$

where $Y(j)$ *are considered as part of the central fibre of a weak Néron model for* $X(j)$ *as in Remark 9.1.9.*

Proof. Since the values of $c(\mathrm{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}), [Z])$ and $\mathrm{ord}_{[Z]}(\omega^{[n]}(\tilde{n}))$ depend only on the connected component containing $[Z]$ and since the generic point of each connected component corresponds to a reduced scheme, we may compute them with the additional assumption that Z is a reduced subscheme of $\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}$. In particular Z is the disjoint union of α_1 orbits of length 1, α_2 orbits of length 2 and so on. There is an equivariant isomorphism

$$T_{[Z]}\,\mathrm{Hilb}^n\big(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}\big) \cong \bigoplus_{p\in\mathrm{supp}\,Z} T_p\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}.$$

Let $\zeta = \zeta_{\tilde{n}}$ be a primitive root of unity and let σ be the unique generator of $\mathrm{Gal}(K(\tilde{n})|K)$ such that σ acts on $R(\tilde{n})$ by multiplying the uniformizing parametre by ζ. Consider an orbit of points $p_0, \ldots, p_{j-1} \in Z$, let e_1, e_2 be two generators of $T_{p_0}\mathfrak{X}(\tilde{n})_k$, so that $\sigma^l(e_1)$, $\sigma^l(e_2)$ will give a basis of $T_{p_l}\mathfrak{X}(\tilde{n})_k$ for each $l = 0, \ldots, j-1$. Notice that $\sigma^j(e_h) = e_h$ for $h = 1, 2$ since $\mathrm{Gal}(K(\tilde{n})|K(j))$ acts trivially on the whole connected component containing p_0.

For $i = 0, \ldots, j-1$, $h = 1, 2$ we have that

$$\big(e_h, \zeta^{i\tilde{n}/j}\sigma(e_h), \zeta^{2i\tilde{n}/j}\sigma^2(e_h), \ldots, \zeta^{(j-1)i\tilde{n}/j}\sigma^{j-1}(e_h)\big) \in T_{p_0}\mathfrak{X}(\tilde{n})_k \oplus \cdots \oplus T_{p_{j-1}}\mathfrak{X}(\tilde{n})$$

is an eigenvector with eigenvalue $\zeta^{-i\tilde{n}/j}$. In total there are $2j$ of such eigenvectors, which constitute a basis for $T_{p_0}\mathfrak{X}(\tilde{n}) \oplus \cdots \oplus T_{p_{j-1}}\mathfrak{X}(\tilde{n})_k$.

The sum of the exponents of this base is

$$2\sum_{i=0}^{j-1} -\frac{i\tilde{n}}{j} = -(j-1)\tilde{n}$$

We construct eigenvectors of $T_{[Z]}\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_k)$ by putting a vector such as the above one at the coordinates corresponding to an orbit and 0 at the other coordinates. Running through all the possible orbits, we get a base of eigenvectors of $T_{[Z]}\operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_k)$. Thus summing the exponents among all the eigenvectors will lead to the desired result for the conductor.

Concerning the order of the volume form, we have that

$$\begin{aligned}
\operatorname{ord}_{[Z]}\big(\omega^{[n]}(\tilde{n})\big) &= \sum_{p\in\operatorname{supp} Z} \operatorname{ord}_p\big(\omega(\tilde{n})\big) \\
&= \sum_{j=1}^{n} \frac{\tilde{n}}{j} \sum_{p\in\operatorname{supp} Z_j} \operatorname{ord}_p\big(\omega(j)\big) \\
&= \sum_{j=1}^{n} \tilde{n} \sum_{p\in Z_j/\mu_j} \operatorname{ord}_p\big(\omega(j)\big) \\
&= \tilde{n} \sum_{j=1}^{n} \operatorname{ord}_{Z_j/\mu_j}\big(\omega(j)^{[\alpha_j]}\big).
\end{aligned}$$

Where the first and the last equality follow from the fact that the stalk of the canonical bundle at a point with reduced support of an Hilbert scheme are the tensor products of the stalks of the canonical bundle of the surface at every point in the support. The second equality follows from Remark 9.1.9. The third equality follow from the fact that $\omega(j)$ has the same order on all the j points of an orbit of $\operatorname{Gal}(K(j)|K)$. ∎

9.2.5. We are able, now, to compute the order of $\omega^{[n]}$ at any point $z \in \mathfrak{X}_k^{[n]}$:

Corollary 9.2.6. *Let* $\pi\colon \mathfrak{X}_k^{[n]} \to \operatorname{Hilb}^n(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}})$ *and assume that*

$$\begin{aligned}
\pi(z) = &\big(\pi^1(z), \pi^2(z), \dots, \pi^n(z)\big) \\
&\in \operatorname{Hilb}^{\alpha_1}\big(Y(1)\big) \times \operatorname{Hilb}^{\alpha_2}\big(Y(2)/\operatorname{Gal}\big(K(2)|K\big)\big) \times\cdots\times \operatorname{Hilb}^{n}\big(Y(n)/\operatorname{Gal}\big(K(n)|K\big)\big).
\end{aligned}$$

We have that

$$\operatorname{ord}_z(\omega^{[n]}) = \sum_{j=1}^{n} \big(-(j-1)\alpha_j + \operatorname{ord}_{\pi^j(z)}\big(\omega(j)^{[\alpha_j]}\big)\big).$$

Proof. As a direct consequence of Lemma 4.4.6 and of the previous lemma we get:

$$\begin{aligned}\operatorname{ord}_z(\omega^{[n]}) &= \frac{\operatorname{ord}_{\pi(z)}\big(\omega(\tilde{n})^{[n]}\big) - c\big(\operatorname{Hilb}^n\big(\mathfrak{X}(\tilde{n})_{k,\mathrm{sm}}\big),[Z]\big)}{\tilde{n}} \\ &= \sum_{j=1}^{n}\big(-(j-1)\alpha_j + \operatorname{ord}_{\pi^j(z)}\big(\omega(j)^{[\alpha_j]}\big)\big).\end{aligned}$$

■

9.3. Motivic integral

9.3.1. We keep the convention on $\mathscr{R}$ being one of the three rings $\mathscr{M}_k^{\widehat{\mu}}$, $\mathscr{M}_k^{\widehat{\mu}}[(\mathbb{L}^r-1)^{-1}\colon 0<r\in\mathbb{N}]$ or $\widehat{\mathscr{M}_k}^{\widehat{\mu}}$; the same computations will hold, a fortiori, in the non-equivariant versions of $\mathscr{R}$. We are now ready to perform the main computation of the manuscript; by using the models we constructed above we are able to compute a generating function for the motivic integrals of all the Hilbert schemes of points of a surface with trivial canonical bundle. More precisely the formula we are going to prove is the content of the following proposition.

Theorem 9.3.2. *The following identity holds true in $\mathscr{R}[\![q]\!]$ if* $\operatorname{char} k=0$, *while it holds true in $\mathscr{R}[q]/(q^p)$ if* $\operatorname{char} k = p > 0$*:*

$$\sum_{n\ge 0}\left(\int_{\operatorname{Hilb}^n(X)}\omega^{[n]}q^n\right) = \prod_{m\ge 1}\left((1-\mathbb{L}^{m-1}q^m)^{-(\int_{X(m)}\omega(m))/\operatorname{Gal}(K(m)|K)}\right).$$

Corollary 9.3.3. *Assume that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > n$*, then the following equation holds:*

$$\int_{\operatorname{Hilb}^n(X)}\omega^{[n]} = \sum_{\alpha\dashv n}\prod_{j=1}^{\infty}\left(\mathbb{L}^{(j-1)\alpha_j}\operatorname{Sym}^{\alpha_j}\left(\left(\int_{X(j)}\omega(j)\right)/\operatorname{Gal}\big(K(j)|K\big)\right)\right).$$

Proof. Since

$$\begin{aligned}&\big(1-\mathbb{L}^{m-1}q^m\big)^{-(\int_{X(m)}\omega(m))/\operatorname{Gal}(K(m)|K)} \\ &\quad = \sum_{l=0}^{\infty}\mathbb{L}^{(m-1)l}\operatorname{Sym}^l\left(\left(\int_{X(m)}\omega(m)\right)/\operatorname{Gal}\big(K(m)|K\big)\right)q^{ml},\end{aligned}$$

and since, for all the sequences

$$\alpha = (\alpha_1,\alpha_2,\dots)$$

such that $\alpha_j=0$ for $j\gg 1$, one has that

$$\deg_q\left(\prod_{j=1}^{\infty}\left(\mathbb{L}^{(j-1)\alpha_j}\operatorname{Sym}^{\alpha_j}\left(\left(\int_{X(j)}\omega(j)\right)/\operatorname{Gal}\big(K(j)|K\big)\right)q^{j\alpha_j}\right)\right) = \sum_{j=1}^{\infty}j\alpha_j.$$

We get the desired result after identifying the coefficients of q^n from Theorem 9.3.2. ■

9.3.4. Before facing the theorem, let us introduce a piece of notation that will help facing the computation more smoothly: If $Z \to \operatorname{Spec} k$ is a scheme and $\Theta\colon Z \to \mathscr{M}_k^{\widehat{\mu}}$ is a locally constant function, we denote by

$$\int_Z \Theta(z)\,\mathrm{d}z := \sum_{\substack{C \subseteq Z \\ \text{connected component}}} [C]\Theta(p) \in \mathscr{M}_k^{\widehat{\mu}},$$

where the p in the sum above is an arbitrary point of the connected component C. We state a lemma that will be useful for the proof of the theorem:

Lemma 9.3.5. *Let $Y \to \operatorname{Spec} k$ be a smooth surface endowed with a locally constant function*

$$\nu\colon |Y| \to \mathscr{R}.$$

Suppose that functions

$$\nu^{[n]}\colon \left|\operatorname{Hilb}^n(Y)\right| \to \mathscr{R} \quad \text{and} \quad \nu'^{[n]}\colon \left|\operatorname{Sym}^n(Y)\right| \to \mathscr{R}$$

are defined in such a way that, for a given subscheme $Z \subseteq Y$ of length n we have

$$\nu^{[n]}(Z) = \prod_{p \in \operatorname{supp}(Z)} \nu(p)^{\operatorname{length}(\mathscr{O}_{Z,p})}$$

and $\nu^{[n]} = \nu'^{[n]} \circ p_n$, where

$$p_n\colon \operatorname{Hilb}^n(Y) \to \operatorname{Sym}^n(Y)$$

is the Hilbert–Chow morphism. Then, for an arbitrary natural number $a \in \mathbb{N}$, the following identity holds:

$$\int_{\operatorname{Hilb}^a(Y)} \nu^{[a]}(z)\,\mathrm{d}z = \sum_{\beta \dashv a} \left(\prod_{l \geq 1} \left(\mathbb{L}^{(l-1)\beta_l} \int_{\operatorname{Sym}^{\beta_l}(Y)} \nu'^{[l\beta_l]}(z)\,\mathrm{d}z \right)\right).$$

Proof. We first suppose that Y is connected and, thus, $\nu \equiv \lambda \in \mathscr{R}$ is constant. Thus we simply have that

$$\int_{\operatorname{Hilb}^a(Y)} \nu^{[a]}(z)\,\mathrm{d}z = \lambda^a \left[\operatorname{Hilb}^a(Y)\right].$$

It follows from a well known result, for instance [22, Section 2.2.3], that

$$\left[\operatorname{Hilb}^a(Y)\right] = \sum_{\beta \dashv a} \left(\prod_{l \geq 1} \left(\mathbb{L}^{(l-1)\beta_l} \left[\operatorname{Sym}^{\beta_l}(Y)\right]\right)\right),$$

thus we deduce the desired statement, at least when Y is connected.

Now suppose that $C \subseteq Y$ is a connected component and that the statement holds for $Y \backslash C$.

Using the fact that $\mathrm{Hilb}^a(Y) = \bigsqcup_{j=0}^{a}(\mathrm{Hilb}^{a-j}(Y\backslash C) \times \mathrm{Hilb}^j(C))$, we deduce that

$$\begin{aligned}
&\int_{\mathrm{Hilb}^a(Y)} \nu^{[a]}(z)\,\mathrm{d}z \\
&= \sum_{j=0}^{a}\left(\int_{\mathrm{Hilb}^{a-j}(Y\backslash C)} \nu^{[a-j]}(z)\,\mathrm{d}z\right)\cdot\left(\int_{\mathrm{Hilb}^j(C)} \nu^{[j]}\,\mathrm{d}z\right) \\
&= \sum_{j=0}^{a}\left(\sum_{\delta^j \dashv a-j}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\delta_l^j}\int_{\mathrm{Sym}^{\delta_l^j}(Y\backslash C)} \nu'^{l[\delta_l^j]}(z)\,\mathrm{d}z\right)\right)\right. \\
&\quad\left.\cdot\sum_{\gamma^j\dashv j}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\gamma_l^j}\int_{\mathrm{Sym}^{\gamma_l^j}(C)} \nu'^{l[\gamma_l^j]}(z)\,\mathrm{d}z\right)\right)\right) \\
&= \sum_{j=0}^{a}\left(\sum_{\substack{\delta^j\dashv a-j\\ \gamma^j\dashv j}}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)(\delta_l^j+\gamma_l^j)}\int_{\mathrm{Sym}^{\beta_l^j}(Y\backslash C)\times\mathrm{Sym}^{\gamma_l^j}(C)} \nu'^{l[\delta_l^j+\gamma_l^j]}(z)\,\mathrm{d}z\right)\right)\right) \\
&= \sum_{\beta\dashv a}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\beta_l}\sum_{i=0}^{\beta_l}\int_{\mathrm{Sym}^{\beta_l-i}(Y\backslash C)\times\mathrm{Sym}^i(C)} \nu'^{l[\beta_l]}(z)\,\mathrm{d}z\right)\right) \\
&= \sum_{\beta\dashv a}\left(\prod_{l\geq 1}\left(\mathbb{L}^{(l-1)\beta_l}\int_{\mathrm{Sym}^{\beta_l}(Y)} \nu'^{l[\beta_l]}(z)\,\mathrm{d}z\right)\right),
\end{aligned}$$

which concludes the proof. ■

Proof of Theorem 9.3.2. We begin our computation using some identities we proved in the previous section.

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n\right) = \sum_{n\geq 0}\left(\int_{\mathfrak{X}_k^{[n]}} \mathbb{L}^{-\mathrm{ord}_z(\omega^{[n]})}\,\mathrm{d}z q^n\right),$$

by decomposing $\mathfrak{X}_k^{[n]}$ as union of its strata $\{\mathfrak{X}_{k,\alpha}^{[n]}\}_{\alpha\dashv n}$, we get:

$$\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n\right) = \sum_{n\geq 1}\left(\sum_{\alpha\dashv n}\left(\int_{\mathfrak{X}_{k,\alpha}^{[n]}} \mathbb{L}^{-\mathrm{ord}_z(\omega^{[n]})}\,\mathrm{d}z\right) q^n\right),$$

by Corollary 9.2.6 we obtain:

$$\begin{aligned}
&\sum_{n\geq 0}\left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n\right) \\
&= \sum_{n\geq 1}\left(\sum_{\alpha\dashv n}\left(\int_{\mathfrak{X}_{k,\alpha}^{[n]}} \prod_{j\geq 1}\left(\mathbb{L}^{(j-1)\alpha_j - \mathrm{ord}_{\pi^j(z)}(\omega^{[\alpha_j]}(j))} q^{j\alpha_j}\right)\mathrm{d}z\right)\right)
\end{aligned}$$

$$= \sum_{\alpha \in \mathbb{N}^{\oplus \mathbb{N}^*}} \left(\prod_{j \geq 1} \left(\mathbb{L}^{(j-1)\alpha_j} q^{j\alpha_j} \int_{\mathrm{Hilb}^{\alpha_j}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(j)^{[\alpha_j]})} \, \mathrm{d}z \right) \right)$$
$$= \prod_{j \geq 1} \left(\sum_{\alpha_j \geq 0} \left(\mathbb{L}^{(j-1)\alpha_j} q^{j\alpha_j} \int_{\mathrm{Hilb}^{\alpha_j}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(j)^{[\alpha_j]})} \, \mathrm{d}z \right) \right).$$

We plug the lemma above in our chain of equalities using $Y = Y(j)/\mathrm{Gal}(K(j)|K)$, $\nu = \mathbb{L}^{-\mathrm{ord}_z(\omega(j))}$, recalling also that $l \,\mathrm{ord}_z(\omega(j)) = \mathrm{ord}_z(\omega(jl))$ and that $Y(j)$ naturally embeds in the central fibre of some weak Néron model of $X(lj)$ (as in Remark 9.1.9 with $d = \tilde{n}/lj$), we get:

$$\sum_{n \geq 0} \left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n \right)$$
$$= \prod_{j \geq 1} \left(\sum_{\alpha_j \geq 0} \left(\mathbb{L}^{(j-1)\alpha_j} \sum_{\beta^j \dashv \alpha_j} \left(q^{j\alpha_j} \cdot \prod_{l \geq 1} \left(\mathbb{L}^{(l-1)\beta_l^j} \int_{\mathrm{Sym}^{\beta_l^j}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(lj)^{[\beta_l^j]})} \, \mathrm{d}z \right) \right) \right) \right)$$
$$= \prod_{j \geq 1} \left(\sum_{\beta^j \in \mathbb{N}^{\oplus \mathbb{N}^*}} \left(\prod_{l \geq 1} \left(\mathbb{L}^{(jl-1)\beta_l^j} q^{jl\beta_l^j} \int_{\mathrm{Sym}^{\beta_l^j}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(lj)^{[\beta_l^j]})} \, \mathrm{d}z \right) \right) \right)$$
$$= \prod_{j,l \geq 1} \left(\sum_{\beta_l^j \geq 0} \left(\mathbb{L}^{(jl-1)\beta_l^j} q^{jl\beta_l^j} \int_{\mathrm{Sym}^{\beta_l^j}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(lj)^{[\beta_l^j]})} \, \mathrm{d}z \right) \right). \tag{9.1}$$

Recalling that, in the sum above, there is only a finite number of nonvanishing coefficients of q^n, for every positive integer n, we are allowed to group such summands in a different order; since the map

$$\mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^* \times \mathbb{N}^*$$
$$(j, l) \mapsto (j \cdot l, j)$$

is injective and its image is $\{(m, j) \colon j | m\}$, after the substitution $\lambda_j^m := \beta_l^j$, we get the equivalent expression:

$$\sum_{n \geq 0} \left(\int_{\mathrm{Hilb}^n(X)} \omega^{[n]} q^n \right)$$
$$= \prod_{m \geq 1} \left(\prod_{j | m} \left(\sum_{\lambda_j^m \geq 0} \left(\mathbb{L}^{(m-1)\lambda_j^m} q^{m\lambda_j^m} \int_{\mathrm{Sym}^{\lambda_j^m}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(m)^{[\lambda_j^m]})} \, \mathrm{d}z \right) \right) \right)$$
$$= \prod_{m \geq 1} \left(\sum_{\lambda^m \in \mathbb{N}^{\mathrm{Div}(m)}} \left((\mathbb{L}^{(m-1)} q^m)^{\sum_{j|m} \lambda_j^m} \cdot \prod_{j|m} \left(\int_{\mathrm{Sym}^{\lambda_j^m}(Y(j)/\mathrm{Gal}(K(j)|K))} \mathbb{L}^{-\mathrm{ord}_z(\omega(m)^{[\lambda_j^m]})} \, \mathrm{d}z \right) \right) \right)$$

$$
\begin{aligned}
&= \prod_{m\geq 1}\Bigg(\sum_{r_m\geq 0}\Bigg(\big(\mathbb{L}^{(m-1)}q^m\big)^{r_m}\int_{\mathrm{Sym}^{r_m}((\sqcup_{j|m}Y(j))/\operatorname{Gal}(K(m)|K))}\mathbb{L}^{-\operatorname{ord}_z(\omega(m)^{[r_m]})}\,\mathrm{d}z\Bigg)\Bigg)\\
&= \prod_{m\geq 1}\Bigg(\sum_{r_m\geq 0}\Bigg(\big(\mathbb{L}^{(m-1)}q^m\big)^{r_m}\operatorname{Sym}^{r_m}\Bigg(\int_{(\sqcup_{j|m}Y(j))/\operatorname{Gal}(K(m)|K)}\mathbb{L}^{-\operatorname{ord}_z(\omega(m))}\,\mathrm{d}z\Bigg)\Bigg)\Bigg)\\
&= \prod_{m\geq 1}\Bigg(\sum_{r_m\geq 0}\Bigg(\big(\mathbb{L}^{(m-1)}q^m\big)^{r_m}\operatorname{Sym}^{r_m}\Big(\big(\textstyle\int_{X(m)}\omega(m)\big)/\operatorname{Gal}(K(m)|K)\Big)\Bigg)\Bigg)\\
&= \prod_{m\geq 1}\Bigg((1-\mathbb{L}^{m-1}q^m)^{-(\int_{X(m)}\omega(m))/\operatorname{Gal}(K(m)|K)}\Bigg). \qquad \blacksquare
\end{aligned}
$$

9.3.6. Applying this identity to every coefficient of the zeta function $Z_{X,\omega}(T)$, we obtain a formula for the motivic zeta function for its Hilbert schemes of points:

Theorem 9.3.7. *Assume that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > n$*, then the following equation holds:*

$$
Z_{\mathrm{Hilb}^n(X),\omega^{[n]}} = \sum_{\alpha\dashv n}\prod_{j=1}^{\infty}\big(\mathbb{L}^{(j-1)\alpha_j}\operatorname{Sym}^{\alpha_j}\big(Z_{X(j),\omega(j)}/\operatorname{Gal}\big(K(j)|K\big)\big)\big). \tag{9.2}
$$

Proof. It follows after an application of Corollary 9.3.3 to each coefficient of the zeta function. ∎

10. Proof of the monodromy property

10.1. Poles of the zeta function

10.1.1. Throughout this section, we denote by $\overline{\mathscr{R}}$ one of the two rings $\mathscr{M}_k^{\hat{\mu}}[(\mathbb{L}^r-1)^{-1}:0<r\in\mathbb{N}]$ or $\widehat{\mathscr{M}}_k^{\hat{\mu}}$, while $\mathscr{R}$ will denote either $\overline{\mathscr{R}}$ or $\mathscr{M}_k^{\hat{\mu}}$. We fix an element $\sigma\in\operatorname{Gal}(\overline{K}|K)$ such that $\sigma|_{K^{\mathrm{tame}}}$ is a topological generator of the tame quotient, so that, throughout this section, the notion of monodromy is interpreted in terms of this choice.

The aim of this section is to study the poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$ in terms of those of $Z_{X,\omega}$ and deduce the following:

Theorem 10.1.2 (Monodromy property for Hilbert schemes). *Let X be a surface with trivial canonical bundle satisfying the monodromy property in $\overline{\mathscr{R}}$. If* $\operatorname{char} k = 0$*, then also* $\mathrm{Hilb}^n(X)$ *satisfies the monodromy property, for $n\in\mathbb{N}$. If* $\operatorname{char} k = p > 0$ *and X admits a model as in Section 9.1, then the monodromy property in $\overline{\mathscr{R}}$ holds for* $\mathrm{Hilb}^n(X)$*, for* $n < \operatorname{char} k$.

Remark 10.1.3. As the computations in the previous section hold also in the non equivariant versions of the Grothendieck rings, also this theorem can be stated in a non-equivariant fashion; in particular, should X be a surface with the monodromy property in a non-equivariant context, but not its corresponding equivariant version, then we infer that also

$\mathrm{Hilb}^n(X)$ has the non-equivariant monodromy property (without any claims for its equivariant version).

10.1.4. Despite not being the aim of our discussion, we report here the following statement, which can be obtained as a byproduct of the arguments we have developed so far:

Proposition 10.1.5. *Let Y, Z be two Calabi–Yau varieties endowed with volume forms ω_1, ω_2 satisfying the monodromy property in $\overline{\mathscr{R}}$. Let ω be the volume form on $Y \times_K Z$ defined as $\omega := \mathrm{pr}_Y^* \omega_1 \wedge \mathrm{pr}_Z^* \omega_2$. Then also $Y \times_K Z$, endowed with the volume form ω, satisfies the monodromy property in $\overline{\mathscr{R}}$.*

10.1.6. For an arbitrary positive integer $l > 0$ we have that

$$l \cdot Z_{X(l),\omega(l)}(T^l) = \sum_{i=0}^{l-1} Z_{X,\omega}(\zeta_l^i T),$$

where we consider the functions as power series with coefficient in an algebraic extensions of $\mathscr{R}$ containing the l-th roots of unity (though after the due cancellations, the above equation involves only elements of $\mathscr{R}$). Thus, by writing $Z_{X,\omega}(T)$ in the form of equation (6.1), with N divisible by l, we see that the set of poles of $Z_{X(l),\omega(l)}(T)$ is contained in $l \cdot \mathcal{P}$.

10.1.7. Using this observation and the results from Section 7.3 we get an upper bound on the set of poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$:

Corollary 10.1.8. *Let X be a surface with trivial canonical bundle and ω a volume form on it. Assume that $Z_{X,\omega}(T) \in \mathscr{R}[\![T]\!]$ can be written as a sum of functions with only one pole. Let $\mathcal{P}$ be the set of poles of $Z_{X,\omega}$. Then all the poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$ are contained in $\Sigma^n \mathcal{P}$.*

Proof. Let us write $Z_{X(j),\omega(j)}/\operatorname{Gal}(K(j)|K)(T) = \sum_{i \geq 0} A_i^{(j)} T^i$. For each $\alpha \dashv n$, let

$$F_\alpha(T) := \sum_{i>0} \big(\mathrm{Sym}^{\alpha_1} A_i^{(1)}\big) \cdots \big(\mathrm{Sym}^{\alpha_n} A_i^{(n)}\big) T^i.$$

According to equation (9.1), we have that $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T) = \sum_{\alpha \dashv n} \mathbb{L}^{n-|\alpha|} F_\alpha(T)$, thus we only need to prove that F_α has only poles inside $\Sigma^n \mathcal{P}$. Lemma 7.3.15 implies that $(\mathrm{Sym}^{\alpha_j} Z_{X(j),\omega(j)}/\operatorname{Gal}(K(j)|K))(T)$ has poles in $\Sigma^{\alpha_j}(j\mathcal{P}) \subseteq \Sigma^{j\alpha_j}\mathcal{P}$; our statement follows from Proposition 7.3.13 and from the identity

$$\Sigma^{\alpha_1}\mathcal{P} + \Sigma^{2\alpha_2}\mathcal{P} + \cdots + \Sigma^{n\alpha_n}\mathcal{P} = \Sigma^n \mathcal{P}.$$ ∎

10.1.9. We are now ready to prove Theorem 10.1.2:

Proof of Theorem 10.1.2. Let q be a pole of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$. Consider poles $q_1, \ldots, q_n$ of $Z_{X,\omega}(T)$ such that $q = q_1 + \cdots + q_n$. Since X has the monodromy property, there are elements $v_1, v_2, \ldots, v_n \in H^*(X_{\overline{K}}, \mathbb{Q}_l)$ such that $\sigma(v_j) = e^{2\pi i q_j} v_j$. Let us consider the Galois-equivariant isomorphism from [8, Theorem 2]:

$$H^*\big(\mathrm{Hilb}^n(X), \mathbb{Q}_l\big) \cong \bigoplus_{\alpha \dashv n} H^*\big(\mathrm{Sym}^{|\alpha|}(X), \mathbb{Q}_l\big)\big(n - |\alpha|\big);$$

focusing on the summand $H^*(\mathrm{Sym}^n(X), \mathbb{Q}_l) \cong H^*(X^n, \mathbb{Q}_l)^{\Sigma_n}$, where the action of Σ_n on $H^*(X^n, \mathbb{Q}_l) \cong H^*(X, \mathbb{Q}_l)^{\otimes n}$ is induced by the usual action $\Sigma_n \curvearrowright X^n$ given by permutation of the factors. Thus the element

$$\sum_{\rho \in \Sigma_n} v_{\rho(1)} \otimes \cdots \otimes v_{\rho(n)}$$

is a non-zero eigenvector of $H^*(\mathrm{Hilb}^n(X), \mathbb{Q}_l)$ for the eigenvalue $\prod_{j=1}^n e^{2\pi i q_j}$. ■

10.1.10. And similarly:

Proof of Proposition 10.1.5. We have that

$$\int_{Y(n)\times Z(n)} \omega(n) = \Big(\int_{Y(n)} \omega_1(n)\Big)\Big(\int_{Z(n)} \omega_2(n)\Big).$$

Hence Proposition 7.3.13 implies that for any pole q of $Z_{Y\times Z,\omega}(T)$ there are a pole q_1 of $Z_{Y,\omega_1}(T)$ and a pole q_2 of $Z_{Z,\omega_2}(T)$ such that $q = q_1 + q_2$.

Since Y and Z satisfy the monodromy property, there are nonzero eigenvectors $v \in H^*(Y, \mathbb{Q}_l)$ with eigenvalue $\exp(2\pi i q_1)$ and $w \in H^*(Z, \mathbb{Q}_l)$ with eigenvalue $\exp(2\pi i q_2)$, so that the element $v \otimes w \in H^*(Y, \mathbb{Q}_l) \otimes H^*(Z, \mathbb{Q}_l) \cong H^*(Y \times Z, \mathbb{Q}_l)$ is an eigenvector with eigenvalue $\exp(2\pi i q)$. ■

10.1.11. We are not able to say much about the monodromy property in $\mathscr{M}_k^{\hat{\mu}}$ for all the Hilbert schemes of points on a surface, since it is not always possible to write $Z_{X,\omega}(T) \in \mathscr{M}_k^{\hat{\mu}}[\![T]\!]$ as a sum of functions with a single pole. However, there are a few remarkable classes of surfaces whose zeta function has a unique pole. In these cases, such a condition is automatically satisfied, so also $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}$ has a unique pole which is n times the pole of $Z_{X,\omega}$ and $\mathrm{Hilb}^n(X)$ will then satisfy the monodromy property.

Example 10.1.12. We list a few classes of surfaces satisfying the property above:

- Assume that X is an abelian surface; according to [10], $Z_{X,\omega}$ has a unique pole which coincides with Chai's basechange conductor of X;
- If $X \to \mathrm{Spec}\, K$ is a $K3$ surface admitting an equivariant Kulikov model after a finite base change with respect to a finite extension F/K, then Halle and Nicaise proved in [11] that $Z_{X,\omega}$ has a unique pole;
- Assume that X is a Kummer surface constructed from an abelian surface A; then Overkamp proved in [20] that $Z_{X,\omega}$ has a unique pole.

Moreover all the surfaces in this list satisfy the monodromy property.

Corollary 10.1.13. *Let X be a surface in the list above, then the monodromy property holds for* $\mathrm{Hilb}^n(X)$*, provided that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > n$*, with the usual assumptions on the models of* X*.*

Similarly, the monodromy property holds for a product $X_1 \times \cdots \times X_n$, where all the X_i are surfaces in the list above.

Proof. The first statement follows from Theorem 10.1.2, while the latter follows from Proposition 10.1.5. ■

10.2. Further remarks

10.2.1. It is still an open question whether all the sums of n poles of $Z_{X,\omega}(T)$ are actually poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$, or if cancellation might occur. It is reasonable to expect that given a very general $K3$ surface X and a positive integer n, if $\mathcal{P}$ is the set of poles of $Z_{X,\omega}$, then the set of poles of $Z_{Hilb^n(X),\omega^{[n]}}$ coincides with $\Sigma^n\mathcal{P}$. We expect this in virtue of the fact that the expression for $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}$, obtained by following the algorithm of Corollary 10.1.8 and Section 7.3, contains terms having a pole in each element of $\Sigma^n\mathcal{P}$ and the cancellation among them "should happen only exceptionally".

Example 10.2.2. Let $K := k((t))$, $R := k[\![t]\!]$, where $\operatorname{char} k = 0$. Let $X \subseteq \mathbb{P}^3_K$ (with homogeneous coordinates $[w : x : y : z]$) be the surface defined by the quartic polynomial:

$$w^2x^2 + w^2y^2 + w^2z^2 + x^4 + y^4 + z^4 + tw^4, \tag{10.1}$$

and let, finally, ω be an arbitrary volume form over it; this example was already studied in [11]. It is possible to prove that $Z_{X,\omega}$ has two poles and satisfies the monodromy property in $\mathscr{M}_k$. A direct computation (relying on a construction we will sketch later) shows that the poles of $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}$ are actually the three expected poles. Let $\mathfrak{Y} \subseteq \mathbb{P}^3_R$ the model of X obtained by equation (10.1) considered as a polynomial with coefficients in R. The model constructed in this way is a regular model whose central fibre is an irreducible surface with only a singular point O of type A_1. After blowing up $O \in \mathfrak{Y}$ one obtains a regular model with strict normal crossing divisor, whose central fibre consists of two components: a regular $K3$ surface (the strict transform of $\mathfrak{Y}_k$) and a copy of $\mathbb{P}^2_k$ with multiplicity 2, their intersection is a rational curve which sits in $\mathbb{P}^2$ as a conic. After semistable reduction, one gets a model $\mathfrak{X}(2) \to R(2)$ of $X(2)$ whose central fibre consists of a smooth $K3$ surface intersecting $\mathbb{P}^1 \times_k \mathbb{P}^1$ along its diagonal. Using the construction of Nagai in [16] one gets a semistable model for $\mathrm{Hilb}^2(X(2))$ over $R(2)$ and after basechanging such model and Weil-restricting it is possible to compute the motivic integral of $\mathrm{Hilb}^2(X(m))$ for all $m \in \mathbb{N}$ and thus $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}} \in \mathscr{M}_k[\![T]\!]$. After specializing the zeta function using the Poincaré polynomial one sees that all the three possible poles are indeed poles for $Z_{\mathrm{Hilb}^2(X),\omega^{[2]}}$.

10.2.3. The proof of rationality of the motivic zeta function relies on the existence of a log-smooth model of any Calabi–Yau variety. In particular, if $\operatorname{char} k = p > 0$, it is not known whether all the Calabi–Yau varieties have a rational zeta function. In our case, by determining an upper bound of the set of poles of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T)$, we proved that for a surface X admitting a log smooth model (with further assumptions described in Section 9), then for $n < p$, the rationality of $Z_{\mathrm{Hilb}^n(X),\omega^{[n]}}(T) \in \widehat{\mathscr{M}}_k[\![T]\!]$ follows independently on the existence of a log-smooth model of $\mathrm{Hilb}^n(X)$.

Acknowledgements. The author is very grateful to his PhD advisor, Prof. Lars Halle, for introducing him to the problem and for many interesting ideas and conversations. He is also grateful to Prof. Johannes Nicaise for many helpful conversations. The author thanks the anonymous referees for their comments that helped improving this paper.

Funding. This work was partially supported by the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 801199.

References

[1] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.* **18** (1983), no. 4, 755–782 (1984) Zbl 0537.53056 MR 730926

[2] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*. Ergeb. Math. Grenzgeb. (3) 21, Springer, Berlin, 1990 Zbl 0705.14001 MR 1045822

[3] E. Bultot, *Motivic integration and Logarithmic geometry*. Ph.D. thesis, Katholieke Universiteit Leuven, 2015

[4] E. Bultot and J. Nicaise, Computing motivic zeta functions on log smooth models. *Math. Z.* **295** (2020), no. 1-2, 427–462 Zbl 1476.14035 MR 4100007

[5] J. Denef and F. Loeser, Geometry on arc spaces of algebraic varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, pp. 327–348, Progr. Math. 201, Birkhäuser, Basel, 2001 Zbl 1079.14003 MR 1905328

[6] B. Edixhoven, Néron models and tame ramification. *Compos. Math.* **81** (1992), no. 3, 291–306 Zbl 0759.14033 MR 1149171

[7] O. Gabber and L. Ramero, *Almost ring theory*. Lecture Notes in Math. 1800, Springer, Berlin, 2003 Zbl 1045.13002 MR 2004652

[8] L. Göttsche and W. Soergel, Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Math. Ann.* **296** (1993), no. 2, 235–245 Zbl 0789.14002 MR 1219901

[9] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points. *Michigan Math. J.* **54** (2006), no. 2, 353–359 Zbl 1122.14003 MR 2252764

[10] L. H. Halle and J. Nicaise, Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Adv. Math.* **227** (2011), no. 1, 610–653 Zbl 1230.11076 MR 2782205

[11] L. H. Halle and J. Nicaise, Motivic zeta functions of degenerating Calabi–Yau varieties. *Math. Ann.* **370** (2018), no. 3-4, 1277–1320 Zbl 1400.14045 MR 3770167

[12] A. Hartmann, On rational points of varieties over local fields having a model with tame quotient singularities. *Trans. Amer. Math. Soc.* **367** (2015), no. 11, 8199–8227 Zbl 1349.14095 MR 3391914

[13] A. Hartmann, Equivariant motivic integration on formal schemes and the motivic zeta function. *Comm. Algebra* **47** (2019), no. 4, 1423–1463 Zbl 1441.14155 MR 3975946

[14] A. Jaspers, The global monodromy property for $K3$ surfaces allowing a triple-point-free model. *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 2, 200–204 Zbl 1360.14100 MR 3612709

[15] F. Loeser and J. Sebag, Motivic integration on smooth rigid varieties and invariants of degenerations. *Duke Math. J.* **119** (2003), no. 2, 315–344 Zbl 1078.14029 MR 1997948

[16] Y. Nagai, On monodromies of a degeneration of irreducible symplectic Kähler manifolds. *Math. Z.* **258** (2008), no. 2, 407–426 Zbl 1140.14008 MR 2357645

[17] J. Nicaise, Geometric criteria for tame ramification. *Math. Z.* **273** (2013), no. 3-4, 839–868 Zbl 1329.11067 MR 3030680

[18] J. Nicaise and J. Sebag, Motivic Serre invariants, ramification, and the analytic Milnor fiber. *Invent. Math.* **168** (2007), no. 1, 133–173 Zbl 1136.14010 MR 2285749

[19] J. Nicaise and C. Xu, Poles of maximal order of motivic zeta functions. *Duke Math. J.* **165** (2016), no. 2, 217–243 Zbl 1366.14008 MR 3457672

[20] O. Overkamp, Degeneration of Kummer surfaces. *Math. Proc. Cambridge Philos. Soc.* **171** (2021), no. 1, 65–97 Zbl 1483.14044 MR 4268804

[21] Z. Qu, *Toric schemes over a discrete valuation ring and tropical compactifications*. Ph.D. thesis, The University of Texas at Austin, 2009

[22] A. T. Ricolfi, Local contributions to Donaldson-Thomas invariants. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 19, 5995–6025 Zbl 1423.14325 MR 3867400

[23] J. Wang, *Equivariant resolution of singularities and semi-stable reduction in characteristic* 0. Ph.D. thesis, Massachusetts Institure of Technology, 1997

Communicated by Takeshi Saito

Received 30 March 2021; revised 3 May 2023.

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